Arindama Singh

MA1101 Classnotes
Functions of Several Variables

Contents

Syllabus

Functions of two and three variables: Regions in plane, level curves and level surfaces, limit, continuity, partial derivatives, directional derivatives and gradient, normal to level curves and tangents, extreme values, Lagrange multipliers, double integral and iterated integral, volume of solids of revolution, approximation of volume, triple integral, change of variables, multiple integrals in cylindrical and spherical coordinates. [T]: Ch 14, §6.1-6.2, Ch. 15.

Vector calculus: Gradient, Divergence, Curl, Line integral, conservative fields, Green's theorem, surface area of solids of revolution, surface area, surface integral, Triple integrals and Gauss Divergence theorem, Stokes' theorem. [T]: §6.5, Ch. 16.

Text:
[T] G.B. Thomas Jr., M.D. Weir and J.R. Hass, Thomas Calculus, Pearson Education, 2009.

References:

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2. N. Piskunov, Differential and Integral Calculus Vol. 1-2, Mir Publishers, 1974.
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1

Differential Calculus

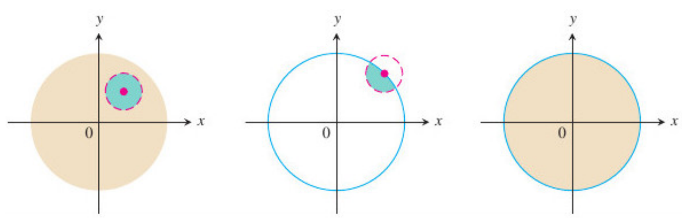
1.1 Regions in the plane and Surfaces

Most of the results we discuss will not hold for arbitrary subsets of $\mathbb{R}^2$, the plane. We thus distinguish special types of subsets of the plane by requiring that certain properties are satisfied.
The **distance** between any two points (a, b) and (c, d) in $\mathbb{R}^2$ is

$$|(a, b) - (c, d)| := \sqrt{(a - c)^2 + (b - d)^2}.$$

Let D be a subset of $\mathbb{R}^2$ and let $(a, b) \in \mathbb{R}^2$ be any point.

Lat $\epsilon > 0$. An ϵ**-disk** around (a, b) is the set of all points $(x, y) \in \mathbb{R}^2$ whose distance from (a, b) is less than ϵ. Such a disk is also called a disk with center as (a, b) and radius ϵ.



$\{(x, y) : x^2 + y^2 < 1\}$ $\{(x, y) : x^2 + y^2 < 1\}$ $\{(x, y) : x^2 + y^2 < 1\}$

D is a **bounded subset** of $\mathbb{R}^2$ iff D is contained in an ϵ-disk around some point of $\mathbb{R}^2$.

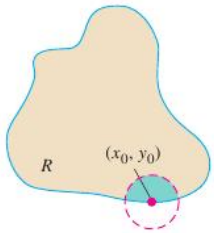
$(a, b) \in \mathbb{R}^2$ is **a boundary point** of D iff *every* ϵ-disk around (a, b) contains points from D *and* points not from D.

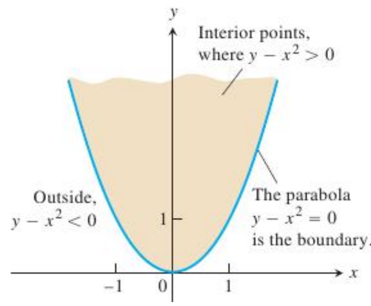
The set of all boundary points of D is called the **boundary** of D, and is written as ∂D. The set $\overline{D} = D \cup \partial D$ is called the **closure** of D.

D is called an **open set** iff D does not contain any of its boundary points.

D is called a **closed set** iff D contains all its boundary points.

$(a, b) \in D$ is **an interior point** of D iff *some* ϵ-disk around (a, b) is contained in D.





$(a, b) \in \mathbb{R}^2$ is called a **limit point** of D iff each ϵ-disk around (a, b) contains a point of $D \setminus \{(a, b)\}$.

$(a, b) \in \mathbb{R}^2$ is called an **isolated point** of D iff (a, b) is the only point of D that is contained in *some* ϵ-disk around (a, b). Thus a point $(c, d) \in D$ is an isolated point iff it is not a limit point of D.

D is called **path connected** if any two points in D can be joined by a polygonal line entirely lying in D.

D is called a **region** in the plane iff D is path-connected, it contains an ϵ-disk around some point of D, and each point of D is a limit point of D. Thus, a region does not contain an isolated point.

D is called an **open region** iff D is a region and D is an open set.

D is called a **closed region** iff D is a region and D is a closed set.

(1.1) *Theorem*

Let D be a region in the plane. Then $\overline{D}$ = the set of all limit points of D.

Proof. Denote the set of all limit points of D by D_L.

Let $(a, b) \in \overline{D}$. If $(a, b) \in D$, then since D is a region, (a, b) is a limit point of D. If $(a, b) \notin D$, then it is a boundary point of D. Now, any ϵ-disk around (a, b) contains a point from D, and such a point is different from (a, b). That is, (a, b) is a limit point of D. In any case, $(a, b) \in D_L$. Therefore, $\overline{D} \subseteq D_L$.

Conversely, let $(a, b) \in D_L$. Suppose $(a, b) \notin D$. Since (a, b) is a limit point of D, any ϵ-disk around (a, b) contains a point from $D \setminus \{(a, b)\}$. Such a disk also contains (a, b), which is outside D. So, (a, b) is a boundary point of D. Hence, $(a, b) \in D \cup \partial D = \overline{D}$. Therefore, $D_L \subseteq \overline{D}$. ∎

Remark 1.2 Let $D \subseteq \mathbb{R}^2$ and let $(a, b) \in \mathbb{R}^2$. Then the following can be proved:

1. Every ϵ-disk around (a, b) is an open set. The boundary of such a disk is the circle with center as (a, b) and radius ϵ.
2. If (a, b) is a limit point of D, then each ϵ-disk around (a, b) contains infinitely many points of D.

3. A finite set has no limit points.
4. If a boundary point of D is not in D, then it is a limit point of D.
5. If a limit point of D is not in D, then it is a boundary point of D.
6. D is open iff $\mathbb{R}^2 \setminus D$ is closed iff $D =$ the set of all interior point of D.
7. D is closed iff $\overline{D} = D$ iff $\mathbb{R}^2 \setminus D$ is open iff D contains all its limit points.
8. If D is a region, then D is closed iff $D = \overline{D} =$ the set of all limit points of D.

We say that D *has* a limit point to mean that there exists a point $(a, b) \in \mathbb{R}^2$ which is a limit point of D. It does not mean that such a limit point (a, b) is in D.

Let D be a region in the plane. Let $f : D \to \mathbb{R}$ be a function.

The **graph** of f is $\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y),\ (x, y) \in D\}$.

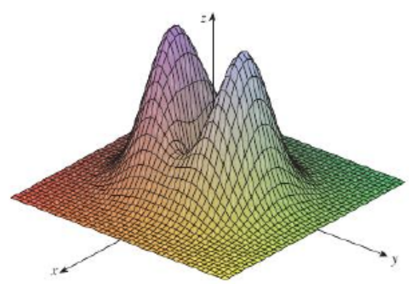
The graph here is also called the **surface** $z = f(x, y)$.

The **domain** of f is D.

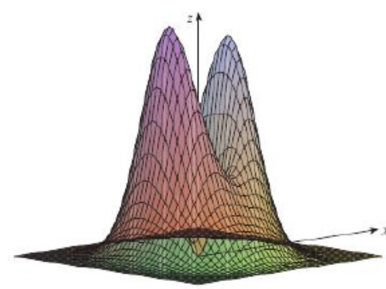
The **co-domain** of f is $\mathbb{R}$.

The **range** of f is $\{z \in \mathbb{R} : z = f(x, y)$ for some $(x, y) \in D\}$.

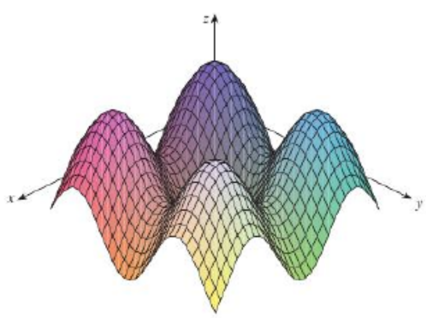
Some examples of surfaces are here:



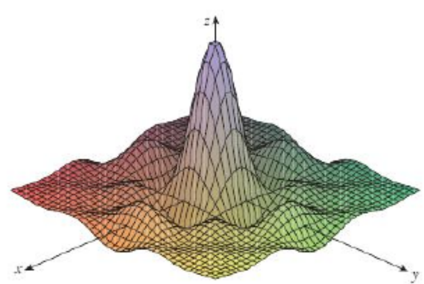
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(c) $f(x, y) = \sin x + \sin y$



(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

Sometimes, we do not fix the domain D of f but ask you to find it. For instance, The function $f(x, y) = \sqrt{y - x^2}$ has domain $D = \{(x, y) : x^2 \le y\}$. Its range is the set of all non-negative reals.

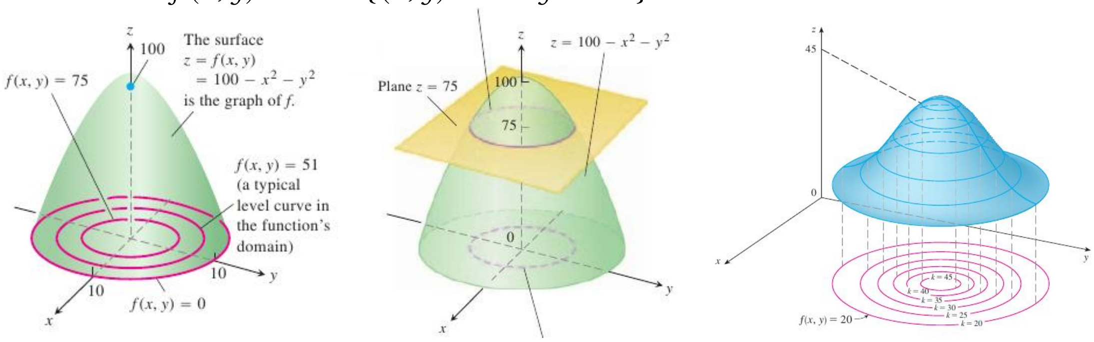
Let $f(x, y)$ be a (real valued) function of two variables. That is, $f : D \to \mathbb{R}$, where D is a region in $\mathbb{R}^2$.

A **contour curve** of f is the curve of intersection of the surface $z = f(x, y)$ and the plane $z = c$ for some constant c in the range of f. It is the curve $f(x, y) = c$ for some constant c in the range of f. Thus, the union of all contour curves is the surface $z = f(x, y)$; it is the graph of f.

A **level curve** of f is the set of points (x, y) in the domain of f for which $f(x, y) = c$ for some constant c in the range of f. Thus, a level curve is the projection of a contour curve on the xy-plane. Thus, the union of all level curves of the surface $z = f(x, y)$ is the domain of f.

(1.3) ***Example***

Consider the function $f(x, y) = 100 - x^2 - y^2$. Its domain is $\mathbb{R}^2$. Its range is the interval $(-\infty, 100]$. The level curve $f(x, y) = 0$ is $\{(x, y) : x^2 + y^2 = 100\}$. The level curve $f(x, y) = 51$ is $\{(x, y) : x^2 + y^2 = 49\}$.



$\square$

Similarly, for a function $f(x, y, z)$ of three variables, the *level surfaces* are the sets of points (x, y, z) such that $f(x, y, z) = c$ for values c in the range of f.

1.2 Limit

Let $f : D \to \mathbb{R}$ be a function, where D is a region in the plane, $(a, b) \in \overline{D}$ and let L be a real number.

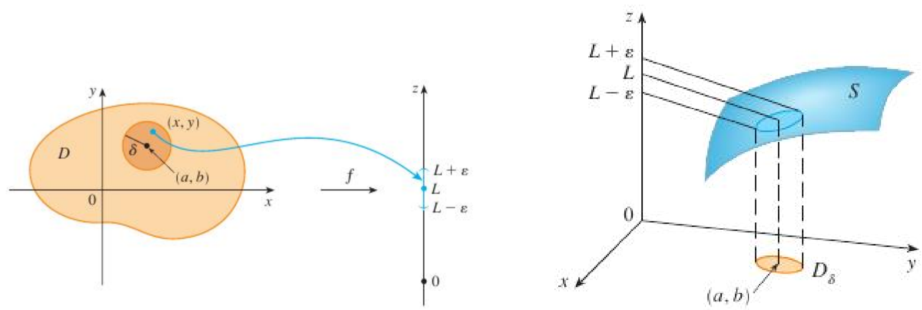
The **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L iff corresponding to each $\epsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \in D$ with $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, we have $|f(x, y) - L| < \epsilon$.

In this case, we write $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ and say that L is the limit of f at (a, b).

If for some real number L, the above happens, we say that **limit of f at (a, b) exists.** If for no real number L, the above happens, then **limit of f at (a, b) does not exist.**

The intuitive understanding of the notion of limit is as follows:

> The distance between $f(x, y)$ and L can be made arbitrarily small by making the distance between (x, y) and (a, b) sufficiently small but not necessarily zero.



When limit exists, we write it in many alternative ways:

$$\text{The limit of } f(x, y) \text{ as } (x, y) \text{ approaches } (a, b) \text{ is } L.$$

$$f(x, y) \to L \text{ as } (x, y) \to (a, b).$$

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L.$$

It is often difficult to show that limit of a function does not exist at a point. We will come back to this question soon.

(1.4) ***Example***

Determine if $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}$ exists.

Observe that the domain D of f is $\mathbb{R}^2 \setminus \{(0, 0)\}$. And $f(0, y) = 0$ for $y \neq 0$; $f(x, 0) = 0$ for $x \neq 0$. We guess that the limit would be 0. To see that it is the case, we start with any $\epsilon > 0$. We want to choose a $\delta > 0$ such that the following sentence becomes true:

$$\text{If } 0 < \sqrt{x^2 + y^2} < \delta, \text{ then } \left|\frac{4xy^2}{x^2 + y^2}\right| < \epsilon.$$

Since $|y^2| = y^2 \le x^2 + y^2$ and $|x^2| = x^2 \le x^2 + y^2$, we have

$$\left|\frac{4xy^2}{x^2 + y^2}\right| \le 4|x| \le 4\sqrt{x^2 + y^2}.$$

So, we choose $\delta = \epsilon/4$. Let us verify whether our choice is all right.

Assume that $0 < \sqrt{x^2+y^2} < \delta$. Then

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \le 4\sqrt{x^2+y^2} < 4\delta = \epsilon.$$

Hence $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$ ∎

(1.5) Observation

Suppose we have obtained a δ corresponding to some ϵ. If we take ϵ_1 which is larger than the earlier ϵ, then the same δ will satisfy the requirement in the definition of the limit. Thus while showing that the limit of a function is such and such at a point, we are free to choose a pre-assigned upper bound for our ϵ.

Similarly, suppose for some ϵ, we have already obtained a δ such that the limit requirement is satisfied. If we choose another δ, say δ_1, which is smaller than δ, then the limit requirement is also satisfied. Thus, we are free to choose a pre-assigned upper bound for our δ provided it is convenient to us and it works.

(1.6) *Example*

Consider $f(x,y) = \sqrt{1-x^2-y^2}$ where $D = \{(x,y) : x^2+y^2 \le 1\}$.

We guess that limit $f(x,y)$ is 1 as $(x,y) \to (0,0)$. To show that the guess is right, let $\epsilon > 0$. Notice that $0 \le f(x,y) \le 1$ on D. Using (1.5), assume that $0 < \epsilon < 1$. Choose $\delta = \sqrt{1-(1-\epsilon)^2}$. Let $|(x,y)-(0,0)| < \delta$. Then

$$0 < x^2+y^2 < 1-(1-\epsilon)^2 \Rightarrow 1-x^2-y^2 > (1-\epsilon)^2 \Rightarrow f(x,y) > 1-\epsilon.$$

That is, $|f(x,y)-1| = 1-f(x,y) < \epsilon$. So, $f(x,y) \to 1$ as $(x,y) \to (0,0)$. ∎

(1.7) *Theorem* (Uniqueness of limit)

Let D be a region in the plane, $(a,b) \in \overline{D}$ and let $f : D \to \mathbb{R}$ be a function. If limit of $f(x,y)$ as (x,y) approaches (a,b) exists, then it is unique.

Proof. Suppose $f(x,y) \to \ell$ and also $f(x,y) \to m$ as $(x,y) \to (a,b)$. Let $\epsilon > 0$. For $\epsilon/2$, we have $\delta_1 > 0$, $\delta_2 > 0$ such that for all $(x,y) \in D$,

$$0 < (x-a)^2 + (y-b)^2 < \delta_1^2 \Rightarrow |f(x,y)-\ell| < \epsilon/2,$$

$$0 < (x-a)^2 + (y-b)^2 < \delta_2^2 \Rightarrow |f(x,y)-m| < \epsilon/2.$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Due to (1.1), (a,b) is a limit point of D. So, there exists a point (α,β) in the δ-disk around (a,b) and $(\alpha,\beta) \ne (a,b)$. This means

$$0 < (\alpha-a)^2 + (\beta-b)^2 < \delta^2, \quad \delta^2 \le \delta_1^2,\ \delta^2 \le \delta_2^2.$$

The above implications now guarantee that $|f(\alpha, \beta) - \ell| < \epsilon/2$ and also $|f(\alpha, \beta) - m| < \epsilon/2$. Then,

$$|\ell - m| \leq |\ell - f(\alpha, \beta)| + |f(\alpha, \beta) - m| < \epsilon/2 + \epsilon/2 = \epsilon.$$

That is, for each $\epsilon > 0$, we have $|\ell - m| < \epsilon$. Hence $\ell = m$. ∎

For a function of one variable, there are only two directions for approaching a point; from left and from right. Whereas for a function of two variables, the limit refers only to the distance between (x, y) and (a, b). It does not refer to any specific direction of approach to (a, b). That is, there are infinitely many directions, and infinite number of paths on which one can vary a point (x, y) to approach this point (a, b). If the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that limit of $f(x, y)$ as (x, y) approaches (a, b) does not exist. We mention this fact as our next theorem.

(1.8) *Theorem*

Let D be a region in the plane, $(a, b) \in \overline{D}$ and let $f : D \to \mathbb{R}$ be a function. Suppose that $f(x, y) \to \ell_1$ as $(x, y) \to (a, b)$ along a path C_1 lying in $\overline{D}$, and $f(x, y) \to \ell_2$ as $(x, y) \to (a, b)$ along a path C_2 lying in $\overline{D}$. If $\ell_1 \neq \ell_2$, then the limit of $f(x, y)$ as $(x, y) \to (a, b)$ does not exist.

Of course, the limit of $f(x, y)$ as (x, y) approaches a point (a, b) along a path $y = g(x)$ means the following:

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along } y=g(x)}} f(x, y) = \lim_{x\to a} f\big(x, g(x)\big), \quad \text{where } \lim_{x\to a} g(x) = b.$$

The last requirement merely says that as x approaches a, y must approach b.

Similarly, the limit of $f(x, y)$ as (x, y) approaches a point (a, b) along a path $x = g(y)$ means that

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along } x=g(y)}} f(x, y) = \lim_{y\to b} f\big(g(y), y\big), \quad \text{where } \lim_{y\to b} g(x) = a.$$

(1.9) *Example*

Consider $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. What is its limit at $(0, 0)$?

When $y = 0$, limit of $f(x, y)$ as $x \to 0$ is $\displaystyle\lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0}(1) = 1$.

That is, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

When $x = 0$, limit of $f(x, y)$ as $y \to 0$ is $\lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$.

That is, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist. $\square$

(1.10) ***Example***

Consider $f(x, y) = \dfrac{xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. What is its limit at $(0, 0)$?

Along the x-axis, $y = 0$; then limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is 0.

Along the y-axis, $x = 0$; then limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is 0.

Does it say that limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is 0?

Along the line $y = x$, limit of $f(x, y)$ as $(x, y) \to 0$ is $\lim\limits_{x\to 0} \dfrac{x^2}{x^2 + x^2} = 1/2$.

Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist. $\square$

(1.11) ***Example***

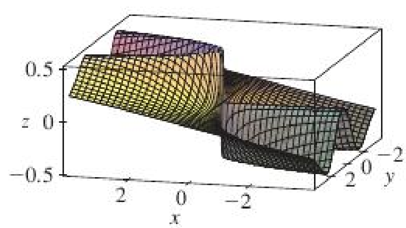
Consider $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ for $(x, y) \neq (0, 0)$. What is its limit at $(0, 0)$?

If $y = mx$, for some $m \in \mathbb{R}$, then $f(x, y) = \dfrac{m^2 x}{1 + m^4 x^2}$.

So, limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along all straight lines is 0.

If $x = y^2$, $y \neq 0$, then $f(x, y) = \dfrac{y^4}{y^4 + y^4} = 1/2$.

That is, as $(x, y) \to (0, 0)$ along the curve $x = y^2$, $f(x, y) \to 1/2$.



Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist. $\square$

Are $\lim\limits_{(x,y)\to(a,b)} f(x, y)$, $\lim\limits_{x\to a} \lim\limits_{y\to b} f(x, y)$, $\lim\limits_{y\to b} \lim\limits_{x\to a} f(x, y)$ all equal?

(1.12) ***Example***

Let $f(x, y) = \dfrac{(y - x)(1 + x)}{(y + x)(1 + y)}$ for $x + y \neq 0, -1 < x,\ y < 1$. Then

$$\lim_{y\to 0} \lim_{x\to 0} f(x, y) = \lim_{y\to 0} \frac{y}{y(1 + y)} = 1.$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\frac{-x(1+x)}{x} = -1.$$

Along $y = mx$, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x(m-1)(1+x)}{x(1+m)(1+mx)} = \frac{m-1}{m+1}.$

For different values of m, we get the last limit value different. For instance, along the path $y = x$, the limit is 0, whereas along the path $y = 2x$, the limit is $1/3$. So, limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist; and the two *iterated limits* exist, though they are not equal. ∎

(1.13) *Example*

Let $f(x,y) = x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}$ for $x \neq 0,\ y \neq 0$. Then

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} y\sin\frac{1}{x} \text{ does not exist.}$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} x\sin\frac{1}{y} \text{ does not exist.}$$

For the limit of $f(x,y)$ as (x,y) approaches (a,b), let $\epsilon > 0$. Take $\delta = \epsilon/2$. Suppose $0 < \sqrt{x^2+y^2} < \delta$. Then,

$$|f(x,y) - 0| \le |x| + |y| = \sqrt{x^2} + \sqrt{y^2} \le 2\sqrt{x^2+y^2} < 2\delta = \epsilon.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

That is, the two iterated limits do not exist, but the limit exists. ∎

Hence existence of the limit of $f(x,y)$ as $(x,y) \to (a,b)$ and the existence of two iterated limits have no connection.

Suppose $f(x,y)$ does not depend on x, say $f(x,y) = g(y)$. If $\lim_{y\to b} g(y) = \ell$, does it follow that $\lim_{(x,y)\to(a,b)} f(x,y) = \ell$? The answer is 'yes' in the following sense.

(1.14) *Theorem*

Let D be a region in the plane, $(a,b) \in \overline{D}$ and let $f : D \to \mathbb{R}$ be a function. Suppose $f(x,y) = g(y)$, $\lim_{y\to b} g(y) = \ell$ and "if $g(y)$ is defined at $y = b$, then $g(b) = \ell$". Then $\lim_{(x,y)\to(a,b)} f(x,y) = \ell$.

Proof. Let $\epsilon > 0$. We have a $\delta > 0$ such that $|y - b| < \delta,\ y \neq b \Rightarrow |g(y) - \ell| < \epsilon$. We plan to use the same δ for proving the conclusion. So, suppose $(x,y) \neq (a,b)$, $f(x,y)$ is defined and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$.

We divide $(x, y) \neq (a, b)$ into two cases:

Case 1: $y \neq b$. Then, $|y - b| \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta$. It implies $|g(y) - \ell| < \epsilon$. That is, $|f(x, y) - \ell| < \epsilon$.

Case 2: $y = b$. Here, $f(x, y) = g(y)$ is defined for $y = b$, and $x \neq a$. We have $|f(x, y) - \ell| = |f(x, b) - \ell| = |g(b) - \ell| = 0 < \epsilon$.

Hence, $\lim\limits_{(x,y)\to(a,b)} f(x, y) = \ell$. ∎

(1.15) ***Example***

1. Let $f(x, y) = \dfrac{\sin y}{y}$ for $y \neq 0$. Here, $\dfrac{\sin y}{y}$ is not defined at $y = 0$.

   We see that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{y\to 0} \dfrac{\sin y}{y} = 1$.

2. Let $f(x, y) = \begin{cases} y & \text{if } y \neq 0 \\ 1 & \text{if } y = 0. \end{cases}$

   Along the line $y = 0$, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,0)\to(0,0)} f(x, 0) = \lim\limits_{x\to 0} 1 = 1$.

   Along the line $y = x$, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,x)\to(0,0)} f(x, x) = \lim\limits_{x\to 0} 0 = 0$.

   Hence, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

The second one shows that the condition "if $g(y)$ is defined at $y = b$, then $g(b) = \ell$" is essential. □

The usual operations of addition, multiplication etc have the expected effects as the following theorem shows. Its proof is analogous to the single variable limits.

(1.16) ***Theorem*** (Algebra of limits)

Let D be a region in the plane, $(a, b) \in \overline{D}$ and let $f : D \to \mathbb{R}$ be a function. Suppose that $L, M, c, r \in \mathbb{R}$, $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ and that $\lim\limits_{(x,y)\to(a,b)} g(x, y) = M$. Then

(1) Constant Multiple*:* $\lim\limits_{(x,y)\to(a,b)} cf(x, y) = cL$.

(2) Sum*:* $\lim\limits_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$.

(3) Product*:* $\lim\limits_{(x,y)\to(a,b)} (f(x, y)\, g(x, y)) = LM$.

(4) Quotient*: If $M \neq 0$ and $g(x, y) \neq 0$ in an open disk around the point (a, b), then* $\lim\limits_{(x,y)\to(a,b)} (f(x, y)/g(x, y)) = L/M$

(5) Power*: If L^r and $(f(x, y))^r$ are well-defined and $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$, then* $\lim\limits_{(x,y)\to(a,b)} (f(x, y))^r = L^r$.

1.3 Continuity

Let $f(x, y)$ be a real valued function defined on a subset D of $\mathbb{R}^2$. We say that $f(x, y)$ is **continuous** at a point $(a, b) \in D$ iff for each $\epsilon > 0$, there exists $\delta > 0$ such that for all points $(x, y) \in D$ with $\sqrt{(x - a)^2 + (y - b)^2} < \delta$ we have $|f(x, y) - f(a, b)| < \epsilon$.

Observe that if $(a, b) \in D$ is an isolated point of D, then f is continuous at (a, b). If D is a region, then $(a, b) \in D$ is not an isolated point of D. In this case, f is continuous at $(a, b) \in D$ iff the following are satisfied:

1. $f(a, b)$ is well defined, that is, $(a, b) \in D$;
2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists; and
3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

The function $f(x, y)$ is said to be **continuous on** a subset of D iff $f(x, y)$ is continuous at all points in the subset.

Unless otherwise mentioned, *we assume that* the domain $D \subseteq \mathbb{R}^2$ of a function $f(x, y)$ is a region, be it open, closed, bounded or unbounded.

It then follows from the algebra of limits that constant multiples, sum, difference, product, quotient, and rational powers of continuous functions defined on a region are continuous provided they are well defined.

Polynomials in two variables are continuous functions.

Rational functions, i.e., ratios of polynomials, are continuous functions provided they are well defined.

(1.17) ***Example***

Is the function $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ continuous on $\mathbb{R}^2$?

At any point other than the origin, $f(x, y)$ is a rational function; therefore, it is continuous. To see that $f(x, y)$ is continuous at the origin, let $\epsilon > 0$ be given. Take $\delta = \epsilon/3$. Assume that $\sqrt{x^2 + y^2} < \delta$. Then

$$\left|\frac{3x^2y}{x^2 + y^2} - f(0, 0)\right| \leq \left|\frac{3(x^2 + y^2)y}{x^2 + y^2}\right| \leq 3|y| \leq 3\sqrt{x^2 + y^2} < 3\delta = \epsilon. \qquad \square$$

(1.18) ***Example***

Is the function $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ continuous on $\mathbb{R}^2$?

Being a rational function, it is continuous at all nonzero points. For the point $(0,0)$, let $\epsilon > 0$ be given. Choose $\delta = \sqrt{\epsilon}$. Notice that $xy \le x^2 + y^2$ and $x^2 - y^2 \le x^2 + y^2$. For any (x, y) with $\sqrt{x^2 + y^2} < \delta$, we have

$$|f(x,y) - 0| = \frac{|x| \cdot |y| \cdot (x^2 - y^2)}{x^2 + y^2} \le \frac{(x^2 + y^2)(x^2 + y^2)}{x^2 + y^2} < \delta^2 = \epsilon.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

That is, the function $f(x, y)$ is continuous. $\square$

(1.19) ***Example***

The function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous on $D = \mathbb{R}^2 \setminus \{(0,0)\}$.
But it is not continuous at $(0,0)$ since $(0,0) \notin D$. Well, we redefine.

Consider the function $g(x, y)$, where

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

By (1.9), $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ does not exist.

Hence $g(x, y)$ is not continuous at $(0,0)$. $\square$

As in the single variable case, composition of continuous functions is continuous. Try to prove the following result.

(1.20) ***Theorem***

Let D be region in the plane, $f : D \to \mathbb{R}$ be continuous at $(a,b) \in D$ with $f(a,b) = c$ and let $g : I \to \mathbb{R}$ be continuous at $c \in I$ for some interval I in $\mathbb{R}$. Then the function $g(f(x,y))$ from D to $\mathbb{R}$ is continuous at (a,b).

For example,

e^{x-y} is continuous at all points in the plane.

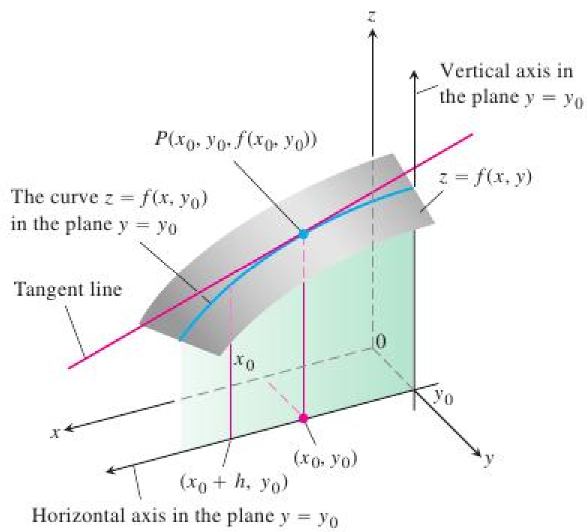
$\cos \dfrac{xy}{1 + x^2}$ and $\ln(1 + x^2 + y^2)$ are continuous on $\mathbb{R}^2$.

$\tan^{-1}(y/x)$ is continuous everywhere except on the y-axis, where it is not defined.

$(x^2 + y^2 + z^2 - 1)^{-1}$ is continuous everywhere except where ever it is not defined, that is, except on the sphere $x^2 + y^2 + z^2 = 1$.

1.4 Partial Derivatives

Let $f(x, y)$ be a real valued function defined on a region $D \subseteq \mathbb{R}^2$. Let $(a, b) \in D$. If C is the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$, then the slope of the tangent line to C at $(a, b, f(a, b))$ is the partial derivative of $f(x, y)$ with respect to x at (a, b).



In the figure take $x_0 = a, y_0 = b$. A formal definition of the partial derivative follows.

The **partial derivative of** $f(x, y)$ **with respect to** x **at the point** (a, b) is

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \left.\frac{df(x,b)}{dx}\right|_{x=a} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h},$$

provided this limit exists. Notice that if $\frac{\partial f}{\partial x}(a, b)$ exists, then $f(x, b)$, as a function of x, is continuous at $x = a$.

The **partial derivative of** $f(x, y)$ **with respect to** y **at the point** (a, b) is

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \left.\frac{df(a,y)}{dy}\right|_{y=b} = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k},$$

provided this limit exists. If $f_y(a, b)$ exists, then $f(a, y)$ is continuous at $y = b$.

Remark 1.21 In the definition of $f_x(a, b)$, the expression $\left.\frac{df(x,b)}{dx}\right|_{x=a}$ is to be interpreted as the value of the limit $\lim\limits_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$. It means that if we write $g(x) = f(x, b)$, then it is equal to $g'(a)$. And, it is not necessarily equal to "$g'(x)$ obtained otherwise and then evaluated at $x = a$". For example, if $g(x) = 1$ for $x \neq 0$

and $g(0) = 0$, then $g'(0) = \lim\limits_{h\to 0} \frac{g(h)-g(0)}{h} = \lim\limits_{h\to 0} \frac{1}{h}$, which does not exist. Whereas if wrongly interpreted, then one would find $g'(x) = \frac{d(1)}{dx} = 0$ for $x \neq 0$, and then evaluated at $x = 0$ would give the spurious result 0. Similar comments go for the expression in determining $f_y(a, b)$.

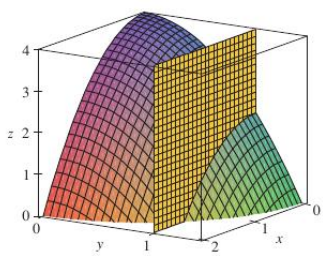
(1.22) ***Example***

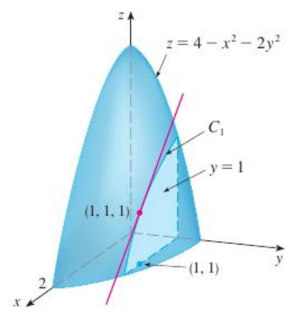
Find $f_x(1, 1)$ where $f(x, y) = 4 - x^2 - 2y^2$.

$$f_x(1,1) = \lim_{h\to 0} \frac{(4 - (1+h)^2 - 2) - (4 - 1 - 2)}{h} = \lim_{h\to 0} \frac{-2h - h^2}{h} = -2.$$

That is, treat y as a constant and differentiate with respect to x.

$$f_x(1,1) = f_x(x,y)\big|_{(1,1)} = -2x\big|_{(1,1)} = -2.$$





The vertical plane $y = 1$ crosses the paraboloid in the curve $C_1 : z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ (which corresponds to $(x, y) = (1, 1)$) is $f_x(1, 1) = -2$. ∎

(1.23) ***Example***

The plane $x = 1$ intersects the surface $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at the point $(1, 2, 5)$.

The asked slope is $\partial z/\partial y$ at $(1, 2)$. It is

$$\frac{\partial(x^2 + y^2)}{\partial y}(1,2) = (2y)(1,2) = 4.$$

Alternatively, the parabola is $z = x^2 + y^2, x = 1$ OR, $z = 1 + y^2$. So, the slope at $(1, 2, 5)$ is

$$\frac{dz}{dy}\bigg|_{y=2} = \frac{d(1+y^2)}{dy}\bigg|_{y=2} = (2y)|_{y=2} = 4. \qquad ∎$$

In general, $f_x(a, b)$ and $\lim\limits_{(x,y)\to(a,b)} f_x(x, y)$ are different. See the following two examples.

(1.24) ***Example***

Let $f(x,y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$

Then $f_x(x,y) = 0$ for all $x > 0$; and $f_x(x,y) = 0$ for all $x < 0$.
So, $\lim_{(x,y)\to(0,0)} f_x(x,y) = 0$. Notice that

$$\lim_{h\to 0-} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0, \quad \lim_{h\to 0+} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}.$$

The latter limit does not exist. Hence, $f_x(0,0)$ does not exist. ∎

(1.25) ***Example***

Let $f(x) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Then, $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{h^3}{h^2 \cdot h} = 1.$

For $(x,y) \ne 0$, $f_x(x,y) = \dfrac{(x^2+y^2)\cdot 3x^2 - x^3 \cdot 2x^2}{(x^2+y^2)^2}$. Thus,

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f_x(x,y) = \lim_{y\to 0} f_x(0,y) = \lim_{y\to 0} 0 = 0.$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=0}} f_x(x,y) = \lim_{x\to 0} f_x(x,0) = \lim_{x\to 0} \frac{3x^4 - 6x^5}{x^4} = 3.$$

Hence, $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist. ∎

Caution: $f_x(a,b)$ is not the same as $\lim_{(x,y)\to(a,b)} f_x(x,y)$. In fact, $f_x(a,b)$ can exist even if $\lim_{(x,y)\to(a,b)} f_x(x,y)$ does not exist.

Of course, if $f_x(x,y)$ is continuous at (a,b), then $f_x(a,b) = \lim_{(x,y)\to(a,b)} f_x(x,y)$.

For a function of one variable, $f'(t)$ exists at $t = a$ implies that $f(t)$ is continuous at $t = a$. Thus, if $f_x(a,b) = df(x,b)/dx$ at $x = a$ exists, then $f(x,b)$ is continuous at $x = a$. Similarly, if $f_y(a,b) = df(a,y)/dy$ at $y = b$ exists, then $f(a,y)$ is continuous at $y = b$. Is it true that if both $f_x(x,y)$ and $f_y(x,y)$ exist at (a,b), then $f(x,y)$ is continuous at (a,b)?

(1.26) *Example*

Consider the function $f(x,y) = \begin{cases} 1 & \text{if } xy \neq 0 \\ 0 & \text{otherwise.} \end{cases}$

We compute its partial derivatives at the origin.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0,$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

So, both the partial derivatives at $(0,0)$, that is, $f_x(0,0)$ and $f_y(0,0)$ exist.

Let us check whether $f(x,y)$ is continuous at (a,b).

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f(x,y) = \lim_{y \to 0} f(0,y) = \lim_{y \to 0} 0 = 0.$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x}} f(x,y) = \lim_{x \to 0} f(x,x) = \lim_{x \to 0} 1 = 1.$$

Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist; $f(x,y)$ is not continuous at $(0,0)$. ∎

Thus, even if both $f_x(a,b)$ and $f_y(a,b)$ exist, $f(x,y)$ need not be continuous at (a,b). For another example, look at (1.29).

It can be shown that if both $f_x(x,y)$ and $f_y(x,y)$ exist in a disk around (a,b) and at least one of these is continuous at (a,b) (or, bounded in a disk around (a,b)), then $f(x,y)$ is continuous at (a,b).

1.5 The Increment Theorem

The partial derivative $f_x(a,b)$ is defined as the limit of the ratio of $f(a+h,b)-f(a,b)$ to h as $h \to 0$. Conventionally, this increment h in the independent variable x is written as Δx. The expression $f(a+\Delta x, b) - f(a,b)$ measures the increment in $f(x,y)$ at (a,b) in the direction of x. Similarly, write the increment k in the variable y as Δy. Then, the expression $f(a, b+\Delta y) - f(a,b)$ measures the increment of $f(x,y)$ at (a,b) in the direction of y. The **total increment** in $f(x,y)$ at (a,b) is then measured by the expression $f(a+\Delta x, b+\Delta y) - f(a,b)$ and it is denoted by $(\Delta f)(a,b)$, or sometimes, by Δf with the understanding that it is taken at the point (a,b). That is,

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b) \quad \text{at the point } (a,b).$$

If the partial derivatives of $f(x, y)$ are continuous, then the total increment Δf can be written in a more suggestive form.

(1.27) ***Theorem*** (Increment Theorem)
Let D be an open region in the plane. Let the function $f : D \to \mathbb{R}$ have continuous partial derivatives. Then f is continuous and the total increment Δf at $(a, b) \in D$ can be written as

$$\Delta f = f_x(a, b)\Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$.

Proof. We prove the formula for Δf, from which continuity of $f(x, y)$ follows. Write h instead of Δx and k in place of Δy for better readability. Now,

$$\Delta f := f(a + h, b + k) - f(a + h, b) + f(a + h, b) - f(a, b).$$

By MVT, there exist $c \in [a, a + h]$ and $d \in [b, b + k]$ such that

$$f(a + h, b) - f(a, b) = h[f_x(c, b) - f_x(a, b)] + hf_x(a, b),$$
$$f(a + h, b + k) - f(a + h, b) = k[f_y(a + h, d) - f_y(a, b)] + kf_y(a, b).$$

Write $\epsilon_1 = f_x(c, b) - f_x(a, b)$ and $\epsilon_2 = f_y(a + h, d) - f_y(a, b)$. When $h \to 0$ and $k \to 0$, we see that $c \to a$ and $d \to b$. Since f_x and f_y are assumed to be continuous, we have $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. Then the total increment can be written as

$$\Delta f = f(a + h, b + k) - f(a, b) = hf_x(a, b) + kf_y(a, b) + \epsilon_1 h + \epsilon_2 k,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $h \to 0, k \to 0$. ∎

Remark 1.28 Let (a, b) be an interior point of a region D. For a function $f : D \to \mathbb{R}$ if the total increment Δf at (a, b) can be written as

$$\Delta f = f_x(a, b)\Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, then we say that f is *differentiable* at (a, b). The following statements give some connections between differentiability, continuity and the partial derivatives.

- Let D be a region in $\mathbb{R}^2$. Let $f : D \to R$ be such that both f_x and f_y exist on D and at least one of them is continuous at an interior point (a, b) of D. Then f is differentiable at (a, b).

- Let D be a region in $\mathbb{R}^2$. Let $f : D \to R$ be differentiable at an interior point (a, b) of D. Then f is continuous at (a, b).

The first statement strengthens the increment theorem. In most of the results that follow, we will assume that both f_x and f_y are continuous. All those results hold true if this assumption is replaced by the weaker assumption that f is differentiable. However, to use differentiability will increase load on terminology, and it is also difficult to verify the condition of differentiability. Instead, we will work with the stronger assumption that f_x and f_y are continuous.

For a function $f(x, y)$, partial derivatives of second order are defined as follows:

$$
\begin{aligned}
f_{xx} &= (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}.\\
f_{xy} &= (f_x)_y = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.\\
f_{yx} &= (f_y)_x = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}.\\
f_{yy} &= (f_y)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}.
\end{aligned}
$$

Similarly, higher order partial derivatives are defined. For instance,

$$f_{xxy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^3 f}{\partial y \partial x \partial x}.$$

The following example shows that some of the higher order partial derivatives at a point may exist but others do not.

(1.29) ***Example***

Let $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

We have $f(x, 0) = 0 = f(0, y)$. Then,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} f(x, y) = \lim_{y\to 0} f(0, y) = 0, \qquad \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} f(x, y) = \lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}.$$

Hence, limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, so that $f(x, y)$ is not continuous at $(0, 0)$. For any x, we have

$$
\begin{aligned}
f_x(x, 0) &= \lim_{h\to 0} \frac{f(x + h, 0) - f(x, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.\\
f_{xx}(x, 0) &= \lim_{h\to 0} \frac{f_x(h, 0) - f_x(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.
\end{aligned}
$$

Similarly, it follows that $f_y(0,y) = 0$ and $f_{yy}(0,y) = 0$ for any y. In particular, we get $f_{xx}(0,0) = 0 = f_{yy}(0,0)$. What about $f_{xy}(0,0)$ and $f_{yx}(0,0)$?

$$f_x(0,y) = \lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0}\frac{y}{h^2+y^2} = \frac{1}{y}.$$
$$f_y(x,0) = \lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k} = \lim_{k\to 0}\frac{x}{x^2+k^2} = \frac{1}{x}.$$
$$f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{1/k-0}{k} \text{ does not exist.}$$
$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{1/h-0}{h} \text{ does not exist.}$$ ∎

(1.30) ***Example***

Consider $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

$$f(x,0) = f(0,y) = f(0,0) = 0.$$
$$f_x(x,0) = f_y(0,y) = f_{xx}(0,0) = f_{yy}(0,0) = 0.$$
$$f_x(0,y) = \lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h} = -y.$$
$$f_y(x,0) = \lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k} = x.$$
$$f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{-k}{k} = -1.$$
$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h}{h} = 1.$$

That is, $f_{xy}(0,0) \neq f_{yx}(0,0)$. ∎

Thus, even if both f_{xy} and f_{yx} exist at a point, they need not be equal. However, if f_x, f_y, f_{xy} and f_{yx} are continuous, then they are equal.

(1.31) ***Theorem*** (Clairaut)

Let D be an open region in the plane. Let $f : D \to \mathbb{R}$ be a function, where f_x, f_y, f_{xy} and f_{yx} are continuous. Then $f_{xy} = f_{yx}$.

Proof. Assume that f_{xy} and f_{yx} are continuous. Let $(a,b) \in D$. Let $h \neq 0$. Write $g(x) = f(x, b+h) - f(x,b)$ and $\tilde{g}(y) = f(a+h, y) - f(a,y)$. Now,

$$\begin{aligned}\phi(h) := g(a+h) - g(a) &= [f(a+h,b+h) - f(a+h,b)] - [f(a,b+h) - f(a,b)] \\ &= [f(a+h,b+h) - f(a,b+h)] - [f(a+h,b) - f(a,b)] \\ &= \tilde{g}(b+h) - \tilde{g}(b).\end{aligned}$$

Notice that $\phi(h)$ is a function of h. Consider the equality $\phi(h) = g(a+h) - g(a)$. Since f_x is continuous, $g'(x)$ is continuous. By the Mean Value Theorem (MVT), we have c between a and $a+h$ such that

$$\phi(h) = g'(c)h = h[f_x(c, b+h) - f_x(c, b)].$$

Now, $f_x(c, y)$ is a function of y. Since f_{xy} is continuous, the function $f_x(c, y)$ as a function of y, is continuously differentiable. Again, applying MVT on $f_x(c, y)$, we get d between b and $b+h$ such that

$$\phi(h) = h \cdot h \cdot f_{xy}(c, d) = h^2 f_{xy}(c, d).$$

Due to continuity of f_{xy}, we have

$$\lim_{h\to 0} \frac{\phi(h)}{h^2} = \lim_{(c,d)\to(a,b)} f_{xy}(c, d) = f_{xy}(a, b).$$

Similarly, considering the equality $\phi(h) = \tilde{g}(b+h) - \tilde{g}(b)$, we obtain

$$\lim_{h\to 0} \frac{\phi(h)}{h^2} = f_{yx}(a, b).$$

Hence, $f_{xy}(a, b) = f_{yx}(a, b)$. ∎

1.6 Chain Rules

We continue to formulate and discuss our results for a function $f(x, y)$ of two variables. Analogously, all these notions and results can be formulated and proved for a function $f(x_1, \ldots, x_n)$ of n variables for $n > 2$.

We apply the increment theorem to partially differentiate composite functions. The results in this section are formulated informally. Precise statements may be formulated so that the compositions make sense.

(1.32) ***Theorem*** (Chain Rule 1)

Let $f(x, y)$ have continuous partial derivatives. If $x(t)$ and $y(t)$ are differentiable functions, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof. Using the increment theorem (1.27) at a point P we obtain

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}.$$

As $\Delta t \to 0$, we have $\Delta x \to 0, \Delta y \to 0, \epsilon_1 \to 0, \epsilon_2 \to 0$. Then the result follows. ∎

For example, if $z = xy$ and $x = \sin t, y = \cos t$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t) = \cos^2 t - \sin^2 t.$$

Check: $z(t) = \sin t \cos t = \frac{1}{2}\sin 2t$. So, $z'(t) = \cos 2t = \cos^2 t - \sin^2 t$.

(1.33) ***Theorem*** (Chain Rule 2)

Let $f(x, y)$, $x(s, t)$ and $y(s, t)$ have continuous partial derivatives. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Its proof is similar to that of (1.33). In subscript notation, it looks like

$$f_s = f_x x_s + f_y y_s, \quad f_t = f_x x_t + f_y y_t.$$

(1.34) ***Example***

Let $z = e^x \sin y,\ x = st^2,\ y = s^2 t$. Then

$$\frac{\partial z}{\partial s} = (e^x \sin y)t^2 + (e^x \cos y)2st = te^{st^2}(t\sin(s^2 t) + 2s\cos(s^2 t)).$$

$$\frac{\partial z}{\partial t} = (e^x \sin y)2st + (e^x \cos y)s^2 = se^{st^2}(2t\sin(s^2 t) + s\cos(s^2 t)).$$

Substitute expressions for x and y to get $z = z(s, t)$ and then check that the results are correct. ∎

Functions can be differentiated implicitly. If F is defined on an open region D in $\mathbb{R}^3$ containing a point (a, b, c), where $F(a, b, c) = 0,\ F_z(a, b, c) \neq 0$, and F_x, F_y, F_z are continuous, then the equation $F(x, y, z) = 0$ defines a function $z = f(x, y)$ in an open ball around (a, b, c) contained in D. Moreover, the function $z = f(x, y)$ can now be differentiated partially with

$$z_x = -F_x/F_z,\ z_y = -F_y/F_z.$$

It is easier to differentiate implicitly than remembering the formula.

(1.35) ***Example***

Find z_x and z_y if $x^3 + y^3 + z^3 + 6xyz = 1$.

We differentiate 'the equation' with respect to x and y as follows:

$$3x^2 + 3z^2 z_x + 6y(z + xz_x) = 0 \Rightarrow z_x = -\frac{(x^2 + 2yz)}{z^2 + 2xy}.$$

$$3y^2 + 3z^2 z_y + 6x(z + xz_y) = 0 \Rightarrow z_y = -\frac{(y^2 + 2xz)}{z^2 + 2xy}.$$ ∎

(1.36) *Example*

Find w_x if $w = x^2 + y^2 + z^2$ and $x^2 + y^2 - z = 0$.

As it looks, $w_x = 2x$. This would have been if x, y, z were independent variables. But this is not so since $x^2 + y^2 - z = 0$.

If z is the dependent variable and x, y are independent variables, then the second equation gives $z = x^2 + y^2$. Then, $w = x^2 + y^2 + (x^2 + y^2)^2$. Thus,

$$w_x = 2x + 4x^3 + 4xy^2.$$

If y is the dependent variable and x, z are independent variables, then $y^2 = z - x^2$ so that $w = y^2 + (z - y^2) + z^2 = z + z^2$. Then $w_x = 0$. $\square$

The correct procedure to get $\partial w/\partial x$ in (1.36) is :

1. w must be dependent variable and x must be independent variable.
2. Decide which of the other variables are dependent or independent.
3. Eliminate the dependent variables from w using the constraints.
4. Then take the partial derivative $\partial w/\partial x$.

(1.37) *Example*

Given that $w = x^2 + y^2 + z^2$ and $z(x, y)$ satisfies $z^3 - xy + yz + y^3 = 1$, evaluate $\partial w/\partial x$ at $(2, -1, 1)$.

Here, z, w are dependent variables and x, y are independent variables. Then,

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}, \quad 3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0.$$

These two together imply $\dfrac{\partial w}{\partial x} = 2x + \dfrac{2yz}{y + 3z^2}$.

Evaluating it at $(2, -1, 1)$ gives $\dfrac{\partial w}{\partial x}(2, -1, 1) = 3$. $\square$

(1.38) *Example*

Suppose $z = f(x, y)$ has continuous second order partial derivatives, $x = r^2 + \theta^2$ and $y = 2r\theta$. Find z_{rr} assuming that r and θ are independent variables.

