PH1020 Physics II **Problem Set 1 - Solutions** **MAR-JUN 23**

1. (a) Consider a square sheet on $xy-$ plane with corners at $(-L/2, -L/2, 0)$, $(L/2, -L/2, 0)$, $(L/2, L/2, 0)$, $(-L/2, L/2, 0)$. The infinitesimal charge element located at $(x, y, 0)$

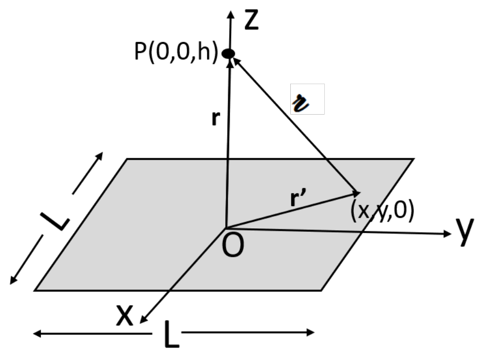


Figure 1: Schematic for problem 1

is $dq = \sigma dxdy$.
The electric filed at point $\mathbf{P} = (0, 0, h)$ due to this infinitesimal charge dq is,

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{|\boldsymbol{\mathscr{r}}|^3}\boldsymbol{\mathscr{r}} \qquad (\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r'}) \tag{1}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma dxdy}{(x^2+y^2+h^2)^{3/2}}(-x\hat{\mathbf{x}} - y\hat{\mathbf{y}} + h\hat{\mathbf{z}}) \tag{2}$$

Hence, the electric field at the point $\mathbf{P}$ due the charged square sheet,

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0}\left(\hat{\mathbf{x}}\int_{-L/2}^{L/2}\frac{-x}{(x^2+y^2+h^2)^{3/2}}dxdy + \hat{\mathbf{y}}\int_{-L/2}^{L/2}\frac{-y}{(x^2+y^2+h^2)^{3/2}}dxdy \right. \\ \left. + \hat{\mathbf{z}}\int_{-L/2}^{L/2}\frac{h}{(x^2+y^2+h^2)^{3/2}}dxdy\right)$$

Notice closely that the integrals associated with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ vanishes as they are odd functions in x and y respectively. And the integral in $\hat{\mathbf{z}}$ is an even function in both x and y and the limit changes to $(0, L/2)$ with a factor of 2 for each integral. Therefore, the net electric filed is along $\hat{\mathbf{z}}$.

$$
\begin{aligned}
\mathbf{E} &= \frac{\sigma h}{4\pi\epsilon_0} 4 \int_0^{L/2} dy \int_0^{L/2} \frac{dx}{(x^2+y^2+h^2)^{3/2}} \hat{\mathbf{z}} \\
&= \frac{\sigma h}{\pi\epsilon_0} \int_0^{L/2} dy \left[\frac{x}{(y^2+h^2)(x^2+y^2+h^2)^{1/2}} \right]_0^{L/2} \hat{\mathbf{z}} \\
&= \frac{\sigma h}{\pi\epsilon_0} \frac{L}{2} \int_0^{L/2} \frac{dy}{(y^2+h^2)(\frac{L^2}{4}+y^2+h^2)^{1/2}} \hat{\mathbf{z}}
\end{aligned}
$$

Now substituting $y = \left(\frac{L^2}{4}+h^2\right)^{1/2} \tan u$. The above integral reduces to

$$
\begin{aligned}
\mathbf{E} &= \frac{\sigma h L}{2\pi\epsilon_0} \int_0^{\tan^{-1}\frac{1}{\sqrt{1+\frac{4h^2}{L^2}}}} \left[\frac{\sec u}{h^2 + \left(h^2+\frac{L^2}{4}\right)\tan^2 u} du \right] \hat{\mathbf{z}} \\
&= \frac{\sigma h L}{2\pi\epsilon_0} \int_0^{\tan^{-1}\frac{1}{\sqrt{1+\frac{4h^2}{L^2}}}} \left[\frac{\sec u}{h^2 \sec^2 u + \frac{L^2}{4}\tan^2 u} du \right] \hat{\mathbf{z}} \\
&= \frac{\sigma h L}{2\pi\epsilon_0} \frac{4}{L^2} \int_0^{\tan^{-1}\frac{1}{\sqrt{1+\frac{4h^2}{L^2}}}} \left[\frac{\cos u}{\frac{4h^2}{L^2} + \sin^2 u} du \right] \hat{\mathbf{z}} \\
&= \frac{\sigma h L}{2\pi\epsilon_0} \frac{4}{L^2} \int_0^{\tan^{-1}\frac{1}{\sqrt{1+\frac{4h^2}{L^2}}}} \left[\frac{d(\sin u)}{\frac{4h^2}{L^2} + \sin^2 u} \right] \hat{\mathbf{z}}
\end{aligned}
$$

