DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II **Problem Set 5 - Solutions** **MAR-JUN 23**

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1. Show that the force on an infinitesimal current loop with dipole moment $\mathbf{m}$, in the presence of a magnetic field $\mathbf{B}$ is

$$\boldsymbol{F} = \nabla\left(\boldsymbol{m}.\boldsymbol{B}\right)$$

Solution:

Let's consider an infinitesimal square loop of side length ϵ in the $y-z$ plane as shown in Figure 1.

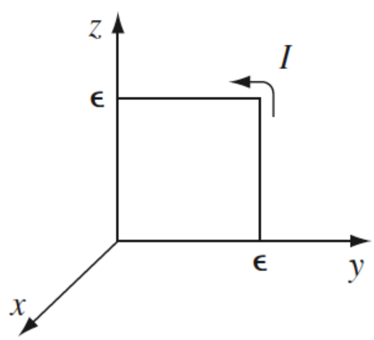

Figure 1: This is an infinitesimal sqaure loop of side length ϵ. The loop is on the $y-z$ plane.

Now, the net force on the loop due to the background field $\boldsymbol{B}$ would be superposition of the forces on each side of the loop , [1]

$$\begin{aligned}
\boldsymbol{F} &= \sum_{i=1}^{4} I\int d\boldsymbol{l}_i \times \boldsymbol{B}_i \\
&= I\int \left\{dy\hat{\boldsymbol{y}} \times \boldsymbol{B}(0,y,0) + dz\hat{\boldsymbol{z}} \times \boldsymbol{B}(0,\epsilon,z) - dy\hat{\boldsymbol{y}} \times \boldsymbol{B}(0,y,\epsilon) - dz\hat{\boldsymbol{z}} \times \boldsymbol{B}(0,0,z)\right\} \\
&= I\int \left\{-dy\hat{\boldsymbol{y}} \times [\boldsymbol{B}(0,y,\epsilon) - \boldsymbol{B}(0,y,0)] + dz\hat{\boldsymbol{z}} \times [\boldsymbol{B}(0,\epsilon,z) - \boldsymbol{B}(0,0,z)]\right\} \\
&= I\int \left\{-\hat{\boldsymbol{y}}\ \ dy \times \epsilon \frac{\partial \boldsymbol{B}}{\partial z}\Big|_{(0,y,0)} + \hat{\boldsymbol{z}}\ \ dz \times \epsilon \frac{\partial \boldsymbol{B}}{\partial y}\Big|_{(0,0,z)}\right\} \\
&= I\,\epsilon^2 \left\{\hat{\boldsymbol{z}} \times \frac{\partial \boldsymbol{B}}{\partial y} - \hat{\boldsymbol{y}} \times \frac{\partial \boldsymbol{B}}{\partial z}\right\} \qquad \left(\because \int dy \frac{\partial \mathbf{B}}{\partial z}\Big|_{(0,y,0)} \approx \epsilon \frac{\partial \mathbf{B}}{\partial z}\Big|_{(0,0,0)}\right) \\
&= m\left\{\hat{\boldsymbol{y}}\frac{\partial B_x}{\partial y} - \hat{\boldsymbol{x}}\frac{\partial B_y}{\partial y} - \hat{\boldsymbol{x}}\frac{\partial B_z}{\partial z} + \hat{\boldsymbol{z}}\frac{\partial B_z}{\partial z}\right\}
\end{aligned}$$

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[1] We have used Taylor expansion in the 4th step: e.g. $\boldsymbol{B}(0,y,\epsilon) = \boldsymbol{B}(0,y,0) + \epsilon\frac{\partial \boldsymbol{B}}{\partial z}\big|_{(0,y,0)}$

