CS1200 — Discrete Mathematics for Computer Science — 2023 (Jan-May)

Assignment 4

Release Date: 09/06/2023 **Due Date: 17/06/2023 — 8:00 AM IST**

Collaborator (if any): *Aditya Srivastava*

Academic Integrity Statement: I, *Hema Venkata Raadhes Chanadaluru*, affirm that I have not given or received any **unauthorized** help (from any source: people, internet, etc.) on this assignment, and that I have written/typed each response on my own, and in my own words.

THE MARKS FOR EACH PROBLEM (1, 2, 3, 4, 5) ARE FIXED. HOWEVER, THE MARKS FOR EACH SUBPROBLEM (1A, 1B, 2A, ETC.) ARE TENTATIVE — THEY MAY BE CHANGED DURING MARKING IF NECESSARY.

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1. [COMBINATORIAL PROOFS] [12]

Recall (from lectures) that we have seen two types of combinatorial proofs (for proving identities): (i) establishing a bijection between two sets (one counted by LHS, and the other counted by RHS), or (ii) using a double counting argument (that is, proving that LHS and RHS both count the same set).

The goal of this exercise is to prove some more identities using combinatorial arguments.

(a) Give a bijective proof to show that, for all $k, n \in \mathbb{N}$, where $0 \le k \le n$, the following holds:

$$\binom{n}{k} = \binom{n}{n-k}$$

Response: Warm up exercise. NO response required.

(b) Give a combinatorial proof to show that, for all $n \in \mathbb{N}$, the following holds:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Response: Let us go with a double counting proof here.

Take a set of 2n elements , the number of ways to choose n elements out of these 2n elements is by definition $\binom{2n}{n}$.
Let us now choose n elements out of these 2n elements in another way:
Partition this set into two sets A and B each with n elements each.
Number of ways to select i $(0 \le i \le n)$ elements from set A is $\binom{n}{i}$.
Number of ways to select n-i elements from B is equal to the number of ways to select i elements from B. Observe this from the results of part (a).

So number of ways to select a total of n elements from two sets A and B each with n elements each would be sum of $\binom{n}{i}^2$ for all i from 0 to n . This covers all the possible ways of choosing.

Note that $\binom{n}{i}^2$ refers to the possible ways of selecting n elements from two sets A and B be n elements each such that we choose exactly i of them from set A. It is possible to choose anywhere from 0 to n elements from A ,hence we sum up $\binom{n}{i}^2$ for those values of i to get the total ways.

Hence we have $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$.

(c) Give a combinatorial proof to show that, for all $n \in \mathbb{N}$, the following holds:

$$\sum_{j=0}^{n} j\binom{n}{j} = n \cdot 2^{n-1}$$

Response: Let us go with a double counting proof.
How many sets A,B exist such that A contains exactly 1 element, B contains an arbitrary number of elements , A and B are disjoint and both those sets are subsets of set C which has n elements.–(i)

(1) Let us first choose the element in set A. n possibilities.
Now choose the elements in set B. We cannot choose the element which we choose to be in A. So each of the remaining n-1 elements may or may not be there, which gives rise to 2^{n-1} possibilities. Total of $n \cdot 2^{n-1}$ ways .

(2) Say that $A \cup B$ has j elements , this number can range from 0 to n.
The number of ways to choose such j elements from set C is $\binom{n}{j}$. Now exactly one of this is in A and the rest are in B. The element in A has j possibilities. So if $A \cup B$ has j elements there are

$j \cdot \binom{n}{j}$ ways of choosing such sets A and B.
Now j can range from 0 to n.
Hence, number of ways of choosing sets A and B with conditions (i) is $\sum_{j=0}^{n} j\binom{n}{j}$.

Hence by double counting we have , $\sum_{j=0}^{n} j\binom{n}{j} = n \cdot 2^{n-1}$.

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2. [PLACING NON-ATTACKING ROOKS ON A HUGE CHESSBOARD] [12]

In your hostel is a 100×100 marble chessboard, and a large number of chess pieces.

(It is very beautiful, and very heavy, and you have no idea what to do with it! So, let's count something for fun!)

(a) Prove that the number of ways to place k non-attacking rooks on this chessboard is $k!\binom{100}{k}^2$.

Note: Two rooks attack each other if they are in the same row or column. We consider any two rooks to be indistinguishable, so we only care about which squares contain rooks and which squares do not. Hence the question is really asking about the number of ways to choose a set of k squares on the chessboard so that no two squares are in the same row or column.