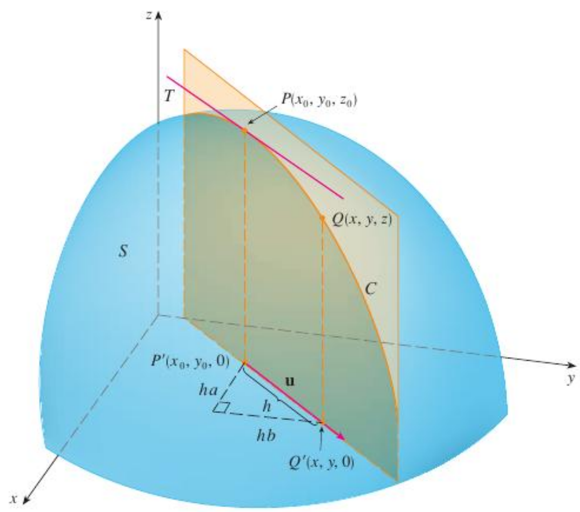
We have $x_r = 2r$, $y_r = 2\theta$. Then

$$\begin{aligned}
z_r &= z_x x_r + z_y y_r = 2rz_x + 2\theta z_y.\\
z_{xr} &= z_{xx} x_r + z_{xy} y_r = 2rz_{xx} + 2\theta z_{xy}.\\
z_{yr} &= z_{yx} x_r + z_{yy} y_r = 2rz_{yx} + 2\theta z_{yy}.\\
z_{rr} &= \frac{\partial z_r}{\partial r} = \frac{\partial}{\partial r}(2rz_x + 2\theta z_y) = 2z_x + 2rz_{xr} + 2\theta z_{yr}\\
&= 2z_x + 2r(2rz_{xx} + 2\theta z_{xy}) + 2\theta(2rz_{yx} + 2\theta z_{yy})\\
&= 2z_x + 4r^2 z_{xx} + 8r\theta z_{xy} + 4\theta^2 z_{yy}.
\end{aligned}$$

$\square$

1.7 Directional Derivative

Recall that if $f(x, y)$ is a function, then $f_x(x_0, y_0)$ is the rate of change in f with respect to change in x, at (x_0, y_0), that is, in the direction $\hat{\imath}$. Similarly, $f_y(x_0, y_0)$ is the rate of change at (x_0, y_0) in the direction $\hat{\jmath}$. How do we find the rate of change of $f(x, y)$ at (x_0, y_0) in the direction of any unit vector $\hat{u}$ in the xy-plane?



Consider a smooth surface S given by the equation $z = f(x, y)$. Let $z_0 = f(x_0, y_0)$. The point $P(x_0, y_0, z_0)$ lies on S. Let $\hat{u}$ be a unit vector in the xy-plane. Translating $\hat{u}$ suitably, we may assume that $\hat{u}$ contains the point P. The vertical plane that passes through P and contains the vector $\hat{u}$ intersects S in a curve C. The slope of the tangent line T to the curve C at the point P is the rate of change of z in the direction of $\hat{u}$.

Convention: A vector is written as $\vec{u}$ whereas a unit vector is written as $\hat{u}$. In the context of the directional derivative $D_u z$, the vector $\hat{u}$ is taken as a unit vector.

Let $f(x, y)$ be a function defined in an open region D. Let $(x_0, y_0) \in D$. The **directional derivative** of $f(x, y)$ in the direction of a unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ at (x_0, y_0) is given by

$$(D_u f)(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

(1.39) *Example*

Find the derivative of $z = x^2 + y^2$ at $(1, 2)$ in the direction $\hat{\imath} + \hat{\jmath}$.

The unit vector in the given direction is $\hat{u} = \frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}$. Then,

$$(D_u z)(1,2) = \lim_{h\to 0} \frac{f\left(1 + \frac{h}{\sqrt{2}}, 2 + \frac{h}{\sqrt{2}}\right) - f(1,2)}{h} = \lim_{h\to 0} \frac{\frac{2h}{\sqrt{2}} + 2 \cdot \frac{2h}{\sqrt{2}}}{h} = 3\sqrt{2}.$$

Notice that $f_x(1,2) \cdot \frac{1}{\sqrt{2}} + f_y(1,2) \cdot \frac{1}{\sqrt{2}} = (2 + 2 \cdot 2) \cdot \frac{1}{\sqrt{2}} = 3\sqrt{2}$. □

(1.40) *Theorem*

Let D be an open region in the plane. Let $f : D \to \mathbb{R}$ have continuous partial derivatives. Then f has a directional derivative at each $(x, y) \in D$ in any direction $\hat{u} = a\hat{\imath} + b\hat{\jmath}$, and it is given by

$$(D_u f)(x,y) = f_x(x,y)a + f_y(x,y)b.$$

Proof. Due to our convention, it is assumed that $\hat{u}$ is a unit vector.

Let $(x_0, y_0) \in D$. Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(h) = f(x_0 + ah, y_0 + bh)$. Then $g(h)$ is a continuously differentiable function of h. Now,

$$(D_u f)(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0,y_0)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0).$$

Using the Chain rule and continuity of f_x, f_y, we have

$$g'(h) = f_x \frac{dx}{dh} + f_y \frac{dy}{dh} = f_x\, a + f_y\, b \Rightarrow g'(0) = f_x(x_0,y_0)\, a + f_y(x_0,y_0)\, b.$$

Therefore, $(D_u f)(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b$. ∎

(1.41) *Example*

Find the directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ in the direction of the line that makes an angle of $\pi/6$ with the x-axis.

The direction is given by the unit vector $\hat{u} = \cos\left(\frac{\pi}{6}\right)\hat{\imath} + \sin\left(\frac{\pi}{6}\right)\hat{\jmath} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$. Thus

$$\begin{aligned} D_u f(x,y) &= \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y = \frac{\sqrt{3}}{2}\left(3x^2 - 3y\right) + \frac{1}{2}\left(-3x + 8y\right) \\ &= \frac{1}{2}\left[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y\right]. \end{aligned}$$ □

The formula for the directional derivative in the direction of the unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ can be written as

$$D_u f = f_x a + f_y b = (f_x\hat{\imath} + f_y\hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath}).$$

The vector operator $\nabla := \dfrac{\partial}{\partial x}\hat{\imath} + \dfrac{\partial}{\partial y}\hat{\jmath}$ is called the **gradient** and the **gradient of** $f(x, y)$ is

$$\nabla f := \operatorname{grad} f := \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}.$$

Therefore, $D_u f = \nabla f \cdot \hat{u}$. That is, at (x_0, y_0), the directional derivative is given by

$$(D_u f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \hat{u}.$$

Caution: This formula has been derived under the assumptions that f_x and f_y are continuous at (x_0, y_0), and $\hat{u}$ is a unit vector.

(1.42) *Example*

Find the directional derivative of $f(x, y) = xe^y + \cos(xy)$ in the direction of the vector $3\hat{\imath} - 4\hat{\jmath}$ at $(2, 0)$.

Here $\hat{u} = \frac{3}{5}\,\hat{\imath} - \frac{4}{5}\,\hat{\jmath}$. The partial derivatives of f are continuous. We have

$$\begin{aligned}\nabla f &= f_x\hat{\imath} + f_y\hat{\jmath} = \big(e^y - y\sin(xy)\big)\hat{\imath} + \big(xe^y - x\sin(xy)\big)\hat{\jmath}.\\ (\nabla f)(2, 0) &= 1 \cdot \hat{\imath} + 2 \cdot \hat{\jmath}.\end{aligned}$$

Thus, the directional derivative of f in the direction of $3\hat{\imath} - 4\hat{\jmath}$ at $(2, 0)$ is

$$(D_u f)(2, 0) = (\nabla f)(2, 0) \cdot \big(\tfrac{3}{5}\,\hat{\imath} - \tfrac{4}{5}\,\hat{\jmath}\big) = (\hat{\imath} + 2\hat{\jmath}) \cdot \big(\tfrac{3}{5}\,\hat{\imath} - \tfrac{4}{5}\,\hat{\jmath}\big) = \tfrac{3}{5} - \tfrac{8}{5} = -1. \qquad \square$$

The directional derivative can be used to approximate a function at a neighboring point. Assume that h is (intuitively) small. Let $\hat{u} = a\hat{\imath} + b\hat{\jmath}$ be a unit vector. Then, $(x_0 + ah, y_0 + bh)$ is taken as a point close to (x_0, y_0) in the direction of $\hat{u}$. The rate of change in $f(x, y)$ at (x_0, y_0) in the direction of $\hat{u}$ is given by the directional derivative $(D_u f)(x_0, y_0)$. Thus the change is approximately equal to $h\,(D_u f)(x_0, y_0)$. We get the approximation

$$f(x_0 + ah, y_0 + bh) \approx f(x_0, y_0) + h\,(D_u f)(x_0, y_0).$$

(1.43) *Example*

How much the value of $y\sin x + 2yz$ change if the point (x, y, z) moves 0.1 units from $(0, 1, 0)$ toward $(2, 2, -2)$?

Let $f(x, y, z) = y\sin x + 2yz$. $P(0, 1, 0)$, $Q(2, 2, -2)$. $\vec{u} = \overrightarrow{PQ} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$. The unit vector in the direction of $\vec{u}$ is $\hat{u} = \frac{1}{3}\,\vec{v}$. We find $D_u f$ at P which requires ∇f.

$$\nabla f = (y\cos x)\hat{\imath} + (\sin x + 2z)\hat{\jmath} + 2y\,\hat{k}.$$

Then $(D_u f)(P) = (\nabla f)(0, 1, 0) \cdot \vec{u} = (\hat{\imath} + 2\hat{k}) \cdot \big(\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}\big) = -\frac{2}{3}$.

The change of $f(x, y, z)$ in the direction of $\vec{u}$ in moving 0.1 units is approximately

$$\Delta f \approx D_u(P) \times 0.1 = -\tfrac{2}{3}\,(0.1) = -0.067 \text{ units.} \qquad \square$$

The formula $D_u f = \nabla f \cdot \hat{u}$ means that the directional derivative is the length of the projection of the gradient in the direction of $\hat{u}$.

(1.44) *Theorem*

Let D be an open region in the plane. Let the function $f : D \to \mathbb{R}$ have continuous partial derivatives. Then the maximum value of the directional derivative $(D_u f)(x, y)$ is $|\nabla f|$, and it is achieved in the direction of ∇f.

Proof. Let $(x, y) \in D$. At the point (x, y) we have

$$D_u f \le |D_u f| = |\nabla f \cdot \hat{u}| = |\nabla f|\,|\hat{u}|\,\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between ∇f and $\hat{u}$. Since maximum of $\cos\theta$ is 1, maximum of $D_u f$ is $|\nabla f|$. The maximum is achieved when $\theta = 0$, that is, when the directions of ∇f and $\hat{u}$ coincide. ∎

From (1.44) we observe the following:

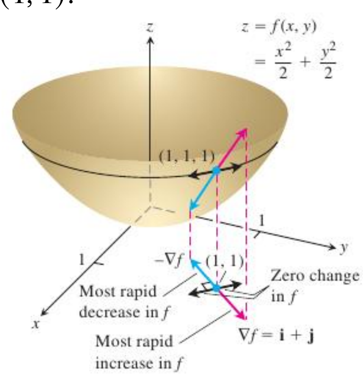
$f(x, y)$ increases most rapidly in the direction of its gradient.

$f(x, y)$ decreases most rapidly in the opposite direction of its gradient.

$f(x, y)$ remains constant in any direction orthogonal to its gradient.

(1.45) *Example*

Find the directions in which the function $f(x, y) = \frac{1}{2}(x^2 + y^2)$ changes most, least, and not at all, at the point $(1, 1)$.



Note that when we ask for a direction, we mean a unit vector. Now,

$$\nabla f = f_x\hat{\imath} + f_y\hat{\jmath} = x\hat{\imath} + y\hat{\jmath}. \quad (\nabla f)(1, 1) = \hat{\imath} + \hat{\jmath}.$$

Thus the function $f(x, y)$ increases most at $(1, 1)$ in the direction $\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}$. It decreases most at $(1, 1)$ in the direction $-\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}}$. And it does not change at $(1, 1)$ in the directions $\pm\left(\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}}\right)$. $\square$

In higher dimensions, if $f(x_1, \ldots, x_n)$ is a function of n independent variables defined on $D \subseteq \mathbb{R}^n$, then its gradient at any point is

$$\nabla f = \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

The directional derivative at any point $\vec{x}$ in the direction of a unit vector $\hat{u} = (u_1, \ldots, u_n)$ is

$$D_u f = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h} = \nabla f \cdot \hat{u} = f_{x_1} u_1 + \cdots + f_{x_n} u_n.$$

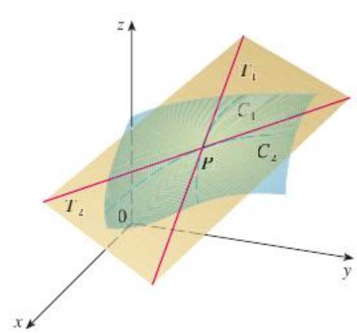
The algebraic rules for the gradient are as follows:

1. Constant multiple: $\nabla(kf) = k\,\nabla f$ for $k \in \mathbb{R}$.
2. Sum: $\nabla(f + g) = \nabla f + \nabla g$.
3. Difference: $\nabla(f - g) = \nabla f - \nabla g$.
4. Product: $\nabla(fg) = f\,\nabla g + g\,\nabla f$.
5. Quotient: $\nabla\Big(\dfrac{f}{g}\Big) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2}$.

1.8 Tangent Planes and Normal Lines

Partial derivatives can be used to derive an equation of the tangent plane to a surface at a point. Let S be a *smooth* surface given by the equation $z = f(x, y)$ for $(x, y) \in D$, an open region in the plane. Here, S is smooth means that both f_x and f_y are continuous on D.

Let $(a, b) \in D$. Write $c = f(a, b)$. The point $P = (a, b, c)$ lies on S. The **tangent plane** to S at P consists of the tangent lines at P to all possible curves that lie on S and pass through P. This plane approximates S at P most closely.



Let C_1 be the curve of intersection of the plane $y = b$ with S and let T_1 be the tangent line to the curve C_1 at the point $P = (a, b, c)$. Similarly, let T_2 be the tangent line at P to the curve C_2 obtained by the intersection of the plane $x = a$ with S. Then, the tangent plane to S at P is the plane that contains the lines T_1 and T_2.

For the equation of the tangent plane, notice that any plane passing through $P = (a, b, c)$ is given by $z - c = A(x - a) + B(y - b)$. When $y = b$, this tangent plane gives us the tangent line T_1 with slope as A. That is, $A = f_x(a, b)$. Similarly, $B = f_y(a, b)$, the slope of the tangent line T_2. Hence the equation of the tangent plane to the smooth surface S given by $z = f(x, y)$ at the point (a, b, c) is

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Recall that the **normal line** to the surface $z = f(x, y)$ at (a, b, c) is the line that passes through the point (a, b, c) and is orthogonal to the tangent plane at the same point. Then the normal line has direction ratios $f_x(a, b),\ f_y(a, b),\ -1$. Thus the equation of the normal line to $z = f(x, y)$ at $(a, b, f(a, b))$ in parametric form is

$$x = a + f_x(a, b)\, t,\ y = b + f_y(a, b)\, t,\ z = f(a, b) - t.$$

(1.46) ***Example***

Find the equation of the tangent plane and the normal line to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1, 1, 3)$.

Here, $z_x = 4x, z_y = 2y$. So, $z_x(1, 1) = 4, z_y(1, 1) = 2$. Then the equation of the tangent plane is $z - 3 = 4(x - 1) + 2(y - 1)$. It simplifies to $z = 4x + 2y - 3$.

The equation of the normal line is $x = 1 + 4t,\ y = 1 + 2t,\ z = 3 - t$. □

An alternative derivation of the equations of tangent planes and normal lines uses the geometric meaning of the gradient. Let $z = f(x, y)$ be a given surface. Assume that f_x and f_y are continuous. Recall that a level curve to this surface is a curve in the plane where $f(x, y)$ is a constant. Fix some constant c in the range of f. On the corresponding level curve, $f(x, y)$ takes the constant value c. Suppose $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ is a parametrization of this level curve. For each point on this level curve, $f(x, y) = c$, that is, $f\big(x(t), y(t)\big) = c$ for all t. Differentiating, we have $\frac{d}{dt} f\big(x(t), y(t)\big) = 0$. Or,

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \nabla f \cdot \frac{d\,\vec{r}(t)}{dt} = 0.$$

We also write $d\vec{r}/dt$ as $\vec{r}\,'(t)$. Since $\vec{r}\,'(t)$ is the tangent to the curve, ∇f is the normal to the level curve, at any point. We thus obtain the following:

> Let D be an open region in the plane, $(a, b) \in D$ and let $f : D \to \mathbb{R}$ have continuous partial derivatives at (a, b). If $(\nabla f)(a, b) \neq 0$, then $(\nabla f)(a, b)$ is the direction of the normal to the level curve $f(x, y) = f(a, b)$ at (a, b).

To determine the tangent planes and normal lines to a surface at a point $P = (a, b, c)$ defined implicitly, say, by $f(x, y, z) = 0$, we consider the function $w = f(x, y, z)$, whose graph is in $\mathbb{R}^4$. Suppose f_x, f_y, f_z are continuous in a **ball** around P. A ball around P is a set of the form $\{(x, y, z) \in \mathbb{R}^3 : (x-a)^2+(y-b)^2+(z-c)^2 < \epsilon^2\}$ for some $\epsilon > 0$. The surface $f(x, y, z) = 0$ is a level surface of the function $w = f(x, y, z)$. Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be a parametrization of a smooth curve on the level surface $f(x, y, z) = 0$ passing through P. Then $f\big(x(t), y(t), z(t)\big) = 0$ for all t. Differentiating this we get $df/dt = 0$. By the Chain rule, it gives

$$\nabla f \cdot \vec{r}\,'(t) = 0.$$

Look at all such smooth curves that pass through P on the level surface. The above equation asserts that the velocity vectors $\vec{r}\,'(t)$ at P to all these smooth curves are orthogonal to the gradient at P. That is, the direction of the normal line to the surface $f(x, y, z) = 0$ at P is $(\nabla f)(P)$. Then, the equation of the **normal line** to the surface $f(x, y, z) = 0$ at $P = (a, b, c)$ in parametric form is

$$x = a + f_x(a, b, c)\, t,\ y = b + f_y(a, b, c)\, t,\ z = c + f_z(a, b, c)\, t.$$

The **tangent plane** at $P = (a, b, c)$ on the level surface $f(x, y, z) = 0$ is the plane through P which is orthogonal to ∇f at P. Then, its equation is

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.$$

If the surface is given in the form $z = f(x, y)$, then write $F(x, y, z) = f(x, y) - z$; the surface is given by $F(x, y, z) = 0$. Now, $F_x = f_x,\ F_y = f_y,\ F_z = -1$. Using the equations derived in case of a surface given in implicit form, we see that the equation of the normal line is

$$x = a + f_x(a, b)\, t,\ y = b + f_y(a, b)\, t,\ z = c - t.$$

Similarly, the equation of the tangent plane is

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

as obtained earlier.

(1.47) *Example*

Find the tangent plane and the normal line of the surface $x^2 + y^2 + z = 9$ at the point $(1, 2, 4)$.

Here, $f(x, y, z) = x^2 + y^2 + z - 9$. First, check that the point $(1, 2, 4)$ lies on the surface. Next, $f_x(1, 2, 4) = 2$, $f_y(1, 2, 4) = 4$ and $f_z(1, 2, 4) = 1$. The tangent plane is given by

$$2(x - 1) + 4(y - 2) + (z - 4) = 0.$$

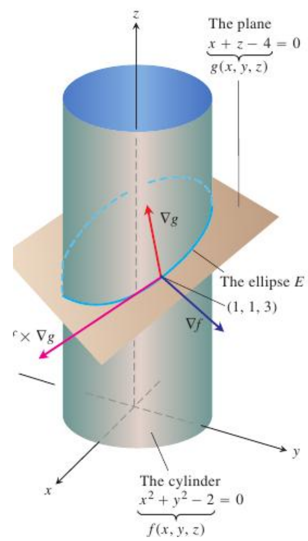
The normal line at $(1, 2, 4)$ is given by

$$x = 1 + 2t, \ y = 2 + 4t, \ z = 4 + t.$$

□

(1.48) ***Example***

Find the tangent line to the curve of intersection of the surfaces $f(x, y, z) := x^2+y^2-2 = 0$ and $g(x, y, z) := x + z - 4 = 0$ at the point $(1, 1, 3)$.



The tangent line lies in the tangent plane to $f(x, y, z) = 0$ and also it lies in the tangent plane to $g(x, y, z) = 0$, at $(1, 1, 3)$. So, the tangent line is orthogonal to both ∇f and ∇g at $(1, 1, 3)$. Thus it is parallel to

$$\nabla f \times \nabla g = (2\hat{\imath} + 2\hat{\jmath}) \times (\hat{\imath} + \hat{k}) = 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}.$$

Then the equation of the tangent line is

$$x = 1 + 2t, \ y = 1 - 2t, \ z = 3 - 2t.$$

□

1.9 Taylor's Theorem

For a function of one variable, a polynomial approximation is given by Taylor's formula. Observe that it is a generalization of the Mean value theorem. In the following, we identify $f^{(0)}(x)$ with $f(x)$.

(1.49) ***Theorem*** (Taylor's Formula for one variable)
Let I be an open interval, $a \in I$ and let a function $f : I \to \mathbb{R}$ have nth order derivative for $n \in \mathbb{N}$. Then, corresponding to each $x \in I, \ x \neq a$ there exists a point c between a and x such that

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}+\frac{f^{(n)}(c)}{n!}(x-a)^n.$$

Proof. Suppose $x > a$ and $x \in I$. Then $[a, x] \subseteq I$. For any $t \in [a, x]$, let

$$p(t) = f(a) + f'(a)(t - a) + \frac{f''(a)}{2!}(t - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(t - a)^{n-1}.$$

Here, we treat x as a certain point, not a variable; and t as a variable. Write

$$g(t) = f(t) - p(t) - \frac{f(x) - p(x)}{(x-a)^n}\,(t-a)^n.$$

We see that $g(a) = 0,\ g'(a) = 0,\ g''(a) = 0,\ \ldots, g^{(n-1)}(a) = 0,$ and $g(x) = 0$.

By Rolle's theorem, there exists $c_1 \in (a, x)$ such that $g'(c_1) = 0$. Since $g(a) = 0$, apply Rolle's theorem once more to get a $c_2 \in (a, c_1)$ such that $g''(c_2) = 0$.

Continuing this way, we get a $c_n \in (a, c_{n-1})$ such that $g^{(n)}(c_n) = 0$.

Since $p(t)$ is a polynomial of degree at most $n-1$, $p^{(n)}(t) = 0$. Then

$$g^{(n)}(t) = f^{(n)}(t) - \frac{f(x) - p(x)}{(x-a)^n}\,n!.$$

Evaluating at $t = c_n$ we obtain $\dfrac{f(x) - p(x)}{(x-a)^n} = \dfrac{f^{(n)}(c_n)}{n!}$. Consequently,

$$g(t) = f(t) - p(t) - \frac{f^{(n)}(c_n)}{n!}\,(t-a)^n.$$

Evaluating it at $t = x$ and using the fact that $g(x) = 0$, we get

$$f(x) = p(x) + \frac{f^{(n)}(c_n)}{n!}\,(x-a)^n.$$

This proves the result for any $x > a$ and $x \in I$. The case $x < a$ and $x \in I$ is proved analogously. ∎

We have a similar result for functions of several variables.

(1.50) ***Theorem*** (Taylor)

Let D be a region in the plane, (a, b) be an interior point of D and let the function $f : D \to \mathbb{R}$ have continuous partial derivatives of order n in some disk D_0 around (a, b) and contained in D for $n \in \mathbb{N}$. Then corresponding to each $(a+h, b+k) \in D_0$, there exists $\theta \in [0, 1]$ such that

$$\begin{aligned} f(a+h, b+k) = f(a, b) &+ \sum_{m=1}^{n-1} \frac{1}{m!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a, b) \\ &+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k). \end{aligned}$$

For example, $\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) = \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{(a,b)}$.

Proof. Due to the Increment theorem, f has continuous partial derivatives up to order n in any disk around (a, b). Let I be an open interval containing $[0, 1]$. Define the function $\phi : I \to \mathbb{R}$ by $\phi(t) = f(a + th, b + tk)$. Then,

$$\phi'(t) = f_x(a + th, b + tk)h + f_y(a + th, b + tk)k = \Big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\Big)f(a + th, b + tk).$$

$$\phi^{(2)}(t) = (f_{xx}h + f_{xy}k)h + (f_{yx}h + f_{yy}k)k = \Big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\Big)^2 f(a + th, b + tk).$$

By induction, we get $\phi^{(m)}(t) = \Big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\Big)^m f(a + th, b + tk).$

Using (1.49) on the single variable function $\phi(t)$, we have

$$\phi(1) = \phi(0) + \sum_{m=1}^{n-1} \frac{\phi^{(m)}(0)}{m!} + \frac{\phi^{(n)}(\theta)}{n!} \quad \text{for some } \theta \in [0, 1].$$

Substituting the expressions for $\phi(1)$, $\phi(0)$, $\phi^{(m)}(0)$ and $\phi^{(n)}(\theta)$, we get the required result. ∎

(1.51) *Example*

Let $f(x, y) = x^2 + xy - y^2$, $a = 1$, $b = -2$.

Here, $f(1, -2) = -5$, $f_x(1, -2) = 0$, $f_y(1, -2) = 5$, $f_{xx} = 2$, $f_{xy} = 1$, $f_{yy} = -2$.

Then $f(x, y) = -5 + 5(y + 2) + \frac{1}{2}\big[2(x - 1)^2 + 2(x - 1)(y + 2) - 2(y + 2)^2\big]$.

This becomes exact, since third (and more) order derivatives are 0. □

Taylor's theorem can be used to approximate $f(a + h, b + k)$ in terms of $f(a, b)$, the increments h, k and the values of the partial derivatives at (a, b). When the terms involving h^2, k^2 and higher powers of h, k are neglected, we obtain a linear approximation such as $f(a, b) + f_x(a, b)h + f_y(a, b)k$ to $f(a + h, b + k)$. Taking $h = x - a$ and $k = y - b$, we define the **standard linearization** of $f(x, y)$ as

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We also call $L(x, y)$ as a **linear approximation** of $f(x, y)$. Here, $L(x, y)$ is taken as an approximation of $f(x, y)$, assuming that the point (x, y) is (intuitively) close to the point (a, b).

The standard linearization is closely related to the tangent plane. Recall that the equation of the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

This formula holds true for all points (x, y, z) on the tangent plane at $(a, b, f(a, b))$. To approximate $f(x, y)$ for a point (x, y) close to (a, b), we may take

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The LHS is the z-component of the point on the surface $z = f(x, y)$ corresponding to the point (x, y). The RHS is the the z-component of the point on the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ corresponding to the point (x, y). Notice that the RHS is same as $L(x, y)$.

In the notation of increments, the standard linearization takes the form

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

Analogous to functions of one variable, we define the RHS as the **total differential** of f, also called the **differential**, and write it as df at (a, b). Conventionally, Δx is written as dx and Δy is written as dy also. That is, for any point (x, y) in the domain of $f(x, y)$ we define the differential as

$$df = f_x(x, y)dx + f_y(x, y)dy.$$

The above approximation formula states that the differential df is a linear approximation to the total increment Δf.

(1.52) *Example*

The dimensions of a rectangular box are measured to be 75cm, 60cm, and 40 cm, and each measurement is correct to within 0.2cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

The volume of the box is $V = xyz$. So,

$$dV = V_x dx + V_y dy + V_z dz \quad \text{at } (75, 60, 40).$$

Now, $|\Delta x|, |\Delta y|, |\Delta z| \le 0.2$cm, $V_x(75, 60, 40) = (yz)$ at $y = 60, z = 40$, which is equal to 60×40. Similarly, $V_y(75, 60, 40) = 75 \times 40$ and $V_z(75, 60, 40) = 75 \times 60$. So, the largest error in cubic cm is

$$|\Delta V| \approx |dV| = 60 \times 40 \times 0.2 + 40 \times 75 \times 0.2 + 75 \times 60 \times 0.2 = 1980.$$

Notice that the relative error is $1980/(75 \times 60 \times 40)$, which is about 1%. □

The **linearization error** in the standard linearization at (a, b) can be written as

$$\begin{aligned} E(x, y) &= f(x, y) - L(x, y) \\ &= \frac{1}{2!}\big((x-a)^2 f_{xx}(c, d) + 2(x-a)(y-b)f_{xy}(c, d) + (y-b)^2 f_{yy}(c, d)\big) \end{aligned}$$

where $c = a + \theta(x - a),\ d = b + \theta(y - b)$ for some $\theta \in [0, 1]$. An estimate of the error $E(x, y)$ is as follows.

(1.53) *Theorem*

Let D be an open region in the plane. Let the function $f : D \to \mathbb{R}$ have continuous second order partial derivatives. Let R be a rectangle containing (a, b) and contained in D. Suppose there exists an $M \in \mathbb{R}$ such that $|f_{xx}|, |f_{xy}|, |f_{yy}| \le M$ for all points in R. Then, for all $(x, y) \in R$,

$$|E(x,y)| \le \frac{M}{2}\big(|x-a| + |y-b|\big)^2.$$

Proof. Taylor's formula says that $f(x,y) = L(x,y) + E(x,y)$, where

$$E(x,y) = \frac{1}{2}\Big[(x-a)^2 f_{xx}(c,d) + 2(x-a)(y-b)f_{xy}(c,d) + (y-b)^2 f_{yy}(c,d)\Big].$$

for some c in between x and a, and some d between y and b. Since $|f_{xx}| \le M$, $|f_{xy}| \le M$, and $|f_{yy}| \le M$ for all points in R,

$$|E(x,y)| \le \frac{M}{2}|(x-a)^2 + 2(x-a)(y-b) + (y-b)^2| \le \frac{M}{2}\big(|(x-a| + |y-b|\big)^2. \quad \blacksquare$$

(1.54) *Example*

Find the standard linearization of $f(x,y) = x^2 - xy + y^2/2 + 3$ at $(3,2)$. Also find an upper bound of the linearization error in the rectangle $|x-3| \le 0.1$, $|y-2| \le 0.1$.

The standard linearization (linear approximation) of $f(x,y)$ at (a,b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Now, $f(3,2) = 8$, $f_x(3,2) = (2x-y)|_{(3,2)} = 4$ and $f_y(3,2) = (-x+y)|_{(3,2)} = -1$. Thus

$$L(x,y) = 8 + 4(x-3) - (y-2) = 4x - y - 2.$$

The error in this linearization is

$$E(x,y) = f(x,y) - L(x,y) = x^2 - xy + y^2/2 + 3 - 4x + y + 2.$$

The rectangle is $R:\ |x-3| \le 0.1, |y-2| \le 0.1$. Here, $f_{xx} = 2$, $f_{xy} = -1$, $f_{yy} = 1$.

So, we take $M = 2$ as an upper bound for their absolute values. Then

$$|E(x,y)| \le |x-3|^2 + |y-2|^2 \le (0.1 + 0.1)^2 = 0.04. \quad \square$$

(1.55) *Example*

For the function $f(x,y,z) = x^2 - xy + 2\sin z$, find the linearization and the maximum error incurred at $P(2,1,0)$ in the cuboid $|x-2| \le 0.01$, $|y-1| \le 0.02$, $|z| \le 0.01$.

$f(P) = f(2,1,10) = 4-2 = 2$, $f_x(P) = (2x-y)|_{(2,1,0)} = 3$, $f_y(P) = (-x)|_{(2,1,0)} = -2$ and $f_z(P) = (2\cos z)|(2,1,0) = 2$. Thus

$$L(x,y,z) = f(P) + f_x(P)(x-2) + f_y(P)(y-1) + f_z(P)z = 3x - 2y + 2z - 2.$$

All double derivatives are bounded above by 2. So,

$$E(x, y, z)|_P \le \frac{2}{2}\big(|x-2| + |y-1| + |z|\big)^2 \le 0.0016. \qquad \square$$

In Taylor's theorem, if we keep h^2, k^2 terms and neglect all higher powers of the increments h, k, we obtain the **quadratic approximation** to $f(x, y)$. Thus, the quadratic approximation of $f(x, y)$, where (x, y) is close to the point (a, b) is given by

$$f(x,y) \approx f(a,b) + (x-a)f_x + (y-b)f_y + \tfrac{1}{2}\big((x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\big).$$

Here, all the partial derivatives are evaluated at the point (a, b). That is, in the above approximation formula, f_x is actually $f_x(a, b)$ and etc. The error $E(x, y)$ in the quadratic approximation is the term that has been left out in Taylor's theorem for $n = 2$. That is, the error in the quadratic approximation is

$$E(x,y) = \frac{1}{6}\Big((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\Big)^3 f\big(a + \theta(x-a), b + \theta(y-b)\big)$$

for some $\theta \in [0, 1]$. This error can be estimated as earlier provided we know some upper bound for the third order partial derivatives of f in a rectangle containing the point (a, b).

Similarly, higher order approximations and their associated errors can be obtained from Taylor's theorem, though those will look quite complicated.

1.10 Extreme Values

Let D be a region in the plane, $(a, b) \in D$, and let $f : D \to \mathbb{R}$ be a function.

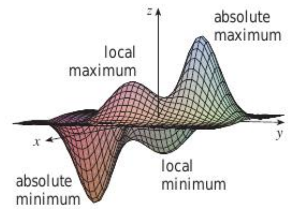
We say that $f(x, y)$ has a **local maximum** at (a, b) iff (a, b) is an interior point of D and $f(x, y) \le f(a, b)$ for all (x, y) in some open disk around (a, b) that is contained in D. Informally, the condition says that $f(x, y) \le f(a, b)$ holds for all $(x, y) \in D$ near (a, b). In such a case, the point (a, b) is called a **point of local maximum** of $f(x, y)$, and the number $f(a, b)$ is called a **local maximum value** of $f(x, y)$.

Replace all $\le$ by $\ge$ in the above definitions; and call all those **minimum** instead of *maximum*. The function $f(x, y)$ has a **local extremum** at (a, b) iff $f(x, y)$ has a local maximum or a local minimum at (a, b).

We say that f has the **absolute maximum** at $(a, b) \in D$ iff for all $(x, y) \in D$, $f(x, y) \le f(a, b)$. In this case, the point (a, b) is called a **point of absolute**

maximum of $f(x, y)$, and the number $f(a, b)$ is called the **(absolute) maximum value** of f. The *points of absolute minimum* and the *(absolute) minimum value* are defined by replacing $\leq$ with $\geq$. Both 'absolute maximum' and 'absolute minimum' are commonly referred to as 'absolute extremum'.

Notice that a local extremum point must be an interior point whereas an absolute extremum point need not be an interior point; it is allowed to be any point from D.



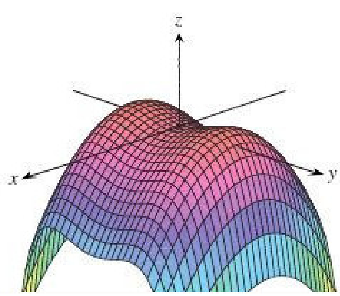
An interior point (a, b) of D is called a **critical point** of $f(x, y)$ iff either $f_x(a, b) = 0 = f_y(a, b)$ or at least one of $f_x(a, b)$, $f_y(a, b)$ does not exist.

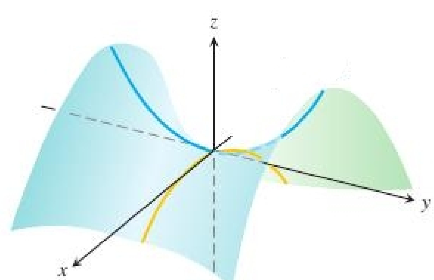
(1.56) *Theorem*

Let D be a region in the plane, (a, b) be an interior point of D and let the function $f : D \to \mathbb{R}$ have partial derivatives at (a, b). If (a, b) is a point of local extremum of f, then it is a critical point of f.

Proof. Suppose (a, b) is a point of local maximum of f. Then the function $g(x) = f(x, b)$ has a local maximum at $x = a$. We have $g'(a) = 0$. It gives $f_x(a, b) = 0$. Similarly, consider $h(y) = f(a, y)$ and conclude that $f_y(a, b) = 0$. Give similar argument if f has a local minimum at (a, b). ∎

Geometrically, it says that if at an interior point (a, b), there exists a tangent plane to the surface $z = f(x, y)$, and if this point (a, b) happens to be an extremum point, then there exists a horizontal tangent plane to the surface at (a, b).





Let D be a region in the plane. Let $f : D \to \mathbb{R}$ have continuous partial derivatives. Let (a, b) be a critical point of $f(x, y)$. The point $(a, b, f(a, b))$ on the surface is called a **saddle point** of $f(x, y)$ if in every disk around (a, b) and contained in D, there are points (x_1, y_1), (x_2, y_2) such that $f(x_1, y_1) < f(a, b) < f(x_2, y_2)$. At a saddle point, the function has neither a local maximum nor a local minimum.

The **Hessian** of a function $f(x, y)$ is defined by $H(f) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$.

(1.57) *Theorem*

Let D be a region in the plane, (a, b) be an interior point of D and let the function $f : D \to \mathbb{R}$ have continuous second order partial derivatives in a disk D_0 around (a, b), where $D_0 \subseteq D$. Assume that (a, b) is a critical point of f.

(1) *If $H(f)(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).*

(2) *If $H(f)(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).*

(3) *If $H(f)(a, b) < 0$ then $f(x, y)$ has a saddle point at (a, b).*

(4) *If $H(f)(a, b) = 0$, then nothing can be said about extrema at (a, b), in general.*

Proof. Let $(a + h, b + k) \in D_0$. By Taylor's formula,

$$f(a+h, b+k) = (f + hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a+\theta h, b+\theta k)}.$$

Since (a, b) is a critical point of f, $f_x(a, b) = 0 = f_y(a, b)$. Then

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a+\theta h, b+\theta k)} \qquad (1.10.1)$$

(1) Suppose $H(f)(a, b) > 0$ and $f_{xx}(a, b) < 0$. Then, multiply both sides of (1.10.1) by $2f_{xx}$, add and subtract $(f_{xy})^2 k^2$, and rearrange to get (All of f_{xx}, f_{xy}, f_{yy} are evaluated at the point $(a + \theta h, b + \theta k)$.)

$$2f_{xx}[f(a+h, b+k) - f(a, b)] = (hf_{xx} + kf_{xy})^2 + k^2 H(f)(a + \theta h, b + \theta k).$$

By continuity of $f_{xx}, f_x y, f_{yy}$, we have $f_{xx} < 0$ and $H(f) > 0$ at $(a + \theta h, b + \theta k)$ in a disk D_1 around (a, b), where $D_1 \subseteq D_0$ and $\theta \in [0, 1]$. The RHS is positive. Therefore, $f(a + h, b + k) - f(a, b) < 0$. That is, (a, b) is a local maximum point.

(2) Let $H(f)(a, b) > 0$ and $f_{xx}(a, b) > 0$. As in (1), we have $f_{xx}(a + \theta h, b + \theta k) > 0$ in a disk D_2 around (a, b), where $D_2 \subseteq D_0$. So, $f(a + h, b + k) - f(a, b) > 0$. That is, (a, b) is a local minimum point.

(3) Let $H(f)(a, b) < 0$. We want to show that $f(a + h, b + k) - f(a, b)$ has opposite signs at different points in any small disk around (a, b). We break this case into three sub-cases:

(3A) $f_{xx}(a, b) \neq 0$. (3B) $f_{yy}(a, b) \neq 0$, (3C) $f_{xx}(a, b) = f_{yy}(a, b) = 0$.

(3A) Let $H(f)(a, b) < 0$ and $f_{xx}(a, b) \neq 0$. First, set $h = t, k = 0$ in (1.10.1) and evaluate the following limit:

$$\lim_{t \to 0} \frac{f(a+h, b+k) - f(a, b)}{t^2} = \lim_{t \to 0} \frac{t^2 f_{xx}(a+t, b)}{2t^2} = \frac{f_{xx}(a, b)}{2}.$$

Next, set $h = -tf_{xy}(a,b),\ k = tf_{xx}(a,b)$. Use (1.10.1) to obtain

$$\lim_{t\to 0}\frac{f(a+h,b+k)-f(a,b)}{t^2} = \lim_{t\to 0}\frac{1}{2}(f_{xy}^2 f_{xx} - 2f_{xx}f_{xy}^2 + f_{xx}^2 f_{yy}) = \frac{f_{xx}(a,b)}{2}H(f)(a,b).$$

Since $H(f)(a,b) < 0$, these two limits have opposite signs. Due to continuity, $f(a+h,b+k)-f(a,b)$ has opposite signs in a disk D_3 around (a,b), where $D_3 \subseteq D_0$.

(3B) Let $H(f)(a,b) < 0$ and $f_{yy}(a,b) \neq 0$. This is similar to (3A).

(3C) Let $H(f)(a,b) < 0$ and $f_{xx}(a,b) = f_{yy}(a,b) = 0$. First, set $h = k = t$. Use (1.10.1) to get

$$\lim_{t\to 0}\frac{f(a+h,b+k)-f(a,b)}{t^2} = \lim_{t\to 0}\frac{1}{2}(f_{xx} + 2f_{xy} + f_{yy})|_{(a+t,b+t)} = f_{xy}(a,b).$$

Next, set $h = t,\ k = -t$. Using (1.10.1) again, we have

$$\lim_{t\to 0}\frac{f(a+h,b+k)-f(a,b)}{t^2} = \lim_{t\to 0}\frac{1}{2}(f_{xx} - 2f_{xy} + f_{yy})|_{(a+t,b+t)} = -f_{xy}(a,b).$$

As in (3A), $f(a+h,b+k)-f(a,b)$ has opposite signs in any disk $D_4 \subseteq D_0$ around (a,b).

(4) Consider $f(x,y) = x^2y^2$. At $(0,0)$, all of $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are 0. So, $(0,0)$ is a critical point and $H(f)(0,0) = 0$. If $(x,y) \neq (0,0)$, then $f(x,y) \geq 0 = f(0,0)$. That is, $H(f)(0,0) = 0$ and $f(x,y)$ has a point of local minimum at $(0,0)$.

For the function $g(x,y) = x^3y^3$, we see that $(0,0)$ is a critical point with $g(0,0) = 0$ and $H(g)(0,0) = 0$. However, in any disk around $(0,0)$, there are points where $g(x,y)$ is positive and also there are points where g is negative. So, $(0,0)$ is a saddle point of $g(x,y)$. ∎

Notice that the case $H(f)(a,b) > 0$ and $f_{xx}(a,b) = 0$ is not possible. Moreover, Under the condition that $H(f)(a,b) > 0$, both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have the same sign. Thus, in (1.57-1 & 2), the sign condition on $f_{xx}(a,b)$ can be replaced by the corresponding sign condition on $f_{yy}(a,b)$. It also says that if $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have opposite signs, then the critical point (a,b) is not a point of extremum of $f(x,y)$. Geometrically, the condition $H(f)(a,b) > 0$ implies that the surface $z = f(x,y)$ curves the same way (either upward or downward) in all directions near (a,b); and if $H(f)(a,b) < 0$, then the surface curves in different directions near (a,b).

Caution: The point (a,b) in (1.57) is assumed to be an interior point of the domain of $f(x,y)$. If the domain has points which are not interior points, then those can also be possible (absolute) extremum points.

(1.58) *Example*

Find the extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Domain of f is the whole plane. Its extreme values are all local extrema. The first and second order partial derivatives of f are continuous. Now,

$$f_x = y - 2x - 2, \quad f_y = x - 2y - 2 \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

The critical points satisfy $f_x = 0 = f_y$. That is, $x = y = -2$.

We get $f_{xx}(-2, -2) = -2$, $f_{xy}(-2, -2) = 1$, $f_{yy}(-2, -2) = -2$.

Then $H(f)(-2, -2) = 3 > 0$.

Since $f_{xx} < 0$, the function f has a local maximum at $(-2, -2)$.

Also f has absolute maximum at $(-2, 2)$; the maximum value is $f(-2, -2) = 8$. ∎

(1.59) ***Example***

Investigate $f(x, y) = x^4 + y^4 - 4xy + 1$ for extreme values.

The function has continuous first and second partial derivatives everywhere.

The critical points are at (x, y) where $f_x = 4x^3 - 4y = 0 = f_y = 4y^3 - 4x$.

That is, when $x^3 = y$ and $y^3 = x$. This gives $x^9 = x$, which has solutions $x = 0, 1, -1$ in $\mathbb{R}$. The corresponding y values are $0, 1, -1$.

The possible points of extremum are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

Now, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Thus $H(f) = 144x^2y^2 - 16$.

$H(f)(0, 0) = -16 < 0$. Thus $f(x, y)$ has a saddle point at $(0, 0)$.

$H(f)(1, 1) = 128 > 0$, $f_{xx}(1, 1) = 12 > 0$. So, f has a local minimum at $(1, 1)$.

$H(f)(-1, -1) = 128$, $f_{xx}(-1, -1) = 12$. Thus f has a local minimum at $(-1, -1)$.

The local minimum values are $f(1, 1)$ and $f(-1, -1)$; here both are equal to -1. Also, the absolute minimum value of $f(x, y)$ is -1, which is achieved at $(1, 1)$ and $(-1, -1)$. ∎

(1.60) ***Example***

Find absolute extrema of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ defined on the triangular region bounded by the straight lines $x = 0$, $y = 0$, and $x + y = 9$.

The critical points are solutions of $f_x = 2 - 2x = 0 = f_y = 2 - 2y$. That is, $x = 1, y = 1$. This accounts for the interior points of the region.

Draw the picture. We should consider the boundary in detail. First, the vertices of the triangle such as $A(0, 0)$, $B(0, 9)$, $C(9, 0)$ are possible extremum points.

Next, on the line segment AB, $x = 0$; thus $f(x, y)$ is given by:

$$g(y) = f(0, y) = 2 + 2y - y^2 \quad \text{for } 0 \le y \le 9.$$

Then $g'(y) = 0 \Rightarrow y = 1$. Thus, a possible extremum point is $(0, 1)$.

Similarly, on the line segment AC, $y = 0$; thus $f(x, y)$ is given by:

$$h(x) = f(x, 0) = 2 + 2x - x^2 \quad \text{for } 0 \le x \le 9.$$

Now, $g'(x) = 0 \Rightarrow x = 1$. Thus $(1, 0)$ is a possible extremum point.

On the line segment BC, $x + y = 9$; thus $f(x, y)$ is given by:

$$\phi(x) = f(x, 9-x) = 2+2x+2(9-x)-x^2-(9-x)^2 = -61+18x-2x^2 \quad \text{for } 0 \le x \le 9.$$

Now, $\phi'(x) = 0 \Rightarrow 18 - 4x = 0 \Rightarrow x = 9/2,\ y = 9 - x = 9/2$. Thus $(9/2, 9/2)$ is a possible extremum point.

Then, all possible points at which $f(x, y)$ may have an extremum are

$$(1, 1),\ (0, 0).\ (0, 9),\ (9, 0),\ (1.0),\ (0, 1),\ (9/2, 9/2).$$

The values of $f(x, y)$ at these points are

$$f(1, 1) = 4, f(0, 0) = 2, f(0, 9) = -61, f(9, 0) = -61,$$
$$f(1, 0) = 3, f(0, 1) = 3, f(9/2, 9/2) = -41/2.$$

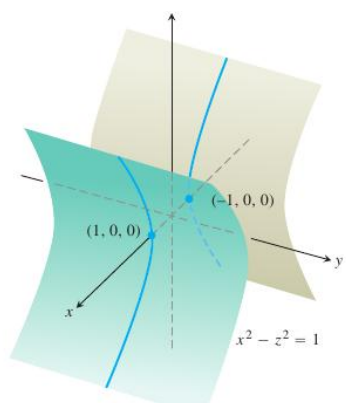
Comparing these values, we see that $f(x, y)$ has absolute minimum at $(0, 9)$ and $(9, 0)$ with the minimum value as -61. It has absolute maximum at $(1, 1)$ with the maximum value as 4. ∎

1.11 Lagrange Multipliers

In some problems we are required to maximize or minimize a function subject to a given constraint. See the following example.

(1.61) *Example*

Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 = 1$.



We seek a point (x, y, z) that minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - z^2 = 1$. Taking $z^2 = x^2 - 1$, we seek (x, y) that minimizes

$$g(x,y) := f\left(x, y, \pm\sqrt{x^2-1}\right) = x^2 + y^2 + x^2 - 1 = 2x^2 + y^2 - 1.$$

Now, $g_x = 4x,\ g_y = 2y$. Equating them to zero gives $x = 0$ and $y = 0$. But $x = 0$ does not correspond to any point on the surface $x^2 - z^2 = 1$. So, the method fails!

Instead of eliminating z, suppose we eliminate x. In that case, we seek to minimize

$$h(y,z) := f(\pm\sqrt{1+z_2}, y, z) = 1 + z^2 + y^2 + z^2 = y^2 + 2z^2 + 1.$$

Then $h_y = 0 = h_z$ implies that $2y = 0 = 4z$. The point so obtained is $y = 0,\ z = 0$. This corresponds to the points $(\pm 1, 0, 0)$ on the surface.

Now, of course, we can proceed as earlier for minimizing $h(y,z)$.

Here, $h_{yy} = 2,\ h_{yz} = 0,\ h_{zz} = 4$.

At $y = 0, z = 0$, we have $H(h)(0,0) = h_{yy}h_{zz} - h_{yz}^2 = 8 > 0$.

Since $h_{yy}(0,0) > 0$, we conclude that $h(y,z)$ has a local minimum at $(0,0)$.

These points $(\pm 1, 0, 0)$ of local minima give the minimum value of the distance $f(x,y,z)$ as 1. ∎

It raises the question: "How do we know which variable is to be eliminated so that a solution may be reached?" We would rather look for alternative ways of solving such *constrained optimization problems*.

(1.62) *Theorem*

Let D be an open region in the plane. Let $f, g : D \to \mathbb{R}$ have continuous partial derivatives. Let f have an extreme value at a point (a,b) on the curve $g(x,y) = 0$, where $\nabla g(a,b) \neq 0$. Then a, b and λ satisfy the equations

$$f_x(a,b) + \lambda g_x(a,b) = 0, \quad f_y(a,b) + \lambda g_y(a,b) = 0, \quad g(a,b) = 0.$$

Proof. The constraint $g(x,y) = 0$ is a level curve of the surface $z = g(x,y)$. Let $f(x,y)$ have an extreme value at (a,b) on this curve. Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ be a prametrization of this curve, where $(a,b) = \vec{r}(t_0)$.

Now, $h(t) = g\big(x(t), y(t)\big) = 0$ for all t. Differentiating with respect to t and evaluating at $t = t_0$, we have $h'(t_0) = 0$. That is,

$$0 = h'(t_0) = g_x(a,b)x'(t_0) + g_y(a,b)y'(t_0) = \nabla g(a,b) \cdot \vec{r}\,'(t_0).$$

On the other hand, the composite function $\phi(t) = f\big(x(t), y(t)\big)$ represents the values that f takes on C. Since f has an extreme value at $(a,b) = \vec{r}(t_0)$, the function $\phi(t)$ has an extreme value at $t = t_0$. Then $\phi'(t_0) = 0$. That is,

$$0 = \phi'(t_0) = f_x(a,b)x'(t_0) + f_y(a,b)y'(t_0) = (\nabla f)(a,b) \cdot \vec{r}\,'(t_0).$$

So, both $(\nabla f)(a,b)$ and $(\nabla g)(a,b)$ are orthogonal to $\vec{r}\,'(t_0)$, and $(\nabla g)(a,b) \neq 0$. Thus, $(\nabla f)(a,b)$ and $(\nabla g)(a,b)$ are parallel. That is, there exists $\lambda \in \mathbb{R}$ such that $(\nabla f + \lambda\,\nabla g)(a,b) = 0$. Breaking into components, we obtain $f_x(a,b) + \lambda g_x(a,b) = 0$ and $f_y(a,b) + \lambda g_y(a,b) = 0$. Further, (a,b) lies on the curve $g(x,y) = 0$; so, $g(a,b) = 0$. Therefore, a, b and λ satisfy the required equations. ∎

Notice that if we set $F(x,y,\lambda) := f(x,y) + \lambda g(x,y)$, then

$$F_x = f_x + \lambda g_x,\ F_y = f_y + \lambda g_y,\ F_\lambda = g(x,y).$$

These are the equations mentioned in (1.62) that a, b and λ satisfy. A result similar to (1.62) holds when f is a function of n variables and there are m number of constraints. We thus reach at the following method of solving a constrained optimization problem, called the method of **Lgarange multipliers**.

