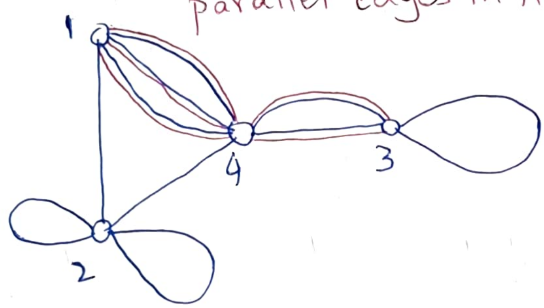
Definitions: $G := (V, E)$: a graph

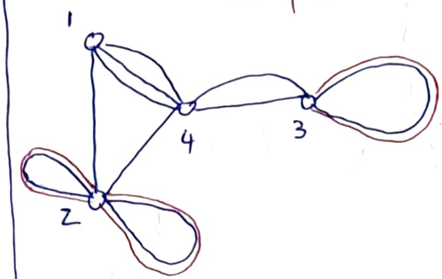
① If for some distinct $u, v \in V$, there is more than one edge joining u & v, all such edges are called **multiple/parallel edges**.

② A **loop** is an edge that joins a vertex v with itself.

multiple/parallel edges in RED

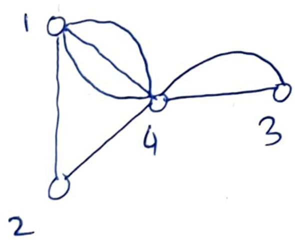


loops in RED



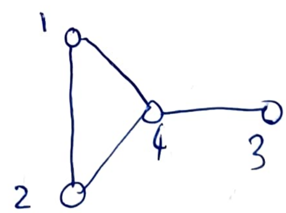
loopless graph:

A graph without loops



simple graph:

A graph without loops & without multiple/parallel edges.

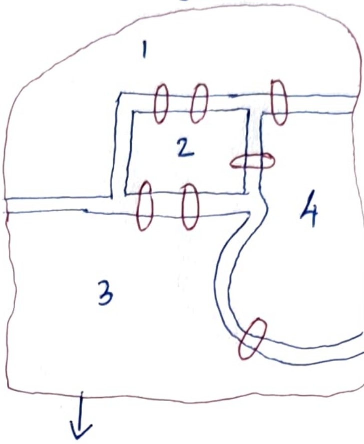


TIY: what are all the graphs that can be thought of as just symmetric relations?

MAIN POINT: Graphs are more general than symmetric relations.

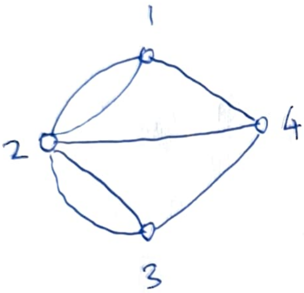
The beginning of Graph Theory (to the best of our knowledge):

The seven bridges of Königsberg



Map of Königsberg (in Prussia) during Euler's time showing layout of the 7 bridges (Pregel river)

converting to a graph



Problem: To devise a walk through the city that would cross each of those bridges exactly once.

(1736) Euler proved (using graph theory) that NO solution exists. Euler had to discover/invent graph theory in order to accomplish this.

Leonhard Euler (1707-1783)
Swiss mathematician, physicist, astronomer, geographer, logician & engineer

a generalization

Problem: Which graphs can be drawn without lifting the chalk?

TIY

(Rules: ① you are NOT allowed to draw same edge twice.
② you may think of vertices as just points in $\mathbb{R}\times\mathbb{R}$.)

Definition: degree of a vertex v in a loopless graph G is the # of edges that have v as an end.

$v \in V(G)$

Notation: $d(v)$

Let's go back to relations:

We have seen that all symmetric relations can be modeled as (or thought of as) just graphs. (finite/infinite)

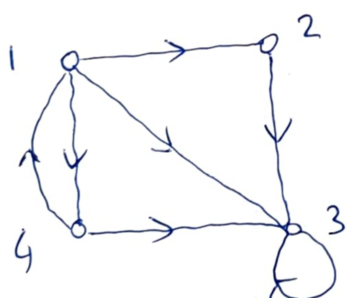
DIY: For an equivalence relation, what does the graph look like?

What about relations that are NOT symmetric? How do we represent them? Clearly, graphs are NOT good enough.

Idea: Let's put an arrow on each edge ab to indicate whether aRb or bRa.

Example: $U = \{1,2,3,4\}$

$R = \{(1,2),(1,3),(1,4),(4,1),(2,3),(4,3),(3,3)\}$



← An example of a directed graph (aka digraph)

Revisiting all special properties of relations using digraphs:

Reflexivity

loop at each element

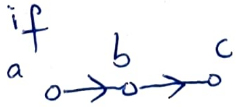
Symmetry

either a o o b

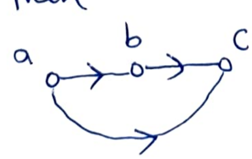
or

Transitivity

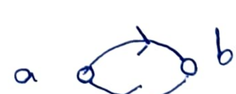
if



then



Antisymmetry



NOT allowed

Definition: A digraph $D := (V, A)$ has:

① $V := V(D)$: a set of vertices (aka vertex set)

② $A := A(D)$: a set of arcs (aka arc set)

each arc is an ordered pair of vertices (not necessarily distinct)

$a \xrightarrow{f} b$

$\boxed{\text{tail}}$ of f — aka initial vertex of f

$\boxed{\text{head}}$ of f — aka terminal vertex of f

Rosen

Arcs are also called directed edges.

For a loopless digraph D, for any vertex $v \in V(D)$, $\boxed{\text{in-degree of } v}$, denoted by $d^{in}(v)$ is the # of arcs for which v is the head; and $\boxed{\text{out-degree of } v}$, denoted by $d^{out}(v)$ is the # of arcs for which v is the tail.

An arc of a digraph is a loop if its tail is the same as its head. loop

A digraph is $\boxed{\text{loopless}}$ if it has NO loops.

Question: (TIY) For which loopless undirected graphs, is it possible to put directions on the edges so that in-degree of v = out-degree of v for each vtx v? Come up with a conjecture.