Examples: ~~Bo~~ Recurrence $a_n = a_{n-1} + a_{n-2}^{2}$ is NOT linear.

Recurrence $b_n = 2b_{n-1} + 1$ is NOT homogeneous.

Recurrence $d_n = n\, d_{n-1}$ does NOT have constant coefficients.

We have seen how to solve linear homogeneous recurrences of degree 2 with constant coefficients when the characteristic equation has 2 distinct roots.

↓

what about degree k with k distinct roots?

↓

Turns out there is an easy generalization of the same recipe.

→ proof omitted

Theorem: (distinct roots case)

$a_n = c_1 a_{n-2} + c_2 a_{n-2} + \dots + c_k a_{n-k}$: recurrence where $c_1, c_2, \dots, c_k$ are real #s & $c_k \neq 0$

$x^k - c_1 x^{k-1} - \dots - c_k = 0$: characteristic equation

Suppose that the characteristic equation has k distinct roots, say $x_1, x_2, \dots, x_k$. Then $\alpha_1 x_1^n + \alpha_2 x_2^n + \dots + \alpha_k x_k^n$ $(\forall n \in \mathbb{N})$ is a closed form formula for the given recurrence — where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

↳ that may be computed using the initial conditions

Let's see an example:

Consider the recurrence

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

$(k = 3)$

Char. Eqn.:

$$x^3 - 6x^2 + 11x - 6 = 0$$

Observe that

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

Char. Roots: $x_1 = 1, x_2 = 2, x_3 = 3$

(3 distinct roots)

By Theorem (on previous page), closed form looks like

$$\alpha_1 x_1^n + \alpha_2 x_2^n + \ldots + \alpha_k x_k^n$$

$$= \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$$

$$= \alpha_1 + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$$

with initial conditions

$$a_0 = 2,\ a_1 = 5\ \&\ a_2 = 15$$

Let's determine the constants α_1, α_2 & α_3 using the above initial conditions:

$n=0$: $2 = \alpha_1 + \alpha_2 + \alpha_3$

$n=1$: $5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$

$n=2$: $15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$

Solving this system of linear equations (DIY), we get: $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2$

Thus the final closed form we obtain is:

$$\boxed{1 - 2^n + 2 \cdot 3^n}$$

In All of the examples we have seen so far: the characteristic equation had k distinct roots (where k is the degree of the recurrence).

What if the characteristic equation does NOT have k distinct roots? Turns out we can deal with those too; we will only consider the case of $k = 2$.

Theorem: ($k=2$; single root (with multiplicity two) case) → proof omitted

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$: recurrence (where $c_1, c_2 \in \mathbb{R}$ and $c_2 \neq 0$)

$x^2 - c_1 x - c_2 = 0$: characteristic eqn.

Suppose that the characteristic eqn. has only one root x_0 (with multiplicity two).

Then $\alpha_1 x_0^n + \alpha_2 n x_0^n$ ($\forall n \in \mathbb{N}$) is a closed form formula for the given recurrence — where α_1 & α_2 are constants.

that may be computed using the initial conditions

Let's see an example:

Consider the recurrence

$a_n = 6a_{n-1} - 9a_{n-2}$

with initial conditions $a_0 = 1$ & $a_1 = 6$

Char. Eqn.: ($k=2$)

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

Char. Roots:

$x_0 = 3$ (with multiplicity two)

By above theorem, closed form looks like:

$\alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$

Let's determine the constants α_1 & α_2 using the initial conditions:

$n=0$: $1 = \alpha_1$

$n=1$: $6 = 3\alpha_1 + 3\alpha_2$

Solving this system of linear equations (DIY), we get: $\alpha_1 = 1$; $\alpha_2 = 1$.

Thus the final closed form we obtain is:

$$\boxed{3^n + n \cdot 3^n}$$

What if $k>2$ and we don't have k distinct roots? See Rosen 519-520. ← Will NOT be tested on end sem exam.