We have been counting a lot for the last few weeks. In particular, we have computed the cardinalities of various *finite* sets, and we have compared them.

What about infinite sets?

In particular, can we compare their cardinalities?

YES, we can! Recall: Two sets are $\boxed{\text{equicardinal}}$ if $\exists$ a bijection between them.

So, ~~[illegible]~~ using this definition, we may compare the cardinalities of various infinite sets.

Here are some infinite sets:

$$\mathbb{N},\ \mathbb{Z},\ \mathbb{Z}^+,\ \mathbb{Q},\ \mathbb{R},\ \mathbb{Z}^+_{odd},\ \mathbb{Z}^+_{even}$$

We will compare the cardinalities of some of them; rest are DIY.

set of odd positive integers

set of even positive integers

Let's begin by comparing $|\mathbb{Z}^+|$ & $|\mathbb{Z}^+_{even}|$.

Can we find/establish a bijection b/w these sets?

$$\mathbb{Z}^+ := \{1, 2, 3, \ldots\}$$
$$\updownarrow \quad \updownarrow \quad \updownarrow$$
$$\mathbb{Z}^+_{even} := \{2, 4, 6, \ldots\}$$

Clearly, we can. Right?

$f(x) = 2x$ where $f: \mathbb{Z}^+ \to \mathbb{Z}^+_{even}$

DIY: Prove that f is a bijection.

Thus $|\mathbb{Z}^+| = |\mathbb{Z}^+_{even}|$.

In words, $\mathbb{Z}^+$ & $\mathbb{Z}^+_{even}$ are equicardinal.

DIY: Show that $|\mathbb{Z}^+| = |\mathbb{Z}^+_{odd}|$.

DIY: Show that $|\mathbb{N}| = |\mathbb{Z}^+|$.

What about $\mathbb{Z}^+$ & $\mathbb{Z}$?

Can we establish a bijection?

Yes, we can ~~order~~ list the elements of $\mathbb{Z}$ in a systematic fashion as follows: (starting from a "first" element)

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots$

Now we can define a bijection from $\mathbb{Z}^+$ to $\mathbb{Z}$:

1	2	3	4	5	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
0	1	-1	2	-2	...

DIY: Write down the bijection $f: \mathbb{Z}^+ \to \mathbb{Z}$.

Surprise Factor: $\mathbb{Z}^+_{even}$ & $\mathbb{Z}^+_{odd}$ are proper subsets of $\mathbb{Z}^+$. In the case of finite sets, a proper subset S of a set T is always smaller. NOT in case of infinite sets!

What about $\mathbb{Z}^+$ & $\mathbb{Q}$ (set of rational #s) ?

Can we compare their cardinalities ?

Intuitively, one may think that $\mathbb{Q}$ should be "much bigger" than $\mathbb{Z}^+$. However, surprisingly, they can also be shown to be equicardinal.

We will do this NOT be explicitly defining a bijection, but simply by proving the existence of a bijection b/w $\mathbb{Z}^+$ & $\mathbb{Q}$.

In fact, we will do $\boxed{\text{THIS}}$ for $\mathbb{Z}^+$ & $\mathbb{Q}^+$.

(Then, a similar method (as in previous example $\mathbb{Z}^+$ & $\mathbb{Z}$) can be used for $\mathbb{Z}^+$ & $\mathbb{Q}$. DIY)

As in previous example, it suffices to find a systematic method to list all rational #s starting from a first element.

A first idea is to list all positive rational #s ($\frac{p}{q}$ where $p, q \in \mathbb{Z}^+$) with denominator $q = 1$, then $q = 2$, ...

Unfortunately, this is already an infinite set (it is precisely $\mathbb{Z}^+$;-)) so we will never reach

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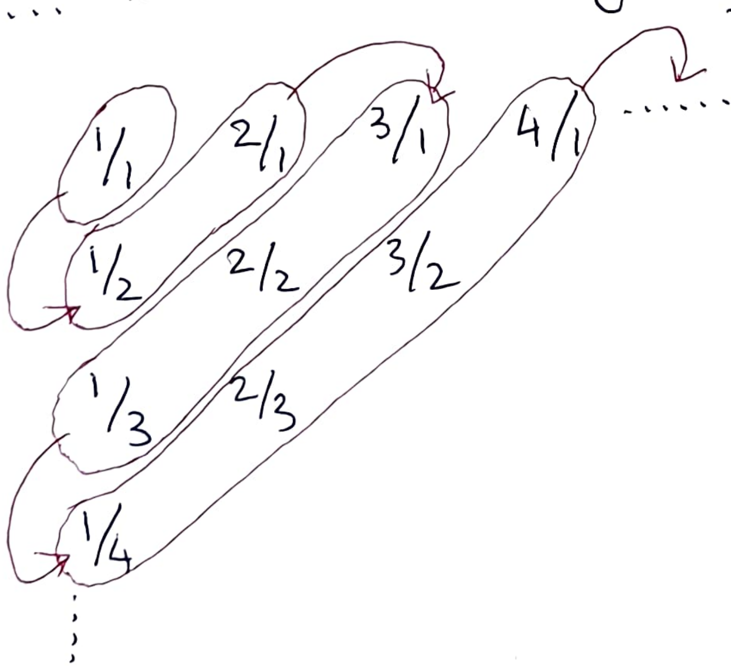
<u>Question</u>: Can we break/partition $\mathbb{Q}$ into infinitely many finite sets?

<u>Answer</u>:
Yes, we can do this based on the sum of p & q!

Let us first list all rational #s with $p+q=2$, then $p+q=3$, ... here is a nice way to visualize this IDEA:



This is a systematic way to list down ALL positive rational #s starting from a "first" element

↓

1/1 (in this case)

We have thus proved existence of a bijection b/w $\mathbb{Z}^+$ & $\mathbb{Q}^+$. Thus $|\mathbb{Z}^+| = |\mathbb{Q}^+|$.

DIY: Prove that $|\mathbb{Z}^+| = |\mathbb{Q}|$.

What about the real #s?

From the previous example, one may think (or get the impression) that any two infinite sets are equicardinal.

This is NOT true!

We will show/discuss (in the next lecture) why the set of real #s is a "bigger infinity" than the set of positive integers using a method called Cantor's Diagonalization Method.

$$\boxed{\text{END OF SYLLABUS FOR END SEM EXAM}}$$