Response: We must place k such non attacking rooks on a 100x100 board.
As we must place each rook in a different row , let us choose k rows out of 100 rows in which the rooks are present. This can be done in $\binom{100}{k}$ ways.
Now each rook also lies in a different column so let us choose k different columns in which the rooks can be placed . This can be done in $\binom{100}{k}$ ways .
Observe that we have now chosen k rows and k columns , there are $\binom{100}{k}^2$ different ways to choose k different rows and k different columns.
More formally , if the rows and columns of the board are represented by sets $A = \{r_1, r_2, ...r_{100}\} and B = \{c_1, c_2...c_{100}\}$ then the number of ways to select 2 sets A_k, B_k where A_k contains k elements of set A and B_k contains k elements of set B is $\binom{100}{k}^2$.

Each square on a chess board can be represented by its coordinates , or in essence one column and one row. Each pair of squares must pairwise have different column and different row.
For a given A_k and B_k we can choose k squares each with different columns and rows in $k!$ ways.
Every permuation from set A_k to B_k represents k squares in different rows and columns, and every set of k squares can be represented a permutation from A_k to B_k.
So number of ways is $k! \cdot \binom{100}{k}^2$.

(b) Prove that the number of ways to place k non-attacking rooks on the chessboard, with all rooks on white squares, is:

$$\sum_{i=0}^{k} i!\binom{50}{i}^2 (k-i)!\binom{50}{k-i}^2.$$

Hint: You may want to color the white squares.

Response: Without loss of generality(i hope this is the appropriate place to use this statement), we can assume the first (top left most) square on the chess board is a white square.
We label coordinates of squares with x coordinate ranging from 1 to 100 from left to right. We label y coordinates ranging from 1 to 100 from top to bottom.
Clearly in this chess board we have that a square is white iff the sum of coordinates is even.
Let us split these white squares into two sets , one set A which has both coordinates even, and other set B has both coordinates odd.
Observe that each of these sets form a 50x50 grid :
In set A , there are squares in precisely 50 rows(note 2 squares in same row iff they have same y coordinate) and precisely 50 columns(note 2 squares in same column iff they have same x coordinate). And for every of these rows and every of these columns there are precisely 50 squares in each. Similar is true for set B also.
And so , both sets of squares essentially form thei own separate grids of size 50x50.
Also observe that there is no common column or common row between any of squares in sets A

and B. The implication of this is that any rook placed in A would not attack a rook in B and vice versa. I.e We can place rooks in A and B independently .

If I want to place i non attacking rooks in squares of set A we can do it in $i! \cdot \binom{50}{i}^2$ ways.(observe this from part (a), we just change the 100 to 50 as it is essentially the same thing)
Our end goal is to choose k squares from sets A and B such that no two squares have same column or row.
Say of these k squares , i squares be from A , where i ranges from 0 to k. And then the remaining k-i will be from set B .
The number of ways to choose i squares from A is $i \cdot \binom{50}{i}^2$ where no two squares are in same row or column . The number of ways to choose k-i squares from B is $(k-i)! \cdot \binom{50}{(k-i)}^2$ where no two squares are in same row or column.
Hence number of ways to choose k squares from A and B such that no two squares are in same row or column would be the sum of $i!\binom{50}{i}^2(k-i)!\binom{50}{k-i}^2$ for all possible i (here i can be from 0 to k clearly , and is stated above). Hence :
$\sum_{i=0}^{k} i!\binom{50}{i}^2(k-i)!\binom{50}{k-i}^2$.

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3. [Another application of the Pigeonhole Principle: Handshakes at a Party] [12]

In this exercise, we will use the pigeonhole principle to prove the following.

Theorem 0.1. *Some $n \in \mathbb{N} - \{0, 1\}$ people attended the CSE@IIT-M Freshers' 2023 Party, and some of them shook hands with others*[1]. *There exist two partygoers — each of whom shook hands with an equal number of people.*

(a) Write down a theorem statement — using the language/terminology of graphs — that has the same meaning as Theorem 0.1.

Response: There are some $n \in \mathbb{N} - \{0, 1\}$ vertices in a simple graph. There exist 2 vertices whose degrees are equal in this graph.

(b) Prove the theorem you wrote in part (a) — using the Pigeonhole Principle (often abbreviated to PHP). In your proof, explain the use of PHP clearly — in particular, state the version of PHP you are using, and clearly state where and how it is applied.

Response:

Pigeonhole Principle : If there are n elements which we are distributing into k boxes then there must exist atleast one box with atleast $\lceil n/k \rceil$ elements.