Now we can use the standard integral $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$. We finally get the expression

$$
\boxed{\mathbf{E} = \frac{\sigma}{\pi\epsilon_0} \left[\tan^{-1}\left(\frac{L^2}{4h} \frac{1}{\left(\frac{L^2}{2}+h^2\right)^{1/2}} \right) \right] \hat{\mathbf{z}}}
$$

(b) As $L \to \infty$,

$$
\begin{aligned}
\mathbf{E} &= \frac{2\sigma}{\pi\epsilon_0} \left[(\arctan(\infty)) \right] \hat{\mathbf{z}} \\
&= \frac{2\sigma}{\pi\epsilon_0} \left(\frac{\pi}{2}\right) \hat{\mathbf{z}} \\
&= \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}
\end{aligned}
$$

(c) We consider a Gaussian surface which runs equal distances above and below the sheet of charge as shown in Figure 2. Area of the lid of the Gaussian box is A. Let's apply the Gauss's law to this surface, (Eq. 2.13 of [1])

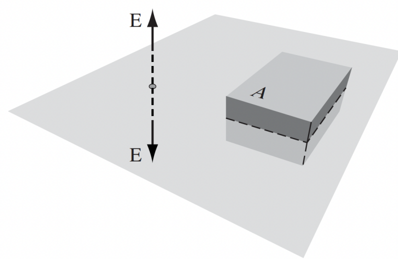


Figure 2: The Gaussian pill box for the infinite sheet of charge on xy plane

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$2A|\mathbf{E}| = \frac{\sigma A}{\epsilon_0} \quad \left(\text{$\mathbf{E}$ directs away normal to plane, hence only upper and lower planes of the box contribute.}\right)$$

$$\boxed{\mathbf{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}} & z > 0 \\ \frac{\sigma}{2\epsilon_0}(-\hat{\mathbf{z}}) & z < 0 \end{cases}}$$

(d) In order to check the boundary condition we compute the $\mathbf{E}$ just below and just above the sheet. In that case (I mean when the point is very close to the sheet), the sheet will behave like an infinite sheet of charge.

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{2\epsilon_0} - \left(-\frac{\sigma}{2\epsilon_0}\right) = \frac{\sigma}{\epsilon_0}$$

Hence,

$$\boxed{\text{The normal component of $\mathbf{E}$ is discontinuous by an amount } \frac{\sigma}{\epsilon_0} \text{ at any boundary.}}$$

2. We consider the long hollow cylindrical tube of radius R and surface charge density σ. We will find the electric field inside and outside the cylinder.

   Assuming s is the point where we calculate the E-field.

   <u>Inside the cylinder $|\mathbf{s}| = s < R$:</u>

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$|\mathbf{E}| 2\pi s L = 0$$

$$\boxed{\mathbf{E} = 0 \quad \text{for} \quad s < R}$$

Outside the cylinder $|\mathbf{s}| = s > R$:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$
$$|\mathbf{E}| 2\pi s L = \frac{\sigma 2\pi R L}{\epsilon_0}$$

$$\boxed{\mathbf{E} = \frac{\sigma R}{\epsilon_0}\frac{1}{s} \quad \text{for} \quad s > R}$$

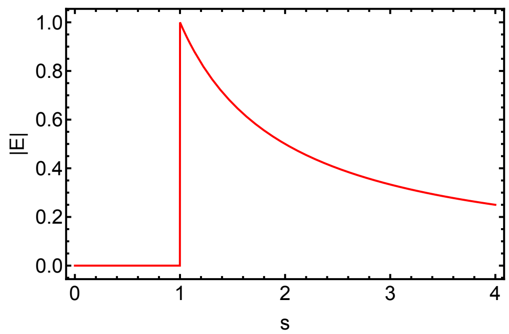


Figure 3: Plot of the magnitude of the electric field as a function of radial distance from the cylinder axis. (Note: $|\mathbf{E}|$ is in the units of $\frac{\sigma R}{\epsilon_0}$)