Using $\nabla \cdot \boldsymbol{B} = 0$ to write $\frac{\partial \boldsymbol{B}}{\partial y} + \frac{\partial \boldsymbol{B}}{\partial z} = -\frac{\partial \boldsymbol{B}}{\partial x}$, we have

$$\boldsymbol{F} = m\left(\hat{\boldsymbol{x}}\frac{\partial B_x}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial B_y}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial B_z}{\partial z}\right)$$

but $\boldsymbol{m} \cdot \boldsymbol{B} = mB_x$. Therefore,

$$\begin{aligned} \boldsymbol{F} &= \hat{\boldsymbol{x}}\frac{\partial\left(m\boldsymbol{B_x}\right)}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial\left(m\boldsymbol{B_x}\right)}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial\left(m\boldsymbol{B_x}\right)}{\partial z} \\ &= \nabla\left(\boldsymbol{m}.\boldsymbol{B}\right) \end{aligned}$$

2. The magnetization of a long cylinder is found to be proportional to square of the distance measured from the axis and is directed along the azimuthal direction. Calculate the magnetic field inside and outside the cylinder.

Solution:

As given in the question magnetization $\boldsymbol{M} \propto \rho^2$, and along the azimuthal direction. Hence,

$$\boldsymbol{M} = k\rho^2\hat{\boldsymbol{\phi}} \ ,$$

where ρ is the radial distance from the axis and k is some arbitrary constant.

The volume bound current is, [2]

$$\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M} = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\right)\hat{\boldsymbol{z}} = 3k\rho\hat{\boldsymbol{z}}$$

Similarly, the surface bound current would be,

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}} = -kR^2\hat{\boldsymbol{z}}$$

Notice that the bound current flows up the cylinder and returns down the surface. The field inside and outside of the cylinder can be easily computed using the cylindrical symmetry of the current distribution.

[2] The curl in cylindrical polar coordinates:

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho A_\phi\right) - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}}$$

Magnetic field inside the cylinder: $(0 < \rho < R)$

Using Ampere's law,

$$\begin{aligned}
\oint \boldsymbol{B} \cdot d\boldsymbol{l} &= \mu_0 I_{\text{enc}} \\
B\ 2\pi\rho &= \mu_0 \int_0^{\rho} J_b da \\
B &= \frac{1}{2\pi\rho}\mu_0 \int_0^{\rho} 3k\rho 2\pi\rho d\rho \\
B &= \frac{3k\mu_0}{\rho}\frac{\rho^3}{3} = \mu_0 k\rho^2
\end{aligned}$$

Therefore,

$$\boldsymbol{B}_{\text{inside}} = \mu_0 k\rho^2\ \hat{\boldsymbol{\phi}}, \qquad 0 < \rho < R$$

The direction of the field has been determined by the Biot-Savart law.

Magnetic field outside the cylinder: $(\rho > R)$

Replicating the steps as in the previous case,

$$\begin{aligned}
\oint \boldsymbol{B} \cdot d\boldsymbol{l} &= \mu_0 I_{\text{enc}} \\
\implies B\ 2\pi\rho &= \mu_0 \left(\int J_b da + \int K_b dl \right) \\
&= \mu_0 \left(\int_0^R (3k\rho)\ (2\pi\rho d\rho) + \int_0^{2\pi} (-kR^2)(Rd\phi) \right) \\
&= \mu_0 \left(2\pi k R^3 - 2\pi k R^3 \right) \\
&= 0
\end{aligned}$$

For any Amperian loop considered in the region $\rho > R$ the net enclosed current is zero, therefore,

$$\boldsymbol{B}_{\text{outside}} = 0$$

3. A very long cylinder of radius R carries a magnetization $\boldsymbol{M} = ks\hat{\boldsymbol{z}}$. Here, k is a constant and s is the distance from the axis which lies along $\hat{\boldsymbol{z}}$. Find the magnetic field inside and outside the cylinder by (a) employing Ampere's law and (b) calculating the bound currents.

