(Let us complete the proof from last lecture.)

(If $k=0$ then $E(G)=\emptyset$ and clearly each vtx. has $\overset{(0)}{\text{even}}$ degree.)

Goal: To show that each vtx. has even degree.

Consider $\boxed{\text{any}}$ vertex $v \in V(G)$.

Since $(C_1, C_2, \ldots, C_k)$ is a partition of $E(G)$, each edge incident at vertex v belongs to (or participates in) some cycle — ⊛ in $C_1, C_2, \ldots, C_k$.

(in fact, exactly one)

Suppose that v participates in l cycles of the set $\{C_1, C_2, \ldots, C_k\}$.
(where $1 \le l \le k$)

Observe that each of these l cycles contributes 2 to the degree of v (that is, to $d(v)$).

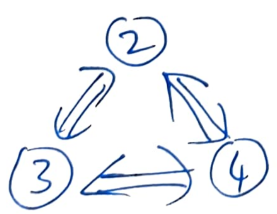
Thus, $d(v) \geq 2l$.

It follows from ⊛ that $d(v) = 2l$. Thus, v has even degree.

(Since v was chosen arbitrarily, each vtx has even degree.) ☒

Remark: Note that we did NOT use the assumption that G is connected.

In fact,



does NOT require the graph to be connected.

However, statement ① requires the graph to be connected.

I mean: for statement ① to be equivalent to any of the other statements (②, ③, ④).

TIY: Prove as many other implications as you can.
(There are 12 implications. We have only proved one in lectures: (4) $\Rightarrow$ (2))

Let us discuss a problem from Tutorial-3:

Every non-negative integer n is "sandwiched between" two consecutive perfect squares. $\longrightarrow$ $\exists$ unique non-negative integer m such that $(m^2 \le n < (m+1)^2)$

determined uniquely by n.

There are many ways to prove that the above ~~statement~~ proposition is TRUE.

Of course, one can consider $\sqrt{n}$, but let's NOT do that.

This is a special case of a much more general statement applicable to all total orders.

The ~~set~~ $\mathbb{N}$ & the relation $\le$ (that is, $(\mathbb{N}, \le)$) form a totally ~~ordered~~ ordered set, and 0 is the smallest element. (Not every totally ordered set has a smallest element. For example, $(\mathbb{Z}, \le)$.)

Here is a much more general statement:

Let $(S, \precsim)$ denote any totally ordered set that has a smallest element, say a, and assume that S is NOT finite.

(An element a such that $a \precsim b$ $\forall\, b \in S - \{a\}$.)

Let T denote any infinite subset of S such that $a \in T$.

Every element n of S is "sandwiched between" two consecutive elements of T that are determined uniquely by n.

SAME AS

$\forall\, n \in S$, $\exists$ a unique pair of consecutive elements $m_1, m_2 \in T$ such that $m_1 \precsim n \prec m_2$.

Two distinct elements m_1, m_2 of a totally ordered set $(T, \precsim)$ are consecutive if $m_1 \prec m_2$ and $\nexists\, k \in T$ such that $m_1 \prec k \prec m_2$.

What does this mean?

Note that $(T, \precsim)$ is also a totally ordered set. (Right?)

What is the point of this discussion?

There is tremendous value in considering generalizations & abstractions of specific examples & concepts that we are already familiar with.

$\downarrow$

This is the theme for the next couple of lectures.