Checking the boundary condition: The boundary is at $s = R$.

$$E^{\perp}_{out} - E^{\perp}_{in} = \frac{\sigma}{\epsilon_0} - 0 = \frac{\sigma}{\epsilon_0}$$

As the parallel component is zero both inside and out side, $\boxed{\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}}$

3. (a) Potential of a uniformly charged disk of radius R at a distance z along the axis,

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\sigma 2\pi r}{\sqrt{r^2 + z^2}} dr = \boxed{\frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2 + z^2} - z\right)}$$

In the limit $z \to 0$

$$\boxed{V(r) = \frac{\sigma R}{2\epsilon_0}}$$

.

(b) The electric field,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial z}\hat{\mathbf{z}} \qquad \left(\because \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0\right)$$
$$= -\frac{\sigma}{2\epsilon_0}\left(\frac{1}{2}\frac{1}{\sqrt{R^2 + z^2}} 2z - 1\right)\hat{\mathbf{z}}$$
$$= \boxed{\frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right)\hat{\mathbf{z}}}$$

In the limit $z \to 0$

$$\boxed{\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}}$$

Therefore, very close to the sheet, the field behaves like the field of an infinite sheet of charge.

4. We know the electrostatic potential $V(r) = -\int_\infty^r \mathbf{E} \cdot d\mathbf{l}$. So let's find the electric field first.

<u>For $r < R$:</u> Applying Gauss's law

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$|E| 4\pi r^2 = \rho_0 \frac{4}{3}\pi r^3 \frac{1}{\epsilon_0}$$
$$\boxed{\mathbf{E} = \frac{\rho_0}{3\epsilon_0} r\hat{\mathbf{r}}}$$

<u>For $r > R$:</u> Applying Gauss's law again

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$|E| 4\pi r^2 = \rho_0 \frac{4}{3}\pi R^3 \frac{1}{\epsilon_0}$$
$$\boxed{\mathbf{E} = \frac{\rho_0 R^3}{3\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}}$$

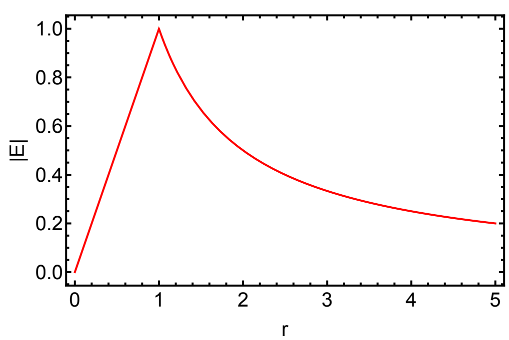


Figure 4: Plot of the magnitude of the electric field as a function of radial distance of uniformly charged solid sphere. (Note: r is in the units of R and $|\mathbf{E}|$ is in the units of $\frac{\rho_0 R}{3\epsilon_0}$)

Now let's find the potential.

For $r > R$:

$$\begin{aligned} V(r) &= -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} \\ &= -\int_{\infty}^{r} \frac{\rho_0 R^3}{3\epsilon_0}\frac{1}{r^2}dr \\ &= \frac{\rho_0 R^3}{3\epsilon_0}\frac{1}{r}\Big|_{\infty}^{r} \\ &= \boxed{\frac{\rho_0 R^3}{3\epsilon_0}\frac{1}{r}} \end{aligned}$$

For $r < R$:

$$\begin{aligned} V(r) &= -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} \\ &= -\int_{\infty}^{R} \frac{\rho_0 R^3}{3\epsilon_0}\frac{1}{r^2}dr - \int_{R}^{r} \frac{\rho_0}{3\epsilon_0} r\,dr \\ &= \frac{\rho_0 R^2}{3\epsilon_0} - \left(\frac{\rho_0}{3\epsilon_0}\frac{r^2}{2}\right)\Big|_{R}^{r} \\ &= \boxed{\frac{\rho_0 R^2}{6\epsilon_0}\left(3 - \frac{r^2}{R^2}\right)} \end{aligned}$$