Requirement: Find extrema of the function $f(x_1, \ldots, x_n)$ subject to the conditions

$$g_1(x_1, \ldots, x_n) = 0,\ \cdots,\ g_m(x_1, \ldots, x_n) = 0.$$

Method: Set the **auxiliary function**

$$F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) := f(x_1, \ldots, x_n) + \lambda_1 g_1(x_1, \ldots, x_n) + \cdots \lambda_m g_m(x_1, \ldots, x_n).$$

Equate to zero the partial derivatives of F with respect to $x_1, \ldots, x_n,\ \lambda_1, \ldots, \lambda_m$.
It results in $m + n$ equations in the unknowns $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$.
Determine the values of $x_1, \ldots, x_n\ \lambda_1, \ldots, \lambda_m$ from these equations.
The points of extremum are among these values of $x_1, \ldots, x_n$.

Caution: Lagrange multipliers method succeeds under the assumption that such extreme values exist where $\nabla g_j \neq 0$ for any j. Further, the points thus obtained are only possible points of extremum. All of those need not be points of extremum. Other considerations may be required to determine whether any such point is an actual maximum or minimum of f.

(1.63) *Example*

Consider the problem in (1.61). We need to minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to $g(x,y,z) = x^2 - z^2 - 1 = 0$. The auxiliary function is

$$F(x,y,z,\lambda) = f + \lambda g = x^2 + y^2 + z^2 + \lambda(x^2 - z^2 - 1).$$

Now, $F_x = 2x + 2\lambda x$, $F_y = 2y$, $F_z = 2z - 2\lambda z$ and $F_\lambda = x^2 - z^2 - 1$. We then equate the partial derivatives of F at (a,b,c) to 0. This gives

$$2a + 2\lambda a = 0,\ 2b = 0,\ 2c - 2\lambda c = 0,\ a^2 - c^2 - 1 = 0.$$

We then have $\big(a = 0 \text{ or } \lambda = -1\big)$, $b = 0$, $\big(c = 0 \text{ or } \lambda = 1\big)$, and $a^2 - c^2 - 1 = 0$.

From these options, $a = 0$ is not possible since for any c, $a^2 - c^2 - 1 = -c^2 - 1 \neq 0$. And, $\lambda = 1$ leads to $a = 0$, which is not possible. We are left with $\lambda = -1, b = 0$ and $c = 0$. Now, $a^2 - c^2 - 1 = 0$ gives $a = \pm 1$.

The corresponding points are $(\pm 1, 0, 0)$. Notice that $\nabla g(\pm 1, 0, 0) = \pm 2\hat{\imath} \neq 0$. At these points $f(x, y, z)$ has value 1. Since $f(x, y, z) = x^2 + y^2 + z^2$ can be made as large as possible by choosing (x, y, z) satisfying $x^2 = z^2 + 1$ suitably, $f(x, y, z)$ does not have a maximum. So, $f(x, y, z)$ at these points is minimum. Therefore, the points closest to the origin on the hyperbolic cylinder are $(\pm 1, 0, 0)$. ▯

(1.64) *Example*

Find the maximum volume of a box of length x, width y and height z given that $x + 2y + 2z = 108$.

We maximize $V(x, y, z) = xyz$ subject to the constraint $x + 2y + 2z - 108 = 0$. using Lagrange multipliers method, we set up the auxiliary function

$$F(x, y, z) = xyz + \lambda(x + 2y + 2z - 108).$$

Now, $F_x = yz + \lambda,\ F_y = xz + 2\lambda,\ F_z = xy + 2\lambda$ and $F_\lambda = x + 2y + 2z - 108$.
A possible point of extremum (a, b, c) satisfies

$$bc + \lambda = 0, \quad ac + 2\lambda = 0, \quad ab + 2\lambda = 0, \quad a + 2b + 2c - 108 = 0.$$

If $a = 0$ or $b = 0$ or $c = 0$, then $\lambda = 0$ and $V(a, b, c) = abc = 0$.
Otherwise, assume that $a \neq 0,\ b \neq 0$ and $c \neq 0$. Then, $\lambda = -bc$, $a = 108 - 2b - 2c$, $ac - 2bc = 0$ and $ab - 2bc = 0$. The latter two equations give $a = 2b = 2c$. Thus $b = c$ and the second equation gives $2b = 108 - 2b - 2b \Rightarrow b = 108/6 = 18$. Then, $c = 18$, $a = 36$ and $\lambda = -364$. Then, $V = abc = 11664$.

The two possible extreme values of V are 0 and 11664, the maximum is 11664. Therefore, the maximum volume is 11644 cubic units, when the box has length 36 units, width as 18 units and height as 18 units.

Notice that $\nabla g(36, 18, 18) = \hat{\imath} + 2\hat{\jmath} + 2\hat{k} \neq 0$. ▯

(1.65) *Example*

Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

With $g(x, y, z) = x - y + z - 1$ and $h(x, y, z) = x^2 + y^2 - 1$, the auxiliary function is

$$F(x, y, z, \lambda, \mu) := f + \lambda g + \mu h = x + 2y + 3z + \lambda(x - y + z - 1) + \mu(x^2 + y^2 - 1).$$

Setting $F_x = F_y = F_z = F_\lambda = F_\mu = 0$, for (a, b, c), we obtain

$$1 + \lambda + 2a\mu = 0,\ 2 - \lambda + 2b\mu = 0,\ 3 + \lambda = 0,\ a - b + c - 1 = 0,\ a^2 + b^2 - 1 = 0.$$

If $\mu = 0$, then $1 + \lambda = 0 = 2 - \lambda$; and it leads to inconsistency. So, let $\mu \neq 0$. We obtain: $\lambda = -3$, $a = 1/\mu$, $b = -5/(2\mu)$, $1/\mu^2 + 25/(4\mu^2) = 1$. That is, $\mu^2 = 29/4$. Then possible extreme points are

$$a = \pm 2/\sqrt{29},\ b = \mp 5/\sqrt{29},\ c = 1 \mp 7/\sqrt{29}.$$

Notice that at these points (a, b, c), $\nabla g(a, b, c) \neq 0$ and $\nabla(h)(a, b, c) \neq 0$. Evaluating $f(a, b, c)$, we see that the maximum of $f(a, b, c) = a + 2b + 3c$ is obtained for $a = -2/\sqrt{29}$, $b = 5/\sqrt{29}$ and $c = 1 + 7/\sqrt{29}$, and it is $3 + \sqrt{29}$. $\square$

1.12 Review Problems

Problem 1.1 Where is the function $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ continuous? What are the limits of f at the points of discontinuity?

The function $f(x, y)$ is defined everywhere in the plane except at the origin. When $(x, y) \neq (0, 0)$, the functions $g(x) = 2xy$ and $h(x, y) = x^2+y^2$ are continuous. Hence $f(x, y)$ is continuous everywhere except at $(0, 0)$. And, at $(0, 0)$,

$$\lim_{\substack{x\to 0\\ \text{along } y=x}} f(x, y) = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1, \qquad \lim_{\substack{x\to 0\\ \text{along } y=-x}} f(x, y) = \lim_{x\to 0} \frac{-2x^2}{2x^2} = -1.$$

Then, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist so that $f(x, y)$ is discontinuous at $(0, 0)$.

Problem 1.2 Determine the directions in which the function $f(x, y)$ has directional derivatives at $(0, 0)$, where

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Consider a unit vector $\hat{u} = a\hat{\imath} + b\hat{\jmath}$. The directional derivative of $f(x, y)$ at $(0, 0)$ in the direction of $\hat{u}$ is

$$(D_u f)(0, 0) = \lim_{h\to 0} \frac{f(ah, bh) - f(0, 0)}{h} = \lim_{h\to 0} \frac{ab^2}{a^2 + bh^2} = \begin{cases} b^2/a & \text{for } a \neq 0 \\ 0 & \text{for } a = 0. \end{cases}$$

Hence the directional derivative of $f(x, y)$ at $(0, 0)$ exists in each direction.

Notice that $(\nabla f)(0,0) = 0\hat{\imath} + 0\hat{\jmath}$. If you use the formula for the directional derivative at $(0,0)$ blindly, then it would give the wrong result that in every direction, the directional derivative of $f(x,y)$ is 0. What is the reason for this anomaly?

Problem 1.3 The hypotenuse c and the side a of a right angled triangle ABC determined with maximum absolute errors $|\Delta c| = 0.2$, $|\Delta a| = 0.1$ are, respectively, $c = 75$, $a = 32$. Determine the angle A and determine the maximum absolute error ΔA in the calculation of the angle A.

$$A(a,c) = \sin^{-1}\frac{a}{c} \Rightarrow \frac{\partial A}{\partial a} = \frac{1}{\sqrt{c^2 - a^2}},\quad \frac{\partial A}{\partial c} = \frac{-a}{c\sqrt{c^2 - a^2}}.$$

Then $|\Delta A| \le \dfrac{1}{\sqrt{(75)^2 - (32)^2}} \times 0.1 + \dfrac{32}{75\sqrt{(75)^2 - (32)^2}} \times 0.2 = 0.00273.$

Therefore $\sin^{-1}\frac{32}{75} - 0.00273 \le A \le \sin^{-1}\frac{32}{75} + 0.00273.$

Problem 1.4 Let $f(x,y,z) = x^2 + y^2 + z^2$. Find the directional derivative of $f(x,y,z)$ in the direction of the vector $\vec{v} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ at $(1,1,1)$.

The unit vector in the direction of $\vec{v}$ is $\hat{u} = \frac{2}{\sqrt{14}}\hat{\imath} + \frac{1}{\sqrt{14}}\hat{\jmath} + \frac{3}{\sqrt{14}}\hat{k}$. The gradient of f at $(1,1,1)$ is $\nabla f(1,1,1) = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k})(1,1,1) = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$. Then

$$(D_u f)(1,1,1) = (\nabla f \cdot \hat{u})(1,1,1) = \frac{12}{\sqrt{14}}.$$

Problem 1.5 Find the total increment Δz and the total differential dz of the function $z = xy$ at $(2,3)$ for $\Delta x = 0.1$, $\Delta y = 0.2$.

At $(2,3)$ with $\Delta x = 0.1$, $\Delta y = 0.2$, we have

$$\Delta z = (x+\Delta x)(y+\Delta y) - xy = y\Delta x + x\Delta y + \Delta x\Delta y = 3\times 0.1 + 2\times 0.2 + 0.1\times 0.2 = 0.72.$$

$$dz = z_x dx + z_y dy = y dx + x dy = y\Delta x + x\Delta y. = 3\times 0.1 + 2\times 0.2 = 0.7.$$

Problem 1.6 It is known that in computing the coordinates of a point (x,y,z,t) certain (small) errors such as $\Delta x, \Delta y, \Delta z, \Delta t$ might have been committed. Find the maximum absolute error so committed when we evaluate a function $f(x,y,z,t)$ at that point.

Let $\Delta u = f(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - f(x,y,z,t)$. We want to find max Δu. By Taylor's formula,

$$\Delta u = (f_x\Delta x + f_y\Delta y + f_z\Delta z + f_t\Delta t)(a,b,c,d)$$

where (a,b,c,d) lies on the line segment joining (x,y,z,t) to $(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t)$. Therefore,

$$|\Delta u| \le |f_x|\,|\Delta x| + |f_y|\,|\Delta y| + |f_z|\,|\Delta z| + |f_t|\,|\Delta t|.$$

Problem 1.7 Find a point in the plane where the function $f(x, y) = \frac{1}{2} - \sin(x^2 + y^2)$ has a local maximum.

We see that at $(0, 0)$, the function has a maximum value of $\frac{1}{2}$. To prove this, consider the neighborhood $B = \{(x, y) : x^2 + y^2 \le \pi/9\}$ of $(0, 0)$. Now, for any point $(a, b) \in B$ other than $(0, 0)$, we have

$$f(a, b) = \frac{1}{2} - \sin(a^2 + b^2) \le \frac{1}{2} = f(0, 0).$$

Problem 1.8 Decompose a given positive number a into three parts to make their product maximum.

Let $a = x + y + (a - x - y)$, for $0 \le x, y, a - x - y \le a$. Then x and y can take values from the region D bounded by the straight lines $x = 0, y = 0$ and $x + y = a$. The function to be maximized is

$$f(x, y) = xy(a - x - y) \quad \text{on } D.$$

On the line $x = 0$, $f(x, y)$ is 0; on the line $y = 0$, $f(x, y) = 0$; and on the line $x + y = a$, $f(x, y) = 0$.

In the interior of D, the partial derivatives of f are continuous. They are

$$f_x = y(a - 2x - y), \; f_y = x(a - x - 2y).$$

The critical points satisfy $y(a - 2x - y) = 0$ and $x(a - x - 2y) = 0$, where $0 < x, y, a - x - y < a$. The solutions of these equations give:

$$P_1 = (0, 0), \; P_2 = (0, a), \; P_3 = (a, 0), \; P_4 = \left(\frac{a}{3}, \frac{a}{3}\right).$$

Of these, the points P_1, P_2, P_3 are on the boundary of D; the only interior point is P_4. At P_4, the value of $f(x, y) = \frac{a^3}{27}$. Comparing $f(P_4)$ and the values of $f(x, y)$ on the boundary of D, we see that $f(P_4)$ is the maximum. The point P_4 corresponds to the decomposition of a as $a = \frac{a}{3} + \frac{a}{3} + \frac{a}{3}$.

Problem 1.9 Find extreme points of the function $z = x^3 + y^3 - 3xy$.

Here, z_x and z_y are continuous. Thus the critical points are obtained by solving

$$z_x = 3x^2 - 3y = 0, \; z_y = 3y^2 - 3x = 0.$$

These are $P_1 = (1, 1)$ and $P_2 = (0, 0)$.

The second derivatives are $z_{xx} = 6x, \; z_{xy} = -3, \; z_{yy} = 6y$.

$H(P_1) = (z_{xx}z_{yy} - z_{xy}^2)(P_1) = 36 - 9 = 27 > 0, \; z_{xx}(P_1) = 6 > 0$. Thus, P_1 is a minimum point and the minimum value of z is $z(1, 1) = -1$.

$H(P_2) = (z_{xx}z_{yy} - z_{xy}^2)(P_2) = -9 < 0$. Hence P_2 is a saddle point.

Problem 1.10 Find the maximum of $w = xyz$ given that $xy + zx + yz = a$ for a given positive number a, and $x > 0,\ y > 0,\ z > 0$.

The auxiliary function is

$$F(x, y, z, \lambda) = xyz + \lambda(xy + zx + yz - a).$$

Equating its partial derivatives to zero, we have

$$yz + \lambda(y + z) = 0,\ xz + \lambda(x + z) = 0,\ xy + \lambda(x + y) = 0.$$

Multiply the first by x, the second by y, and subtract to obtain:

$$\lambda x(y + z) - \lambda y(x + z) = 0 \Rightarrow \lambda z(x - y) = 0.$$

If $\lambda = 0$, then $xy + \lambda(x + y) = 0$ would imply $x = 0$ or $y = 0$. But $x > 0$ and $y > 0$. So, $\lambda \neq 0$. Also, $z > 0$. Therefore, $x = y$. Similarly, using the second and third equations, we get $y = z$. Therefore, $x = y = z$. Then

$$xy + zx + yz = a \text{ gives } x = y = z = \sqrt{a/3}.$$

The corresponding value of w is $(a/3)^{3/2}$. This cannot be the minimum value of w; reason: by reducing x, y close to 0, and taking a large value of z so that $xy + zx + yz = a$ is satisfied, w can be made as small as possible. Hence w has a maximum at $(\sqrt{a/3}, \sqrt{a/3}, \sqrt{a/3})$ and the maximum value is $(a/3)^{3/2}$.

Problem 1.11 Determine the maximum value of $z = (x_1 \cdots x_n)^{1/n}$ provided that $x_1 + \cdots + x_n = a$, where a is a given positive number.

Maximizing z is equivalent to maximizing $f(x_1, \ldots, x_n) = z^n = x_1 x_2 \cdots x_n$. Set up the auxiliary function

$$F(x_1, \ldots, x_n, \lambda) = x_1 x_2 \cdots x_n + \lambda(x_1 + \cdots x_n - a).$$

Equate the partial derivatives F_{x_i} to zero to obtain

$$x_1 \cdots x_{i-1} x_{i+1} \cdots x_n + \lambda = 0 \text{ for } i = 1, 2, \ldots, n.$$

Notice that $\lambda \neq 0$. Then multiplying by x_i, we see that $-\lambda x_i = x_1 x_2 \cdots x_n$ for each i. Therefore, $x_1 = x_2 = \cdots = x_n = a/n$. In that case, $f = (a/n)^n$ and $z = a/n$. This value is not a minimum value of z since z can be made arbitrarily small by choosing x_1 close to 0. Thus, the maximum of z is a/n.

This gives an alternative proof that the geometric mean of n positive numbers is no more than the arithmetic mean of those numbers.

Problem 1.12 A function $f(x, y)$ is called **homogeneous** of degree n in an open region $D \subseteq \mathbb{R}^2$ if for all $(x, y) \in D$, and for each positive λ, $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. For example,

$f(x, y) = (x^2 + y^2)^{3/2}$ is homogeneous of degree 3 in the whole plane;

$f(x, y) = x^{1/3}y^{-4/3}\tan^{-1}(y/x)$ is homogeneous of degree -1 in the region D, which is any quadrant without the axes.

Prove the following theorem by Euler:

Let D be an open region in the plane. Let $f : D \to \mathbb{R}$ have continuous partial derivatives. Then f is a homogeneous function of degree n iff $xf_x + yf_y = nf$.

Proof of Euler's theorem: Let f be a homogeneous function of degree n. Then $f(\lambda x, \lambda y) - \lambda^n f(x, y) = 0$. Differentiate it partially with respect to λ to obtain:

$$xf_x(\lambda x, \lambda y) + yf_y(\lambda x, \lambda y) = n\lambda^{n-1}f(x, y).$$

Here, f_x means the partial derivative of f with respect to the first variable, and f_y means the partial derivative of f with respect to the second variable.

Then set $\lambda = 1$ to get $xf_x(x, y) + yf_y(x, y) = nf(x, y)$.

Conversely, let $(a, b) \in D$. Define $\phi(\lambda) = f(\lambda a, \lambda b)$. Differentiate with respect to λ to get

$$\phi'(\lambda) = af_x(\lambda a, \lambda b) + bf_y(\lambda a, \lambda b).$$

However, $nf(\lambda a, \lambda b) = \lambda af_x(\lambda a, \lambda b) + \lambda bf_y(\lambda a, \lambda b)$. That is, $\lambda\phi'(\lambda) = n\phi(\lambda)$.

Now, differentiate $\lambda^{-n}\phi(\lambda)$ with respect to λ to obtain

$$[\phi(\lambda)\lambda^{-n}]' = \phi'(\lambda)\lambda^{-n} - n\phi(\lambda)\lambda^{-n-1} = 0.$$

Therefore, $\phi(\lambda)\lambda^{-n} = c$ for some constant c. Set $\lambda = 1$ to get $c = f(a, b)$. Then

$$f(\lambda a, \lambda b) = \lambda^n f(a, b).$$

Since (a, b) is any arbitrary point in D, we have $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. This completes the proof.

For our earlier examples, you can check that

$$x\frac{\partial[x^{1/3}y^{-4/3}\tan^{-1}(y/x)]}{\partial x} + y\frac{\partial[x^{1/3}y^{-4/3}\tan^{-1}(y/x)]}{\partial y} + x^{1/3}y^{-4/3}\tan^{-1}(y/x) = 0.$$

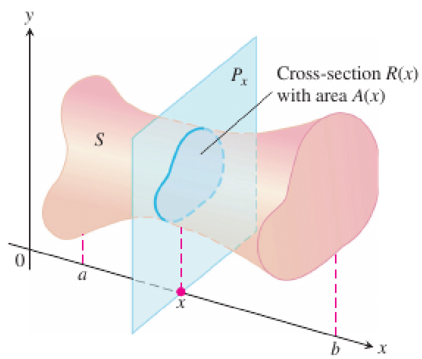
$$x\frac{\partial(x^2 + y^2)^{3/2}}{\partial x} + y\frac{\partial(x^2 + y^2)^{3/2}}{\partial y} = 3(x^2 + y^2)^{3/2}.$$

2

Multiple Integrals

2.1 Volumes by Slicing

We know that the integral of a continuous function $f(x)$ from a to b is the area of the region enclosed between the graph of $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$. Analogously, an integral can be used to compute the volume of a solid in certain cases. Given a solid, let us choose our axes in such a way that the solid is bounded between and touching the planes $x = a$ and $x = b$, where $a < b$. Suppose the plane at any point $x \in [a, b]$ parallel to the yz-plane crosses the solid at a region $R(x)$ whose area is $A(x)$. Assume that $A(x)$ is a continuous function of x.



We divide the interval $[a, b]$ into n sub-intervals. Let the partition be

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

On the ith subinterval we approximate the slice of the solid by $A(x_i^*)$ for a point $x_i^* \in [x_{i-1}, x_i]$. Write $\Delta x_i = x_{i+1} - x_i$. Then the volume of the solid is approximated by the sum

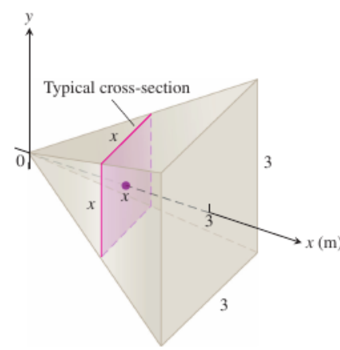
$$\sum_{i=1}^{n} A(x_i^*)\Delta x_i.$$

The volume of the solid is the limit of the above sum where each Δx_i approaches 0 (necessarily $n \longrightarrow \infty$). Since $A(x)$ is a continuous function of x, the volume is,

$$V = \int_a^b A(x)\, dx.$$

(2.1) ***Example***

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square with each side x m. Find the volume of the pyramid.

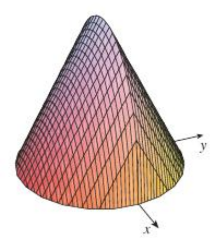


The cross section of the pyramid made by the plane at $x \in [0, 3]$ parallel to the yz-plane has area $A(x) = x^2$.

So, the volume $V = \displaystyle\int_0^3 A(x)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 = 9\,\mathrm{m}^3.$ □

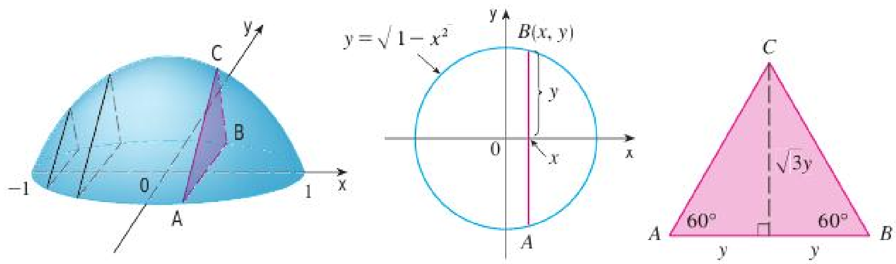
(2.2) ***Example***

In the figure is shown a solid with a circular base of radius 1. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Take the base of the solid as the disk $x^2 + y^2 \leq 1$.

The solid, its base, and a typical triangle at a distance x from the origin are shown in the figure below.



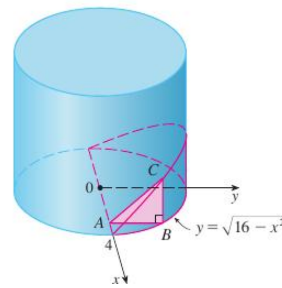
The point B lies on the circle $y = \sqrt{1 - x^2}$. So, the length of AB is $2\sqrt{1 - x^2}$. Since the triangle is equilateral, its height is $\sqrt{3}\sqrt{1 - x^2}$. The cross sectional area is

$$A(x) = \frac{1}{2}\, 2\, \sqrt{1 - x^2}\, \sqrt{3}\, \sqrt{1 - x^2} = \sqrt{3}\,(1 - x^2).$$

The volume of the solid is $V = \displaystyle\int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\, dx = \frac{4}{\sqrt{3}}.$ □

(2.3) ***Example***

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.



If we place the x-axis along the diameter where the planes meet, then the base of the solid is the semicircle with equation

$$y = \sqrt{16 - x^2}, \quad -4 \le x \le 4.$$

A cross-section perpendicular to the x-axis at a distance x from the origin is the triangle ABC, whose base is $y = \sqrt{16 - x^2}$; its height is $|BC| = y \tan 30° = \sqrt{16 - x^2}/\sqrt{3}$. Thus the cross sectional area is

$$A(x) = \frac{1}{2}\sqrt{16 - x^2}\,\frac{\sqrt{16 - x^2}}{\sqrt{3}} = \frac{16 - x^2}{2\sqrt{3}}.$$

Then the required volume of the wedge is

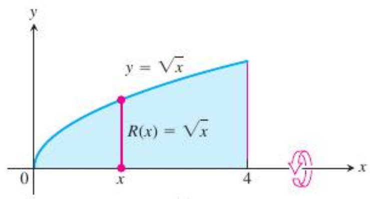
$$V = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx = \frac{128}{3\sqrt{3}}.$$

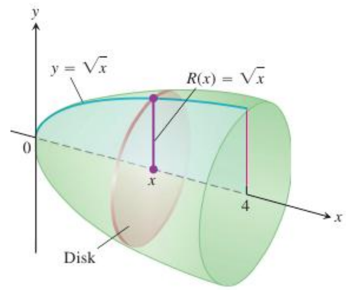
Aliter: Consider the cross section of the solid made by the plane at a point y on the y-axis parallel to the zx-plane. The cross section is a rectangle $R(y)$. Now, the length of $R(y)$ is the length of the chord of the circle $x^2 + y^2 = 4$ where the perpendicular distance from the origin to the chord is y. Thus, it is $2\sqrt{16 - y^2}$. The width of $R(y)$ is $y \tan 30° = y/\sqrt{3}$. Then, the volume is

$$V = \int_0^4 2\sqrt{16 - y^2} \cdot \frac{y}{\sqrt{3}}\,dy = -\frac{1}{\sqrt{3}}\int_0^4 \sqrt{16 - y^2}\,d(16 - y^2) = \frac{128}{3\sqrt{3}}. \qquad \square$$

2.2 Volume of a solid of revolution

We can use the method of slicing for computing the volume of a solid of revolution. The solid obtained by rotating a plane region about a straight line in the same plane is called a **solid of revolution**. The line is called the **axis of revolution**.





Suppose the region is bounded above by the curve $y = f(x)$ and below by the x-axis, where $a \leq x \leq b$. Suppose the axis of revolution is the x-axis. The cross section of the solid made by the plane at the point $x \in [a, b]$ parallel to the yz-plane is a disk of radius $f(x)$. Thus, $A(x) = \pi[f(x)]^2$. The volume of the solid of revolution is

$$V = \int_a^b A(x)\, dx = \int_a^b \pi[f(x)]^2\, dx.$$

This method of computing the volume of a solid of revolution is sometimes called the **disk method**.

If the axis of revolution is a straight line other than the x-axis, similar formulas can be obtained for the volume.

(2.4) *Example*

The region between the curves $y = \sqrt{x},\ 0 \leq x \leq 4$ and $y = 0$ is revolved around the x-axis. Find the volume of the solid of revolution.

As shown in the above figure, the required volume is

$$V = \int_0^4 \pi\big(\sqrt{x}\big)^2\, dx = \int_0^4 \pi\, x\, dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi. \qquad \square$$

(2.5) *Example*

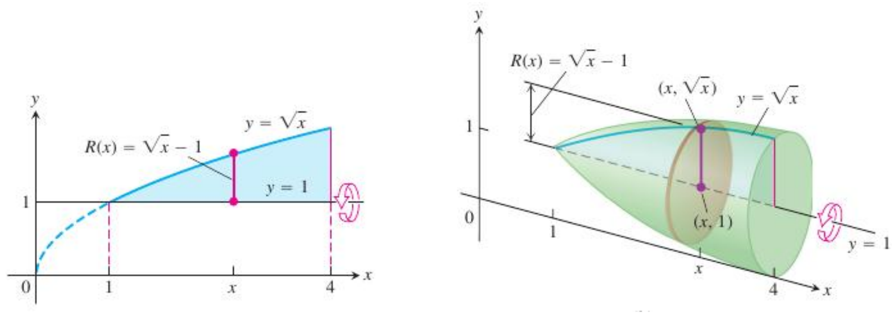
Find the volume of the sphere of radius $a > 0$.

We think of the sphere as the solid of revolution of the region bounded by the upper semi-circle $x^2 + y^2 = a^2,\ y \geq 0$ with $-a \leq x \leq a$. The curve is thus $y = \sqrt{a^2 - x^2}$. Then the volume of the sphere is

$$V = \int_{-a}^{a} \pi\left(\sqrt{a^2 - x^2}\right)^2\, dx = \int_{-a}^{a} \pi\big(a^2 - x^2\big)\, dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3. \qquad \square$$

(2.6) *Example*

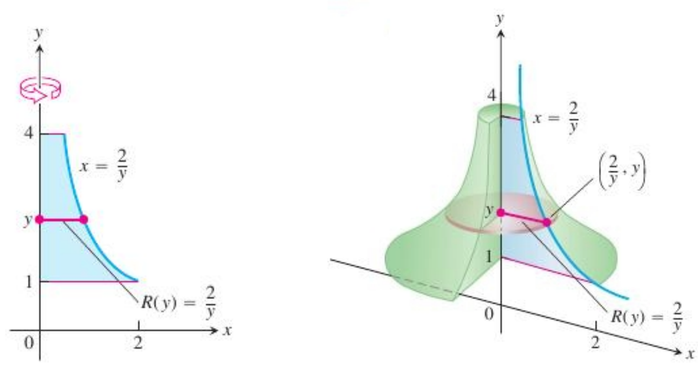
Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1,\ x = 4$ about the line $y = 1$.



$$V = \int_1^4 \pi[R(x)]^2\,dx = \int_1^4 \pi\left(\sqrt{x}-1\right)^2 dx = \int_1^4 \pi\left(x - 2\sqrt{x} + 1\right) dx = \frac{7\pi}{6}. \qquad \square$$

(2.7) ***Example***

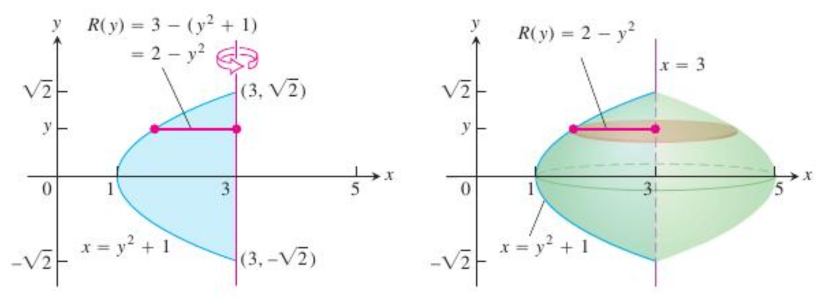
Find the volume of the solid generated by revolving the region between the y-axis and the curve $xy = 2,\ 1 \le y \le 4$, about the y-axis.



The volume is $V = \displaystyle\int_1^4 \pi[R(y)]^2\,dy = \pi\int_1^4 \frac{4}{y^2}\,dy = 3\pi.$ $\square$

(2.8) ***Example***

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.



Notice that the cross sections are perpendicular to the axis of revolution: $x = 3$.

The volume is $V = \displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2 dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\big(2-y^2\big)^2 dy = \frac{64\pi\sqrt{2}}{15}.$ □

(2.9) *Example*

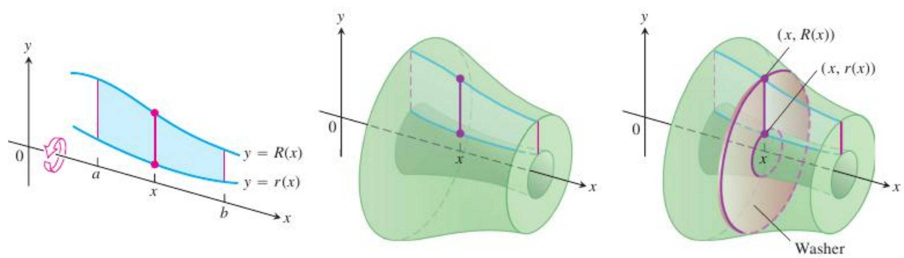
Find the volume of the solid generated by revolving the region bounded by the lines $x = 0,\ x = 2,\ y = 0$ and the curve $y = 4/(x^2+4)$, about the x-axis.

The volume is $V = \displaystyle\int_0^2 \pi \frac{16}{(x^2+4)^2}\, dx.$

Substitute $x = 2\tan t$. $dx = 2\sec^2 t\, dt$, $(x^2+4)^2 = 16\sec^4 t$ for $0 \le t \le \pi/4$. So,

$$V = \int_0^{\pi/4} 16\pi \frac{2\sec^2 t}{16\sec^4 t}\, dt = \int_0^{\pi/4} 2\pi\cos^2 t\, dt = \pi\Big(\frac{\pi}{4} + \frac{1}{2}\Big).$$ □

If the region which revolves does not border the axis of revolution, then there are **holes** in the solid. In this case, we subtract the volume of the hole to obtain the volume of the solid of revolution. This is sometimes called the **washer method**.



Thus, the volume of the the solid of revolution is given by

$$V = \int_a^b A(x)\, dx = \int_a^b \pi\big[(R(x))^2 - (r(x))^2\big]\, dx.$$

(2.10) *Example*

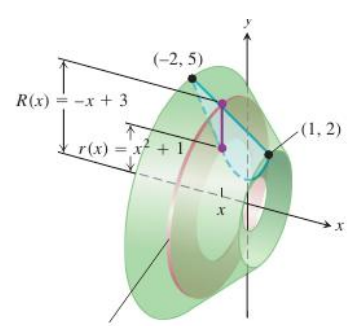
The region bounded by the curve $y = x^2+1$ and the line $x + y = 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

The outer radius of the washer is $R(x) = -x+3$ and the inner radius is $r(x) = x^2+1$. The limits of integration are obtained by finding the points of intersection of the given curves:

$$x^2+1 = -x+3 \Rightarrow x = -2,\ 1.$$

The required volume is

$$V = \int_{-2}^{1} \pi\big[(-x+3)^2 - (x^2+1)^2\big]\, dx = \frac{117\pi}{5}.$$

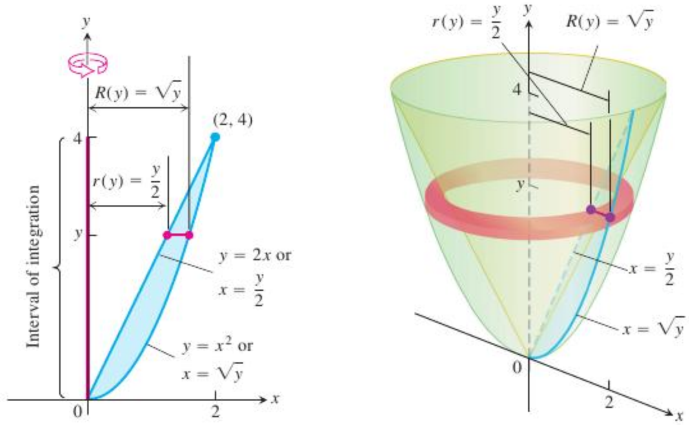


□

(2.11) *Example*

Find the volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = 2x$, about the y-axis.

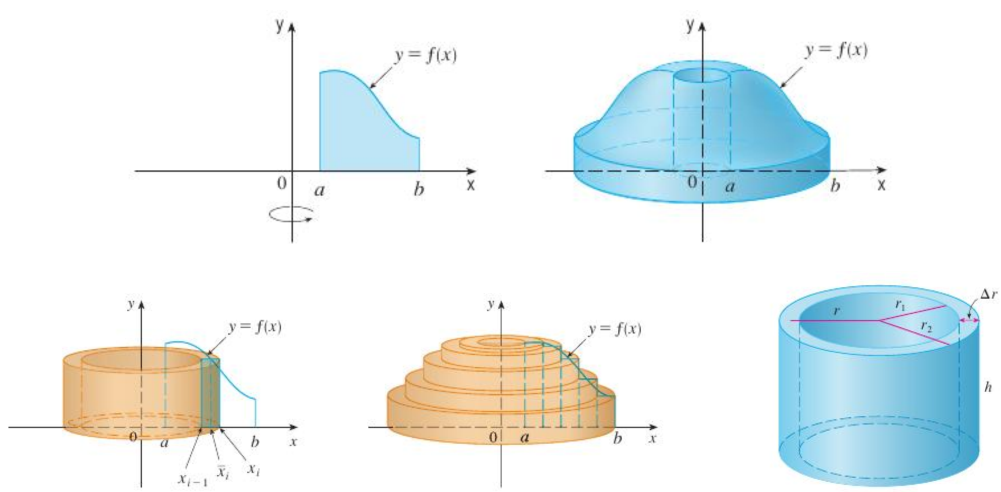
The given curves intersect at $y = 0$ and $y = 4$. The required volume is



$$V = \int_0^4 \pi\left[(R(y))^2 - (r(y))^2\right] dy = \int_0^4 \pi\left[(\sqrt{y})^2 - (y/2)^2\right] dy = \frac{8\pi}{3}. \qquad \square$$

2.3 The Cylindrical Shell Method

Let $a \geq 0$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(x) \geq 0$. Let S be the solid obtained by revolving about the y-axis the region bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = a$ and $x = b$.



In *Cylindrical Shell method*, we slice the solid into cylindrical shells as shown in the figures. The sum of these cylindrical shells is taken as an approximation to the volume of the solid. When the width of each cylindrical shell approaches zero, the limit of this sum gives the volume.

To carry out this idea, we divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and take $\overline{x}_i$ as the mid-point of the subinterval. The ith cylindrical shell is obtained by revolving the rectangle with base $[x_{i-1}, x_i]$ and average height $f(\overline{x}_i)$ about the y-axis. The base of this cylindrical shell has inner radius x_i, outer radius $x_{i+1} = x_i + \Delta x$, and height $f(\overline{x}_i)$. So, its volume is

$$V_i = \pi\big[x_{i+1}^2 - x_i^2\big] f(\overline{x}_i) = \pi\big[x_{i+1} + x_i\big]\Delta x f(\overline{x}_i) = 2\pi\overline{x}_i\, f(\overline{x}_i)\,\Delta x.$$

Then an approximation to the volume V is given by

$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\overline{x}_i\, f(\overline{x}_i)\,\Delta x.$$

When n approach ∞, we get the volume as

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} V_i = \int_a^b 2\pi x f(x)\,dx.$$

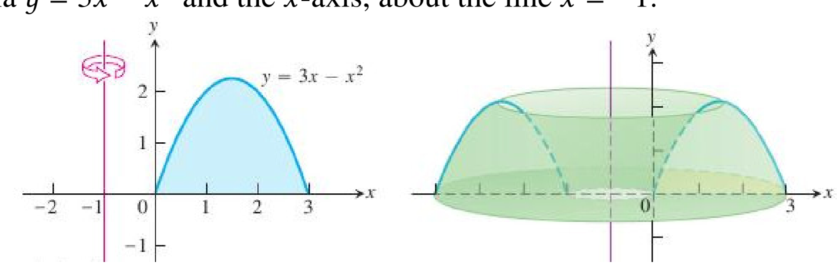
In the integrand of this formula, the term x is called the **shell radius** and the term $f(x)$ is called the **shell height**.

Instead of taking the axis of revolution as the y-axis, we may take the vertical line $x = \ell$. In that case, the shell radius will be $x - \ell$ instead of $x = x - 0$. With the assumptions used, we may state the result as follows:

> The volume of the solid generated by revolving the region between the x-axis and the curve $y = f(x)$, $a \le x \le b$ with $f(x) \ge 0$ about a vertical line $x = \ell$, where $\ell \le a$, is
>
> $$V = \int_a^b 2\pi(x - \ell) f(x)\,dx = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx.$$

(2.12) ***Example***

Find the volume of the solid generated by revolving the region bounded by the parabola $y = 3x - x^2$ and the x-axis, about the line $x = -1$.

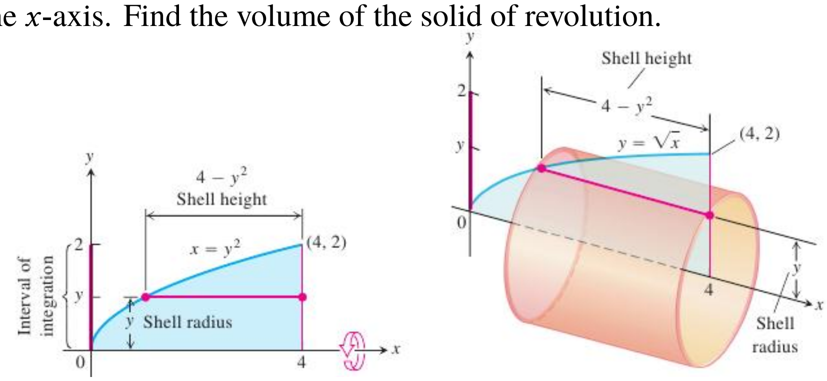


The parabola intersects the x-axis at $x = 0$ and $x = 3$. The required volume is

$$V = \int_0^3 2\pi(x+1)(3x - x^2)\,dx = 2\pi \int_0^3 \left(2x^2 + 3x - x^3\right)dx = \frac{45\pi}{2}. \qquad \square$$

(2.13) ***Example***

The region bounded by the x-axis, the line $x = 4$ and the curve $y = \sqrt{x}$ is revolved about the x-axis. Find the volume of the solid of revolution.



Since the revolution is about the x-axis, that is, the line $y = 0$, the variable of integration is y. The limits of integration are $y = 0$ and $y = 2$. The shell radius is y and the shell height is $4 - y^2$. Thus the volume of the solid of revolution is

$$V = \int_0^2 2\pi\, y(4 - y^2)\,dy = 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \qquad \square$$

2.4 Approximating Volume

We now consider solids which are not necessarily solids of revolution. First, we consider a simple case, when the given solid has all plane faces except one, which is a portion of a surface given by $z = f(x, y)$.

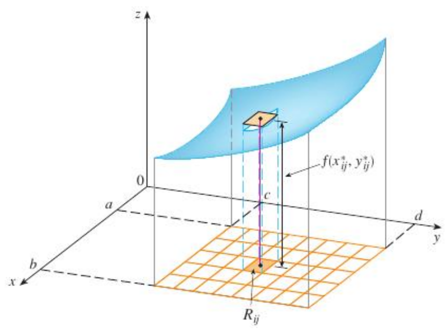
Let $f(x, y)$ be defined on the rectangle $R: \; a \le x \le b, \; c \le y \le d$. For simplicity, let $f(x, y) \ge 0$. The graph of f is the surface $z = f(x, y)$. Consider the solid

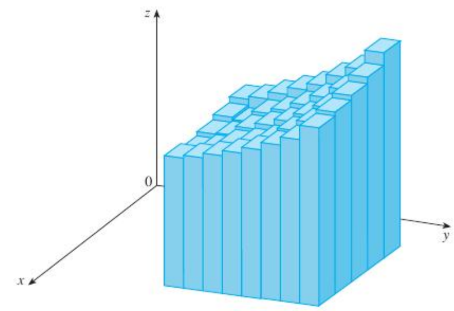
$$S: \; \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R, \; 0 \le z \le f(x, y)\}.$$

We approximate the volume of S by partitioning R and then adding up the volumes of the solid rods. So, consider a **partition** of R as in the following:

$$P: \; R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \; 1 \le i \le m, \; 1 \le j \le n, \; a = x_0, \; b = x_m, \; c = y_0, \; d = y_n.$$

Let $A(R_{ij})$ denote the area of the rectangle R_{ij}. Write $\|P\| = \max A(R_{ij})$ and call it the **norm** of P.





Choose **sample points** $(x_i^*, y_j^*) \in R_{ij}$. An approximation to the volume of S is the **Riemann sum**

$$S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) A(R_{ij}).$$

If limit of S_{mn} exists as $\|P\| \to 0$, then this limit is called the **double integral** of $f(x, y)$. It is denoted by

$$\iint_R f(x, y)dA.$$

Whenever the integral exists, it is also enough to consider **uniform partitions**, that is, $x_i - x_{i-1} = (b - a)/m = \Delta x$ and $y_j - y_{j-1} = (d - c)/n = \Delta y$. In this case, we write $A(R_{ij}) = \Delta A = \Delta x \Delta y$. Then

$$\iint_R f(x, y)dA = \lim_{\|P\| \to 0} S_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta x \Delta y.$$

Since $f(x, y) \ge 0$, the value of this integral is the volume of the solid S bounded by the rectangle R and the surface $z = f(x, y)$, $a \le x \le b$, $c \le y \le d$.

When the integral of $f(x, y)$ exists, we say that f is **Riemann integrable** or just **integrable**.

Riemann sum is well defined even if $f(x) \ge 0$ does not hold for each $(x, y) \in R$. However, the double integral computes the signed volume. Analogous to the single variable case, we have the following result; we omit its proof.

(2.14) *Theorem*

Each continuous function defined on a closed bounded rectangle is integrable.

Volumes of solids can also be calculated by using *iterated integrals*.

(2.15) *Theorem*

Let R be the rectangle $[a, b] \times [c, d]$. Let $f : R \to \mathbb{R}$ be a continuous function. Then

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

An expression such as $\int_c^d \int_a^b f(x,y)dxdy$ is called an **iterated integral**.

(2.16) *Example*

Evaluate $\iint_R (1-6x^2y)dA$, where $R = [0,2]\times[-1,1]$.

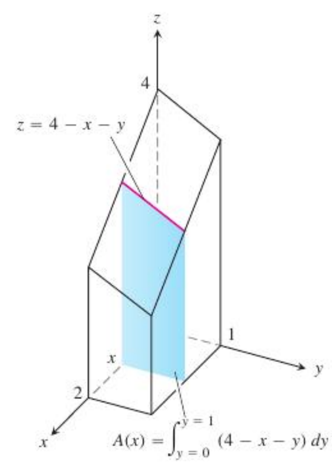
$$\iint_R (1-6x^2y)dA = \int_{-1}^{1}\int_0^2 (1-6x^2y)dxdy = \int_{-1}^{1}(2-16y)dy = 4.$$

Also, reversing the order of integration, we have

$$\iint_R (1-6x^2y)dA = \int_0^2\int_{-1}^{1}(1-6x^2y)dydx = \int_0^2 2dx = 4. \qquad \square$$

(2.17) *Example*

Find the volume V of the solid raised over the rectangle $R:\ [0,2]\times[0,1]$ and bounded above by the plane $z = 4-x-y$.



To use the method of slicing, suppose $A(x)$ is the cross sectional are at x. Then $V = \int_0^2 A(x)dx$.

Now, $A(x) = \displaystyle\int_0^1 (4-x-y)dy.$

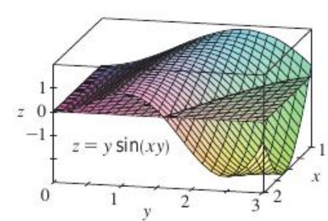
Thus, $V = \displaystyle\int_0^2\int_0^1 (4-x-y)dydx.$

Alternatively, as a double integral,

$$\iint_R (4-x-y)dA = \int_0^2\int_0^1 (4-x-y)dydx = \int_0^2 (4-x-1/2)\,dx = 5. \qquad \square$$

(2.18) *Example*

Evaluate $\iint_R y\sin(xy)dA$, where $R = [1,2]\times[0,\pi]$.



$$\iint_R y\sin(xy)dA = \int_0^\pi\int_1^2 y\sin(xy)\,dxdy$$

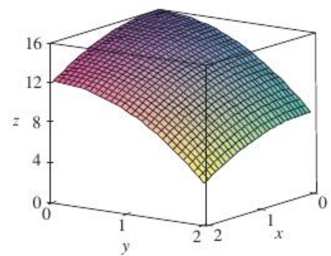
$$= \int_0^\pi (-\cos 2y + \cos y)dy = 0.$$

The volume of the solid above R and below the surface $z = y\sin(xy)$ is the same as the volume below R and above the surface. Since they have opposite signs, the net volume is zero. $\square$

(2.19) ***Example***

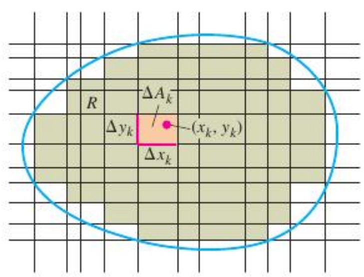
Find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, planes $x = 2$ and $y = 2$, and the three coordinate planes.

Let R be the rectangle $[0, 2] \times [0, 2]$. The solid is above R and below the surface defined by $z = f(x, y) = 16 - x^2 - 2y^2$, where f is defined on R. Then,



$$V = \iint_R (16 - x^2 - 2y^2)dA$$
$$= \int_0^2 \int_0^2 \left(16 - x^2 - 2y^2\right)dxdy \int_0^2 \left(16x - \tfrac{1}{3}x^3 - 2xy^2\right) dy = 48. \qquad \square$$

The double integrals can be extended to functions defined on non-rectangular regions. Essentially, the approach is the same as earlier. We partition the region into smaller rectangles, form the Riemann sum, take its limit as the norm of the partition goes to zero.



When the region R is of a specific type, a double integral can be computed by an iterated integral.

(2.20) ***Theorem*** (Fubini)

Let R be a region in the plane and let $f : R \to \mathbb{R}$ be a continuous function.

(1) *If R is given by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, where $g_1, g_2 : [a, b] \to \mathbb{R}$ are continuous, then*

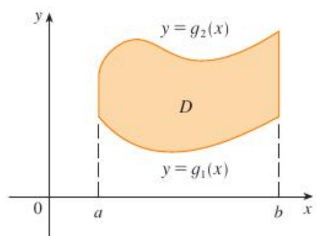
$$\iint_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx.$$

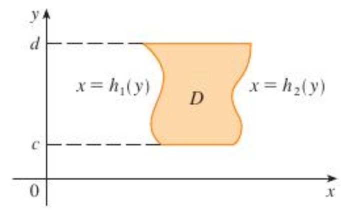
(2) *If R is given by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, where $h_1, h_2 : [c, d] \to \mathbb{R}$ are continuous, then*

$$\iint_R f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dxdy.$$

Fubini's theorem is applicable when the region R on which the function $f(x, y)$ is defined is given in one of the following ways:

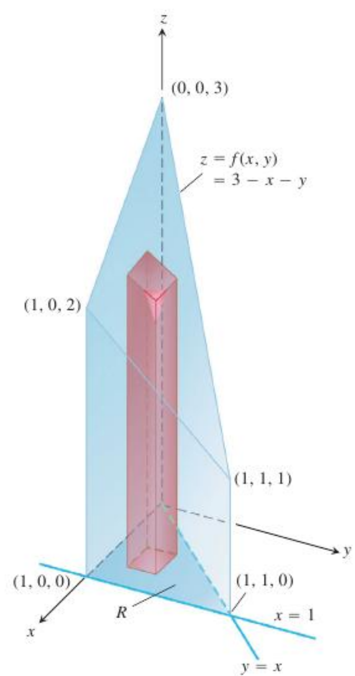
1. R is bounded by the lines $x = a$, $x = b$ and the curves $y = g_1(x)$, $y = g_2(x)$; or
2. R is bounded by the lines $y = c$, $y = d$ and the curves $x = h_1(y)$, $x = h_2(y)$.

