Department of Mathematics, IIT Madras
MA1102 Series & Matrices
Solutions to Assignment-5 (Matrix Eigenvalue Problem)

1. Find the eigenvalues and the associated eigenvectors for the matrices given below.

   (a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 3 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

   (d) Call the matrix A. Its characteristic polynomial is $-(2+t)(3-t)(5-t)$.
   So, the eigenvalues are $\lambda = -2,\ 3,\ 5$.
   For $\lambda = -2$, $A(a,b,c)^T = -2(a,b,c)^T \Rightarrow -2a+3c = -2a,\ -2a+3b = -2b,\ 5c = -2c$.
   One of the solutions for $(a,b,c)^T$ is $(5,2,0)^T$. It is an eigenvector for $\lambda = -2$.
   For $\lambda = 3$, $A(a,b,c)^T = 3(a,b,c)^T \Rightarrow -2a+3c = 3a,\ -2a+3b = 3b,\ 5c = 3c$.
   One of the solutions for $(a,b,c)^T$ is $(0,1,0)^T$. It is an eigenvector for $\lambda = 3$.
   For $\lambda = 5$, $A(a,b,c)^T = 5(a,b,c)^T \Rightarrow -2a+3c = 5a,\ -2a+3b = 5b,\ 5c = 5c$.
   One of the solutions for $(a,b,c)^T$ is $(3,-3,7)^T$. It is an eigenvector for $\lambda = 5$.
   Similarly, solve others.

2. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.
   If each row sums to α, then $A(1,1,\ldots,1)^T = \alpha(1,1,\ldots,1)^T$. Thus α is an eignevlaue with an eigenvector as $(1,1,\ldots,1)^T$.
   If each column sums to α, then each row sums to α in A^T. Thus A^T has an eigenvalue as α. However, A^T and A have the same eigenvalues. Thus α is also an eigenvalue of A.

3. Let $A \in \mathbb{C}^{n\times n}$ be invertible. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.
   Since A is invertible, its determinant is nonzero. As $\det(A)$ is the product of eigenvalues of A, no eigenvalue of A is 0.
   Also, for any nonzero λ, $Av = \lambda v$ iff $\lambda^{-1}A^{-1}Av = \lambda^{-1}A^{-1}\lambda v$ iff $\lambda^{-1}v = A^{-1}v$.
   This shows that λ is an eigenvalue of A iff λ^{-1} is an eigenvlaue of A^{-1}.

4. Show that eigenvectors corresponding to distinct eigenvalues of a unitary (or orthogonal) matrix are orthogonal to each other.
   Let α and β be distinct eigenvalues of a unitary matrix A with corresponding eigenvectors x and y. That is, we have: $A^*A = AA^* = I,\ Ax = \alpha x,\ Ay = \alpha y,\ x \neq 0,\ y \neq 0$ and $\alpha \neq \beta$.
   We need to show that $x \perp y$. Now,
   $(Ax)^*(Ay) = (\alpha x)^*(\beta y) \Rightarrow x^*A^*Ay = \overline{\alpha}\beta x^*y \Rightarrow \big(\overline{\alpha}\beta - 1\big)x^*y = 0.$
   Since A is unitary, any eigenvalue of A has absolute value 1.
   So, $|\alpha|^2 = 1 \Rightarrow \alpha\,\overline{\alpha} = 1 \Rightarrow \overline{\alpha} = 1/\alpha$.
   Then $\big(\overline{\alpha}\beta - 1\big)x^*y = 0 \Rightarrow \big(\beta/\alpha - 1\big)x^*y = 0 \Rightarrow (\beta - \alpha)x^*y = 0.$
   Since $\alpha \neq \beta$, we get $x^*y = 0$. That is, $x \perp y$.

5. Give an example of an $n \times n$ matrix that cannot be diagonalized.
   Take $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ with $a_{12} = 1$ and all other entries as 0. Its eigenvalue is 0 with

algebraic multiplicity as n. If A is diagonalizable, then A is similar to the zero matrix. But the only matrix similar to the zero matrix is the zero matrix!

6. Find the matrix $A \in \mathbb{R}^{3\times 3}$ that satisfies the given condition. Diagonalize it if possible.

   (a) $A(a,b,c)^T = (a+b+c,\ a+b-c,\ a-b+c)^T$ for all $a,b,c \in \mathbb{R}$.

   (b) $Ae_1 = 0, \quad Ae_2 = e_1, \quad Ae_3 = e_2$.

   (c) $Ae_1 = e_2, \quad Ae_2 = e_3, \quad Ae_3 = 0$.

   (d) $Ae_1 = e_3, \quad Ae_2 = e_2, \quad Ae_3 = e_1$.

   (a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Its characteristic polynomial is $-(t+1)(t-2)^2$.

   So, eigenvalues are -1 and 2. Solving $A(a,b,c)^T = \lambda(a,b,c)^T$ for $\lambda = -1, 2$, we have
   $\lambda = -1\ :\ a+b+c = -a,\ a+b-c = -b,\ a-b+c = -c \Rightarrow a = -c, b = c$.
   Thus a corresponding eigenvector is $(-1,1,1)^T$.
   $\lambda = 2\ :\ a+b+c = 2a,\ a+b-c = 2b,\ a-b+c = 2c \Rightarrow a = b+c$.
   Thus two linearly independent corresponding eigenvectors are $(1,1,0)^T$ and $(1,0,1)^T$.

   Take the matrix $P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then verify that $P^{-1}AP = diag(-1,2,2)$.

   (b) The eigenvalue 0 has algebraic multiplicity 3. If it is diagonalizable, then it is similar to 0. But the only matrix similar to 0, is 0. So, A is not diagonalizable.

   (c) Similar to (b).

   (d) Proceed as in (a) to get $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ and verify $P^{-1}AP = diag(-1,1,1)$.

7. Which of the following matrices is/are diagonalizable? If one is diagonalizable, then diagonalize it.

   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

   (a) It is a real symmetric matrix; so diagonalizable. Its eigenvalues are $-2, -1, 2$. Also since the 3×3 matrix has three distinct eigenvalues, it is diagonalizable.
   Proceed like 6(a).

   (b) 1 is an eigenvalue with algebraic multipliity 3. If it is diagonalizable, then it is similar to I. But the only matrix similar to I is I. Hence, it is not diagonalizable.

   (c) Its eigenvalues are 2, $(1 \pm \sqrt{3}\, i)/2$. Since three distinct eigenvalues; it is diagonalizable. Here, P will be a complex matrix. Proceed as in 6(a).

   (d) $(1,0,-1)^T$ and $(1,-1,0)^T$ are two linearly independent eigenvectors associated with the eigenvalue -1.
   $(1,1,1)^T$ is an eigenvector for the eigenvalue 2.

   Hence taking $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, we have $P^{-1}AP = diag(-1,-1,2)$.