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**PH1020 Physics II** **Problem Set 4 - Solutions** **MAR-JUN 23**

1. Let's consider the field point $\mathbf{r} = s\hat{\mathbf{x}}$.[1] The magnetic field for a given surface current density is given by (equation 5.42 of [1])

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|^3} da' \\ &= \frac{\mu_0 K_0}{4\pi}\int \frac{\hat{\mathbf{e}}_\phi \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|^3} da'\end{aligned}$$

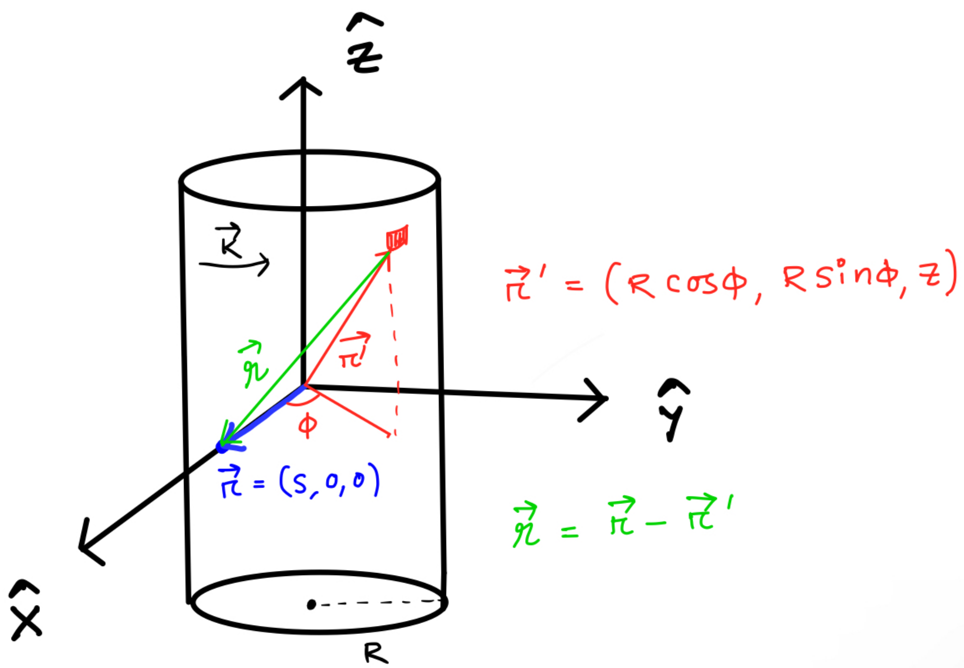


Figure 1: This is the picture of the cylinder. The red colored patch is the source. The blue vector is the field point. The green vector is the position vector of the field point w.r.t the source.

[1] Students are encouraged to take any general point, i.e. $\mathbf{r} = s\cos\phi_0\hat{\mathbf{x}} + s\sin\phi_0\hat{\mathbf{y}} + z_0\hat{\mathbf{z}}$ and verify the qualitative results are same.

Notice, the $x$ and $y$ component of the magnetic field vanishes due to symmetry [2] Now,

$$\begin{aligned} B_z &= \frac{\mu_0 K_0}{4\pi} R \int_{\phi=0}^{2\pi} \int_{z=-L/2}^{L/2} \frac{R - s\cos\phi}{(z^2+d^2)^{3/2}} d\phi dz, \qquad d^2 = R^2 + s^2 - 2Rs\cos\phi \\ &= \frac{\mu_0 K_0}{4\pi} R \int_0^{2\pi} (R - s\cos\phi) \frac{2L}{d^2\sqrt{(L^2+4d^2)}} d\phi \end{aligned}$$

In the limit $L \gg s$

$$B_z = \frac{\mu_0 K_0}{4\pi} R \int_0^{2\pi} (R - s\cos\phi) \frac{2}{d^2} d\phi$$

Now substituting $d^2 = R^2 + s^2 - 2Rs\cos\phi$ and using the formulae $\int_0^{2\pi} \frac{d\phi}{a+b\cos\phi} = \frac{2\pi}{\sqrt{a^2-b^2}}$ we get,

$$\mathbf{B} = \frac{\mu_0 K_0}{2} \left( \frac{R^2 - s^2}{|R^2 - s^2|} + 1 \right) \hat{\mathbf{z}}$$