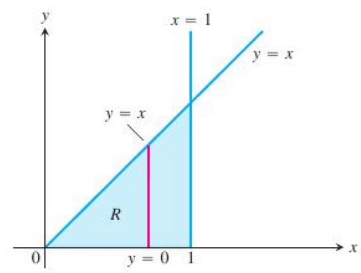




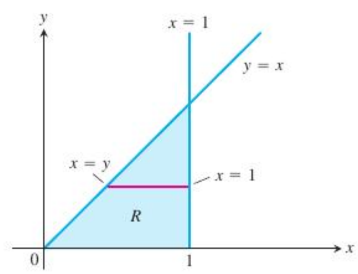
(2.21) *Example*

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the lines $y = 0,\ x = 1$ and $y = x$, and whose top lies in the plane $z = 3 - x - y$.





(b)



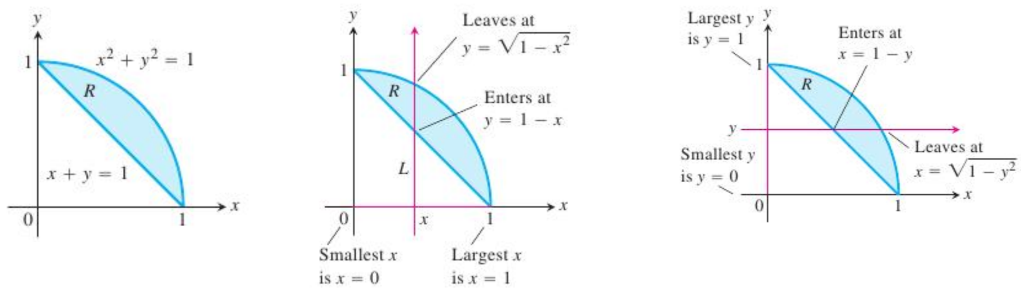
$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 (3x - 3x^2/2)\,dx = 1.$$

Also, $V = \displaystyle\int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 (5/2 - 4y + 3y^2/2)\,dy = 1.$ ∎

(2.22) ***Example***

Let R be the region bounded by the line $x + y = 1$ and the portion of the circle $x^2 + y^2 = 1$ in the first quadrant. Express $\iint_R f(x, y)\, dA$ as in iterated integral.

We sketch the region, find the limits, and then write the appropriate integrals.

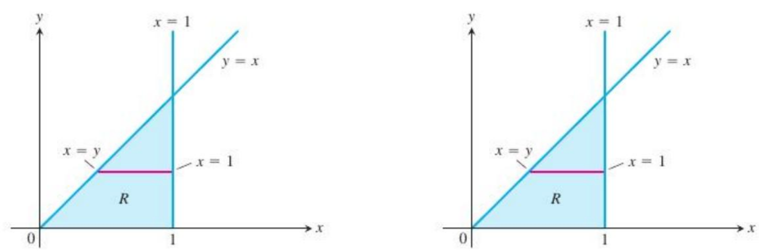


$$\iint_R f(x, y) dA = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x, y) dy dx = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) dx dy. \qquad \square$$

For evaluating a double integral as an iterated integral, choose some order: first x, next y. If it does not work, or if it is complicated, you may have to choose the reverse order.

(2.23) ***Example***

Evaluate $\iint_R (\sin x)/x\, dA$, where R is the triangle in the xy-plane bounded by the lines $y = 0$, $x = 1$, and $y = x$.



The triangular region R can be expressed as $\{(x, y) : 0 \le y \le 1,\ y \le x \le 1\}$. So,

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_y^1 \frac{\sin x}{x} dx dy.$$

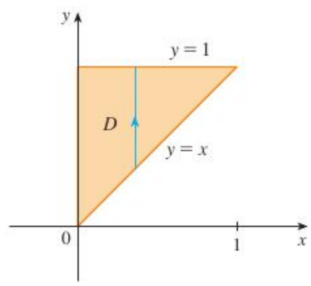
We do not see any way to proceed further. Instead, we choose a different order to express the same region. $R = \{(x, y) : 0 \le y \le x,\ 0 \le x \le 1\}$. Then

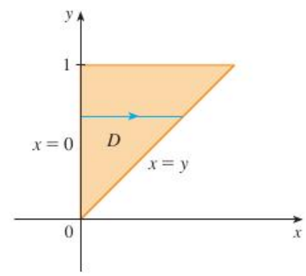
$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_0^x \frac{\sin x}{x} dy dx = \int_0^1 \left(\frac{\sin x}{x} \int_0^x dy \right) dx$$
$$= \int_0^1 \sin x dx = -\cos(1) + 1. \qquad \square$$

(2.24) *Example*

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)dydx$.

Write $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ x \leq y \leq 1\}$. We plan to change the order of integration. we have $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y,\ 0 \leq y \leq 1\}$. Then,





$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)dydx &= \iint_D \sin(y^2)dA = \int_0^1 \int_0^y \sin(y^2)dxdy \\ &= \int_0^1 y\sin(y^2)dy = \frac{1}{2}(1 - \cos(1).\end{aligned}$$

$\square$

Properties of double integrals with respect to addition, multiplication etc. are similar to integrals with a single variable. They are as follows.

(2.25) *Theorem* (Algebra of Integrals)

Let D be a region in the plane and let $f, g : D \to \mathbb{R}$ be integrable functions. Then the following are true:

(1) Constant Multiple: $\iint_D cf(x, y)dA = c\iint_D f(x, y)dA$ *for any* $c \in \mathbb{R}$.

(2) Sum-Difference: $\iint_D [f(x, y) \pm g(x, y)]dA = \iint_D f(x, y)dA \pm \iint_D g(x, y)dA.$

(3) Additivity: $\iint_{D\cup R} f(x, y)dA = \iint_D f(x, y)dA + \iint_R f(x, y)dA,$
provided $f(x, y)$ is continuous on a region R also, and $D \cap R = \emptyset$.

(4) Domination: *If $f(x, y) \leq g(x, y)$ in D, then* $\iint_D f(x, y)dA \leq \iint_D g(x, y)dA.$

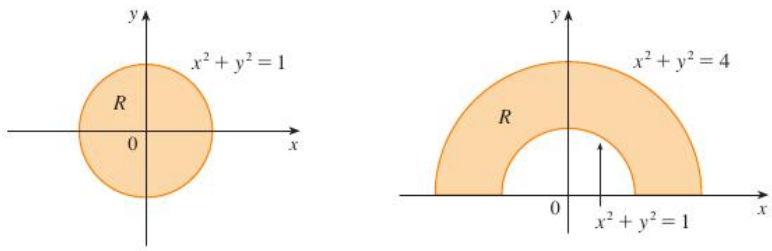
(5) Area: $\iint_D 1\, dA = \text{Area of } D.$

(6) Boundedness: *If $m \leq f(x, y) \leq M$ in D for some $m, M \in \mathbb{R}$, then*

$$m\,(\text{Area of } D) \leq \iint_D f(x, y)dA \leq M\,(\text{Area of } D).$$

2.5 Riemann Sum in Polar coordinates

Suppose R is one of the following regions in the plane:

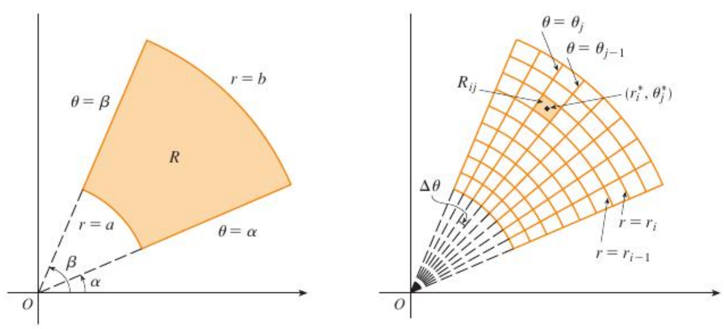


(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

It is easy to describe such regions in polar coordinates. Using polar coordinates, we define a **polar rectangle** as a region given in the form:

$$R = \{(r, \theta) : a \leq r \leq b,\ \alpha \leq \theta \leq \beta,\ 0 \leq \beta - \alpha < 2\pi\}.$$

We can divide a polar rectangle into polar subrectangles as in the following:



$$R_{ij} = \{(r, \theta) : r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}.$$

Suppose f is a real valued function defined on a polar rectangle R. Let P be a partition of R into smaller polar rectangles R_{ij}. The area of R_{ij} is

$$A(R_{ij}) = \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j - \theta_{j-1}) = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})(\theta_j - \theta_{j-1}).$$

Take a uniform grid dividing $r = a$ into m equal parts and $\theta = \beta - \alpha$ into n equal parts. Write $r_i - r_{i-1} = \Delta r$ and $\theta_j - \theta_{j-1} = \Delta\theta$. Also write the mid-point of r_{i-1} and r_i as $r_i^* = \frac{1}{2}(r_i + r_{i-1})$, similarly, $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$. Then the Riemann sum for $f(r, \theta)$ can be written as

$$S = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*, \theta_j^*)A(R_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*, \theta_j^*)r_i^*\Delta r\Delta\theta.$$

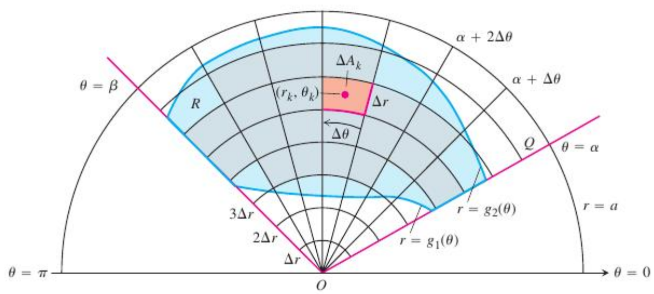
Therefore, if $f(r,\theta)$ is continuous on the polar rectangle R, then

$$\iint_R f(r,\theta)dA = \iint_R f(r,\theta)\, r\, dr\, d\theta$$

It means that if $f(x,y)$ is given in Cartesian coordinates, which is continuous on the polar rectangle $R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta,\ 0 \le \beta - \alpha < 2\pi\}$, then converting its double integral into polar form, we have

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

The double integral in polar form can be generalized to functions defined on regions other than polar rectangles.



Let f be a continuous function defined over the region bounded by the rays $\theta = \alpha,\ \theta = \beta$ and the continuous curves $r = g_1(\theta),\ r = g_2(\theta)$. Then

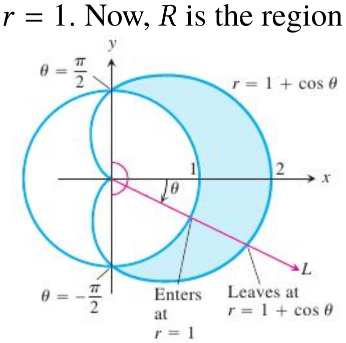
$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

Caution: Do not forget the r on the right hand side.

(2.26) *Example*

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $x^2 + y^2 = 1$.

Better write the circle as $r = 1$. Now, R is the region shown in blue:

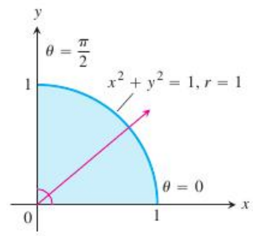


Then, $\displaystyle\iint_R f(r,\theta)dA = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} f(r,\theta)r\,dr\,d\theta.$ ∎

(2.27) ***Example***

Evaluate $I = \displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)dydx.$

The limits of integration say that the region is the quarter of the unit disk in the first quadrant:

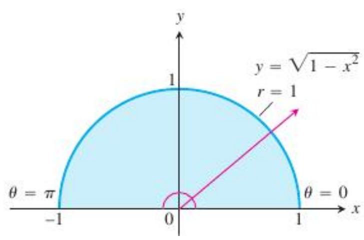


The region in polar coordinates is $R:\ 0 \le \theta \le \pi/2,\ 0 \le r \le 1.$
Changing to polar coordinates, we have $x = r\cos\theta, y = r\sin\theta$ and then

$$I = \int_0^{\pi/2}\int_0^1 r^2\,r\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{\pi}{8}.$$ ∎

(2.28) ***Example***

Evaluate $I = \displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}dydx.$



The region is the upper semi-unit-disk, whose polar description is

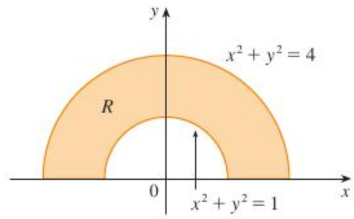
$$R = \{(r,\theta) : 0 \le \theta \le \pi,\ 0 \le r \le 1\}.$$

Then $I = \displaystyle\iint_R e^{x^2+y^2}dA$. Using integration in polar form,

$$I = \int_0^{\pi}\int_0^1 e^{r^2}r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta = \int_0^{\pi}\frac{e-1}{2}\,d\theta = \frac{\pi}{2}(e-1).$$ ∎

(2.29) ***Example***

Evaluate $\displaystyle\iint_R (3x+4y^2)dA$, where R is the region in the upper half plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$R = \{(r, \theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$. Therefore,

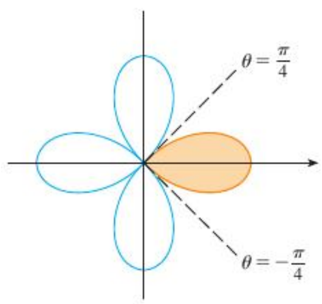
$$\begin{aligned}\iint_R (3x + 4y^2)dA &= \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2 \sin^2\theta) r\, dr\, d\theta \\ &= \int_0^{\pi} \left[r^3 \cos\theta + r^4 \sin^2\theta \right]_1^2 d\theta \\ &= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta = \frac{15\pi}{2}.\end{aligned}$$

□

(2.30) *Example*

Find the area enclosed by one of the four leaves of the curve $r = |\cos(2\theta)|$.

The region is $R = \{(r, \theta) : -\pi/4 \le \theta \le \pi/4,\ 0 \le r \le \cos(2\theta)\}$.



Then the required area is

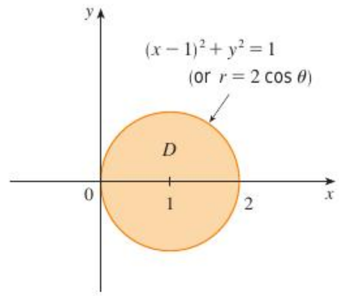
$$\begin{aligned}\iint_R dA &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta) - 1}{2}\, d\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{cos(4\theta) - 1}{4}\, d\theta = \frac{\pi}{8}.\end{aligned}$$

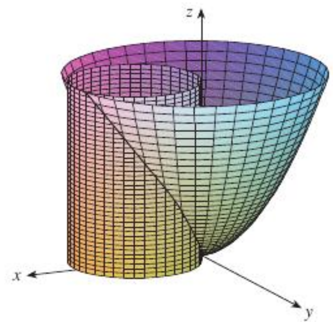
□

(2.31) *Example*

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

The solid lies above the disk D whose boundary has equation $x^2 + y^2 = 2x$, or in polar coordinates, $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$.





The disk $D = \{(r, \theta) : -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}$.
Then the required volume V is given by

$$V = \iint_D (x^2 + y^2)dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} (3 + \cos 4\theta + 4\cos 2\theta)\, d\theta = \frac{3\pi}{2}.$$

$\square$

2.6 Triple Integral

Similar to the double integrals, we introduce the triple integral. Let $f(x, y, z)$ be a real valued function defined on a bounded region D in $\mathbb{R}^3$. As earlier we divide the region into smaller cubes enclosed by planes parallel to the coordinate planes. The set of these smaller cubes is called a partition P. The norm of the partition is the maximum volume enclosed by any smaller cube. Then form the Riemann sum S and take its limit as the cubes become smaller and smaller. If the limit exists, we say that the limit is the triple integral of the function over the region D.

$$\iiint_D f(x, y, z)dV = \lim_{\|P\|\to 0} \sum f(x_i^*, y_j^*, z_k^*)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}),$$

where (x_i^*, y_j^*, z_k^*) is a point in the (i, j, k)-th cube in the partition.

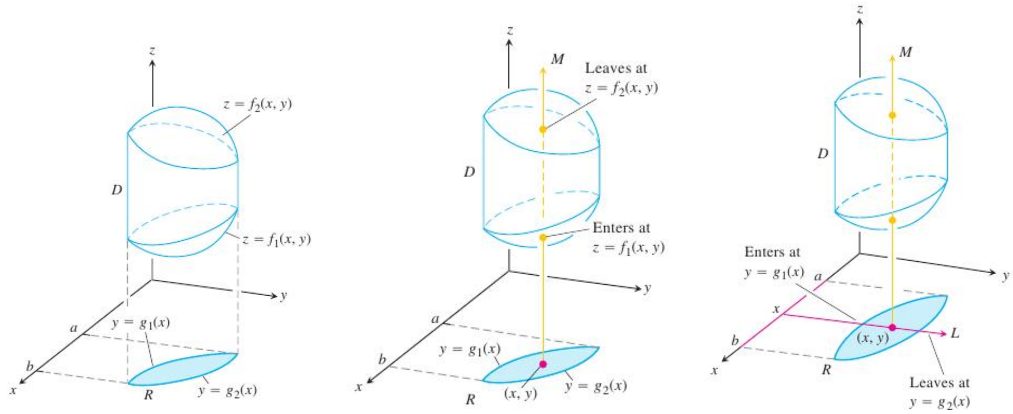
As earlier, Fubuni's theorem says that for continuous functions, if the region D can be written as

$$D = \{(x, y, z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x, y) \le z \le h_2(x, y)\},$$

then the triple integral can be written as an iterated integral:

$$\iiint_D f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)dz\, dy\, dx.$$

To find the limits of integration, we first sketch the region D along with its shadow on the xy-plane. Next, we find the z-limits, then y-limits and then x-limits.



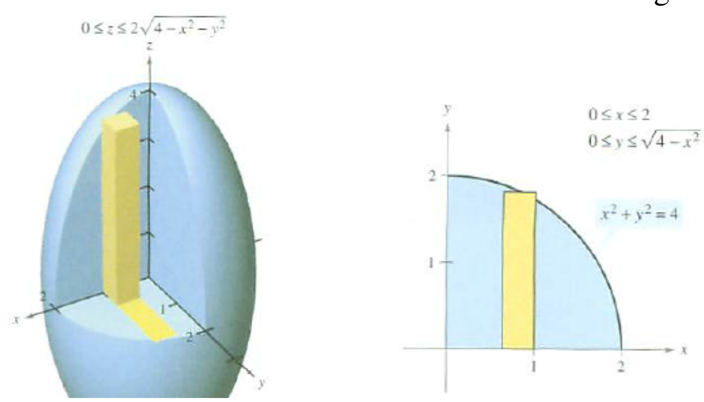
Observe that the volume of D is $\iiint_D 1\, dV$.

All properties for double integrals hold analogously for triple integrals.

(2.32) *Example*

Find the volume of the ellipsoid given by $4x^2 + 4y^2 + z^2 = 16$.

Due to symmetry, the volume is 8 times the volume of D, where D is the portion of the ellipsoid in the first octant. Let us choose the order of integration as $dzdydx$.



Now, $D = \{(x, y, z) : 0 \le x \le 2,\ 0 \le y \le \sqrt{4 - x^2},\ 0 \le z \le 2\sqrt{4 - x^2 - y^2}\}$. Therefore, Volume of the ellipsoid

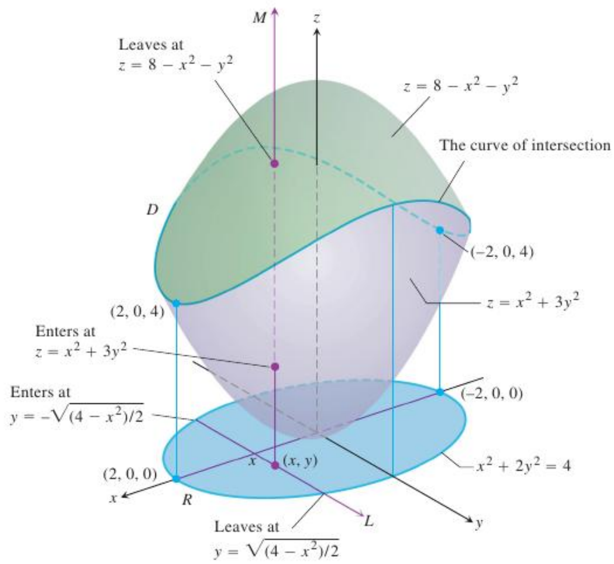
$$= 8 \iiint_D dV = 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} dzdydx$$

$$= 16 \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{(4 - x^2) - y^2}\, dydx = 16 \int_0^2 \frac{\pi}{4}(4 - x^2)\, dx = \frac{64\pi}{3}.$$

Notice that $\displaystyle\int_0^{\sqrt{4-x^2}} \sqrt{(4 - x^2) - y^2}\, dy$ is the area of the quarter circle of radius $\sqrt{4 - x^2}$; so it is equal to $\frac{\pi}{4}(4 - x^2)$. You can also get the same answer by using the formula $\displaystyle\int \sqrt{a^2 - t^2}\, dt = \frac{t}{2}\sqrt{a^2 - t^2} + \frac{a^2}{2}\sin^{-1}\frac{t}{a} + C.$ $\square$

(2.33) ***Example***

Find the volume of the solid enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



Eliminating z from the two equations, we get the projection of the curve of intersection of the two surfaces on the xy-plane, which is $x^2 + 2y^2 = 4$. Here, the projection of the solid on the xy-plane is the region $\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 \le 4\}$. The curve of intersection of the two surfaces is given by the function $z = x^2 + 3y^2$ where (x, y) varies over this region. Since we have equated the z-values on the two surfaces, the same curve of intersection is also given by $z = 8 - x^2 - y^2$ where (x, y) varies over this region. Now, the projected region gives the limits of integration for y as $\pm\sqrt{(4 - x^2)/2}$. Clearly, $-2 \le x \le 2$. Therefore,

$$
\begin{aligned}
V = \iiint_D dV &= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \\
&= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx \\
&= \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\, dx = 8\pi\sqrt{2}.
\end{aligned}
$$

$\square$

Notice that changing the order of integration involves expressing the region by choosing different order of the limits of values in the axes.

(2.34) ***Example***

Evaluate $\int_0^1 \int_0^z \int_0^y e^{(1-x)^3}\, dx\, dy\, dz$ by changing the order of integration.

Here, the region is $D = \{(x, y, z) : 0 \le z \le 1, 0 \le y \le z, 0 \le x \le y\}$. Sketch the region. Notice that for any point $(x, y, z) \in D$, we have $0 \le x, y, z \le 1$.

We plan to change the order of integration from $dxdydz$ to $dzdydx$. Towards this, we project D on the xy-plane. The projection is the triangle R bounded by the lines $x = 0,\ y = 1$ and $y = x$ in the xy-plane. So, $R = \{(x, y, 0) : 0 \le x \le 1,\ x \le y \le 1\}$.

Next, for any point $(x, y, 0) \in R$, we draw a line parallel to z-axis. It enters D at $z = y$ and leaves at $z = 1$. So, $D = \{(x, y, z) : 0 \le x \le 1,\ x \le y \le 1,\ y \le z \le 1\}$. Therefore,

$$\int_0^1 \int_0^z \int_0^y e^{(1-x)^3}\, dx\, dy\, dz = \int_0^1 \int_x^1 \int_y^1 e^{(1-x)^3}\, dz\, dy\, dx$$

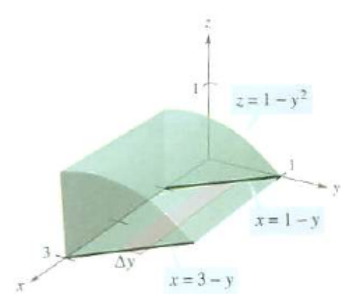
$$= \int_0^1 \int_x^1 (1-y) e^{(1-x)^3}\, dy\, dx = \int_0^1 \frac{(1-x)^2}{2} e^{(1-x)^3}\, dx$$

$$= -\int_{(1-0)^3}^0 \frac{e^t}{6}\, dt = \frac{e-1}{6}. \qquad \text{with } t = (1-x)^3. \qquad \square$$

(2.35) ***Example***

Write the volume of the solid in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the planes $x + y = 1$ and $x + y = 3$ as an iterated integral.

Projection of the solid D to the plane $z = 0$ gives a parallelogram with sides on the lines $y = 0,\ y = 1$ and $x = 1 - y,\ x = 3 - y$, in the xy-plane. Further, D is bounded below by the plane $z = 0$ and above by the cylinder $z = 1 - y^2$. So, $0 \le z \le 1 - y^2$. Then, $D = \{(x, y, z) : 0 \le y \le 1,\ 1 - y \le x \le 3 - y,\ 0 \le z \le 1 - y^2\}$. Then, the volume of D is



$$V = \iiint_D dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dzdxdy.$$

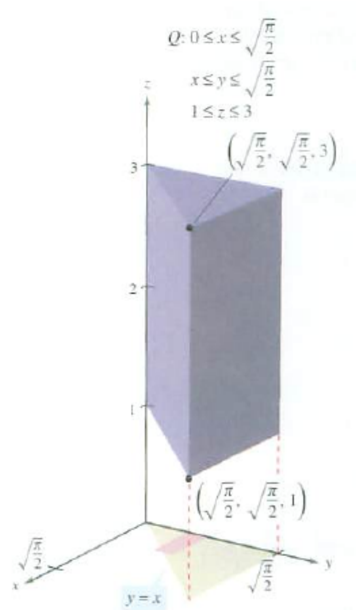
$\square$

(2.36) ***Example***

Evaluate $I = \int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\, dzdydx$.

We change the order of integration to $dzdxdy$ so that y is the outer variable of integration. Now, $D = \{(x, y, z) : 0 \le x \le \sqrt{\pi/2},\ 0 \le x \le y \le \sqrt{\pi/2},\ 1 \le z \le 3\}$.

Projection of D on the xy-plane gives the region $R = \{(x, y, 0) : 0 \le y \le \sqrt{\pi/2},\ 0 \le x \le y\}$. The limits of z are still 1 and 3. Therefore,



$$I = \int_0^{\sqrt{\pi/2}} \int_0^y \int_1^3 \sin(y^2)\, dzdxdy$$

$$= \int_0^{\sqrt{\pi/2}} \int_0^y \sin(y^2)\, dxdy$$

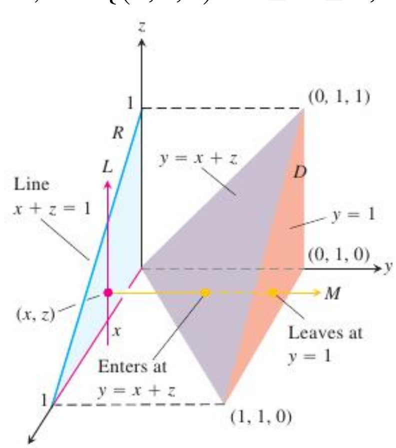
$$= 2\int_0^{\sqrt{\pi/2}} y \sin(y^2)\, dy = 1.$$

□

(2.37) ***Example***

Write the triple integral of $f(x, y, z)$ over the tetrahedron with vertices at $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$ as an iterated integral.

First, sketch the region D to see the limits geometrically. The right hand side bounding surface of D lies in the plane $y = 1$. The projection of D on the zx-plane is the triangle R with vertices at $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$ in the zx-plane. The upper boundary of R is the line $z = 1 - x$ and the lower boundary of R is the line $z = 0$, in the zx-plane. Thus, $R = \{(x, 0, z) : 0 \le x \le 1,\ 0 \le z \le 1 - x\}$.



To find the y-limits for D, we consider a typical point $(x, 0, z)$ in R and a line through this point parallel to y-axis. It enters D at the left side plane bounding D, whose equation is $y = x + z$, and leaves at the right hand side plane with equation as $y = 1$. Thus, $D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le z \le 1 - x,\ x + z \le y \le 1\}$.

Then the triple integral of a function $f(x, y, z)$ over D is given by

$$\iiint_D f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_{x+z}^1 f(x, y, z)\, dy\, dz\, dx.$$

If we interchange the orders of y and z, then first we consider limits for z and then of y. In this case, we project D on the xy-plane. A line parallel to z-axis through (x, y) in the xy-plane enters D at $z = 0$ and leaves D through the upper plane $z = y - x$. So, $0 \le z \le y - x$.

For the y-limits, on the xy-plane, where $z = 0$, the sloped side of D crosses the plane along the line $y = x$. A line through (x, y) parallel to y-axis enters the xy-plane at $y = x$ and leaves at $y = 1$. So, $x \le y \le 1$. The x-limits are as earlier, that is, $0 \le x \le 1$. So, $D = \{(x, y, z) : 0 \le x \le 1,\ x \le y \le 1,\ 0 \le z \le y - x\}$.

Then, the same triple integral is rewritten as the following iterated integral:

$$\iiint_D f(x, y, z)\, dV = \int_0^1 \int_x^1 \int_0^{y-x} f(x, y, z)\, dz\, dy\, dx. \qquad \Box$$

Analogous to a function of a single variable, we can define the average value over a planar region. It is defined as follows.

The **average value** of a function $f(x, y)$ over a planar region D is

$$\frac{\iint_D f(x, y)\, dA}{\text{Area of } D} = \frac{\iint_D f(x, y)\, dA}{\iint_D dA}.$$

Similarly, the average value of $f(x, y, z)$ over a region D in $\mathbb{R}^3$ is

$$\frac{\iiint_D f(x, y, z)\, dV}{\iiint_D dV}.$$

Average value lies within the minimum and maximum values of f.

(2.38) *Example*

Find the average value of $f(x, y, z) = xyz$ over the cube bounded by the planes $x = 0, x = 2, y = 0, y = 2, z = 0$, and $z = 2$.

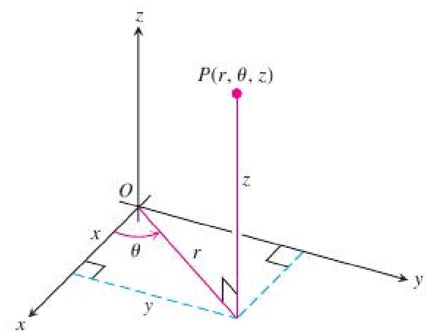
The required average value is

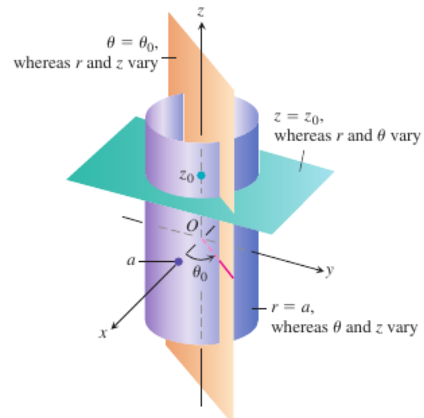
$$\frac{1}{V} \int_0^2 \int_0^2 \int_0^2 xyz\, dz\, dy\, dx = \frac{1}{8} \int_0^2 \int_0^2 2xy\, dy\, dx = \frac{1}{8} \int_0^2 4x\, dx = 1. \qquad \Box$$

2.7 Triple Integral in Cylindrical coordinates

Cylindrical coordinates express a point P in space as a triple (r, θ, z), where (r, θ) is the polar representation of the projection of P on the xy-plane. If P has Cartesian representation (x, y, z) and cylindrical representation (r, θ, z), then

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ r^2 = x^2 + y^2,\ \tan\theta = y/x.$$

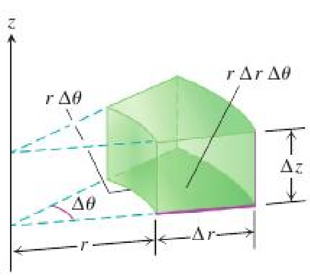




In cylindrical coordinates,

$r = a$ describes a cylinder with axis as the z-axis.
$\theta = \alpha$ describes a plane containing the z-axis.
$z = b$ describes a plane perpendicular to the z-axis.

The Riemann sum of $f(r, \theta, z)$ uses a partition of D into cylindrical wedges:

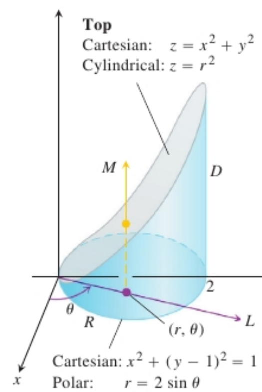


The volume element $dV = r\,dr\,d\theta\,dz$. Thus the triple integral is

$$\iiint_D f(r, \theta, z)dV = \iiint_D f(r, \theta, z) r\,drd\theta dz.$$

Its conversion to iterated integrals uses a similar technique of determining the limits of integration.

(2.39) *Example*



Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

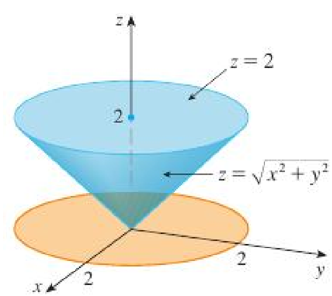
The projection of D onto the xy-plane gives the disk R enclosed by the circle $x^2 + (y - 1)^2 = 1$. It simplifies to $x^2 + y^2 = 2y$. Its polar form is $r^2 = 2r \sin \theta$ or, $r = 2 \sin \theta$. Here, $0 \le \theta \le \pi$ and $0 \le r \le 2 \sin \theta$.
A line through a point $(r, \theta) \in R$ enters D at $z = 0$ and leaves D at $z = x^2 + y^2 = r^2$. Hence

$$\iiint f(r, \theta, z) dV = \int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z) r \, dz \, dr \, d\theta. \qquad \square$$

(2.40) *Example*

Evaluate $I = \displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2) \, dz \, dy \, dx$ using cylindrical coordinates.

The z-limits show that the solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 2$. Its projection on the xy-plane is the disk $x^2 + y^2 = 4$. The limits for y also confirm this. A sketch of the solid looks as follows:



The disk $x^2 + y^2 = 4$ (projected region in the xy-plane) is described in the cylindrical coordinates by $0 \le \theta \le 2\pi$, $0 \le r \le 2$. For any point (r, θ) in this region, we have $\sqrt{x^2 + y^2} = r \le z \le 2$ for the solid. Thus

$$I = \iiint_D (x^2 + y^2) dV = \int_0^{2\pi} \int_0^{2} \int_r^{2} r^2 \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} (r^3 (2 - r) \, dr \, d\theta = \int_0^{2\pi} \left(2\frac{2^4}{4} - \frac{2^5}{5}\right) d\theta = \int_0^{2\pi} \frac{8}{5} \, d\theta = \frac{16}{5}\pi. \qquad \square$$

(2.41) ***Example***

Let D be the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the iterated integrals in cylindrical coordinates for computing $\iiint_D f(r, \theta, z)\, dV$ using the following orders of integration:
(a) $dz\, dr\, d\theta$ (b) $dr\, dz\, d\theta$ (c) $d\theta\, dz\, dr$.

In cylindrical coordinates, the equation of the cone is $r = z$ and the equation of the paraboloid is $z = 2 - r^2$ or $r = \sqrt{2 - z}$. (Since $r \geq 0$.) Further, eliminating z from these equations, we have $r = 2 - r^2 \Rightarrow (r + 2)(r - 1) = 0 \Rightarrow r = 1$. The projection of D onto $r\theta$-plane is the unit disk: $0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1$. For any $(r, \theta, z) \in D$, neither z nor r depends on θ; also θ does not depend on z and r.

(a) For any $(r, \theta, z) \in D$, $0 \leq r \leq 1$. For any such r, we have $r \leq z \leq 2 - r^2$. Hence,

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} f(r, \theta, z)\, r\, dz\, dr\, d\theta.$$

(b) For any $(r, \theta, z) \in D$, $0 \leq z \leq 2$. To get the dependence of r on z, we see that $D = D_1 \cup D_2$, where D_1 is the portion of D inside the cone bounded by the plane $z = 1$, and D_2 is the rest of D. For any $(r, \theta, z) \in D_1$, $0 \leq z \leq 1$ and $0 \leq r \leq z$. For any $(r, \theta, z) \in D_2$, $1 \leq z \leq 2$ and $0 \leq r \leq \sqrt{2 - z}$. Then

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{2\pi} \int_0^1 \int_0^z f(r, \theta, z) r dr dz d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} f(r, \theta, z) r dr dz d\theta.$$

(c) Since θ is independent of r and z, from (a) we get the volume as

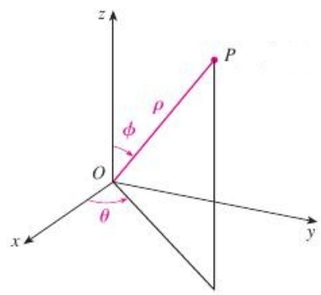
$$\iiint_D f(r, \theta, z)\, dV = \int_0^1 \int_r^{2-r^2} \int_0^{2\pi} f(r, \theta, z)\, r\, d\theta\, dz\, dr.$$

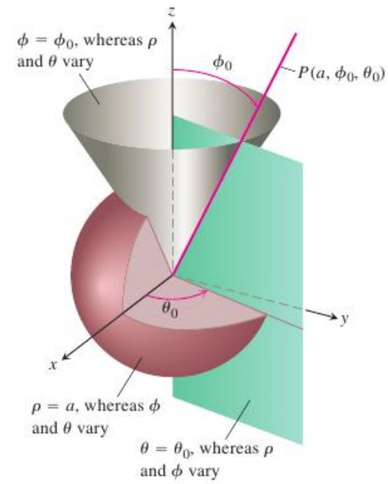
By taking $f(r, \theta, z) = 1$, you can verify that all the three iterated integrals in (a)-(c) give the volume of D as $5\pi/6$. □

2.8 Triple Integral in Spherical coordinates

Spherical coordinates express a point P in space as a triple (ρ, ϕ, θ), where ρ is the distance of P from the origin O, ϕ is the angle between +ve z-axis and the line OP, and θ is the angle between the projected line of OP on the xy-plane and the +ve x-axis. This θ is the same as the 'cylindrical' θ. Moreover, $\rho \geq 0,\ 0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$. If $P(x, y, z)$ has spherical representation (ρ, ϕ, θ), then

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi,\ \rho = \sqrt{x^2 + y^2 + z^2}.$$

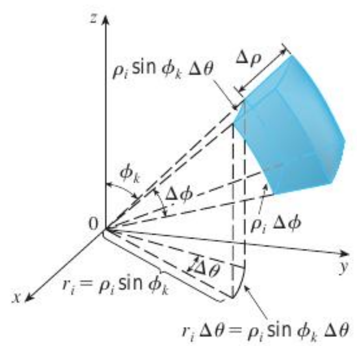




The 'cylindrical' r is equal to $\rho \sin \phi$. In spherical coordinates,

$\rho = a$ describes the sphere of radius a centered at the origin.
$\phi = \phi_0$ describes a cone with its axis as the z-axis.
$\theta = \theta_0$ describes a plane containing the z-axis.

When computing a triple integral over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by the changes $\Delta\rho_k, \Delta\phi_k, \Delta\theta_k$ in ρ, ϕ, θ.



Such a spherical wedge has one edge a circular arc of a circle with radius ρ_k and angle $\Delta\phi_k$, whose length is $\rho_k \Delta\phi_k$; another edge a circular arc of a circle with radius $\rho_k \sin \phi_k$ and angle $\Delta\theta_k$, whose length is $\rho_k \sin \phi_k \Delta\theta_k$; and thickness $\Delta\rho_k$. The volume of such a spherical wedge is approximately that of a rectangular box with dimensions ρ_k, $\rho_k \Delta\phi_k$ and $\rho_k \sin \phi_k \Delta\theta_k$. Thus

$$\Delta V_k \approx \rho_k^2 \sin \phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k.$$

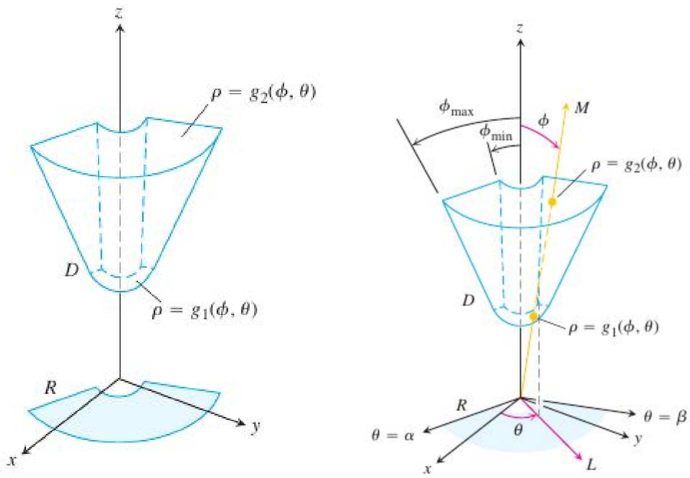
The corresponding Riemann sum is $S = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin \phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$.

Accordingly,

$$\iiint_D f(\rho,\phi,\theta)\,dV = \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho d\phi d\theta.$$

The procedure in computing a triple integral in spherical coordinates is similar to that in cylindrical coordinates:

Sketch the region D and its projection on the xy-plane. Then find the ρ limit, ϕ limit and θ limit.

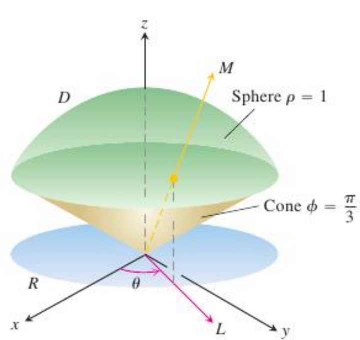


$$\iiint_D f(\rho,\phi,\theta)dV = \int_\alpha^\beta \int_{\phi-min}^{\phi-max} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

(2.42) ***Example***

Find the volume of the solid D cut from the ball $\rho \le 1$ by the cone $\phi = \pi/3$.

Draw a ray M through D from the origin making an angle ϕ with the z-axis. Draw also its projection L on the xy-plane. The line L makes an angle θ with the x-axis. Let R be the projected region of D in the xy-plane.



M enters D at $\rho = 0$ and leaves D at $\rho = 1$.

Angle ϕ runs through 0 to $\pi/3$, since D is bounded by the cone $\phi = \pi/3$.

L sweeps through R as θ varies from 0 to 2π. Thus

$$V = \iiint_D \rho^2 \sin\phi \, d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \left[\frac{-\cos\phi}{3} \right]_0^{\pi/3} = \frac{1}{6} 2\pi = \frac{\pi}{3}. \qquad \square$$

(2.43) ***Example***

Evaluate $I = \displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} dz \, dy \, dx.$

Notice that $I = \displaystyle\iiint_D e^{(x^2+y^2+z^2)^{3/2}} dV$, where D is the unit ball. In spherical coordinates, $I = \displaystyle\iiint_D e^{\rho^3} dV$. Converting I to an iterated integral, we have

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

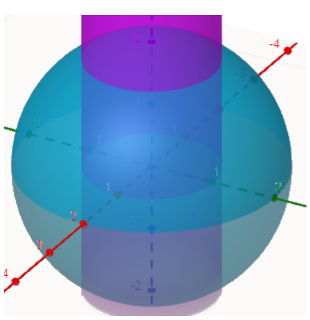
The integrand is a product of a function of ρ, a function of ϕ, and a function of θ. Thus,

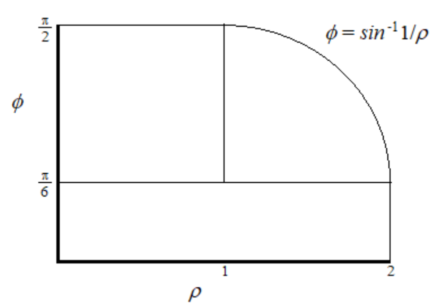
$$I = \int_0^1 e^{\rho^3} \rho^2 \, d\rho \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta = \left[\frac{e^{\rho^3}}{3} \right]_0^1 \Big[-\cos\phi \Big]_0^{\pi} (2\pi) = \frac{4\pi}{3}(e-1). \qquad \square$$

(2.44) ***Example***

A solid is bounded below by the xy-plane, above by the sphere $x^2+y^2+z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the iterated integrals in spherical coordinates for computing the volume of the solid by using the following order of integration: (a) $d\rho \, d\phi \, d\theta$ (b) $d\phi \, d\rho \, d\theta$.

The projection of the solid in the xy-plane is the unit disk. So, for any point (ρ, ϕ, θ) in the solid, $0 \le \theta \le 2\pi$; the variables ρ and ϕ do not depend on θ, and θ does not depend on ρ and ϕ. The equation of the sphere in spherical coordinates is $\rho = 2$, and the equation of the cylinder is $\rho^2 \sin^2\phi = 1$, or $\rho \sin\phi = 1$.





Notice that $0 \le \phi \le \pi/2$. Eliminating ρ, we have $2\sin\phi = 1 \Rightarrow \phi = \pi/6$. This gives the curve of intersection of the surfaces, which is a circle.

(a) We think of the solid as the disjoint union of D_1 and D_2, where D_1 is the spherical sector with the curve of intersection as the bounding circle, and D_2 is the rest of the solid. The top of D_1 is the portion of the sphere cut by the bounding circle, and the bounding wall of D_2 is the cylinder. If the point $(\rho, \phi, \theta) \in D_1$, then $0 \le \phi \le \pi/6$ and $0 \le \rho \le 2$. If the point $(\rho, \phi, \theta) \in D_2$, then $\pi/6 \le \phi \le \pi/2$ and $0 \le \rho \le \operatorname{cosec}\phi$. Thus, the volume of the solid is

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\operatorname{cosec}\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

(b) We think of the solid as the disjoint union of two solids D_3 and D_4, where D_3 is the portion of the solid cut-out by the unit ball, and D_4 is the rest. If the point $(\rho, \phi, \theta) \in D_3$, then $0 \le \rho \le 1$ and $0 \le \phi \le \pi/2$. If the point $(\rho, \phi, \theta) \in D_4$, then $1 \le \rho \le 2$ and $0 \le \phi \le \sin^{-1}(1/\rho)$. Notice that in this case, $0 \le \phi \le \pi/6$ describes the top portion of D_4 up to the circle of intersection, and $\pi/6 \le \phi \le \sin^{-1}(1/\rho)$ describes the side portion of D_4 from the circle of intersection to the xy-plane along the cylindrical wall. (This relationship of ρ and ϕ is depicted schematically in the second figure above.) Hence, the volume of the solid is

$$\int_0^{2\pi}\int_0^{1}\int_0^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi}\int_1^{2}\int_0^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta. \qquad \square$$

2.9 Change of Variables

The change of coordinate system from Cartesian to Cylindrical or to Spherical are examples of change of variables. Let us consider what happens when a different type of change of variables occurs.

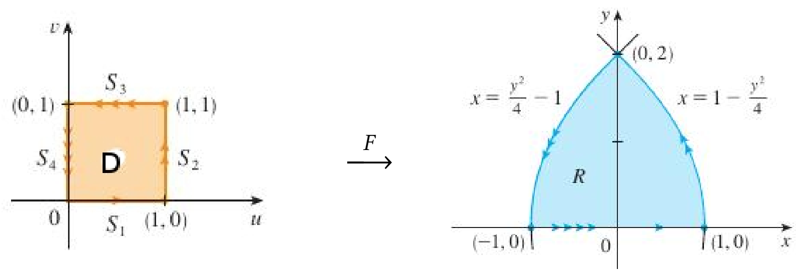
Suppose F maps a region D in $\mathbb{R}^2$ onto a region R in $\mathbb{R}^2$ in a one-one manner. For convenience, we say that D is a region in the uv-plane and R is a region in the xy-plane; and F maps (u, v) to (x, y). Then F can be thought of as a pair of maps: (F_1, F_2). That is, $x = F_1(u, v)$ and $y = F_2(u, v)$. We often show this dependence implicitly by writing

$$x = x(u, v), \quad y = y(u, v).$$

(2.45) ***Example***

What is the image of $D = \{(u, v) : 0 \le u \le 1, 0 \le v \le 1\}$ under the map given by $x = u^2 - v^2,\ y = 2uv$?

The region D in the uv-plane is a square of side 1. Its lower boundary is the line segment joining the point $(0,0)$ to $(1,0)$ in the uv-plane. Here, u varies from 0 to 1 while $v = 0$. It is transformed to the line segment $x = u^2, y = 0$. In the xy-plane it is the line segment joining $(0,0)$ to $(1,0)$. Here, x varies from 0 to 1 while $y = 0$.



The right side boundary of D is the line segment joining $(1,0)$ to $(1,1)$ in the uv-plane. Here, $u = 1$ and v varies from 0 to 1. This is transformed to $x = 1 - v^2, y = 2v$. Eliminating v from these equations we have the arc of the curve $x = 1 - \frac{y^2}{4}$ joining the points $(1,0)$ to $(0,2)$ in the xy-plane.

The upper boundary of D is the line segment joining $(1,1)$ to $(1,0)$ in the uv-plane. Here, u varies from 1 to 0 while $v = 1$. This is transformed to $x = u^2 - 1, y = 2u$. Eliminating u from these equations, we get the arc of the curve $x = \frac{y^2}{4} - 1$ joining the points $(0,2)$ to $(-1,0)$ in the xy-plane.

The left side boundary of D is the line segment joining $(0,1)$ to $(0,0)$. Here, $u = 0$ and v varies from 1 to 0. This is transformed to $x = -v^2, y = 0$. In the xy-plane, it is the line segment joining $(-1,0)$ to $(0,0)$. Here, x varies from -1 to 0 while $y = 0$.

Moreover, the interior of D is mapped onto the interior of the region R in the xy-plane whose boundaries are the line segments and the arcs. $\square$

We would like to see how a double integral over a region D in the uv-plane and the double integral over the corresponding region R in the xy-plane are related when $(u,v) \mapsto (x,y)$. For this, let us first find out the change in a small rectangle in the uv-plane under the same map.

Suppose x_u, x_v, y_u, y_v are continuous. Let a typical small rectangle D' contained in D have vertices at

$$A_1 = (a,b),\ A_2 = (a + \Delta u, b),\ A_3 = (a + \Delta u, b + \Delta v),\ A_4 = (a, b + \Delta v).$$

That is, D' has sides as Δu and Δv; so its area is $\Delta u \Delta v$. Suppose the image of this rectangle under the map $(u,v) \mapsto (x,y)$ is the region R' in the xy-plane. We wish to approximate the area of R'. For this purpose, we write the images of the points A_k under the map $(u,v) \mapsto (x,y)$ as $B_k = (a_k, b_k)$ for $k = 1,2,3,4$. We then approximate the numbers a_k, b_k using the partial derivatives and call those numbers as c_k, d_k, respectively. Then, we take the area of the parallelogram P' with vertices at (c_k, d_k) as an approximation to the area of R'. Now, the approximations of the numbers a_k are as follows:

$$\begin{aligned}
a_1 &= x(a,b) = c_1,\\
a_2 &= x(a+\Delta u, b) \approx x(a,b) + x_u \Delta u = c_2,\\
a_3 &= x(a+\Delta u, b+\Delta v) \approx x(a,b) + x_u \Delta u + x_v \Delta v = c_3\\
a_4 &= x(a, b+\Delta v) \approx x(a,b) + x_v \Delta v = c_4.
\end{aligned}$$

Here, $x_u = x_u(a,b)$ and $x_v = x_v(a,b)$. Similar approximations hold for b_1, b_2, b_3, b_4. That is,

$$\begin{aligned}
b_1 &= d_1 = y(a,b), & b_2 &\approx d_2 = y(a,b) + y_u \Delta u,\\
b_3 &\approx d_3 = y(a,b) + y_u \Delta u + y_v \Delta v, & b_4 &\approx d_4 = y(a,b) + y_v \Delta v.
\end{aligned}$$

Here again, $y_u = y_u(a,b)$ and $y_v = y_v(a,b)$.

