PH-1020
Solution of Problem Set - 7
Department of Physics, IIT Madras
Magnetic Fields in Matter
March-June 2023 Semester

Notation:

- Notation throughout follows that of Griffiths, Electrodynamics.
- Bold face characters, such as $\boldsymbol{v}$, represent three-vectors.

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1. **Write the real component of electric and magnetic fields for a monochromatic plane wave (Amplitude = E_0, frequency = ω and phase angle $\delta = 0$) which is**

   (a) **travelling in the negative x-direction and polarized in the z-direction.**

   (b) **travelling along (1,1,1) with polarization parallel to the xz-plane.**

   As given in the question

$$\text{Amplitude} = E_0, \quad \text{Frequency} = \omega \quad \text{and} \quad \text{Phase angle}(\delta) = 0 \,.$$

   The real part of the electric and the magnetic field, as discussed in [1], is

$$\begin{aligned} \boldsymbol{E_R}(r,t) &= E_0 \cos(\boldsymbol{k}.\boldsymbol{r} - \omega t)\, \hat{\boldsymbol{n}} \\ \boldsymbol{B_R}(r,t) &= \frac{E_0}{c} \cos(\boldsymbol{k}.\boldsymbol{r} - \omega t) \left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}\right) \end{aligned}$$

   (a) As the wave is travelling in the negative x-direction and polarized in the z-direction,

$$\boldsymbol{k} = \frac{\omega}{c}(-\hat{\boldsymbol{x}}) \quad \text{and} \quad \hat{\boldsymbol{n}} = \hat{\boldsymbol{z}} \,.$$

   Now, it is easy to verify $\boldsymbol{k}.\boldsymbol{r} = -\frac{\omega}{c}x$ and $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{y}}$. The real part of the electric and magnetic field is

$$\begin{aligned} \boldsymbol{E_R}(r,t) &= E_0 \cos\left(\frac{\omega}{c}[x + ct]\right) \hat{\boldsymbol{z}} \\ \boldsymbol{B_R}(r,t) &= \frac{E_0}{c} \cos\left(\frac{\omega}{c}[x + ct]\right) \hat{\boldsymbol{y}} \end{aligned}$$

   (b) As wave travelling along$(1,1,1)$ then [1] ,

$$\boldsymbol{k} = \frac{\omega}{c}\left(\frac{\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}}{\sqrt{3}}\right) \quad \textit{and} \quad \hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}}{\sqrt{2}} \,.$$

   Now, it is easy to verify $\boldsymbol{k}.\boldsymbol{r} = \frac{\omega}{\sqrt{3}c}(x + y + z)$ and $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = \frac{1}{\sqrt{6}}(-\hat{\boldsymbol{x}} + \boldsymbol{2}\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})$. The real part of the electric and magnetic field is

$$\begin{aligned} \boldsymbol{E_R}(r,t) &= E_0 \cos\left(\frac{\omega}{\sqrt{3}c}\left[x + y + z - \sqrt{3}ct\right]\right) \left(\frac{\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}}{\sqrt{2}}\right) \\ \boldsymbol{B_R}(r,t) &= \frac{E_0}{c} \cos\left(\frac{\omega}{\sqrt{3}c}\left[x + y + z - \sqrt{3}ct\right]\right) \left(\frac{-\hat{\boldsymbol{x}} + \boldsymbol{2}\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}}{\sqrt{6}}\right) \end{aligned}$$

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[1] While deriving $\hat{\boldsymbol{n}}$ parallel to xz-plane we start with the general expression of $\hat{\boldsymbol{n}} = \alpha\hat{\boldsymbol{x}} + \beta\hat{\boldsymbol{z}}$ and using the condition $\hat{\boldsymbol{n}}.\boldsymbol{k} = 0$ results in $\alpha = -\beta$.

