CS1200 — Discrete Mathematics for Computer Science — 2023 (Jan-May)

Assignment 2

Release Date: 01/05/2023 **Due Date: 10/05/2023 — 11:00 PM IST**

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Academic Integrity Statement: I, Chandaluru Hema Venkata Raadhes, affirm that I have not given or received any **unauthorized** help (from any source: people, internet, etc.) on this assignment, and that I have written/typed each response on my own, and in my own words.

THE MARKS FOR EACH PROBLEM (1, 2, 3, 4, 5) ARE FIXED. HOWEVER, THE MARKS FOR EACH SUBPROBLEM (1A, 1B, 2A, ETC.) ARE TENTATIVE — THEY MAY BE CHANGED DURING MARKING IF NECESSARY.

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1. [AN EXERCISE IN LOGIC: REVISITING BASIC SET THEORY USING PROPOSITIONS] [9]

Throughout this exercise, we let S and T denote two sets (in some universe U).

Several statements are written below, and they become propositions when specific values of S and T are substituted.

Two statements are said to be *logically equivalent* if, for all values of S, T and U, the truth values of the corresponding propositions are the same (that is, either both propositions are true, or both propositions are false). For example: the two statements $\forall\, x \in U,\ x \in S \cap T$ and $\forall\, x \in U,\ x \in S \wedge x \in T$ are logically equivalent.

(a) $S \subseteq T$

(b) $(\forall\, x \in U,\ x \in S \implies x \in T) \wedge (\exists\, x \in U, x \in T \wedge x \notin S)$

(c) $T \subseteq S$

(d) $\forall\, x \in U,\ x \in T \implies x \in S$

(e) $S \cap T = \emptyset$

(f) $\forall\, x \in U, (x \in S) \oplus (x \in T)$

(g) $T \subset S$

(h) $\forall\, x \in U,\ x \in S \iff x \in T$

(i) $S = T$

(j) $\forall\, x \in U, x \in (S \oplus T)$

(k) $(\forall\, x \in U,\ x \in T \implies x \in S) \wedge (\exists\, x \in U, x \in S \wedge x \notin T)$

(l) S and T are disjoint

(m) $\forall\, x \in U,\ (x \notin S) \vee (x \notin T)$

(n) $S \subset T$

(o) $(S \subseteq T) \wedge (T \subseteq S)$

(p) (S, T) is a partition of U

(q) $\forall\, x \in U,\ (x \notin T) \vee (x \in S)$

(r) $\forall\, x \in U, \neg((x \in S) \wedge (x \in T))$

Let Z denote the set comprising the above 18 statements: from (a) to (r). That is, $Z := \{(a), (b), \ldots, (r)\}$.

Observe that logical equivalence is an equivalence relation on Z; **list all** of the equivalence classes (defined in the next exercise for your convenience).

No justifications required.

Response: Equivalence Classes are given below:(note that i have given brief explanation for every equivalence class, which is not required as per instructions, i have done it purely for the reason of easier understanding) .
1. S is subset of T.
$\{(a),\}$
2. S is proper subset of T.
$\{(b),(n)\}$
3. T is subset of S.
$\{(c),(d),(q)\}$
check q before submitting
4. S is disjoint with T.
$\{(e),(l),(m),(r)\}$
check r before submitting

5. T is proper subset of S.
$\{(g),(k)\}$
6. S and T sets have same elements. The sets are same.
$\{(h),(i),(o)\}$
7. S and T have no common elements and their union makes up the Universal Set. (S,T) is a partition of U.
$\{f),(j),(p)\}$

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2. [EQUIVALENCE RELATIONS AND EQUIVALENCE CLASSES] [6]

The goal of this exercise is to prove the following theorem:

Theorem 0.1. *Let R denote an equivalence relation on a set S. Then the equivalence classes (with respect to R) comprise a partition of the set S.*

Throughout this exercise, let R denote an equivalence relation on a set S. Recall that, for any element $a \in S$, the *equivalence class of a (with respect to R)*, denoted by $[a]_R$, is (the set) defined as follows:

$$[a]_R := \{b \in S \mid aRb\}$$

(a) **<u>Prove that</u>**, for any $x \in S$, the equivalence class $[x]_R$ is nonempty (that is, $[x]_R \neq \emptyset$).

