1. Let R be a homogeneous relation on the set $S = \{1, 2, 3, 4, 5, 6\}$.
   $R = \{(a, b) \mid a, b \in S, a$ and b are coprimes.$\}$.
   (a) What are the properties of R?
   (b) Draw the graph representing the relation R.

2. Consider the set $S = \{1, 2, \ldots, n\}$ and the homogeneous relation $R_<$ (as defined in Assignment-1) on S. Let D be a directed graph representing R and A be the adjacency matrix representing D.
   $$f(n) = \sum_{i=0}^{n}\sum_{j=0}^{n} A_{ij} \text{ and } g(n) = \sum_{i=0}^{n} A_{ii}$$
   Find a closed-form expression for $f(n)$ and $g(n)$.

3. Consider the adjacency matrix A_G for a graph G and the incidence matrix B_H for a graph H.
   $$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } B_H = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$
   (a) Draw the graphs G and H.
   (b) The *complement* of a graph G (denoted by $\overline{G}$) is a graph on the same set of vertices as of G such that there will be an edge between two vertices (u, v) in $\overline{G}$, whenever there is no edge (u, v) in G for all $u \neq v$.
   Construct the complement of graph G and write the adjacency and incidence matrix of $\overline{G}$.
   (c) Find the matrix C such that $A_{\overline{G}} = C - A_G$. What kind of graph does C represent?
   (d) Construct $\overline{\overline{G}}$. What is the relation between G and $\overline{\overline{G}}$?

4. (a) Consider the homogeneous relation $R = \{(a, b) \mid sign(a) = sign(b)\}$ on the set $S = \{-2, -1, 0, 1, 2\}$. The $sign$ function is defined as follows.
   $$sign(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$
   (i) What kind of a relation is R?
   (ii) Draw the graph G representing the relation R.
   (iii) How many connected components are there in G?
   (iv) What do these connected components represent in terms of R?
   (b) In general, given the graph of an equivalence relation, what does any connected component in the graph correspond to in the relation?

5. A vertex whose removal produces a graph with more connected components is called a *cut vertex*.
   (a) The maximum number of cut-vertices in a graph with n vertices, where $n \geq 2$, is $n - 2$.
   Draw a graph with 8 vertices such that it has the maximum number of cut vertices.

(b) Given a graph G, a graph H is obtained from G by deleting one of the cut vertices of G. What is the maximum number of connected components in H?

(c) Given a graph G with n vertices, we remove all the edges from G to obtain a graph H. What is the number of connected components in H?

6. (a) Construct a graph G with exactly one walk.

   (b) Find the number of walks, paths, trails and cycles for the simple complete graph K_3.

7. (a) Let D be a digraph with at least one arc and for each vertex $indegree = outdegree$. Show that there is at least one cycle in the underlying undirected graph of D.

   (b) Let H be a digraph such that for each vertex $indegree \neq outdegree$. Determine whether the following statements are true or false. If true, explain why; if false, give a concrete example to illustrate your point.

      i) H has at least one cycle.

      ii) H has no cycles.

8. Given a graph G, an *Eulerian tour* is defined as a trail containing every edge of G that starts and ends at the same vertex. Armed with this definition answer the following.

   (a) A complete graph K_n is given with a guarantee that K_n has an Eulerian tour. What are the possible values of n?

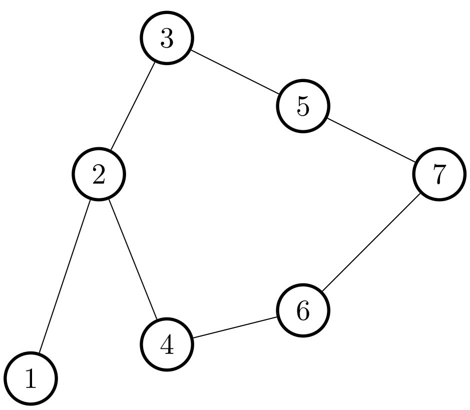
   (b) A complete graph K_n is given with a guarantee that K_n has at least one Eulerian trail. Does K_n also have an Eulerian tour?

   (c) Given a graph G with an Eulerian trail, you are allowed to add additional edges to G to get a graph H.

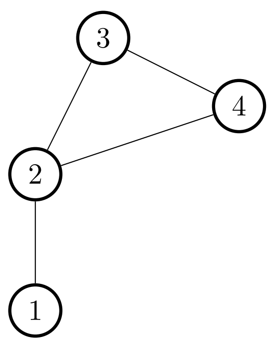
      i) What is the minimum number of additional edges to be added to G such that H has an Eulerian tour?

      ii) Where should the additional edges be added?

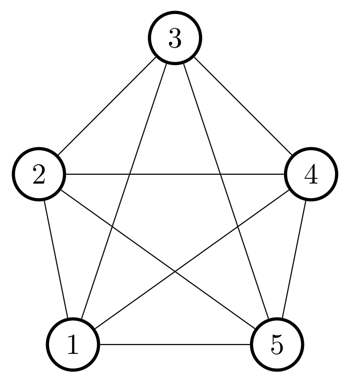
9. Consider the following simple graphs.



