CS1200 — Discrete Mathematics for Computer Science — 2023 (Jan-May)

Assignment 2

Release Date: 01/05/2023 **Due Date: 12/05/2023 — 11:00 PM IST**

Collaborator (if any): Aditya Srivastava

Academic Integrity Statement: I, *Chandaluru Hema Venkata Raadhes*, affirm that I have not given or received any **unauthorized** help (from any source: people, internet, etc.) on this assignment, and that I have written/typed each response on my own, and in my own words.

THE MARKS FOR EACH PROBLEM (1, 2, 3, 4, 5) ARE FIXED. HOWEVER, THE MARKS FOR EACH SUBPROBLEM (1A, 1B, 2A, ETC.) ARE TENTATIVE — THEY MAY BE CHANGED DURING MARKING IF NECESSARY.

---

1. [AN EXERCISE IN LOGIC: REVISITING BASIC SET THEORY USING PROPOSITIONS] [9]

Throughout this exercise, we let S and T denote two sets (in some universe U).

Several statements are written below, and they become propositions when specific values of S and T are substituted.

Two statements are said to be *logically equivalent* if, for all values of S, T and U, the truth values of the corresponding propositions are the same (that is, either both propositions are true, or both propositions are false). For example: the two statements $\forall\, x \in U,\ x \in S \cap T$ and $\forall\, x \in U,\ x \in S \wedge x \in T$ are logically equivalent.

(a) $S \subseteq T$

(b) $(\forall\, x \in U,\ x \in S \implies x \in T) \wedge (\exists\, x \in U, x \in T \wedge x \notin S)$

(c) $T \subseteq S$

(d) $\forall\, x \in U,\ x \in T \implies x \in S$

(e) $S \cap T = \emptyset$

(f) $\forall\, x \in U, (x \in S) \oplus (x \in T)$

(g) $T \subset S$

(h) $\forall\, x \in U,\ x \in S \iff x \in T$

(i) $S = T$

(j) $\forall\, x \in U, x \in (S \oplus T)$

(k) $(\forall\, x \in U,\ x \in T \implies x \in S) \wedge (\exists\, x \in U, x \in S \wedge x \notin T)$

(l) S and T are disjoint

(m) $\forall\, x \in U,\ (x \notin S) \vee (x \notin T)$

(n) $S \subset T$

(o) $(S \subseteq T) \wedge (T \subseteq S)$

(p) (S, T) is a partition of U

(q) $\forall\, x \in U,\ (x \notin T) \vee (x \in S)$

(r) $\forall\, x \in U, \neg((x \in S) \wedge (x \in T))$

Let Z denote the set comprising the above 18 statements: from (a) to (r). That is, $Z := \{(a), (b), \ldots, (r)\}$.

Observe that logical equivalence is an equivalence relation on Z; **list all** of the equivalence classes (defined in the next exercise for your convenience).

No justifications required.

Response: Equivalence Classes are given below:(note that i have given brief explanation for every equivalence class, which is not required as per instructions, i have done it purely for the reason of easier understanding) .
1. S is subset of T.
$\{(a)\}$
2. S is proper subset of T.
$\{(b), (n)\}$
3. T is subset of S.
$\{(c), (d), (q)\}$
check q before submitting
4. S is disjoint with T.
$\{(e), (l), (m), (r)\}$
check r before submitting

5. T is proper subset of S.
$\{(g), (k)\}$
6. S and T sets have same elements. The sets are same.
$\{(h), (i), (o)\}$
7. S and T have no common elements and their union makes up the Universal Set. (S,T) is a partition of U.
$\{f), (j), (p)\}$

---

2. [Equivalence Relations and Equivalence Classes] [6]

The goal of this exercise is to prove the following theorem:

Theorem 0.1. *Let R denote an equivalence relation on a set S. Then the equivalence classes (with respect to R) comprise a partition of the set S.*

Throughout this exercise, let R denote an equivalence relation on a set S. Recall that, for any element $a \in S$, the *equivalence class of a (with respect to R)*, denoted by $[a]_R$, is (the set) defined as follows:

$$[a]_R := \{b \in S \mid aRb\}$$

(a) **<u>Prove that</u>**, for any $x \in S$, the equivalence class $[x]_R$ is nonempty (that is, $[x]_R \neq \emptyset$).

