My grades for **Quiz-1**

P1 9

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Problem No.	Marks
1	9
2	6
3	8
4a	2
4b	Bonus: 3

1. [Posets and Chains] [9]

   a) **Prove that** the divides (|) relation is a partial order on the set of positive integers $\mathbb{N} - \{0\}$.

   b) **Explain why** $(\mathbb{N} - \{0\}, |)$ is a poset but it is not a totally ordered set (aka chain).

   c) **Give an example** of an infinite subset S (of $\mathbb{N} - \{0\}$) such that the divides (|) relation is a total order on S — in other words, such that $(S, |)$ is a chain.

a) For a relation to be a POSET, it must be reflexive, antisymmetric and transitive

*We have for all $a \in N - \{0\}$ that a divides itself

So $\forall a \in N - \{0\}$, ~~(a, a) is an elemen~~ aRa.

Hence, it is reflexive.

* If we have $a \neq b$, and a divides b, that means that b does not divide a, (because $b > a$ clearly): Antisymmetric

Also, it means $b = na$. $n \in \mathbb{N}$, $n \geq 2$ (for this problem)

*Let's say $b \neq c$, and then b divides c. That means $c = mb$. $m \in \mathbb{N}$, $m \geq 2$.

Please write the other case of anti-symmetric as well. At most a | b or b |a holds 5

[Extra page for Problem 1]

*So we can also say $c = ka$, $k \in \mathbb{N}$. Hence a divides c.
So if (a,b) and (b,c) are part of the relation, (a,c) is also too: Transitive

Hence, it is a POSET. Hence, proved.

b) It is a POSET but not a TOSET. Because for a TOSET, exactly one of either aRb or bRa MUST be true for ALL a, b in $\mathbb{N}-\{0\}$ (when $a \neq b$). However, here there will be cases where this is not true. For example $b=7$, $a=3$.

Score 2

c) ~~If we have have a set S let's say which contains all even numbers greater than or equal to 2, then $(S, |)$ is a TOSET~~

If we have a set S, which contains all numbers of form 2^n where $n \in \mathbb{N}-\{0\}$, then $(S, |)$ is a TOSET.

It is reflexive, Antisymmetric and transitive.

Score 2

And ~~aRb~~ exactly one of aRb or bRa will be present (when $a \neq b$). for all $a, b \in S$.

P2 4.5

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2. [Injections, Surjections and Bijections] [6]

For each of the following functions, **indicate whether** it is:
(i) a bijection,
(ii) an injection (aka 1-to-1 function) but not a surjection,
(iii) a surjection (aka onto function) but not an injection, OR
(iv) neither an injection (1-to-1) nor a surjection (onto).

Provide explanation in each case.

a) $f_1 : \mathbb{Z} \to \mathbb{Z}$ is defined as $f_1(x) = x + 1$.
b) $f_2 : \mathbb{N} \to \mathbb{N}$ is defined as $f_2(x) = x + 1$.
c) $f_3 : \mathbb{Z} \to \mathbb{N}$ is defined as $f_3(x) = x^2$.

a) A bijection.

* For every element $a \in \mathbb{Z}$ in codomain, there is a preimage $(a-1)$ which is in domain of ~~funct~~ $(\mathbb{Z})$

Score: 2

preimage is $b \in \mathbb{Z}$ such that $f(b) = a$.

* It is a one-one function because, for each element $a \in \mathbb{Z}$ in codomain, there is atmost ~~exactly~~ one element $b \in \mathbb{Z}$ in domain such that $f(b) = a$. $b = a - 1$.

b) Injection but not surjection.

* It is not a surjection because for the element 0 in the codomain there is no element $a \in \mathbb{N}$ in domain ~~such~~ such that $f(a) = 0$.

Score: 2

So it cannot be surjective function

* It is one-one because for every $a \in \mathbb{N}$ (codomain) there is atmost one element $b \in \mathbb{N}$ (domain) such that $f(b) = a$.
For all $a \geq 1$, such b exists $b = a - 1$.

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[Extra page for Problem 2]

Score: 0.5

(c) Surjective, but not injective.

For every $a \in \mathbb{N}$ in codomain, we have an element $b \in \mathbb{Z}$ in domain such that $f(b) = a$. $b = \pm\sqrt{a}$. So it is surjective.

* For every $a \in \mathbb{N}$ in codomain there are ~~for more~~ sometimes more than one $b \in \mathbb{Z}$ (domain) such that $f(b) = a$.

Ex: $a = 4$, then $f(2) = 4$, $f(-2) = 4$

Hence it is not one-one.

f_3 is neither an injection nor a surjection.

Not injection because: $f_3(-1) = f(1) = 1$.

Not surjection because there no element in the domain of f_3 whose image is 2.

P3a 4

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3. [A symmetric (and stronger) notion of reachability in digraphs] [8]

Recall from Assignment-1: Given a digraph $D := (V, A)$, for two (not necessarily distinct) vertices $u, w \in V$, we say that w *is reachable from* u (or that u *is able to reach* w) if there is a directed walk Q (in D) with u as the start vertex, and w as the end vertex.