The area of the parallelogram P' with vertices at (c_1, d_1), (c_2, d_2), (c_3, d_3) and (c_4, d_4) is twice the area of the triangle T with vertices at (c_1, d_1), (c_2, d_2) and (c_3, d_3). By shifting the origin to the point (c_1, d_1), we see that the area of T is equal to the area of the triangle T' with vertices at $(0,0)$, $(c_2 - c_1, d_2 - d_1)$ and $(c_3 - c_1, d_3 - d_1)$. Recall that the area of a triangle with vertices at $(0,0)$, (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}|x_2 y_3 - x_3 y_2|$. Using this, we obtain

$$\begin{aligned}
\text{Area of } R' \approx \text{Area of } P' &= 2 \times \text{Area of } T'\\
&= |(c_2 - c_1)(d_3 - d_1) - (c_3 - c_1)(d_2 - d_1)|\\
&= |x_u \Delta u (y_u \Delta u + y_v \Delta v) - (x_u \Delta u + x_v \Delta v) y_u \Delta u|\\
&= |x_u y_v - x_v y_u|\, \Delta u \Delta v = \left|\det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\right| (a,b) \Delta u \Delta v.
\end{aligned}$$

This determinant is called the **Jacobian** of the map $(u,v) \mapsto (x,y)$; and we write

$$J = J(x(u,v), y(u,v)) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

We thus see that

$$\text{Area of } R' \approx |J|(a,b)\, \Delta u \Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| (a,b)\, \Delta u \Delta v. \tag{2.9.1}$$

That is, the Area of the image of a rectangle contained in D with one corner at (a,b) and sides of length Δu and Δv is approximately equal to $|J|(a,b) \Delta u \Delta v$.

We now look at the double integrals. Let the region D in the uv-plane be in one-one correspondence with the region R in the xy-plane by the map $(u,v) \mapsto (x,y)$. Assume that $x = x(u,v)$ and $y = y(u,v)$ have continuous partial derivatives with respect to u and v in D. Let $f(x, y)$ be a real valued continuous function defined on the region R. Then we have the map $\tilde{f}(u, v) = f(x(u, v), y(u, v))$, which is a continuous function defined on D.

Divide D in the uv-plane into smaller rectangles D_i, and call their images in the xy-plane as R_i. By using (2.9.1), forming the Riemann sum, and taking the limit, we obtain:

$$\iint_R f(x, y)dA = \iint_D \tilde{f}(u, v) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$

Similarly, when $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, we write the Jacobian as

$$J = J(x(u, v, w), y(u, v, w), z(u, v, w)) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}.$$

If R is the region in $\mathbb{R}^3$ on which f has been defined and D is the region in the uvw-space so that the functions x, y, z map D onto R in a one-one manner, then

$$\iiint_R f(x, y, z)dV = \iiint_D f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw.$$

Of course, these formulas hold under the continuity assumptions mentioned earlier.

(2.46) *Example*

(1) In case of polar coordinates, $x = x(r, \theta) = r\cos\theta, y = y(r, \theta) = r\sin\theta$. Thus, the Jacobian is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = x_r y_\theta - x_\theta y_r = \cos\theta(r\cos\theta) - (-r\sin\theta)\sin\theta = r.$$

Therefore, the double integral in polar coordinates for a function $f(x, y)$ takes the form

$$\iint_R f(x, y)dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, drd\theta.$$

(2) In case of cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Then

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = \left|\det \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix}\right| = r.$$

Therefore, $\displaystyle\iiint_R f(x, y, z)dV = \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\, drd\theta dz.$

(3) For the spherical coordinates, we see that

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$$

Then, $\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \left|\det\begin{bmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{bmatrix}\right| = \rho^2 \sin\phi$. And, the triple integral is given by

$$\iiint_R f(x,y,z)dV = \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho d\phi d\theta.$$

We had already derived these results independently. ∎

The formula relating integrals over D and R helps us in evaluating a multiple integral over R by using a suitable transformation from D to R.

(2.47) *Example*

Evaluate the double integral $\iint_R (y-x)dA$, where R is the region bounded by the lines $y-x=1,\ y-x=-3,\ 3y+x=7,\ 3y+x=15$.

Take $u = y-x,\ v = 3y+x$. That is, $x = \frac{1}{4}(v-3u), y = \frac{1}{4}(u+v)$. Then

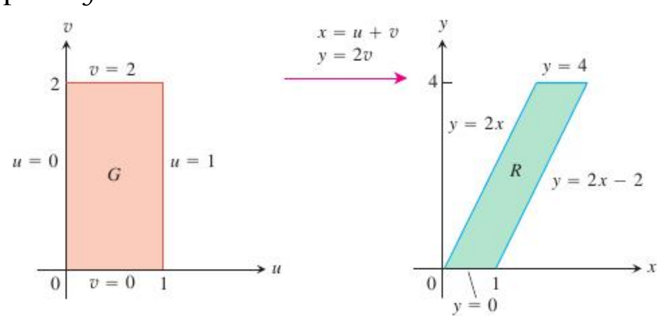
$$D = \{(u,v) : -3 \le u \le 1,\ 7 \le v \le 15\}.$$

The Jacobian is $J = x_u y_v - x_v y_u = -\frac{3}{4}\cdot\frac{1}{4} - \frac{1}{4}\cdot\frac{1}{4} = -\frac{1}{4}$. Therefore,

$$\begin{aligned}\iint_R (y-x)dA &= \iint_D u\,|J|\,dudv = \iint_D \frac{u}{4}\,dudv \\ &= \int_{-3}^{1}\int_{7}^{15} \frac{u}{4}\,dv\,du = \int_{-3}^{1} 2u\,du = -8.\end{aligned}$$ ∎

(2.48) *Example*

Compute $\displaystyle\iint_R \frac{2x-y}{2}\,dA$ by using the transformation $u = x - y/2,\ v = y/2$, where R is the region enclosed by the lines $y=0$ on the bottom, $y = 2x-2$ on the right, $y=4$ on the top and $y=2x$ on the left.



The transformation has the inverse given by $x = u+v,\ y = 2v$. The regions R in the xy-plane and G in the uv-plane are

$$R = \{(x,y) : 0 \le y \le 4, y/2 \le x \le 1 + y/2\},\ G = \{(u,v) : 0 \le u \le 1, 0 \le v \le 2\}.$$

$$f(x,y) = \frac{2x-y}{2} = u, \ \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |x_u y_v - x_v y_u| = |(1)(2) - (0)(1)| = 2.$$

So,

$$\iint_R \frac{2x-y}{2} dA = \iint_G 2u\,dudv = \int_0^2 \int_0^1 2u\,du\,dv = \int_0^2 1^2\,dv = 2.$$

Also, looking at R, direct integration gives

$$\iint_R \frac{2x-y}{2}\,dA = \int_0^4 \int_{y/2}^{1+y/2} \frac{2x-y}{2}\,dx\,dy = \int_0^4 \left(\frac{x^2}{2}\Big|_{y/2}^{1+y/2} - \frac{y}{2}\cdot x\Big|_{y/2}^{1+y/2}\right) dy$$

$$= \int_0^4 \left(\frac{1}{2}\left[\left(1+\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^2\right] - \frac{y}{2}\right) dy$$

$$= \int_0^4 \left(\frac{1}{2}\left[1\cdot(1+y)\right] - \frac{y}{2}\right) dy = \int_0^4 \frac{1}{2}\,dy = 2. \qquad \square$$

The change of variables formula turns an xy-integral into a uv-integral. But the map that changes the variables goes from uv-region onto xy-region. This map must be one-one on the interior of the uv-region. Sometimes it is easier to get such a map from xy-region to uv-region. Then we will be tackling with the inverse of such an easy map. However, for computing the integral it is enough to compute the Jacobian of the inverse map instead of the map itself. The following fact helps us:

The Jacobian of the inverse map is the inverse of the Jacobian of the original map.

This may be expressed as

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}.$$

Similarly, triple integrals undergo change of variables by using the inverse of the Jacobian.

(2.49) *Example*

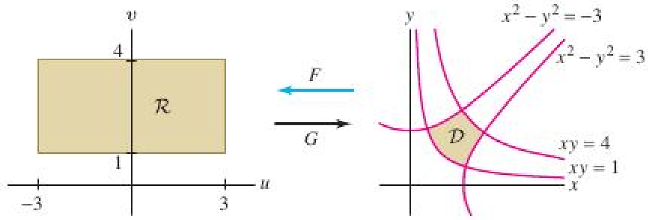
Integrate $f(x,y) = xy(x^2+y^2)$ over the region

$$D = \{(x,y) \in \mathbb{R}^2 : -3 \le x^2 - y^2 \le 3,\ 1 \le xy \le 4\}.$$

There is a simple map that goes in the *wrong direction*: $u = x^2 - y^2,\ v = xy$. Then the image of D, which we denote as R in the uv-plane is the rectangle

$$R = \{(u,v) : -3 \le u \le 3,\ 1 \le v \le 4\}.$$

We have $F : D \to R$ defined by $F(x,y) = (u,v) = (x^2-y^2, xy)$. And its inverse is $G = F^{-1}$, where $G : R \to D$.



We need not compute the map G. Instead, we go for the Jacobian.

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det\begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2(x^2+y^2).$$

Therefore, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2(x^2+y^2)}$. Then

$$I = \iint_D xy(x^2+y^2)\,dA = \iint_R \left[xy(x^2+y^2)\times\frac{1}{2(x^2+y^2)}\right] du\,dv.$$

The integral on the right side is in the uv-plane and the term inside $[\,\cdot\,]$ is a function of (u,v), which simplifies to $xy/2 = v/2$. Thus,

$$I = \iint_R \frac{v}{2}\,dudv = \frac{1}{2}\int_{-3}^{3}\int_1^4 v\,dv\,du = \frac{1}{2}\int_{-3}^{3}\frac{4^2-1^2}{2}\,du = \frac{15}{4}\big[3-(-3)\big] = \frac{45}{2}. \quad \square$$

2.10 Review Problems

Problem 2.1 Find the area of the region bounded by the curves $y=x$ and $y=2-x^2$.

The points of intersection of the curves satisfy $y=x$ and $x=2-x^2$. The last equation is same as $(x+2)(x-1)=0$. Thus the points of intersection are $(-2,-2)$ and $(1,1)$. Hence the area is

$$\left|\int_{-2}^{1}\int_x^{2-x^2} dydx\right| = \left|\int_{-2}^{1}(2-x^2-x)dx\right| = \left|\left[2x-\frac{x^3}{3}-\frac{x^2}{2}\right]_{-2}^{1}\right| = \frac{9}{2}.$$

Since the significant portion of the curve $y=2-x^2$ lies above the portion of the line $y=x$, there is no need to take the absolute value. The calculation also confirms this.

Problem 2.2 Evaluate $I = \iint_D (4-x^2-y^2)\,dA$ if D is the region bounded by the straight lines $x=0,\ x=1,\ y=0$ and $y=3/2$.

$$I = \int_0^{3/2}\int_0^1 (4-x^2-y^2)dxdy = \int_0^{3/2}\left[4x-x^3/3-y^2x\right]_0^1 dy = \int_0^{3/2}\left(\frac{11}{3}-y^2\right)dy = \frac{35}{8}.$$

Problem 2.3 Evaluate the double integral of $f(x, y) = 1 + x + y$ over the region bounded by the lines $y = -x,\ y = 2$ and the parabola $x = \sqrt{y}$.

Draw the region. The integral is equal to

$$\int_0^2 \int_{-y}^{\sqrt{y}} (1 + x + y)dx\,dy = \int_0^2 \big(\sqrt{y} + \frac{y}{2} + \sqrt{y}\,y - \big(-y + \frac{y^2}{2} - y^2\big)\,dy$$

$$= \int_0^2 \big(\sqrt{y} + \frac{3y}{2} + y\sqrt{y} + \frac{y^2}{2}\big)dy = \Big[\frac{2}{3}y^{3/2} + \frac{3}{4}y^2 + \frac{5}{2}y^{5/2} + \frac{1}{6}y^3\Big]_0^2 = \frac{1}{3}(13 + 44\sqrt{2}).$$

Problem 2.4 Change the order of integration in $\int_0^1 \int_x^{\sqrt{x}} f(x, y)dydx$.

The region D of integration is bounded by the straight line $y = x$ and the parabola $y = \sqrt{x}$. Every straight line parallel to x-axis cuts the boundary of D in no more than two points, and it remains in between y^2 to y. Also, y lies between 0 and 1. Hence

$$\int_0^1 \int_x^{\sqrt{x}} f(x, y)dydx = \int_0^1 \int_{y^2}^{y} f(x, y)dx\,dy.$$

Problem 2.5 Evaluate $\iint_D e^{y/x}\,dA$, where D is a triangle bounded by the straight lines $y = x$, $y = 0$, and $x = 1$.

In D, the variable x remains in between 0 and 1, and y lies between 0 and x. Hence

$$\iint_D e^{y/x}\,dA = \int_0^1 \int_0^x e^{y/x}dy\,dx = \int_0^1 x(e - 1)\,dx = \frac{e - 1}{2}.$$

Problem 2.6 Find $I = \iint_D e^{x+y}\,dA$, where D is the annular region bounded by two squares of sides 2 and 4, each having center at $(0, 0)$ and sides parallel to the axes.

Draw the picture. D is not a simply connected region. Divide D into four simply connected regions by drawing lines $x = -1$ and $x = 1$. Let D_1 be the rectangle to the left of the inner square; D_2 be the square on top of the inner square; D_3 be the square below the inner square; and D_4 be the rectangle to the right of the inner square; so that D is the disjoint union of D_1, D_2, D_3, D_4. Then

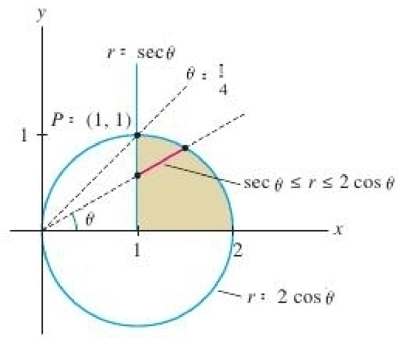
$$I = \iint_{D_1} e^{x+y}\,dA + \iint_{D_2} e^{x+y}\,dA + \iint_{D_3} e^{x+y}\,dA + \iint_{D_4} e^{x+y}\,dA.$$

Converting each integral to an iterated integral, we have

$$I = \int_{-2}^{-1} \int_{-2}^{2} e^{x+y}dy\,dx + \int_{-1}^{1} \int_{1}^{2} e^{x+y}dy\,dx + \int_{-1}^{1} \int_{-2}^{-1} e^{x+y}dy\,dx$$

$$+ \int_1^2 \int_{-2}^2 e^{x+y} dy\, dx = e^4 - e^2 - e^{-2} + e^{-4}.$$

Problem 2.7 Evaluate $\iint_D (x^2 + y^2)^{-2}\, dA$, where D is the shaded region in the figure below:



The integrand in polar coordinates is $f(r,\theta) = r^{-4}$. The region D is given by $0 \le \theta \le \pi/4$, $\sec\theta \le r \le 2\cos\theta$. Thus

$$\iint_D (x^2 + y^2)^{-2}\, dA = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-4}\, r\, dr\, d\theta = \frac{1}{8} \int_0^{\pi/4} (4\cos^2\theta - \sec^2\theta)\, d\theta = \frac{\pi}{16}.$$

Problem 2.8 Calculate the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

The volume $V = \iint_D (1 - x - y) dA$, where D is the base of the solid on the xy-plane. We see that D is the triangular region bounded by the straight lines $x = 0$, $y = 0$ and $x + y = 1$. Thus,

$$V = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx = \int_0^1 \frac{1}{2}(1 - x)^2 dx = \frac{1}{6}.$$

Problem 2.9 Compute the volume V of the solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$, the cylinder $x^2 + y^2 = 2ay$, where $a > 0$, and which is inside the cylinder.

The region of integration is the base of the cylinder. It is the circle $x^2+y^2-2ay = 0$, whose centre is $(0, a)$ and radius a. We calculate $V/4$, the volume of the portion of the solid in the first octant. Now, the region of integration D is the semicircular disk whose boundaries are given by

$$x = g_1(y) = 0,\ x = g_2(y) = \sqrt{2ay - y^2},\ y = 0,\ y = 2a.$$

The integrand is $z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$. Then

$$\frac{V}{4} = \int_0^{2a} \int_0^{\sqrt{2ay-y^2}} \sqrt{4a^2 - x^2 - y^2}\, dx\, dy.$$

To evaluate this, use polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$. For the limits of integration, use $x^2 + y^2 = r^2,\ y = r\sin\theta$ to get:

$$x^2 + y^2 - 2ay = 0 \Rightarrow r^2 - 2ar\sin\theta = 0 \Rightarrow r = 2a\sin\theta.$$

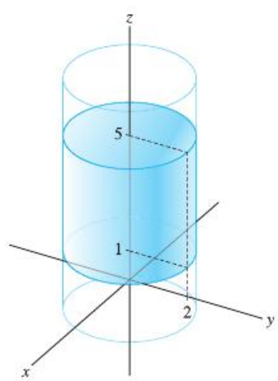
That is, in polar coordinates, the boundaries of D are given by

$$r = g_1(\theta) = 0,\ r = g_2(\theta) = 2a\sin\theta,\ 0 \le \theta \le \pi/2.$$

The integrand is $f(r,\theta) = \sqrt{4a^2 - r^2}$. Hence,

$$\begin{aligned} V &= 4\int_0^{\pi/2}\int_0^{2a\sin\theta} \sqrt{4a^2 - r^2}\, r\, dr\, d\theta \\ &= \frac{-4}{3}\int_0^{\pi/2} \left[(4a^2 - 4a^2\sin^2\theta)^{3/2} - (4a^2)^{3/2}\right] d\theta = \frac{16}{9}a^3(3\pi - 4). \end{aligned}$$

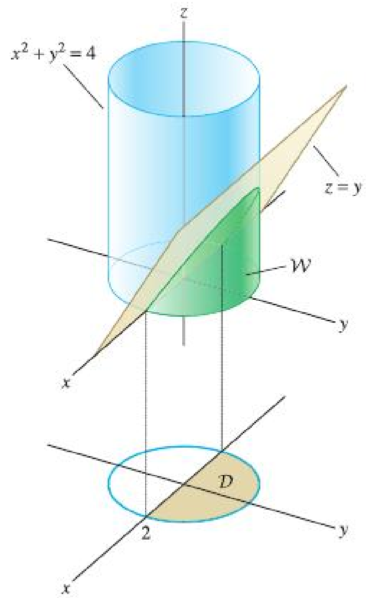
Problem 2.10 Integrate $f(x,y,z) = z\sqrt{x^2+y^2}$ over the solid cylinder $x^2+y^2 \le 4$ for $1 \le z \le 5$.



The region of integration D in cylindrical coordinates is given by $0 \le \theta \le 2\pi$, $0 \le r \le 2$, $1 \le z \le 5$. The integrand is zr. Thus

$$\iiint_D z\sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^2\int_1^5 (zr)\, r\, dz\, dr\, d\theta = 64\pi.$$

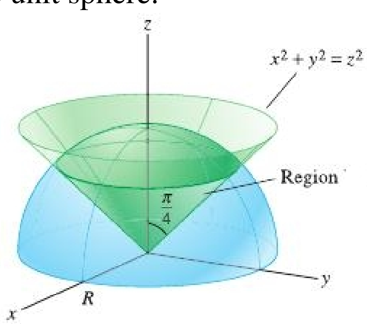
Problem 2.11 Integrate $f(x,y,z) = z$ over the part of the solid cylinder $x^2+y^2 \le 4$ for $0 \le z \le y$.



The region W has the projection D on the xy-plane as the semicircle depicted in the figure. The z-coordinate varies from 0 to y and $y = r\sin\theta$. Thus W is given by $0 \le \theta \le \pi$, $0 \le r \le 2$, $0 \le z \le r\sin\theta$. In cylindrical coordinates,

$$\iiint_W z\,dV = \int_0^\pi \int_0^2 \int_0^{r\sin\theta} z\,r\,d\theta\,dr\,dz = \int_0^\pi \int_0^2 \frac{1}{2}\big(r\sin\theta\big)^2\,r\,d\theta\,dr = \pi.$$

Problem 2.12 Compute $\iiint_D z\,dV$, where D is the solid lying above the cone $x^2 + y^2 = z^2$ and below the unit sphere.



The upper branch of the cone, which is relevant to D, has the equation $\phi = \pi/4$ in spherical coordinates. The sphere has the equation $\rho = 1$. Thus D is given by

$$D:\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le 1.$$

Since $z = \rho\cos\phi$, the required integral is

$$\begin{aligned}\iiint_D z\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi\int_0^{\pi/4}\int_0^1 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi = \frac{\pi}{2}\int_0^{\pi/4}\cos\phi\sin\phi\,d\phi = \frac{\pi}{8}.\end{aligned}$$

Problem 2.13 Evaluate $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

$$\begin{aligned} I^2 &= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2 = \lim_{a\to\infty}\left[\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)\right] \\ &= \lim_{a\to\infty}\left[\int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy\right] = \lim_{a\to\infty}\iint_R e^{-x^2-y^2}\,dA \end{aligned}$$

where R is the square $[-a,a]\times[-a,a]$ for $a>0$.

Let $D = B(0,a)$ and $S = B(0,\sqrt{2}\,a)$, the balls centred at 0 and with radii a and $\sqrt{2}\,a$, respectively. Then $D \subseteq R \subseteq S$. Since $e^{-x^2-y^2} > 0$ for all $(x,y)\in\mathbb{R}^2$, we have

$$\iint_D e^{-x^2-y^2}\,dA \le \iint_R e^{-x^2-y^2}\,dA \le \iint_S e^{-x^2-y^2}\,dA.$$

Now,

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta = \pi(1-e^{-a^2}).$$

Similarly, $\iint_S e^{-x^2-y^2}\,dA = \pi(1-e^{-2a^2})$. We see that

$$\lim_{a\to\infty}\iint_D e^{-x^2-y^2}\,dA = \pi, \quad \lim_{a\to\infty}\iint_S e^{-x^2-y^2}\,dA = \pi.$$

Therefore, by Sandwich theorem, we have

$$I^2 = \lim_{a\to\infty}\iint_R e^{-x^2-y^2}\,dA = \pi \Rightarrow I = \sqrt{\pi}.$$

Problem 2.14 Compute the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

Projection of this solid on the xy-plane is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Therefore, the required volume is

$$V = \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx = 2c\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx.$$

Substitute $y = b(1-x^2/a^2)^{1/2}\sin t$. Then $dy = b(1-x^2/a^2)\cos t\,dt$ and $-\pi/2 \le t \le \pi/2$. Therefore,

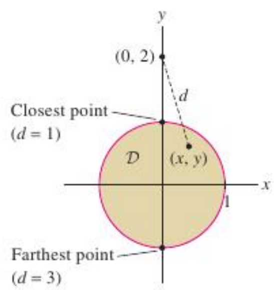
$$\begin{aligned} V &= 2c\int_{-a}^{a}\int_{-\pi/2}^{\pi/2}\left[\left(1-\frac{x^2}{a^2}\right)-\left(1-\frac{x^2}{a^2}\right)\sin^2 t\right]^{1/2} b\left(1-\frac{x^2}{a^2}\right)\cos t\,dt\,dx \\ &= \frac{bc\pi}{a^2}\int_{-a}^{a}\left(a^2-x^2\right)dx = \frac{4\pi abc}{3}. \end{aligned}$$

Problem 2.15 Evaluate $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$ for $a > 0, b > 0$.

$$\begin{aligned}\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx &= \int_0^\infty \int_a^b e^{-yx}\,dy\,dx \\ &= \int_a^b \int_0^\infty e^{-yx}\,dx\,dy = \int_a^b \frac{1}{y}\,dy = \ln\frac{b}{a}.\end{aligned}$$

Notice the change in order of integration above.

Problem 2.16 Show that $\displaystyle\frac{\pi}{3} \le \iint_D \frac{dA}{\sqrt{x^2 + (y-2)^2}} \le \pi$, where D is the unit disc.



The quantity $f(x, y) = \sqrt{x^2 + (y-2)^2}$ is the distance of any point (x, y) from $(0, 2)$. For $(x, y) \in D$, maximum of $f(x, y)$ is thus 3 and minimum is 1. Therefore,

$$\frac{1}{3} \le \frac{1}{\sqrt{x^2 + (y-2)^2}} \le 1.$$

Integrating over D, we have

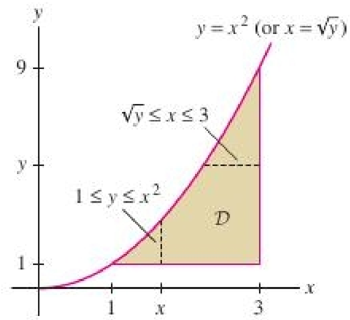
$$\iint_D \frac{1}{3}\,dA \le \iint_D \frac{1}{\sqrt{x^2 + (y-2)^2}}\,dA \le \iint_D 1\,dA.$$

Since $\displaystyle\iint_D dA$ = area of D, we obtain

$$\frac{\pi}{3} \le \iint_D \frac{dA}{\sqrt{x^2 + (y-2)^2}} \le \pi.$$

Problem 2.17 Evaluate $\displaystyle\int_1^9 \int_{\sqrt{y}}^3 xe^y\,dx\,dy$.

The region of integration is given by $1 \le y \le 9$, $\sqrt{y} \le x \le 3$.

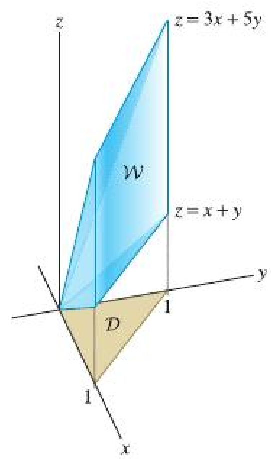


The same is expressed as $1 \le x \le 3,\ 1 \le y \le x^2$. Changing the order of integration, we have

$$\int_1^9 \int_{\sqrt{y}}^3 xe^y\,dx\,dy = \int_1^3 \int_1^{x^2} xe^y\,dx\,dy = \int_1^3 (xe^{x^2} - e\,x)dx = \frac{1}{2}(e^9 - 9e).$$

Problem 2.18 Evaluate $\iiint_W z\,dV$, where W is the solid bounded by the planes $x = 0$, $y = 0$, $x + y = 1$, $z = x + y$, and $z = 3x + 5y$ in the first octant.

W lies over the triangle D in the xy-plane defined by $0 \le x \le 1,\ 0 \le y \le 1 - x$. Hence



$$\iiint_D z\,dV = \int_0^1 \int_0^{1-x} \int_{x+y}^{3x+5y} z\,dz\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} (4x^2 + 14xy + 12y^2)\,dy\,dx = \int_0^1 (4 - 5x + 2x^2 - x^3)\,dx = \frac{23}{12}.$$

Problem 2.19 (Fun Problem) The n-dimensional cube with side a has volume a^n. What is the volume of an n-dimensional ball?

Denote by $V_n(r)$ the volume of the n-dimensional ball with radius r. Also, write $A_n = V_n(1)$. For $n = 1$, we have the interval $[-1, 1]$, whose volume we take as its length, that is, $A_1 = 2,\ V_1(r) = 2r$. For $n = 2$, we have the unit disk, whose volume is its area; that is, $A_2 = \pi,\ V_2 = \pi r^2$. For $n = 3$, we know that $A_3 = 4\pi/3$ and $V_3(r) = 4\pi r^3/3$.

Exercise 1: Show by induction that $V_n(r) = A_n\, r^n$.

Suppose $V_{n-1}(r) = A_{n-1} r^{n-1}$. The slice of the n-dimensional ball $x_1^2 + \cdots x_{n-1}^2 + x_n^2 = r^2$ at the height $x_n = c$, has the equation

$$x_1^2 + \cdots + x_{n-1}^2 + c^2 = r^2.$$

This slice has the radius $\sqrt{r^2 - c^2}$. Thus

$$V_n(r) = \int_{-r}^{r} V_{n-1}\sqrt{r^2 - x_n^2}\, dx_n = A_{n-1}\int_{-r}^{r} \left(r^2 - x_n^2\right)^{(n-1)/2} dx_n.$$

Substitute $x_n = r\sin\theta$. So, $dx_n = r\cos\theta$ and $-\pi/2 \le \theta \le \pi/2$. Then

$$V_n(r) = A_{n-1} r^n \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta = A_{n-1} C_n r^n,$$

where $C_n = \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta$. This says that $A_n = A_{n-1} C_n$.

Exercise 2: Prove that $C_3 = 4/5, C_4 = 3\pi/8$ and $C_n = \frac{n-1}{n} C_{n-2}$.

Exercise 3: Prove that $A_{2m} = \dfrac{\pi^m}{m!}$ and $A_{2m+1} = \dfrac{2^{m+1}\pi^m}{1\cdot 3\cdots(2m+1)}$.

This sequence of numbers have a curious property: A_n increases up to $n = 5$ and then it decreases to 0 as $n \to \infty$.

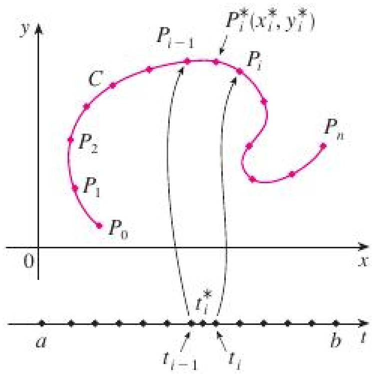
3

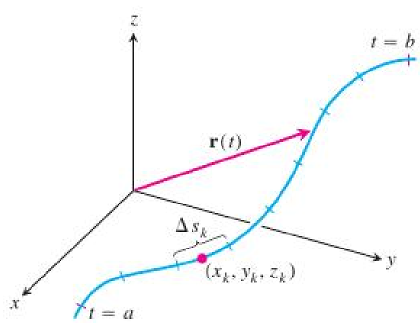
Vector Calculus

3.1 Line Integral

Line integrals are single integrals which are obtained by integrating a function over a curve instead of integrating over an interval.

Let $f(x, y, z)$ be a real valued function with region D as its domain. Let C be a curve that lies in D. Then, the function f is defined at every point (x, y, z) on the curve C. We would like to define the integral of f when (x, y, z) varies over C.





For this purpose, we partition C into n sub-arcs so that when n approaches ∞, the lengths of the sub-arcs approach 0. We choose a point (x_k, y_k, z_k) on the kth sub-arc. Suppose the kth sub-arc has length Δs_k. We then form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k.$$

If $\lim_{n\to\infty} S_n$ exists, then we call this limit as the **line integral of** f **over the curve** C; and write

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} S_n.$$

Suppose C is given in parametric form by

$$C:\ \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \quad a \le t \le b.$$

The values of f on the curve C are given by the composite function $f\big(x(t), y(t), z(t)\big)$. The following result helps in computing the line integral of f over C.

(3.1) *Theorem*

Let $C: \ \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \ a \le t \le b$, be a parametrization of the curve C lying in a region $D \subseteq \mathbb{R}^3$. Let $f: D \to \mathbb{R}$ have continuous partial derivatives and let the component functions $x(t), y(t), z(t)$ have continuous derivatives. Then the line integral of f over C exists and is given by

$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

We also write $\dot{x} = \dfrac{dx}{dt}, \ \dot{y} = \dfrac{dy}{dt}, \ \dot{z} = \dfrac{dz}{dt}$ and $ds = |\vec{r}\,'(t)|dt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt$.

When the curve C is given without parameterization, we first parameterize it, and then evaluate the line integral.

Informally, we say that the curve C given by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \ a \le t \le b$, *is smooth* when $x(t), \ y(t)$ and $z(t)$ have continuous derivatives.

Notice that the function f here takes a vector (x, y, z) as it argument and returns a real number. Such functions are called *scalar functions* or *scalar fields*. Accordingly, the line integral introduced here is referred to as the *line integral of a scalar function* and also as the *line integral of a scalar field*.

(3.2) *Example*

Evaluate $\displaystyle\int_C (2 + x^2 y)\,ds$, where C is the upper half of the unit circle in the xy-plane.

A parametrization of C is $x(t) = \cos t, \ y(t) = \sin t$ for $0 \le t \le \pi$. Then,

$$\int_C (2 + x^2 y)\,ds = \int_0^\pi (2 + \cos^2 t\,\sin t)\sqrt{\dot{x}^2 + \dot{y}^2}\,dt = 2\pi + \frac{2}{3}. \qquad \square$$

(3.3) *Example*

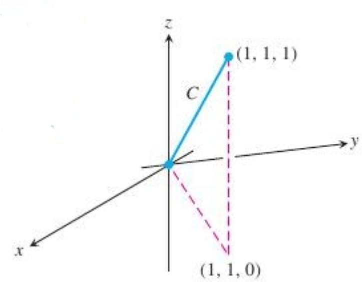
Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment from $(0,0,0)$ to $(1,1,1)$.

The curve C here is the line segment from $(0,0,0)$ to $(1,1,1)$. Its parametrization is

$C: \ \vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath} + t\,\hat{k}, \quad 0 \le t \le 1.$

That is, $x(t) = y(t) = z(t) = t$. So,

$|\vec{r}\,'(t)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. Then,



$$\begin{aligned}\int_C f\,ds &= \int_0^1 \left[x(t) - 3y^2(t) + z(t)\right]\sqrt{3}\,dt \\ &= \int_0^1 [t - 3t^2 + t]\sqrt{3}\,dt = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0.\end{aligned} \qquad \square$$

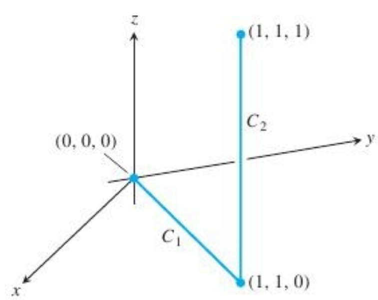
If C is a piecewise smooth curve, i.e., it is a join of finite number of smooth curves, written as $C = C_1 \cup \cdots \cup C_m$, then we define

$$\int_C f(x,y,z)ds = \int_{C_1} f(x,y,z)\,ds + \cdots + \int_{C_m} f(x,y,z)\,ds.$$

(3.4) ***Example***

Let C be the curve consisting of line segments joining $(0,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$. Evaluate $\int_C \left(x - 3y^2 + z\right) ds$.

The curve C is the join of C_1 and C_2, whose parametrization are given by



$C_1: \ \vec{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath}, \quad 0 \le t \le 1.$

$C_2: \vec{r}(t) = \hat{\imath} + \hat{\jmath} + t\,\hat{k}, \quad 0 \le t \le 1.$

On C_1, $|\vec{r}\,'(t)| = \sqrt{2}$; on C_2, $|\vec{r}\,'(t)| = 1$.

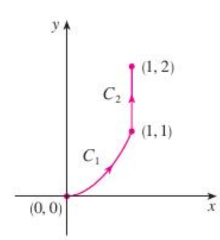
Then,

$$\begin{aligned}\int_C (x - 3y^2 + z)\,ds &= \int_{C_1} (x - 3y^2 + z)\,ds + \int_{C_2} (x - 3y^2 + z)\,ds \\ &= \int_0^1 f(t,t,0)\sqrt{2}\,dt + \int_0^1 f(1,1,t)\,1\,dt \\ &= \int_0^1 (t - 3t^2 + 0)\sqrt{2}\,dt + \int_0^1 (1 - 3 + t)\,dt = \frac{-3 - \sqrt{2}}{2}. \quad \square\end{aligned}$$

(3.5) ***Example***

Evaluate $\int_C 2x\,ds$, where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment joining $(1,1)$ to $(1,2)$.

Here, $C = C_1 \cup C_2$, where



$C_1: x = x, \ y = x^2, \quad 0 \le x \le 1.$
$C_2: x = 1, \ y = y, \quad 1 \le y \le 2.$

Choosing $x = t$ for C_1 and $y = t$ for C_2, we have

$C_1: x = t, \ y = t^2, \quad 0 \le t \le 1$
$C_2: x = 1, \ y = t, \quad 1 \le t \le 2.$

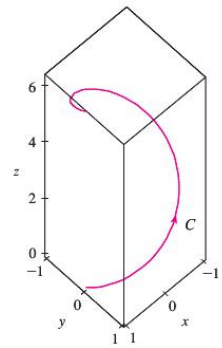
On C_1, $dx = 1\,dt$, $dy = 2t\,dt \Rightarrow \ ds = \sqrt{1 + 4t^2}\,dt$. Similarly, on C_2, $ds = dt$. Then

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \int_0^1 2t\sqrt{1 + 4t^2}\,dt + \int_1^2 2(1)\,dt = \frac{5\sqrt{5} - 1}{6} + 2. \ \square$$

(3.6) ***Example***

Evaluate $\int_C y \sin z\, ds$, where C is the circular helix given by

$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = t,\ 0 \le t \le 2\pi.$



$$\begin{aligned}&\int_C y \sin z\, ds\\ &= \int_0^{2\pi} \sin t \cdot \sin t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt\\ &= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \sqrt{2}\pi.\end{aligned}$$

$\square$

(3.7) Observation A line integral is independent of parameterization of the curve, and it is also independent of the orientation of the curve. These may be seen as follows.

In the first case, we may think of the new parameterization as

$$C(s):\ x(t(s))\hat{\imath} + y(t(s))\hat{\jmath} + z(t(s))\hat{k} \quad \text{for } c \le s \le d.$$

Then you can complete the argument.

For the second, suppose C is parameterized by

$$C(t) : x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \le t \le b.$$

Then $-C$ is parameterized by

$$-C(t) : x(a+b-t)\hat{\imath} + y(a+b-t)\hat{\jmath} + z(a+b-t)\hat{k}, \quad a \le t \le b.$$

The line integrals are given by

$$\int_C f(x, y, z)\, ds = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt.$$

$$\int_{-C} f(x, y, z)\, ds = \int_a^b f\big(x(a+b-t), y(a+b-t), z(a+b-t)\big)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt.$$

And these two integrals have the same value.

If the curve C happens to be a line segment on the x-axis, then $ds = dx$. In that case, the line integral over the curve becomes

$$\int_C f(x, y, z)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta x_k.$$

Then,

$$\int_C f(x,y,z)\,ds = \int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\dot{x}\,dt.$$

We generalize this observation and define the following three line integrals. As earlier, we require some conditions to hold so that the line integrals become meaningful. If $f(x,y,z)$ has continuous partial derivatives and C is a smooth curve with parametrization as $C:\ \vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k},\ a \le t \le b$, then we define

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\dot{x}\,dt,$$
$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\dot{y}\,dt,$$
$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\dot{z}\,dt.$$

These line integrals are called as the **line integrals of** f **over** C **with respect to** x, y, z, respectively.

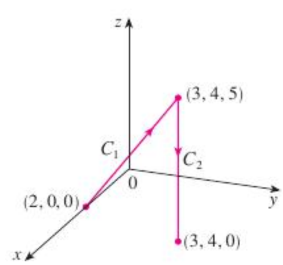
We also make it a convention to write

$$\int_C f_1\,dx + \int_C f_2\,dy + \int_C f_3\,dz = \int_C f_1\,dx + f_2\,dy + f_3\,dz$$

for functions f_1, f_2, f_3 of x, y, z.

(3.8) *Example*

Evaluate $\int_C ydx + zdy + xdz$, where C is the curve joining the line segments from $(2,0,0)$ to $(3,4,5)$ to $(3,4,0)$.



Parameterize the curve: $C = C_1 \cup C_2$, where

$C_1:\ x = 2+t,\ y = 4t,\ z = 5t,\ 0 \le t \le 1;$

$C_2:\ x = 3,\ y = 4,\ z = 5-5t,\ 0 \le t \le 1.$

Then $\int_C ydx + zdy + xdz$

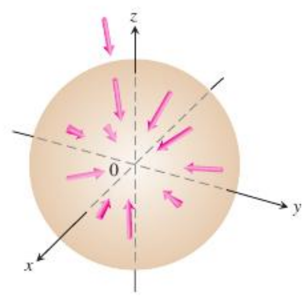
$$= \int_{C_1} ydx + zdy + xdz + \int_{C_2} ydx + xdz + zdx$$
$$= \int_0^1 (4t)dt + (5t)4dt + (2+t)5dt \ + \int_0^1 3(-5)dt = 49/2 - 15 = 19/2. \quad \square$$

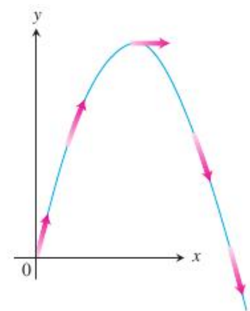
3.2 Line Integral of Vector Fields

Until now we have considered functions defined on regions in $\mathbb{R}^2$ or $\mathbb{R}^3$ which return real numbers. Recall that such functions are called scalar fields. We want to generalize line integrals to functions defined over such regions but which return vectors.

A **vector field** is a function defined on a region D in the plane or space that assigns a vector to each point in D. If D is a region in space, a vector field on D may be written as

$$\vec{F}(x,y,z) = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}.$$





For example, vectors in a gravitational field point toward the center of mass that gives the source of the field. The velocity vectors on a projectile's motion make a vector field along the trajectory.

Let $\vec{F}(x,y,z)$ be a continuous vector field defined over a curve C given by

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k} \quad \text{for } a \le t \le b.$$

The **line integral of $\vec{F}$ along** C, also called the **work done by moving a particle on** C **under the force field** $\vec{F}$ is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\hat{T}\,ds,$$

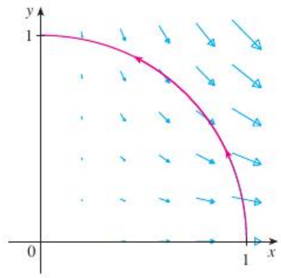
where $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the unit tangent vector at a point on C.

(3.9) Observation As in the line integral of a scalar field, the line integral of a vector field is independent of the parameterization of the curve C. However, the above formula shows that the line integral of the vector field is minus of the line integral when the orientation of the curve is changed. That is,

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}\,.$$

(3.10) ***Example***

Evaluate the line integral of the vector field $\vec{F}(x, y, z) = x^2\,\hat{\imath} - xy\,\hat{\jmath}$ along the first quarter unit circle in the first quadrant.



The curve C is given by $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath},\ 0 \le t \le \pi/2$. Then

$$\vec{F}(\vec{r}(t)) = \cos^2 t\,\hat{\imath} - \cos t \sin t\,\hat{\jmath}, \quad \vec{r}\,' = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}.$$

The work done is

$$\int_C \vec{F}\cdot d\,\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r})\cdot\vec{r}\,'\,dt = \int_0^{\pi/2} -2\cos^2 t \sin t\,dt = -\frac{2}{3}. \qquad \square$$

Let the vector filed be $\vec{F}(x, y, z) = M(x, y, z)\,\hat{\imath} + N(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k}$. Let C be the curve given by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ for $a \le t \le b$. Then

$$\int_C \vec{F}\cdot d\,\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b [M\,\dot{x} + N\,\dot{y} + P\,\dot{z}]\,dt = \int_C Mdx + Ndy + Pdz.$$

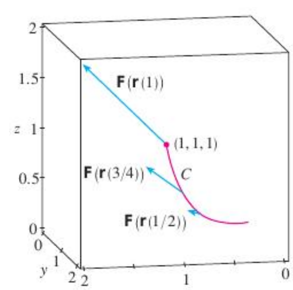
(3.11) ***Example***

Evaluate $\int_C \vec{F}\cdot d\,\vec{r}$, where $\vec{F} = xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$ and C is the twisted cube given by $x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$.

$$\int_C Mdx = \int_0^1 t\,t^2\,1\,dt = \frac{1}{4},$$

$$\int_C Ndy = \int_0^1 t^2\,t^3\,2t\,dt = \frac{2}{7},$$

$$\int_C Pdz = \int_0^1 t^3\,t\,3t^2\,dt = \frac{3}{7}.$$



So, $\int_C \vec{F}\cdot d\,\vec{r} = \frac{1}{4} + \frac{2}{7} + \frac{3}{7} = \frac{27}{28}.$

Also, $\int_C \vec{F}\cdot d\,\vec{r} = \int_0^1 [xy\dot{x} + yz\dot{y} + zx\dot{z}]dt = \int_0^1 [t^3 + 2t^6 + 3t^6]dt = \frac{27}{28}.$ $\square$

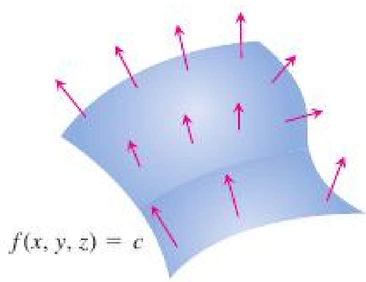
3.3 Conservative Fields

Let $f(x, y, z)$ be a function from a region in $\mathbb{R}^3$ to $\mathbb{R}$. Recall that if f_x, y_y, f_z exist, then the gradient of f, also called the **gradient field** of $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}.$$

The gradient field of the surface $f(x, y, z) = c$ may be drawn as follows:

> At each point on the surface, we have a vector, the gradient vector, which is normal to the surface. And we draw it there itself to show it.



For example, the gradient field of $f(x, y, z) = xyz$ is

$$\nabla f = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}.$$

Notice that $f(x, y, z)$ has a continuous gradient iff f_x, f_y, f_z are continuous on the domain of definition of f.

A vector field $\vec{F}$ is called **conservative** if there exists a scalar field (function) f such that $\vec{F} = \nabla f$. In such a case, the scalar field f is called the **potential** of the vector field $\vec{F}$.

For example, consider the gravitational force field $\vec{F} = -\dfrac{mMG}{|r|^3}\,\vec{r}$. It is also written in the form:

$$\vec{F}(x, y, z) = -\frac{mMG}{(x^2 + y^2 + z^2)^{3/2}}\,[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}]$$

Here, $\vec{F}$ is a conservative field. Reason?

Define $f(x, y, z) = \dfrac{mMG}{(x^2 + y^2 + z^2)^{1/2}}$. Then

$$\nabla f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k} = \vec{F}.$$

Physically, the law of conservation of energy holds in every conservative field.

Recall: $\int_a^b f'(t)dt = f(b) - f(a)$ for a function $f(t)$. In case of line integrals, the gradient acts as a sort of derivative; see the following theorem.

(3.12) ***Theorem***

Let C be a smooth curve given by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ for $a \le t \le b$. Suppose C joins points (x_1, y_1, z_1) to (x_2, y_2, z_2). That is,

$$\vec{r}(a) = x_1\,\hat{\imath} + y_1\,\hat{\jmath} + z_1\,\hat{k} \quad \text{and} \quad \vec{r}(b) = x_2\,\hat{\imath} + y_2\,\hat{\jmath} + z_2\,\hat{k}.$$

Let $f(x, y, z)$ be a scalar field whose gradient vector is continuous on a region containing C. Then

$$\int_C \nabla f \cdot d\,\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

Proof.

$$\begin{aligned}\int_C \nabla f \cdot d\,\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right] dt \\ &= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(t))\Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a)).\end{aligned}$$ ∎

It means that if $\vec{F}$ is a conservative vector field with potential f, then the line integral of $\vec{F}$ over any smooth curve joining points A to B can be evaluated from the potential f by:

$$\int_C \vec{F} \cdot d\,\vec{r} = f(B) - f(A).$$

In such a case, the line integral is independent of path of C; it only depends on the initial point and the end point of C.

We say that a line integral $\int_C \vec{F} \cdot d\,\vec{r}$ is **independent of path** iff for any curve C' that lies in the domain of $\vec{F}$ and has the same initial and end points as that of C, we have

$$\int_C \vec{F} \cdot d\,\vec{r} = \int_{C'} \vec{F} \cdot d\,\vec{r}\,.$$

Thus, if $\vec{F}$ is conservative, then the line integral $\int_C \vec{F} \cdot d\,\vec{r}$ is independent of path.

(3.13) ***Example***

Find the line integral of the field $\vec{F} = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}$ along any smooth curve joining the points $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Notice that $\vec{F}$ is conservative since $\vec{F} = \nabla(xyz)$. That is, $\vec{F} = \nabla f$ with $f = xyz$. Let C be any curve joining the points $A(-1, 3, 9)$ to $B(1, 6, -4)$. Then

$$\int_C \vec{F} \cdot d\,\vec{r} = \int_A^B \nabla f \cdot d\,\vec{r} = f(B) - f(A) = 3. \qquad \square$$

A curve C is called a **closed curve** iff C has the same initial and end points. When C is a closed curve, the line integral over C is written as

$$\oint_C \vec{F} \cdot d\,\vec{r}\,.$$

The following result is a corollary to (3.12).

(3.14) *Theorem*

Let $\vec{F}$ be a continuous vector field defined on a region D. Let C be any piecewise smooth curve lying in D. The line integral $\int_C \vec{F} \cdot d\,\vec{r}$ is independent of path iff $\oint_{C'} \vec{F} \cdot d\,\vec{r} = 0$ for every closed curve C' lying in D.

Path independence implies that the field is conservative. We state this theorem without proof.

(3.15) *Theorem*

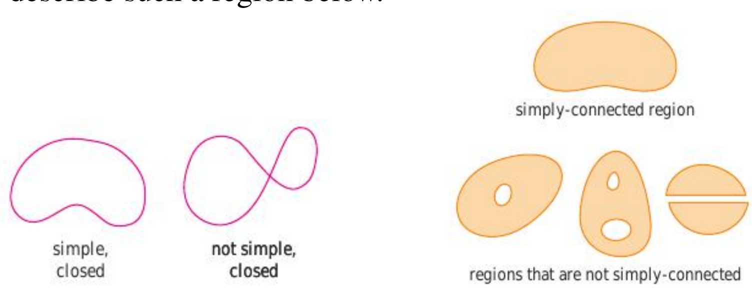
Let $\vec{F}$ be a continuous vector field defined on a region D. If $\displaystyle\int_C \vec{F} \cdot d\,\vec{r}$ is path independent for each piecewise smooth curve C lying in D, then $\vec{F}$ is conservative.

If $\vec{F}(x,y) = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ is conservative, then we have a scalar function $f(x,y)$ such that $f_x = M,\ f_y = N$. Suppose M_y and N_x are continuous. Then using Clairaut's theorem, we have $f_{xy} = M_y = f_{yx} = N_x$. That is, if $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$ is conservative, then $M_y = N_x$. Similar result holds in three dimensions.

(3.16) *Theorem*

Let $\vec{F}(x,y,z) = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}$, where the gradients of the component functions M, N, P are continuous on a region D. If $\vec{F}$ is conservative, then we have $M_y = N_x,\ N_z = P_y,\ P_x = M_z$ on D.

The converse of Theorem 3.16 holds if the region of $\vec{F}$ is a simply connected region. We describe such a region below.



A **simple curve** is a curve which does not intersect itself. A region D is said to be a **simply connected region** iff every simple closed curve lying in D encloses only points from D.

(3.17) *Theorem*

Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ be a vector field on a simply connected region D, where gradients of M, N, P are continuous. If $M_y = N_x$, $N_z = P_y$, and $P_x = M_z$ hold on D, then $\vec{F}$ is conservative.

Proof of this can be done here, but it follows from Green's theorem in the plane and from Stokes' theorem in space, which we will do later.

The equations $M_y = N_x$, $N_z = P_y$, $P_x = M_z$ help in determining the potential function of a conservative field.

(3.18) *Example*

Are the following vector fields conservative?

