DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II **Problem Set 3 - Solutions** **MAR-JUN 23**

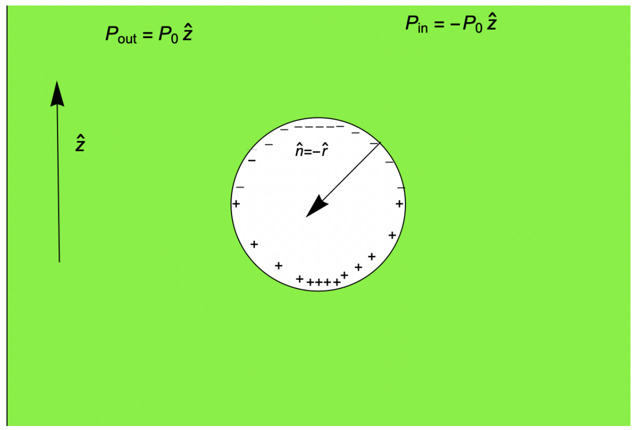

Figure 1: The green colored area is the bulk material and the white region is the cavity.

1. (a) It is given that polarization is $\mathbf{P}_0$. Without loss of generality we take the z axis along the direction of polarization such that $\mathbf{P}_0 = |\mathbf{P}_0|\hat{\mathbf{z}} = P_0\hat{\mathbf{z}}$. Then, the bound surface charge density

$$\boxed{\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = -|\mathbf{P}_0|\cos\theta}$$

"θ" is the usual spherical polar angle. Here, we have expressed the bound charge in terms of the polarization of the bulk material $\mathbf{P}_0$. Observe here that the unit normal vector $\hat{\mathbf{n}}$ is radially inward.

(b) The electric field at the center of the cavity due to the above surface charge density

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_b dA}{|\mathbf{r}|^2}(-\hat{\mathbf{r}}) \\
&= \frac{1}{4\pi\epsilon_0}\int_{\theta=0}^{\pi} d\theta \int_{\phi=0}^{2\pi} d\phi \frac{(-|\mathbf{P}_0|\cos\theta)\,|\mathbf{r}|^2\sin\theta}{|\mathbf{r}|^2}\,(-\sin\theta\cos\phi\hat{\mathbf{x}} - \sin\theta\sin\phi\hat{\mathbf{y}} - \cos\theta\hat{\mathbf{z}}) \\
&= \frac{|\mathbf{P}_0|}{4\pi\epsilon_0}\int_{\theta=0}^{\pi} d\theta \int_{\phi=0}^{2\pi} d\phi \left(\hat{\mathbf{x}}\left(\sin^2\theta\cos\theta\cos\phi\right) + \hat{\mathbf{y}}\left(\sin^2\theta\cos\theta\sin\phi\right) + \hat{\mathbf{z}}(\cos^2\theta\sin\theta)\right) \\
&= \frac{|\mathbf{P}_0|}{4\pi\epsilon_0}\int_{\theta=0}^{\pi} d\theta \int_{\phi=0}^{2\pi} d\phi \cos^2\theta\sin\theta\hat{\mathbf{z}} \\
&= \frac{\mathbf{P}_0}{4\pi\epsilon_0}\times 2\pi\times\frac{2}{3} \\
&= \boxed{\frac{\mathbf{P}_0}{3\epsilon_0}}
\end{aligned}$$

(c) In order to find the total electric field at the center of the cavity, we can consider the cavity as superposition of two systems: (i) The whole dielectric as if there is no cavity and then (ii) a sphere with exactly negative polarization and as exact size of the cavity. Hence,

$$\boxed{\mathbf{E}_{\text{total}} = \mathbf{E}_0 + \frac{\mathbf{P}_0}{3\epsilon_0}}$$

(d) The subscript "out" means outside cavity and "in" means inside cavity.

