DEPARTMENT OF PHYSICS
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PH1020 Physics II **Problem Set 6 - Solutions** **MAR-JUN 23**

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1. A square loop of wire with side length a lies at a distance s from a very long straight wire carrying current I. Assume the loop and the wire on the same plane.

   (a) Find the magnetic flux through the loop.

   (b) When the loop moves directly away from the wire, at a speed v, calculate the emf generated. Also indicate the direction of the current flow in the loop.

   (c) What will happen when the loop moves with speed v in the direction of the current flow?

   (d) Evaluate the mutual inductance between the wire and the loop.

   Solution:

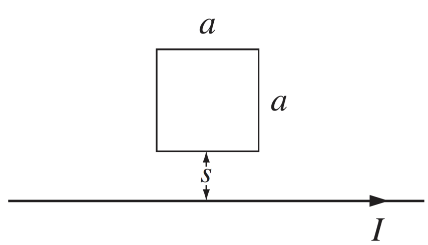

Figure 1: A square loop of side length a is kept at a distance s from an infinite wire carrying current I.

(a) Magnetic field at a distance s above the wire is

$$|\mathbf{B}| = \frac{\mu_0 I}{2\pi s}$$

in a direction out of the page. In order to find out the total flux through the square loop , let's first consider a thin rectangular strip at a distance y from the wire. The magnetic flux through the thin strip would be

$$d\Phi_B = \frac{\mu_0 I}{2\pi y}\, a\, dy$$

Therefore the net magnetic flux through the square loop is

$$\begin{aligned}\Phi_B &= \int d\Phi_B \\ &= \int_s^{s+a} \frac{\mu_0 I}{2\pi y}\, a\, dy \\ &= \boxed{\frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{s}\right)}\end{aligned}$$

(b) The induced emf

$$\begin{aligned}
\mathcal{E} &= -\frac{d\Phi_B}{dt} \\
&= -\frac{d}{dt}\left(\frac{\mu_0 I a}{2\pi}\ln\left(\frac{s+a}{s}\right)\right) \\
&= -\frac{\mu_0 I a}{2\pi}\,\frac{s}{s+a}\left(\frac{-a}{s^2}\right)\frac{ds}{dt} \\
&= \boxed{\frac{\mu_0 I a^2 v}{2\pi s(s+a)}}
\end{aligned}$$

The field value decreases as we move away from the wire so as the out of the page flux. The induced current would oppose the process and try to increase the out of the page flux. Therefore, the induced current will flow $\boxed{\textit{counterclockwise}}$.

(c) When the loop moves parallel to the current flow in the wire the net magnetic flux does not change with time. Hence, the induced emf would be zero.

(d) From part (a) we have,

$$\Phi_B = \left(\frac{\mu_0 a}{2\pi}\ln\left(\frac{s+a}{s}\right)\right) I$$

Following equation (7.22) of [1], i.e. $\Phi_2 = M_{21} I_1$, we conclude the mutual inductance in this case,

$$\boxed{M = \frac{\mu_0 a}{2\pi}\ln\left(\frac{s+a}{s}\right)}$$

2. Consider a square loop of wire with side length a lying in a region with uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. Initially if the square loop is co-planar to the $xy-$ plane and rotates with respect to either x-axis or y-axis with an angular speed ω, plot the induced emf as a function of time.

Solution:
It is given that $\mathbf{B} = B_0\hat{\mathbf{z}}$. Let the loop rotates about the $\hat{\mathbf{y}}$ and $\theta(t)$ be the angle between the loop and the $xy-$ plane at time t. Then, the magnetic flux would be,

$$\Phi_B = B_0 a^2 \cos\theta$$

Therefore, the induced emf would be,

$$\begin{aligned}
\mathcal{E} &= -\frac{d\Phi_B}{dt} \\
&= B_0 a^2 \omega \sin\omega t
\end{aligned}$$

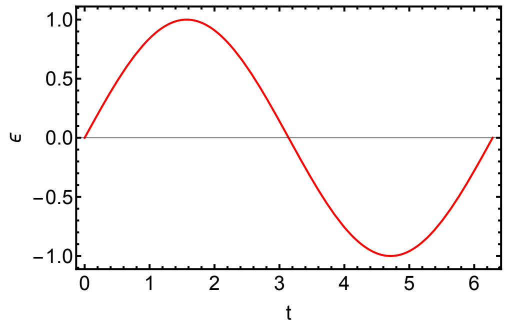

Figure 2: Plot of induced emf vs time for a single rotation of the loop. x axis is in the units of ω and y axis is in the units of $B_0 a^2 \omega$

3. (a) Determine the self-inductance per unit length of a long coaxial cable of inner radius a and outer radius b. The region between the inner and the outer conductor is filled with a linear magnetic material with relative permeability μ_r.

