Thus all (not necessarily symmetric) relations can be modeled as (or thought of as, or represented as) finite/infinite digraphs.

Let's go back to relations:

A relation R on a set U is any subset of the Cartesian product $U \times U$.

multiple ways of thinking / viewing

$\mathbb{N}, R(\leq)$

(3,3), (3,4), (3,5), ...

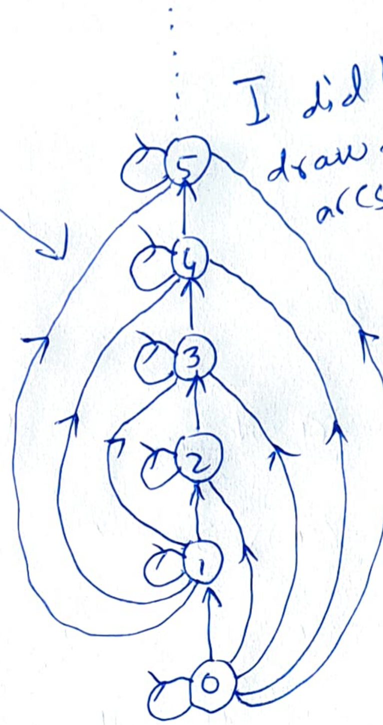
(4,4), (4,5), (4,6), ...

(0,0), (0,1), (0,2), ...

$3 \leq 3, 3 \leq 4, 3 \leq 5, \ldots$

$4 \leq 4, 4 \leq 5, 4 \leq 6, \ldots$

$0 \leq 0, 0 \leq 1, 0 \leq 2, \ldots$



I did NOT draw all arcs!

In day-to-day life, we often think about relations as "relations between humans", but is that always the case?

How about the relation "is a pet of"?

The same can be done for mathematical objects.

relates animals (mostly, dogs & cats) to humans

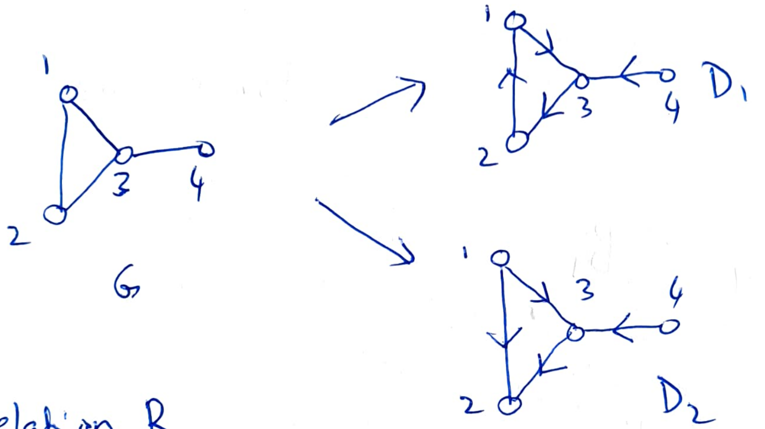
In general, a relation need NOT be defined on one set ; we may define a relation between two different sets.

For example : $\mathcal{D}$: set of all finite digraphs

$\mathcal{G}$: set of all graphs

Given a graph G, an $\boxed{\text{orientation of } G}$ is any digraph D obtained from G by putting directions on the edges of G.

Example:



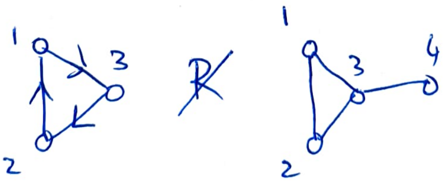
Consider the relation R "is an orientation of" from $\mathcal{D}$ to $\mathcal{G}$.

Two different orientations of G

$D_1 \; R \; G$

$D_2 \; R \; G$

However :



In general, a relation R (from set A to set B) is any subset of the Cartesian Product $A \times B$.

Why just two sets? Can we define relations on more than 2 sets?

Examples:

① Consider the relation that relates each graph with its # of vertices and # of edges.

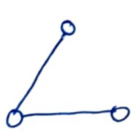
$R \subseteq \mathcal{G} \times \mathbb{N} \times \mathbb{N}$ defined as:

$(G, n, m) \in R$ if

G has n vertices & m edges;

otherwise $(G, n, m) \notin R$.

For example:

$\Big($  $, 3, 2\Big) \in R$

but $\Big($  $, 3, 1\Big) \notin R$

& $\Big($  $, 4, 2\Big) \notin R$

② Consider the relation that contains all triples $a, b, c \in \mathbb{N}$ such that $a+b=c$.

In order words, $R \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ defined as:

$(a,b,c) \in R$ if

$a+b=c$;

otherwise $(a,b,c) \notin R$.

For example:

$(17, 18, 35) \in R$

but $(17, 2, 20) \notin R$

& $(17, 18, 34) \notin R$.

$A_1, A_2, \ldots, A_n$: sets $(n \in \mathbb{N} - \{0\})$

The $\boxed{\text{Cartesian Product of } A_1, A_2, \ldots, A_n}$ is the set that contains all ordered pairs $(a_1, a_2, \ldots, a_n)$ where each $a_i \in A_i$ (for all i in $\{1, 2, \ldots, n\}$).

A $\boxed{\text{relation on sets } A_1, A_2, \ldots, A_n}$ (order matters) is any subset of the Cartesian Product $A_1 \times A_2 \times \ldots \times A_n$.

Such a relation is called $\boxed{\text{n-ary relation}}$ & n is called its $\boxed{\text{arity}}$.

# of arguments

A "real world" example:

The ~~new~~ batch CS23B is going to join IIT-M in July 2023. In order to make students feel more welcome the CS22B batch has decided to pick them at the airport / railway-station / bus-station. They need to create an Excel sheet as follows:

Name of CS23B student	Flight # / train # / bus #	Arrival date	Arrival hour	Name of CS22B volunteer

Example contd:

This Excel sheet may be viewed as a relation on:

A, B, C, D, E

- A: set containing CS23B student names
- B: set containing flight #s, train #s, bus #s
- C: set containing all dates in July 2023
- D: set containing $\{0, 1, \ldots, 23\}$ for each hour of the day
- E: set containing CS22B student names

Each row of the Excel sheet is an element of the Cartesian Product $A \times B \times C \times D \times E$.

Most commonly studied relations (in mathematics) are 2-ary (aka binary) relations.

Among the binary relations (subsets of $A \times B$) the most commonly studied case is **homogeneous binary relations** (subsets of $A \times A$).

heterogenous binary relations: subsets of $A \times B$ where $A \neq B$.

All the special properties (reflexivity, symmetry, antisymmetry, transitivity) we studied make sense (are defined) ONLY for homogeneous binary relations!