What exactly do we mean by partitioning a set?

May appear on Assignment-2.

A $\boxed{\text{partition of a set } S}$ is a collection of pairwise disjoint nonempty subsets of S that have S as their union.

$\boxed{\text{pairwise disjoint}}$: any two are disjoint

TIY: Convince yourself. (we will write a formal proof in Module-2.)

<u>Theorem</u>: Let R be an equivalence relation defined on a set U. Then the equivalence classes of R form a partition of U.

Let's define this clearly.

<u>Definition</u>: Let R be an equivalence relation on a set U.

For any element a of U,

the $\boxed{\text{equivalence class (w.r.t. } R) \text{ of } a}$, denoted by $[a]_R$,

is the set of all elements that are related to a.

(<u>same as</u>: "that a is related to")

why? (because of symmetry)

<u>For example</u>: if $U = \mathbb{N}$ and R is the parity ($\equiv$) relation:

$[17]_R$ = set of all odd natural #s.

$[38]_R$ = set of all even natural #s

<u>Example</u>: if $U = \mathbb{N}$ and R is the equality ($=$) relation:

$[17]_R = \{17\}$ and $[38]_R = \{38\}$.

To summarize: $\leq, \geq, |$ are NOT equivalence relations, on $\mathbb{N}$

whereas $\equiv$ (and $=$) are equivalence relations.

Observe that $\leq, \geq$ & $|$ are transitive. (Right?)

also, reflexive. reflexive on positive integers.

So, $\leq, \geq$ & $|$ are NOT equivalence relations

mainly because they are NOT symmetric!

In fact, if $a \leq b$ (and if $a \neq b$) then $b \nleq a$.

(same is true for $\geq$ and $|$).

Special Property 4: Antisymmetry

U: universe

R: a relation (defined on U)

We say that R is $\boxed{\text{antisymmetric}}$

(or that R $\boxed{\text{satisfies antisymmetry}}$).

if (for any two distinct members a & b of U) (means $a \neq b$)

at most one of aRb & bRa holds.

For example: $\leq, \geq, |$ (on $\mathbb{N}$) are antisymmetric.

($\equiv$ is NOT antisymmetric.)

DIY: Can a relation be both symmetric & antisymmetric?

If so, what is an example of such a relation?

If NOT, why NOT?

Observe that, on the set $\mathbb{N}-\{0\}$, the relations

$\leq$, $\geq$ & $|$ are reflexive, anti-symmetric & transitive.

Question: However, there is one key (important) difference between $\leq$ & $\geq$ AND $|$. What is that?

Hint: It is related to antisymmetry.

Answer:

For any two distinct members a & b of $\mathbb{N}-\{0\}$ exactly one of $a \leq b$ & $b \leq a$ holds; but this is NOT true for $|$. (For example: $3 \nmid 7$ & $7 \nmid 3$)

In other words, any two distinct elements can be compared using $\leq$ (and also using $\geq$); however, if we use $|$, some elements are incomparable.

$\boxed{\text{Partial Order}}$: a relation that is reflexive, antisymmetric & transitive.

why partial? because some pairs may be incomparable.

$\boxed{\text{Total Order}}$: Partial order where any two elements are comparable

Let's write down the key definitions so far:

U: some set (maybe universe; doesn't matter)

R: a relation defined on U

We say that R is an $\boxed{\text{equivalence relation}}$ if R is reflexive, symmetric & transitive.

We say that R is a $\boxed{\text{partial order}}$ (aka order) if R is reflexive, antisymmetric & transitive.

We say that a partial order R is a $\boxed{\text{total order}}$ if for any two distinct elements a & b (of U) exactly one of aRb & bRa holds. (aka linear order)

---

another way of writing this

For a partial order R (defined on a set U), we say that two distinct elements a & b (of U) are $\boxed{\text{comparable}}$ if either aRb or bRa holds.

both can NOT hold by definition of partial order, right?

A $\boxed{\text{total order}}$ is a partial order with the additional property that any two distinct elements are comparable.

Commonly used notation for equivalence relation: $\boxed{\sim}$, $\approx$, $\equiv$

we will use this to refer to an arbitrary (in general) equivalence relation.

Commonly used notation for partial order: $\precsim$

(like $\leq$ but curved)

Notation: $a \prec b$ means that $a \precsim b$ AND that a & b are distinct elements.

Definition: A set U with a partial order $\precsim$ defined on U, is called a $\boxed{\text{partially ordered set}}$ (abbreviated to $\boxed{\text{poset}}$), and is denoted by $(U, \precsim)$. Members of U are called $\boxed{\text{elements of the poset}}$ $(U, \precsim)$.

For example, $(\mathbb{N}-\{0\}, |)$ is a poset whereas $(\mathbb{N}, \leq)$ & $(\mathbb{N}, \geq)$ are totally ordered sets.

Definition: A poset $(U, \precsim)$ is called a $\boxed{\text{totally ordered set}}$ if $\precsim$ is a total order (on U).

(aka linearly ordered set)
(aka chain)

abbreviated to toset?
Almost no one does that >.<