In the case of a long solenoid (axis is parallel to $\hat{\mathbf{z}}$) consisting $n$ closely wound turns per unit length on a cylinder of radius $R$ and carrying a steady current $I$ the magentic field is given by (see Example 5.9 of [1])

$$\mathbf{B} = \begin{cases} \mu_0 n I \hat{\mathbf{z}} & \text{inside the solenoid} \\ 0 & \text{outside solenoid} \end{cases}$$

This perfectly makes sense as a closely wound solenoid can be considered as a cylinder having surface current density $\mathbf{K}$, where $|\mathbf{K}| = nI$

2. **Magnetic field B due to an infinite surface current:** Consider a uniform surface current density on $xy$ plane as $\mathbf{K} = K\hat{\mathbf{x}}$. Notice that, $B_x = 0$ (as $\mathbf{B}$ is perpendicular to $\mathbf{K}$). Moreover, $B_z = 0$ due to the symmetry of the problem (any contribution along $\hat{\mathbf{z}}$ from the differential current element strip located at $+y$ will be canceled by the contribution from a similar strip at $-y$).

   Considering the fact that the magnetic field is along $\hat{\mathbf{y}}$ direction we draw an Amperian loop as shown in Figure 2. Now, Applying Ampère's law

$$\begin{aligned} \oint \mathbf{B} \cdot \mathbf{dl} &= \mu_0 I_{\text{enclosed}} \\ \implies 2Bl &= \mu_0 K l \end{aligned}$$

[2] We will leave it up to the student to explicitly write down the $x$ and $y$ component and verify that integration w.r.t variable "$z$" vanishes if the field point is on the $x$ axis. Note that it is not necessarily true for any general point.

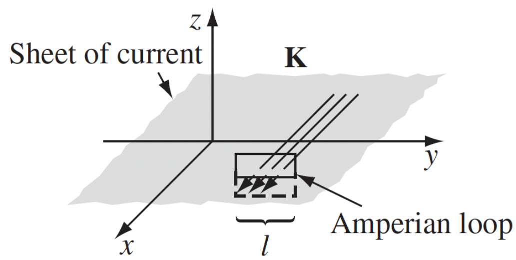


Figure 2:

Therefore,

$$\mathbf{B} = \begin{cases} +\frac{\mu_0}{2} K\hat{\mathbf{y}} & z < 0 \\ -\frac{\mu_0}{2} K\hat{\mathbf{y}} & z > 0 \end{cases}$$

The direction is determined using the Biot-Savart law.

(a) Having this idea in mind we approach to compute magnetic field of the original system consists of a parallel plate capacitor on $xy$ plane. Note that, with the surface charge density $\sigma$ moving with a velocity $V_0\hat{\mathbf{x}}$ the surface current of a sheet would be

$$\mathbf{K} = \sigma V_0 \hat{\mathbf{x}}$$

Hence, using the principle of superposition

$$\mathbf{B}_T = \mathbf{B}_L + \mathbf{B}_U$$

where, $\mathbf{B}_T$ is the total magnetic field, $\mathbf{B}_L$ is the magnetic field due to the lower plate and $\mathbf{B}_U$ is the magnetic field due to the upper plate.

$$\mathbf{B}_T = \begin{cases} +\frac{\mu_0}{2} K\hat{\mathbf{y}} - \frac{\mu_0}{2} K\hat{\mathbf{y}} & \text{above the upper plate} \\ +\frac{\mu_0}{2} K\hat{\mathbf{y}} + \frac{\mu_0}{2} K\hat{\mathbf{y}} & \text{between the two plates} \\ -\frac{\mu_0}{2} K\hat{\mathbf{y}} + \frac{\mu_0}{2} K\hat{\mathbf{y}} & \text{below the lower plate} \end{cases}$$

Therefore,

$$\mathbf{B} = \begin{cases} 0 & \text{above the upper plate} \\ +\mu_0 \sigma V_0 \hat{\mathbf{y}} & \text{between the two plates} \\ 0 & \text{below the lower plate} \end{cases}$$

