**Recall:** We started our deep dive into the theory of posets — with the intention — of proving some very general results (about some more abstract mathematical structures) — and applying them to prove cool facts about other specific mathematical objects (such as $\mathbb{N}$, $\mathbb{Z}$, etc.)

Let us see an example of this:

**Next Goal:** To prove the following.

**Erdos-Szekeres Theorem:**

Any sequence of $n^2+1$ (not necessarily distinct) integers contains a monotone subsequence of length $n+1$.

monotone means non-increasing or non-decreasing

**Examples:** ($n=3$. So, $n^2+1=10$.)

① 17, 21, 13, 41, 15, 60, 19, 23, 9, 18

a monotone subsequence of length $n+1=4$

② −3, 15, 5, 18, 5, 6, 2, 3, 1, 25

a monotone subsequence of length $n+1=4$

We will prove the Erdos-Szekeres Theorem using the machinery/theory/language of posets.

For that, we need some more concepts in poset theory.

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$(S, \preceq)$: poset

A subset $C \subseteq S$ is called a $\boxed{\text{chain}}$ if $(C, \preceq)$ is a totally ordered set (aka chain!).

In other words, a $\boxed{\text{chain}}$ is any nonempty subset $C \subseteq S$ whose elements are pairwise comparable.

In the same spirit, we can now define $\boxed{\text{antichain}}$:

An $\boxed{\text{antichain}}$ is any nonempty subset $A \subseteq S$ whose elements are pairwise incomparable.

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<u>Examples:</u> $(\{1,2,\dots,10\}, \mid)$: poset

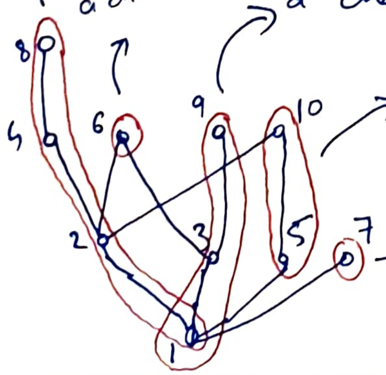
a chain: $\{1,2,4,8\}$

a chain: $\{6\}$

a chain: $\{1,3,9\}$

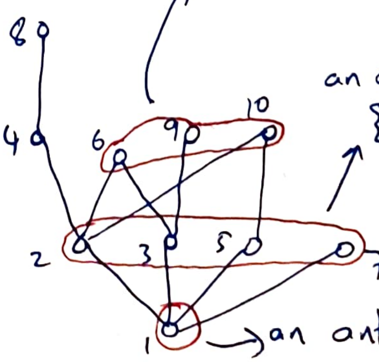
a chain: $\{5,10\}$

a chain: $\{7\}$




an antichain: $\{6,9,10\}$

an antichain: $\{2,3,5,7\}$



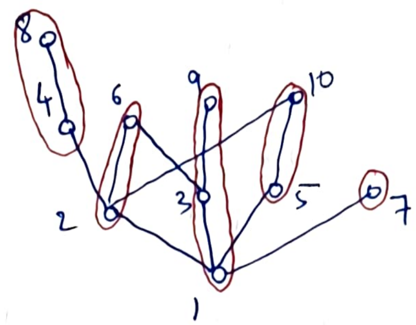

an antichain: $\{1\}$

We define one more concept: $\boxed{\text{chain partition of a poset}}$.

For a poset $(S, \preceq)$, a $\boxed{\text{chain partition}}$ is any partition of $S$ whose each part/member is a chain.

(recall definition from Module-1)

Example: poset $(\{1,2,\ldots,10\}, |)$




A chain partition: $\{\{4,8\},\{2,6\},\{5,10\},\{7\},\{1,3,9\}\}$

In other words, a $\boxed{\text{chain partition}}$ is a collection of disjoint chains $\mathcal{C} := \{c_1, c_2, \ldots\ldots\}$ such that

$$\underbrace{\bigcup_{c \in \mathcal{C}}}_{\text{union of all chains in } \mathcal{C}} c = S$$

Now, I would like you to think of two quantities in a finite poset $(S, \preceq)$:

① Among all chain partitions of $(S, \preceq)$, a chain partition of smallest cardinality (aka a smallest chain partition) and its cardinality.

② Among all antichains of $(S, \preceq)$, an antichain of largest cardinality (aka a largest antichain) and its cardinality.

Is there any relation between these?

TIY: