Department of Mathematics, IIT Madras
MA1102 Series & Matrices
Assignment-4 (Row Reduced Echelon Form)

1. Convert the following matrices into RREF and determine their ranks.
(a) $\begin{bmatrix} 5 & 2 & -3 & 1 & 7 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 & -3 & 1 & 30 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$

2. Determine linear independence of $\{(1,2,2,1),\ (1,3,2,1),\ (4,1,2,2),\ (5,2,4,3)\}$ in $\mathbb{C}^{1\times 4}$.

3. Compute the A^{-1} using RREF and also using determinant, where $A = \begin{bmatrix} 4 & -7 & -5 \\ -2 & 4 & 3 \\ 3 & -5 & -4 \end{bmatrix}$.

4. Solve the following system by Gauss-Jordan elimination:
$$\begin{array}{cccccc} x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = 6 \\ 2x_1 & +3x_2 & +x_3 & +4x_4 & -9x_5 & = 17 \\ x_1 & +x_2 & +x_3 & +2x_4 & -5x_5 & = 8 \\ 2x_1 & +2x_2 & +2x_3 & +3x_4 & -8x_5 & = 14 \end{array}$$

5. Check if the system is consistent. If so, determine the solution set.
   (a) $x_1 - x_2 + 2x_3 - 3x_4 = 7,\ 4x_1 + 3x_3 + x_4 = 9,\ 2x_1 - 5x_2 + x_3 = -2,$ $3x_1 - 2x_2 - 2x_3 + 10x_4 = -12.$
   (b) $x_1 - x_2 + 2x_3 - 3x_4 = 7, 4x_1 + 3x_3 + x_4 = 9, 2x_1 - 5x_2 + x_3 = -2,$ $3x_1 - 2x_2 - 2x_3 + 10x_4 = -14.$

6. Using Gauss-Jordan elimination determine the values of $k \in \mathbb{R}$ so that the system of linear equations
$$x + y - z = 1,\ 2x + 3y + kz = 3,\ x + ky + 3z = 2$$
has (a) no solution, (b) infinitely many solutions, (c) exactly one solution.

7. Let A be an $n \times n$ matrix with integer entries and $\det(A^2) = 1$. Show that all entries of A^{-1} are also integers.

8. Let $A \in \mathbb{F}^{m\times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:
   (a) The equation $Ax = 0$ has a non-zero solution iff $A_1, \ldots, A_n$ are linearly dependent.
   (b) The equation $Ax = b$ has at least one solution iff $b \in \operatorname{span}\{A_1, \ldots, A_n\}$.
   (c) Let u be a solution of $Ax = b$. Then, u is the only solution of $Ax = b$ iff $A_1, \ldots, A_n$ are linearly independent.
   (d) The equation $Ax = b$ has a unique solution iff $\operatorname{rank} A = \operatorname{rank}[A|b]$ = number of unknowns.

9. Let $A \in \mathbb{F}^{m\times n}$ have rank r. Give reasons for the following:
   (a) $\operatorname{rank}(A) \le \min\{m, n\}$.
   (b) If $n > m$, then there exist $x, y \in \mathbb{F}^{n\times 1}$ such that $x \ne y$ and $Ax = Ay$.
   (c) If $n < m$, then there exists $y \in \mathbb{F}^{m\times 1}$ such that for no $x \in \mathbb{F}^{n\times 1}$, $Ax = y$.
   (d) If $n = m$, then the following statements are equivalent:
      i. $Au = Av$ implies $u = v$ for all $u, v \in \mathbb{F}^{n\times 1}$.
      ii. Corresponding to each $y \in \mathbb{F}^{n\times 1}$, there exists $x \in \mathbb{F}^{m\times 1}$ such that $y = Ax$.