Let us look at an application of the Principle of Inclusion-Exclusion:

<u>Question</u>: All CS22B students attend a concert in Chennai in November, and it is raining heavily (of course)! They are all carrying an umbrella — but the umbrellas are all black/identical. At the end of the concert, each student picks up an umbrella randomly.

(a zombie by the end of the concert)

What is the probability that NO student picks up their own umbrella?

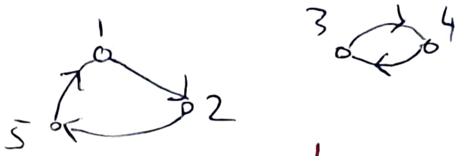
We will solve this by first translating the problem into the language/terminology of permutations.

Given a permutation $\pi : S \to S$, a $\boxed{\text{fixed point}}$ is any element $a \in S$ such that $\pi(a) = a$.

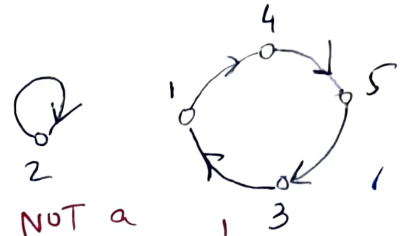
If you consider the cycle representation of the permutation π, a fixed point $a \in S$ looks like: 

A $\boxed{\text{derangement}}$ is any permutation that does NOT have a fixed point.

For example:



a derangement



NOT a derangement

Here is a translation of the "umbrellas at a concert" problem:

If you pick a permutation in $\boxed{S_n}$ (for $n \in \mathbb{N} - \{0\}$) randomly, what is the probability of picking a derangement?

set of all permutations of $\{1, 2, \ldots, n\}$

symmetric group of order n

Of course, it is simply

$$\frac{\#\text{ of derangements in } S_n}{\#\text{ of permutations in } S_n}$$

$$= \frac{\#\text{ of derangements in } S_n}{n!}$$

Next Goal: To compute THIS using the Principle of Inclusion-Exclusion.

Interestingly, we will find it easier to compute the cardinality of the complement — that is, the # of permutations that have ≥ 1 fixed point.

(in other words: # of permutations that are NOT derangements)

Let's ask some simpler questions:

Let $i \in \{1, 2, \dots, n\}$ for $n \in \mathbb{N} - \{0\}$.

How many permutations are there (in S_n) that have i as a fixed point? $(n-1)!$ Right?

Now suppose $i, j \in \{1, 2, \dots, n\}$ and i, j are distinct.

How many permutations are there (in S_n) that have both i & j as fixed points? $(n-2)!$ Right?

Let $A_i := \{\pi \in S_n : \pi(i) = i\}$

Thus $|A_i| = (n-1)!$ (where $i \in \{1, 2, \dots, n\}$)

$|A_i \cap A_j| = (n-2)!$ (where $i, j \in \{1, 2, \dots, n\}$ are distinct)

$\vdots$

$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = (n-k)!$ (where $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$ are distinct)

Note that $A_1 \cup A_2 \cup \dots \cup A_n$ is precisely the set of all permutations (in S_n) that have at least one fixed point.

By the Principle of Inclusion-Exclusion:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} \left[(-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} \left| \bigcap_{i \in I} A_i \right| \right]$$

$$= \sum_{k=1}^{n} \left[(-1)^{k-1} \binom{n}{k} (n-k)! \right]$$ (why?)

$$= \sum_{k=1}^{n} \left[(-1)^{k-1} \frac{n!}{k!} \right] = n! \left[\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} \right]$$

$$= n! \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!} \right]$$

Thus the # of derangements (in S_n) =

$$n! - n! \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!} \right]$$

$$= \boxed{n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots - (-1)^{n-1} \frac{1}{n!} \right)} \approx \frac{n!}{e}$$

using some calculus

↳ approximately

↳ answer as far as CS1200 is concerned

Thus, the probability that NO student picks up their own umbrella $\approx \dfrac{\left(\frac{n!}{e}\right)}{n!} = \boxed{\frac{1}{e}}$

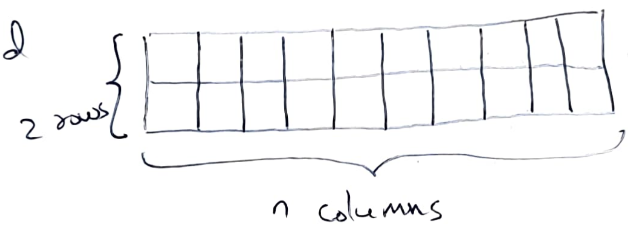
approximately

Does not depend on n ....

cool, eh?

Now, let's look at another tiling problem:

Question: How many ways of tiling a $(2 \times n)$-grid



using brick-shaped tiles?

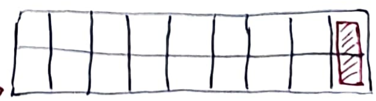
placed horizontally

placed vertically

Let $T(n)$ denote the # of tilings of a $(2 \times n)$-grid using brick-shaped tiles.

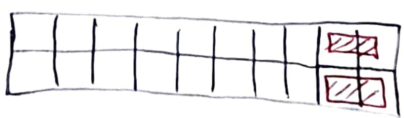
Claim: $T(n) = T(n-1) + T(n-2)$

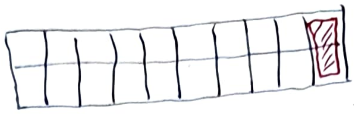
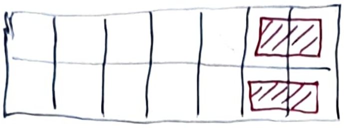
Proof: Observe that in any tiling of a $(2 \times n)$-grid, say B,



we have one of these two situations in the last 1 or 2 columns

OR



The number of such () tilings is precisely $T(n-1)$ and the number of such () tilings is precisely $T(n-2)$.

Thus $T(n) = T(n-1) + T(n-2)$.

Such an equation is called a recurrence

This, however, does NOT answer our question completely. Right?

This recurrence is useful only if we know the first two terms

$T(1)$ & $T(2)$

These can be calculated easily, right?

$T(1) = 1$

$T(2) = 2$

THIS is enough information to uniquely determine $T(n)\ \forall n \in \mathbb{N} - \{0\}$:

1, 2, 3, 5, 8, 13, 21, ...

The famous Fibonacci sequence (an Italian mathematician)

However, this sequence was known to Pingala & Hemachandra (in Indian subcontinent) much earlier.

discovered in the study of poetry! Wiki it!