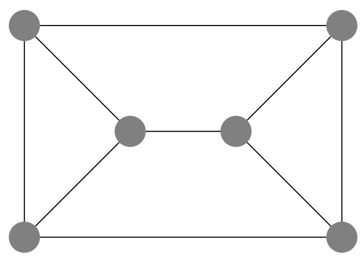
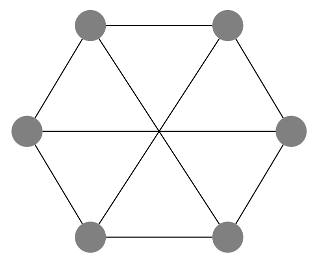
1. A graph is 2-*colorable* if each of its vertices can be colored using two colors such that no two adjacent vertices have the same color.
   (a) Define 2-colorable graphs using a function from $V(G)$ to $\{0, 1\}$.
   (b) A graph G is *bipartite* if its vertex set admits (has) a partition (A, B) such that each edge has one end in A and the other end in B. Which of the following two graphs are bipartite? Give a partition of the vertex set for those graphs that are bipartite.

   (c) Prove that a graph G is bipartite if and only if it is 2-colorable.

2. Prove that: Every simple graph G has a path of length $\delta(G)$ where $\delta(G)$ is the minimum degree among all the vertices of G.

3. Prove the following theorem.

   Theorem 1 *The following are equivalent for a graph* G*:*

   (1) G *is a tree.*
   (2) For any two vertices u *and* v*, there exists a unique* uv*-path in* G*.*
   (3) G *is connected and for each edge* $e := uv$*,* $G - e$ *has at least two connected components such that* u *and* v *belong to distinct components of* $G - e$*.*

4. Let $a, b \in \mathbb{Z}$. Prove that if $a + b$ is even, then a and b are NOT *consecutive*. [1]

5. For each *immediate predecessor relation* I (of some poset) given below, draw the Hasse diagram and write the corresponding poset. Finally, verify whether it is a lattice or not:
   (a) $I = \{e \lhd b, e \lhd c, b \lhd a, c \lhd a, f \lhd d, f \lhd e\}$ over the set $S := \{a, b, c, d, e, f\}$
   (b) $I = \{b \lhd a, c \lhd a, d \lhd b, e \lhd c, f \lhd b, g \lhd c, f \lhd e, g \lhd d, h \lhd f, h \lhd g\}$ over the set $S := \{a, b, c, d, e, f, g, h\}$.

6. For a set S, the *power set* $P(S) := \{A \mid A \subseteq S\}$, i.e., it is the collection of all subsets of S.
   (a) For $S := \{1, 2, 3, 4\}$, write the set $P(S)$. What is the cardinality of $P(S)$?
   (b) Draw the Hasse Diagram for $(P(S), \subseteq)$.
   (c) Find a "nice" description for the immediate predecessor relation of $(P(S), \subseteq)$? Prove that it is indeed the immediate predecessor relation for $(P(S), \subseteq)$.

[1] Two integers a and b are said to be consecutive if $|a - b| = 1$.

7. Recall the *inverse* of a relation, defined in Assignment 1. For a lattice $L := (S, \preceq)$, its *inverse* $L^{-1} := (S, \preceq^{-1})$.

   (a) Prove that L^{-1} is a lattice.

   (b) Let $L := (\{1, 2, 4, 6, 12\}, |)$ where $a \mid b$ denotes that a divides b. Prove that L is a lattice. Draw the Hasse diagram for L and L^{-1}.

   (c) Write the minimal and maximal elements for L, and likewise for L^{-1}.

   (d) Describe L^{-1} using words/notation.

   (e) Given a lattice L, what can we say about the Hasse diagrams of L and L^{-1}. (Informal descriptions are allowed)

8. For a lattice $L := (S, \preceq)$ and some nonempty subset $T \subseteq S$, we say that $(T, \preceq)$ is a sublattice (of L) if it is a lattice.

   Prove that for a lattice L, the following are equivalent:

   (1) L is a chain.

   (2) For any nonempty subset $T \subseteq S$, $(T, \preceq)$ is a sublattice of L.

   (3) For any 2-element subset $T \subseteq S$, $(T, \preceq)$ is a sublattice of L.