Response: An equivalence relation is reflexive , symmetric and transitive. We shall use the reflexive property here to prove the given statement. By the reflexivity property of equivalence relations we have for $\forall x \in S$ that $x\ R\ x$.
Equivalence class of an element a is given as

$$[a]_R := \{b \in S \mid aRb\}$$

i.e set of all elements which a is related to.
Hence $[x]_R$ must contain the element x itself.
So as the equivalence class of any element must contain atleast one element (the element itself) , we can say that for any $x \in S$, the equivalence class $[x]_R$ is nonempty (that is, $[x]_R \neq \emptyset$).
Hence ,Proved.

(b) **<u>Prove that</u>**, for any $x, y \in S$, either $[x]_R = [y]_R$, or otherwise $[x]_R \cap [y]_R = \emptyset$ (that is, either the two sets $[x]_R$ and $[y]_R$ are the same, or otherwise they are disjoint).

Response: Let us start with this. For an equivalent class of x it is given that :

$$[x]_R := \{x \in S \mid xRa\}$$

We can say that for all $a \in [x]_R$ we also have that $a\ R\ x$ because of the symmetric nature of equivalence relations.

Now let us take 2 cases . $x\ R\ y$ and $x \not R\ y$.
If $x\ R\ y$ then:
For all $b \in [y]_R$ we have $y\ R\ b$. By transitivity we can now say that $x\ R\ b$ (xRy and yRb) and hence we can state that b is in $[x]_R$ also because $x\ R\ b$. So basically for all $b \in [y]_R$ we have $b \in [x]_R$.$[y]_R \subseteq [x]_R$.
Similarly we can say that for all $a \in [x]_R$ we have $a \in [y]_R.[x]_R \subseteq [y]_R$.
As both as appearing to be subsets of one another it implies that both are equal sets.
$[x]_R = [y]_R$

If $x \not R\ y$ then:
For all $b \in [y]_R$ we can say that $x \not R\ b$; because if $x\ R\ b$ were true for some $b \in [y]_R$ then x would have to be related to y (as xRb and bRy and transitivity condition) which is a contradiction to the initial condition. Hence there are no elements $b \in [y]_R$ such that $x\ R\ b$ which also means that $b \notin [x]_R$. This means that there are no elements which are in both $[y]_R$ and $[x]_R$. As if there were some element in both then there would exist an element $b \in [y]_R$ such that $b \in [x]_R$ also which was disproven just above. $[x]_R \cap [y]_R = \phi$

(c) Convince yourself that you have proved Theorem 0.1. (No response required.)

3. [PATHS IN A FOREST] [13]

The objective of this exercise is to gain a deeper understanding of paths in forests — so that we don't lose our way in a forest ;-).

In general, we view paths as subgraphs. However, when convenient (inside a proof), you may also view a path as a sequence of vertices and edges.

A path with (not necessarily distinct) ends u and v is called a *uv-path.*

(a) **Prove the following:**

Lemma 0.2. *Let G be a graph, and let $u, v \in V(G)$. If there exist two distinct uv-paths P and Q then G has a cycle.*

Response: There is given to be two paths between vertices u and v. Let these paths be P_1 and P_2, where P_2 is longer or equal length to P_1.
As the paths are distinct , there must be atleast one vertice which is not common to both paths. Take one vertice that is in P_2 but not in P_1 to be x. (possible because P_2 is longer or equal legnth to $P1$)
Choose the shortest path Q_2 which contains vertice x which is a made of a set of contiguous edges from P_2 whose end vertices are also in path P_1. Let the end vertices of this path be u_1 and v_1. (note that this is possible because there is atleast one pair of vertices common to paths P_1 and P_2 .u and v are common to both paths.)
Let the path constructed by the edges between vertices u_1 and v_1 in path P_1 be Q_1.Let the path constructed by the edges between vertices u_1 and v_1 in path P_2 be Q_2.There are no vertices that are common to the paths Q_1 and Q_2 except endpoints because if there were such common vertices then Q_2 would not be the shortest such path satisfying conditions we require. (we can find a shorter path if there are common vertice in Q1 and Q2)
Join paths Q_1 and Q_2 by merging them at end vertice u. ex : if Q_1 is represented by u e1 a e2 b e3 c e4 v and Q_2 is represented by u e5 d e6 f e7 v then join them to form v e7 f e6 d e5 u e1 a e2 b e3 c e4 v. This is a cycle as it has no repeating vertice except end vertices. Hence graph G must contain atleast one cycle.

