We have used the following observation/fact many times in our arguments:

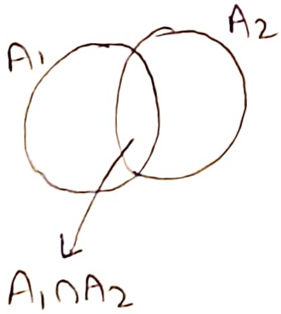
Observation: If $(A_1, A_2, \ldots, A_n)$ is a weak partition of a finite set A then $|A| = |A_1| + |A_2| + \ldots + |A_n|$.

(In particular, if A is the disjoint union of A_1 & A_2 — that is, if $A = A_1 \dot\cup A_2$ then $|A| = |A_1| + |A_2|$.)

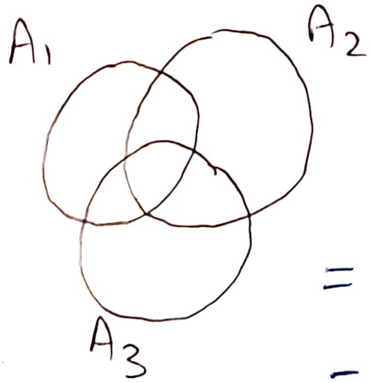
Clearly, this is NOT true for union (in general).

However, for two finite sets A_1 & A_2 (not necessarily disjoint), we all know (and it is easy to prove) DIY:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



What about 3 sets?



DIY

It is easy to prove that $|A_1 \cup A_2 \cup A_3|$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

There seems to be a pattern, right?

Turns out it is NOT a coincidence!

Theorem: (Principle of Inclusion-Exclusion)

For any collection of finite sets $A_1, A_2, \dots, A_n$:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}\left[(-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} \left|\bigcap_{i \in I} A_i\right|\right]$$

what is this notation?

For a set S, $\binom{S}{k}$ is the collection of all k-subsets of S.

Thus for a finite set S:

$$\underbrace{\left|\binom{S}{k}\right|}_{\text{a set}} = \underbrace{\binom{|S|}{k}}_{\text{a number}}$$

Proof: By counting.

Observe that each element $x \in \bigcup_{i=1}^{n} A_i$ contributes exactly 1 to LHS. (Right?)

GOAL: To show that each element $x \in \bigcup_{i=1}^{n} A_i$ contributes 1 to RHS.

(This will prove LHS = RHS, right?)

Let $x \in \bigcup_{i=1}^{n} A_i$. Thus, x belongs to some j of the sets in $A_1, A_2, \dots, A_n$ (where $j \geq 1$).

We label the sets so that $x \in A_1$, $x \in A_2$, ..., $x \in A_j$ and x does NOT belong to any of the other sets.

Now let us count the contribution of x to RHS.

Observe that x belongs to the intersection of any $k \geq 1$ sets chosen from $A_1, A_2, \dots, A_j$ and that x does NOT belong to any other "intersection of sets" considered in RHS.

Thus contribution of x to RHS =

$$\sum_{k=1}^{j}\left[(-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,j\}}{k}} \overset{\text{why?}}{1}\right] \underset{\text{why?}}{=} \sum_{k=1}^{j}\left[(-1)^{k-1}\binom{j}{k}\right]$$

$$= \binom{j}{1} - \binom{j}{2} + \binom{j}{3} + \dots + (-1)^{j-1}\binom{j}{j}$$

$$= \binom{j}{0} = \boxed{1.}$$

Recall:

Binomial Thm

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

put $x = -1$:

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots$$

Thus x contributes 1 to RHS.

Thus LHS = RHS. This proves the Principle of Inclusion-Exclusion. ☒