There are a total of n vertices in a graph.
Say m of the vertices in this graph have degree 0.
(1) If $m \geq 2$ then we can choose 2 vertices which have degree equal to 0, they have equal degree . We are done.
Otherwise if $m \leq 1$ let us continue .
(2)In this case take graphs with $m \leq 1$.
The maximum degree of any vertice is clearly n-m-1.
This is because in a simple graph there cannot be any self loops or multiple edges between two vertices , so a vertice can atmost have n-m-1 edges connected to it ,when it is connected to each of the other n-m-1 vertices whose degrees are not zero. i.e have atmost degree n-m-1. (Do observe that for every simple graph that has $m \leq 1$ and $n \geq 2$ we have $n - m - 1 \geq 1$)

Note that :
For $n = 2$ there are only two possible graphs . One graph has $m = 2$ (two vertices of degree 0) . This graph comes in case (1). The other graph has $m = 0$. (one edge connecting the two vertices). $n - m - 1 = 1$ for this case .
For $n \geq 3$ we have $n - m - 1 \geq 1$ considering $m \leq 1$.

There are n-m vertices with degree not equal to 0 , and each of these vertices can have degree lying from 1 to n-m-1.
This is analogous to having n-m-1 boxes , and filling them with n-m elements. By PHP we can say that there will be atleast one box with atleast $\lceil (n-m)/(n-m-1) \rceil$ elements. In other words there will atleast be one degree such that there are atleast $\lceil (n-m)/(n-m-1) \rceil$ vertices with that degree.
For $n = 2$ graph with $m = 0$ we have atleast one box with atleast $\lceil 2/1 \rceil = 2$ elements.
For $n \geq 3$ and $m \leq 1$ we have atleast one box with atleast $\lceil (n-m)/(n-m-1) \rceil = 2$ elements also.
This is equivalent to saying that for any graph with greater than equal to 2 vertices and atmost 1 vertice of degree 0 , we will 2 vertices with equal degree.

[1]At this party, **no one** shakes hands with themself!

Combining cases (1) and (2) we can say that for any simple graph of atleast 2 vertices we will be able to find 2 vertices of equal degree.

4. [AN APPLICATION OF THE PRINCIPLE OF INCLUSION-EXCLUSION] [12]

We started Module-3 by counting the number of one-to-one functions between two finite sets.

In this exercise, we will count the number of onto functions — using the principle of inclusion-exclusion. As we did in the case of derangements (in lectures), it will be useful to first count the number of functions that are **not** onto, and then subtract that from the total number of functions.

Throughout this exercise, let A and B denote two finite sets with $|A| = m$ and $|B| = n$ where $n \leq m$.

(a) Prove that the total number of functions from A to B is n^m.

Response: The cardinality of set A is m and the cardinality of set B is n.
Base Case : If m=1 then clearly the number of possible functions is n. Say the element in A is called a, then we have n possibilities for $b \in B$ where f(a)=b.

If $m \geq 2$, then we shall prove it below :
Let us assume inductively that for any proper subset $A_1 \subset A$ which has k elements where $k < m$, the number of functions from A_1 to B is n^k.
Take the set A, remove one element , let us say a.
The new set say $X \subset A$, where $X := A - \{a\}$. The cardinality of X is m-1 clearly.
So the number of functions from X to B is n^{m-1} by the induction assumption.
There are n possibilities for the value of $f(a)$, hence we can say that the number of functions from A to B will be n times the number of functions from X to B. Hence the number of functions from A to B is n^m.

Hence proved inductively.

(b) Use the Principle of Inclusion-Exclusion (often abbreviated to PIE) to prove that the total number of functions from A to B that are **not** onto is:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m$$

Explain clearly how and where PIE is being used.

Response: An onto function is a function in which for every $b \in B$ we have $a \in A$ such that $f(a) = b$ where $f : A -> B$.
So for a function to not be onto there should exist atleast one element $b \in B$ such that there exist no $a \in A$ such that $f(a) = b$.
Say that $B = \{b_1, b_2 ... b_n\}$.
Let F_i be the set of all functions such that there is no $a \in A$ such that $f(a) = b_i$.
$F_i \cup F_j$ would be the set of functions such that for atleast one $b \in \{b_1, b_2\}$ there is no $a \in A$ such that $f(a) = b$.
Similarly we can say that :
$\bigcup_{i=1}^{n} F_i$ would be the set of functions such that for atleast one $b \in \{b_1, b_2 ... b_n\} = B$ there is no $a \in A$ such that $f(a) = b$. The cardinality of this set is required.

PIE: For n sets, $A_1, A_2 \ldots A_n$ we have that:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{S}{k}} \left|\bigcap_{i \in I} A_i\right|$$

Where, $S = \{1, 2 \ldots n\}$ and $\binom{S}{k}$ denotes set which is a collection of all subsets of S which have k elements.