2. **Consider a linearly polarized plane EM waves propagating in z-directions, with their plane of polarization along the x direction. The electric field's amplitude is given by $|E_0|$, the frequency of the wave is ω, and its wave number is k. Find the value of**

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{S} \ ,$$

where u is the energy density and $\boldsymbol{S}$ is the Poynting vector.

For the EM wave having arbitrary phase ϕ is

$$\boldsymbol{E} = |E_0| \cos(kz - wt + \phi) \ \hat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{B} = \frac{k|E_0|}{\omega} \cos(kz - wt + \phi) \ \hat{\boldsymbol{y}} \ .$$

As we know, the energy density can be written as

$$u = \frac{\epsilon_0}{2}|\boldsymbol{E}|^2 + \frac{1}{2\mu_0}|\boldsymbol{B}|^2 \ .$$

For the above solution

$$u = \epsilon_0 |E_0|^2 \cos^2(kz - wt + \phi) \ \ ,$$

and

$$\frac{\partial u}{\partial t} = \omega \epsilon_0 |E_0|^2 \sin(2[kz - wt + \phi]) \ . \tag{1}$$

Now the Poynting vector $\boldsymbol{S}$ for first wave is

$$\begin{aligned} \boldsymbol{S} &= \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} \\ &= \frac{k|E_0|^2}{\mu_0 \omega} \cos^2(kz - wt + \phi) \, \hat{\boldsymbol{z}} \\ \boldsymbol{\nabla}.\boldsymbol{S} &= -\frac{k^2|E_0|^2}{\mu_0 \omega} \sin(2[kz - wt + \phi]) \ . \end{aligned} \tag{2}$$

By adding eq.(1) and eq.(2) we get

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{S} = 0$$

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3. **The intensity of sunlight hitting the earth is about 1300 W/m^2. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?**

As we know, the pressure exerted is

$$P = \frac{I}{c} = \frac{1.3 \times 10^3}{3 \times 10^8} = 4.3 \times 10^{-6} N/m^2 \ .$$

For a perfect reflector, the pressure is $P = 8.6 \times 10^{-6} N/m^2$. Fraction of atmospheric pressure amounts in

$$\frac{\text{Pressure}}{\text{Atmospheric Pressure}} = \frac{8.6 \times 10^{-6}}{1.03 \times 10^5} \approx 8.3 \times 10^{-11}$$

Note : Please follow the discussion made in section: 9.2.3

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4. **A He-Ne laser emits a plane wave which is polarized along $\hat{\boldsymbol{x}}$ and propagating in yz-plane at an angle $\pi/3$ to the y-axis in a medium of refractive index 1.5. The wavelength and intensity of the plane wave are 633 nm and $1\ W/m^2$, respectively. Calculate the electric and magnetic field associated with the plane waves.**
Given,

$$\begin{aligned} \lambda &= 6.33 \times 10^{-7}\ m \\ \text{Intensity} &= 1\ W/m^2 \end{aligned}$$

It is easy to verify $E_0 \approx 22.40\ V/m$. (As we know, $I = \frac{1}{2}\epsilon \frac{c}{n} E_0^2$)

As the wave is propagating in the yz-plane with an angle $\pi/3$ to the y-axis

$$\begin{aligned} \boldsymbol{k} &= \frac{2\pi}{\lambda}\left[\cos\frac{\pi}{3}\hat{\boldsymbol{y}} + \sin\frac{\pi}{3}\hat{\boldsymbol{z}}\right] \\ &= 7.445 \times 10^6 \left[\hat{\boldsymbol{y}} + \sqrt{3}\hat{\boldsymbol{z}}\right]\ . \end{aligned} \tag{3}$$

The associated E-field can be written as

$$\boldsymbol{E} = \boldsymbol{E_0}\ e^{i(\boldsymbol{k}.\boldsymbol{r} - \omega\ t)}\ . \tag{4}$$

