Quick Recap:

A function $f: A \to B$ is a $\boxed{\text{bijection}}$ (aka 1-to-1 correspondence)

if it is 1-to-1 & onto.

1-to-1 same as injective; onto same as surjective

We have discussed two graph-theoretical properties: ① connected & ② Eulerian

We have observed that these properties do NOT depend on the representation (drawing or matrix) and do NOT depend on labels of vertices and/or edges.

This leads to the question:

When are two simple graphs $G_1 := (V_1, E_1)$ & $G_2 := (V_2, E_2)$ "the same"?

Such a bijection is called an $\boxed{\text{isomorphism}}$ from G_1 to G_2.

Boring answer:

- We say that G_1 & G_2 are $\boxed{\text{equal}}$ if $V_1 = V_2$ & $E_1 = E_2$.

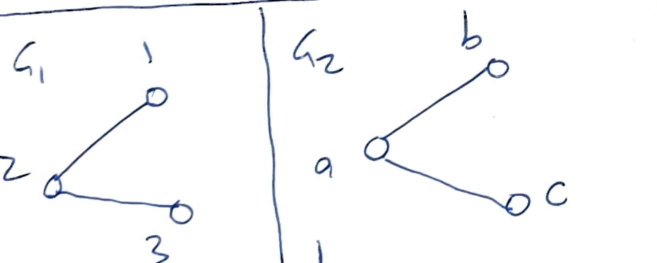
Exciting answer: We say that G_1 $\boxed{\text{is isomorphic to}}$ G_2 if there IS a bijection $f: V_1 \to V_2$ such that for any two distinct $u, v \in V_1$:

① if $uv \in E_1$ then $f(u)f(v) \in E_2$

edges/adjacencies are preserved

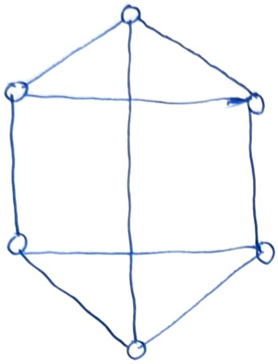
② if $uv \notin E_1$ then $f(u)f(v) \notin E_2$

nonadjacences/non-edges are preserved

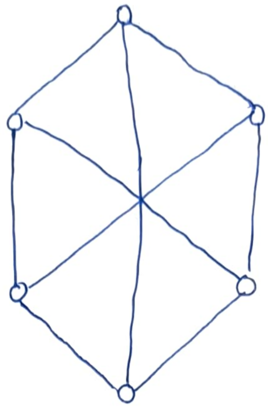


G_1 & G_2 are NOT EQUAL but they are ISOMORPHIC.

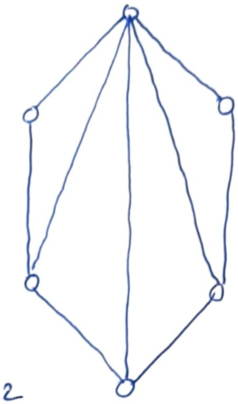
Example:



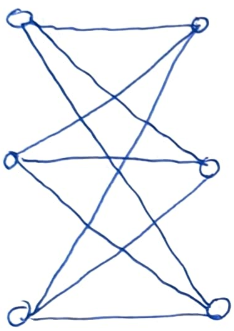
G_0



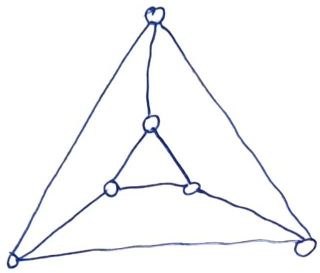
G_1



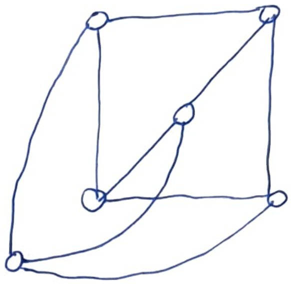
G_2



G_3

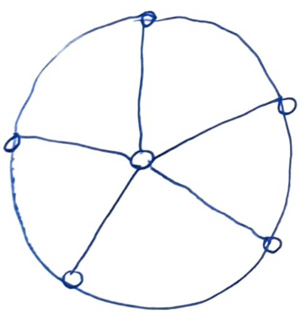


G_4

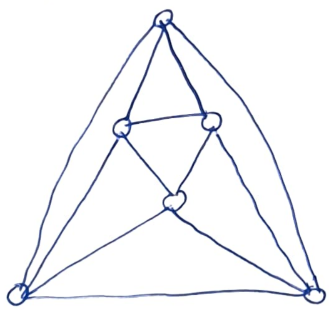


G_5

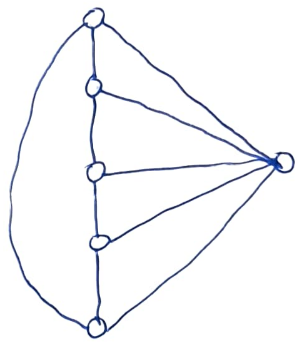
G_6



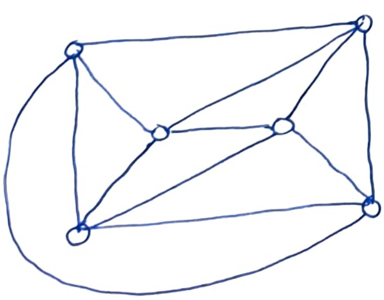
G_7



G_8



G_9



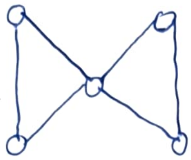
DIY: For these 10 graphs, consider each pair (G & H) and figure out if G is isomorphic to H.