1. $\vec{F}(x, y) = (x - y)\,\hat{\imath} + (x - 2)\,\hat{\jmath}$
2. $\vec{F}(x, y) = (3 + 2xy)\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$.
3. $\vec{F}(x, y, z) = (2x - 3)\,\hat{\imath} + z\,\hat{\jmath} + \cos z\,\hat{k}$.

1. $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$, where $M = x - y,\ N = x - 2$. $M_y = -1,\ N_x = 1$. Since $M_y \neq N_x$, the vector field F is not conservative.
2. Here, $M = 3 + 2xy,\ N = x^2 - 3y^2$. $M_y = 2x = N_x$. The vector field is defined on $\mathbb{R}^2$, which is a simply connected region. The partial derivatives of M and N are continuous. Therefore, $\vec{F}$ is a conservative field.
3. $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, where $M = 2x - 3,\ N = z,\ P = \cos z$.

   $M_y = 0,\ N_x = 0,\ N_z = 1, P_y = 0,\ P_x = 0,\ M_z = 0.$

   Since $N_z \neq P_y$, the field $\vec{F}$ is not conservative. ∎

(3.19) *Example*

Find a potential for the vector field $\vec{F} = (3 + 2xy)\,\hat{\imath} + (x^2 - 3y^2)\,\hat{\jmath}$. Then evaluate $\int_C \vec{F} \cdot d\,\vec{r}$, where C is given by $\vec{r}(t) = e^t \sin t\,\hat{\imath} + e^t \cos t\,\hat{\jmath},\ 0 \leq t \leq \pi$.

To determine a scalar function $f(x, y, z)$ such that $\vec{F} = \nabla f$, we start with

$$f_x = 3 + 2xy, \quad f_y = x^2 - 3y^2.$$

Integrate the first one with respect to x and integrate the second with respect to y to obtain:

$$f(x, y) = 3x + x^2 y + g(y), \quad f(x, y) = x^2 y - y^3 + h(x).$$

Taking $g(y) = -y^3 + \text{const.}$ and $h(x) = 3x + \text{const.}$, we have

$$f(x, y) = 3x + x^2 y - y^3 + k \quad \text{for any constant } k.$$

Verify to see that f is a potential for F.

Next, $\displaystyle\int_C \vec{F} \cdot d\,\vec{r} = f\big(x(\pi), y(\pi)\big) - f\big(x(0), y(0)\big) = e^{3\pi} + 1.$ ∎

(3.20) ***Example***

Find a potential for the vector field $\vec{F} = y^2\,\hat{\imath} + (2xy + e^{3z})\,\hat{\jmath} + 3ye^{3z}\,\hat{k}$.

Denote the potential by $f(x, y, z)$. Then

$$f_x = y^2, \quad f_y = 2xy + e^{3z}, \quad f_z = 3ye^{3z}.$$

Integrate with respect to suitable variables:

$$f = xy^2 + g(y, z), \quad f = xy^2 + ye^{3z} + h(x, z), \quad f = ye^{3z} + \phi(x, y).$$

Taking $g(y, z) = ye^{3z}$, $\phi(x, y) = xy^2$, $h(x, z) = k$, a constant, we get one such f.

Sometimes matching may not be obvious. So, differentiate the first:

$$f_y = 2xy + g_y(y, z) = 2xy + e^{3z}.$$

Thus, $g_y(y, z) = e^{3z}$. Integrate: $g(y, z) = ye^{3z} + \psi(z)$. Then

$$f = xy^2 + ye^{3z} + \psi(z).$$

This gives $f_z = 3e^{3z} + \psi'(z) = 3ye^{3z}$. Thus, $\psi(z) = k$, a const. Therefore,

$$f(x, y, z) = xy^2 + ye^{3z} + k.$$ ∎

(3.21) ***Example***

Show that the vector field $\vec{F} = (e^x \cos y + yz)\,\hat{\imath} + (xz - e^x \sin y)\,\hat{\jmath} + (xy + z)\,\hat{k}$ is conservative by finding a potential for it.

Let the potential be $f(x, y, z)$. Then

$$f_x = e^x \cos y + yz, \quad f_y = xz - e^x \sin y, \quad f_z = xy + z.$$

Integrate the first w.r.t. x to get

$$f = e^x \cos y + xyz + g(y, z).$$

Differentiate w.r.t. y to get

$$f_y = -e^x \sin y + xz + g_y(y, z) = xz - e^x \sin y \Rightarrow g_y(y, z) = 0.$$

Thus $g(y,z) = h(z)$. And then $f = e^x \cos y + xyz + h(z)$. Differentiate w.r.t. z to obtain

$$f_z = xy + h'(z) = xy + z \Rightarrow h'(z) = z \Rightarrow h(z) = z^2/2 + k.$$

Then $f(x,y,z) = e^x \cos y + xyz + z^2/2 + k$. ∎

If M, N, P are functions of x, y, z, on a region D in space, then the expression

$$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$$

is called a **differential form**. The differential form is called **exact** iff there exists a function $f(x,y,z)$ such that

$$M(x,y,z) = \frac{\partial f}{\partial x}, \quad N(x,y,z) = \frac{\partial f}{\partial y}, \quad P(x,y,z) = \frac{\partial f}{\partial z}.$$

Notice that if the differential form is exact, then

$$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz = df,$$

which is an exact differential. In that case, if C is any curve joining points A to B in the region D, then

$$\int_C [Mdx + Ndy + Pdz] = \int_C \nabla f \cdot d\vec{r} = \int_A^B df = f(B) - f(A).$$

Therefore, the differential form $M\,dx + N\,dy + P\,dz$ is exact iff $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ is conservative. In this case, the scalar function $f(x,y,z)$ is the potential of the field $\vec{F}$ and $df = M\,dx + N\,dy + P\,dz$.

(3.22) *Example*

Show that the differential form $ydx + xdy + 4dz$ is exact. Then evaluate the integral $\int_C (ydx + xdy + 4dz)$ over the line segment C joining the points $(1,1,1)$ to $(2,3,-1)$.

$M = y,\ N = x,\ P = 4$. Then $M_y = 1 = N_x,\ N_z = 0 = P_y,\ P_x = 0 = M_z$.

Therefore, the differential form is exact.

Also, notice that $ydx + xdy + 4dz = d(xy + 4z + k)$. Hence it is exact.

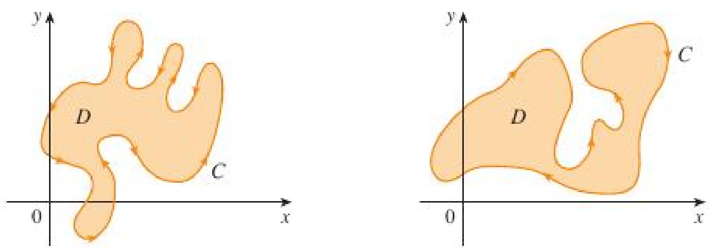
In case, f is not obvious, we can determine it as earlier by differentiating and integrating etc. Next,

$$\int_C (ydx + xdy + 4dz) = \int_{(1,1,1)}^{(2,3,-1)} d(xy + 4z + k) = (xy + 4z + k)\Big|_{(1,1,1)}^{(2,3,-1)} = -3.$$

∎

3.4 Green's Theorem

Let C be a simple closed curve in the plane. The **positive orientation** of C refers to a single counter-clockwise traversal of C. If C is given by $\vec{r}(t),\ a \le t \le b$, then its positive orientation refers to a traversal of C keeping the region D bounded by the curve to the left.

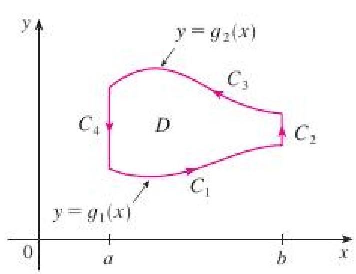


(3.23) ***Theorem*** (Green's Theorem)

Let C be a positively oriented simple closed piecewise smooth curve in the plane. Let D be the region with boundary as C. If $M(x,y)$ and $N(x,y)$ have continuous partial derivatives on an open region containing D, then

$$\oint_C (Mdx + Ndy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA, \quad \oint_C (Mdy - Ndx) = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

Proof. We only prove for a special kind of regions to give an idea of how it is proved.



Consider the region $D = \{(x,y) : a \le x \le b,\ f(x) \le y \le g(x)\}$. In the picture, f is shown as g_1 and g is shown as g_2. Assume that f, g are continuous functions. Then

$$\iint_D \frac{\partial M}{\partial y} dA = \int_a^b \int_{f(x)}^{g(x)} M_y dy dx = \int_a^b [M(x, g(x)) - M(x, f(x))] dx.$$

Now we compute $\int_C Mdx$ by breaking C into four parts C_1, C_2, C_3 and C_4.

The curve C_1 is given by $x = x,\ y = f(x),\ a \le x \le b$. Thus

$$\int_{C_1} M dx = \int_a^b M(x, f(x)) dx.$$

On C_2 and also on C_4, the variable x is a single point. So,

$$\int_{C_2} M dx = \int_{C_4} M dx = 0.$$

As x increases, C_3 is traversed backward. That is, $-C_3$ is given by $x = x$, $y = g(x)$, $a \le x \le b$. So,

$$\int_{C_3} M dx = -\int_{-C_3} M dx = -\int_a^b M(x, g(x)) dx.$$

Therefore, $\displaystyle\iint_D \frac{\partial M}{\partial y} dA = -\int_C M dx.$

Similarly, express D using the variable of integration as y. We get $\displaystyle\iint_D \frac{\partial N}{\partial x} dA = \int_C N dy.$ Next, add the two results obtained to get

$$\int_C (M dx + N dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

The second form follows similarly. ∎

Green's theorem helps in evaluating an integral of the type $\int_a^b \vec{F} \cdot d\vec{r}$ in a non-conservative vector field $\vec{F}$. It gives a relationship between a line integral around a simple closed curve C and the double integral over the plane region D bounded by this closed curve.

(3.24) *Example*

Verify Green's theorem for the field $\vec{F} = (x - y)\,\hat{\imath} + x\,\hat{\jmath}$, where C is the unit circle oriented positively.

Here, we have $C:\ \vec{r}(t) = x\,\hat{\imath} + y\,\hat{\jmath} = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath},\ 0 \le t \le 2\pi$. The region D is the unit disk.

$$M = \cos t - \sin t,\ N = \cos t, \quad dx = -\sin t\, dt,\ dy = \cos t\, dt.$$

$$M_x = 1,\ M_y = -1,\ N_x = 1,\ N_y = 0.$$

Now,

$$\oint_C (M dy - N dx) = \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)] dt = \pi.$$

$$\iint_D (M_x + N_y)dA = \iint_D (1+0)dA = \text{Area of } D = \pi.$$

Similarly,

$$\oint_C (Mdx + Ndy) = \int_0^{2\pi} [(\cos t - \sin t)(-\sin t) + \cos^2 t]dt = 2\pi.$$

$$\iint_D (N_x - M_y)dA = \iint_D (1 - (-1))dA = 2 \times \text{Area of } D = 2\pi. \qquad \square$$

(3.25) ***Example***

Evaluate the integral $I = \oint_C xy\,dy + y^2\,dx$, where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$, with positive orientation.

Take $M = y^2,\ N = xy,\ D$ as the region bounded by C. Then

$$I = \oint_C (Mdx + Ndy) = \iint_D (N_x - M_y)dA = \int_0^1 \int_0^1 (y - 2y)dxdy = -1/2.$$

Also, taking $M = xy$, and $N = -y^2$, we have

$$I = \oint_C (Mdy - Ndx) = \iint_D (M_x + N_y)dA = \int_0^1 \int_0^1 (y - 2y)dxdy = -1/2. \qquad \square$$

(3.26) ***Example***

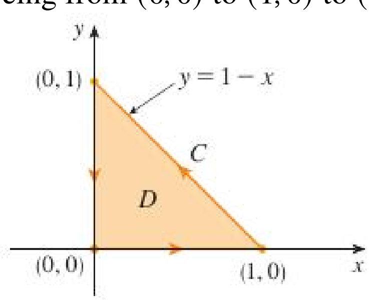
Evaluate the integral $I = \oint_C (3y - e^{\sin x})dx + (7x + \sqrt{1 + y^4})dy$, where C is the positively oriented circle $x^2 + y^2 = 9$.

Take D as the disk $x^2 + y^2 \le 9$. By Green's theorem,

$$I = \iint_D \big[(7x + \sqrt{1+y^4})_x - (3y - e^{\sin x})_y\big]dA = \iint_D (7-3)dA = 36\pi. \qquad \square$$

(3.27) ***Example***

Evaluate $I = \oint_C x^4\,dx + xy\,dy$, where C is the triangle with vertices at $(0,0),\ (0,1)$ and $(1,0)$; its orientation being from $(0,0)$ to $(1,0)$ to $(0,1)$ to $(0,0)$.



The triangle is positively oriented. Let D be the region bounded by the triangle. Take $M = x^4,\ N = xy$. Then

$$I = \iint_D [(xy)_x - (x^4)_y]\,dA = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{2}\int_0^1 (1-x^2)dx = \frac{1}{6}. \qquad \Box$$

(3.28) ***Example***

Evaluate $\int_C (xdy - y^2dx)$, where C is the positively oriented square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Here, $M = x,\ N = y^2$, and D is the region bounded by C. By Green's theorem,

$$\oint_C (Mdy - Ndx) = \iint_D (M_x + N_y)dA = \int_{-1}^1 \int_{-1}^1 (1+2y)dxdy = 4. \qquad \Box$$

(3.29) Observation Consider the formula $\iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy)$.

1. Take $M = 0$ and $N = x$. Then $N_x - M_y = 1$ so that

$$\text{Area of } D = \iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy) = \oint_C xdy.$$

2. Take $M = -y$ and $N = 0$. Then $N_x - M_y = 1$ so that

$$\text{Area of } D = \iint_D (N_x - M_y)dA = \oint_C (Mdx + Ndy) = -\oint_C ydx.$$

3. Combining the two above, we also have

$$\text{Area of } D = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

(3.30) ***Example***

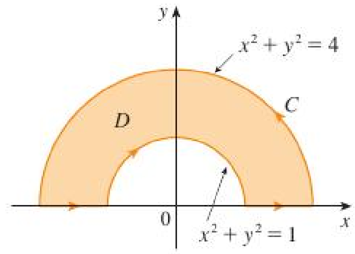
Compute the area enclosed by the ellipse $C:\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

We parameterize C as $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$. And then the area is

$$\begin{aligned}\frac{1}{2}\oint_C (x\,dy - y\,dx) &= \frac{1}{2}\int_0^{2\pi} [(a\cos t\, b\cos t) - (b\sin t\,(-b\sin t))]dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi\,ab.\end{aligned} \qquad \Box$$

(3.31) ***Example***

Evaluate $\oint_C (y^2\,dx + xy\,dy)$, where C is the boundary of the semi-annular region between the semicircles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the upper half plane.

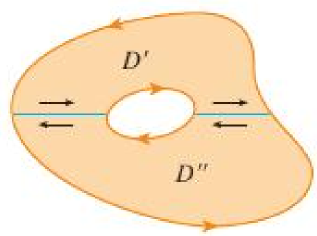


In polar coordinates, $D = \{(r, \theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$. Then

$$\oint_C (y^2\,dx + xy\,dy) = \iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(y^2)\right] dA = -\iint_D y\,dA$$
$$= -\int_1^2 \int_0^\pi r\sin\theta\, r\,dr\,d\theta = -\int_1^2 r^2 dr \int_0^\pi \sin\theta\,d\theta = -\frac{14}{3}.$$

□

In fact, Green's theorem can be applied to regions having holes, provided the region can be divided into simply connected regions.



The boundary C of the region D consists of two simple closed curves C_1 (Outer) and C_2 (inner). Assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.

Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. Divide D into two regions D' and D'' as shown in the figure. Green's theorem on D' and D'' gives

$$\iint_D (N_x - M_y)dA = \iint_{D'} (N_x - M_y)dA + \iint_{D''} (N_x - M_y)dA$$
$$= \int_{\partial D'} (Mdx + Ndy) + \int_{\partial D''} (Mdx + Ndy) = \int_C (Mdx + Ndy).$$

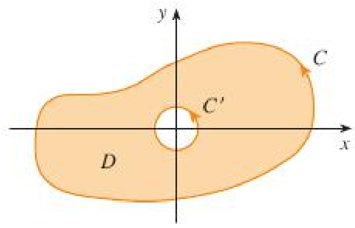
This is a general version of Green's Theorem.

(3.32) *Example*

Show that if C is any positively oriented simple closed path that encloses the origin, then

$$\oint_C \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy = 2\pi,$$

No idea how to show it for every such curve. So, take a positively oriented circle C', of radius a, around origin that lies entirely in the region bounded by C. Let D be the annular region bounded by C and C'. Take $\vec{F}(x, y) = (-y\,\hat{\imath} + x\,\hat{\jmath})/(x^2 + y^2)$.



Then the positively oriented boundary of D is $\partial D = C \cup (-C')$. Here, $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$ gives $N_x = M_y = (y^2 - x^2)/(x^2 + y^2)^2$. Green's theorem on D gives

$$\oint_C (Mdx + Ndy) + \oint_{-C'} (Mdx + Ndy) = \iint_D (N_x - M_y)dA = 0.$$

Then

$$\oint_C (Mdx + Ndy) = \oint_{C'} (Mdx + Ndy).$$

But C' is parameterized by $x(t) = a\cos t,\ y(t) = a\sin t,\ 0 \le t \le 2\pi$. So,

$$\begin{aligned}\oint_C (Mdx + Ndy) &= \int_{C'} (Mdx + Ndy)\\ &= \int_0^{2\pi} (a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}) \cdot (a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath})' dt = 2\pi.\end{aligned}$$

Generalize this example by taking the constraint $N_x = M_y$ on the vector field. $\square$

3.5 Curl and Divergence of a vector field

If $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ is a vector field in $\mathbb{R}^3$, where the partial derivatives of the component functions exist, then curl $\vec{F}$ is a vector field given by

$$\text{curl } \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\imath} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

Using $\nabla = \nabla = \hat{\imath}\dfrac{\partial}{\partial x} + \hat{\jmath}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$ we write curl $\vec{F}$ as follows:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}.$$

For example, if $\vec{F} = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$, then curl $\vec{F} = -y(2+x)\,\hat{\imath} + x\,\hat{\jmath} + yz\,\hat{k}$.

Intuitively, curl $\vec{F}$ measures how quickly a tiny peddle (at a point) in some fluid in a vector field moves around itself. If curl $\vec{F} = 0$, then there is no rotation of such a tiny peddle.

(3.33) ***Theorem***

Let $\vec{F}$ be a vector field defined over a simply connected region D whose component functions have continuous partial derivatives. Then $\vec{F}$ is conservative iff curl $\vec{F} = 0$.

Proof. ($\Rightarrow$) If $\vec{F}$ is conservative, then $\vec{F} = \nabla f$ for some f, where f is some scalar function defined on D. Now,

$$\text{curl } \nabla f = \nabla \times (\nabla f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (f_{yz} - f_{zy})\,\hat{\imath} + (f_{zx} - f_{xz})\,\hat{\jmath} + (f_{xy} - f_{yx})\,\hat{k} = 0.$$

The converse follows from Stokes' theorem, which we will discuss later. ∎

Remember: The curl of gradient of any scalar function is zero: curl $\nabla f = 0$.

(3.34) ***Example***

Are the following vector fields conservative?

1. $\vec{F} = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$ 2. $\vec{F} = y^2z^3\,\hat{\imath} + 2xyz^3\,\hat{\jmath} + 3xy^2z^2\,\hat{k}$

1. Here, curl $\vec{F} = -y(2+x)\,\hat{\imath} + x\,\hat{\jmath} + yz\,\hat{k} \neq 0$. So, $\vec{F}$ is not conservative.
2. Here, $\vec{F}$ is defined on $\mathbb{R}^2$ and

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$
$$= (6xy^2z^2 - 6xy^2z^2)\,\hat{\imath} - (3y^2z^2 - 3y^2z^2)\,\hat{\jmath} + (2yz^3 - 2yz^3)\,\hat{k} = 0.$$

Hence $\vec{F}$ is conservative. In fact, $\vec{F} = \nabla f$, where $f(x,y,z) = xy^2z^3$. □

If $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ is a vector field defined on a region, where its component functions have first order partial derivatives, then

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The **divergence** is also called **flux** or **flux density**.

For example, if $\vec{F} = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k}$, then div $\vec{F} = z + xz$.

The divergence of the vector field $\vec{F} = (x^2 - y)\,\hat{\imath} + (xy - y^2)\,\hat{\jmath}$ is

$$\frac{\partial(x^2 - y)}{\partial x} + \frac{\partial(xy - y^2)}{\partial y} = 3x - 2y.$$

Intuitively, div $\vec{F}$ measures the tendency of the fluid to diverge from the point (a, b). When the gas (fluid) is expanding, divergence is positive; and when it is compressing, the divergence is negative. The fluid is said to be **incompressible** iff div $\vec{F} = 0$.

(3.35) ***Theorem***
Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ be a vector field defined on a simply connected region $D \subseteq \mathbb{R}^3$, where M, N, P have continuous second order partial derivatives. Then div curl $\vec{F} = 0$.

Proof. div curl $\vec{F} = \nabla \cdot (\nabla \times \vec{F})$

$$= \frac{\partial}{\partial x}\Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) = 0$$

due to Clairaut's Theorem. ∎

(3.36) ***Example***
Does there exist a vector field $\vec{G}$ such that $\vec{F} = zx\,\hat{\imath} + xyz\,\hat{\jmath} - y^2\,\hat{k} =$ curl $\vec{G}$?
div $\vec{F} = z + xz \neq 0$. Hence there is no such $\vec{G}$. □

The operator $\nabla^2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the **Laplacian**.
Divergence of ∇f is the Laplacian of a scalar function f since

$$\text{div } \nabla f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

Using divergence and curl, Green's theorem may be stated in two different forms.

Green's Theorem - Tangent form
Let D be a simply connected region in the plane whose boundary is the simple closed curve C. Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath}$ be a vector field defined on D. Let C be parameterized by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$. Let $\hat{T}(t)$ be the unit tangent vector to C at the point $(x(t), y(t))$. Then

$$\vec{F} \cdot \hat{T}(t) ds = \vec{F} \cdot d\,\vec{r} = M dx + N dy.$$

The line integral of $\vec{F}$ over C is

$$\oint \vec{F} \cdot \hat{T}(t)\, ds = \oint_C \vec{F} \cdot d\,\vec{r} = \oint_C (M\, dx + N\, dy).$$

Consider $\vec{F}$ as a vector field on $\mathbb{R}^3$ with $P = 0$. Then

$$\text{curl } \vec{F} = (N_x - M_y)\,\hat{k} \Rightarrow \text{curl } \vec{F}\cdot\hat{k} = N_x - M_y.$$

Thus Green's theorem takes the form

$$\oint_C \vec{F}\cdot\hat{T}(t)\,ds = \oint_C \vec{F}\cdot d\,\vec{r} = \iint_D (\text{curl } \vec{F}\cdot\hat{k})\,dA. \tag{3.5.1}$$

Green's Theorem - Normal form

Let a curve C in the plane be given by $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$. Then

$$\hat{T} = \frac{\dot{x}}{|\,\vec{r}\,'(t)|}\,\hat{\imath} + \frac{\dot{y}}{|\,\vec{r}\,'(t)|}\,\hat{\jmath}, \qquad \hat{n}(t) = \frac{\dot{y}}{|\,\vec{r}\,'(t)|}\,\hat{\imath} - \frac{\dot{x}}{|\,\vec{r}\,'(t)|}\,\hat{\jmath}.$$

Now, $\vec{F}\cdot\hat{n} = [M\,\dot{y} - N\,\dot{x}]/|\,\vec{r}\,'(t)|$. And,

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \int_a^b \vec{F}\cdot\hat{n}\,|\,\vec{r}\,'(t)|\,dt = \oint_C (Mdy - Ndx).$$

Also,

$$\iint_D \text{div } \vec{F}\;dA = \iint_D (M_x + N_y)dA.$$

Hence Green's theorem takes the form

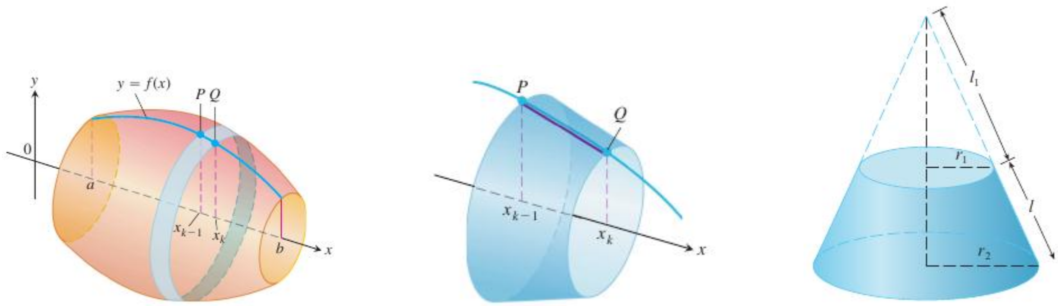
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \text{div } \vec{F}\;dA. \tag{3.5.2}$$

The tangent form of Green's theorem (3.5.1) is also called the **circulation-curl** form and the normal form (3.5.2) is called the **flux-divergence** form. Both the tangent form and the normal form of Green's theorem are together referred to as **vector forms** of Green's theorem.

3.6 Surface Area of solids of Revolution

Suppose a smooth curve is given by $y = f(x)$, where $f(x) \geq 0$. Its arc when $a \leq x \leq b$ is revolved about the x-axis to generate a solid. How do we compute the area of the surface of this solid?

We follow a strategy similar to computing the volume of revolution. Partition $[a, b]$ into n subintervals $[x_{k-1}, x_k]$. When each Δx_k is small, the surface area corresponding to this subinterval is approximately same as the area on the frustum of a right circular cone.



If a right circular cone has base radius R and slant height ℓ, then its surface area is given by $\pi R\ell$. Now, for the frustum, we subtract the smaller cone surface area from the larger. Look at the figure. The area of the frustum is

$$A = \pi r_2(\ell_1 + \ell) - \pi r_1 \ell_1 = \pi[(r_2 - r_1)\ell_1 + r_2\ell].$$

Using similarity of triangles, we have $\dfrac{\ell_1}{r_1} = \dfrac{\ell_1 + \ell}{r_2}$.

This gives $r_2\ell_1 = r_1\ell_1 + r_1\ell \Rightarrow (r_2 - r_1)\ell_1 = r_1\ell$. Therefore,

$$A = \pi(r_1\ell + r_2\ell) = 2\pi\, r\, \ell, \text{ where } r = \frac{r_1 + r_2}{2}.$$

To use this formula on the frustum obtained on the subinterval $[x_{k-1}, x_k]$, we notice that the slant height ℓ is approximated by $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = f(x_k) - f(x_{k-1})$. Next, the average radius

$$r = \frac{r_1 + r_2}{2} = \frac{f(x_{k-1}) + f(x_k)}{2}.$$

Thus the area of the frustum is approximated by

$$A_k = 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Due to MVT, we have $c_k \in [x_{k-1}, x_k]$ such that

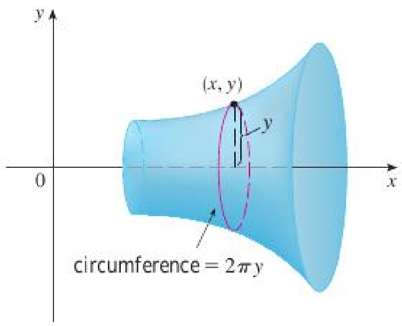
$$\Delta y_k = f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1}) = f'(c_k)\Delta x_k.$$

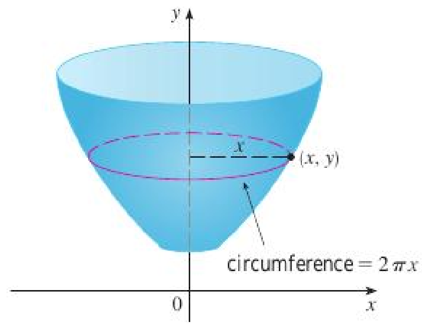
So, $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + (f'(c_k))^2}\,\Delta x_k$. The surface of revolution is approximated by

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} 2\pi\, \frac{f(x_{k-1}) + f(x_k)}{2}\, \sqrt{1 + (f'(c_k))^2}\,\Delta x_k.$$

Its limit as $n \to \infty$ is the Riemann sum of an integral, which is the required area:

$$S = \int_a^b 2\pi\, y\, \sqrt{1 + (f'(x))^2}\, dx = \int_a^b 2\pi f(x)\, \sqrt{1 + (f'(x))^2}\, dx.$$





We summarize.

1. If the arc of the curve $y = f(x)$ for $a \le x \le b$ is revolved about the x-axis, then write $y' = f'(x)$ an $ds = \sqrt{1 + (y')^2}\, dx$. The area of the surface of the solid of revolution is given by
$$S = \int 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1 + (y')^2}\, dx.$$
2. If the arc of the curve $x = g(y)$ for $c \le y \le d$ is revolved about the y-axis, then write $x' = g'(y)$ and $ds = \sqrt{1 + (x')^2}\, dy$. The area of the surface of the solid of revolution is given by
$$S = \int 2\pi x\, ds = \int_c^d 2\pi x \sqrt{1 + (x')^2}\, dy.$$

For parameterized curves, suppose the smooth curve is given by $x = x(t),\ y = y(t)$ for $a \le t \le b$. If the curve is traversed exactly once while t increases from a to b, then the surface area of the solid generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the x-axis $(y \ge 0)$: $S = \displaystyle\int_a^b 2\pi\, y(t) \sqrt{\dot{x}^2 + \dot{y}^2}\, dt.$
2. Revolution about the y-axis $(x \ge 0)$: $S = \displaystyle\int_a^b 2\pi\, x(t) \sqrt{\dot{x}^2 + \dot{y}^2}\, dt.$

(3.37) *Example*

Find the surface area of the solid obtained by revolving about the x-axis, the arc of the curve $y = 2\sqrt{x}$, $1 \le x \le 2$.

Since $y = 2\sqrt{x},\ y' = 1/\sqrt{x},\ \sqrt{1 + (y')^2} = \sqrt{1 + 1/x}$. Then
$$S = \int_1^2 2\pi y \left(1 + [y']^2\right)^{1/2} dx = \int_1^2 2\pi\, 2\sqrt{x} \sqrt{1 + \frac{1}{x}}\, dx = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2}). \qquad \square$$

(3.38) *Example*

The arc of the parabola $y = x^2,\ 1 \le x \le 2$ is revolved about the y-axis. Find the surface area of revolution.

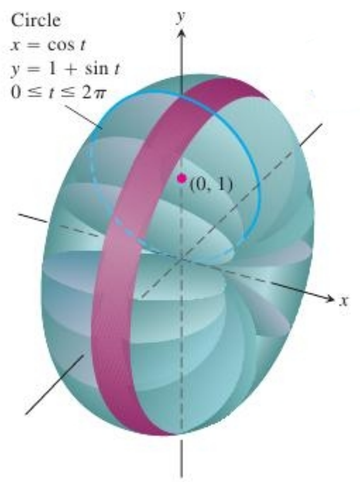
Since $x = \sqrt{y},\ 1 \le y \le 4$, the surface area is

$$\begin{aligned} S &= \int_1^2 2\pi\, x\, \sqrt{1+(x')^2}\, dy = 2\pi \int_1^4 \sqrt{y}\, \sqrt{1+1/(4y)}\, dy \\ &= \pi \int_1^4 \sqrt{1+4y}\, dy = \frac{\pi}{4}\int_1^4 \sqrt{1+4y}\, d(1+4y) = \frac{\pi}{4}\cdot\frac{2}{3}\Big[(1+4y)^{3/2}\Big]_1^4 \\ &= \frac{\pi}{6}\big(17^{3/2} - 5^{3/2}\big) \end{aligned}$$

□

(3.39) ***Example***

The circle of radius 1 centered at (0, 1) is revolved about the x-axis. Find the surface area of the solid so generated.



The circle can be parameterized as $x = \cos t,\ y = 1 + \sin t,\ 0 \le t \le 2\pi$. Then $\dot{x}^2 + \dot{y}^2 = 1$. Thus the area is

$$S = \int_0^{2\pi} 2\pi\,(1+\sin t)\, dt = 4\pi^2.$$

□

3.7 Surface area

As we know, smooth surfaces can be given by a function such as $z = f(x, y)$. More generally, a smooth surface is given parametrically by

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),$$

where (u, v) varies over a given parameter region. Normally, we say that the point (u, v) varies over a region in uv-plane. The parametric equation is also written in vector form as

$$\vec{r} = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}.$$

(3.40) ***Example***

Some examples of surfaces in parametric form are as follows.

1. The cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$ can be parametrized by

   $$x = r\cos\theta,\ y = r\sin\theta,\ z = r,\ \text{where } 0 \le r \le 1 \text{ and } 0 \le \theta \le 2\pi.$$

   Then its vector form is

   $$\vec{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\,\hat{k}.$$

2. The sphere $x^2 + y^2 + z^2 = a^2$ can be parametrized by

   $$x = a\cos\theta\sin\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\phi \text{ for } 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

   In vector form the parametrization is

   $$\vec{r}(\theta,\phi) = a\cos\theta\sin\phi\,\hat{\imath} + a\sin\theta\sin\phi\,\hat{\jmath} + a\cos\phi\,\hat{k}.$$

3. The cylinder $x^2 + y^2 = a^2$, $0 \le z \le b$ can be parametrized by

   $$\vec{r}(\theta,z) = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath} + z\,\hat{k},\ \text{for } 0 \le \theta \le 2\pi,\ 0 \le z \le b.$$

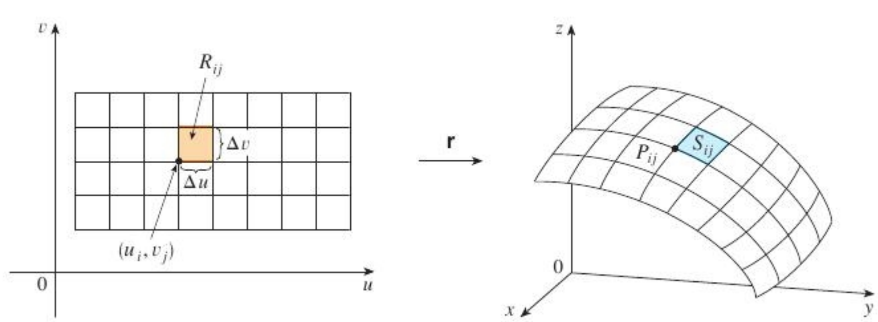
□

Let S be a smooth surface given parametrically by

$$x = x(u,v),\ y = y(u,v),\ z = z(u,v),$$

where (u,v) ranges over a parameter region D in the uv-plane. Suppose that S is covered exactly once as (u,v) varies over D. For simplicity, assume that D is a rectangle. We write S in vector form:

$$\vec{r} = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}.$$



Divide D into smaller rectangles R_{ij} with the lower left corner point as $P_{ij} = (u_i, v_j)$. For simplicity, let the partition be uniform with u-lengths as Δu and v-lengths as Δv. The part S_{ij} of S that corresponds to R_{ij} has the corner P_{ij} with position vector $\vec{r}(u_i, v_j)$. The tangent vectors to S at P_{ij} are given by

$$\vec{r}_u^{\,*} := \vec{r}_u(u_i, v_j) = x_u(u_i, v_j)\hat{\imath} + y_u(u_i, v_j)\hat{\jmath} + z_u\hat{k}(u_i, v_j)$$

$$\vec{r}_v^{\,*} := \vec{r}_v(u_i, v_j) = x_v(u_i, v_j)\hat{\imath} + y_v(u_i, v_j)\hat{\jmath} + z_v\hat{k}(u_i, v_j)$$

The tangent plane to S is the plane that contains the two tangent vectors $\vec{r}_u(u_i, v_j)$ and $\vec{r}_v(u_i, v_j)$. The normal to S at P_{ij} is the vector $\vec{r}_u(u_i, v_j) \times \vec{r}_v(u_i, v_j)$. Notice that since S is assumed to be smooth, the normal vector is non-zero.

The part S_{ij} is a curved parallelogram on S whose sides can be approximated by the vectors $\vec{r}_u^{\,*}\Delta u$ and $\vec{r}_v^{\,*}\Delta v$. Then the area of S_{ij} can be approximated by

$$\text{Area of } S_{ij} \simeq |\vec{r}_u^{\,*} \times \vec{r}_v^{\,*}|\Delta u \Delta v.$$

Then an approximation to the area of S is obtained by summing over both indices i and j:

$$\text{Area of } S \simeq \sum_j \sum_i |\vec{r}_u^{\,*} \times \vec{r}_v^{\,*}|\Delta u \Delta v.$$

We thus define the surface area by taking the limit of the above approximated quantity. It is as follows:

Let S be a smooth surface given parametrically by

$$\vec{r} = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k},$$

where $(u, v) \in D$, a region in the uv-plane. Suppose that S is covered exactly once as (u, v) varies over D. Then the **surface area** of S is given by

$$\text{Area of } S = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$$

where $\vec{r}_u = x_u\hat{\imath} + y_u\hat{\jmath} + z_u\hat{k}$ and $\vec{r}_v = x_v\hat{\imath} + y_v\hat{\jmath} + z_v\hat{k}$.

In case, the surface S is given by the graph of a function such as $z = f(x, y)$, where $(x, y) \in D$, then we take the parameters as $u = x,\ v = y$ and $z = z(u, v) = f(x, y)$. That is, S is given by

$$\vec{r} = u\hat{\imath} + v\hat{\jmath} + z\hat{k}.$$

We see that

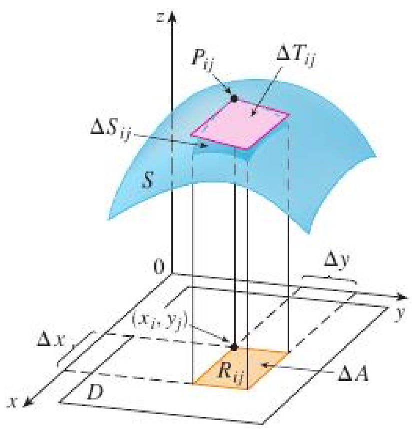
$$\vec{r}_u = \hat{\imath} + z_u\hat{k} = \hat{\imath} + f_x\hat{k}, \quad \vec{r}_v = \hat{\jmath} + z_v\hat{k} = \hat{\jmath} + f_y\hat{k}.$$

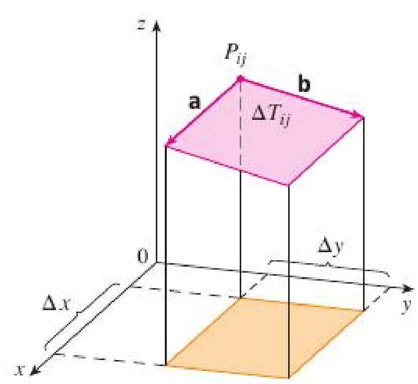
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k}.$$

Therefore,

$$\text{Area of } S = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA.$$

This formula can also be derived from the first principle as we had done for the parametric form. For this, suppose that S is given by the equation $z = f(x, y)$ for $(x, y) \in D$. Divide D into smaller rectangles R_{ij} with area $\Delta(R_{ij}) = \Delta x \Delta y$. For the corner (x_i, y_j) in R_{ij}, closest to the origin, let P_{ij} be the point $(x_i, y_j, f(x_i, y_j))$ on the surface. The tangent plane to S at P_{ij} is an approximation to S near P_{ij}.





The area T_{ij} of the portion of the tangent plane that lies above R_{ij} approximates the area of S_{ij}, the portion of S that is directly above R_{ij}. Therefore, we define the **area of the surface** S as

$$\Delta(S) = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} T_{ij}.$$

Let $\vec{a}$ and $\vec{b}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram whose area is T_{ij}. Then $T_{ij} = |\,\vec{a} \times \vec{b}\,|$. However, $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of $\vec{a}$ and $\vec{b}$, respectively. Therefore,

$$\vec{a} = \Delta x\, \hat{\imath} + f_x(x_i, y_j)\Delta x\, \hat{k}, \quad \vec{b} = \Delta y\, \hat{\jmath} + f_y(x_i, y_j)\Delta y\, \hat{k}.$$

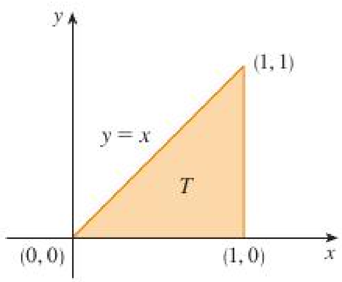
$$\begin{aligned} T_{ij} = |\,\vec{a} \times \vec{b}\,| &= |-f_x(x_i, y_j)\,\hat{\imath} - f_y(x_i, y_j)\,\hat{\jmath} + k|\,\Delta(R_{ij}) \\ &= \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1}\,\Delta(R_{ij}). \end{aligned}$$

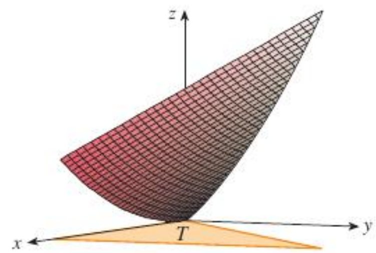
Summing over these T_{ij} and taking the limit, we obtain:

$$\text{Area of } S = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA = \iint_D \sqrt{1 + z_x^2 + z_y^2} \, dA.$$

(3.41) ***Example***

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region in the xy-plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.





$$T = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}, \quad f(x, y) = x^2 + 2y.$$

The required surface area is

$$\iint_T \sqrt{1 + (2x)^2 + 2^2}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy dx = \frac{1}{12}(27 - 5\sqrt{5}). \qquad \square$$

Surface Area - a generalized form

Recall that for a surface S which is given by $f(x, y) = z$, the surface area is

$$\iint_D \sqrt{f_x^2 + f_y^2 + 1} dA.$$

Here, D is the rectangle on the xy-plane obtained by projecting S onto the plane.

Look at this surface as $f(x, y) - z = 0$. Then $\nabla f = f_x\, \hat{\imath} + f_y\, \hat{\jmath} - 1\, \hat{k}$. If $\vec{p}$ is the unit normal to the projected rectangle, then $\vec{p} = \hat{k}$. Then

$$\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|} = \frac{\sqrt{1 + f_x^2 + f_y^2}}{1^2},$$

which is the integrand in the surface area formula.

Warning: $\nabla f \cdot \vec{p}$ must not be ZERO.

A derivation similar to the surface area formula gives the following.

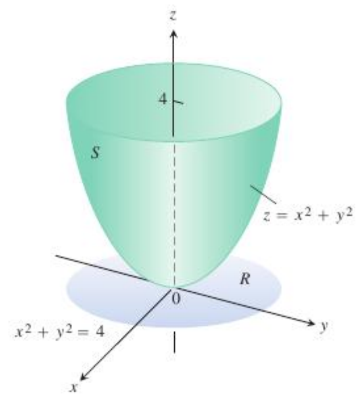
Let the surface S be given by $f(x, y, z) = c$. Let R be a closed bounded region which is obtained by projecting the surface to a plane whose unit normal is $\vec{p}$. Suppose that ∇f is continuous on R and $\nabla f \cdot \vec{p} \neq 0$ on R. Then

$$\text{The surface area of } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|} dA.$$

Of course, whenever possible, we project onto the coordinate planes.

(3.42) ***Example***

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 = z$ by the plane $z = 4$.



Surface S is given by $f(x, y, z) = x^2 + y^2 - z = 0$. Project it onto xy-plane to get the region R as $x^2 + y^2 \le 4$. Then

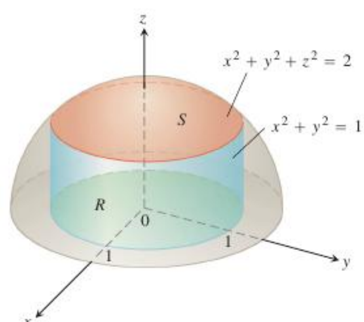
$$\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - \hat{k}, \quad |\nabla f| = \sqrt{1 + 4x^2 + 4y^2}, \quad \vec{p} = \hat{k}, \quad |\nabla f \cdot \vec{p}\,| = 1.$$

R is given by $x = r\cos\theta,\ y = r\sin\theta,\ 0 \le \theta \le 2\pi, 0 \le r \le 2$. So, the surface area is

$$\iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\, drd\theta = \frac{\pi}{6}(17\sqrt{17} - 1). \qquad \square$$

(3.43) ***Example***

Find the surface area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2,\ z \ge 0$ by the cylinder $x^2 + y^2 = 1$.



The surface projected on xy-plane gives R as the disk $x^2 + y^2 \le 1$. The surface is $f(x, y, z) = 2$, where $f(x, y, z) = x^2 + y^2 + z^2$. Then

$$\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k},\ |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2},\ \vec{p} = k,\ |\nabla f \cdot \vec{p}\,| = |2z| = 2z.$$

Thus the surface area is

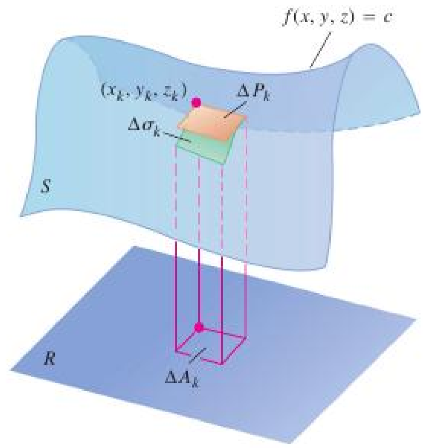
$$\Delta = \iint_R \frac{2\sqrt{2}}{2z}\,dA = \sqrt{2}\iint_R z^{-1}\,dA = \sqrt{2}\iint_R (2 - x^2 - y^2)^{-1/2}\,dA.$$

R is given by $x = r\cos\theta, y = r\sin\theta,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1$. So,

$$\Delta = \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{2 - r^2}} = 2\pi(2 - \sqrt{2}). \qquad \square$$

3.8 Integrating over a surface

Suppose a function $g(x, y, z)$ is defined over a surface S given by $f(x, y, z) = c$. To compute the integral of g, where the area elements are taken over the surface, we look at the region R on which this surface is defined as a function. Divide the region R into smaller rectangles ΔA_k. Consider the corresponding surface areas $\Delta\sigma_k$.



Then $\Delta\sigma_k \approx \left(\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\right)_k \Delta A_k$. Assuming that g is nearly constant on the smaller surface fragment σ_k, we form the sum

$$\sum_k g(x_k, y_k, z_k)\Delta\sigma_k \approx \sum_k g(x_k, y_k, z_k)\left(\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\right)_k \Delta A_k.$$

If this sum converges to a limit, then we define that limit as the integral of g over the surface S.

Let S be a surface S given by $f(x, y, z) = c$. Let the projection of S onto a plane with unit normal $\vec{p}$ be the region R. Let $g(x, y, z)$ be defined over S. Then the **surface integral of g over S** is

$$\iint_S g\,d\sigma = \iint_R g(x, y, z)\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\,dA.$$

Also, we write the surface differential as

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\, dA.$$

Warning: $|\nabla f \cdot \vec{p}\,|$ must not be ZERO.

If the surface S can be represented as a union of non-overlapping smooth surfaces $S_1, \ldots, S_n$, then

$$\iint_S g\, d\sigma = \iint_{S_1} g\, d\sigma + \cdots + \iint_{S_n} g\, d\sigma.$$

If $g(x, y, z) = g_1(x, y, z) + \cdots + g_m(x, y, z)$ over the surface S, then

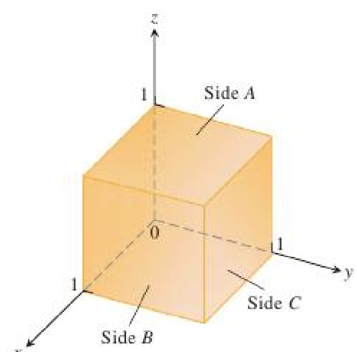
$$\iint_S g\, d\sigma = \iint_S g_1\, d\sigma + \cdots + \iint_S g_m\, d\sigma.$$

Similarly, if $g(x, y, z) = k\, h(x, y, z)$ holds for a constant k, over S, then

$$\iint_S g(x, y, z)\, d\sigma = k \iint_S h(x, y, z)\, d\sigma.$$

(3.44) ***Example***

Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1$, and $z = 1$.



We integrate g over the six surfaces and add the results. As $g = xyz$ is zero on the coordinate planes, we need integrals on sides A, B and C.

Side A is the surface defined on the region $R_A : 0 \le x \le 1,\ 0 \le y \le 1$ on the xy-plane. For this surface and the region,

$$\vec{p} = \hat{k},\ \nabla f = \hat{k},\ |\nabla f| = 1,\ |\nabla f \cdot \vec{p}\,| = |\,\hat{k} \cdot \hat{k}| = 1,\ g(x, y, z) = xyz\big|_{z=1} = xy.$$

Therefore,

$$\iint_A g(x, y, z)\, d\sigma = \iint_{R_1} xy \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|} dA = \int_0^1 \int_0^1 xy dx dy = \int_0^1 \frac{y}{2} = \frac{1}{4}.$$

Similarly,

$$\iint_B g(x,y,z)\,d\sigma = \frac{1}{4} = \iint_C g(x,y,z)\,d\sigma.$$

Thus, $\displaystyle\iint_S g\,d\sigma = \frac{3}{4}$. □

(3.45) *Example*

Evaluate the surface integral of $g(x,y,z) = x^2$ over the unit sphere.