$$\mathbf{D}_{\text{out}} = \epsilon_0 \mathbf{E}_0 + \mathbf{P}_0$$
$$\mathbf{D}_{\text{in}} = \epsilon_0 \mathbf{E} = \epsilon_0 \left(\mathbf{E}_0 + \frac{\mathbf{P}_0}{3\epsilon_0} \right) = \epsilon_0 \mathbf{E}_0 + \frac{\mathbf{P}_0}{3}$$

(e)

$$E_{out}^{\perp} - E_{in}^{\perp} = \frac{\sigma}{\epsilon_0}$$
$$-\frac{P_0 \cos\theta}{3\epsilon_0} = \frac{\sigma_b + \sigma_f}{\epsilon_0}$$
$$\sigma_f = \frac{2P_0 \cos\theta}{3}$$

Now.

$$D_{out}^{\perp} - D_{in}^{\perp} = P_0 \cos\theta - \frac{P_0 \cos\theta}{3} = \frac{2P_0 \cos\theta}{3} = \sigma_f$$

This is consistent with equation (4.26) of [1].

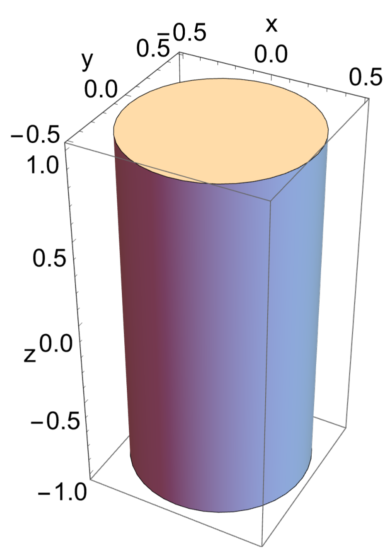

Figure 2: The cylinder whose symmetry axis is along $\hat{\mathbf{z}}$ and in units of L.

2. (a) The cylinder runs from $-L$ to L along the z axis being the symmetric axis of the cylinder, see Figure 2.

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} +P_0 & \text{on the top circular surface at } z = +L \\ -P_0 & \text{on the bottom circular surface at } z = -L \\ 0 & \text{on the curved surface} \end{cases}$$

(b) The electric field at a point z above the origin,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{\text{due to plate at } z=L} + \mathbf{E}_{\text{due to plate at } z=-L} \\ &= \left[\frac{P_0}{2\epsilon}\left(1 - \frac{L-z}{\sqrt{a^2+(L-z)^2}}\right) + \frac{P_0}{2\epsilon}\left(1 - \frac{L+z}{\sqrt{a^2+(L+z)^2}}\right)\right](-\hat{\mathbf{z}}) \qquad \text{for} \quad 0 < z < L \\ &= \boxed{\frac{P_0}{2\epsilon}\left(2 - \frac{L-z}{\sqrt{a^2+(L-z)^2}} - \frac{L+z}{\sqrt{a^2+(L+z)^2}}\right)(-\hat{\mathbf{z}})} \qquad \text{for} \quad 0 < z < L \end{aligned}$$

Verifying the boundary condition at $z = L$: As there is no source, therefore, $\sigma_f = 0$. Thus, verifying the boundary condition for the normal component of $\mathbf{D}$,

$$\begin{aligned} D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} &= \left(\epsilon_0 E^{\perp}_{\text{above}}\right) - \left(\epsilon_0 E^{\perp}_{\text{below}} + P^{\perp}_0\right) \\ &= \epsilon_0\left(E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}}\right) - P^{\perp}_0 \qquad \text{(perpendicular direction is along } \hat{\mathbf{z}}) \\ &= \epsilon_0 \frac{\sigma}{\epsilon_0} - P_0 \qquad (\sigma = \sigma_b + \sigma_f \quad \text{but} \quad \sigma_f = 0) \\ &= \sigma_b - P_0 \\ &= P_0 - P_0 \\ &= 0 \end{aligned}$$

(c) Case I: $\frac{a}{L} \to 0$: Substituting $a/L \to 0$ in the above expression of $\mathbf{E}$ we see the electric field vanishes; which of course make sense because when the radius of the cylinder is much much smaller compared to the length of it then, the observer at the origin sees the charge distribution at $\pm L$ as point charges. Hence the whole system behaves like a dipole. See Figure 3

Case II : $\frac{a}{L} \to \infty$: In this case

$$\mathbf{E} = \frac{P_0}{\epsilon_0} = \frac{\text{surface charge density}}{\epsilon_0}$$

.

