Completing the direct proof of Erdos-Szekeres Theorem: (using Dilworth's Theorem)

(as per the plan we discussed)

Recall: Given a sequence of n^2+1 integers

$a_1, a_2, \ldots\ldots, a_{n^2+1}$, we define a ("poset") relation:

$(S, \precsim)$ as follows: needs proof

$S := \{1, 2, \ldots, n^2+1\}$

For $i, j \in S$, we write $i \precsim j$ IF: ① $i \leq j$ AND ② $a_i \leq a_j$

usual less than or equal to for integers

Claim 1: $(S, \precsim)$ is a poset.

Proof: DIY.

If $T \subseteq S$ is a chain in $(S, \precsim)$ then $\{a_k : k \in T\}$ is a non-decreasing subsequence.

what does this mean?

Claim 2: Every chain in $(S, \precsim)$ corresponds to a nondecreasing subsequence (in the given sequence $a_1, a_2, \ldots, a_{n^2+1}$)

Proof: DIY.

Claim 3: Every antichain in $(S, \precsim)$ corresponds to a decreasing subsequence (in the given sequence $a_1, a_2, \ldots, a_{n^2+1}$).

Proof: DIY.

Now, we will apply Dilworth's Theorem to the finite poset $(S, \preceq)$:

If there is an antichain of cardinality (at least) $n+1$, then we have a decreasing subsequence of length (at least) $n+1$ by Claim 3, and we are done.

Now suppose that there is NO antichain of cardinality at least $n+1$.

In other words, every antichain has cardinality at most n.

In particular, the cardinality of a largest ~~antichain~~ antichain is at most n.

Now, let $\mathcal{C} := \{C_1, C_2, \ldots, C_k\}$ denote a smallest chain partition.

This completes the proof of Erdos–Szekeres Theorem.

By Dilworth's Theorem: $|\mathcal{C}| \leq n$.

Dilworth's Theorem: In any finite poset, the cardinality of a largest antichain equals the cardinality of a smallest chain partition.

Claim 4: There exists a chain $C \in \mathcal{C}$ of cardinality at least $n+1$.

Proof: Suppose NOT. Then cardinality of each chain in $\mathcal{C}$ is at most n.

By definition of chain partition,

$|S| \leq |\mathcal{C}| \cdot n \leq n^2$ (since $|\mathcal{C}| \leq n$);

contradiction since $|\mathcal{C}| = n^2+1$. $\boxtimes$ Let C denote any chain in $\mathcal{C}$ of cardinality at least $n+1$. By Claim 2, C corresponds to a nondecreasing subsequence of length at least $n+1$. $\boxtimes$