IIT M-CS1200 : Discrete Math (Mar - Jul 2023) Release Date: 27-06-2023

Tutorial No: 5 Tutorial Date: 01-06-2023

1. Prove using induction that every tree T has at least $\Delta(T)$ leaves, where $\Delta(T)$ is the maximum degree among all the vertices of T.
2. Prove that for a finite set S of cardinality $|S|$, the number of subsets of S is $2^{|S|}$.
3. Suppose that for every pair of cities in a country, there is a direct one-way road connecting them in one direction or the other (but not both). Then the following holds:

   Theorem 1 *There exists a route such that it starts from some city and visits each city (in the country) exactly once.*

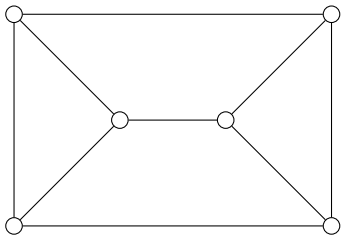
   (a) Convert the above problem into the digraph problem.

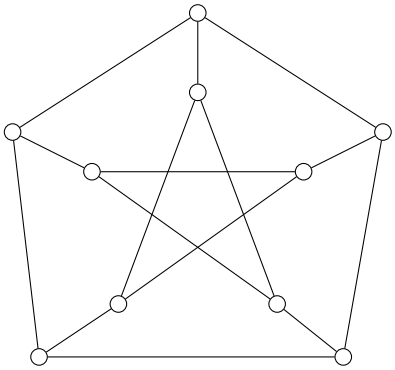
   (b) Prove the digraph version of Theorem 1.
4. Prove by induction that every natural number $n \geqslant 2$ can be written as a product of prime numbers.
5. Prove by induction that every natural number $n \geqslant 1$ can be written as a sum of distinct powers of 2.
6. The goal of this question is to come up with an alternative definition for lattices.

   (a) Prove the following using induction: Let $P = (S, \preceq)$ be a partial order. If any 2-element subset T of S has a least upper bound and a greatest lower bound, then any non-empty finite subset T of S has a least upper bound and a greatest lower bound.

   (b) Using (a), come up with an alternative definition for lattices and prove it's equivalence with the one given below (i.e. as it was stated in lectures). A partial order $P = (S, \preceq)$ is said to be a lattice if any 2-element subset T of S has a least upper bound and a greatest lower bound.
7. Let H be a subgraph of a graph G.
   H is said to be a *spanning subgraph* of G if $V(H) = V(G)$.
   T is said to be a *spanning tree* of a graph G if T is a spanning subgraph of G and T is a tree.

   (a) Draw 2 spanning trees for each of the following graphs.





   (b) Prove using induction that every connected graph G has a spanning tree.

   (Hint: Recall from Tutorial 4 that if a connected graph G is a tree then, for each edge $e \in E(G)$, $G - e$ has at least 2 connected components.)