   (b) Calculate the energy stored (per unit length) in this inductor if a current I flows along the inner conductor and returns along the outer conductor.

   (c) Calculate the power transported down the cables when both the conductors are held at a potential difference V.

Solution:

(a) Let the cable runs along $\hat{\mathbf{z}}$. The inner radius is a and outer radius is b. We will use Ampere's law for $\mathbf{H}$ and then find B. From equation (6.20) of [1],

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\mathrm{f}_{enc}}$$

$$\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad (a < s < b)$$

Therefore,

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H} = \mu_0 \mu_r \frac{I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad (a < s < b)$$

Magnetic flux per unit length,

$$\begin{aligned} \frac{d\Phi_B}{dz} &= \mu_0 \mu_r \frac{I}{2\pi} \int_a^b \frac{1}{s} ds \\ &= \frac{\mu_0 \mu_r}{2\pi} \ln \frac{b}{a} I \end{aligned}$$

Therefore, self inductance per unit length,

$$L = \frac{\mu_0 \mu_r}{2\pi} \ln \frac{b}{a}$$

(b) Energy stored per unit length,

$$E = \frac{1}{2}LI^2 = \frac{\mu_0\mu_r}{4\pi}\ln\left(\frac{b}{a}\right)I^2$$

(c) The power (P) transported down the cables when both the conductors are held at a potential difference V can be written as,

$$P = VI$$

4. The electric and magnetic fields are given by

$$\mathbf{E}(r,t) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\hat{\mathbf{r}}, \qquad \mathbf{B}(r,t) = 0,$$

Where

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Show that these fields satisfy the four Maxwell's equation. Find the charge density ρ and the current density $\mathbf{J}$. What kind of physical situation gives rise these fields?

Solution:
It is given,

$$\mathbf{E}(r,t) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\hat{\mathbf{r}}, \qquad \mathbf{B}(r,t) = 0,$$

Let's verify the Maxwell's equation one by one and determine the corresponding charge density, ρ and current density, $\mathbf{J}$.

- $\underline{\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}}$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= -\frac{q}{4\pi\epsilon_0}\boldsymbol{\nabla}\cdot\left(\frac{}{r^2}\theta(vt-r)\hat{\mathbf{r}}\right) \\
&= -\frac{q}{4\pi\epsilon_0}\left[\theta(vt-r)\,\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2}\cdot\boldsymbol{\nabla}(\theta(vt-r))\right] \\
&= -\frac{q\delta^3(\mathbf{r})}{\epsilon_0}\theta(vt-r) + \frac{q\delta(vt-r)}{4\pi\epsilon_0 r^2}
\end{aligned}$$

Therefore, $\boxed{\rho(\mathbf{r}) = -q\delta^3(\mathbf{r})\theta(vt-r) + \frac{q\delta(vt-r)}{4\pi r^2}}$

- $\underline{\boldsymbol{\nabla}\cdot\mathbf{B} = 0}$ This is trivial as $\mathbf{B}(r,t) = 0$

- $\underline{\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}}$ [1]

$$\nabla \times \mathbf{E} = 0 = -\frac{\partial \mathbf{B}}{\partial t}$$

- $\underline{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$
  It is obvious to see that, $\nabla \times \mathbf{B} = 0$. We will use this to find the current distribution.

$$\begin{aligned}
\frac{\partial \mathbf{E}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{-q}{4\pi\epsilon_0}\frac{1}{r^2}\theta(vt-r)\hat{\mathbf{r}}\right) \\
&= \frac{-q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\,\delta(vt-r)\,v
\end{aligned}$$

  Therefore,

$$\begin{aligned}
\mu_0 \mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} &= 0 \\
\mathbf{J} &= \frac{q}{4\pi r^2} v\delta(vt-r)\hat{\mathbf{r}}
\end{aligned}$$

5. A transformer has two coils wrapped around an iron core in such a way that the same flux passes through every turn of both coils. The primary coil has n_1 turnsand the secondary coil has n_2 turns.

   (a) If the current I_1 through the primary coil is changing, what is the ratio of the emfs in the primary and secondary coils.

   (b) Show that, $M^2 = L_1 L_2$ where M is the mutual inductance and L_1,L_2 are individual self inductances.

