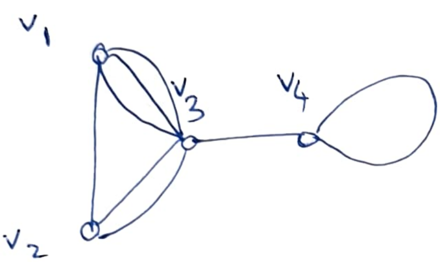
Using adjacency matrices to represent (finite) graphs:

Example: $G := (V, E)$



$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & & & & \\ v_2 & & & & \\ v_3 & & & & \\ v_4 & & & & \end{array} \quad \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Adjacency Matrix A_G

$|V| \times |V|$ matrix

if vertices are labeled using $\{1, 2, \ldots, n\}$

then $A_{ij} := \#$ of edges with ends i & j

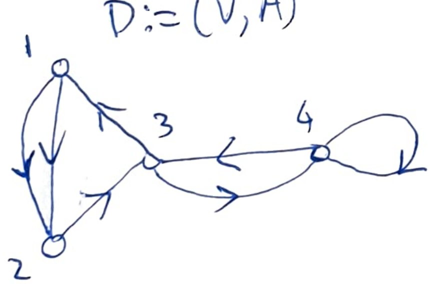
Observe that A is a symmetric matrix.

$A = A^T$

$A_{ij} = A_{ji}$ for all pairs i, j

Using adjacency matrices to represent (finite) digraphs:

$D := (V, A)$



$$A_D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

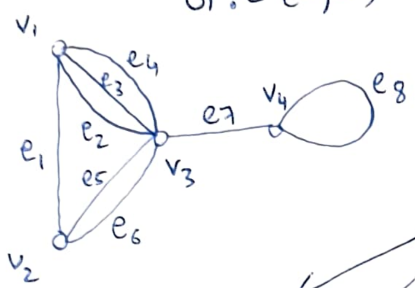
$A_D \rightarrow$ Need NOT be a symmetric matrix

if vertices are labeled using $\{1, 2, \ldots, n\}$

then $A_{ij} := \#$ of arcs with tail i & head j

Using incidence matrices to represent (finite) graphs:

$G := (V, E)$



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & ? \end{array}\right] \end{array}$$

$|V| \times |E|$ matrix

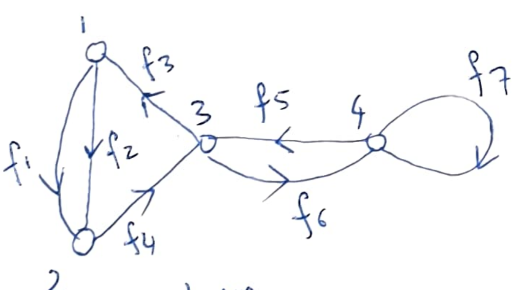
1? or 2?

TIY: How can you decide the degree of each vertex by looking at the incidence matrix?

For each edge e that is NOT a loop: two 1's in the corresponding column — one for each end of edge e; 0 everywhere else

Using incidence matrices to represent (finite) digraphs:

$D := (V, A)$



$|V| \times |A|$ matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \end{array} \\ \left[\begin{array}{ccccccc} -1 & -1 & +1 & 0 & 0 & 0 & 0 \\ +1 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +1 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 & ? \end{array}\right] \end{array}$$

-1? or $+1$?

For each arc (column $\updownarrow$ f_i) that is NOT a loop,

-1 in the row corresponding to tail of f;

$+1$ in the row corresponding to head of f;

0 everywhere else.

Walks, Trails & Paths in Graphs:

Given a graph $G := (V, E)$, a $\boxed{\text{walk}}$ is any sequence of vertices and edges, say $v_1 e_1 v_2 e_2 v_3 \ldots\ldots\ldots v_{k-1} e_{k-1} v_k$, such that each edge e_i (in this sequence) has ends v_i & v_{i+1}.

Trail & Path are defined similarly but with additional restrictions:

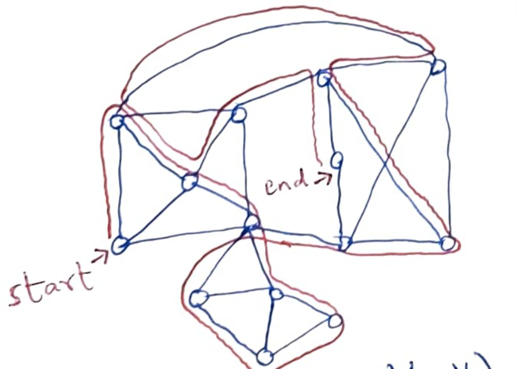
Additional restrictions:	Walk	Trail	Path
Repetition of vertices allowed?	YES	YES	NO
Repetition of edges allowed?	YES	NO	NO

Remarks: (think why)

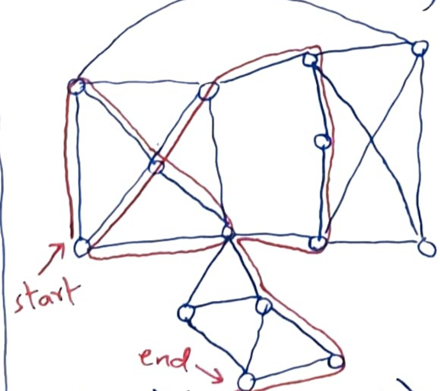
① In any graph G, sequence of edges can be used to get the entire sequence.

② In a simple graph, sequence of vertices can be used to get the entire sequence.

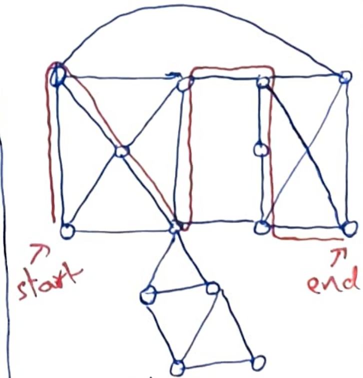
Examples: (carefully follow the red line from start to end)



walk (but NOT trail/path)



trail (but NOT path)



path

DIY: Write down definitions of trails & paths (using the previous discussion).

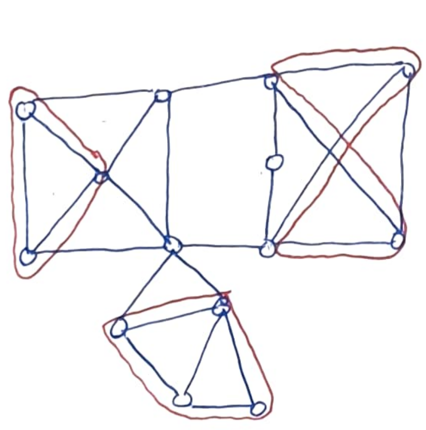
In particular, any walk/trail/path is a sequence $v_1, e_1, v_2, \ldots, e_{k-1}, v_k$ (which satisfies some properties/rules).

v_1 → start vertex; v_k → end vertex

If $\underbrace{\text{start vertex is same as end vertex}}_{\text{NOT possible in path}}$, we call it a closed walk or closed trail.

Cycle: same definition as path but $v_1 = v_k$ (= is same as).

Example: 3 cycles shown in red color



Recall Euler's Problem: which graphs can be drawn without lifting the chalk? (Rules: ① each vertex can be thought of as a point. ② No edge should be drawn twice or more.)

A trail is called an $\boxed{\text{Eulerian trail}}$ if each edge appears once (and exactly once - by definition of trail).

Euler's Problem: which graphs have an Eulerian trail?