PH-1020
Problem Set - 3
Department of Physics, IIT Madras
Field in Matter
March-June 2023 Semester

Notation:

- Notation throughout follows that of Griffiths, Electrodynamics.
- Bold face characters, such as $\boldsymbol{v}$, represent three-vectors.
- The suggested question (Q. 5) may not be discussed in the tutorial class. Students are encouraged to solve on their own or in group. The solution will be provided later.

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1. Consider a dielectric medium with uniform polarisation $\boldsymbol{P}_0$. A spherical cavity is removed from inside this medium.
   (a) Find the bound surface charge density on the surface of the cavity.
   (b) Find the electric field at the center of the cavity due to the above surface charge.
   (c) Find the *total* electric field within the cavity, if the electric field in the bulk of the material is $\boldsymbol{E_0}$.
   (d) Find the displacement vector $\boldsymbol{D}$ everywhere.
   (e) Verify that the normal component of $\boldsymbol{D}$ satisfies the correct boundary condition at the surface of the cavity.

2. Consider a cylinder of length $2L$ and radius a, centred at the origin, and with it's symmetry axis being the z-axis. The cylinder carries a uniform polarisation $\boldsymbol{P} = P_0\hat{\boldsymbol{z}}$, with $P_0 =$ constant.
   (a) Find the bound charge densities ρ_b and σ_b.
   (b) Find the electric field on the positive z-axis. Check that it satisfies appropriate boundary condition at $z = L$.
   (c) Sketch the magnitude of the electric field at the origin as a function of a/L.

3. A linear, but inhomogeneous, dielectric is inserted between plates of a parallel plate capacitor with the plates kept at $x = 0$ and $x = d$ (and lying in the $y - z$ plane), and carrying charge density $\sigma/(-\sigma)$ on the left/right plates. The relative permittivity is given by

$$\epsilon_r = 1 + \alpha \left(\frac{x}{d}\right)^2 \tag{1}$$

   (a) Find the displacement vector $\boldsymbol{D}$, electric field $\boldsymbol{E}$, and polarization $\boldsymbol{P}$ in this system.
   (b) Sketch the magnitudes of $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{P}$ as functions of x.
   (c) Calculate the bound surface and volume charge densities everywhere.

(d) Calculate the capacitance C of this system. Verify that you obtain the correct $\alpha \to 0$ limit.

4. (a) Consider a spherical capacitor with inner radius r_a and outer radius r_c.The total charge at the outer surface ($r = r_c$) is $-Q$ and at the inner surface ($r = r_a$) is Q. The space between these two surfaces is filled with a linear dielectric material with dielectric constant ϵ_{r_1} out to radius r_b, $r_a < r_b < r_c$, and another material with dielectric constant ϵ_{r_2} from r_b to r_c.

   i. Find the displacement vector $\boldsymbol{D}$ everywhere.

   ii. Calculate the total bound charges on each of the boundaries. Hence verify the boundary conditions for normal component of the *electric field* at the interface of the two dielectrics.

   iii. Find the total capacitance of this system.

   (b) Repeat problem 4 above with

$$\epsilon_r = 1 + \alpha \left(\frac{x}{d} \right) \tag{2}$$

Suggested Question

5. (a) Consider a sphere of dielectric material characterized by dielectric constant $\epsilon_r = 1 + \chi_e$, put into a homogeneous field $\boldsymbol{E_0}$. (For example, you may put the sphere between the parallel plates of a capacitor, and assume the plates are so far away from the sphere that the sphere does not affect the charge distribution on the plates significantly. Obviously, the presence of the dielectric sphere will distort the electric field, and the electric field outside will therefore no longer be $\boldsymbol{E_0}$ except at large distance from the sphere.) Show that:

   i. the electric field inside the sphere is given by

$$\boldsymbol{E_{in}} = \left(\frac{3}{\epsilon_r + 2} \right) \boldsymbol{E_0} \tag{3}$$

   ii. the sphere in this case acquires a uniform polarization given by

$$\boldsymbol{P} = 3\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) \boldsymbol{E_0} \tag{4}$$

   Note: You will need the following result, which you can use without proof for the sake of this problem (you are, of course, encouraged to derive it from expression for the potential): The potential inside a sphere centered at the origin, with surface charge density $\sigma(\theta) = k \cos\theta$, is given by

$$V(r, \theta) = \frac{k}{3\epsilon_0} r \cos\theta \tag{5}$$

   Calculate the resultant electric field (the expression should look familiar from one of the previous tutorials!), and take it from there.

(b) i. Let a dielectric consist of N atoms per unit volume, and let p be the dipole moment per atom. The atomic polarizability α is defined by $\boldsymbol{p} = \alpha \boldsymbol{E_0}$, where $\boldsymbol{E_0}$ is a specified external field (that is, total electric field minus the field produced by the dipole itself). Using the result above, show that:

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \tag{6}$$

This important relation between atomic polarizability and dielectric constant is known as the *Clausius-Mossotti* formula. It also goes by the name of *Lorentz-Lorenz* equation in optics, where it relates the refractive index of a substance to it's polarizability.

ii. Show that, for $N\alpha/\epsilon_0 << 1$, the above relation reduces to

$$\epsilon_r - 1 \approx \frac{N\alpha}{\epsilon_0} \tag{7}$$