So, finally

$$\begin{aligned} \boldsymbol{E}_{\text{Real}} &= 22.40\left\{\cos\left[7.45 \times 10^6\left(\hat{\boldsymbol{y}} + \sqrt{3}\hat{\boldsymbol{z}}\right).\boldsymbol{r} - 2.98 \times 10^{15} t\right]\right\}\hat{\boldsymbol{x}} \\ \boldsymbol{H}_{\text{Real}} &= 0.089\left\{\cos\left[7.45 \times 10^6\left(\hat{\boldsymbol{y}} + \sqrt{3}\hat{\boldsymbol{z}}\right).\boldsymbol{r} - 2.98 \times 10^{15} t\right]\right\}\left(\sqrt{3}\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}\right)\ . \end{aligned} \tag{5}$$

Note: While deriving $\boldsymbol{H}$, we have use the formula $\boldsymbol{H} = \frac{\boldsymbol{K} \times \boldsymbol{E}}{\omega \mu_0}$

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5. **Considering an EM wave, traveling in the air, with amplitude $5\ V/m$ and polarized along $\hat{\boldsymbol{y}}$, incident normally on a dielectric of refractive index 2.5. The free space wavelength is $6 \times 10^{-7}\ m$.**

   (a) **Find the reflecting and transmitting waves (i.e., express E_R, H_R, E_T, H_T).**

   (b) **Calculate the Poynting vectors associated with the incident, reflected, and transmitted wave and show that $R + T = 1$.**

As we know,

$$\begin{aligned} \boldsymbol{E_{0R}} &= \left|\frac{n_1 - n_2}{n_1 + n_2}\right| \boldsymbol{E_{0I}} = 0.429\ \boldsymbol{E_{0I}} \\ \boldsymbol{E_{0T}} &= \frac{2\ n_1}{n_1 + n_2}\ \boldsymbol{E_{0I}} = 0.571\ \boldsymbol{E_{0I}} \end{aligned}$$

According to the question

$$\boldsymbol{E_{0I}} \;=\; 5\;V/m \quad ; \quad \hat{\boldsymbol{n}} = \hat{\boldsymbol{y}} \quad ; \quad \lambda = 6 \times 10^{-7} m\ .$$

By using the formulae: $\boldsymbol{B_{0R}} = \frac{\boldsymbol{E_{0R}}}{c}$, $\boldsymbol{H_{0R}} = \frac{\boldsymbol{B_{0R}}}{\mu_0}$ and $\boldsymbol{H_{0T}} = \frac{\boldsymbol{E_{0T}}}{\boldsymbol{v}}$. Again taking only real components can be written as

- Incident Wave:

$$\boldsymbol{E_I} \;=\; 5\left\{\cos\left(\boldsymbol{k_1 z} - \omega t\right)\right\}\;\; \hat{\boldsymbol{y}} \quad ; \quad \boldsymbol{H_I} = 1.33 \times 10^{-2}\left\{\cos\left(\boldsymbol{k_1 z} - \omega t\right)\right\}\;\; \hat{\boldsymbol{x}}$$

- Transmitted Wave:

$$\boldsymbol{E_T} \;=\; 2.86\left\{\cos\left(\boldsymbol{k_2 z} - \omega t\right)\right\}\;\; \hat{\boldsymbol{y}} \quad ; \quad \boldsymbol{H_T} = 1.9 \times 10^{-2}\left\{\cos\left(\boldsymbol{k_2 z} - \omega t\right)\right\}\;\; \hat{\boldsymbol{x}}$$

- Reflected Wave:

$$\boldsymbol{E_R} \;=\; 2.146\left\{\cos\left(\boldsymbol{k_1 z} + \omega t\right)\right\}\;\; \hat{\boldsymbol{y}} \quad ; \quad \boldsymbol{H_R} = 5.7 \times 10^{-3}\left\{\cos\left(\boldsymbol{k_2 z} + \omega t\right)\right\}\;\; \hat{\boldsymbol{x}}$$

where $k_1 = \frac{\omega}{c} n_1$ and $k_2 = \frac{\omega}{c} n_2$.