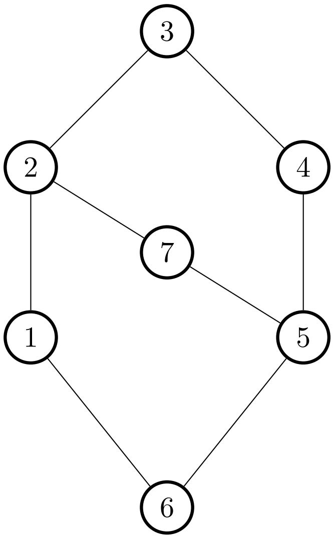
G_1



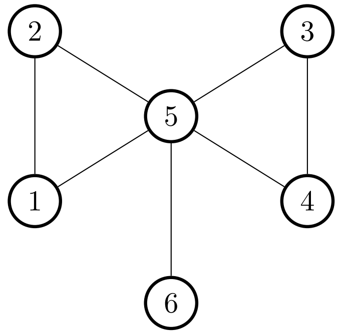
G_2



G_3



G_4



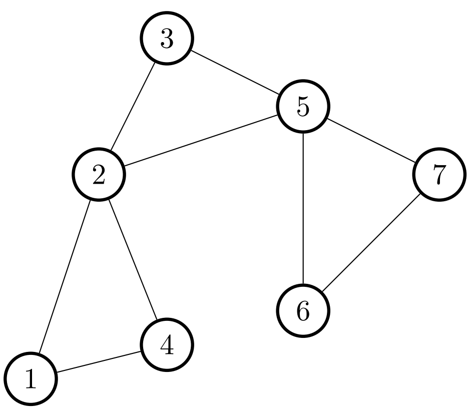
G_5

(a) A *Hamiltonian path* P of a graph G is a path that visits each vertex of G exactly once.
Which of the given graphs $G_1, G_2, \ldots, G_5$ admit a Hamiltonian path? If it admits, show a Hamiltonian path.

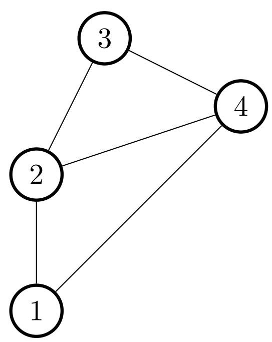
(b) A *Hamiltonian cycle* C of a graph G is a cycle that visits each vertex of G exactly once (excluding the starting and ending vertices of C).
Which of the given graphs $G_1, G_2, \ldots, G_5$ admit a Hamiltonian cycle? If it admits, show a Hamiltonian cycle.

(c) Construct $\overline{G_i}$, for all $i \in \{1, 2, \ldots, 5\}$. Do they admit a Hamiltonian path and a Hamiltonian cycle?

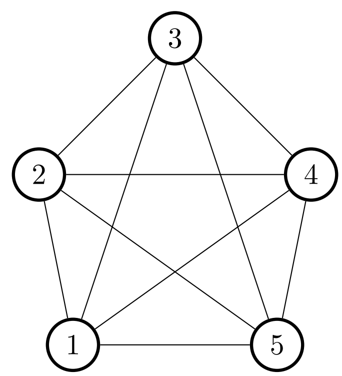
10. Consider the following simple graphs.



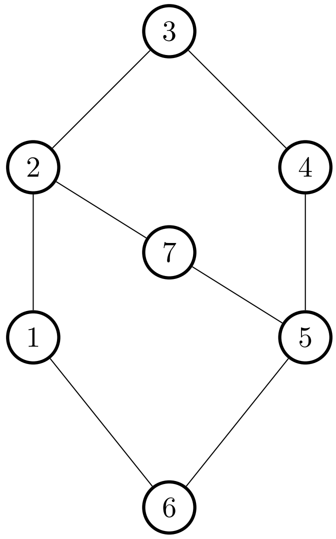
G_1



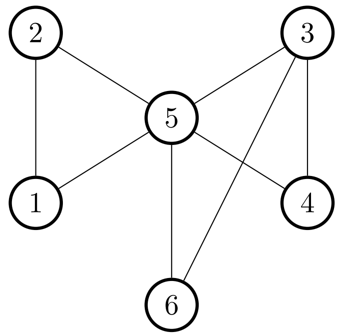
G_2



G_3



G_4



G_5

(a) Which of the given graphs $G_1, G_2, \ldots, G_5$ admit an Eulerian trail? If it admits, show an Eulerian trail.

(b) Which of the given graphs $G_1, G_2, \ldots, G_5$ admit an Eulerian tour? If it admits, show an Eulerian tour.

(c) Construct $\overline{G_i}$, for all $i \in \{1, 2, \ldots, 5\}$. Do they admit an Eulerian trail and an Eulerian tour?