The sphere can be divided into the upper hemisphere and the lower hemisphere. Let S be the upper hemisphere $f(x,y,z) := x^2 + y^2 + z^2 = 1,\ z \ge 0$. Its projection on the xy-plane is the region

$$R:\ x = r\cos\theta, y = r\sin\theta,\ 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

Here,

$$\vec{p} = \hat{k},\ |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2,$$

$$|\nabla f \cdot \vec{p}\,| = 2|z| = 2\sqrt{1-(x^2+y^2)} = 2\sqrt{1-r^2}.$$

Hence

$$\begin{aligned}\iint_S x^2\,d\sigma &= \iint_R x^2 \frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|}\,dA = \iint_R \frac{x^2}{\sqrt{1-r^2}}dA \\ &= \int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}}\,r\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 \frac{r^3}{\sqrt{1-r^2}}dr = \frac{2\pi}{3}.\end{aligned}$$

Since the integral of x^2 on the upper hemisphere is equal to that on the lower hemisphere, the required integral is $2 \times \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$. □

Recall that when $\vec{p} = \hat{k}$, that is, when the region R is obtained by projecting the surface S onto the xy-plane,

$$\frac{|\nabla f|}{|\nabla f \cdot \vec{p}\,|} = \sqrt{1 + z_x^2 + z_y^2}.$$

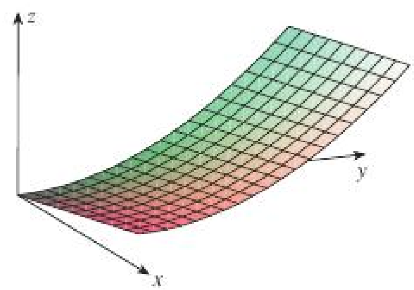
1. If the surface $f(x,y,z) = c$ can be written explicitly by $z = h(x,y)$, then the surface integral takes the form

$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(x,y,h(x,y))\,\sqrt{1 + z_x^2 + z_y^2}\,dA.$$

2. If the surface can be written as $y = h(x,z)$ and R is obtained by projecting S onto the xz-plane, then
$$\iint_S g(x,y,z)\, d\sigma = \iint_R g(x,h(x,z),z)\,\sqrt{1+y_x^2+y_z^2}\, dA.$$
3. If the surface can be written as $x = h(y,z)$ and R is obtained by projecting S onto the yz-plane, then
$$\iint_S g(x,y,z)\, d\sigma = \iint_R g(h(y,z),y,z)\,\sqrt{1+x_y^2+x_z^2}\, dA.$$

(3.46) ***Example***

Evaluate $\iint_S y\, d\sigma$, where S is the surface $z = x + y^2,\ 0 \le x \le 1,\ 0 \le y \le 2.$



Projecting the surface onto xy-plane, we obtain the region R as the rectangle
$$R:\ 0 \le x \le 1,\ 0 \le y \le 2.$$
Here, the surface is given by $z = h(x,y) = x + y^2$. So,
$$\iint_S y\, d\sigma = \iint_R y\sqrt{1+1+(2y)^2}\, dA = \int_0^1\int_0^2 \sqrt{2}y\sqrt{(1+2y^2)}\, dy\, dx = \frac{13\sqrt{2}}{3}.$$
$\square$

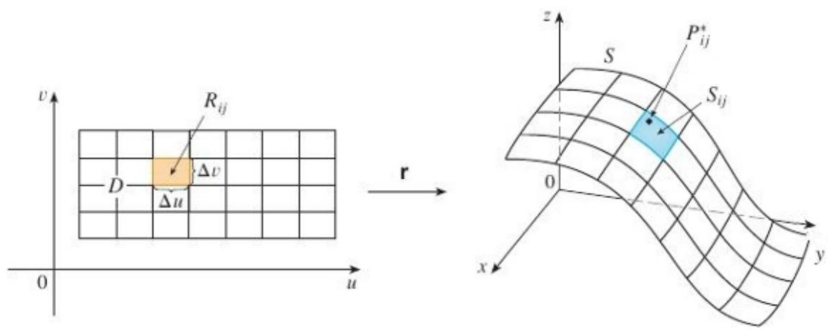
Suppose the surface S is given in the parametric form:
$$\vec{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k},$$
where (u,v) ranges over the region D in the uv-plane. Here, a change of variable happens. The Jacobian is simply $\vec{r}_u \times \vec{r}_v$. Then
$$d\sigma = |\,\vec{r}_u \times \vec{r}_v|\, dA,$$
where $\vec{r}_u = x_u\,\hat{\imath} + y_u\,\hat{\jmath} + z_u\,\hat{k}$ and $\vec{r}_v = x_v\,\hat{\imath} + y_v\,\hat{\jmath} + z_v\,\hat{k}$. Then
$$\iint_S g(x,y,z)\, d\sigma = \iint_D g(\vec{r}(u,v))|\,\vec{r}_u \times \vec{r}_v|\, dA.$$

Also this formula can directly be derived as we had done for computing surface area when a surface is given parametrically. It is as follows.

Suppose the smooth surface S has the parametric equation in vector form as

$$\vec{r} = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}.$$

Assume that the parameter region D is a rectangle. Divide D into smaller rectangles R_{ij} by taking grid lengths Δu and Δv.



Then the surface S is divided into corresponding patches S_{ij}. We evaluate g at a point P_{ij} in S_{ij} and form the Riemann sum $\sum_i \sum_j g(P_{ij})\Delta S_{ij}$, where ΔS_{ij} is the area of the patch S_{ij}. Taking limit as the number of sub-rectangles approach ∞, we obtain the surface integral of f over S as

$$\iint_S g(x,y,z)\,d\sigma = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} g(P_{ij})\Delta S_{ij}.$$

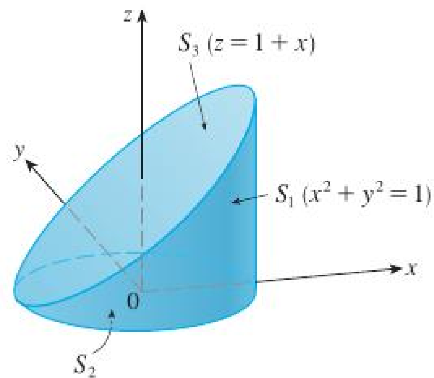
However, $\Delta S_{ij} = |\vec{r}_u(P_{ij}) \times \vec{r}_v(P_{ij})|\Delta u\Delta v$. Therefore, the surface integral is given by

$$\iint_S g(x,y,z)\,d\sigma = \iint_D g(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA.$$

Observe that the surface area of S is simply $\iint_S 1\,d\sigma$ as it should be. The relation between a surface integral and surface area is much the same as that between a line integral and the arc length of a curve.

(3.47) *Example*

Evaluate $\displaystyle\iint_S z\,d\sigma$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2 = 1$, bottom S_2 is the disk $x^2+y^2 \le 1,\ z = 0$, and whose top S_3 is part of the plane $z = 1+x$ that lies above S_2.



S_1 is given by $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ with $x = \cos\theta,\ y = \sin\theta,\ z = z$, where D is given by $0 \le \theta \le 2\pi$ and $0 \le z \le 1 + x = 1 + \cos\theta$. Then

$$|\vec{r}_\theta \times \vec{r}_z| = |\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}| = 1;$$

$$\iint_{S_1} z\,d\sigma = \iint_D z\,|\vec{r}_\theta \times \vec{r}_z|\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta = \frac{3\pi}{2}.$$

S_2 lies in the plane $z = 0$. Hence $\iint_{S_2} z\,d\sigma = 0.$

S_3 lies above the unit disk and lies in the plane $z = 1 + x$.

Here, $u = x, v = y$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z(x, y)\hat{\jmath}$. Then

$$|\vec{r}_u \times \vec{r}_v| = |(\hat{\imath} + z_x\hat{k}) \times (\hat{\jmath} + z_y\hat{k})| = \sqrt{1 + z_x^2 + z_y^2}.$$

So,

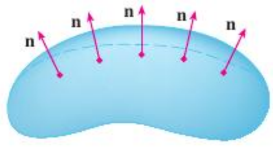
$$\begin{aligned}\iint_{S_3} z\,d\sigma &= \iint_D (1+x)\sqrt{1 + z_x^2 + z_y^2}\,dA \\ &= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{1+1+0}\,r\,dr\,d\theta = \sqrt{2}\pi.\end{aligned}$$

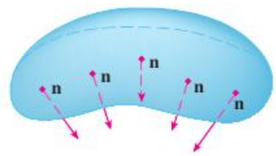
Hence,

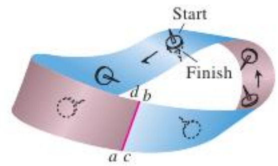
$$\iint_S z\,d\sigma = \iint_{S_1} z\,d\sigma + \iint_{S_2} z\,d\sigma + \iint_{S_3} z\,d\sigma = \frac{3\pi}{2} + \sqrt{2}\pi. \qquad \square$$

3.9 Surface Integral of a Vector Field

A smooth surface is called **orientable** iff it is possible to define a vector field of unit normal vectors $\hat{n}$ to the surface which varies continuously with position. Once such normal vectors are chosen, the surface is considered an **oriented** surface.







Conventionally, the outward direction is taken as the positive direction.

If the surface S is given by $z = f(x, y)$, then we take its orientation by considering the unit normal vectors $\hat{n} = \dfrac{-f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$.

If S is a part of a level surface $g(x, y, z) = c$, then we may take $\hat{n} = \dfrac{\nabla g}{|\nabla g|}$.

If S is given parametrically as $\vec{r}(u, v) = x(u, v)\,\hat{\imath} + y(u, v)\,\hat{\jmath} + z(u, v)\,\hat{k}$, then

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

This uses the convention that the unit normal $\hat{n}$ is positive, that is outward. Sometimes we may take negative sign if it is required.

Let $\vec{F}$ be a continuous vector field defined over an oriented surface S with unit normal $\hat{n}$. The **surface integral of $\vec{F}$ over** S, also called, the **flux of $\vec{F}$ across** S is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma.$$

The flux is the integral of the scalar component of $\vec{F}$ along the unit normal to the surface. Thus in a flow, the flux is the net rate at which the fluid is crossing the surface S in the chosen positive direction.

1. If S is part of a level surface $g(x, y, z) = c$, which is defined over the region D, then $d\sigma = \dfrac{|\nabla g|}{|\nabla g . \vec{p}\,|}\, dA$. So, the flux across S is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_S \vec{F} \cdot \frac{\pm\nabla g}{|\nabla g|} d\sigma = \iint_D \vec{F} \cdot \frac{\pm\nabla g}{|\nabla g \cdot \vec{p}\,|} dA.$$

   Here, the + sign is taken for computing the outward flux and − sign is taken for the inward flux.

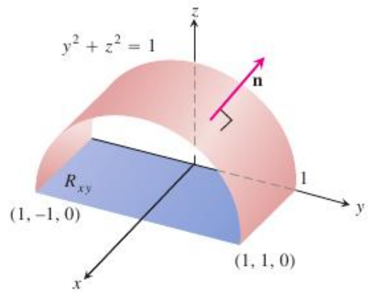
2. If S is parametrized by $\vec{r}(u, v)$, where D is the region in the uv-plane, then $d\sigma = |\vec{r}_u \times \vec{r_v}|\, du\, dv$. So, flux across S is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r_v}}{|\vec{r}_u \times \vec{r_v}|} d\sigma = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r_v})\, dA.$$

   This is the outward flux; for inward flux we take the negative sign with $\vec{r}_u \times \vec{r_v}$.

(3.48) ***Example***

Find the outward flux of $\vec{F} = yz\,\hat{\jmath} + z^2\,\hat{k}$ through the surface S which is cut from the cylinder $y^2 + z^2 = 1, z \geq 0$ by the planes $x = 0$ and $x = 1$.



S is given by $g(x, y, z) := y^2 + z^2 - 1 = 0$, defined over the rectangle $R = R_{xy}$ as in the figure.

The outward unit normal is $\hat{n} = +\dfrac{\nabla g}{|\nabla g|} = y\,\hat{\jmath} + z\,\hat{k}$.

Here, $\vec{p} = \hat{k}$. So, $d\sigma = \dfrac{|\nabla g|}{|\nabla g \cdot \hat{k}|} dA = \dfrac{\sqrt{y^2 + z^2}}{z} = \dfrac{1}{2z}\,dA$.

$\vec{F} \cdot \hat{n}$ on S is $y^2 z + z^3 = z(y^2 + z^2) = z$. Therefore, outward flux through S is

$$\iint_S \vec{F} \cdot \hat{n}\,d\sigma = \iint_R z\,\frac{1}{2z}\,dA = \iint_R dA = \tfrac{1}{2}\text{Area of } R = 1.$$

∎

(3.49) ***Example***

Find the flux of the vector field $\vec{F} = z\,\hat{\imath} + y\,\hat{\jmath} + x\,\hat{k}$ across the unit sphere.

If no direction of the normal vector is given and the surface is a closed surface, we take $\hat{n}$ in the positive direction, which is directed outward.

Using the spherical coordinates, the unit sphere S is parametrized by

$$\vec{r}(\phi, \theta) = \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k},$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ give the region D. Then

$$\vec{F}(\vec{r}(\phi, \theta)) = \cos\phi\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \sin\phi\cos\theta\,\hat{k}.$$
$$\vec{r}_\phi \times \vec{r}_\theta = \sin^2\phi\cos\theta\,\hat{\imath} + \sin^2\phi\sin\theta\,\hat{\jmath} + \sin\phi\cos\phi\,\hat{k}.$$

Consequently,

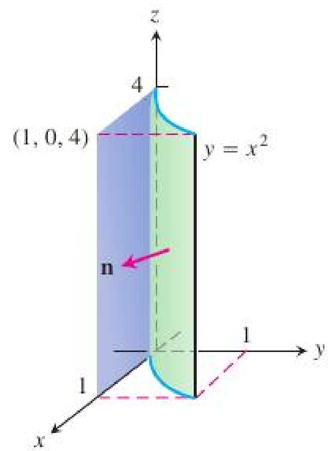
$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n}\,d\sigma &= \iint_D \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\phi\,d\theta \\ &= 2\int_0^{\pi} \sin^2\phi\cos\phi\,d\phi \int_0^{2\pi} \cos\theta\,d\theta + \int_0^{\pi} \sin^3\phi\,d\phi \int_0^{2\pi} \sin^2\theta\,d\theta\end{aligned}$$

$$= 0 + \int_0^{\pi} \sin^3 \phi \, d\phi \int_0^{2\pi} \sin^2 \theta \, d\theta = \frac{4\pi}{3}. \qquad \square$$

(3.50) ***Example***

Find the surface integral of the vector field $\vec{F} = yz\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}$ over the portion of the parabolic cylinder given by $y = x^2,\ 0 \le x \le 1,\ 0 \le z \le 4$.

We assume the positive direction of the normal $\hat{n}$. On the surface, we have $x = x,\ y = x^2,\ z = z$ giving the parametrization as $\vec{r}(x, z) = x\,\hat{\imath} + x^2\,\hat{\jmath} + z\,\hat{k}$ where D is given by $0 \le x \le 1,\ 0 \le z \le 4$.



On the surface $\vec{F} = x^2 z\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}$. So,

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_z)\, dx\, dz = \iint_D (x^2 z\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}) \cdot (2x\,\hat{\imath} - \hat{\jmath})\, dx\, dz$$
$$= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz = \int_0^4 \frac{z - 1}{2}\, dz = 2. \qquad \square$$

If S is given by $z = f(x, y)$, then think of x, y as the parameters u and v. We have

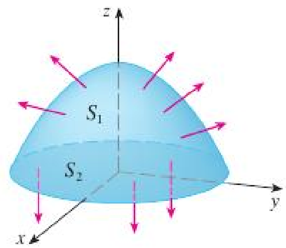
$$\vec{F} = M(x, y)\,\hat{\imath} + N(x, y)\,\hat{\jmath} + P(x, y)\,\hat{k}, \quad \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + f(x, y)\,\hat{k}.$$

Then $\vec{r}_x \times \vec{r}_y = (\hat{\imath} + f_x\,\hat{k}) \times (\hat{\jmath} + f_y\,\hat{k}) = -f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \hat{k}$. So, the flux is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dxdy = \iint_D (-Mf_x - Nf_y + P)\, dxdy.$$

(3.51) ***Example***

Evaluate $\displaystyle\iint_S \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F} = y\,\hat{\imath} + x\,\hat{\jmath} + z\,\hat{k}$ and S is the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



The surface S has two parts: the top portion S_1 and the base S_2. Since S is a closed surface, we consider its outward unit normal $\hat{n}$. Projections of both S_1 and S_2 on xy-plane are D, the unit disk.

By the simplified formula for the flux, we have

$$\begin{aligned}\iint_{S_1} \vec{F}\cdot\hat{n}\,d\sigma &= \iint_D (-Mf_x - Nf_y + P)dxdy\\ &= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]dxdy\\ &= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\left(\frac{1}{4} + \cos\theta\sin\theta\right)d\theta = \frac{\pi}{2}.\end{aligned}$$

The disk S_2 has positive direction, when $\hat{n} = -\hat{k}$. Thus

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,d\sigma = \iint_{S_2} (-\vec{F}\cdot\hat{k})\,d\sigma = \iint_D (-z)dxdy = 0$$

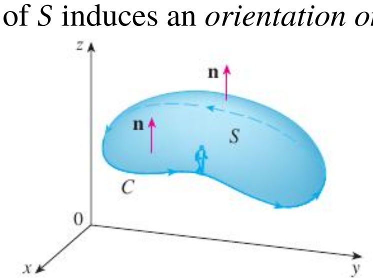
since on $D = S_2$, $z = 0$. Then

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{S_1} \vec{F}\cdot\hat{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\hat{n}\,d\sigma = \frac{\pi}{2}.$$

□

3.10 Stokes' Theorem

Consider an oriented surface with a unit normal vector $\hat{n}$. Call the boundary curve of S as C. The orientation of S induces an *orientation on C*.



We say that C is **positively oriented** iff whenever you walk in the positive direction of C keeping your head pointing towards $\hat{n}$, S will be to your left.

Recall that Green's theorem relates a double integral in the plane to a line integral over its boundary. We will have a generalization of this to 3 dimensions. Write the boundary curve of a given smooth surface as ∂S. The boundary is assumed to be a closed curve, positively oriented unless specified otherwise.

(3.52) ***Theorem*** (Stokes' Theorem)

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve ∂S with positive orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_{\partial S} \vec{F} \cdot d\,\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, d\sigma.$$

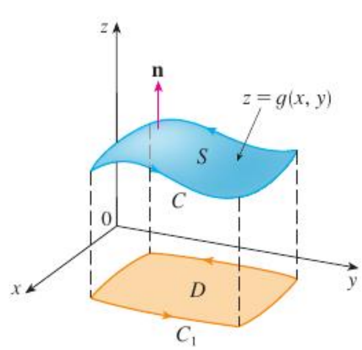
In particular, if S is a bounded region D in the xy-plane, $\partial S = C$, the smooth boundary of D, then $\hat{n} = \hat{k}$ and $d\sigma = dA$. We obtain

$$\oint_C \vec{F} \cdot d\,\vec{r} = \iint_D \text{curl } \vec{F} \cdot \hat{k}\, dA = \iint_D (N_x - M_y)\, dA.$$

as Green's theorem states. In fact, we can use Green's theorem to prove Stokes' theorem in case S is the graph of a smooth function $z = f(x, y)$ with a smooth boundary, and the vector field $\vec{F}$ is smooth.

Proof. Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$. We see that $\oint_{\partial S} \vec{F} \cdot d\,\vec{r} = \oint_{\partial S} M\,dx + N\,dy + P\,dz$. And

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, d\sigma = \iint_S \text{curl } (M\,\hat{\imath}) \cdot \hat{n}\, d\sigma + \iint_S \text{curl } (N\,\hat{\jmath}) \cdot \hat{n}\, d\sigma + \iint_S \text{curl } (P\,\hat{k}) \cdot \hat{n}\, d\sigma.$$



We show that the M-, N- and P- components in both are equal.

Suppose S is given by $z = f(x,y)$ for $(x,y) \in D$. Orient ∂D positively, i.e., counter-clock-wise. Choose a parameterization for this. Suppose ∂D is given by

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} \quad \text{for } a \le t \le b.$$

Then ∂S has the parameterization as

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + f(x(t), y(t))\,\hat{\jmath} \quad \text{for } a \le t \le b.$$

Thus

$$\oint_{\partial S} M(x,y,z)\,dx = \int_a^b M(x(t), y(t), f(x(t), y(t)))\frac{dx}{dt}\,dt.$$

Or that

$$\oint_{\partial S} M(x,y,z)\,dx = \int_{\partial D} M(x,y,z)\,dx.$$

Next, we apply Green's theorem on the integral on the right to obtain:

$$\oint_{\partial S} M(x,y,z)\,dx = -\iint_D M_y(x,y,f(x,y))\,dA.$$

Apply Chain rule on the right side integrand to obtain

$$\oint_{\partial S} M(x,y,z)\,dx = -\iint_D \Big[M_y(x,y,f(x,y)) + M_z(x,y,f(x,y))f_y\Big]\,dA.$$

We now compute $\iint_S \text{curl }(M\,\hat{\imath})d\sigma$. For this, notice that S has the parameterization:

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + f(x,y)\,\hat{k}.$$

So, $\hat{n} = \dfrac{-f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \hat{k}}{c}$, where $c = |-f_x\,\hat{\imath} - f_y\,\hat{\jmath} + \hat{k}|$. Then

$$\text{curl }(M\,\hat{\imath}) \cdot \hat{n} = (0\,\hat{\imath} + M_z\,\hat{\jmath} + M_y\,\hat{k}) \cdot \hat{n} = [-M_z f_y - M_y]/c.$$

$$\iint_S \text{curl }(M\,\hat{\imath}) \cdot \hat{n}\,d\sigma = -\iint_D \big[M_y(x,y,f(x,y))dy + M_z(x,y,f(x,y))\big]/c\,(c\,dA),$$

since $c = |\nabla(z - f(x,y))|/|\nabla(z - f(x,y)) \cdot \hat{k}|$. Therefore,

$$\iint_S \text{curl }(M\,\hat{\imath}) \cdot \hat{n} = \oint_{\partial S} M(x,y,z)\,dx.$$

Similarly, other components become respectively equal. ∎

(3.53) *Example*

Let S be the hemisphere $x^2 + y^2 + z^2 = 9,\ z \ge 0$. Let $\vec{F}(\vec{r}) = y\,\hat{\imath} - x\,\hat{\jmath}$. The bounding curve for S in the xy-plane is ∂S, which is given by $x^2 + y^2 = 9,\ z = 0$.

Parameterization of ∂S is $\vec{r}(\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath}$ for $0 \le \theta \le 2\pi$. Then

$$\oint_{\partial S} \vec{F}\cdot d\,\vec{r} = \int_0^{2\pi} [(3\sin\theta)\,\hat{\imath} - (3\cos\theta)\,\hat{\jmath}]\cdot[(-3\sin\theta)\,\hat{\imath} + (3\cos\theta)\,\hat{\jmath}]\,d\theta$$
$$= \int_0^{2\pi} [-9\sin^2\theta - 9\cos^2\theta]\,d\theta = -18\pi.$$

This is the line integral in Stokes' theorem. For the surface integral, we have

$$\text{curl }\vec{F} = (P_y - N_z)\,\hat{\imath} + (M_z - P_x)\,\hat{\jmath} + (N_x - M_y)\,\hat{k} = -2\,\hat{k}.$$

Since on the surface $g := x^2 + y^2 + z^2 - 9$, we have

$$\hat{n} = \frac{\nabla g}{|\nabla g|} = \frac{1}{3}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}).$$

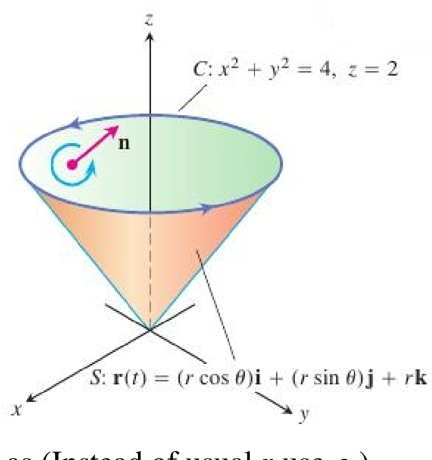
$$\vec{p} = \hat{k}, \quad d\sigma = \frac{|\nabla g|}{|\nabla g\cdot\ \vec{p}\,|}\,dA = \frac{2\times 3}{2z}dA = \frac{3}{z}dA,$$

where dA is the differential in the projected region $D:\ x^2 + y^2 \le 9$. Then

$$\iint_S \text{curl }\vec{F}\cdot\hat{n}\,d\sigma = \iint_S \frac{-2z}{3}\,d\sigma = \iint_D \frac{-2z}{3}\,\frac{3}{z}\,dA = \iint_D (-2)\,dA = -18\pi. \quad \square$$

(3.54) ***Example***

Evaluate $\oint_C ((x^2 - y)\,\hat{\imath} + 4z\,\hat{\jmath} + x^2\,\hat{k})\cdot d\,\vec{r}$, where C is the intersection of the plane $z = 2$ and the cone $z = \sqrt{x^2 + y^2}$.



Parameterize the cone as (Instead of usual r use ρ.)

$$\vec{r}(\rho,\theta) = \rho\cos\theta\,\hat{\imath} + \rho\sin\theta\,\hat{\jmath} + \rho\,\hat{k} \quad \text{for } 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi.$$

Then

$$\begin{aligned}
\vec{F} &= (x^2 - y)\,\hat{\imath} + 4z\,\hat{\jmath} + x^2\,\hat{k}.\\
\hat{n} &= \frac{\vec{r}_\rho \times \vec{r}_\theta}{|\vec{r}_\rho \times \vec{r}_\theta|} = \frac{1}{\sqrt{2}}(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} + \hat{k}).\\
\text{curl } \vec{F} &= (P_y - N_z)\,\hat{\imath} + (M_z - P_x)\,\hat{\jmath} + (N_x - M_y)\,\hat{k} = -4\,\hat{\imath} - 2r\cos\theta\,\hat{\jmath} + \hat{k}.\\
\text{curl } \vec{F}\cdot\hat{n} &= \frac{1}{\sqrt{2}}(4\cos\theta + \rho\sin(2\theta) + 1)\\
d\sigma &= \rho\sqrt{2}\,d\rho\,d\theta.
\end{aligned}$$

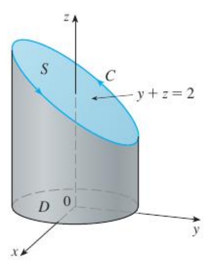
By Stokes' theorem,

$$\oint_C \vec{F}\cdot d\,\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\,d\sigma = \int_0^{2\pi}\int_0^2 \tfrac{1}{\sqrt{2}}(4\cos\theta + \rho\sin(2\theta) + 1)\rho\sqrt{2}\,d\rho\,d\theta = 4\pi.$$

∎

(3.55) ***Example***

Evaluate $\oint_C (-y^2\,\hat{\imath} + x\,\hat{\jmath} + z^2\,\hat{k})\cdot d\,\vec{r}$, where C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clock-wise when looked from above.



$$\begin{aligned}
\vec{F} &= M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}, \quad \text{where } M = -y^2,\ N = x,\ P = z^2.\\
\text{curl } \vec{F} &= (P_y - N_z)\,\hat{\imath} + (M_z - P_x)\,\hat{\jmath} + (N_x - M_y)\,\hat{k} = (1 + 2y)\,\hat{k}.
\end{aligned}$$

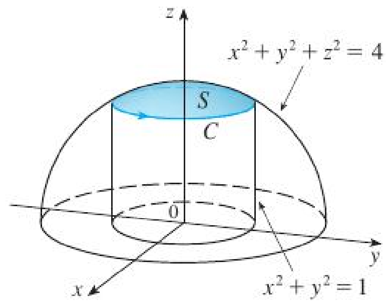
Here, there are many surfaces with boundary C. We choose a convenient one: the surface S on the plane $y + z = 2$ with boundary as C. Its projection on the xy-plane is the disc $D:\ x^2 + y^2 \le 1$. Then $\vec{p} = \hat{k}$. With $g(x, y) = y + z - 2$, we have $\hat{n} = (\nabla g)/|\nabla g| = (\hat{\jmath} + \hat{k})/\sqrt{2}$, $\nabla g\cdot\vec{p} = 1$, and $d\sigma = \sqrt{2}\,dA$. Stokes' theorem gives

$$\begin{aligned}
\oint_C \vec{F}\cdot d\,\vec{r} &= \iint_S \text{curl } \vec{F}\cdot\hat{n}\,d\sigma = \iint_D \frac{1+2y}{\sqrt{2}}\sqrt{2}\,dA\\
&= \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\Big(\frac{1}{2} + \frac{2}{3}\sin\theta\Big)\,d\theta = \pi.
\end{aligned}$$

∎

(3.56) ***Example***

Compute $\iint_S \text{curl } \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F} = xz\,\hat{\imath} + yz\,\hat{\jmath} + xy\,\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.



The boundary curve C is obtained by solving the two equations. We get $z^2 = 3$. Since $z > 0$, we have the curve C as $x^2 + y^2 = 1,\ z = \sqrt{3}$. In vector parametric form,

$$C:\ \vec{r}(\theta) = \cos\theta\ \hat{\imath} + \sin\theta\,\hat{\jmath} + \sqrt{3}\ \hat{k} \text{ for } 0 \le \theta \le 2\pi.$$

Then

$$\vec{F}(\vec{r}(\theta)) = \sqrt{3}\cos\theta\,\hat{\imath} + \sqrt{3}\sin\theta\,\hat{\jmath} + \cos\theta\sin\theta\,\hat{k}.$$

By Stokes' theorem,

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, d\sigma = \oint_C \vec{F} \cdot d\,\vec{r} = \int_0^{2\pi} \vec{F} \cdot \vec{r}\,'(\theta)\, d\theta$$
$$= \int_0^{2\pi} (-\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\sin\theta\cos\theta)\, d\theta = 0.$$

□

Stokes' theorem can be generalized to piecewise smooth surfaces like union of sides of a polyhedra. Here, we take the integral over the sides as the sum of integrals over each individual side.

Similarly, Stokes' theorem can be generalized to surfaces with holes. The line integrals are to be taken over all the curves which form the boundaries of the holes.

The surface integral over S of the normal component of curl $\vec{F}$ is equal to the sum of the line integrals around all the boundary curves of the tangential component of $\vec{F}$. Here, the curves are traced in the direction induced by the orientation of S.

Recall that a conservative field is one which can be expressed as a gradient of another scalar field. In such a case, curl $\vec{F} = 0$. Then from Stokes' theorem, it follows that $\oint_C \vec{F} \cdot d\,\vec{r} = 0$.

(3.57) ***Theorem***

If curl $\vec{F} = 0$ *at each point of an open simply connected region D in space, then on any piecewise smooth closed path C lying in D,* $\oint_C \vec{F} \cdot d\,\vec{r} = 0$.

3.11 Gauss' Divergence Theorem

We have seen how to relate an integral of a function over a region with the integral of possibly some other related function over the boundary of the region.

For definite integrals on intervals: $\displaystyle\int_a^b f'(t)\,dt = f(b) - f(a)$.

For a path from a point P to a point Q in $\mathbb{R}^3$, $\displaystyle\int_C \nabla f \cdot ds = f(Q) - f(P)$.

For a region D in $\mathbb{R}^2$, $\displaystyle\iint_D (N_x - M_y)\,dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$.

For a surface S in $\mathbb{R}^3$, $\displaystyle\iint_S \text{curl } \vec{F} \cdot \hat{n}\,d\sigma = \int_C \vec{F} \cdot d\vec{r}$.

It suggests a generalization to three dimensions; and we use the divergence of a vector field for this purpose.

Recall that div $\vec{F} = \nabla \cdot \vec{F}$. That is, the **divergence** of a vector field

$$\vec{F} = M(x, y, z)\,\hat{\imath} + N(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k}$$

is the scalar function div $\vec{F} = M_x + N_y + P_z$.

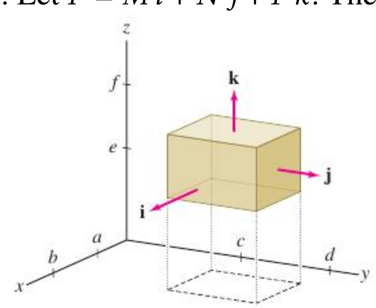
Our generalization is $\displaystyle\iiint_D \text{div } \vec{F}\,dV = \iint_S \vec{F} \cdot \hat{n}\,d\sigma$.

(3.58) ***Theorem*** (Gauss' Divergence Theorem)

Let S be a piecewise smooth simple closed bounded surface that encloses a solid region D in $\mathbb{R}^3$. Suppose S is oriented positively by its outward normals. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains D. Then

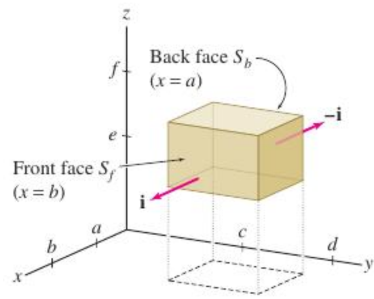
$$\iint_S \vec{F} \cdot \hat{n}\,d\sigma = \iiint_D \text{div } \vec{F}\,dV.$$

Proof. We prove this in the special case, where D is a box in $\mathbb{R}^3$ given by $D = [a, b] \times [c, d] \times [e, f]$. Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$. Then



$$\iiint_D \text{div } \vec{F}\, dV = \iiint_D \text{div } M\, dV + \iiint_D \text{div } N\, dV + \iiint_D \text{div } \vec{F}\, dV.$$
$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_S M \cdot \hat{n}\, d\sigma + \iint_S N \cdot \hat{n}\, d\sigma + \iint_S P \cdot \hat{n}\, d\sigma.$$

We prove that the respective components are equal. We thus consider only the $\hat{\imath}$-component. That is, we take $\vec{F} = M\,\hat{\imath}$ and prove the divergence theorem in this case.



So, let $\vec{F} = M\,\hat{\imath}$. The solid has six faces. The surface integral over S is the sum of integrals over these faces. A simplification occurs. $\vec{F} = M\,\hat{\imath}$ we have $\vec{F} \cdot \hat{\jmath} = F \cdot \hat{k} = 0$. That is, $\vec{F}$ is orthogonal to the normals of the top, bottom, and the two side faces.

Writing the remaining faces as S_f and S_b, we have

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_{S_f} \vec{F} \cdot \hat{n}\, d\sigma + \iint_{S_b} \vec{F} \cdot \hat{n}\, d\sigma.$$

Parameterization of these faces give

$$S_f : \ \vec{r} = b\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}, \ S_b : \ \vec{r} = a\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}.$$

for $c \le y \le d, \ e \le z \le f$. The outward normal to S_f is $\hat{\imath}$, and to S_b is $-\hat{\imath}$. Then

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n}\, d\sigma &= \int_e^f \int_c^d M(b, y, z)\, dydz - \int_e^f \int_c^d M(a, y, z)\, dydz \\
&= \int_e^f \int_c^d [M(b, y, z) - M(a, y, z)]\, dydz \\
&= \int_e^f \int_c^d \int_a^b M_x(x, y, z)\, dxdydz \\
&= \iiint_D \text{div } \vec{F}\, dV,
\end{aligned}$$

since $\vec{F} = M\,\hat{\imath} \Rightarrow \text{div } \vec{F} = \text{div } M = M_x$. ∎

(3.59) *Example*

Consider the field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ over the sphere $S: \; x^2 + y^2 + z^2 = a^2$.

The outer unit normal to S computed from ∇f, with $f = x^2 + y^2 + z^2 - a^2$, is

$$\hat{n} = \frac{2(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{1}{a}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}).$$

Hence on the given surface, $\vec{F} \cdot \hat{n}\, d\sigma = \dfrac{1}{a}(x^2 + y^2 + z^2)\, d\sigma = a\, d\sigma$.

Therefore, $\displaystyle\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_S a\, d\sigma = a \times \text{Area of } S = 4\pi a^3$.

Now, for the triple integral, $\text{div } \vec{F} = M_x + N_y + P_z = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3$.

Therefore, with D as the ball bounded by S,

$$\iiint_D \text{div } \vec{F}\; dV = \iiint_D 3\, dV = 3 \times \text{Volume of } D = 4\pi a^3. \qquad \Box$$

(3.60) *Example*

Find the outward flux of the vector field $xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$ through the surface cut from the first octant by the planes $x = 1,\; y = 1$ and $z = 1$.

The solid D is a cube having six faces. Call the surface of the cube as S. Instead of computing the surface integral, we use Divergence theorem.

With $\vec{F} = xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{k}$, we have

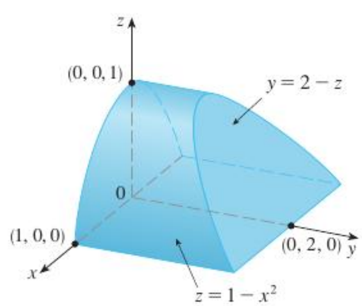
$$\text{div } \vec{F} = \frac{\partial(xy)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(zx)}{\partial z} = y + z + x.$$

Therefore the required flux is

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \text{div } \vec{F}\; dV = \int_0^1 \int_0^1 \int_0^1 (y + z + x)\, dxdydz = \frac{3}{2}. \qquad \Box$$

(3.61) *Example*

Evaluate $\iint_S \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F} = xy\,\hat{\imath} + y^2 + e^{xz^2}\,\hat{\jmath} + \sin(xy)\,\hat{k}$ and S is the surface of the solid D bounded by the parabolic cylinder $z = 1 - x^2$, and the planes $y = 0$, $z = 0$, and $y + z = 2$.



S has four sides. Instead of computing the surface integrals, we use Divergence theorem. We have

$$\operatorname{div}\ \vec{F} = (xy)_x + (y^2 + e^{xz^2})_y + (\sin(xy))_z = 3y.$$

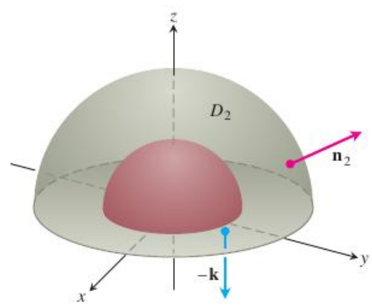
And D is given by $-1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z$. Therefore,

$$\begin{aligned}\iint_S \vec{F}\cdot\hat{n}\,d\sigma &= \iiint_D \operatorname{div}\ \vec{F}\ dV = \iiint_D 3y\,dV \\ &= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\,dy\,dz\,dx = \int_{-1}^{1}\int_0^{1-x^2} \frac{(2-z)^2}{2}\,dz\,dx \\ &= -\frac{1}{2}\int_{-1}^{1}[(x^2+1)^3 - 8]\,dx = \frac{184}{35}.\end{aligned}$$

$\square$

(3.62) ***Example***

Find the outward flux of the vector field $\vec{F}$ across the boundary of the solid D where $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}$ and $D:\ 0 < a^2 \le x^2 + y^2 + z^2 \le b^2$.



Write $\rho = \sqrt{x^2+y^2+z^2}$. Then $\dfrac{d\rho}{dx} = \dfrac{x}{\rho}$. With $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, we have

$$M_x = \frac{\partial(x\rho^{-3})}{\partial x} = \rho^{-3} - 3x\rho^{-4}\frac{\partial\rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}.$$

Similarly, $N_y = \dfrac{1}{\rho^3} - \dfrac{3y^2}{\rho^5}$ and $P_z = \dfrac{1}{\rho^3} - \dfrac{3z^2}{\rho^5}$.

Then $\operatorname{div}\ \vec{F} = \dfrac{3}{\rho^3} - \dfrac{3x^2+3y^2+3z^2}{\rho^5} = 0$.

Thus the required flux is $\displaystyle\iiint_D \operatorname{div}\ \vec{F}\ dV = 0$. $\square$

In the above example, flux through the inner surface and flux through the outer surface are in opposite directions. Are their magnitudes equal?

(3.63) ***Example***

Consider the vector field $\vec{F} = \dfrac{1}{a^3}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$ on the sphere S of radius a centered at the origin. Show that the flux through S is a constant.

We compute the flux directly. Let S be the sphere $x^2 + y^2 + z^2 = a^2$ for any $a > 0$. The gradient computed from $f = x^2 + y^2 + z^2 - a^2$ gives the outward unit normal to S as

$$\hat{n} = \frac{2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{a}.$$

Therefore, on the sphere S with $\vec{F} = (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})/(x^2 + y^2 + z^2)^{3/2}$,

$$\vec{F}\cdot\hat{n} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{1}{a^2}.$$

Then $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S \frac{1}{a^2}\,d\sigma = \frac{1}{a^2} \times \text{Area of } S = 4\pi.$ $\square$

3.12 Review Problems

Problem 3.1 Compute the line integral of the vector function $x^3\,\hat{\imath} + 3zy^2\,\hat{\jmath} - x^2y\,\hat{k}$ along the straight line segment L from the point $(3, 2, 1)$ to $(0, 0, 0)$.

The parametric equation of the line segment joining these points is

$$x = -3t,\ y = -2t,\ z = -t \text{ for } -1 \le t \le 0.$$

The derivatives of these with respect to t are

$$x_t = -3,\ y_t = -2,\ z_t = -1.$$

Then the required line integral is

$$\begin{aligned}&\int_L x^3\,dx + 3zy^2\,dy - x^2y\,dz \\ &= \int_{-1}^{0} [(-3t)^3(-3) + 3(-t)(-2t)^2(-2) - (-3t)^2(-2t)(-1)]\,dt = \frac{-87}{4}.\end{aligned}$$

Problem 3.2 Let C be the portion of the curve $y = x^3$ from $(1, 1)$ to $(2, 8)$. Compute

$$\int_C (6x^2y\,dx + 10xy^2\,dy).$$

C is parametrized as $x = t,\ y = t^3,\ 1 \le t \le 2$. Then $x_t = 1,\ y_t = 3t^2$. The line integral is

$$\int_C (6x^2y\,dx + 10xy^2\,dy) = \int_1^2 (6t^5 \cdot 1 + 10t^7 \cdot 3t^2)\,dt = 3132.$$

Problem 3.3 Evaluate $\displaystyle\int_C (-y\,\hat\imath - xy\,\hat\jmath) \cdot d\vec r$, where C is the circular arc joining $(1,0)$ to $(0,1)$ of a circle centered at the origin.

Prameterize C by $\vec r(\theta) = \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath$, for $0 \le \theta \le \pi/2$. Thus $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$. Then

$$\begin{aligned}\int_C \vec F \cdot d\vec r &= \int_0^{\pi/2} \vec F(\vec r(\theta)) \cdot \vec r\,'(\theta)\,d\theta \\ &= \int_0^{\pi/2} (-\sin\theta\,\hat\imath - \cos\theta\sin\theta\,\hat\jmath) \cdot (-\sin\theta\,\hat\imath + \cos\theta\,\hat\jmath)\,d\theta \\ &= \int_0^{\pi/2} (\sin^2\theta - \cos^2\theta\sin\theta)\,d\theta = \frac{\pi}{4} - \frac{1}{3}.\end{aligned}$$

Problem 3.4 Let $\vec F = 5z\,\hat\imath + xy\,\hat\jmath + x^2z\,\hat k$. Is $\displaystyle\int_C \vec F \cdot d\vec r$ the same if C is a curve joining $(0,0,0)$ to $(1,1,1)$, given by

(a) $\vec r(t) = t\,\hat\imath + t\,\hat\jmath + t\,\hat k$ for $0 \le t \le 1$; (b) $\vec r(t) = t\,\hat\imath + t\,\hat\jmath + t^2\,\hat k$ for $0 \le t \le 1$?

(a) $\vec F(\vec r(t)) = 5t\,\hat\imath + t^2\,\hat\jmath + t^3\,\hat k$. $d\vec r(t) = \hat\imath + \hat\jmath + \hat k$. Thus

$$\int_C \vec F \cdot d\vec r = \int_0^1 (5t + t^2 + t^3)dt = \frac{37}{12}.$$

(b) $\vec F(\vec r(t)) = 5t\,\hat\imath + t^2\,\hat\jmath + t^3\,\hat k$. $d\vec r(t) = \hat\imath + \hat\jmath + 2t\,\hat k$. Thus

$$\int_C \vec F \cdot d\vec r = \int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{28}{12}.$$

As we see the line integral is not path-independent.

Problem 3.5 Let D be a simply connected region containing a smooth curve C from $(0,0,0)$ to $(1,1,1)$. Evaluate $\displaystyle\int_C (2xdx + 2ydy + 4zdz)$.

$\vec F = 2x\,\hat\imath + 2y\,\hat\jmath + 4z\,\hat k = \nabla f$, where $f = x^2 + y^2 + 2z^2$. Therefore, the line integral is independent of path C. Hence its value is $f(1,1,1) - f(0,0,0) = 4$.

Problem 3.6 Evaluate $\displaystyle\iint_S (7x\,\hat\imath - z\,\hat k) \cdot \hat n\,d\sigma$ over the surface $S:\ x^2+y^2+z^2 = 4$.

div $\vec F$ = div $(7x\,\hat\imath - z\,\hat k) = 7 - 1 = 6$. So, the integral = $6 \times$ volume of $S = 64\pi$.

Problem 3.7 Evaluate $I = \int_C (3x^2\,dx + 2yz\,dy + y^2\,dz)$, where C is a smooth curve joining $(0, 1, 2)$ to $(1, -1, 7)$ by showing that $\vec{F}$ has a potential.

In order that $\vec{F} = \nabla f$, we should have

$$f_x = M = 3x^2, \quad f_y = N = 2yz, \quad f_z = P = y^2.$$

To obtain such a possible f, we use integration and differentiation:

$$f = x^3 + g(y, z), \quad f_y = g_y = 2yz, \quad g = y^2 z + h(z),$$

$$f_z = y^2 + h'(z) = y^2, \quad h'(z) = 0, \quad h(z) = \text{ constant.}$$

Then $f = x^3 + y^2 z$. We verify that $\vec{F} = \nabla f$. Therefore, $I = \vec{F}(1, -1, 7) - f(0, 1, 2) = 6$.

Problem 3.8 Determine whether the line integral

$$I = \int_C (2xyz^2\,dx + (x^2z^2 + z\cos(yz))\,dy + (2x^2yz + y\cos(yz)\,dz)$$

is independent of path. Evaluate I, where C is the line segment joining $(0, 0, 1)$ to $(1, \pi/4, 2)$.

Here, $M = 2xyz^2,\ N = x^2z^2 + z\cos(yz),\ P = 2x^2yz + y\cos(yz)$. Then

$$M_y = 2xz^2 = N_x,\ N_z = 2x^2z + \cos(yz) - yz\sin(yz) = P_y,\ P_x = 4xyz = M_z.$$

Hence the line integral is independent of path.

This suggests we find f such that $\vec{F} = \nabla f$. Now,

$$f = \int N dy = x^2z^2y + \sin(yz) + g(x, z),\ f_x = 2xz^2y + g_x = M = 2xyz^2.$$

$g_x = 0,\ g = h(z),\ f_z = 2x^2yz + y\cos(yz) + h'(z) = P = 2x^2yz + y\cos(yz),\ h'(z) = 0.$

Taking $h(z) = 0$, we get $f(x, y, z) = x^2yz^2 + \sin(yz)$ as a possible potential. Then

$$I = f(1, \pi/4, 2) - f(0, 0, 1) = \pi + 1.$$

Problem 3.9 Use Green's theorem to compute the area of the region

(a) bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) bounded by the cardioid $r = a(1 - \cos\theta)$ for $0 \le \theta \le 2\pi$.

(a) Recall: Green's theorem gave Area of $D = \frac{1}{2}\oint_{\partial D}(x\,dy - y\,dx)$. The ellipse $x^2/a^2 + y^2/b^2 = 1$ has the parameterization $x(t) = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$. Then its area is

$$\frac{1}{2}\int_0^{2\pi}(x\dot{y} - y\dot{x})\,dt = \frac{1}{2}\int_0^{2\pi}(ab\cos^2 t - (-ab\sin^2 t))\,dt = \pi ab.$$

(b) In polar form, $x = r\cos\theta,\ y = r\sin\theta$. Then $dx = \cos\theta\,dr - r\sin\theta\,d\theta$ and $dy = \sin\theta\,dr + r\cos\theta\,d\theta$. Consequently the area is equal to

$$\frac{1}{2}\oint_{\partial D}(x\,dy - y\,dx) = \frac{1}{2}\oint_{\partial D} r^2\,d\theta = \frac{a^2}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}a^2.$$

Problem 3.10 Compute the flux of the water through the parabolic cylinder $S:\ y = x^2,\ 0 \le x \le 2, 0 \le z \le 3$ if the velocity vector $\vec{F} = 3z^2\,\hat{\imath} + 6\,\hat{\jmath} + 6zx\,\hat{k}$, speed being measured in m/sec.

Write $x = u,\ z = v$. We have $y = x^2 = u^2$. The surface is

$$S:\ \vec{r} = u\,\hat{\imath} + u^2\,\hat{\jmath} + v\,\hat{k},\ \text{for } 0 \le u \le 2,\ 0 \le v \le 3.$$

Then

$$\vec{n} = \vec{r}_u \times \vec{r}_v = (\,\hat{\imath} + 2u\,\hat{\jmath}) \times \hat{k} = 2u\,\hat{\imath} - \hat{\jmath}.$$

On S,

$$\vec{F}(\vec{r}(u,v)) = 3v^2\,\hat{\imath} + 6\,\hat{\jmath} + 6uv\,\hat{k}.$$

Hence $\vec{F}(\vec{r}(u,v)) \cdot \vec{n} = 6uv^2 - 6$. Consequently the flux is

$$\iint_S \vec{F}\cdot\vec{n}\ d\sigma = \int_0^3\int_0^2 (6uv^2 - 6)\,du\,dv = \int_0^3 (12v^2 - 12)\,dv = 72\,\mathrm{m}^3/\mathrm{sec}.$$

Problem 3.11 Find the area of the portion of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + z^2 = a^2$.

One-eighth of the required surface area is in the first octant. This portion of the surface has the equation $y = \sqrt{a^2 - x^2}$. This gives

$$\frac{\partial y}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2}},\ \frac{\partial y}{\partial z} = 0 \ \Rightarrow\ \sqrt{1 + y_x^2 + y_z^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}.$$

The region of integration is a quarter of a disk given by

$$x^2 + x^2 \le a^2 \le a^2, \quad x \ge 0,\ z \ge 0.$$

Therefore, the required area is

$$8 \times \int_0^a \Big[\int_0^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}}\,dz\Big]\,dx = 8a\int_0^a dx = 8a^2.$$

Problem 3.12 A torus is generated by rotating a circle C about a straight line L in space so that C does not intersect or touch L. If L is the z-axis and C has radius b and its centre has distance $a\ (> b)$ from L, then compute the surface area of the torus.

The surface S of the torus is represented by

$$\vec{r}(u,v) = (a + b\cos v)\cos u\,\hat{\imath} + (a + b\cos v)\ \sin u\ \hat{\jmath} + b\sin v\ \hat{k}.$$

Here, v is the angle in describing the circle and u is the angle of rotation. Thus $0 \le u, v \le 2\pi$. Projection onto the uv-plane shows that

$$\begin{aligned}\vec{r}_u &= -(a + b\cos v)\sin u\,\hat{\imath} + (a + b\cos v)\cos u\ \hat{\jmath}\\ \vec{r}_v &= -b\sin v\cos u\,\hat{\imath} - b\sin v\sin u\ \hat{\jmath} + b\cos v\ \hat{k}\\ \vec{r}_u \times \vec{r}_v &= b(a + b\cos v)(\cos u\cos v\,\hat{\imath} + \sin u\cos v\ \hat{\jmath} + \sin v\ \hat{k})\end{aligned}$$

Hence $|\vec{r}_u \times \vec{r}_v| = b(a + b\cos v)$ and the area is

$$\iint_C |\vec{r}_u \times \vec{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)\,du\,dv = 4\pi^2 ab.$$

Problem 3.13 Let S be the closed surface consisting of the cylinder $x^2 + y^2 = a^2,\ 0 \le z \le b$ and the circular disks $x^2 + y^2 \le a^2,\ z = 0$ and $x^2 + y^2 \le a^2,\ z = b$. By transforming to a triple integral evaluate

$$I = \iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy).$$

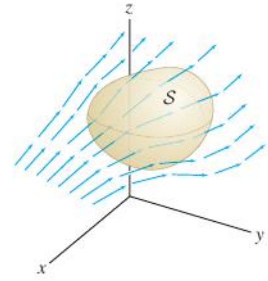
$\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$, where $M = x^3,\ N = x^2y,\ P = x^2z$. Then div $\vec{F} = 5x^2$. Let D be the solid bounded by S. In cylindrical coordinates, using Gauss' divergence theorem,

$$I = \iiint_D 5x^2 dV = 5\int_0^b\int_0^a\int_0^{2\pi} r^2\cos^2\theta\, r\,dr\,d\theta\,dz = \frac{5}{4}\pi a^4 b.$$

Problem 3.14 Compute the flux of the vector field

$$\vec{F} = (z^2 + xy^2)\,\hat{\imath} + \cos(x + z)\,\hat{\jmath} + (e^{-y} - zy^2)\,\hat{k}$$

through the boundary of the surface given in the following figure:



$$\text{div } (F) = \frac{\partial}{\partial x}(z^2 + xy^2) + \frac{\partial}{\partial y}\cos(x+z) + \frac{\partial}{\partial z}(e^{-y} - zy^2) = 0.$$

Let D be the region enclosed by S. By the Divergence theorem,

$$\text{Flux through } S = \iiint_D \text{div } \vec{F}\, dV = 0.$$

Problem 3.15 Let a closed smooth surface S be such that any straight line parallel to the z-axis cuts it in no more than two points. Let n_3 denote the z-component of the unit outward normal $\hat{n}$ to the surface S. Then what is $\iint_S z\, n_3\, d\sigma$?