Observe that, here $\left|\bigcap_{i \in I_k} F_i\right| = (n-k)^m$ ~~because :-~~

Where I_k represents a set of k elements of $\{1, 2 \ldots n\}$.

- Say $I_k = \{i_1, i_2 \ldots i_k\}$
- Then we can say that the number of functions from A to $B - \{b_{i_1}, b_{i_2} \ldots b_{i_k}\}$ is equal to the cardinality of $\bigcap_{i \in I_k} F_i$.
- This is because ~~for any function in~~ $\bigcap_{i \in I_k} F_i$ represents the set of functions for which for all $b \in \{b_1, b_2 \ldots b_k\}$ there is no $a \in A$ such that $f(a) = b$.

Basically it is the same set of function as the set of all functions from A to $B - \{b_1, b_2 \ldots b_k\}$.

- By part (a), number of functions from A to $B - \{b_1, b_2 \ldots b_k\}$ is $(n-k)^m$.
- For a certain k, we have number of subsets of k elements of $S = \{1, 2 \ldots n\}$ be $\binom{n}{k}$

And we have $\left|\bigcap_{i \in I} F_i\right| = (n-k)^m$, when I has k elements

$$\left|\bigcup_{i=1}^{n} F_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)^m$$ Hence, proved.

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(c) Use parts (a) and (b), and some easy manipulations, to prove that the total number of onto functions from A to B is:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$$

Response: The total number of functions from A to B is n^m where $|A| = m$ and $|B| = n$.
An onto function is a function for which , for all $b \in B$ we have atleast one $a \in A$ such that

$f(a) = b$.
We have found number of functions which are not onto from A to B has been found in part (b).
Hence the number of onto functions from A to B would be total number of function from A to B minus the number of non onto functions from A to B.
Number of onto functions :

$$n^m - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)^m = n^m + \sum_{k=1}^{n} (-1)^{k} \binom{n}{k} (n-k)^m = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)^m$$

Hence proved. $\square$

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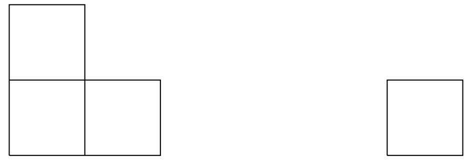
5. [USING AND SOLVING RECURRENCES: YET ANOTHER TILING PROBLEM] [12]

In this exercise, we our goal is to compute the number of tilings[2] of a $(2 \times n)$-grid for all $n \in \mathbb{N} - \{0\}$ using two types of tiles: L-shaped tiles (drawn below on the left) and box-shaped (1×1) tiles (drawn below on the right):



(a) For convenience, we denote by T_{n-1} the number of tilings of a $(2 \times n)$-grid — using L-shaped and box-shaped tiles. Compute T_0, T_1 & T_2, and explain your answers clearly — with drawings (if required).

Response:

(1) $T_0 = 1$. We cannot use the L-shaped tiles because T_2 is 2 squares large . The L-shaped tile is itself 3 squares large so it is not possible to tile it with L-shaped tiles. We can tile it with just 2 1x1 tiles stacked one on top of another.

(2) $T_1 = 5$. It is impossible to tile a T_1 with only L-shaped tiles because L-shaped tiles are 3 squares large. T_1 is 4 squares large . If it is possible to tile it only with L -shaped tiles the number of tiles must be a multiple of 3.

We must use atleast one 1x1 tile.

Say we use ONLY one 1x1 tile , we can place it in four places. We can fill the remaining with one L-shaped tile in each case.

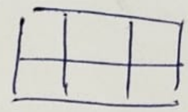
It is impossible to use exactly 2,3 1x1 tiles, because then the remaining squares to tile would not be a multiple of 3.

It is possible to use 4 1x1 tiles to tile.

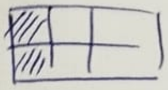
Hence 5 possible cases.

[2] As per examples discussed in lectures, tiling means: "no two tiles should overlap" and "each square of the grid should be covered by a tile".

(3) $T_2 = 11$. Let us take 2×3 grid.



Case 1: 2 1×1 tiles in leftmost column.



T_1 ways to fill remaining squares

$T_1 = 5$

Case 2: 1 1×1 tile in leftmost column

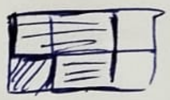
Subcase 1:

• Place 1×1 tile on top position, then L-shaped tile as shown

T_0 ways to tile remaining squares

$T_0 = 1$

Subcase 2:



• Also $T_0 = 1$ ways, similar to before.

*Case 2 has total 2 ways to tile

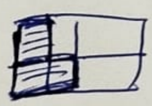
Case 3: No 1×1 tile in left column.