   Solution:

   (a) Let Φ_B be the magnetic flux due to $\mathbf{B}$ through each turn of either coil. Therefore,

$$\Phi_1 = n_1\Phi_B, \quad \text{and } \Phi_2 = n_2\Phi_B$$

   and

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{-n_1\frac{d\Phi_B}{dt}}{-n_2\frac{d\Phi_B}{dt}} = \frac{n_1}{n_2}$$

[1] In spherical polar coordinate $\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$

(b) Suppose current I_1 and I_2 passes through the primary and secondary coil respectively. Then,

$$\Phi_1 = I_1 L_1 + I_2 M = n_1 \Phi_B$$
$$\Phi_2 = I_2 L_2 + I_1 M = n_2 \Phi_B$$

Therefore,

$$\Phi_B = I_1 \frac{L_1}{n_1} + I_2 \frac{M}{n_1} = I_2 \frac{L_2}{n_2} + I_1 \frac{M}{n_2}$$

In case $I_1 = 0$, we have $\frac{M}{n_1} = \frac{L_2}{n_2}$ and in case $I_2 = 0$ then, $\frac{L_1}{n_1} = \frac{M}{n_2}$. Dividing these two results we get $M^2 = L_1 L_2$

6. Consider a solenoid whose axis of symmetry is along z axis and a constant current I runs through it. A metal coin is allowed to fall under gravity along the axis of the solenoid. Find the induced emf and hence, the force on the coin due to the magnetic interaction when the coin is at a distance b from the top of the solenoid. (Neglect the mutual inductance for simplicity.)

   (Useful parameters: radius of the coin is a, thickness of the coin is τ, conductivity of the coin is σ, cross-sectional area of the solenoid is A. You can neglect the mutual inductance)

   Solution:

   Let's first analyze the problem qualitatively. Consider the current in the solenoid is counter-clockwise when one look from the top. As the coin approaches the solenoid upward magnetic flux through the coin will increase. Hence, it will induce a current in the coin clockwise which in result exert an upward force opposite to the gravity. We can see it as a drag force and at the end we will be able to find exactly the drag coefficient.

   The idea to approach the problem is as follows: we consider an infinitesimal ring at a radial distance r. We will find the induced emf on the ring and then the force on the same. Then we will integrate from 0 to the radius of the coin.

   The induced emf in the infinitesimal ring at a distance r, (following equation (7.11) of [1])

$$\mathcal{E}_r = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = 2\pi r v(t) B_\rho(r, b).$$

   where $v(t) = \frac{dz}{dt}$ (the relative speed of the coin w.r.t. the solenoid) and $B_\rho(r,t)$ is the radial component of the magnetic field due to the solenoid. Notice, the vertical

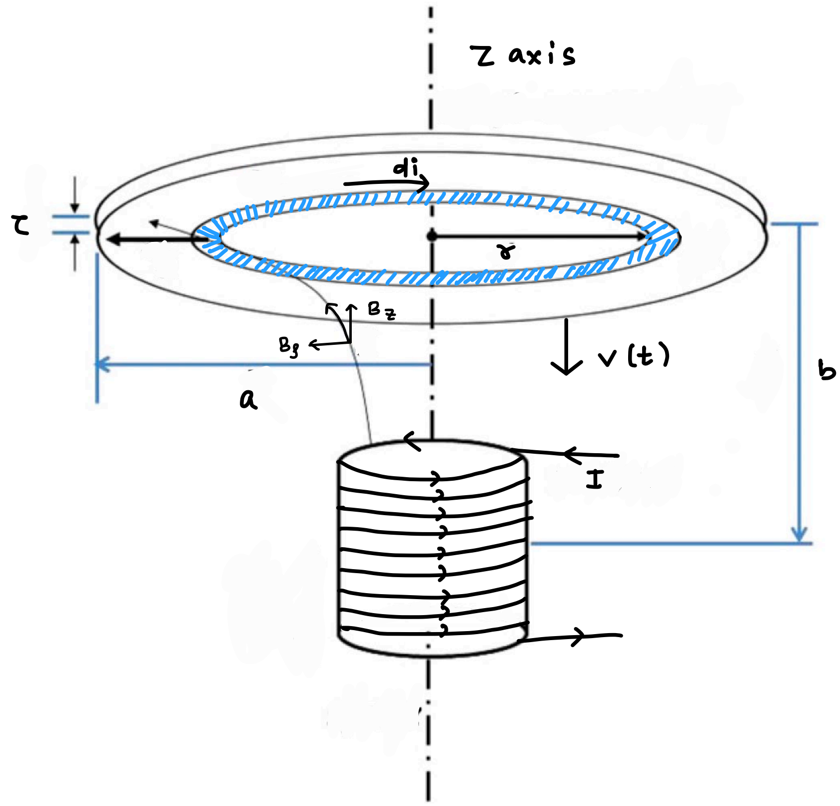

Figure 3:

component of the magnetic field due to the solenoid will not contribute to the induced emf.