In this case, S has an upper part and a lower part. Suppose they are given, respectively, by the equations

$$z = f_u(x,y), \quad z = f_b(x,y).$$

Let D be the projection of S on the xy-plane. Then

$$\iint_S z\, n_3\, d\sigma = \iint_D f_u(x,y)\, dA - \iint_D f_b(x,y)\, dA.$$

Alternatively, take $\vec{F} = z\hat{k}$. Then div $\vec{F} = 1$. By the Divergence theorem,

$$\iint_S z\, n_3\, d\sigma = \iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_B \text{div } \vec{F}\, dV = \text{volume of } B.$$

Problem 3.16 Prove that the integral of the Laplacian over a planar region is the same as the integral, over the boundary curve, of the directional derivative in the direction of the unit normal to the boundary curve.

We rephrase: Let $f(x,y)$ be a function defined over a simply connected region D in the xy-plane. Let C be the boundary curve of D. Denote by $D_n f(x,y)$ the directional derivative of f in the direction of the unit outer normal $\hat{n}$ to C. Show that $\iint_D (f_{xx} + f_{yy}) dA = \int_C D_n f\, ds$.

Let θ be the angle between $\hat{n}$ and $\hat{\imath}$, the x-axis. Then $\hat{n} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$. If α is the angle between the tangent line to C and the x-axis, then $\cos\alpha = -\sin\theta$ and $\sin\alpha = \cos\theta$. Then

$$dx = \cos\alpha\, ds = -\sin\theta\, ds \text{ and } dy = \sin\alpha\, ds = \cos\theta\, ds.$$

Consequently, the directional derivative $D_n f$ is given by

$$D_n f(x,y) = (f_x\,\hat{\imath} + f_y\,\hat{\jmath}) \cdot \hat{n} = f_x \cos\theta + f_y \sin\theta.$$

For the vector function $\vec{F} = f_x\,\hat{\imath} + f_y\,\hat{\jmath}$, by Green's theorem, we obtain

$$\iint_D (f_{xx} + f_{yy})dA = \int_C f_x dy - f_y dx = \int_C (f_x \cos\theta + f_y \sin\theta)ds = \int_C D_n f\, ds.$$

Problem 3.17 Let f and g be functions with continuous partial derivatives up to second order on a region D in space, which has a smooth boundary ∂D. Denote by Δf and Δg their Laplacians. Prove the Green's formula:

$$\iiint_D (g\Delta f - f\Delta g)dV = \iint_{\partial D} \left(g\frac{\partial f}{\partial \hat{n}} - f\frac{\partial g}{\partial \hat{n}} \right) d\sigma.$$

Let $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$. Gauss' divergence theorem says that

$$\iiint_D \text{div}\ \vec{F}\ dV = \iint_{\partial D} \vec{F}\cdot\hat{n}\, d\sigma.$$

Suppose the unit normal $\hat{n}$ has the components a, b, c in the x, y, z-directions, respectively. Then

$$\iiint_D (M_x + N_y + P_z)\, dV = \iint_{\partial D} (aM + bN + cP)\, d\sigma.$$

Substitute $M = gf_x - fg_x,\ N = gf_y - fg_y,\ P = gf_z - fg_z$. Then

$$M_x + N_y + P_z = g(f_{xx} + f_{yy} + f_{zz}) - f(g_{xx} + g_{yy} + g_{zz}) = g\Delta f - f\Delta g.$$

$$aM + bN + cP = g(af_x + bf_y + cf_z) - f(ag_x + bg_y + cg_z) = g\frac{\partial f}{\partial \hat{n}} - f\frac{\partial g}{\partial \hat{n}}.$$

Now Green's formula follows from Gauss' divergence theorem.

A

One Variable Summary

This appendix is devoted to summarizing some results and formulas from calculus of functions of one real variable that we may use in the class. For details, see *Functions of One Variable - A Survival Guide*.

A.1 Graphs of Functions

The **absolute value** of $x \in \mathbb{R}$ is defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x| = \sqrt{x^2}$. And $|-a| = a$ or $a \geq 0$; $|x - y|$ is the distance between real numbers x and y. Moreover, if $a, b \in \mathbb{R}$, then

$$|-a| = |a|,\ |ab| = |a|\,|b|,\ \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ if } b \neq 0,\ |a+b| \leq |a|+|b|,\ \big|\,|a|-|b|\,\big| \leq |a-b|.$$

Let $x \in \mathbb{R}$ and let $a > 0$. The following are true:

1. $|x| = a$ iff $x = \pm a$.
2. $|x| < a$ iff $-a < x < a$ iff $x \in (-a, a)$.
3. $|x| \leq a$ iff $-a \leq x \leq a$ iff $x \in [-a, a]$.
4. $|x| > a$ iff $-a < x$ or $x > a$ iff $x \in (-\infty, -a) \cup (a, \infty)$ iff $x \in \mathbb{R} - [-a, a]$.
5. $|x| \geq a$ iff $-a \leq x$ or $x \geq a$ iff $x \in (-\infty, -a] \cup [a, \infty)$ iff $x \in \mathbb{R} - (-a, a)$.

Therefore, for $a \in \mathbb{R}, \delta > 0, \quad |x - a| < \delta$ iff $a - \delta < x < a + \delta$.

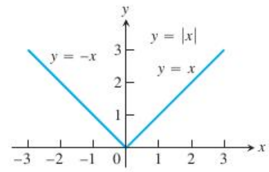
The following statements are useful in proving equalities from inequalities:

Let $a, b \in \mathbb{R}$.

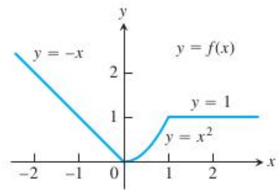
1. If for each $\epsilon > 0$, $|a| < \epsilon$, then $a = 0$.
2. If for each $\epsilon > 0$, $a < b + \epsilon$, then $a \leq b$.

Graphs of some known functions including $|\cdot|$, are as follows:

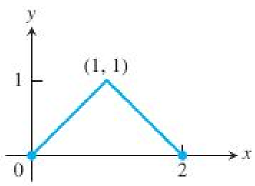
1. $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



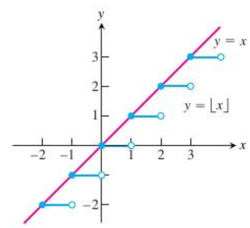
2. $y = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$



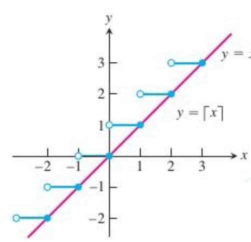
3. $y = f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$



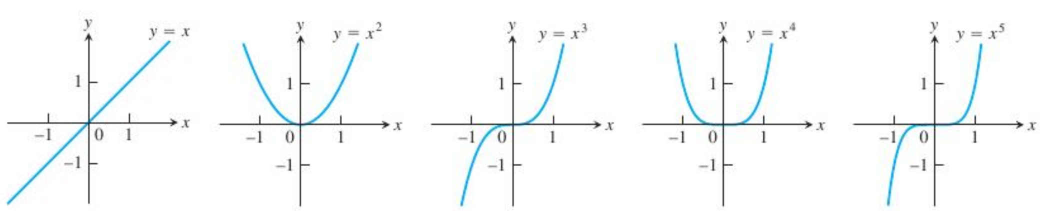
4. $y = \lfloor x \rfloor = n$ if $n \leq x < n + 1$ for $n \in \mathbb{N}$. It is the largest integer less than or equal to x.
The largest integer function or the **floor** function.
Sometimes we write $\lfloor\ \rfloor$ as $[\ ]$.



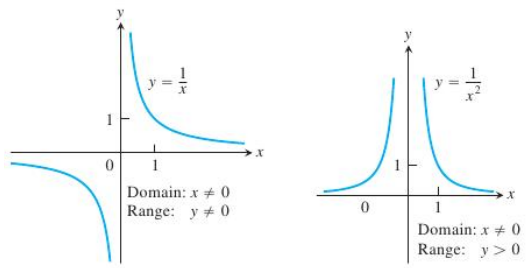
5. $y = \lfloor x \rfloor = n + 1$ if $n < x \leq n + 1$ for $n \in \mathbb{N}$. It is the smallest integer greater than or equal to x.
The smallest integer function or the **ceiling** function.



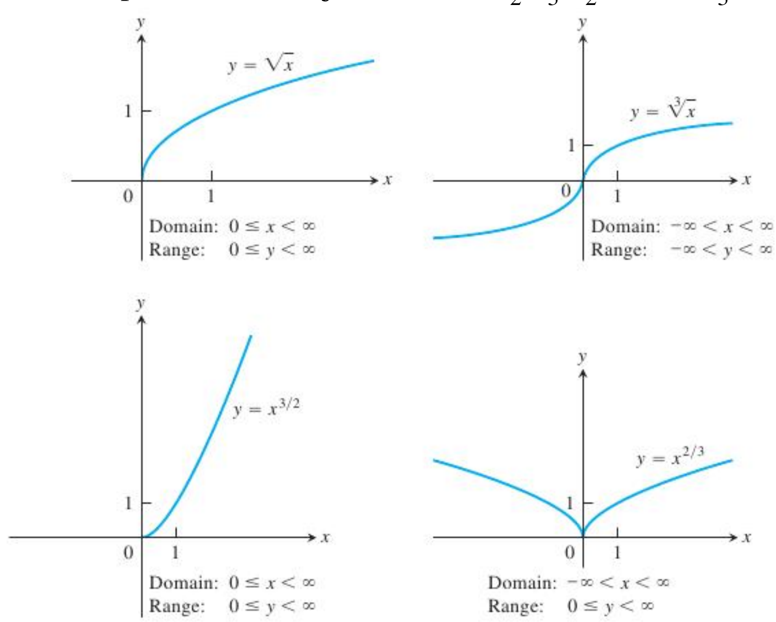
6. The **power function** $y = x^n$ for $n = 1, 2, 3, 4, 5$ look like



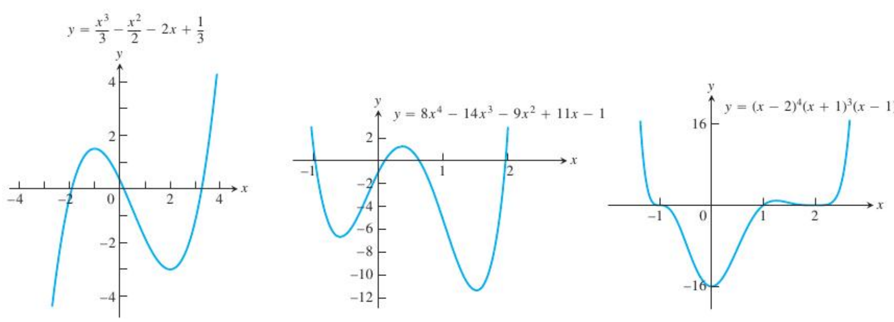
7. The power function $y = x^n$ for $n = -1$ and $n = -2$ look like



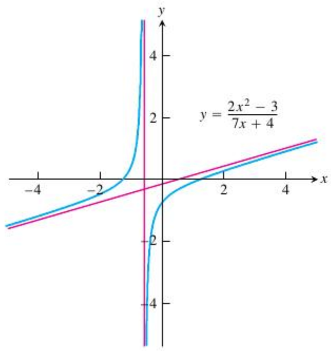
8. The graphs of the power function $y = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $a = \frac{2}{3}$ are

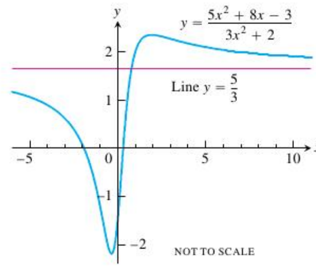


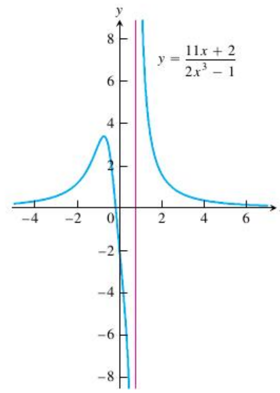
9. **Polynomial** functions are $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$ for some $n \in \mathbb{N} \cup \{0\}$. Here, the coefficients of powers of x are some given real numbers $a_0, \ldots, a_n$ and $a_n \neq 0$. The highest power n in the polynomial is called the **degree** of the polynomial. Graphs of some polynomial functions are as follows:



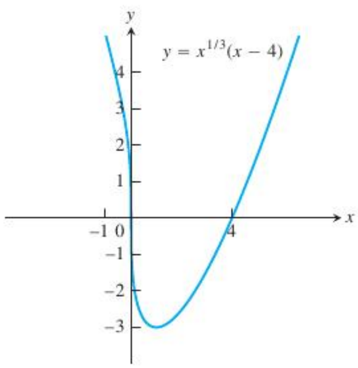
10. A **rational function** is a ratio of two polynomials; $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, may or may not be of the same degree. Graphs of some rational functions are as follows:

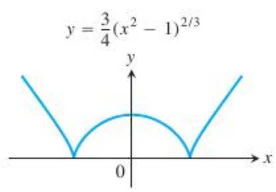


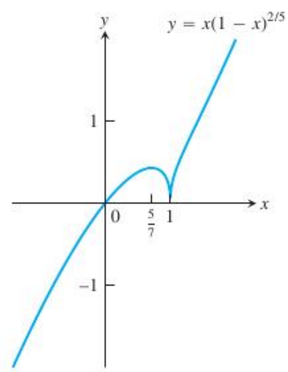




11. **Algebraic functions** are obtained by adding subtracting, multiplying, dividing or taking roots of polynomial functions. Rational functions are special cases of algebraic functions. Some graphs of alhebraic functions:







12. **Trigonometric functions** come from the ratios of sides of a right angled triangle. The angles are measured in radian. The trigonometric functions have a period. That is, $f(x + p) = f(x)$ happens for some $p > 0$. The **period** of $f(x)$ is the minimum of such p. The period for $\sin x$ is 2π.

The functions $\cos x$ and $\sec x$ are **even functions** and all others are **odd functions**. Recall that $f(x)$ is even if $f(-x) = f(x)$ and it is odd if $f(-x) = -f(x)$ for each x in the domain of the function. Some of the useful inequalities are as follows:

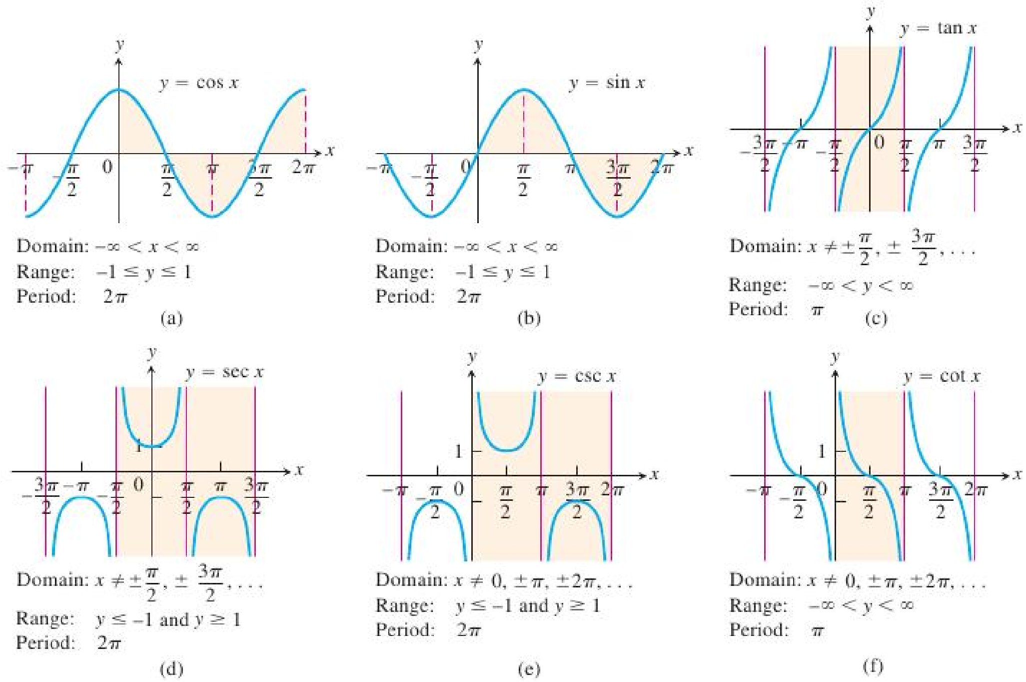
For each $x \in \mathbb{R}$,

$$-|x| \le \sin x \le |x|, \quad -1 \le \sin x,\ \cos x \le 1, \quad 0 \le 1 - \cos x \le |x|.$$

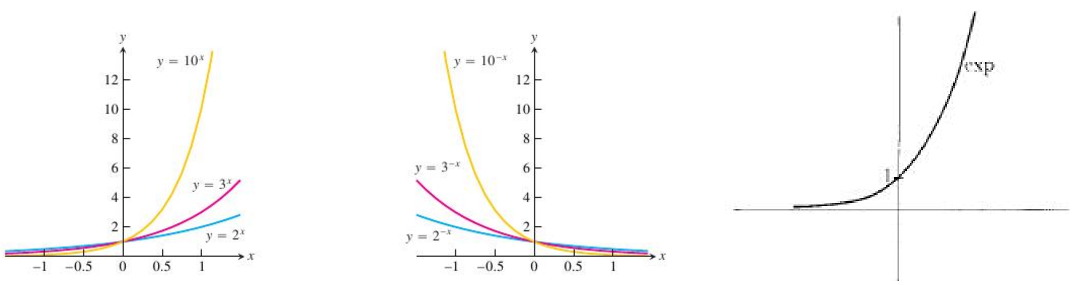
$$\sin x \le x \le \tan x \quad \text{for } x \in (0, \pi/2).$$

$$\sin x < |x| \quad \text{for } x \ne 0.$$

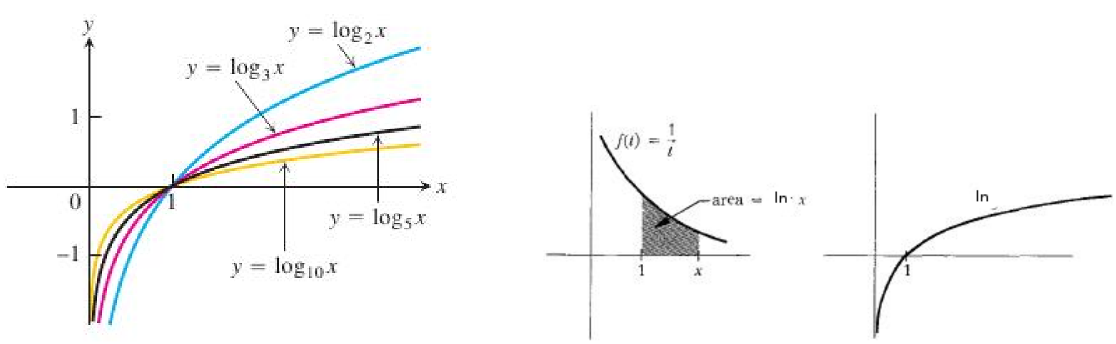
Graphs of the trigonometric functions are as follows:



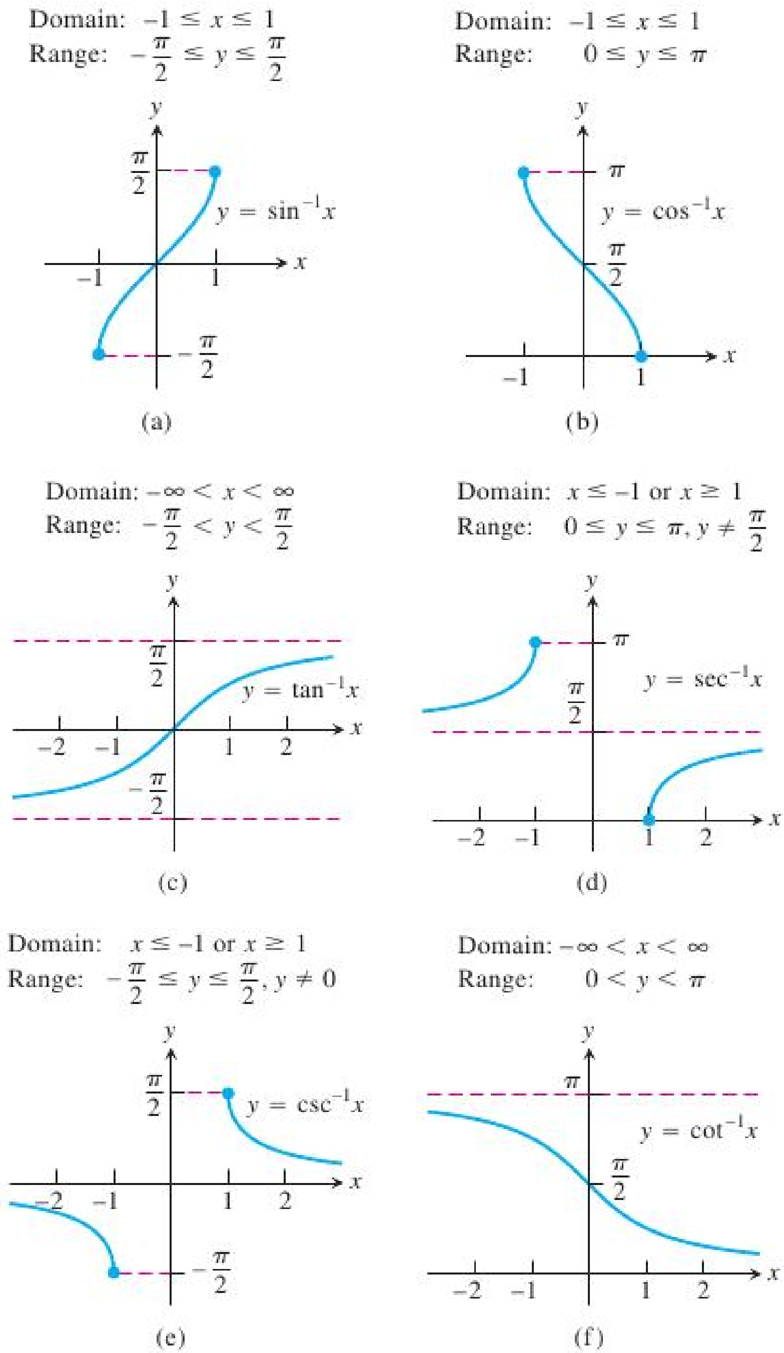
13. **Exponential functions** are in the form $y = a^x$ for some $a > 0$ and $a \neq 1$. All exponential functions have domain $(-\infty, \infty)$ and co-domain $(0, \infty)$. They never assume the value 0. Graphs of some exponential functions:



14. **Logarithmic functions** are inverse of exponential functions. That is, $a^{\log_a x} = \log_a(a^x) = x$. Some examples:



15. **Inverse trigonometric functions**:



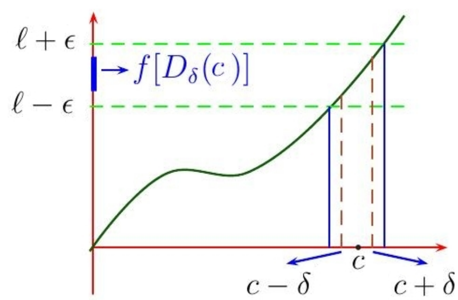
Functions that are not algebraic are called transcendental functions. Trigonometric functions, exponential functions, logarithmic functions and inverse trigonometric functions are examples of transcendental functions.

A.2 Concepts and Facts

Let $a < c < b$. Let $f : D \to R$ be a function whose domain D contains the union $(a, c) \cup (c, b)$. Let $\ell \in \mathbb{R}$. We say that **the limit of** $f(x)$ **as** x **approaches** c **is** ℓ and write it as

$$\lim_{x\to c} f(x) = \ell$$

iff for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each $x \in (a, c) \cup (c, b)$ with $0 < |x - c| < \delta$, we have $|f(x) - \ell| < \epsilon$.



Limit Properties: Let k be a constant; or a constant function.

1. $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$.
2. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$.
3. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$.
4. $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \ \lim_{x\to c} g(x)$.
5. $\lim_{x\to c}[f(x)/g(x)] = \left[\lim_{x\to c} f(x)\right] / \left[\lim_{x\to c} g(x)\right]$ if $\lim_{x\to c} g(x) \neq 0$.
6. $\lim_{x\to c}(f(x))^r = (\lim_{x\to c} f(x))^r$ if taking powers are meaningful.
7. $\lim_{x\to c} f(x)$ is a unique real number if it exists.
8. If $\lim_{x\to c} g(x) = 0$, and $\lim_{x\to c}[f(x)/g(x)]$ exists, then $\lim_{x\to c} f(x) = 0$.
9. (**Sandwich**) Let f, g, h be functions whose domain include $(a, c) \cup (c, b)$ for $a < c < b$. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \in (a, c) \cup (c, b)$. If $\lim_{x\to c} g(x) = \ell = \lim_{x\to c} h(x)$, then $\lim_{x\to c} f(x) = \ell$.
10. (**Domination**) Let f, g be functions whose domains include $(a, c) \cup (c, b)$ for $a < c < b$. Suppose that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. If $f(x) \leq g(x)$ for all $x \in (a, c) \cup (c, b)$, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.

Let I be (a, ∞) or $[a, \infty)$ for some $a \in \mathbb{R}$. Let $f : I \to \mathbb{R}$. Let $\ell \in \mathbb{R}$. We say that $\lim_{x\to\infty} f(x) = \ell$ if for each $\epsilon > 0$, there exists an $m > 0$ such that if x is any real number greater then m, then $|f(x) - \ell| < \epsilon$.

Let $f(x)$ have a domain containing (a, c). Then $\lim_{x \to c-} f(x) = \infty$ iff for each $m > 0$, there exists a $\delta > 0$ such that for every x with $c - \delta < x < c$, we have $f(x) > m$.

That is, $\lim_{x \to c-} f(x) = \infty$ iff, "as x increases to c, $f(x)$ increases without bound".

Let $f : D \to \mathbb{R}$ be a function. Let c be an interior point of D. We say that $f(x)$ is **continuous at** c if $\lim_{x \to c} f(x) = f(a)$.

If $D = [a, b)$ or $D = [a, b]$, then $f(x)$ is called continuous at the left end-point a if $\lim_{x \to a+} f(x) = f(a)$.

If $D = (a, b]$ or $D = [a, b]$, then $f(x)$ is called continuous at the right end-point b if $\lim_{x \to b-} f(x) = f(a)$.

$f(x)$ is called **continuous** if it is continuous at each point of its domain D.

The sum, multiplication by a constant, and product of continuous functions is continuous. In addition, the following are some properties of continuous functions:

1. Let $f(x)$ be continuous at $x = c$, where the domain of $f(x)$ includes a neighborhood of c. If $f(c) > 0$, then there exists a neighborhood $(c - \delta, c + \delta)$ such that $f(x) > 0$ for each point $x \in (c - \delta, c + \delta)$.
2. Let $f(x)$ be a continuous function, whose domain contains $[a, b]$ for $a < b$. Then there exist $\alpha,\ \beta \in \mathbb{R}$ such that $\{f(x) : x \in [a, b]\} = [\alpha, \beta]$.
3. *(Extreme Value Theorem)* Let $f(x)$ be continuous on a closed bounded interval $[a, b]$. Then there exist numbers $c, d \in [a, b]$ such that $f(c) \le f(x) \le f(d)$ for each $x \in [a, b]$.
4. *(Intermediate Value Theorem)* Let $f(x)$ be continuous on a closed bounded interval $[a, b]$. Let d be a number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = d$.

Let $f(x)$ be a function whose domain includes an open interval (a, b). Let $c \in (a, b)$. If the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists, we say that $f(x)$ is **differentiable at** $x = c$; and we write the limit as $f'(c)$ and call it the derivative of $f(x)$ at $x = c$. If $f'(c)$ exists for each $c \in (a, b)$, then we write $f'(x)$ as $\frac{df}{dx}$.

Also, derivative of f defined on a closed interval $[a, b]$ at the end-point a is taken as the left hand derivative, where in the defining limit of the derivative we take $h \to a-$. Similarly, derivative at b is taken as the limit of that ratio for $h \to 0+$.

Let $f(x)$ be a function defined on an interval I.
We say that $f(x)$ is **increasing on** I if for all $s < t \in I,\ f(s) < f(t)$.
Similarly, we say that $f(x)$ is **decreasing on** I if for all $s < t \in I,\ f(s) > f(t)$.
A **monotonic** function on I is one which either increases on I or decreases on I.

The sum, multiplication by a constant, and product of differentiable functions is differentiable. In addition, the following are some properties of differentiable functions:

1. Each function differentiable at $x = c$ is continuous at $x = c$.
2. Derivatives of Sum, product etc. are respectives equal to sum, product etc of derivatives.
3. *(Chain Rule)* $\frac{dg(f(x))}{dx} = \frac{dg(f(x))}{df(x)} \cdot \frac{df(x)}{dx}$.
4. *(Rolle's Theorem)* Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, $f(x)$ is differentiable on (a, b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.
5. *(Mean value Theorem)* Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x)$ is differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.
6. Let I be an interval containing at least two points. Let $f : I \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for each $x \in I$, iff $f(x)$ is a constant function.
7. *(Cauchy Mean Value Theorem)* Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ on (a, b), then there exists $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
8. *(L'Hospital's Rule)* Let $f(x)$ and $g(x)$ be differentiable on a neighborhood of a point $x = a$. Suppose $f(a) = g(a) = 0$ but $g(x) \neq 0,\ g'(x) \neq 0$ in the deleted neighborhood of $x = a$. If $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.
9. Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).

   (a) If $f'(x) > 0$ on (a, b), then $f(x)$ is increasing on $[a, b]$.

   (b) If $f'(x) < 0$ on (a, b), then $f(x)$ is decreasing on $[a, b]$.

Let a function $f(x)$ have domain D. The function $f(x)$ has a **local maximum** at a point $d \in D$ if $f(x) \leq f(d)$ for every x in some neighborhood of d contained in D. in such a case, we also say that the point $x = d$ is a **point of local maximum** of the function $f(x)$.

Similarly, $f(x)$ has an **local minimum** at $b \in D$ if $f(b) \leq f(x)$ for every x in some neighborhood of b contained in D. In this case, we say that the point $x = b$ is a **point of local minimum** of the function $f(x)$.

The points of local maximum and local minimum are commonly referred to as **local extremum points**; and the function is said to have **local extrema** at those points.

Let $f(x)$ have domain D. A point $c \in D$ is called a **critical point** of $f(x)$ if c is not an interior point of D, or if $f(x)$ is not differentiable at $x = c$, or if $f'(c) = 0$.

If $f(x)$ has an extremum at $x = c$, then c is a critical point of $f(x)$.

Test for Local Extrema:
Let c be an interior point of the domain of $f(x)$ with $f'(c) = 0$.

1. $f'(x)$ changes sign from + to − at $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.
2. If $f''(c) < 0$, then $x = c$ is a point of local maximum of $f(x)$.
3. $f'(x)$ changes sign from − to + at $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.
4. If $f''(c) > 0$, then $x = c$ is a point of local minimum of $f(x)$.

Let $x = c$ be a left end-point of the domain of $f(x)$.

1. $f'(x) < 0$ on the immediate right of $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.
2. $f'(x) > 0$ on the immediate right of $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

Let $x = c$ be a right end-point of the domain of $f(x)$.

1. $f'(x) > 0$ on the immediate left of $x = c$ iff $x = c$ is a point of local maximum of $f(x)$.
2. $f'(x) < 0$ on the left of $x = c$ iff $x = c$ is a point of local minimum of $f(x)$.

The graph of a function $y = f(x)$ is **concave up** on an open interval I if $f'(x)$ is increasing on I. The graph of $y = f(x)$ is **concave down** on an open interval I if $f'(x)$ is decreasing on I.
A **point of inflection** is a point where $y = f(x)$ has a tangent and the concavity changes.

Second derivative test for concavity:
Let $y = f(x)$ be twice differentiable on an interval I.

1. If $f''(x) > 0$ on I, then the graph of $y = f(x)$ is concave up over I.
2. If $f''(x) < 0$ on I, then the graph of $y = f(x)$ is concave down over I.
3. If $f''(x)$ is positive on one side of $x = c$ and negative on the other side, then the point $(c, f(c))$ on the graph of $y = f(x)$ is a point of inflection.

Let $f : [a, b] \to \mathbb{R}$. Divide $[a, b]$ into smaller sub-intervals by choosing the break points as

$$a = x_0 < x_1 < \ldots < x_n = b.$$

The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a **partition of** $[a, b]$.
Now P divides $[a, b]$ into n sub-intervals: $[x_0, x_1], \cdots, [x_{n-1}, x_n]$. Here, the kth sub-interval is $[x_{k-1}, x_k]$. The area under the curve $y = f(x)$ raised over the kth

sub-interval is approximated by $f(c_k)(x_k - x_{k-1})$ for some choice of the point $c_k \in [x_{k-1}, x_k]$.

Write the choice points (also called **sample points**) as a set $C = \{c_1, \ldots, c_n\}$.

Then the **Riemann sum**

$$S(f, P, C) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

is an approximation to the whole area raised over $[a, b]$ and lying between the curve $y = f(x)$ and the x-axis. By taking the **norm** of the partition as $\|P\| = \max_k (x_k - x_{k-1})$, we would say that when the norm of the partition approaches 0, the Riemann sum would approach the required area. Thus, we define the area of the region bounded by the lines $x = a,\ x = b,\ y = 0$, and $y = f(x)$ as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

provided that this limit exists. We define this limit (which is the mentioned area here) as the definite integral of f on the interval $[a, b]$. That is,

$$\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $\int_a^b f(x)\, dx$ exists.

The definite integral has the following properties:

(Properties of Definite Integral)

1. Let $f(x)$ have domain $[a, b]$. Let $c \in (a, b)$. Then $f(x)$ is integrable on $[a, b]$ iff $f(x)$ is integrable on both $[a, c]$ and $[c, b]$. In this case,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2. Let $f(x)$ and $g(x)$ be integrable on $[a, b]$. Then $(f + g)(x)$ is integrable on $[a, b]$ and

$$\int_a^b (f + g)(x)dx = \int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

3. Let $f(x)$ be integrable on $[a, b]$. Let $c \in \mathbb{R}$. Then $(cf)(x)$ is integrable on $[a.b]$ and

$$\int_a^b (cf)(x)\, dx = \int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx.$$

4. Let $f(x)$ and $g(x)$ be integrable on $[a, b]$. If for each $x \in [a, b]$, $f(x) \leq g(x)$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

5. Let $f(x)$ be integrable on $[a, b]$. If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

6. (*Average Value Theorem*) Let $f(x)$ be continuous on $[a, b]$. Then there exists $c \in [a, b]$ such that
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

7. Let $f(x)$ be continuous on $[a, b]$. If $f(x)$ has the same sign on $[a, b]$ and $\displaystyle\int_a^b f(x)\,dx = 0$, then $f(x)$ is the zero function, i.e., $f(x) = 0$ for each $x \in [a, b]$.

We extend the integral even when $a \not< b$ by the following:

1. If $a = b$, then we take $\displaystyle\int_a^b f(x)\,dx = 0$.
2. If $a > b$, then we take $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Also, for any real number c; even when c is outside the interval (a, b) we have
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In all these extensions, we assume that the definite integrals exist.

The main result that shows that differentiation and integration are reverse processes is the following:

(Fundamental Theorem of Calculus) Let $f(x)$ be continuous on $[a, b]$.

1. If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
2. The function $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

The chain rule for differentiation is translated to integration as follows. We call it the method of *Substitution*:

1. Let $u = g(x)$ be a differentiable function whose range is an interval I. Let $f(x)$ be continuous on I. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

2. Let $u = g(x)$ be a continuously differentiable function on $[a, b]$ whose range is an interval I. Let $f(x)$ be continuous on I. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

The rpoduct rule for differntiation gives the *integration by parts* formula.

$$\int f(x)h(x)\,dx = f(x)\int h(x)\,dx - \int \left[f'(x)\int h(x)\,dx\right] dx + C.$$

We remember it as follows (Read F as first and S as second):

Integral of $F \times S = F\times$ integral of $S -$ integral of (derivative of $F \times$ integral of S).

The *natural logarithm* $\ln x$ is defined as follows:

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0.$$

The *exponential function* is the inverse of the natural logarithm. That is,

$$\exp : R \longrightarrow (0, \infty); \quad y = \exp(x) \quad \text{iff} \quad x = \ln y.$$

Since $\exp(x)\ \exp(y) = \exp(x + y)$ and $\exp(0) = 1$, we write

$$\exp(x) = e^x, \quad \text{where} \quad e = \exp(1).$$

Then hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

Notice that $\cosh^{-1}$ has domain as $x \geq 1$ and $\tanh^{-1}$ has domain as $-1 < x < 1$.

Let C be a curve given parametrically by $x = f(t), y = g(t),\ a \leq t \leq b$. Assdume that both $f(t)$ and $g(t)$ are continuously differentiable.

$$\text{Length of the curve} = L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt.$$

If the curve is given as a function $y = f(x),\ a \le x \le b$, then take $x = t$ and $y = f(t)$ as its parameterization. We then have the length as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + (y')^2}\, dx.$$

Notice that this formula is applicable when $f'(x)$ is continuous on $[a, b]$.

We write $L = \int_a^b ds$ with limits a and b for the variable of integration, which may be $x,\ y$ or t. Here,

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Suppose that a curve is given in polar coordinates by $r = f(\theta)$ for a continuous function $f(\theta)$, where $\alpha \le \theta \le \beta$. Then the area of the sector and the arc length of the curve are

$$\text{Area } = \int_\alpha^\beta r^2\, d\theta, \quad \text{Length } = \int_\alpha^\beta \sqrt{r^2 + (r')^2}\, d\theta.$$

A.3 Formulas

Here are some formulas for the exponential and the logarithm functions:

$$\lim_{t\to\infty} \frac{t^p}{a^t} = 0 \text{ for } p \in \mathbb{N} \text{ and } a > 1.$$

$$\ln e = 1 = e^0, \quad e^{\ln x} = x, \quad \ln(e^x) = x, \quad a^x = e^{x \ln a},$$

$$\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0+} \ln x = -\infty, \quad \lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0,$$

$$(\ln x)' = \frac{1}{x}, \quad (e^x)' = e^x, \quad (a^x)' = (\ln a)a^x, \quad \int_1^e \frac{1}{t}\, dt = 1, \quad \int e^x\, dx = e^x.$$

$$\lim_{h\to 0} \frac{\ln(1 + xh)}{xh} = 1 \text{ for } x \ne 0, \quad \lim_{h\to 0} \frac{e^h - 1}{h} = 1, \quad \lim_{h\to 0} (1 + xh)^{1/h} = e^x,$$

Below are given some integrals, which are obtained from the derivatives by following the simple rule that if $g'(x) = f(x)$, then $\int f(x)\, dx = g(x) + C$.

1. $\displaystyle\int u\, dv = uv - \int v\, du$
2. $\displaystyle\int a^u\, du = \frac{a^u}{\ln a} + C, \quad a \ne 1,\ a > 0$

3. $\displaystyle\int \cos u\, du = \sin u + C$

4. $\displaystyle\int \sin u\, du = -\cos u + C$

5. $\displaystyle\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq 1$

6. $\displaystyle\int (ax+b)^{-1}\, dx = \frac{1}{a}\ln|ax+b| + C$

7. $\displaystyle\int x(ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a^2}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, \quad n \neq -1,\ -2$

8. $\displaystyle\int x(ax+b)^{-1}\, dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$

9. $\displaystyle\int x(ax+b)^{-2}\, dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

10. $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

11. $\displaystyle\int \left(\sqrt{ax+b}\right)^n dx = \frac{2}{a}\frac{\left(\sqrt{ax+b}\right)^{n+2}}{n+2} + C, \quad n \neq -2$

12. $\displaystyle\int \frac{\sqrt{ax+b}}{x}\, dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

13. $\displaystyle\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C$

14. $\displaystyle\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C$

15. $\displaystyle\int \frac{\sqrt{ax+b}}{x^2}\, dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} + C$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{b}{2a}\int \frac{dx}{x\sqrt{ax+b}} + C$

17. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

18. $\displaystyle\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

19. $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

20. $\displaystyle\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{x+a}{x-a}\right| + C$

21. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C = \ln\left(x + \sqrt{a^2+x^2}\right) + C$

22. $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+C$

23. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\ln\left(x+\sqrt{a^2+x^2}\right)+C$

24. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|+C$

25. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = \ln\left(x+\sqrt{a^2+x^2}\right) - \frac{\sqrt{a^2+x^2}}{x}+C$

26. $\displaystyle\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = -\frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + \frac{x\sqrt{a^2+x^2}}{2}+C$

27. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|+C$

28. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x}+C$

29. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$

30. $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

31. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{8}\sin^{-1}\frac{x}{a} - \frac{x}{8}\sqrt{a^2-x^2}\,(a^2-2x^2)+C$

32. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|+C$

33. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x}+C$

34. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}+C$

35. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|+C$

36. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}+C$

37. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}+C = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

38. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

39. $\displaystyle\int \left(\sqrt{x^2-a^2}\right)^n dx = \frac{x\left(\sqrt{x^2-a^2}\right)^n}{n+1} - \frac{na^2}{n+1}\int \left(\sqrt{x^2-a^2}\right)^{n-2} dx, \quad n \neq 1$

40. $\displaystyle\int \frac{dx}{\left(\sqrt{x^2-a^2}\right)^n} = \frac{x\left(\sqrt{x^2-a^2}\right)^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2}\int \frac{dx}{\left(\sqrt{x^2-a^2}\right)^{n-2}}, \quad n \neq 2$

41. $\displaystyle\int x^2\left(\sqrt{x^2-a^2}\right)^n dx = \frac{\left(\sqrt{x^2-a^2}\right)^{n+2}}{n+2} + C, \quad n \neq -2$

42. $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

43. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$

44. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| - \frac{\sqrt{x^2-a^2}}{x} + C$

45. $\displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{x}{2}\sqrt{x^2-a^2} + C$

46. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$

47. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

48. $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

49. $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

50. $\displaystyle\int \left(\sqrt{2ax-x^2}\right)^n dx = \frac{(x-a)\left(\sqrt{2ax-x^2}\right)^n}{n+1} + \frac{na^2}{n+1}\int \left(\sqrt{2ax-x^2}\right)^{n-2} dx$

51. $\displaystyle\int \frac{dx}{\left(\sqrt{2ax-x^2}\right)^n} = \frac{(x-a)\left(\sqrt{2ax-x^2}\right)^{2-n}}{(n-2)a^2} + \frac{n-3}{(n-2)a^2}\int \frac{dx}{\left(\sqrt{2ax-x^2}\right)^{n-2}}$

52. $\displaystyle\int x\sqrt{2ax-x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax-x^2}}{6} + \frac{a^3}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

53. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\left(\frac{x-a}{a}\right) + C$

54. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$

55. $\displaystyle\int \frac{x\,dx}{\sqrt{2ax-x^2}} = a\sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{2ax-x^2} + C$

56. $\displaystyle\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\sqrt{\frac{2a-x}{x}} + C$

57. $\displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$

58. $\displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$

59. $\displaystyle\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$

60. $\displaystyle\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C$

61. $\displaystyle\int \sin^n(ax)\,dx = -\frac{\sin^{n-1}(ax)\cos(ax)}{na} + \frac{n-1}{n}\int \sin^{n-2}(ax)\,dx$

62. $\displaystyle\int \cos^n(ax)\,dx = \frac{\cos^{n-1}(ax)\sin(ax)}{na} + \frac{n-1}{n}\int \cos^{n-2}(ax)\,dx$

63. $\displaystyle\int \sin(ax)\,\cos(bx)\,dx = -\frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} + C, \quad a^2 \neq b^2$

64. $\displaystyle\int \sin(ax)\,\sin(bx)\,dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)} + C, \quad a^2 \neq b^2$

65. $\displaystyle\int \cos(ax)\,\cos(bx)\,dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)} + C, \quad a^2 \neq b^2$

66. $\displaystyle\int \sin(ax)\,\cos(ax)\,dx = -\frac{\cos(2ax)}{4a} + C$

67. $\displaystyle\int \sin^n(ax)\,\cos(ax)\,dx = \frac{\sin^{n+1}(ax)}{(n+1)a} + C, \quad n \neq -1$

68. $\displaystyle\int \frac{\cos(ax)}{\sin(ax)}\,dx = \frac{1}{a}\ln|\sin(ax)| + C$

69. $\displaystyle\int \cos^n(ax)\,\sin(ax)\,dx = -\frac{\cos^{n+1}(ax)}{(n+1)a} + C, \quad n \neq -1$

70. $\displaystyle\int \frac{\sin(ax)}{\cos(ax)}\,dx = -\frac{1}{a}\ln|\cos(ax)| + C$

71. $\displaystyle\int \cos^m(ax)\,\sin^n(ax)\,dx = -\frac{\cos^{m+1}(ax)\sin^{n-1}(ax)}{a(m+n)} + \frac{n-1}{m+n}\int \cos^m(ax)\sin^{n-2}(ax)\,dx, \quad n \neq -m$

72. $\displaystyle\int \cos^m(ax)\,\sin^n(ax)\,dx = -\frac{\cos^{m-1}(ax)\sin^{n+1}(ax)}{a(m+n)} + \frac{m-1}{m+n}\int \cos^{m-2}(ax)\sin^n(ax)\,dx, \quad n \neq -m$

73. $\displaystyle\int \frac{dx}{b + c\sin(ax)} = \frac{-2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\,\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right] + C, \quad b^2 > c^2$

74. $\displaystyle\int \frac{dx}{b + c\sin(ax)} = \frac{-1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c + b\sin(ax) + \sqrt{c^2-b^2}\cos(ax)}{b + c\sin(ax)}\right| + C, \quad b^2 < c^2$

75. $\displaystyle\int \frac{dx}{1 + \sin(ax)} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

76. $\displaystyle\int \frac{dx}{1 - \sin(ax)} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + C$

77. $$\int \frac{dx}{b + c\cos(ax)} = \frac{2}{a\sqrt{b^2 - c^2}} \tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\ \tan\frac{ax}{2}\right] + C, \quad b^2 > c^2$$

78. $$\int \frac{dx}{b + c\cos(ax)} = \frac{1}{a\sqrt{c^2 - b^2}} \ln\left|\frac{c + b\cos(ax) + \sqrt{c^2 - b^2}\sin(ax)}{b + c\cos(ax)}\right| + C, \quad b^2 < c^2$$

79. $$\int \frac{dx}{1 + \cos(ax)} = \frac{1}{a}\tan\frac{ax}{2} + C$$

80. $$\int \frac{dx}{1 - \cos(ax)} = -\frac{1}{a}\cot\frac{ax}{2} + C$$

81. $$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) + C$$

82. $$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

83. $$\int x^n\sin(ax)\,dx = -\frac{x^n}{a}\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx$$

84. $$\int x^n\cos(ax)\,dx = \frac{x^n}{a}\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx$$

85. $$\int \tan(ax)\,dx = \frac{1}{a}\ln|\sec(ax)| + C$$

86. $$\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$$

87. $$\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x + C$$

88. $$\int \cot^2(ax)\,dx = -\frac{1}{a}\cot(ax) - x + C$$

89. $$\int \tan^n(ax)\,dx = \frac{\tan^{n-1}(ax)}{a(n-1)} - \int \tan^{n-2}(ax)\,dx, \quad n \neq 1$$

90. $$\int \cot^n(ax)\,dx = -\frac{\cot^{n-1}(ax)}{a(n-1)} - \int \cot^{n-2}(ax)\,dx, \quad n \neq 1$$

91. $$\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax) + \tan(ax)| + C$$

92. $$\int \operatorname{cosec}(ax)\,dx = -\frac{1}{a}\ln|\operatorname{cosec}(ax) + \cot(ax)| + C$$

93. $$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + C$$

94. $$\int \operatorname{cosec}^2(ax)\,dx = -\frac{1}{a}\cot(ax) + C$$

95. $$\int \sec^n(ax)\,dx = \frac{\sec^{n-2}(ax)\tan(ax)}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2}(ax)\,dx, \quad n \neq 1$$

96. $\displaystyle\int \operatorname{cosec}^{n}(ax)\,dx = -\frac{\operatorname{cosec}^{n-2}(ax)\cot(ax)}{a(n-1)} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2}(ax)\,dx, \quad n \neq 1$

97. $\displaystyle\int \sec^{n}(ax)\,\tan(ax)\,dx = \frac{\sec^{n}(ax)}{na} + C, \quad n \neq 0$

98. $\displaystyle\int \operatorname{cosec}^{n}(ax)\,\cot(ax)\,dx = -\frac{\operatorname{cosec}^{n}(ax)}{na} + C, \quad n \neq 0$

99. $\displaystyle\int \sin^{-1}(ax)\,dx = x\sin^{-1}(ax) + \frac{1}{a}\sqrt{1-a^2x^2} + C$

100. $\displaystyle\int \cos^{-1}(ax)\,dx = x\cos^{-1}(ax) - \frac{1}{a}\sqrt{1-a^2x^2} + C$

101. $\displaystyle\int \tan^{-1}(ax)\,dx = x\tan^{-1}(ax) - \frac{1}{2a}\ln(1+a^2x^2) + C$

102. $\displaystyle\int x^n \sin^{-1}(ax)\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}(ax) - \frac{a}{n+1}\int \frac{x^{n+1}}{\sqrt{1-a^2x^2}}\,dx, \quad n \neq -1$

103. $\displaystyle\int x^n \cos^{-1}(ax)\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}(ax) + \frac{a}{n+1}\int \frac{x^{n+1}}{\sqrt{1-a^2x^2}}\,dx, \quad n \neq -1$

104. $\displaystyle\int x^n \tan^{-1}(ax)\,dx = \frac{x^{n+1}}{n+1}\tan^{-1}(ax) - \frac{a}{n+1}\int \frac{x^{n+1}}{\sqrt{1+a^2x^2}}\,dx, \quad n \neq -1$

105. $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

106. $\displaystyle\int b^{ax}\,dx = \frac{b^{ax}}{a\ln b} + C, \quad b > 0,\ b \neq 1$

107. $\displaystyle\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$

108. $\displaystyle\int x^n\,e^{ax}\,dx = \frac{1}{a}x^n\,e^{ax} - \frac{n}{a}\int x^{n-1}\,e^{ax}\,dx$

109. $\displaystyle\int x^n\,b^{ax}\,dx = \frac{x^n\,b^{ax}}{a\ln b} - \frac{n}{a\ln b}\int x^{n-1}\,b^{ax}\,dx, \quad b > 0,\ b \neq 1$

110. $\displaystyle\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\,\sin(bx) - b\,\cos(bx)\big) + C$

111. $\displaystyle\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\,\cos(bx) + b\,\sin(bx)\big) + C$

112. $\displaystyle\int \ln(ax)\,dx = x\ln(ax) - x + C$

113. $\displaystyle\int x^m\,\big(\ln(ax)\big)^n\,dx = \frac{x^{m+1}\big(\ln(ax)\big)^n}{m+1} - \frac{n}{m+1}\int x^m\,\big(\ln(ax)\big)^{n-1}\,dx, \quad m \neq -1$

114. $\displaystyle\int x^{-1}\big(\ln(ax)\big)^n\,dx = \frac{\big(\ln(ax)\big)^{n+1}}{n+1} + C, \quad n \neq -1$

115. $\displaystyle\int \frac{dx}{x\,\ln(ax)} = \ln\big|\ln(ax)\big| + C$

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