Solution:

(a) Given the magnetization vector $\boldsymbol{M} = ks\ \hat{\boldsymbol{z}}$, the volume bound current is

$$\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M} = -k\hat{\boldsymbol{\phi}}$$

and the surface bound current is

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}} = kR\hat{\boldsymbol{\phi}}$$

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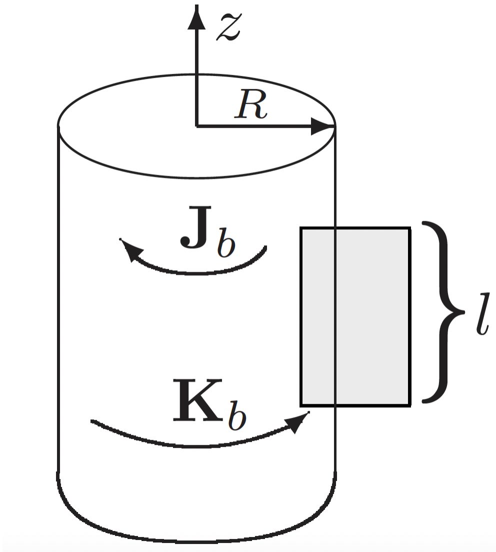

Figure 2: The cylinder with volume bound current $\boldsymbol{J}_b$ and surface bound current $\boldsymbol{K}_b$. The Amperian loop for of length l is shown.

It is evident from the bound current computation that the system is basically superposition of two solenoids. Therefore, [3]

$$\boldsymbol{B}_{\text{outside}} = 0$$

Now for inside the cylinder,

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \left[\int J_b \, da + \int K_b \, dl \right]$$
$$B \; l = \mu_0 \left[-k(R-s)l + kRl \right]$$

Therefore,

$$\boldsymbol{B}_{\text{inside}} = \mu_0 ks \; \hat{\boldsymbol{z}} \; .$$

(b) It is obvious from the symmetry that $\boldsymbol{H}$ directs along $\hat{\boldsymbol{z}}$.
Using the same Amperian loop as in Figure 2

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \mu_0 I_{\text{fenc}}$$
$$H \; l = 0 \qquad \text{(the right hand side is zero as there is no free current)}$$

[3] Students are encouraged to revisit Example 5.9 of [1]

Therefore, $\boldsymbol{H} = 0$.
Using equation (6.18) of [1] ($\boldsymbol{H} \equiv \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$) we get,

$$\begin{aligned}\boldsymbol{B}_{\text{outside}} &= 0 \\ \boldsymbol{B}_{\text{inside}} &= \mu_0 k s \hat{\boldsymbol{z}}\end{aligned}$$

4. A large piece of material have "frozen-in" magnetization $\boldsymbol{M}$. The field measured inside the material is $\boldsymbol{B_0}$. Now a cavity is hollowed out of the material. Calculate the fields $\boldsymbol{B}$ and $\boldsymbol{H}$ at the centre of the cavity, if it is

   (a) a small spherical cavity,
   (b) a long needle-shaped cavity running parallel to $\boldsymbol{M}$,
   (c) a thin wafer-shaped cavity perpendicular to $\boldsymbol{M}$.

   Solution:

   Hollowing out of a material is equivalent to filling the cavity with same material and material with negative magnetization. Remember, we have followed the similar kind of approach while solving such "cavity problems" in electrostatics.

   (a) The magnetic field of a uniformly magnetized sphere is $\frac{2}{3}\mu_0\boldsymbol{M}$ (refer equation 6.16 [1]). If we remove the sphere, the magnetic field is

$$\boldsymbol{B} = \boldsymbol{B_0} - \frac{2}{3}\mu_0\boldsymbol{M}\ .$$

   Similarly,

$$\begin{aligned}\boldsymbol{H} &= \frac{1}{\mu_0}\boldsymbol{B} \\ &= \frac{1}{\mu_0}\left[\boldsymbol{B_0} - \frac{2}{3}\mu_0\boldsymbol{M}\right] \\ &= \boldsymbol{H_0} + \frac{1}{3}\boldsymbol{M}\end{aligned}$$

