Department of Mathematics, IIT Madras

MA1102 Series & Matrices

Solutions to Assignment-4 (Row Reduced Echelon Form)

1. Convert the following matrices into RREF and determine their ranks.

(a) $\begin{bmatrix} 5 & 2 & -3 & 1 & 7 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 & -3 & 1 & 30 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$

(a) RREF of the matrix is $\begin{bmatrix} 1 & 0 & -5/17 & -1/17 & 0 \\ 0 & 1 & -13/17 & 11/17 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. So its ranks is 3.

(b) RREF of the matrix is $\begin{bmatrix} 1 & 0 & -5/17 & -1/17 & 112/17 \\ 0 & 1 & -13/17 & 11/17 & -25/17 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So its rank is 2.

2. Determine linear independence of $\{(1,2,2,1),\ (1,3,2,1),\ (4,1,2,2),\ (5,2,4,3)\}$ in $\mathbb{C}^{1\times 4}$.

The RREF of the matrix whose rows are the given vectors is $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since a zero row has appeared in the RREF, the vectors are linearly dependent. Moreover, there had been no row exchanges in this reduction, and the fourth vector has been reduced to the zero row. Thus the fourth vector is a linear combination of the three previous ones.

Alternative: Take the matrix, where the given vectors are taken as column vectors. Reduce it to RREF. You obtain: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus the fourth vector is a linear combination of the earlier ones, whose coefficients are $2, -1, 1$, respectively. You can verify that $(5,2,4,3) = 2(1,2,2,1) - 1(1,3,2,1) + 1(4,1,2,2)$. So, the set is linearly dependent.

3. Compute the A^{-1} using RREF and also using determinant, where $A = \begin{bmatrix} 4 & -7 & -5 \\ -2 & 4 & 3 \\ 3 & -5 & -4 \end{bmatrix}$.

Compute and see that $A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$.

4. Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rrrrrl} x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = 6 \\ 2x_1 & +3x_2 & +x_3 & +4x_4 & -9x_5 & = 17 \\ x_1 & +x_2 & +x_3 & +2x_4 & -5x_5 & = 8 \\ 2x_1 & +2x_2 & +2x_3 & +3x_4 & -8x_5 & = 14 \end{array}$$

We reduce the augmented matrix to its RREF:

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right] \to \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 2 & -1 & 0 & 1 \\ 0 & \boxed{1} & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 2 & 0 & -2 & 3 \\ 0 & \boxed{1} & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Find out the row operations used in each step. Since no pivot is on the b portion, the system is consistent. To solve this system, we consider only the pivot rows, ignoring the bottom zero rows. The basis variables are x_1, x_2, x_4 and the free variables are x_3, x_5. Write $x_3 = \alpha$ and $x_5 = \beta$. Then

$$x_1 = 3 - 2\alpha + 2\beta, \quad x_2 = 1 + \alpha - \beta, \quad x_3 = \alpha, \quad x_4 = 2 + 2\beta, \quad x_5 = \beta.$$

5. Check if the system is consistent. If so, determine the solution set.

   (a) $x_1 - x_2 + 2x_3 - 3x_4 = 7,\ 4x_1 + 3x_3 + x_4 = 9,\ 2x_1 - 5x_2 + x_3 = -2,$
   $3x_1 - 2x_2 - 2x_3 + 10x_4 = -12.$

   (b) $x_1 - x_2 + 2x_3 - 3x_4 = 7, 4x_1 + 3x_3 + x_4 = 9, 2x_1 - 5x_2 + x_3 = -2,$
   $3x_1 - 2x_2 - 2x_3 + 10x_4 = -14.$

   (a) RREF of $[A|b]$ is $\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -7/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Thus inconsistent.

   (b) RREF of $[A|b]$ is $\begin{bmatrix} 1 & 0 & 0 & 2 & -10/9 \\ 0 & 1 & 0 & 1/3 & 23/27 \\ 0 & 0 & 1 & -7/3 & 121/27 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   Thus solution is $x_1 = -\frac{10}{9} + 2\alpha,\ x_2 = \frac{23}{27} + \frac{\alpha}{3},\ x_3 = \frac{121}{27} + \frac{7\alpha}{3},\ x_4 = \alpha.$

6. Using Gauss-Jordan elimination determine the values of $k \in \mathbb{R}$ so that the system of linear equations

   $$x + y - z = 1,\ 2x + 3y + kz = 3,\ x + ky + 3z = 2$$

   has (a) no solution, (b) infinitely many solutions, (c) exactly one solution.

   Gauss-Jordan elimination on $[A|b]$ yields the matrix $\begin{bmatrix} \boxed{1} & 0 & -k-3 & 0 \\ 0 & \boxed{1} & k+2 & 1 \\ 0 & 0 & (k+3)(2-k) & 2-k \end{bmatrix}$.

   (a) The system has no solution when $(k+3)(2-k) = 0$ but $2-k \neq 0$, that is, when $k = -3$.
   (b) It has infinitely many solutions when $(k+3)(2-k) = 0 = 2-k$, that is, when $k = 2$.
   (c) It has exactly one solution when $(k+3)(2-k) \neq 0$, that is, when $k \neq -3, k \neq 2$.

