Observation: $(S, \preceq)$: finite poset

$\mathcal{C}$: chain partition of $(S, \preceq)$

A: antichain of $(S, \preceq)$

Then $|\mathcal{C}| \geq |A|$.

Why? Observe that each element of A belongs to a distinct chain in $\mathcal{C}$. So, $\mathcal{C}$ must have at least as many chains as cardinality of A.

Corollary (of above observation):

means (an immediate/easy consequence)

$(S, \preceq)$: finite poset

Then: $\min\{|\mathcal{C}| : \mathcal{C} \text{ is a chain partition of } (S, \preceq)\}$

$$\geq$$

$\max\{|A| : A \text{ is an antichain of } (S, \preceq)\}$

Dilworth's Theorem: $(S, \preceq)$: finite poset

$\min\{|\mathcal{C}| : \mathcal{C} \text{ is a chain partition of } (S, \preceq)\}$

$= \max\{|A| : A \text{ is an antichain of } (S, \preceq)\}$.

$(\geq)$ easy (we just proved)

$(\leq)$ difficult.

OR

$(=)$ difficult.

proofs of such theorems generally establish $(\geq)$ and $(\leq)$ separately. One of them is generally easy; other one is difficult.

(1950)

Dilworth's Theorem: (stated differently; same meaning)

In any finite poset, the minimum cardinality of a chain partition equals the maximum cardinality of an antichain.

We will see a proof later.

discussion beyond CS1200:

There are many "such theorems" in combinatorics (a branch of discrete mathematics).

two closely related viewpoints

Min-Max Results/Theorems	Packing & Covering Problems.
There is a minimization problem (in Dilworth's Theorem, one tries to minimize the cardinality among all chain partitions) and a maximization problem (in Dilworth's Theorem, one tries to maximize the cardinality among all antichains) and the theorem says that equality holds.	There is a packing problem (in Dilworth's Theorem, one tries to "pack" as many elements as possible in an antichain) and a covering problem (in Dilworth's Theorem, one tries to cover — in fact, partition — the entire poset using chains) and the theorem says that the two problems are closely related. (maybe equality holds)

Our Goal is to use Dilworth's Theorem to prove Erdos-Szekeres Theorem:

Erdos-Szekeres Theorem	Dilworth's Theorem
Any sequence of n^2+1 integers contains a monotone subsequence of length $n+1$.	In any finite poset, maximum cardinality of an antichain equals minimum cardinality of a chain partition.

How does one achieve this? Let's think....

① Clearly, given a sequence of n^2+1 integers, we need to define some poset, say $(S, \precsim)$.

② Erdos-Szekeres Theorem is about establishing existence of a monotone subsequence (nonincreasing OR nondecreasing) whereas Dilworth's Theorem is about chain (partitions) and antichains.

We will DO (almost) exactly this!

③ It would make sense to have a correspondence between:

nonincreasing subsequences & nondecreasing subsequences AND antichains & chains

Proving Erdos-Szekeres Theorem using Dilworth's Theorem

PROOF PLAN:

(I)
Given a sequence $a_1, a_2, \ldots\ldots, a_{n^2+1}$ of integers,

we will define a finite poset $(S, \preceq)$ so that:

(1) decreasing ~~nonincreasing~~ subsequences will correspond to antichains in $(S, \preceq)$ **AND**

(2) nondecreasing subsequences will correspond to chains in $(S, \preceq)$

(II) Then we will use Dilworth's Theorem on $(S, \preceq)$ and prove Erdos-Szekeres Theorem.

that is, show existence of monotone subsequence of length $n+1$.

IS THE PLAN CLEAR?

(I) Let us define the finite poset $(S, \preceq)$. (Clearly, S should correspond to the elements of the sequence. Right?)

Let $S := \{1, 2, \ldots\ldots, n^2+1\}$.

How should we define the partial order $\preceq$?

Focus on THIS: $i \preceq j$ IF: (1) $i \leq j$

AND (2) $a_i \leq a_j$

usual less than or equal to (for integers)

DIY: 1) Prove that $(S, \precsim)$ is a poset.

↓

(defined on previous page)

2) Prove that each chain in $(S, \precsim)$ corresponds to a nondecreasing subsequence (in given sequence), (and vice versa). → NOT required for proof

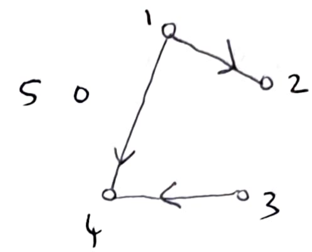
3) Prove that each antichain in $(S, \precsim)$ corresponds to a ~~nonincreasing~~ decreasing subsequence (in given sequence), (and vice versa). → NOT required for proof

TIY: Apply Dilworth's Theorem to $(S, \precsim)$ and try completing proof of Erdos-Szekeres Theorem.

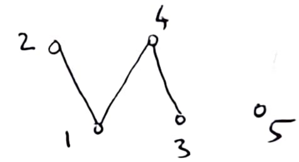
Example: $n = 2$

So, $n^2+1 = 5$.

a_1	a_2	a_3	a_4	a_5
17	19	15	18	13



Hasse diagram:



Observe:

$\{1,2\}$ is a chain

$(a_1, a_2) = (17, 19)$ is a nondecreasing subsequence.

$\{1,3,5\}$ is an antichain

$(a_1, a_3, a_5) = (17, 15, 13)$ is a nonincreasing subsequence. → in fact, decreasing