Subcase 1:

• 2 ways to tile remaining region → either 1 L-shaped tile

→ or 3 1×1 tiles

Subcase 2:



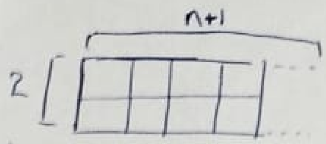
• Also 2 ways to tile similar to above

*Case 3 total 4 ways

$T_2 = 5 + 2 + 4 = 11$

(b) Write down a recurrence for T_n, and explain clearly why this recurrence is correct.

Response: $T_n = T_{n-1} + 4T_{n-2} + 2T_{n-3}$



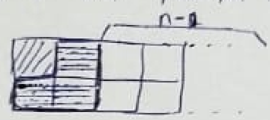
Let us take cases: (with respect to filling of leftmost 2 squares (column))

(1) There are 2 1x1 tiles in 2 left squares

In this case the number of ways to tile is $\underline{T_{n-1}}$ (as remaining grid to tile is $2\times(n-1)$)

(2) There is exactly 1 1x1 tile in 2 left squares.

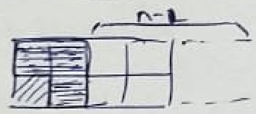
(a) Subcase 1:



Observe that we must place L-shaped tile as such, because ~~other~~ square below 1x1 tile must not contain 1x1 tile.

In this case $\underline{T_{n-2}}$ ways to tile remaining grid.

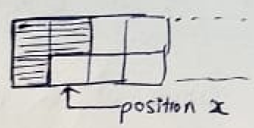
(b) Subcase 2:



As similar to above, here also there are $\underline{T_{n-2}}$ ways.

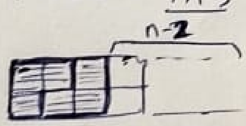
(3) There is no 1x1 tile in the leftmost column.
Hence we must place L-shaped tile in that column.

(a) Subcase 1:



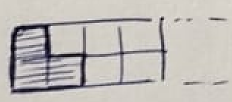
(i) If position x is filled by 1x1 tile, then remaing grid can be tiled in $\underline{T_{n-2}}$ ways

(ii) Else we must place L-shaped tile as shown and There are $\underline{T_{n-3}}$ ways to tile remaining region



• So in this subcase, there are total $T_{n-2}+T_{n-3}$ ways to tile

(b) Subcase 2:



• Symmetrical sub case as above. Just flipped ~~to~~ Vertically.
Without loss of generality there are $T_{n-2}+T_{n-3}$ ways to tile here as well.

Adding all the cases we will obtain the recurrence relation stated in the first line of the response.
Note : Slanted lines correspond to 1x1 tiles and horizontal lines correspond to L-shaped tiles in the above depictions.

(c) Solve the recurrence obtained in part (b) — using the initial conditions from part (a) — and write down a closed form formula for T_n. Explain the steps followed clearly.

Response: The solution for a recurrence $T_n = c_1T_n - 1 + c_2T_n - 2 + c_3T_n - 3$ where x_1, x_2, x_3 are distinct roots of the equation $x^3 - c_1x^2 - c_2x - c_3 = 0$ would be $T_n = ax_1{}^n + bx_2{}^n + cx_3{}^n$ where the constants a,b,c are determined by the initial conditions.
Here the cubic equation to obtain x_1, x_2 and x_3 would be:
$x^3 - x^2 - 4x - 2 = 0$
The roots of this cubic are $x_1 = -1$, $x_2 = 1 + \sqrt{3}$ and $x_3 = 1 - \sqrt{3}$.
We have : (using $T_n = ax_1{}^n + bx_2{}^n + cx_3{}^n$ and substituting the values for T_0, T_1 and T_2 we will get the below)
$T_0 = 1 = a + b + c$
$T_1 = 5 = -a + (b + c) + \sqrt{3}(b - c)$
$T_2 = 11 = a + 4(b + c) + 2\sqrt{3}(b - c)$
Solving the above linear equations we get the solutions for the constants as follows:
$a = -1, b = 1 + 1/\sqrt{3}, c = 1 - 1/\sqrt{3}$
Hence the closed form is :
$T_n = (-1)(-1)^n + (1 + 1/\sqrt{3})(1 + \sqrt{3})^n + (1 - 1/\sqrt{3})(1 - \sqrt{3})^n$
Simplifying we have closed form to be :

$$T_n = (-1)^{n+1} + (1 + \sqrt{3})^{n+1}/\sqrt{3} - (1 - \sqrt{3})^{n+1}/\sqrt{3}$$