My grades for **Quiz-2**

P1 8

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Problem No.	Marks
1	8
2	9
3	8

1. [What's so special about leaves in a tree?] [8]

Recall that, in a tree[1], a *leaf* is a vertex of degree one.

<u>Prove the following.</u>

Theorem: *Let T be a tree on two or more vertices, and let $v \in V(T)$. The graph $T - v$ is a tree if and only if v is a leaf.*

($T - v$ denotes the graph obtained from T by deleting the vertex v.)

A tree is defined as a connected acyclic graph.

Claim: There is exactly one path between every vertex pair $u, v \in V(T)$ as T is a tree.
[note we have proved this in assignment]

(1) If ~~u~~ v is not a leaf it has degree more than 1. It is adjacent to ≥ 2 vertices. Take 2 of these vertices as a, b.
Because T is a tree, there is exactly one path between a and b.

[1]You may ask for definition of tree for a penalty of 1 mark.

2

[Extra page for Problem 1]

$a e_1 v e_2 b$ is the path where e_1 is edge joining a to v, e_2 is edge joining v to b.

→ If we remove vertex v from this tree, then there will be no other path from a to b clearly

→ Hence If T-v is a tree, then v is a leaf (vertex degree ≤ 1)

(2) ~~If v is a leaf~~,

We have proved in assignment that if there is exactly one path between every $u, v \in V(G)$, then G is a tree.

→ If v is a leaf (degree = 1), then if T ~~satisfies~~ has exactly one path between every vertice pair, T-v also has exactly one path between every vertice pair. So T-v is a tree

→ This is true, because for any $a, b \in V(T) - v$ the path between a and b does not contain vertice v. In T and $T - v$

This is true because if it did contain vertice v, then vertice v would have degree ≥ 2 clearly

ex: $a e_1 v e_2 b$. One edge leading to v, one leading out of v.

Hence T-v is a tree if and only if v is a leaf.

~~(As both is)~~

P2 6.5

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2. [The 'subset of' partial order] [9]

Recall that, for a set S, its *power set* $\mathcal{P}(S)$ is the collection of all subsets of S.

Let S denote a set, and let $\mathcal{P}(S)$ denote its power set.

a) Consider the poset $(\mathcal{P}(S), \subseteq)$. **List all** of its minimal, maximal, minimum and maximum elements (if any of those exist), and **provide explanation** (for each).

b) **Prove that** the poset $(\mathcal{P}(S), \subseteq)$ is a lattice[2].

a) The null set ϕ: This is ~~the~~ the only minimal and minimum element of ~~the~~ ~~ϕ~~ $P(S)$.

→This is true because every set can only be a superset of ϕ

→The only subset of ϕ would be itself.

minimal element: An element a is minimal if ~~fo~~ there exist no element $b \in S - a$ such that $b \leq a$.

minimum element: An element a is minimum if for all elements $b \in S$, $a \leq b$. [note in definitions here S is not set in the q above]

☆ ϕ is minimal because, there is no set which is a subset of it except itself

☆ ϕ is minimum because ϕ is a subset of every other set

[note that all other elements cannot be minimal or ~~maxim~~ minimum as ϕ would be subset of them]

[2]You may ask for definition of lattice for a penalty of 1 mark.

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[Extra page for Problem 2]

The set S : This is the only maximal and ~~max~~ maximum.

Maximal : An element a is maximal if there exist no element $b \in S - a$ such that $b \geq a$

Maximum : An element a is maximum if for all $b \in S$, we have $b \leq a$.

→ The set S is maximal as the powerset $P(S)$ is a set of subsets of S. Hence there would be no set $x \in P(S)$ such that ~~x~~ x is superset of S. (so S is not subset of any x)

→ The set S is maximum because every set in $P(S)$ is a subset of S.

b) $(P(S), \subseteq)$ is a lattice because, the $GLB(a,b)$ and $LUB(a,b)$ for all $a, b \in P(S)$ exists.

$\phi \leq a, \quad \phi \leq b$

$S \geq a, \quad S \geq b$

Score 6

Score 0.5

Please look at the definition of lattice. For proving a poset is a lattice, we need to prove Greatest Lower Bound (GLB) and Least Upper Bound (LUB) exists for every pair of elements of the poset. 0

P3 4.5

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3. [Let's test our understanding of Posets and Dilworth's Theorem] [8]

Recall Dilworth's Theorem: In any finite poset, the cardinality of a largest antichain equals the cardinality of a smallest chain partition.

One of the following is True, and the other is False. For the one that is True, **prove it** using Dilworth's Theorem. For the other one, **disprove** it by providing a counterexample.

a) Let $p, q \in \mathbb{N}$. Every sequence of integers of length $pq+1$ either contains a nondecreasing subsequence of length $p+1$, or contains a decreasing subsequence of length $q+1$, possibly both.

b) Let $n \in \mathbb{N}$. Every sequence of integers of length n^2+1 contains a subsequence of length $n+1$ that satisfies one of the following properties:

- each member (except for the last one) divides the next member, or otherwise
- each member (except for the last one) is divisible by the next member.

b) I shall disprove the statement by providing a counterexample.
Let us take a sequence of integers of length 5.
(n=2)

Sequence : 2, 3, 5, 7, 9

By the statement we will be able to find a subsequence of length 3 such that every element (except for last one) ~~either~~ divides next member or is divisible by next member.

In the given sequence none of the elements divide any of the others clearly as all are prime numbers.

Hence, we cannot find any such subsequence

9 is not a prime number.

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[Extra page for Problem 3]

because 2^{nd} element of such subsequence would have to ~~be~~ divide or be divisible by 3^{rd} element which is not possible.

a) Let us take a sequence of integers of length $pq+1$.
Each element has indice. Let's say starting from 1.
Let us define an element $a_i \geq a_j$ if and only if a_i is greater than ~~or equal~~ or equal to a_j and i is greater than or equal to j.

We have shown in class that if some ~~sble~~ sequence is a chain, it is non decreasing. (All elements comparable to one another)
We have also shown that if ~~such~~ a sequence is an antichain, it is a decreasing. (All element non-comparable to one another)

→ Let us say that there exists an antichain of length $\geq q+1$. Then we can take a $q+1$ length part of this antichain. This would be a decreasing subsequence.

→ If there exists no antichain of length $q+1$; i.e. longest antichain has length $\leq q$.
Then we can say that the ~~cardina~~ smallest cardinality of a chain partition is also $\leq q$. (By Dilworth's theorem)
There must be a chain of length $p+1$ because if every chain has $\leq p$ elements then total elements would be $\leq pq$ which is not true.

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You need to prove this is a poset with respect to your relation defined here.

It is not done in class.

It is given as DIY, and you shouldn't use DIY/TIY directly here, but you can use the results which are completely proved in class.

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[Extra page: may be used for any problem]

A chain of p+1 elements means a subsequence which is non decreasing exists.

Hence there ~~will~~ must be a decreasing subsequence or non decreasing subsequence.

Hence, proved.

[Extra page: may be used for <u>any</u> problem]

[Extra page: may be used for <u>any</u> problem]