Recall: Binomial Theorem: $\forall n \in \mathbb{N}$:

an equality of two polyonomials in two variables x & y ← $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(so, in particular, it holds for any two real #s x & y)

A special case of Binomial Theorem: $\forall\, n \in \mathbb{N}$ (*)

$$\boxed{(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k}$$

↓

Let us apply this to prove that the # of subsets of odd cardinality (for an n-element set with $n \in \mathbb{N} - \{0\}$) is 2^{n-1}.

Substitute $x=1$ in (*):

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots\cdots \quad \left(\text{up to } \binom{n}{n}\right) \quad -①$$

substitute $x=-1$ in (*):

$$0^n = \binom{n}{0} - \binom{n}{1} + \cdots\cdots \quad \left(\text{up to } \binom{n}{n}\right) \quad -②$$

Let's add ① & ②:

$$2^n = 2\binom{n}{0} + 2\binom{n}{2} + \cdots\cdots$$

Thus, $\binom{n}{0} + \binom{n}{2} + \cdots\cdots = 2^{n-1}$

→ LHS is counting # of subsets of even cardinality. Thus, # of subsets of odd cardinality $= 2^n - 2^{n-1} = 2^{n-1}$ ☒

Let's look at one final example of double counting:

Nerds at a Party

There are n nerds at a party, and each nerd counts the # of handshakes they participated in. At the end of the party, the host (a nerd too!) asks everyone: how many handshakes did you (including themself!) participate in?

aka Handshaking Lemma/Theorem

For example:

if $n=7$ and nerds answer:

2
3
3
4
4
5
6

for instance, is this possible?

NO.

why?

SUM = 27 (ODD #)

Theorem: Let G be a graph.

Then $$\sum_{v \in V(G)} d(v) = 2|E|.$$

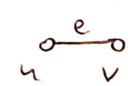
(EVEN #)

Proof: Each ~~on~~ edge contributes 2 to the LHS — why?

Alternative way of looking at it:

LHS & RHS both count # of vertex-edge incidences → pairs (v, e) such that e is incident with v ($e \in E$, $v \in V$) — if e is a loop, we count two such pairs

Case 1: non-loop edge



contributes 1 to degree of u & 1 to degree of v

Case 2: loop edge

contributes 2 to degree of v. ∎

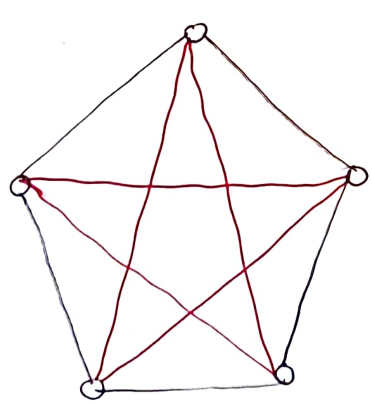
Corollary (of Hand-shaking Lemma): The number of ODD degree vertices is an EVEN number (in any (finite) graph).

Now that we are talking about parties/partying, let's go back to FRIENDS & STRANGERS at a PARTY:

Recall: Consider a complete graph on n vertices with edges colored PINK and BLUE. Is there $\boxed{\text{always}}$ a PINK K_3 $\boxed{\text{OR}}$ a BLUE K_3?

The answer depends on n.

Let's see why the answer is NO for $n \leq 5$:



There is NO BLUE K_3

$\boxed{\text{AND}}$

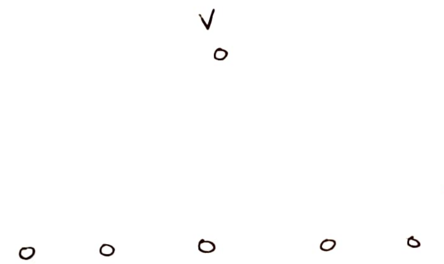
There is NO PINK K_3

Let's see why the answer is YES for $n \geq 6$:

First, let us focus on $n = 6$: ▷ Consider a complete graph G on 6 vertices with edges colored BLUE & PINK.

Let us look at the whole graph from the point of view of a specific vertex v:

This idea will be useful later too.



remaining 5 vertices

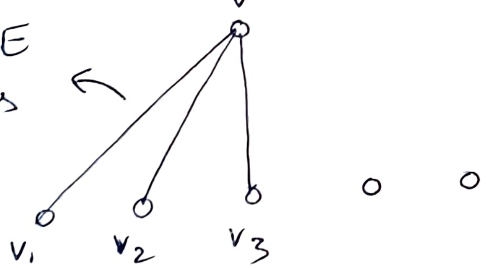
v is joined with each of them using an edge that is either RED or otherwise BLUE.

Since there are 5 edges & ONLY 2 colors, $\exists$ 3 edges that have the SAME color. (Right?)

Suppose WITHOUT LOSS OF GENERALITY that $\exists$ 3 edges that are all colored BLUE.

CAUTION: what does this mean? It means that there are 2 (or more) cases, but any case is general enough to handle all cases.

3 BLUE edges



say $e_i := vv_i$ $\forall i \in \{1, 2, 3\}$.

Now, if $v_i v_j$ (for any $1 \leq i < j \leq 3$) is a BLUE edge then vv_iv_j is a BLUE K_3 and we are DONE.

Now suppose that $v_i v_j$ is a PINK edge ($\forall\ 1 \le i < j \le 3$). In this case $v_1 v_2 v_3$ is a PINK K_3 and we are DONE. $\blacksquare$

what have we proved

Theorem: In any BLUE PINK complete graph on 6 vertices, $\exists$ BLUE K_3 OR $\exists$ PINK K_3. $\blacksquare$

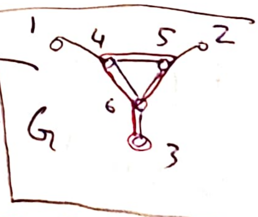
What about more than 6 vertices?

Clearly, we can just focus on the complete subgraph formed/induced by any 6 vertices, and apply above theorem. Right?

Theorem: In any BLUE PINK complete graph on 6 or more vertices, $\exists$ BLUE K_3 OR $\exists$ PINK K_3. $\blacksquare$

Definition: Induced subgraph

$G[\{3,4,5,6\}]$ is shown in RED.



Let $G := (V, E)$ be a graph and let $T \subseteq V$.
The subgraph of G induced by T, denoted $G[T]$, is the subgraph with vertex set T and edge set $\{e \in E : \text{each end of } e \text{ belongs to } T\}$.