Let's go back to our theorem:

Theorem: Let G be a graph.
If each vertex (of G) has degree two or more then G has a cycle.

Let's consider the following statements:

"if G does NOT have a cycle then G has at least one vertex of degree at most one"

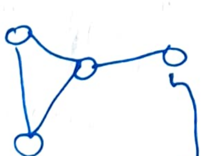
"if G has a cycle then each vertex (of G) has degree two or more"

observe that these have the same meaning (Right?)

these two have different meanings (Right?)

In fact, this is FALSE for some/many graphs.

Example:



has a cycle

but this vertex has degree < 2

Contrapositive of $P \Rightarrow Q$:

$\neg Q \Rightarrow \neg P$ SAME MEANING

Converse of $P \Rightarrow Q$:

$Q \Rightarrow P$ DIFFERENT MEANINGS

What do I mean by $P \Rightarrow Q$ & $\neg Q \Rightarrow \neg P$ having the SAME MEANINGs ?

proving $P \Rightarrow Q$ proves $\neg Q \Rightarrow \neg P$ and vice versa.

Truth Table

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

SAME COLUMN

Note: If you are trying to prove $P \Rightarrow Q$ (in some situation) and you are unable to make progress. Consider proving $\underbrace{\neg Q \Rightarrow \neg P}_{\text{the contrapositive}}$; sometimes, this is easier.

Theorem: Let G be a graph. If each vertex of G has degree two or more then G has a cycle.

contrapositive form

Theorem: Let G be a graph. If G does NOT have a cycle then G has at least one vertex of degree at most one.

SAME AS

Theorem: Every forest has a vertex of degree at most one.

example: converse does NOT hold (in general)

Theorem: Every tree has a vertex of degree at most one. $\boxed{}$

↓

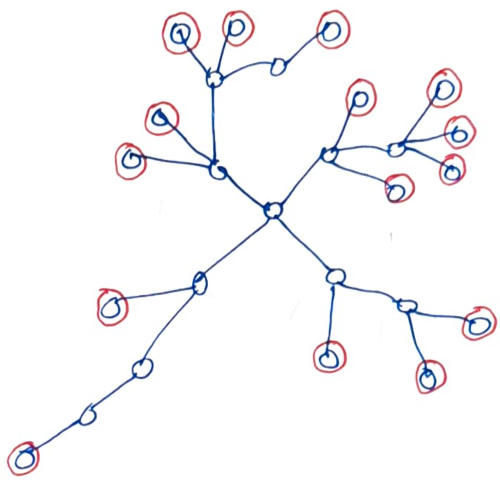
(we have proved this, right?)

Theorem: Every tree, except K_1, has a leaf. → follows immediately from above theorem

DIY: ↓ prove it yourself (from scratch) using similar ideas

TIY: Can you prove something stronger/more?

A $\boxed{\text{LEAF}}$ in a tree is any vertex of degree ONE.



LEAVES of a TREE IN RED

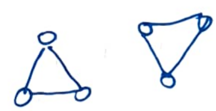
Let's go back to another theorem:

Theorem: Let G be a graph.
If G is connected & Eulerian then G has 0 or 2 vertices of ODD degree.

→ what about the converse?

FALSE in general.

For example:



NOT connected

Can we write a different version where converse is also TRUE?

Theorem: Let G be a connected graph.
G is Eulerian ⟺ (read: if and only if) G has 0 or 2 vertices of ODD degree.

→ Meaning: **Theorem:** Let G be a connected graph.
If G is Eulerian then G has 0 or 2 vertices of ODD degree.
Also, if G has 0 or 2 vertices of ODD degree then G is Eulerian.

Combining two propositions using $\Longleftrightarrow$ (two-way implication) (if and only if):

P, Q : propositions

$P \Leftrightarrow Q$ is a new proposition that is TRUE whenever either P & Q are both TRUE or P & Q are both FALSE.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Some examples:

7 is prime $\Leftrightarrow$ 17 is prime   T

7 is NOT prime $\Leftrightarrow$ 17 is NOT prime   T

7 is prime $\Leftrightarrow$ 17 is NOT prime   F

7 is NOT prime $\Leftrightarrow$ 17 is prime.   F

Why "if and only if"?

↓

$\boxed{P \text{ if } Q}$ means if Q then P that is $\boxed{Q \Rightarrow P}$

$\boxed{P \text{ only if } Q}$ means if NOT Q then NOT P.

that is

if $\neg Q$ then $\neg P$

contrapositive of

if P then Q that is $\boxed{P \Rightarrow Q}$

DIY: $P \Leftrightarrow Q$ is SAME AS → same column in truth table

$(P \Rightarrow Q)$ AND $(Q \Rightarrow P)$