This is exactly the field inside a parallel plate capacitor which can be intuitively seen from the geometry. See Figure 3

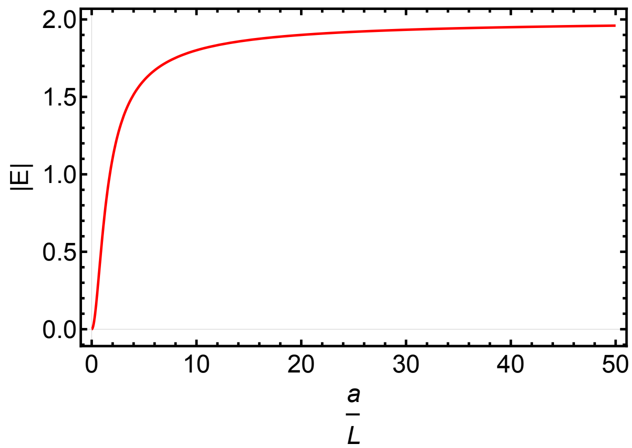

Figure 3: This is the plot of magnitude of electric field of the configuration vs a/L. The y axis is in the units of $P_0/2\epsilon_0$. One can clearly see the field vanishes at the origin which indicates the observer at the origin sees the charge distribution behaves as point charges. Similarly, at large a/L the curve approaches 2, which implies the system behaves as a parallel plate capacitor.

3. (a) The displacement vector can be found using the Gauss' law for **D**.

$$\boxed{\mathbf{D} = \sigma\hat{\mathbf{x}}}$$

The electric field

$$\boxed{\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\sigma}{\epsilon}\hat{\mathbf{x}} = \frac{\sigma}{\epsilon_0\epsilon_r}\hat{\mathbf{x}} = \frac{\sigma}{\epsilon_0\left(1+\alpha\frac{x^2}{d^2}\right)}\hat{\mathbf{x}}}$$

The polarization vector

$$\boxed{\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \sigma\left(1 - \frac{1}{\epsilon_r}\right)\hat{\mathbf{x}}}$$

(b) The plots are as follows, Figure4

(c) The bound surface charge density

$$\boxed{\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} 0 & \text{at} \quad x = 0 \\ \sigma\frac{\alpha}{1+\alpha} & \text{at} \quad x = d \end{cases}}$$

The bound volume charge density

$$\boxed{\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\sigma\frac{2\alpha x}{d^2}\left(1 + \frac{\alpha x^2}{d^2}\right)^{-2}}$$

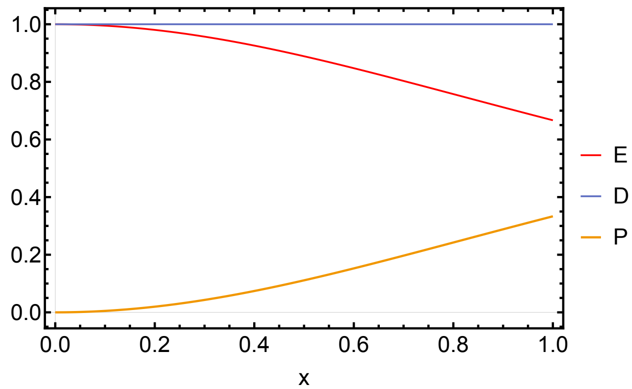

Figure 4: Plot of the magnitude of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{P}$ in the units of σ and x is in the units of d. We have taken $\epsilon_0 = \alpha = 1$

