Discussion regarding proofs:

Many mathematical theorems are stated using "if .... then ...." and "if and only if".

For example:

Theorem: Let G be a graph. If each vertex has degree 2 or more then G has a cycle.

of the form $P \Rightarrow Q$

A proof of such a theorem statement typically requires one to assume that P is TRUE, and then use logical arguments to conclude that Q is also TRUE.

alternatively

one may prove that $\neg Q \Rightarrow \neg P$

alternatively

one may prove that $\neg Q \Rightarrow \neg P$ and that $\neg P \Rightarrow \neg Q$

Theorem: Let G be a conn. graph. $\boxed{\text{Then}}$ G is Eulerian $\Longleftrightarrow$ (if and only if) G has precisely 0 or 2 vertices of ODD degree.

NOT part of if .... then ... just a word used loosely

of the form $P \Longleftrightarrow Q$

SAME AS $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

A proof of such a theorem statement requires one to prove two directions/implications

$P \Rightarrow Q$ ("only if") → assume that P is TRUE and conclude that Q is TRUE

$Q \Rightarrow P$ ("if") → assume that Q is TRUE and conclude that P is TRUE.

A different viewpoint of theorems stated using $\Leftrightarrow$:

<u>Theorem</u>: Let G be a conn. graph.
Then G is Eulerian $\Leftrightarrow$ G has precisely 0 or 2 vertices of odd degree.

let's break into two parts

G is Eulerian $\Rightarrow$ G has precisely 0 or 2 vertices of ODD degree

same meaning as

For a conn. graph to be Eulerian, it is $\boxed{\text{necessary}}$ that the graph has precisely 0 or 2 vertices of ODD degree.

G has precisely 0 or 2 vertices of ODD degree $\Rightarrow$ G is Eulerian.

same meaning as

For a conn. graph, the condition of having precisely 0 or 2 vertices of ODD degree is $\boxed{\text{sufficient}}$ for the graph to be Eulerian.

Thus,
"having precisely 0 or 2 vertices of odd degree" is a $\boxed{\text{necessary as well as sufficient}}$ condition for a graph to be Eulerian.

A lot of theorems provide necessary & sufficient conditions for properties one may care about.

for example: (a conn. graph being) Eulerian

(a conn. graph being) Hamiltonian

same as having a Hamiltonian cycle (defined in Tutorial-2)

Let us compare these two properties

having an Eulerian tour — same as — a closed walk that uses each edge exactly once

having a Hamiltonian cycle — a closed walk that uses each vertex exactly once

think about it

they sound very similar, right?

Eulerian tour: A closed Eulerian trail

closed means first vtx = last vtx

conn. means connected

Theorem: A conn. graph has an Eulerian tour $\iff$ each vertex has even degree.

let us appreciate this

If I give you some conn. graph & ask you whether it has an Eulerian tour, you simply need to check that degree of each vertex is EVEN.

related to algorithms

So, it is "very easy" to check/decide whether some conn. graph has an Eulerian tour (or NOT)

just check degree of each vtx.

if degree of each vtx. is even then say YES

otherwise (that is, if any vtx has odd degree) then say NO.

There are NO "nice" necessary & sufficient conditions for the Hamiltonian cycle property.

same as

On the other hand, there is NO "easy way" to check/decide whether a conn. graph has a Hamiltonian cycle.

All known methods require "LOTS OF TIME."

It is beyond the scope of this course to formalize these points. You will see these points formalized in your future courses related to ALGORITHMS & COMPLEXITY THEORY.

Theorem: Let G be a conn. graph.

Then $\boxed{G \text{ has an Eulerian tour}} \iff \boxed{G \text{ has precisely } 0 \text{ vertices of odd degree}}$.

$\updownarrow$ same as

each vtx has even degree

The theorem is saying that $\boxed{\text{these two}}$ properties are (in some sense) "the same".

$\downarrow$

a conn. graph has one property if and only if it has the other property

What if we have $\boxed{\text{three or more}}$ such properties? How are such theorems stated? Let's see an example.

Theorem: Let G be a conn. graph. The following are equivalent:

(1) G has an Eulerian tour.

(2) Each vtx. of G has even degree.

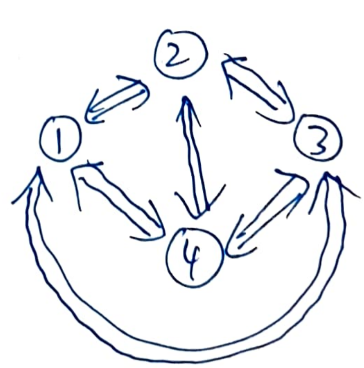
(3) A directed graph D may be obtained from G (by putting direction on each edge) such that $d^{in}(v) = d^{out}(v)$ for each $v \in V(D)$.

(4) $E(G)$ can be partitioned into cycles. → see next page for example

→ what does this mean?

It means (1) $\iff$ (2), (2) $\iff$ (3), (3) $\iff$ (4), (1) $\iff$ (3), (2) $\iff$ (4), (1) $\iff$ (4).

Visually:



So, a proof of this theorem needs to prove each implication, and there are six implications!

TIY: Does one really need to prove each of the six implications?

If not, what is the minimum number of implications one must prove, and what is an example of a minimum set of implications to be proved?

minimum set
means
(set of minimum cardinality)

We will prove one of the six implications (for now):

Proof of ④ ⇒ ②:

Assume that $E(G)$ can be partitioned into cycles.

Let $(C_1, C_2, C_3, \ldots, C_k)$ be such a partition.

Recall definition from Module-1.

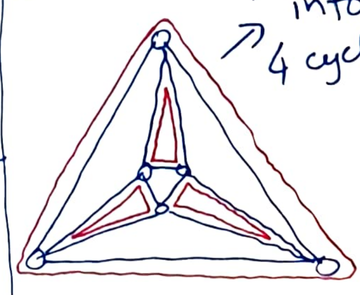
TIY: Complete the proof yourself. (Argue that each vtx has even degree.)

Recall:

④ $E(G)$ can be partitioned into cycles.

② Each vtx. of G has even degree.

Example:



partition into 4 cycles