Deep dive into posets:

Let us "draw" the poset $(\{1,2,3,\dots,15\}, |)$:

↓

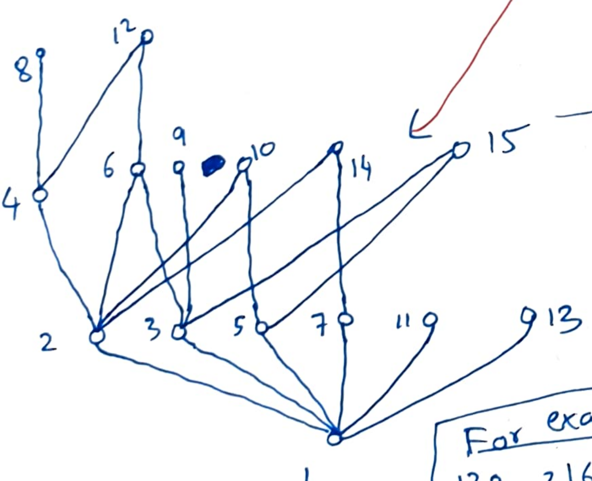
I mean: draw the corresponding digraph

↓

will have too many arcs!

Do we need ALL of them?

No; consider the following drawing:



For example:

2|6

6|12

So, 2|12.

This drawing (called Hasse diagram of given poset) has sufficient information to infer the entire poset.

How to read this drawing?

- Since partial orders are reflexive, imagine a loop at each element.

- Think of each line segment pointing upwards. (anti-symmetry)

For example: 2|6

- Use transitivity to infer all other relations.

We have seen an example of a Hasse Diagram, and we have seen how to infer the entire poset from its Hasse Diagram.

Let us formalize this:

Given any poset $(S, \preceq)$, we will use the following terminology (borrowed from $\leq$ relation as defined for numbers):

Notation:	read as:	also read as:
$a \preceq b$	a is less than or equal to b	b is greater than or equal to a
$a \prec b$	a is less than b	b is greater than a
meaning ① a & b are distinct ② $a \preceq b$		Example continued: on the other hand, $2 \nmid 15$ and $15 \nmid 2$. We say that 2 and 15 are $\boxed{\text{incomparable}}$.

For example: Consider the poset $(\mathbb{N} - \{0\}, |)$:

$3 \mid 15$ — so, 3 is less than or equal to 15
— 15 is greater than or equal to 3

Since 3 & 15 are distinct, 3 is less than 15, and 15 is greater than 3.

For two distinct elements a & b of a poset $(S, \preceq)$, we say that $\boxed{a \text{ \& } b \text{ are incomparable}}$ if $a \npreceq b$ and $b \npreceq a$.

To summarize: for a poset $(S, \preceq)$ and distinct $a, b \in S$, exactly one of the following 3 possibilities holds:

1) $a \preceq b \Rightarrow a \prec b$ (since they are distinct is given)
2) $b \preceq a \Rightarrow b \prec a$ (since they are distinct is given)
3) neither $a \preceq b$ nor $b \preceq a$.

$\Updownarrow$ means

$a \npreceq b$ and $b \npreceq a$

we say that $\boxed{a \text{ \& } b \text{ are incomparable}}$.

Now, given a poset $(S, \preceq)$, we define a new relation on S (called the $\boxed{\text{immediate predecessor}}$ relation):

For distinct $a, b \in S$, we say that

$\boxed{a \text{ is an immediate predecessor of } b}$ if:

① $a \prec b$ and

② $\nexists\ c \in S$ such that $a \prec c \prec b$.

read as $\rightarrow$ there does NOT exist any element ....

Notation: $a \lhd b$

Hasse diagram for a poset $(S, \preceq)$ can now be defined as follows:

It is the digraph that represents the immediate precedence ($\triangleleft$) relation on S;

furthermore, if $a \triangleleft b$ then we put a below b in the drawing and imagine all "edges" pointing upwards;

this can be done because $\triangleleft$ is also anti-symmetric (why?).

Theorem: Let $(S, \preceq)$ denote any finite poset, and let $\triangleleft$ denote the corresponding immediate predecessor relation.

finite poset means: S is finite

Then for any two distinct $a, b \in S$:

$a \preceq b$ if and only if $\exists\ a_1, a_2, \ldots, a_k \in S$ such that $a \triangleleft a_1 \cdots\cdots \triangleleft a_k \triangleleft b$.

(we allow $k=0$. In this case: $a \triangleleft b$)

which of these implications ($\Rightarrow$ & $\Leftarrow$) is easier?

Proof of ($\Leftarrow$): Assume $\exists\ a_1, a_2, \ldots, a_k \in S$ such that $a \triangleleft a_1 \triangleleft \cdots\cdots \triangleleft a_k \triangleleft b$. By definition of $\triangleleft$, it follows that $a \preceq a_1 \preceq a_2 \cdots\cdots \preceq a_k \preceq b$. By transitivity of $\preceq$, $a \preceq b$. ∎

(we will prove ($\Rightarrow$) later.)