Response: An equivalence relation is reflexive , symmetric and transitive. We shall use the reflexive property here to prove the given statement. By the reflexivity property of equivalence relations we have for $\forall x \in S$ that $x \; R \; x$.
Equivalence class of an element a is given as

$$[a]_R := \{b \in S \mid aRb\}$$

i.e set of all elements which a is related to.
Hence $[x]_R$ must contain the element x itself.
So as the equivalence class of any element must contain atleast one element (the element itself) , we can say that for any $x \in S$, the equivalence class $[x]_R$ is nonempty (that is, $[x]_R \neq \emptyset$).
Hence ,Proved.

(b) **<u>Prove that</u>**, for any $x, y \in S$, either $[x]_R = [y]_R$, or otherwise $[x]_R \cap [y]_R = \emptyset$ (that is, either the two sets $[x]_R$ and $[y]_R$ are the same, or otherwise they are disjoint).

Response: Let us start with this. For an equivalent class of x it is given that :

$$[x]_R := \{x \in S \mid xRa\}$$

We can say that for all $a \in [x]_R$ we also have that $a \; R \; x$ because of the symmetric nature of equivalence relations.

Now let us take 2 cases . $x \; R \; y$ and $x \not R \; y$.
If $x \; R \; y$ then:
For all $b \in [y]_R$ we have $y \; R \; b$. By transitivity we can now say that $x \; R \; b$ (xRy and yRb) and hence we can state that b is in $[x]_R$ also because $x \; R \; b$. So basically for all $b \in [y]_R$ we have $b \in [x]_R$.$[y]_R \subseteq [x]_R$.
Similarly we can say that for all $a \in [x]_R$ we have $a \in [y]_R$.$[x]_R \subseteq [y]_R$.
As both as appearing to be subsets of one another it implies that both are equal sets.
$[x]_R = [y]_R$

If $x \not R \; y$ then:
For all $b \in [y]_R$ we can say that $x \not R \; b$; because if $x \; R \; b$ were true for some $b \in [y]_R$ then x would have to be related to y (as xRb and bRy and transitivity condition) which is a contradiction to the initial condition. Hence there are no elements $b \in [y]_R$ such that $x \; R \; b$ which also means that $b \notin [x]_R$. This means that there are no elements which are in both $[y]_R$ and $[x]_R$. As if there were some element in both then there would exist an element $b \in [y]_R$ such that $b \in [x]_R$ also which was disproven just above. $[x]_R \cap [y]_R = \phi$

(c) Convince yourself that you have proved Theorem 0.1. (No response required.)

3. [PATHS IN A FOREST] [13]

The objective of this exercise is to gain a deeper understanding of paths in forests — so that we don't lose our way in a forest ;-).

In general, we view paths as subgraphs. However, when convenient (inside a proof), you may also view a path as a sequence of vertices and edges.

A path with (not necessarily distinct) ends u and v is called a *uv-path.*

(a) **Prove the following:**

Lemma 0.2. *Let G be a graph, and let $u, v \in V(G)$. If there exist two distinct uv-paths P and Q then G has a cycle.*

Response: There is given to be two paths between vertices u and v. Let these paths be P_1 and P_2, where P_2 is longer or equal length to P_1.
As the paths are distinct , there must be atleast one vertice which is not common to both paths. Take one vertice that is in P_2 but not in P_1 to be x. (possible because P_2 is longer or equal legnth to $P1$ and there are no repeating vertices in paths)
Choose the shortest subpath(discussed in class) Q_2 which contains vertice x from P_2 whose end vertices are also in path P_1. Let the end vertices of this path be u_1 and v_1. (note that this is possible because there is atleast one pair of vertices common to paths P_1 and P_2 .u and v are common to both paths.)
Let the path constructed by the edges between vertices u_1 and v_1 in path P_1 be Q_1.The path constructed by the edges between vertices u_1 and v_1 in path P_2 is Q_2(already defined).There are no vertices that are common to the paths Q_1 and Q_2 except endpoints because if there were such common vertices then Q_2 would not be the shortest such path satisfying conditions we require. (we can find a shorter path if there are some other common vertices except u1 and v1 in Q1 and Q2)
Join paths Q_1 and Q_2 by merging them at end vertice u1. ex : if Q_1 is represented by u1 e1 a e2 b e3 c e4 v1 and Q_2 is represented by u1 e5 d e6 f e7 v1 then join them to form v1 e7 f e6 d e5 u1 e1 a e2 b e3 c e4 v1. This is a cycle as it has no repeating vertice except end vertices. Hence graph G must contain atleast one cycle.

(b) For each of the following propositions, **state** and **prove** whether it is True or False.

