Our first complete proof using induction:

Theorem: Let G be a graph.

G has a cycle partition $\iff$ each vertex (of G) has even degree.

($\Rightarrow$) We have already proved this (using a simple counting argument).

↳ EASY direction

TODAY, we will prove ($\Leftarrow$). Let us restate differently.

Let $\mathscr{E}$ (read: script E) denote the class of ALL graphs where each vertex has even degree.

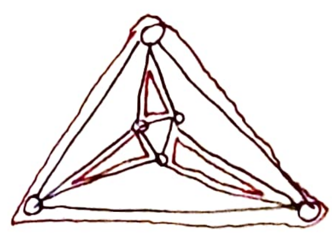
(finite, of course!)

We can restate ($\Leftarrow$) direction as follows:

Theorem: (only ($\Leftarrow$) direction as per above theorem statement).

Each graph in $\mathscr{E}$ admits (has) a cycle partition.

We will prove this theorem using induction.



Main ingredients:

1. Specify induction parameter.
2. Base case(s)
3. State the Induction Hypothesis (I.H.)
4. Induction Step (must use I.H.)

Theorem: Each graph in $\mathcal{C}$ admits (has) a cycle partition.

Proof: We will prove using induction on # of edges. → Induction parameter

Let $G \in \mathcal{C}$.

First suppose that G is an empty graph. Observe that $\emptyset$ is a cycle partition of G. ↔ BASE CASE

INDUCTION STEP:

Now suppose that G is NOT an empty graph, AND assume inductively that any graph in $\mathcal{C}$ with fewer edges than G admits a cycle partition. ↔ INDUCTION HYPOTHESIS

Let S denote the set of vertices of G whose degree is ZERO.

Let $H := G - S$.

Observe that $H \in \mathcal{C}$ and that each vertex (of H) has degree ≥ 2.

By Lemma 1 (since each vtx. has degree ≥ 2) lemma applied to H the graph H has a cycle, say C.

Observe that C is a cycle in G as well. (Why?)

Let $J := G - E(C)$. By Lemma 2, $J \in \mathcal{C}$.

Also, $|E(J)| < |E(G)|$. By INDUCTION HYPOTHESIS, the graph J admits a cycle partition, say $\mathcal{C}$.

Observe that $\mathcal{C} \cup \{C\}$ is a cycle partition of G. (This completes Induction Step and proof.) ∎

Concepts / Ingredients:

Empty graph: A graph with NO edges. Also, complement of complete graph.

$\overline{K_1}$ ∘	$\overline{K_2}$ ∘ ∘
$\overline{K_3}$ ∘ ∘ ∘	$\overline{K_4}$ ∘ ∘ ∘ ∘

Lemma 1: Let G be a graph. If each vtx. of G has degree ≥ 2 then G has a cycle. ∎

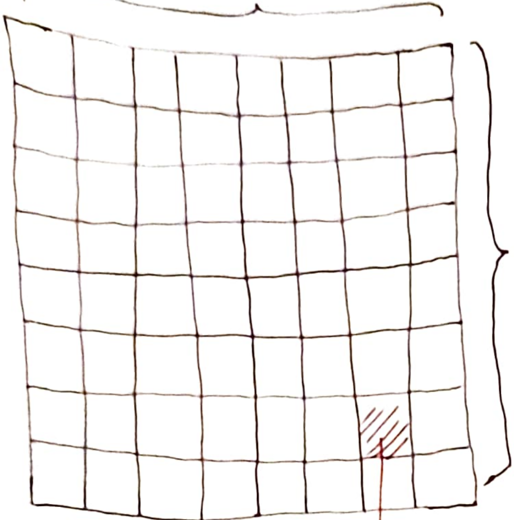
transitivity of subgraph relation.

Lemma 2: Let $G \in \mathcal{C}$ and let C denote a cycle of G. Then $G - E(C) \in \mathcal{C}$. (DIY) ∎

$n \in \mathbb{N} - \{0\}$.

A tiling problem: (we will see other such/similar problems in the future)

2^n columns

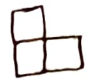


2^n rows

one square

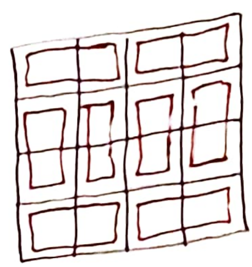
$2^n \times 2^n$ - GRID

L-shaped tiles:



What does it mean to tile a grid using tiles?

Example:



Here, we have used I-shaped tiles: 

So: ① all squares have to be covered using tiles
② Any two tiles can NOT overlap.

Question: Is it possible to tile a $2^n \times 2^n$ - GRID using L-shaped tiles?

Answer: NO. Because $2^n \times 2^n$ is NOT divisible by 3.

Interestingly, $(2^n \times 2^n) - 1$ is a multiple of 3. → This can be proved using induction. (DIY)

So, is it possible to tile a $2^n \times 2^n$ - GRID EXCEPT for any arbitary square using L-shaped tiles? 

TRY → Either prove using induction OR construct a counterexample?