Reachability relation (in Graphs)

$G := (V, E)$: graph

$u, w \in V(G)$

We say that $\boxed{w \text{ is reachable from } u}$
(or that $\boxed{u \text{ is able to reach } w}$) if
there is a walk in G that starts at u & ends at w.

Let's discuss properties of this relation:

(1) Is this relation reflexive?

Technical point: a single vertex is a walk!

But, more intuitively, it makes sense for a vertex to be reachable from itself, right?

Okay, so it is reflexive. ✓

(2) Is this relation symmetric?

Well, if $\underset{u}{\overset{\shortparallel}{v_1}} e_1 v_2 e_2 \ldots\ldots v_{k-1} e_{k-1} \underset{w}{\overset{\shortparallel}{v_k}}$ is a walk from u to w

then the reverse sequence is a walk from w to u, right?

Okay, so it is symmetric. ✓

③ Is this relation transitive?

Suppose that u is able to reach w
and w is able to reach y

Then is y reachable from u?

Intuitively: yes, right?

↓

just go from u to w, and then go from w to y.

↓

How do we formalize this?

↓

Let φ_1 be a walk from u to w
and φ_2 be a walk from w to y.

↓

Now, we walk from u to w using φ_1, and
then we walk from w to y using φ_2;
the resulting sequence is a walk from u to y.
(Right?)

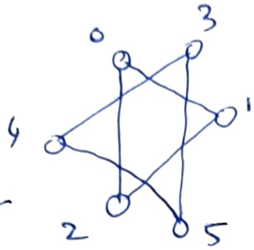
quod erat demonstrandum	Latin
"which was to be demonstrated"	

We have proved our first theorem:

Theorem: The reachability relation (on any graph) is
an equivalence relation. ▨

(\qed in LaTeX)

Example:



G: graph on 6 vertices and 6 edges

(each straight line segment joining 2 vertices is an edge)

Equivalence classes: $\{0,1,2\}, \{3,4,5\}$

Connected components:

Connected Components of a Graph: $G := (V, E)$

Let $\{V_1, V_2, \ldots, V_k\}$ be the partition of the vertex set V induced by the reachability relation (on V).

For any V_i, the graph with vertex set V_i and all edges (of G) with both ends in V_i is called a **connected component of G**. Thus G has k connected components.

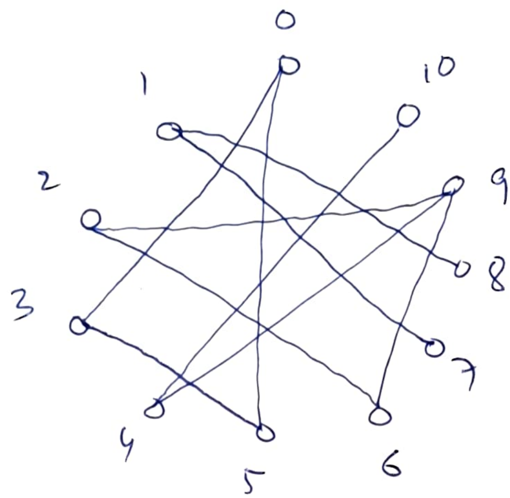
If $k = 1$, we say that G is a **connected graph**.

If $k \geq 2$, we say that G is a **disconnected graph**.

Another example:

$G := (V, E)$

(each straight line segment joining 2 vertices is an edge)



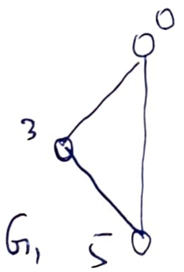
partition of V induced by the reachability relation:

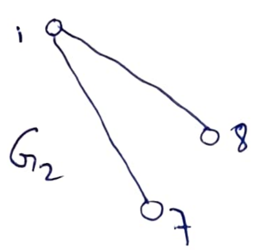
$$\{\overset{V_1}{\{0,3,5\}}, \overset{V_2}{\{1,7,8\}}, \overset{V_3}{\{2,4,6,9,10\}}\}$$

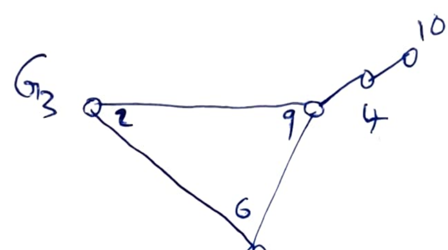
corresponding connected component

corresponding connected components

corresponding connected components







In this example, the graph G has 3 connected components G_1, G_2, G_3.

So: G is disconnected graph.

the drawing does NOT matter, it is just a representation of the graph.

Clearly, each connected component of G is also a graph (and is of course a connected graph).

Let's return to Euler's Problem:

Which graphs can be drawn without lifting the chalk? (isolated vertices NOT allowed)

"almost" same as

Which graphs have an Eulerian trail? (isolated vertices allowed)

why "almost"?

A vertex v of a graph G is an **isolated vertex** if $\{\{v\}, \emptyset\}$ is a connected component of G.

But this is a minor detail!

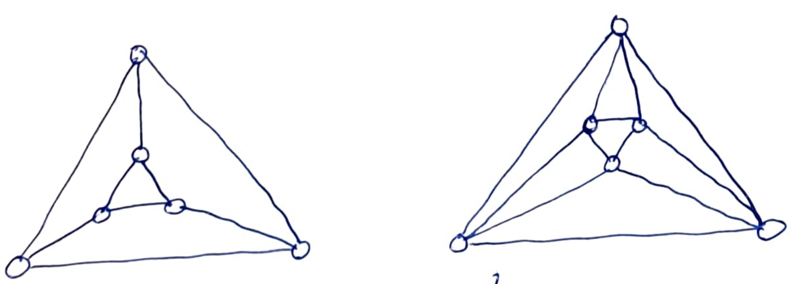
(Right?)

Euler's Problem (rephrased):

Which connected graphs have an Eulerian trail?

Clearly, All connected graphs do NOT have an Eulerian trail.

Remember these graphs?



which of them has an Eulerian trail?