(P1) "In a forest, for any pair of vertices, say u and v, there is at least one uv-path."
Response: False.
A forest is just a simple acyclic graph.
Let us take a forest which consists of 2 vertices which dont have an edge between them. Clearly there is no path between those two vertices in this forest. Hence in a forest there is not necessarily a path between 2 every vertices. Example : Let G be a forest . $V(G) = \{1, 2\} E(G) = \phi$ There is no path between the two vertices 1 and 2 in this case.

(P2) "In a forest, for any pair of vertices, say u and v, there is at most one uv-path."
Response: True
Forests are graphs with no cycles.
Lemma 0.2 .If there is more than one path between some 2 vertices in a graph then there will be atleast one cycle in the graph which will imply the graph is not a forest. Hence we can say that to be a forest there can be atmost one path between any 2 vertices in the graphs ; Because otherwise there will be cycle .

(P3) "In a tree, for any pair of vertices, say u and v, there is at most one uv-path."
Response: True
All trees are forests because trees are just connected acyclic graphs.
This implies that any condition that applies for a graph to be a forest will also apply for a graph to be a tree. (although there may be more conditions also for that graph to be a tree).
By the previous response (P2) , we showed that "In a forest, for any pair of vertices, say u

and v, there is at most one uv-path."
Hence we can also say "In a tree, for any pair of vertices, say u and v, there is at most one uv-path."

(P4) "In a tree, for any pair of vertices, say u and v, there is at least one uv-path."
Response: True
A tree is a connected acyclic graph . i.e a tree is a connected forest.
Statement 1: If there is a walk from a to b, there is a path from a to b.
Proof : Take the shortest walk W1 from a to b , this will be a path because there must not be any repeating vertices. This is because if there were any repeating vertices we could remove the edges between the repeated vertices to create a shorter walk than W1 which would contradict the statement that the walk W1 itself was the shortest walk. Hence it must imply that walk W1 is also a path. Do note that if there are no repeating vertices there cannot be any repeating edge because a edge is defined between 2 vertices ; if those vertices are different the edge must be different.
The above proof was done in tutorial-3 also.
A connected graph is a graph in which there is atleast one walk between every 2 vertices, this implies there is atleast one path between every 2 vertices. Hence we can say that BECAUSE trees are connected graphs , there will be atleast one path betweeen every two vertices in the tree.

(P5) "A simple graph G is a tree if and only if there is precisely one uv-path for all $u, v \in V(G)$."
Response: True
(1) To prove , if a graph G is a tree then there is precisely one uv path for all $u, v \in G$.
A tree is connected so there must be atleast one uv path for every $u, v \in G$. By problem P4.
A tree is acyclic so there are atmost one uv path for every $u, v \in G$. By problem P3.
This means if a graph is a tree there must be precisely one uv path for all $u, v \in G$.

(2) To prove, if there is precisely one uv path for every $u, v \in G$ then G is a tree.
We can say that if there is a cycle in a graph , then there must be a a u,v vertice pair such that there are two distinct paths between them.
Take the cycle represented by a sequence of continuguous edges and split the cycle anywhere in the middle. This will create two distinct paths . The end vertices of both these paths are the same.
example: cycle e1 e2 e3 e4
Edge e1 starts at vertice u , edge e2 ends at vertice v . Edge e3 starts at v , edge e4 ends at u. Now take the paths P2 : e1 e2
and path P2 : e3 e4 .
They both are different paths who have same end vertices , hence if there is a cycle in the graph then there is atleast one $u, v \in G$ such that there are atleast 2 paths between them.
This means that if there are 1 or less paths between all $u, v \in G$ then there must be no cycle in the graph.
Now , if there are atleast 1 path between every u,v in G then the graph is connected.
If there is precisely 1 path between every u,v in G then the graph is connected and acyclic (a tree) .

As we showed each side of the implication ,the if and only if statement is also true.

4. [INFINITELY MANY PRIMES] [16]

The goal of this exercise is to prove the following deep result.

Theorem 0.3. *There are infinitely many primes.*

(a) Prove the following.

Lemma 0.4. *Let n denote any natural number greater than 1. Prove that either n is prime, or otherwise there exists a prime p (distinct from n) that divides n, but not both.*

Response: Case 1 : n is a prime.
If n is a prime then the only factors of n are 1 and n itself. So there is no prime number p that can divide n distinct from n .

Case 2 : n is not a prime.
If n is not a prime it has some factors other than 1 and n.
Let us take the smallest of these factors(except 1) and call it a. This factor a , must be prime ; because if it were not prime , then there would exist some number b except 1 and a , which would divide a .
AND THEN : we would have $b < a$ and b divides n(by transitivity of divisibility relation), Which is a contradiction to the assumption that a is the smallest factor of n (except 1).
Hence we can say that a is a prime. There exists a prime p = a , such that n is divisible by p.
Note : for a number x to be divisible by another number y , it is compulsary that $y \leq x$.