(b) For each of the following propositions, **state** and **prove** whether it is True or False.

(P1) "In a forest, for any pair of vertices, say u and v, there is at least one uv-path."
Response: False.
A forest is just a simple acyclic graph.
Let us take a forest which consists of 2 vertices which dont have an edge between them. Clearly there is no path between those two vertices in this forest. Hence in a forest there is not necessarily a path between 2 edges. Example : Let G be a forest . $V(G) = \{1, 2\}E(G) = \phi$
There is no path between the two vertices 1 and 2 in this case.

(P2) "In a forest, for any pair of vertices, say u and v, there is at most one uv-path."
Response: True
Forests are graphs with no cycles.
Lemma 0.2 .If there is more than one path between some 2 vertices in a graph then there will be atleast one cycle in the graph which will imply the graph is not a forest. Hence we can say that to be a forest there can be atmost one path between any 2 vertices in the graphs ; Because otherwise there will be cycle .

(P3) "In a tree, for any pair of vertices, say u and v, there is at most one uv-path."
Response: True
All trees are forests because trees are just connected acyclic graphs.
This implies that any condition that applies for a graph to be a forest will also apply for a graph to be a tree. (although there may be more conditions also for that graph to be a tree).
By the previous response (P2) , we showed that "In a forest, for any pair of vertices, say u and v, there is at most one uv-path."

Hence we can also say "In a tree, for any pair of vertices, say u and v, there is at most one uv-path."

(P4) "In a tree, for any pair of vertices, say u and v, there is at least one uv-path."
Response: True
A tree is a connected acyclic graph . i.e a tree a connected forest.
Statement 1: If there is a walk from a to b, there is a path from a to b.
Proof : Take the shortest walk W1 from a to b , this will be a path because there must not be any repeating vertices. This is because if there were any repeating vertices we could remove the edges between the repeated vertices to create a shorter walk than W1 which would contradict the statement that the walk W1 itself was the shortest walk. Hence it must imply that walk W1 is also a path. Do note that if there are no repeating vertices there cannot be any repeating edge because a edge is defined between 2 vertices ; if those vertices are different the edge must be different.
The above proof was done in tutorial-3 in detail , I am not writing in detail as the motive of this question is to prove the implication of this.
A connected graph is a graph in which there is atleast one walk between every 2 vertices, this implies there is atleast one path between every 2 vertices. Hence we can say that BECAUSE trees are connected graphs , there will be atleast one path betweeen every two vertices in the tree.

(P5) "A simple graph G is a tree if and only if there is precisely one uv-path for all $u, v \in V(G)$."
Response: True
(1) To prove , if a graph G is a tree then there is precisely one uv path for all $u, v \in G$.
A tree is connected so there must be atleast one uv path for every $u, v \in G$. By problem P4.
A tree is acyclic so there are atmost one uv path for every $u, v \in G$. By problem P3.
This means for a tree to be graph there must be exactly one uv path for all $u, v \in G$.

(2) To prove, if there is precisely one uv path for every $u, v \in G$ then G is a tree.
We can say that if there is a cycle in a graph , then there must be a a u,v vertice pair such that there are two distinct paths between them.
Take the cycle represented by a sequence of continuguous ordered edges and split the cycle anywhere in the middle. This will create two paths . The end vertices of both these paths are the same.
example: cycle e1 e2 e3 e4
Edge e1 starts at vertice u , edge e2 ends at vertice v . Edge e3 starts at v , edge e4 ends at u. Now take the paths P2 : e1 e2
and path P2 : e3 e4 .
They both are different paths who have same end vertices , hence if there is a cycle in the graph then there is atleast one $u, v \in G$ such that there are atleast 2 paths between them.
This means that if there are 1 or less paths between all $u, v \in G$ then there must be no cycle in the graph.

As we showed each side of the implication ,the if and only if statement is also true.