Department of Mathematics, IIT Madras
MA1102 Series & Matrices
Assignment-1 (Series of Numbers)

1. Show the following:

   (a) $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$ (b) $\lim_{n\to\infty} n^{1/n} = 1.$ (c) $\lim_{n\to\infty} x^n = 0$ for $|x| < 1.$

   (d) $\lim_{n\to\infty} \frac{n^p}{x^n} = 0$ for $x > 1.$ (e) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (f) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

2. Prove the following:
   (a) It is not possible that a series converges to a real number ℓ and also diverges to $-\infty$.
   (b) It is not possible that a series diverges to ∞ and also to $-\infty$.

3. Prove the following:
   (a) If both the series $\sum a_n$ and $\sum b_n$ converge, then the series $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ and $\sum ka_n$ converge; where k is any real number.
   (b) If $\sum a_n$ converges and $\sum b_n$ diverges to $\pm\infty$, then $\sum(a_n + b_n)$ diverges to $\pm\infty$, and $\sum(a_n - b_n)$ diverges to $\mp\infty$.
   (c) If $\sum a_n$ diverges to $\pm\infty$, and $k > 0$, then $\sum ka_n$ diverges to $\pm\infty$.
   (d) If $\sum a_n$ diverges to $\pm\infty$, and $k < 0$, then $\sum ka_n$ diverges to $\mp\infty$.

4. Give examples for the following:
   (a) $\sum a_n$ and $\sum b_n$ both diverge, but $\sum(a_n + b_n)$ converges to a nonzero number.
   (b) $\sum a_n$ and $\sum b_n$ both diverge, and $\sum(a_n + b_n)$ diverges to ∞.
   (c) $\sum a_n$ and $\sum b_n$ both diverge, and $\sum(a_n + b_n)$ diverges to $-\infty$.

5. Show that the sequence $1,\ 1.1,\ 1.1011,\ 1.10110111, \ldots$ converges.

6. Determine whether the following series converge:

   (a) $\sum_{n=1}^{\infty} \frac{-n}{3n+1}$ (b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ (c) $\sum_{n=1}^{\infty} \frac{1 + n \ln n}{1 + n^2}$

7. Test for convergence the series $\frac{1}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots + \left(\frac{n}{2n+1}\right)^n + \cdots$.

8. Is the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ convergent?

9. Is the area under the curve $y = (\ln x)/x^2$ for $1 \le x < \infty$ finite?

10. Evaluate (a) $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ (b) $\int_0^3 \frac{dx}{x-1}$

11. Show that $\int_1^{\infty} \frac{\sin x}{x^p}\, dx$ converges for all $p > 0$.

12. Show that $\int_0^{\infty} \frac{\sin x}{x^p}\, dx$ converges for $0 < p \le 1$.

13. Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\alpha}}$ converges for $\alpha > 1$ and diverges to ∞ for $\alpha \leq 1$.

14. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$ converge?

15. Does the series $1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} - \cdots$ converge?

16. Let (a_n) be a sequence of positive terms. Show that if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

17. Let (a_n) be a sequence of positive non-increasing terms. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then the sequence (na_n) converges to 0.