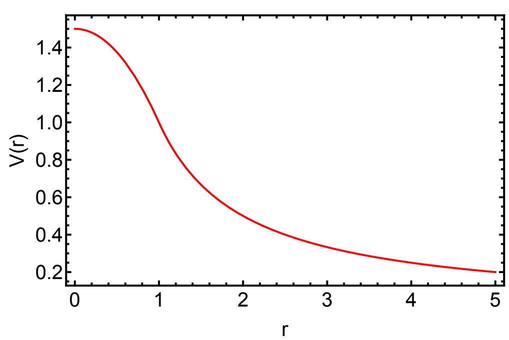


Figure 5: Plot of the potential as a function of radial distance of uniformly charged solid sphere. (Note: r is in the units of R and $V(r)$ is in the units of $\frac{\rho_0 R^2}{3\epsilon_0}$)

Work done to create such a sphere: Following the equation (2.43) of [1]

$$
\begin{aligned}
W &= \frac{1}{2}\int \rho V d\tau \\
&= \frac{1}{2}\frac{\rho_0^2 R^2}{6\epsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^R \left(3 - \frac{r^2}{R^2}\right) r^2 dr \\
&= \frac{1}{2}\frac{\rho_0^2 R^2}{6\epsilon_0} 4\pi \left[r^3 - \frac{r^5}{5R^2}\right]_0^R \\
&= \frac{1}{2}\frac{\rho_0^2 R^2}{6\epsilon_0} 4\pi \frac{4R^3}{5} \\
&= \boxed{\frac{4\pi}{15\epsilon_0}\rho_0^2 R^5}
\end{aligned}
$$

Also, an alternative method

$$
\begin{aligned}
W &= \frac{1}{2}\epsilon_0 \int E^2 d\tau \\
&= \frac{\epsilon_0}{2}\int_0^R \frac{\rho_0^2 r^2}{9\epsilon_0^2} 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^{\infty} \frac{\rho_0^2 R^6}{9\epsilon_0^2 r^4} 4\pi r^2 dr \\
&= \boxed{\frac{4\pi}{15\epsilon_0}\rho_0^2 R^5}
\end{aligned}
$$

5. (a) Given $V(r) = A\frac{e^{-kmr}}{r}$

$$
\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-kmr}}{r}\right)\hat{\mathbf{r}} = \boxed{Ae^{-kmr}(1+kmr)\frac{\hat{\mathbf{r}}}{r^2}}
$$

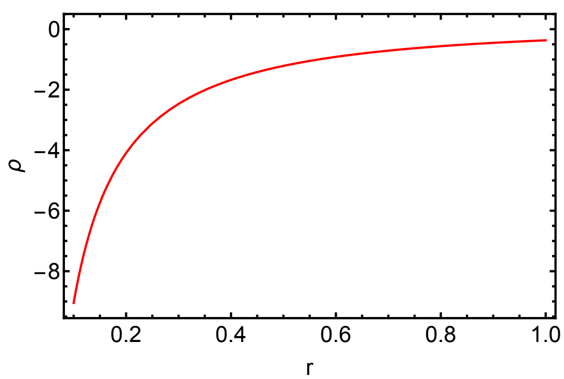
Therefore,

$$
\begin{aligned}
\rho &= \epsilon_0 \nabla\cdot\mathbf{E} \\
&= \epsilon_0 A\left(e^{-kmr}(1+kmr)\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \nabla(e^{-kmr}(1+kmr))\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= \epsilon_0 A\left(e^{-kmr}(1+kmr)4\pi\delta^3(\mathbf{r}) + \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(e^{-kmr}(1+kmr\right)\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= \boxed{\epsilon_0 A\left(4\pi\delta^3(\mathbf{r}) - \frac{k^2m^2}{r}e^{-kmr}\right)}
\end{aligned}
$$

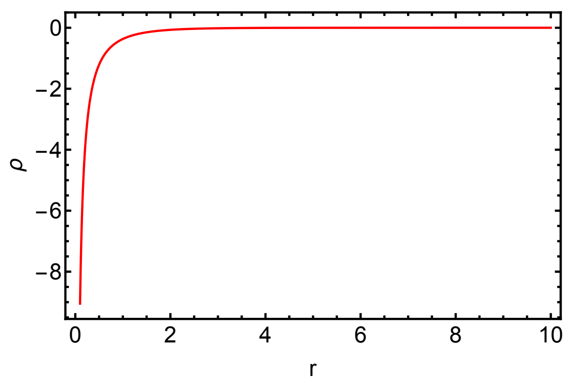
In the above expression we use the identity $f(x)\delta(x) = f(0)\delta(x)$ or more generally $f(x)\delta(x-a) = f(a)\delta(x-a)$ [1].