(b) Prove the following.

Lemma 0.5. *Let m and n denote any natural numbers greater than 1. Then m divides at most one of n and $n + 1$.*

Response: Let us prove it by contradiction : Assume m divides both n and n+1.
This implies that $n = a \times m$ and $n + 1 = b \times m$. Where a and b are natural numbers greater than or equal to 1.
Subtract both to get $1 = (b - a) \times m$.
This is impossible , because if $b = a$ then RHS equals 0; if $b - a \neq 0$ then RHS would have an absolute value greater than 1 (do remember that m is greater than 1 given in question AND that a,b are natural).
Hence it is not possible that m divides both n and n+1.

(c) Prove the following using Lemma 0.5 (even if you are **not** able to prove the lemma).

Corollary 0.6. *Let $n_1, n_2, \ldots, n_k$ denote $k \geq 1$ natural numbers, each of which is greater than 1, and let $n := n_1 \times n_2 \times \cdots \times n_k$. Then, for each $i \in \{1, 2, \ldots, k\}$, the number n_i does **not** divide $n + 1$.*

Response: For each n_i, we have that n_i divides n because $n = a \times n_i$. Where a is product of all the natural numbers given other than n_i.
By lemma0.5 we can say that because all n_i divides n , it cannot divide n+1 also at the same time.

(d) Let $P := \{p_1, p_2, \ldots, p_k\}$ denote any finite set of primes (for some $k \geq 1$). Use Corollary 0.6 and Lemma 0.4 to prove that there exists a prime q that does **not** belong to the set P.

Response: Do note that the smallest prime number is is 2.
Take the smallest number n greater than 1 which is not divisible by any of the primes in set P .
Why can we do it ? How do we know such a number even exists ?
This is because using Corollary 0.6 we have proved if we take a number N is a product of some numbers (each number > 1), then N+1 is not divisible by any of those numbers.
Hence we can say that there is atleast one number greater than the largest prime number in P , such that it is not divisible by any of the primes in P. $N + 1 = 1 + p_1 \times p_2 \times ...p_k$. And clearly N+1

is greater than every prime in set P (primes are $\geq 2 => N+1 > 2p_i$ for all $i \in \{1, 2, \ldots, k\}$) .
Take the smallest number n greater than 1 which is not divisible by any of the primes in set P. We have already proven that there is atleast one such number , just choose the smallest such.
Apply Lemma0.4
>In case n is a prime number we are done :)
>In case prime q divides n , then observe that q is not part of set P. Because n is not divisible by any the primes in set P .
Hence proved.

(e) Use part (d) to prove Theorem 0.3.

Response: Let us use contradiction.
Assume there are a finite number of primes and so we can write the number of primes is some number finite number n .
This assumption must be wrong because if we take this set of primes and apply the process of part (d) of this question then we observe that there are atleast n+1 primes.
Hence we by contradiction , we can say that there are not a finite number of primes. Which means there are infinite primes .

5. [FINITE POSETS, DAGS AND EXISTENCE OF MINIMAL & MAXIMAL ELEMENTS] [16]

A (binary homogeneous) relation R on a set S is said to be *irreflexive* if $(a, a) \notin R$ for each $a \in S$. Furthermore, R is said to be a *strict partial order* if it is irreflexive, antisymmetric and transitive; in this case, the pair (S, R) is called a *strict poset*.

For instance, $(\mathbb{Z}, <)$ is a strict poset, but it is not a poset.

The goals of this exercise are: (i) to establish a connection between finite strict posets and DAGs, and (ii) to establish the existence of minimal and maximal elements in finite posets.

(a) Recall from Module-1 that each (binary homogeneous) relation R on a finite set S may be represented as a (finite) digraph on the vertex set S. Prove the following.

Theorem 0.7. *Let R denote an irreflexive and transitive relation on a finite set S, and let D denote the digraph that represents this relation. Prove that (S, R) is a strict poset if and only if D is a DAG (that is, D is acyclic).*

Response: (1) If R is not anti symmetric we can say that there exists two distinct elements a,b in the set S such that aRb and bRa. If edge from a to b is e1 and edge from b to a is e2 , then these two edges make up a cycle. This means the graph is not acyclic. If it is acyclic then we can say that R is anti symmetric.