If YES, find an isomorphism.

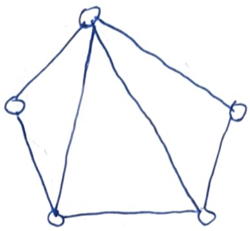
If NO, come up with some explanation. Example on next page.

Example:

G_1



G_2



G_1 & G_2 are NOT isomorphic. Why?

Explanation 1: In G_1, each vertex has degree 2 or 4.

In G_2, there are vertices of degree 3.

So: G_1 is NOT isomorphic to G_2, right? (Think)

Explanation 2: G_1 has 6 edges.

G_2 has 7 edges.

So: G_1 is NOT isomorphic to G_2, right? (Think)

Let us discuss properties of the relation "is isomorphic to"

$\mathcal{G}$: set of all simple graphs

We write $\boxed{G_1 \cong G_2}$ to mean that G_1 is isomorphic to G_2.

① Reflexivity

$(f(v) = v$ for each vertex $v)$

Given any simple graph $G := (V, E)$ the $\boxed{\text{identity bijection } f}$ (from V to itself) is an isomorphism from G to itself, right? (Think...)

② Symmetry

Given two simple graphs $G_1 := (V_1, E_1)$ & $G_2 := (V_2, E_2)$ such that G_1 is isomorphic to G_2 we would expect that G_2 is isomorphic to G_1, right?

If f is an isomorphism from G_1 to G_2 then inverse of f (denoted by f^{-1}) is a bijection from G_2 to G_1.

How do we formalize this? We need to consider the "inverse of a bijection".

DIY: Write a proof by simply following the definitions carefully.

Given a bijection $f: A \to B$, the $\boxed{\text{inverse of } f}$, denoted by f^{-1}, maps each element of B, say b, to the (unique) element $a \in A$ such that $f(a) = b$.

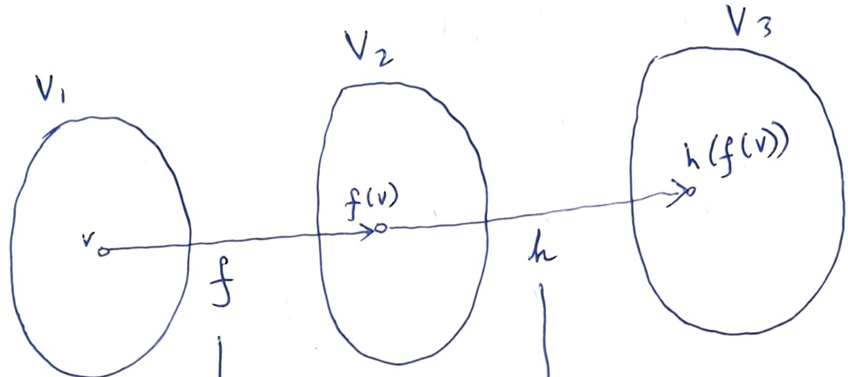
In other words, $f^{-1}(b) = a$ IF $f(a) = b$.

③ Transitivity

Given three simple graphs $G_1 := (V_1, E_1)$, $G_2 := (V_2, E_2)$ and $G_3 := (V_3, E_3)$, we would expect that if

$G_1 \cong G_2$ & $G_2 \cong G_3$ then $G_1 \cong G_3$, right?

Intuitively:



f → an isomorphism from G_1 to G_2

h → an isomorphism from G_2 to G_3

↓

The function that maps each $v \in V_1$ to $h(f(v)) \in V_3$ should be an isomorphism from G_1 to G_3, right?

Composition of two functions:

A, B, C : sets

$f : A \to B$

$h : B \to C$

Then their $\boxed{\text{composition}}$, denoted by $\boxed{h \circ f}$ (read: h after f) is a function from A to C defined as: $(h \circ f)(a) := h(f(a))$ for each $a \in A$.

DIY: Prove that if f is an isomorphism from G_1 to G_2 and h is an isomorphism from G_2 to G_3 then $h \circ f$ is an isomorphism from G_1 to G_3.

DIY (from Kenneth Rosen): Consider the following two functions f & h (from $\mathbb{Z}$ to $\mathbb{Z}$):

$$f(x) := 2x+3 \quad \& \quad h(x) := 3x+2$$

Perform substitutions & find out what $f \circ h$ & $h \circ f$ are. (Are they the same?)

Theorem: Isomorphism ($\cong$) is an equivalence relation on the set of all simple graphs. $\blacksquare$

DIY: Generalize the definition of isomorphism to all (NOT just simple) graphs, and prove that this generalization is also an equivalence relation (on the set of ALL graphs).
