So far, we have seen two proofs using induction:

1. **Theorem:** Let $G \in \mathcal{E}_e$ → (class of graphs where each vtx. has even degree.)
   Then G has (admits) a cycle partition. ▨

2. **Theorem:** Let L be a $(2^n \times 2^n)$-grid and let s denote any arbitrary square (in L). $(n \in \mathbb{N} - \{0\})$
   Then $\underbrace{L - s}_{\text{the entire grid } L \text{ except square } s}$ can be tiled using L-shaped (⊞) tiles. ▨

TODAY, we will prove two more theorems using induction:

3. **Theorem:** The sum of the $\underline{\text{first } n \text{ odd natural \#s}}$ $(n \in \mathbb{N}-\{0\})$ is a perfect square. $\underbrace{\{1, 3, 5, \ldots\ldots\}}_{n \text{ of them}}$

   same as: $\sum_{k=1}^{n}(2k-1) = s^2$ for some $s \in \mathbb{Z}$

4. **Theorem:** Every tree on n vertices $(n \geq 1)$ has precisely $n-1$ edges.

Let us try proving 3: By induction on n.

For $n=1$: $\sum_{k=1}^{1}(2k-1) = 1 = 1^2$. DONE.

Now suppose that $n \geq 2$ and assume inductively that the desired conclusion holds for all positive integers less than n.

$$\sum_{k=1}^{n}(2k-1) = \left(\sum_{k=1}^{n-1}(2k-1)\right) + (2n-1) = s^2 + (2n-1)$$

for some integer s

Observe:

set of first n natural #s $= \{2k-1 : 1 \leq k \leq n\}$

$$\sum_{k=1}^{n}(2k-1) = \left(\sum_{k=1}^{n-1}(2k-1)\right) + (2n-1) = s^2 + (2n-1)$$

for some integer s

By Induction Hypothesis

$=$ .... HOW DO WE DO THIS "MAGIC"? .... $= t^2$ for some integer t

I DON'T KNOW HOW!

Luckily, we can do a different type of MAGIC — a trick called "stronger induction hypothesis".

$s^2 + (2n-1) =$ we don't have enough information to show this $= t^2$ for some integer t

(where s is some integer)

Let's see if we can "guess" value of s based on n:

$1 = 1$

$1+3 = 4$

$1+3+5 = 9$

$\vdots$

$$\underbrace{1+3+5+\dots+(2n-1)}_{n \text{ numbers}} \overset{???}{=} n^2$$

What if we try to prove a stronger theorem?

<u>Theorem</u>: The sum of first n odd natural #s is n^2.

Clearly, this is stronger than original statement (on previous page).

Theorem: The sum of the first n odd natural #s is n^2. ($n \in \mathbb{N} - \{0\}$)

Proof: (Let's see if this works...)

By induction on n. For n=1: $\sum_{k=1}^{1}(2k-1) = 1 = 1^2 = n^2$. DONE.

Now suppose that $n \geq 2$ and assume inductively that the desired conclusion holds for all positive integers less than n.

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n-1}(2k-1) + (2n-1) = (n-1)^2 + (2n-1)$$

By (new) Induction Hypothesis (which is stronger than earlier ; S2)

$= (n^2 - 2n + 1) + (2n-1)$

$= n^2$. (the desired conclusion)

This completes the proof. ∎

It is somewhat surprising that it is easier to prove something stronger! This is a very useful trick when you are unable to make progress in a proof using induction.

Now let's prove:

Theorem: Every tree on n vertices ($n \geq 1$) has precisely n-1 edges.

Proof: By induction on n (that is, # of vertices).
Let T be a tree on n vertices.

If $n=1$, clearly $T \cong K_1$ (o), and # of edges $= 0 = n-1$. DONE.

Now suppose that $n \geq 2$ and assume inductively that the desired conclusion holds for all trees with fewer than n vertices.

By theorem proved* in earlier lectures, since $n \geq 2$, T has a leaf, say v. (Recall: leaf is a vertex of degree 1)

By Quiz-2** Problem-1, $T-v$ is a tree.

Observe that $|V(T-v)| = n-1$ (since we deleted 1 vtx. from T)

and $|E(T-v)| = |E(T)| - 1$ (by defn. of leaf).

So, $|E(T)| = |E(T-v)| + 1 = \left(|V(T-v)| - 1\right) + 1$

By Induction Hypothesis

$= ((n-1)-1)+1$

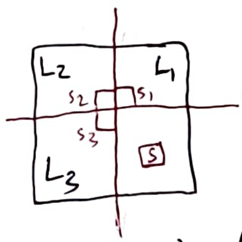
$= n-1$ $(= |V(T)|-1)$. This proves the desired conclusion. $\boxtimes$

We used two earlier results:

* Every tree, on two or more vertices, has a leaf.

** If T is a tree and v is a leaf of T then $T-v$ is a tree.

We have proved 4 theorems using induction.
In each case, we considered a "big" object (in the class of objects we care about) and did some operation to obtain one (or more) "smaller" objects (in the same class)

Let's take another look and see what induction tools we used in each case:	such an operation (and its validity) is called an $\boxed{\text{induction tool}}$:
Theorem:	
① Let $G \in \mathcal{E}$. Then G admits a cycle partition.	Existence of cycle & removal of edges of a cycle.
② L: $(2^n \times 2^n)$-grid ; s : arbitrary square. Then $L - s$ can be tiled using L-shaped tiles.	 Breaking into 4 smaller grids & systematically identifying squares in each of them.
③ Sum of first n odd natural #s is n^2.	Subtracting 1 to get a smaller natural #.
④ Every tree on n vertices has precisely $n-1$ edges.	Existence & removal of a leaf.

Most proofs using induction can be rewritten as proofs using contradiction using a strategy called the "minimal criminal" method jokingly — formally known as the minimal/smallest counterexample method!

We will see one example of this method:

Theorem: Let $G \in \mathcal{C}_e$. Then G admits a cycle partition.

Proof: We prove by contradiction. Suppose $\exists$ a graph in $\mathcal{C}_e$ that does NOT admit a cycle partition. Among all such ~~Ba~~ graphs (aka counterexamples to the theorem), let G denote one that has as few edges as possible.

choice of G

In other words, G is a smallest counterexample, or a minimal/smallest criminal.

Observe that G is NOT an empty graph (since empty graphs admit a cycle partition: $\emptyset$). Thus, G has ≥ 1 edge, say e. Let H denote the component of G that contains e. Observe that, in H, each vertex has even degree AND each vtx. has degree ≥ 2. Since $d_H(v) \geq 2 \ \forall v \in V(H)$, H has a cycle, say C. Observe that C is a cycle in G. Now, let $J := G - E(C)$. By previously proved lemma, $J \in \mathcal{C}_e$. Also, $|E(J)| < |E(G)|$. By choice of G, J admits a cycle partition, say $\mathcal{C}$. Observe that $\mathcal{C} \cup \{C\}$ is a cycle partition of G; contradicts choice of G again. Thus there are NO counterexamples. Proves theorem. ☺

Technical point: smallest versus minimal

among all counterexamples (to theorem), choose one G with as few edges as possible

choose a counterexample (to theorem), say G, such that each proper subgraph of G is NOT a counterexample.

$\boxed{\text{This}}$ would also work in our proof since J is obtained from G by removing edge set of a cycle.