(2) If the graph is not acyclic (note as R is given to be irreflexive the smallest cycle will consist of atleast 2 vertices and 2 edges), We can take 2 vertices a,b from the one of the cycle in the graph . Then we take path from a to b in the cycle and we take the path from b to a in the cycle. Due to transitivity we can say that there is also a direct edge from a to b and also a direct edge from b to a.
So we can say that R is not anti symmetric.
If graph is not acyclic it implies the relation R is not antisymmetric. So : If relation R is anti symmetric then graph is acyclic.

As both sides of implications are proven we can say : The relation R is anti symmetric if and only if D is a DAG. Here relation R being anti symmetric is equivalent to saying it is a strict POSET because it is already given that R is irreflexive and transitive.
Hence The POSET (S,R) is a strict POSET if and only if D is a DAG.

(b) Recall the definition of R^{ref} from Assignment-1. If (S, R) is a strict poset, what can you say about (S, R^{ref})?

Response: We can say that (S, R^{ref}) is a POSET , because it is reflexive , antisymmetric and transitive.
(S, R) is given to be strict POSET , so it is irreflexive , antisymmetric and transitive. Now by definition of R^{ref} we add the elements (a,a) for all elements a in S to R to form R^{ref}. This will not change the transitive or anti symmetric property of the relation. Anti symmetric property only talks about (a,b) when a is distinct from b. Transitive property for a relation also doesnt change when any element (a,a) is added.

(c) For a poset $(S, \preccurlyeq)$, and (not necessarily distinct) elements $a, b \in S$, we say that:

- b is a lower bound of a if $b \preccurlyeq a$, and that
- b is an upper bound of a if $a \preccurlyeq b$.

We use LB(a) to denote the *set of lower bounds* of a, and we use UB(a) to denote the *set of upper bounds* of a. Draw the Hasse diagram for the poset $(\{2, 3, \ldots, 24\}, |)$, and write down the following sets (with respect to this poset):

(i) $\text{LB}(24) = \{2, 3, 4, 6, 8, 12, 24\}$

(ii) $\text{UB}(8) = \{8, 16, 24\}$

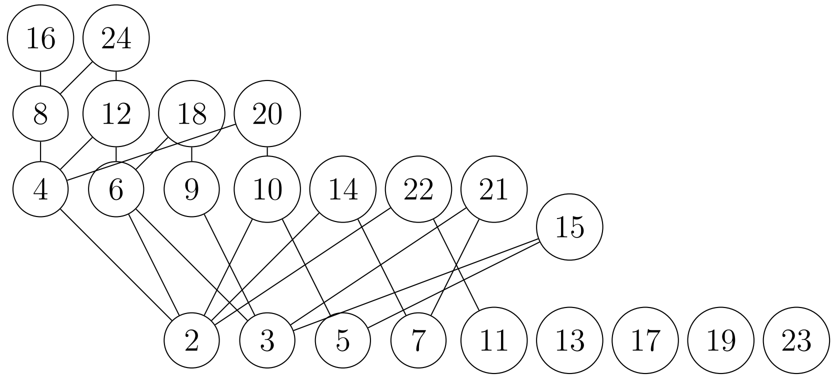
(iii) LB(13) = $\{13\}$

(iv) UB(2) = $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$

(v) UB(24) = $\{24\}$

(vi) LB(12) = $\{2, 3, 4, 6, 8, 12\}$

Response:



(d) Use the concepts of *set of lower bounds* and *set of upper bounds*, defined in part (c), to give a direct proof of the following.

Theorem 0.8. *Let $(S, \preccurlyeq)$ denote a finite poset. Prove that there exists a minimal and a maximal element.*

Response: Set of Upperbounds are the set of elements that are greater than or equal to an element. If an element is maximal there should be no element greater than or equal to it except for itself. i.e upperbound set contains only the element itself if it is a maximal element.
Statement : The cardinality of UB(b) , where b ∈UB(a) and b is distinct from a , is less than the cardinality of UB(a).
Proof : Every element in $x \in$ UB(b) , must also be in UB(a) , because $x \geq b$ *and* $b > a$ *so* $x > a$. Every element of UB(b) is in UB(a) and , the element a itself is not in UB(b) but is in UB(a) (because $b > a$). Hence we can conclude cardiniality of UB(a) greater than UB(b).

Take an element y in set S with a least cardinality of upperbound set , if its upperbound set contains any element except for y itself then the upperbound set for that other element will have cardinality even less which will contradict that y was element with least cardinality of upperbound set. Hence we can say that the upperbound set for y must contain only itself. Hence we can say that it is maximal.

Similarly we can prove that there must also be a minimal , by taking an element with set of lower bounds having least cardinality. That element will have set of lowerbounds which contains only itself. Which implies it is minimal.