7. Let A be an $n \times n$ matrix with integer entries and $\det(A^2) = 1$. Show that all entries of A^{-1} are also integers.

   $\det(A^2) = [\det(A)]^2 = 1 \Rightarrow \det(A) = \pm 1$. So, A is invertible. Since A has integer entries, $\mathrm{adj}(A)$ has also integer entries. Now, $A^{-1} = [\det(A)]^{-1}\mathrm{adj}(A)$ has inteeger entries.

8. Let $A \in \mathbb{F}^{m\times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:

   (a) The equation $Ax = 0$ has a non-zero solution iff $A_1, \ldots, A_n$ are linearly dependent.

   (b) The equation $Ax = b$ has at least one solution iff $b \in \mathrm{span}\{A_1, \ldots, A_n\}$.

   (c) Let u be a solution of $Ax = b$. Then, u is the only solution of $Ax = b$ iff $A_1, \ldots, A_n$ are linearly independent.

   (d) The equation $Ax = b$ has a unique solution iff $\mathrm{rank}A = \mathrm{rank}[A|b] =$ number of unknowns.

   (a) We have scalars $\alpha_1, \ldots, \alpha_n$ not all 0 such that $\sum \alpha_i A_i = 0$. But each $A_i = Ae_i$.
   So, $A(\sum \alpha_i e_i) = 0$. Here, take $x = \sum \alpha_i e_i$. See that $x \neq 0$.

   (b) If b is a linear combination of the columns of A, then that linear combination provides a solution. Conversely, a solution provides a linear combination of columns of A which is equal to b.

   (c) We have $Au = b$. Assume that $A_1, \ldots, A_n$ are linearly independent. If $Av = b$, then $A(u - v) = 0$. Let $u - v = (\alpha_1, \ldots, \alpha_n)^T$. Then $A(u - v) = 0$ can be rewritten as $\alpha_1 A_1 + \cdots \alpha_n A_n = 0$. Since $A_1, \ldots, A_n$ are linearly independent, each α_i is 0. That is, $u - v = 0$. Conversely, if $A_1, \ldots, A_n$ are linearly dependent, then scalars $\beta_1, \ldots, \beta_n$ not all zero exist such that $\beta_1 A_1 + \cdots \beta_n A_n = 0$. That is, $Av = 0$ with $v = (\beta_1, \ldots, \beta_n)^T$. Then, u and $u + v$ are two solutions of $Ax = b$.

   (d) Let $A \in \mathbb{F}^{m\times n}$.
   If the system $Ax = b$ has a unique solution, then it is a consistent system and $\mathrm{rank}(A) = n$. That is, $\mathrm{rank}(A) = \mathrm{rank}[A|b]$ and $\mathrm{rank}(A) = n =$ number of unknowns.

9. Let $A \in \mathbb{F}^{m\times n}$ have rank r. Give reasons for the following:

   (a) $\mathrm{rank}(A) \leq \min\{m, n\}$.

   (b) If $n > m$, then there exist $x, y \in \mathbb{F}^{n\times 1}$ such that $x \neq y$ and $Ax = Ay$.

   (c) If $n < m$, then there exists $y \in \mathbb{F}^{m\times 1}$ such that for no $x \in \mathbb{F}^{n\times 1}$, $Ax = y$.

   (d) If $n = m$, then the following statements are equivalent:

      i. $Au = Av$ implies $u = v$ for all $u, v \in \mathbb{F}^{n\times 1}$.

      ii. Corresponding to each $y \in \mathbb{F}^{n\times 1}$, there exists $x \in \mathbb{F}^{m\times 1}$ such that $y = Ax$.

   (a) $\mathrm{rank}(A)$ is the number of pivots in the RREF. So, it is less than or equal to the number of rows, and also less than or equal to the number of columns.
   (b) Suppose $n > m$. Then the RREF has at most m pivots. And, there are $n - m \geq 1$ number of non-pivotal columns. These non-pivotal columns are linear combinations of pivotal columns in A. So, there exist scalars $\alpha_1, \ldots, \alpha_n$ not all zero such that $\alpha_1 C_1 + \cdots \alpha_n C_n = 0$ where C_i is the ith column of A. Then $(\alpha_1, \ldots, \alpha_n)$ is a nonzero solution to $Ax = 0$. Now, $A0 = 0$ and $Au = 0$, where $u = (\alpha_1, \ldots, \alpha_n) \neq 0$.

(c) Suppose $n < m$. Let EA be the RREF of A. Consider the equation $Ax = E^{-1}e_{n+1}$. This has the same solutions as the system $EAx = e_{n+1}$. But $[EA|e_{n+1}]$ has a pivot in the right most column, which has no solution.
(d) Suppose $n = m$.
Assume (i). Then $Ax = 0$ has a unique solution. Then number of basic variables is n. So RREF of A is I. That is, A is invertible. Then $Ax = y$ has a solution for each y, namely, $x = A^{-1}y$. This proves (ii).
(ii) Conversely, assume (ii). That is, for each y, $Ax = y$ has a solution. In particular, $Ax = e_i$ has a solution for each i. Thus, A is invertible. Then $Ax = Ay$ implies $x = y$.