(d) The potential difference between the plates,

$$\begin{aligned} V &= \int_0^d \mathbf{E}\cdot\mathbf{dx} \\ &= \int_0^d \frac{\sigma}{\epsilon_0}\frac{1}{\left(1+\frac{\alpha x^2}{d^2}\right)}dx \\ &= \frac{\sigma d}{\epsilon_0}\int_0^1 \frac{1}{1+\alpha y^2}dy \qquad \left(y = \frac{x}{d}\right) \\ &= \frac{\sigma d}{\epsilon_0}\frac{\arctan\sqrt{\alpha}}{\sqrt{\alpha}} \end{aligned}$$

The capacitance ,

$$\boxed{C = \frac{Q}{V} = \frac{\sigma A}{\frac{\sigma d}{\epsilon_0}\frac{\arctan\sqrt{\alpha}}{\sqrt{\alpha}}} = \frac{\epsilon_0 A}{d}\frac{1}{\frac{\arctan\sqrt{\alpha}}{\sqrt{\alpha}}}}$$

When $\alpha \to 0$,

$$\frac{1}{\sqrt{\alpha}}\arctan\sqrt{\alpha} = \frac{1}{\sqrt{\alpha}}\left(\sqrt{\alpha} - \frac{\alpha^{3/2}}{3} + \frac{\alpha^{5/2}}{5}\right) = 1 - \frac{\alpha}{3} + \frac{\alpha^2}{5} + ... \approx 1 + \mathcal{O}(\alpha)$$

Therefore, $\alpha \to 0$, $C = \frac{\epsilon_0 A}{d}$ which is the result in vacuum.

4. (a) There is $+Q$ charge on the innermost surface i.e. at $r = r_a$ and $-Q$ at the outermost surface i.e. at $r = r_c$.

i. Using the Gauss' law for the displacement vector $\mathbf{D}$ we get

$$\boxed{\mathbf{D} = \begin{cases} \frac{Q}{4\pi r^2}\hat{\mathbf{r}} & r_a < r < r_c \\ \mathbf{0} & \text{elsewhere} \end{cases}}$$

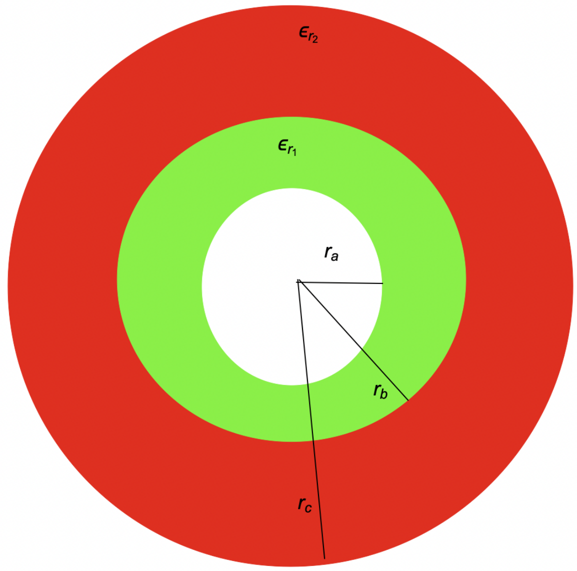

Figure 5:

Hence the electric field, ($\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E}$)

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0\epsilon_{r_1}} \frac{Q}{r^2} \hat{\mathbf{r}} & r_a < r < r_b \\ \frac{1}{4\pi\epsilon_0\epsilon_{r_2}} \frac{Q}{r^2} \hat{\mathbf{r}} & r_b < r < r_c \\ \mathbf{0} & \text{elsewhere} \end{cases}$$

ii. To compute the bound charges we need the polarization first. Using $\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}$,

$$\mathbf{P} = \begin{cases} \frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon_{r_1}}\right) \hat{\mathbf{r}} & r_a < r < r_b \\ \frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon_{r_2}}\right) \hat{\mathbf{r}} & r_b < r < r_c \end{cases}$$