(b) Total charge

$$
\begin{aligned}
Q &= \int \rho d\tau \\
&= \epsilon_0 A \left(4\pi \int \delta^3(\mathbf{r}) d\tau - k^2 m^2 \int \frac{e^{-kmr}}{r} 4\pi r^2 dr \right) \\
&= \epsilon_0 A \left(4\pi - k^2 m^2 4\pi \int_0^\infty r e^{-kmr} dr \right) \\
&= \epsilon_0 A \left(4\pi - k^2 m^2 4\pi \frac{1}{k^2 m^2} \right) \qquad \left(\because \int_0^\infty x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} \right) \\
&= 0
\end{aligned}
$$



(a)



(b)

Figure 6: Plot of the charge density as a function of radial distance of Yukawa potential. (a) The plot in the scale of 0 to 1. (b) The plot is in the scale of 0 to 10.(Note: r is in the units of km and ρ is in the units of $\epsilon_0 A$)

(c) For $A \sim \frac{q}{4\pi\epsilon_0}$ and $m \rightarrow 0$ The Yukawa potential behave like Coulomb potential (Figure 7).

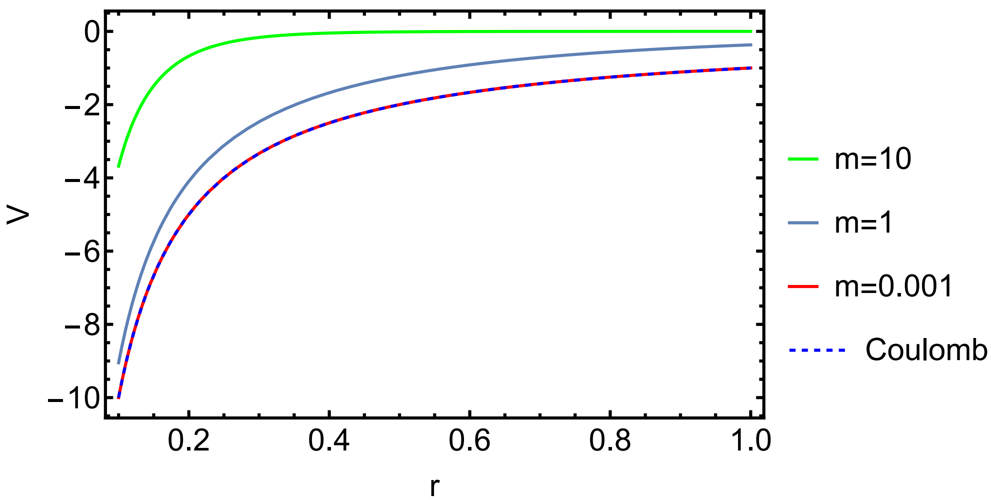


Figure 7: The different colored solid lines are plots of the Yukawa potential for different m. Dashed blue plot is the standard Coulomb potential.You can see, as m approaches zero the Yukawa plot collapses to the coulomb plot.

6. As computed in problem 4 above,

$$
\begin{aligned}
U &= \frac{4\pi}{15\epsilon_0}\rho_0^2 R^5 \\
&= \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{e^2}{r_0} \qquad \text{(substitute} \quad R = r_0 \quad \text{and} \quad \rho_0 = \frac{e}{\frac{4}{3}\pi r_0^3})
\end{aligned}
$$

Now using the on-shell mass energy relation, [1]

$$
r_0 = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{e^2}{m_e c^2} = \boxed{1.68843 \times 10^{-15}\text{m}}
$$

[1] $\epsilon_0 = 8.85418782 \times 10^{-12}$Farad.m^{-1}, $e = 1.60217663 \times 10^{-19}$ C, $m_e = 9.1093837 \times 10^{-31}$Kg, $c = 3 \times 10^8$m/s

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.