The radial component of the magnetic field can be written as (refer Appendix A):

$$B_\rho(r,b) = \frac{\mu_0 m}{4\pi}\frac{3rb}{(r^2+b^2)^{5/2}}$$

where $m = NIA$, N is the total turns in the solenoid.

Now, the current in the ring due to this induced emf would be,

$$\begin{aligned} dI &= \mathcal{E}_r dG && (dG \quad \text{is the conductance of the infinitesimal ring}) \\ &= (2\pi r v(t) B_\rho(r,b))\left(\sigma\tau\frac{dr}{2\pi r}\right) && \left(dG = \sigma\tau\frac{dr}{2\pi r}, \quad \text{refer Appendix B}\right) \\ &= v(t)\sigma\tau\ B_\rho(r,b)dr \end{aligned}$$

The magnetic interaction force,

$$
\begin{aligned}
F_z &= \int I(d\mathbf{l} \times \mathbf{B}) \\
&= 2\pi v(t)\sigma\tau \int_0^a B_\rho^2(r,b)\ r\,dr \\
&= 2\pi v(t)\sigma\tau \int_0^a \left(\frac{\mu_0 m}{4\pi}\frac{3rb}{(r^2+b^2)^{5/2}}\right)^2 r dr \\
&= 2\pi v(t)\sigma\tau \left(\frac{9b^2\mu_0^2 m^2}{16\pi^2}\right)\int_0^a \frac{r^3}{(r^2+b^2)^5} dr && [\text{substitute } u = r/b] \\
&= \left[\frac{3}{4}\pi\sigma\tau\left(\frac{\mu_0 m}{4\pi}\right)^2 g\left(\frac{a}{b}\right)\right] v(t) && \left[g(x) = 1 - \frac{1+4x^2}{(1+x^2)^4}\right]
\end{aligned}
$$

Here you see $F \propto v$, which acts like a drag force. We can call $\frac{3}{4}\pi\sigma\tau\left(\frac{\mu_0 m}{4\pi}\right)^2 g\left(\frac{a}{b}\right)$ as the drag coefficient. So, the coin will fall under gravity with a drag force form the magnetic interaction. Given the numbers, one can work out the exact point where the coin stops. Students are encouraged to study the system considering the inductance. In that case, instead of a purely resistive circuit the coin will behave like an LR circuit. Further, one can study the dynamics for a sinusoidal current instead of a constant current through the solenoid.

Appendix A:

The components of the magnetic filed of a magnetic dipole in spherical polar coordinates (r, θ, ϕ) can be written as, (refer equation (5.88) of [1])

$$
B_r = \frac{2\mu_0 m}{4\pi}\frac{\cos\theta}{r^3}, \qquad B_\theta = \frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^3}
$$

Now, expressing the same result in cylindrical polar coordinates (ρ, ϕ, z), where $\rho = r\sin\theta$ and $z = r\cos\theta$, we have

$$
\begin{aligned}
B_\rho &= B_r \sin\theta + B_\theta \cos\theta \\
&= \frac{2\mu_0 m}{4\pi}\frac{\cos\theta\sin\theta}{r^3} + \frac{\mu_0 m}{4\pi}\frac{\sin\theta\cos\theta}{r^3} \\
&= \frac{\mu_0 m}{4\pi r^3}\left(\frac{2\rho z}{r^2} + \frac{\rho z}{r^2}\right) && \left(\sin\theta\cos\theta = \frac{\rho z}{r^2}\right) \\
&= \frac{\mu_0 m}{4\pi}\frac{3\rho z}{(\rho^2+z^2)^{5/2}} && (r = \sqrt{\rho^2+z^2})
\end{aligned}
$$

Appendix B:

The conductance of a material is,

$$G = \frac{1}{R} = \sigma \frac{A}{L}$$

where, R is the resistance, A is the area of cross-section perpendicular to the current flow, and L is the length along the current flow. Similarly, the infinitesimal conductance of the ring (the shaded strip in the Fig 3)

$$\begin{aligned} dG &= \sigma \frac{dA}{L} \\ &= \sigma \frac{\tau dr}{2\pi r} \end{aligned}$$

You can cut the shaded strip and think of it as a pipe of length $2\pi r$ with tiny rectangular cross-section τdr.

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.