Therefore,

$$\sigma_b = \begin{cases} -\frac{Q}{4\pi r_a^2} \left(1 - \frac{1}{\epsilon_{r_1}}\right) & \text{at} \quad r = r_a \\ \frac{Q}{4\pi r_b^2} \left(\frac{1}{\epsilon_{r_2}} - \frac{1}{\epsilon_{r_1}}\right) & \text{at} \quad r = r_b \\ \frac{Q}{4\pi r_c^2} \left(1 - \frac{1}{\epsilon_{r_2}}\right) & \text{at} \quad r = r_c \end{cases}$$

Following the boundary condition (4.40) of [1]

$$\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \begin{cases} \frac{Q}{4\pi r^2} = \sigma_f & \text{at} \quad r = r_a \\ 0 & \text{at} \quad r = r_b \\ -\frac{Q}{4\pi r^2} = \sigma_f & \text{at} \quad r = r_c \end{cases}$$

iii. The potential difference between the plates,

$$V = \frac{Q}{4\pi\epsilon_0}\left(\int_{r_a}^{r_b}\frac{1}{\epsilon_{r_1} r^2}dr + \int_{r_b}^{r_c}\frac{1}{\epsilon_{r_2} r^2}dr\right)$$
$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\epsilon_{r_1}}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)+\frac{1}{\epsilon_{r_2}}\left(\frac{1}{r_b}-\frac{1}{r_c}\right)\right)$$

Capacitance,

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0}{\left(\frac{1}{\epsilon_{r_1}}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)+\frac{1}{\epsilon_{r_2}}\left(\frac{1}{r_b}-\frac{1}{r_c}\right)\right)}$$

(b) Using the Gauss' law for $\mathbf{D}$, the displacement vector

$$\boxed{\mathbf{D} = \sigma\hat{\mathbf{x}}}$$

The electric field

$$\boxed{\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\sigma}{\epsilon}\hat{\mathbf{x}} = \frac{\sigma}{\epsilon_0\epsilon_r}\hat{\mathbf{x}} = \frac{\sigma}{\epsilon_0\left(1+\alpha\frac{x}{d}\right)}\hat{\mathbf{x}}}$$

The polarization vector

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \sigma\left(1-\frac{1}{\epsilon_r}\right)\hat{\mathbf{x}}$$

The bound surface charge density,

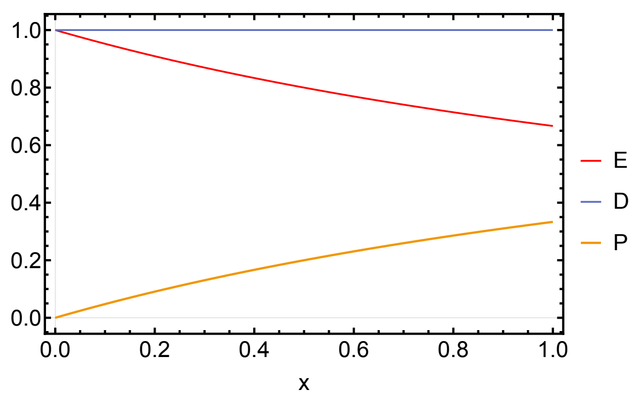

Figure 6: Plot of the magnitude of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{P}$ in the units of σ and x is in the units of d. We have taken $\epsilon_0 = \alpha = 1$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} 0 & \text{at} \quad x = 0 \\ \sigma\frac{\alpha}{1+\alpha} & \text{at} \quad x = d \end{cases}$$

The bound volume charge density

$$\rho_b = -\mathbf{\nabla} \cdot \mathbf{P} = -\sigma \frac{\alpha}{d}\left(1 + \frac{\alpha x}{d}\right)^{-2}$$