(b) We know from the Lorentz force law that

$$\mathbf{F}_B = \int (\mathbf{K} \times \mathbf{B})\, da$$

Then, magnetic force per unit area for individual plate would be

$$\begin{aligned}
\frac{d\mathbf{F}_B}{da} = \mathbf{P}_B =& \mathbf{K}_{\text{individual}} \times \mathbf{B}_{\text{other}} \\
&= \begin{cases} \mathbf{K}_{\text{upper}} \times \mathbf{B}_{\text{lower}} & \text{on upper plate} \\ \mathbf{K}_{\text{lower}} \times \mathbf{B}_{\text{upper}} & \text{on lower plate} \end{cases} \\
&= \begin{cases} (\sigma V_0 \hat{\mathbf{x}}) \times \left(\frac{\mu_0 \sigma V_0 \hat{\mathbf{y}}}{2}\right) & \text{on upper plate} \\ (-\sigma V_0 \hat{\mathbf{x}}) \times \left(\frac{\mu_0 \sigma V_0 \hat{\mathbf{y}}}{2}\right) & \text{on lower plate} \end{cases} \\
&= \begin{cases} \left(\frac{\mu_0 \sigma^2 V_0^2}{2}\right) \hat{\mathbf{z}} & \text{on upper plate} \\ -\left(\frac{\mu_0 \sigma^2 V_0^2 \hat{\mathbf{y}}}{2}\right) \hat{\mathbf{z}} & \text{on lower plate} \end{cases}
\end{aligned}$$

(c) For this problem we will consider a single plate (let's say upper plate) and find the $V_0$ for which electric force balances the magnetic force. The electric force per unit area on the upper plate is

$$\begin{aligned}
\mathbf{P}_E =& \text{charge density of upper plate} \cdot \text{electric field due to lower plate} \\
=& -\frac{\sigma^2}{2\epsilon_0} \hat{\mathbf{z}}
\end{aligned}$$

Equating $|\mathbf{P}_B|$ and $|\mathbf{P}_E|$ on the upper plate we get

$$\mu_0 V_0^2 = \frac{1}{\epsilon_0}$$

Hence,

$$V_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

This is nothing but the speed of light $c$ in the vacuum.

3. It is given, the axis of the cavity runs parallel at a distance $d$ from the axis of the cylinder, as shown in Figure(3).

Now, we consider this system as superposition of two systems, (i) A complete solid cylinder of radius $R$ without any cavity and (ii) an exactly negative current density where the cavity would have been.The net magnetic field is the superposition of system (i) and system (ii). We take the vector joining the centres of the circles (the cross-sectional view of the cylinders) as the $\mathbf{x}$ axis.

Magnetic field due to system (i): We will compute the magnetic field at a distance $r$ from $O$ (see Figure 3). Note that for our case "$d - b < r < d + b$" region is relevant. Applying the Ampere's law,

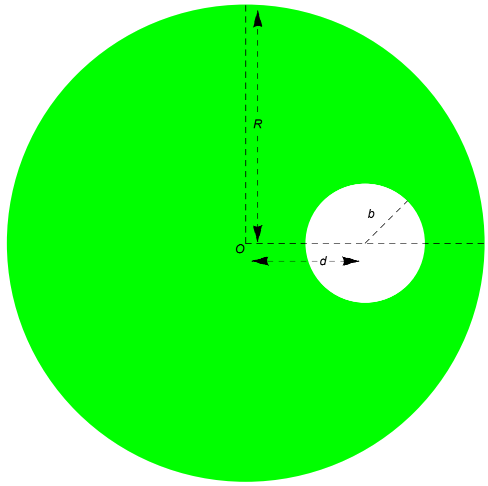


Figure 3: This is the cross sectional view of the cylinder and the cavity. $O$ is the centre of the solid cylinder. $d$ inter-centre distance. $R$ is the radius of the solid cylinder. $b$ is the radius of the cavity.