Now the Poynting vector

$$\begin{aligned}
\langle \boldsymbol{S_I} \rangle &= \langle \boldsymbol{E_I} \times \boldsymbol{H_I} \rangle = 3.325 \times 10^{-2}\;\; J/m^2 \\
\langle \boldsymbol{S_T} \rangle &= \langle \boldsymbol{E_T} \times \boldsymbol{H_T} \rangle = 2.717 \times 10^{-2}\;\; J/m^2 \\
\langle \boldsymbol{S_R} \rangle &= \langle \boldsymbol{E_R} \times \boldsymbol{H_R} \rangle = -6.105 \times 10^{-3}\;\; J/m^2
\end{aligned}$$

Now,

$$R \;=\; \frac{\langle S_R \rangle}{\langle S_I \rangle} \approx 0.183 \quad ; \quad T = \frac{\langle S_T \rangle}{\langle S_I \rangle} \approx 0.817$$

It is easy to see $R + T = 1$.

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6. **The refractive index of diamond is 2.42. Plot the graph of $\frac{E_{0T}}{E_{0I}}$ vs θ_I and $\frac{E_{0R}}{E_{0I}}$ vs θ_I for the air/diamond interface. Here, θ_I is the angle of incidence and consider $\mu_1 = \mu_2 = \mu_0$. Also, calculate**

   (a) **the amplitude of normal incidence,**

   (b) **Brewster's Angle, and**

   (c) **the angle at which the reflected and transmitted amplitudes are equal.**

   Some of the equations already discussed in the class are

$$\alpha = \frac{1}{\cos\theta_I}\sqrt{1 - \left[\frac{n_1}{n_2}\sin\theta_I\right]^2} \quad ; \quad \frac{E_{0R}}{E_{0I}} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \quad ; \quad \frac{E_{0T}}{E_{0I}} = \left(\frac{2}{\alpha + \beta}\right)\ ,$$

   where α and β are amplitudes.

(a) As given $n_2 = 2.42$, it is easy to see $\beta = 2.42$.

$$\alpha = \frac{1}{\cos\theta_I}\sqrt{1 - \left[\frac{1}{2.42}\sin\theta_I\right]^2} \ .$$

For $\theta_I = 0$

$$\frac{E_{0R}}{E_{0I}} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) = -0.415 \quad ; \quad \frac{E_{0T}}{E_{0I}} = \left(\frac{2}{\alpha + \beta}\right) = 0.585$$

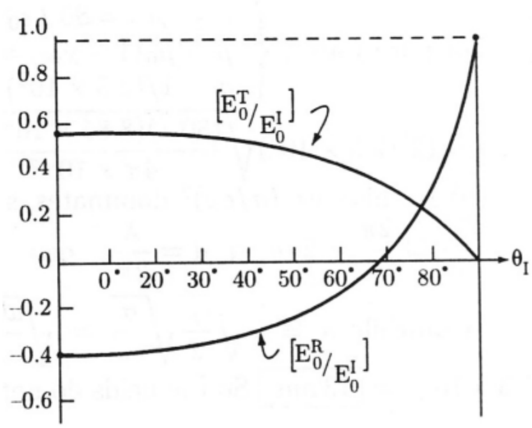

Figure 1: graph of $\frac{E_{0T}}{E_{0I}}$ vs θ_I and $\frac{E_{0R}}{E_{0I}}$ vs θ_I.

(b) Using the relation for Brewster's Angle

$$\theta_B = \tan^{-1}(2.42) \approx 67.5^o$$

(c) For $E_{0R} = E_{0T}$ we have

$$\alpha - \beta = 2$$
$$\alpha = 4.42 \ .$$

Now using the above relation

$$4.42 = \frac{1}{\cos\theta_I}\sqrt{1 - \left[\frac{1}{2.42}\sin\theta_I\right]^2}$$

By solving for θ_I it is easy to verify $\theta_I \approx 78.3^o$

References

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.