The potential difference between the plates,

$$\begin{aligned} V &= \int_0^d \mathbf{E} \cdot \mathbf{dx} \\ &= \int_0^d \frac{\sigma}{\epsilon_0} \frac{1}{\left(1 + \frac{\alpha x}{d}\right)} dx \\ &= \frac{\sigma d}{\epsilon_0} \int_0^1 \frac{1}{1 + \alpha y} dy \qquad \left(y = \frac{x}{d}\right) \\ &= \frac{\sigma d}{\epsilon_0} \frac{ln(1+\alpha)}{\alpha} \end{aligned}$$

The capacitance,

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d} \frac{1}{\frac{ln(1+\alpha)}{\alpha}}$$

When $\alpha \to 0$, $\frac{ln(1+\alpha)}{\alpha} \approx 1 - \mathcal{O}(\alpha)$, hence $C = \frac{\epsilon_0 A}{d}$.

Solution to the suggested question

5. (a) In order to find the electric field we need to find the Potential by solving the Laplace's equation with appropriate boundary condition. From equation (3.65) of [1] we already know the solution of the Laplace's equation for an azimuthal symmetric system.

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

and we will use the following boundary conditions

i. $V_{\text{in}} = V_{\text{out}}$ at $r = R$

ii. $\epsilon \frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}$ at $r = R$

iii. $V_{\text{out}} = -E_0 r \cos\theta$ at $r \gg R$

Now let's fix A_l and B_l using the above boundary conditions (B.C.).

Inside the sphere

$$V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Outside the sphere using B.C.(iii)

$$V_{\text{out}}(r, \theta) = E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Now, using B.C. (i) it is easy to see,

$$\begin{cases} A_l R^l = \frac{B_l}{R^{l+1}} & \text{for} \quad l \neq 1 \\ A_1 R = -E_0 R + \frac{B_1}{R^2} \end{cases}$$

Meanwhile, B.C. (ii) yileds,

$$\begin{cases} \epsilon_r l A_l R^{l-1} = -\frac{l+1}{R^{l+2}} & \text{for} \quad l \neq 1 \\ \epsilon_r A_1 = -E0 - \frac{2B-1}{R^3} \end{cases}$$

It follows,

$$\begin{cases} A_l = B_l = 0 & \text{for} \quad l \neq 1 \\ A_1 = -\frac{3}{\epsilon_r+2} E_0, & \qquad B_1 = \frac{\epsilon_r - 1}{\epsilon_r+2} R^3 E_0 \end{cases}$$

Therefore,

$$V_{\text{in}}(r,\theta) = -\frac{3E_0}{\epsilon_r + 2} r\cos\theta = -\frac{3E_0}{\epsilon_r + 2} z$$

Hence, the electric field ($\mathbf{E}_0$ is along z axis)

$$\boxed{\mathbf{E}_{\text{in}} = \frac{3}{\epsilon_r + 2}\mathbf{E}_0}$$

Next, from equation (4.30) of [1]

$$\boxed{\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}_{\text{in}} = \epsilon_0 (\epsilon_r - 1) \frac{3}{\epsilon_r + 2}\mathbf{E}_0 = 3\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\mathbf{E}_0}$$

(b) We will consider a spherical system with radius R. The net electric field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_{\text{self}}$. Now, following the problem (3.47) of [1]

$$\mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$$

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_{\text{self}} \\ &= \mathbf{E}_0 \left(1 - \frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\right) \\ &= \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_0 \qquad \qquad \left(\because N = \frac{1}{\frac{4}{3}\pi R^3}\right) \end{aligned}$$

We know,

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}_0 = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}}\mathbf{E} = \epsilon_0\chi_e\mathbf{E}$$

Solving the above equation for χ_e and using the relation $\chi_e = \epsilon_r - 1$ we get

$$\boxed{\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)}$$

While solving for α in the above we get

$$\chi_e - \chi_e \frac{N\alpha}{3\epsilon_0} = \frac{N\alpha}{\epsilon_0}$$

Then in the limit $\frac{N\alpha}{\epsilon_0} \ll 1$ it reduces to

$$\boxed{\chi_e = \epsilon_r - 1 \approx \frac{N\alpha}{\epsilon_0}}$$

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.