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I_{enc}$$
$$B_1 \cdot 2\pi r = \mu_0 \pi r^2 J$$
$$\mathbf{B}_1 = \frac{\mu_0 J r}{2}\hat{\mathbf{e}}_\phi = \frac{\mu_0 J}{2}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

Magnetic field due to system (ii): Using the above result, the magnetic field due the system (ii) at the same point would be

$$\mathbf{B}_2 = -\frac{\mu_0 J r}{2}\hat{\mathbf{e}}_\phi = \frac{\mu_0 J}{2}\left(\tilde{y}\hat{\mathbf{x}} - \tilde{x}\hat{\mathbf{y}}\right)$$

where $(\tilde{x}, \tilde{y})$ are the coordinates from the centre of the cavity. But notice $\tilde{y} = y$ and $\tilde{x} = x - d$. Hence,

$$\mathbf{B}_2 = \frac{\mu_0 J}{2}\left(y\hat{\mathbf{x}} - (x-d)\hat{\mathbf{y}}\right)$$

Therefore, the net magnetic field

$$\mathbf{B} = \frac{\mu_0 J d}{2}\hat{\mathbf{y}}$$

4. Let the magnetic vector potential

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

From equation (5.66) of [1] it is evident that for the given system we have $A_y = A_z = 0$ ($\mathbf{A}$ is parallel (antiparallel) to $\mathbf{K}$).[3] Hence, $A_x$ is the only non zero component. Now,

$$\begin{aligned}
\nabla \times \mathbf{A} &= \mathbf{B} \\
\frac{\partial A_x}{\partial z}\hat{\mathbf{y}} &= \begin{cases} -\frac{\mu_0 K}{2}\hat{\mathbf{y}} & z > 0 \\ +\frac{\mu_0 K}{2}\hat{\mathbf{y}} & z < 0 \end{cases} \\
\mathbf{A} &= \begin{cases} \left(-\frac{\mu_0 K}{2} z + f(y)\right)\hat{\mathbf{x}} & z > 0 \\ \left(\frac{\mu_0 K}{2} z + f(y)\right)\hat{\mathbf{x}} & z < 0 \end{cases}
\end{aligned}$$

[4] Verifying the magnetostatistic boundary condition (equation 5.78 of [1])

$$\frac{\partial \mathbf{A}}{\partial z}|_{z>0} - \frac{\partial \mathbf{A}}{\partial z}|_{z<0} = -\mu_0 \mathbf{K}$$

5. The dipole moment of the system is

$$\mathbf{m} = I\pi R^2 \hat{\mathbf{z}}$$

Given the dipole moment of a system the magnetic field is given by (Equation (5.89) of [1])

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right]$$

Substituting $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right]$$

**Suggested Question:**

6. (a) We know $\mathbf{B} = \nabla \times \mathbf{A}$ and it is given that $\mathbf{A} = \nabla \times \mathbf{W}$, then

$$\begin{aligned}
\mathbf{B} &= \nabla \times \left(\nabla \times \mathbf{W}\right) \\
&= \nabla\left(\nabla \cdot \mathbf{W}\right) - \nabla^2 \mathbf{W}
\end{aligned}$$

Fixing the gauge freedom of $\mathbf{W}$ via Lorenz gauge i.e. $\nabla \cdot \mathbf{W} = 0$ we have now,

$$\mathbf{B} = -\nabla^2 \mathbf{W}$$

Oh great! This is nothing but the Poisson's equation. That's something we know from electrostatics. In fact that's the central dogma of electrostatics. In particular, if $\mathbf{B}$ vanishes at infinity then the solution is as follows (equation (2.29) of [1])

$$\mathbf{W}(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\mathbf{B}(\mathbf{r}')}{\tilde{r}} d\tau'$$

[3] We get such a form after the gauge fixing. The author of the book has used the Lorenz gauge in equation 5.63 of [1]

[4] When we solve for $A_x$ we get $A_x = h(z) + g(x, y)$ but the Lorenz gauge ensures $A_x$ is independent of $x$.

(b) In case of uniform magnetic field

$$\mathbf{W} = \frac{1}{4\pi}\mathbf{B}(\mathbf{r}') \int \frac{1}{\tilde{r}} d\tau'$$

This integral can be uniquely computed only if the geometry is given. But more interesting question is just looking at the relation $\mathbf{B} = -\nabla^2 \mathbf{W}$ can we kind of guess $\mathbf{W}$? If so, whether it is unique or not? Let's take an attempt.