   (b) The needle can be considered as a long solenoid. The field inside the long solenoid is $\mu_0\boldsymbol{K}_b$. But we have just seen in the previous problem that $|\boldsymbol{K}_b| = |\boldsymbol{M}|$, so magnetic field of the bound current on the inside surface of the cavity is $\mu_0\boldsymbol{M}$, pointing downward.Therefore,

$$\boldsymbol{B} = \boldsymbol{B_0} - \mu_0\boldsymbol{M}\ .$$

   Then,

$$\begin{aligned}\boldsymbol{H} &= \frac{1}{\mu_0}\boldsymbol{B} \\ &= \mu_0\left(\boldsymbol{B_0} - \mu_0\boldsymbol{M}\right) \\ &= \boldsymbol{H}_0\end{aligned}$$

(c) In this case, the bound currents are small and far away from the center so

$$\boldsymbol{B} = \boldsymbol{B_0} \ .$$

It is easy to check $\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{M}$

Comment: In the wafer, $\boldsymbol{B}$ is the field in the medium. In the needle, $\boldsymbol{H}$ is $\boldsymbol{H}$ in the medium. Both $\boldsymbol{B}$ and $\boldsymbol{H}$ are modified in the sphere.

5. A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface (see Figure 3). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.

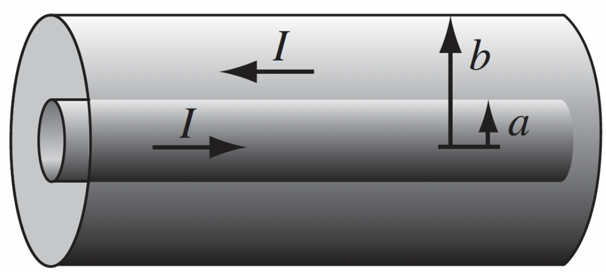

Figure 3:

<u>Solution:</u>

Using Ampere's law,

$$\oint \boldsymbol{H} \cdot \boldsymbol{l} = I_{f_{enc}}$$
$$H \ 2\pi s = I$$

Therefore,

$$\boldsymbol{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}, \qquad a < s < b$$

Now, using equation (6.30) of [1],

$$\boldsymbol{B} = \mu_0 \left(1 + \chi_m\right) \boldsymbol{H} = \mu_0 \left(1 + \chi_m\right) \frac{I}{2\pi s}\hat{\boldsymbol{\phi}} \ .$$

Let's compute magnetic field by obtaining the bound currents.

Given the field $\boldsymbol{H}$ the magnetization can be determined using equation (6.29) of the [1],

$$\boldsymbol{M} = \chi_m \boldsymbol{H} = \frac{\chi_m I}{2\pi s} \hat{\boldsymbol{\phi}} \ .$$

Now,

$$\begin{aligned}
\boldsymbol{J_b} &= \boldsymbol{\nabla} \times \boldsymbol{M} = 0 \\
\boldsymbol{K_b} &= \boldsymbol{M} \times \hat{\boldsymbol{n}} = \begin{cases} \frac{\chi_m I}{2\pi a} \hat{\boldsymbol{z}}, & \text{if } s = a \\ \frac{-\chi_m I}{2\pi b} \hat{\boldsymbol{z}}, & \text{if } s = b \ . \end{cases}
\end{aligned}$$

The total enclosed current inside the Amperian loop considered between the cylinders,

$$I_{\text{enc}} = I + \frac{\chi_m I}{2\pi a} .2\pi a = (1 + \chi_m) I$$

Thus, using the Ampere's law

$$\boldsymbol{B} = \mu_0 \left(1 + \chi_m\right) \frac{I}{2\pi s} \hat{\boldsymbol{\phi}} \ .$$

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.