The Hemachandra recurrence $h_n = h_{n-1} + h_{n-2}$ (with initial conditions) is NOT very good to compute these numbers. In order to compute h_n, one first needs to compute h_{n-1} & h_{n-2}

in order to compute this, one needs to compute $(h_{n-2}\ \&)\ h_{n-3}$

in order to compute this one needs to compute h_{n-4} $(\&\ h_{n-3})$

Clearly, this is NOT very efficient!

One can write a recursive function to compute h_n (as per above discussion), but IS there a better way to compute this numbers?

YES, there IS.

Given a recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

where $c_1, c_2, \dots, c_k$ are real #s and $c_k \neq 0$.

<u>Characteristic Equation:</u>

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_{k-1} x - c_k = 0$$

<u>Characteristic Roots</u>: solutions to above (char.) equation

<u>Example:</u>

$$h_n = h_{n-1} + h_{n-2}$$

$k=2$
$c_1=1$
$c_2=1$

<u>Characteristic Equation:</u>

$$x^2 - x - 1 = 0 \qquad \boxed{ax^2+bx+c=0}$$

<u>Characteristic Roots:</u>

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{5}}{2}$$

2 distinct roots

Theorem: ($k=2$; distinct roots case) $c_1, c_2 \in \mathbb{R}$, $c_2 \neq 0$

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$: recurrence

$x^2 - c_1 x - c_2 = 0$: characteristic eqn.

proof in Rosen 515-516

Suppose that the characteristic equation has 2 distinct roots, say x_1 & x_2, then $\alpha_1 x_1^n + \alpha_2 x_2^n$ ($\forall n \in \mathbb{N}$) is a closed form formula for the given recurrence — where α_1 & α_2 are constants that may be computed using the initial conditions.

A recurrence of this form:

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$
(where $c_1, c_2, \dots, c_k \in \mathbb{R}$ & $c_k \neq 0$)

is called a linear homogeneous recurrence of degree k with constant coefficients.

linear means every term on RHS is a multiple of some previous element like $c_i a_{n-i}$

homogeneous means that RHS does NOT have any terms like a_{n-1}^2, a_{n-1}^3, etc.

degree k means that nth term a_n depends on previous k terms ($a_{n-1}, a_{n-2}, \dots, a_{n-k}$)

constant coefficients → all c_is are constants

Example continued:

By Theorem:

$$\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

is a closed form formula for the Hemachandra recurrence.

Let's consider initial conditions: $h_0 = 0$ & $h_1 = 1$

We may now compute $\alpha_1 + \alpha_2$ using the initial conditions.

$n=0$: $h_0 = 0 = \alpha_1 + \alpha_2$

$n=1$: $h_1 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$

Thus $2 = (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2$

Solving these linear equations, we get

$\alpha_1 = \frac{1}{\sqrt{5}}$ & $a_2 = \frac{-1}{\sqrt{5}}$

Thus

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is the desired closed form formula for the Hemachandra/Fibonacci sequence.