Now, we introduce a new relation (that is clearly reflexive and symmetric):

We say that u and w are *reachable from each other* if:
(i) w is reachable from u, and (ii) u is reachable from w.

a) **Prove that** "reachable from each other" is a transitive relation (for any digraph $D := (V, A)$).

→ We say uRw, if w is reachable from u and v is reachable from w.

If Q_1 is directed walk from u to w
And Q_2 is directed walk from w to u.

→ Let's say wRv. And Q_3 is directed walk from w to v, Q_4 is directed walk from v to w.

→ Then combination of Q_1 and Q_3 will be the directed walk from u to v.

Nice :) 4

Combination of Q_2 and Q_4 will be directed walk from v to u.
Hence uRv.

→ So if uRw and wRv, then uRv.
Hence it is transitive. Hence, proved.

P3b 4

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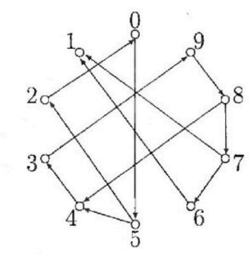
b) Observe that "reachable from each other" is clearly a reflexive and symmetric relation; as per part (a), it is also transitive. So, it is an equivalence relation on V (for any digraph $D := (V, A)$).

For a digraph $D := (V, A)$ and any equivalence class $X \subseteq V$ (with respect to the relation "reachable from each other"), the digraph with vertex set X and all those arcs (in A) that have both tail and head in X, is called a *strongly connected component of* D.

For the following digraph:
(i) **write** the vertex set and arc set of each strongly connected component, and
(ii) **draw** each strongly connected component.



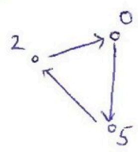
0, 2, 5
~~1, 7, 6~~ 1
3, 4, 9, 8

6

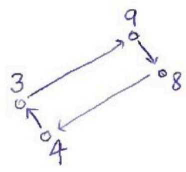
7

(i) Let Vertex sets of strongly connected components be
(ii) V_1, V_2, V_3, V_4, V_5 There are 5 strongly connected componets by the way.
Arc sets be A_1, A_2, A_3, A_4, A_5, Respectively.

→ $V_1 = \{0, 2, 5\}$
$A_1 = \{(0,5), (5,2), (2,0)\}$



→ ~~$V_2 = \{1, 6, 7\}$~~ $V_2 = \{3, 4, 9, 8\}$
~~$A_2 = \{$~~ $A_2 = \{(4,3), (3,9), (9,8), (8,4)\}$



→ $V_3 = \{1\}$
$A_3 = \phi$

→ $V_4 = \{6\}$
$A_4 = \phi$

→ $V_5 = \{7\}$
$A_5 = \phi$

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[Extra page for Problem 3]

P4a 2

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4. [Complement of a simple graph]

Recall that for a simple graph $G := (V, E)$, the *complement of* G is another simple graph with vertex set V and edge set defined as follows:

For any two distinct $u, w \in V$:

- if u and w are adjacent in G then u and w are not adjacent in $\overline{G}$, and
- if u and w are not adjacent in G then u and w are adjacent in $\overline{G}$.

a) **Prove that** G is isomorphic to the complement of its complement (that is, $G \cong \overline{\overline{G}}$). [2]

- We can say that $\boxed{\overline{G} =}$

How can we add/subtract two graphs?

If you want to do so, first you need to define what does this notation means and then you can use (:-

Where C

On how may vertices ?

- We can say

it won't be in G

So $\overline{\overline{G}} = C - (C - \overline{G}) \Rightarrow \overline{\overline{G}}$

~~Hence, proved~~.

- This happens because :-
  - If ~~it~~ u adjacent to v in G
    $\Rightarrow$ u not adjacent to v in $\overline{G}$
    $\Rightarrow$ u adjacent to v in $\overline{\overline{G}}$
  - If u not adjacent to v in G
    $\Rightarrow$ u adjacent to v in $\overline{G}$
    $\Rightarrow$ u not adjacent to v in $\overline{\overline{G}}$

This is true for all $u, v \in V$, hence G isomorphic to $\overline{\overline{G}}$

Hence, proved.

P4b 1.5

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b) For a positive integer k, a graph G is *k-regular* if the degree of each vertex is equal to k. (For example: cycle graphs are precisely the 2-regular connected graphs.)

Prove that there are exactly two 3-regular simple graphs on 6 vertices up to isomorphism, and **draw** these two graphs. [Bonus: 3]

(You may ask an invigilator for a hint for a penalty of 1.5 marks.)

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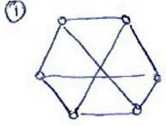

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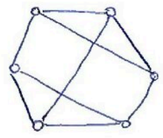

* These are the only 2 such graphs because let us take a cycle graph of 6 vertices

Each vertex has degree 2.

* Then draw

, and then there are only 2 more possibilities to draw remaining edges

(any 2 vertices connection is same upto isomorphism).

you need to prove that the graph has a cycle on 6 vertices. this is NOT at all obvious. if you assume this then reaching the desired conclusion is not too difficult (as you have demonstrated to some extent).

[Extra page: may be used for **any** problem]

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[Extra page: may be used for **any** problem]