PH-1020
Problem Set - 0
Department of Physics, IIT Madras
Vector Calculus
March-June 2023 Semester

Notation:

- Notation throughout follows that of Griffiths, Electrodynamics.
- Bold face characters, such as $\boldsymbol{v}$, represent three-vectors.

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1. Consider the $\boldsymbol{\nabla}$ operator in cartesian coordinates

$$\boldsymbol{\nabla} \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \tag{1}$$

   (a) For any two vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$, calculate $(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}$
   Make sure you understand the meaning of this expression properly; we will encounter it in several places, particularly in the study of the physical aspects of electric and magnetic fields in material media.

   (b) Compute $(\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{r}}$

2. Let $\boldsymbol{r}$ be the position vector of a point with respect to some origin, with magnitude $r = |\boldsymbol{r}|$, and let $\boldsymbol{n} = \boldsymbol{r}/r$ be the corresponding unit vector. Prove the following identities:

$$\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3 \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{r} = \boldsymbol{0} \tag{3}$$

$$\boldsymbol{\nabla}\cdot[\boldsymbol{n}f(r)] = \frac{2f}{r} + \frac{\partial f}{\partial r} \tag{4}$$

$$\boldsymbol{\nabla}\times[\boldsymbol{n}f(r)] = \boldsymbol{0} \tag{5}$$

   where $f(r)$ is some well behaved function that depends only on r.

3. (a) Consider the vector field

$$\boldsymbol{v} = \frac{\hat{\boldsymbol{r}}}{r^2} \tag{6}$$

   Sketch this vector field, and calculate its divergence. From your understanding of divergence (recall **Physics I**), does the answer make sense *intuitively*? If not, what do you think is the origin of the problem? If yes, please refresh your understanding of divergence and think again!

   (b) Recall the divergence theorem for a vector field $\boldsymbol{v}$:

$$\int_V dV(\boldsymbol{\nabla}\cdot\boldsymbol{v}) = \int_{S=\partial V} \boldsymbol{v}\cdot d\boldsymbol{A} \tag{7}$$

   where V is the volume over which the integration on LHS is done, dV is the small volume element in your chosen coordinate system (for e.g., in cartesian coordinates $dV = dxdydz$

etc.), $S = \partial V$ is the notation for the boundary of the region V, and $d\boldsymbol{A}$ represents a small area element of this boundary (of course, this area element is a vector).
Apply the divergence theorem to the vector field v in 3(a) above by choosing V as a sphere of radius r_0, say. That is, calculate the RHS above and see whether it matches with the LHS evaluated using your result for 3(a). Does the answer depend on r_0? And once again, can you think of the origin of the problem?

Note:
This problem represents a peculiar property of the scalar function $f(\boldsymbol{x}) = \boldsymbol{\nabla} \cdot \boldsymbol{v}$. It is zero (almost) everywhere, but it's volume integral over a region that includes the origin is finite! Obviously, an object such as $f(\boldsymbol{x})$ can not be regarded as an ordinary function. It is something called as a **Dirac delta "function"**, represented in three dimensions as $\delta^3(\boldsymbol{x})$. Its properties would be discussed in the lectures, since we will encounter this object repeatedly throughout the course.
(The double quotes around "function" above is intended for the more mathematically inclined; mathematicians prefer to call such an object a distribution rather than a function.)

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4. The *Dirac delta function in electrostatics*
The Dirac delta function can be used to write a formal expression for volume charge density $\rho(\boldsymbol{r})$ associated with point charges, as well as with line and surface charge densities. Express $\rho(\boldsymbol{r})$ using the delta function for following configuration of charges. To fix all the factors correctly, check that $\int \rho dV$ over all volume indeed gives the total charge as expected from the given point/line/surface charge configurations.

   (a) A set of N discrete point charges $q_i,\ i = 1 \cdots N$, located at positions $\boldsymbol{r}_i$.

   (b) A line charge density $\lambda(\phi)$ pasted on a circle of radius R located in the X-Y plane with center at the origin.