We continue our deep dive into the theory of posets:

$(S, \preceq)$: poset

Recall: for any two (not necessarily distinct) elements $a, b \in S$:

- either $a \preceq b$
  In this case, we say that $\underline{a}$ is a lower bound of $\underline{b}$ & that $\underline{b}$ is an upper bound of $\underline{a}$.
- or $b \preceq a$
  In this case, we say that $\underline{b}$ is a lower bound of $\underline{a}$ & that $\underline{a}$ is an upper bound of $\underline{b}$.
- otherwise a & b are incomparable.

Both of these hold if and only if a & b are the same.

So, $\underline{a}$ is a lower bound & upper bound of itself.

For an element $a \in S$:

$\underbrace{LB(a)}_{\text{read as}} := \{b \in S : b \preceq a\}$

the set of lower bounds of a

$\underbrace{UB(a)}_{\text{read as}} := \{b \in S : b \succeq a\}$

the set of upper bounds of a

→ Let us generalize to a set of elements:

For $T \subseteq S$:

$\underbrace{LB(T)}_{\text{read as}} := \bigcap_{a \in T} LB(a)$

the set of lower bounds of T

$\underbrace{UB(T)}_{\text{read as}} := \bigcap_{a \in T} UB(a)$

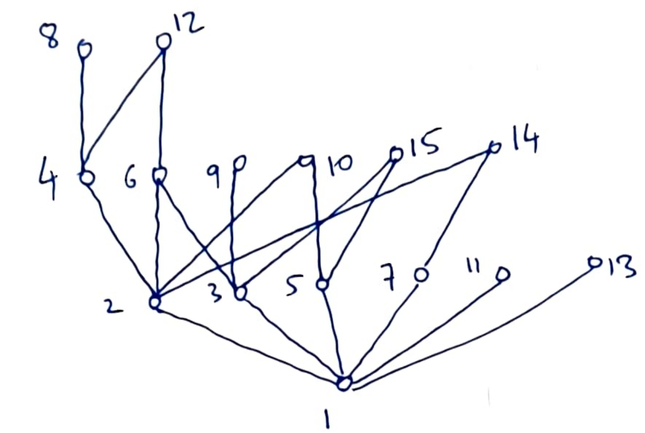
the set of upper bounds of T

contains those elements that are lower bounds for each element in T

contains those elements that are upper bounds for each element in T

Let's go back to our example: $(\{1,2,\dots,15\}, |)$

(and apply these definitions to some elements/sets)



Hasse Diagram

$LB(6) = \{1,2,3,6\}$

$UB(6) = \{6,12\}$

$LB(15) = \{1,3,5,15\}$

$UB(15) = \{15\}$

$LB(11) = \{1,11\}$

$UB(11) = \{11\}$

$LB(\{6,12\}) = \{1,2,3,6\}$

$UB(\{6,12\}) = \{12\}$

$LB(\{11,13\}) = \{1\}$

$UB(\{11,13\}) = \phi$

$LB(\{4,6\}) = \{1,2\}$

$UB(\{4,6\}) = \{12\}$

$LB(\{1,3,5\}) = \{1\}$

$UB(\{1,3,5\}) = \{15\}$

$LB(\{3,4,6\}) = \{1\}$

$UB(\{3,4,6\}) = \{12\}$

Similarly: Does UB(T) always contain an element that is lesser than all others in UB(T)?

→ what about "bigger" sets?

For $T \subseteq S$, does LB(T) always contain an element that is greater than all other elements in LB(T)?

DIY

NO; construct examples yourself

Observations:

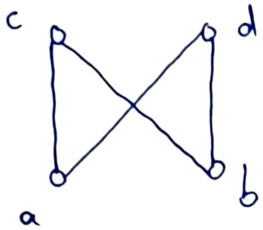
$(S, \preceq)$: poset

$a \in S$

① The set $LB(\underline{a})$ contains $\underline{a}$; furthermore:
$\forall\, b \in LB(\underline{a}) - \underline{a},\ b \prec \underline{a}$.

② The set $UB(\underline{a})$ contains $\underline{a}$; furthermore:
$\forall\, b \in UB(\underline{a}) - \underline{a},\ \underline{a} \prec b$.

For example, consider the following Hasse diagram (of a poset):



NOT a Lattice

$LB(\{c,d\}) = \{a,b\}$

but these are incomparable

$UB(\{a,b\}) = \{c,d\}$

Lattices:

A poset $(S, \preceq)$ is called a lattice if $\forall$ distinct $a, b \in S$: $GLB(\{a,b\})$ exists AND $LUB(\{a,b\})$ exists.

Why do we care about 2-element subsets?

We will prove (later) using induction that this definition implies that $GLB(T)$ & $LUB(T)$ exist $\forall\, T \subseteq S$.

GLB & LUB:

$(S, \preceq)$: poset

$T \subseteq S$

An element $a \in LB(T)$ is a greatest lower bound of T if $b \preceq a \;\forall\, b \in LB(T)$.

An element $a \in UB(T)$ is a least upper bound of T if $b \succeq a \;\forall\, b \in UB(T)$.

May NOT always exist. When it exists, it is unique. (DIY).

(proofs required but beyond scope of CS1200)

FACTS: The divisibility poset for positive integers — that is — $(\mathbb{N}-\{0\}, |)$ is a lattice.

Furthermore, for any two distinct positive integers a,b:

① $GLB(a,b) = GCD(a,b)$

② $LUB(a,b) = LCM(a,b)$

↓

least common multiple

↓ definition

among the common multiples of a & b, consider the $\underbrace{\text{smallest}}$ one

↓

(as per ≤ total order)

↓

NOT the SAME definition as LUB (applied to $\mathbb{N}-\{0\}, |$)

↓

greatest common divisor

↓ definition

among the common divisors of a & b, consider the $\underbrace{\text{largest}}$ one

↓

(as per ≤ ~~partial~~ total order)

↓

NOT the SAME definition as GLB (applied to $\mathbb{N}-\{0\}, |$)