It is evident from the above relation that $\mathbf{W}$ has to be in the direction of $\mathbf{B}$ and it should be quadratic in $\mathbf{r}$. So we will consider the following form

$$\mathbf{W} = \alpha \mathbf{r}\left(\mathbf{r}\cdot\mathbf{B}\right) + \beta r^2 \mathbf{B}$$

and try to fix the $\alpha$ and $\beta$ using the relations $\nabla \cdot \mathbf{W} = 0$, $\nabla \times \mathbf{W} = \mathbf{A}$.

$$\nabla \cdot \mathbf{W} = \alpha\left[\left(\mathbf{r}\cdot\mathbf{B}\right)\left(\nabla\cdot\mathbf{r}\right) + \mathbf{r}\cdot\nabla\left(\mathbf{r}\cdot\mathbf{B}\right)\right] + \beta\left[r^2\left(\nabla\cdot\mathbf{B}\right) + \mathbf{B}\cdot\nabla r^2\right]$$

$$\begin{aligned}
&\nabla\cdot\mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3\\
&\nabla\left(\mathbf{r}\cdot\mathbf{B}\right) = \left(\mathbf{B}\cdot\nabla\right)\mathbf{r} = \mathbf{B}\\
&\nabla\cdot\mathbf{B} = 0\\
&\nabla r^2 = 2\mathbf{r}
\end{aligned}$$

Substituting these in the previous equation

$$\begin{aligned}
\nabla\cdot\mathbf{W} &= 2\left(2\alpha+\beta\right)\left(\mathbf{r}\cdot\mathbf{B}\right) = 0\\
&\implies 2\alpha + \beta = 0
\end{aligned}$$

Similarly, $\nabla \times \mathbf{W} = \mathbf{A}$ results

$$\alpha - 2\beta = \frac{1}{2}$$

Solving both the equation gives

$$\begin{aligned}
\alpha &= \frac{1}{10}\\
\beta &= -\frac{1}{5}
\end{aligned}$$

(c)

$$\begin{aligned}
&\nabla\times\mathbf{W} = \mathbf{A}\\
\implies &\int\left(\nabla\times\mathbf{W}\right)\cdot d\mathbf{a} = \int \mathbf{A}.d\mathbf{a}\\
\implies &\oint \mathbf{W}\cdot d\mathbf{l} = \int \mathbf{A}.d\mathbf{a}
\end{aligned}$$

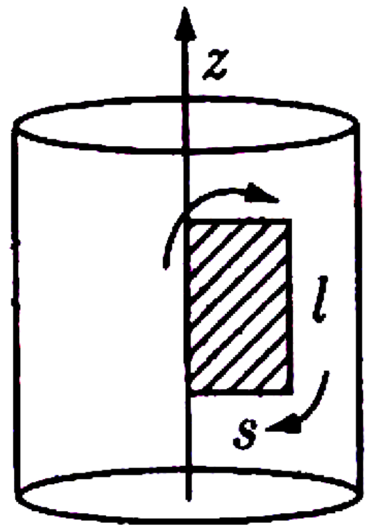


Figure 4:

Integrate around the Amperian loop shown, taking $\mathbf{W}$ to point parallel to the axis, and choosing $\mathbf{W} = 0$ on the axis:

$$-Wl = \int_0^s \frac{\mu_0 n I}{2} \tilde{s} l d\tilde{s}$$

Hence,

$$\mathbf{W} = -\frac{\mu_0 n I s^2}{4} \hat{\mathbf{k}}, \quad s < R$$

Similarly for $s > R$

$$-Wl = -\frac{\mu_0 n I R^2}{4} \int_R^s \frac{\mu_0 n I}{2} \frac{R}{\tilde{s}} l d\tilde{s}$$

Therefore,

$$\mathbf{W} = -\frac{\mu_0 n I R^2}{4} \left(1 + 2\ln\left(\frac{s}{R}\right)\right) \hat{\mathbf{k}} \quad s > R$$

# Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.