PH-1020
Solution of Problem Set - 8
Department of Physics, IIT Madras
Introduction to Quantum Mechanics
March-June 2023 Semester

Notation:

- Notation follows that of **Concepts Of Modern Physics By Arthur Beiser**.

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1. **What is the classical assumption adapted by Rayleigh and Jeans to derive the formula of the spectral energy density of blackbody radiation? How is it modified by Planck to make the theoretically calculated spectral density agreeable with the experimental results?**

Rayleigh and Jeans Idea

The Electromagnetic energy density in the frequency range ν to $\nu + d\nu$ can be written as

$$u(\nu, T) = N(\nu)\,\langle E\rangle = \frac{8\pi\nu^2}{c^3}\,\langle E\rangle \ ,$$

where $\langle E\rangle$ is the average energy of the oscillator present in the walls of the cavity.

By using the Equipartition theorem of classical thermodynamics all oscillations in the cavity have the same mean energy irrespective of their frequencies in terms of Boltzmann constant (k) at temperature T and can be calculated as

$$\begin{aligned}\langle E\rangle &= \frac{\int_0^\infty E\ \ e^{\frac{-E}{kT}}\ \ dE}{\int_0^\infty e^{\frac{-E}{kT}}\ \ dE} \\ &= kT\ . \end{aligned} \tag{1}$$

The energy density is

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} kT\ . \tag{2}$$

It results in Ultraviolet catastrophe, i.e., diverges for high frequencies.

Planck's Modification

He suggested that each atom consisting of an Electromagnetic radiation oscillator has discrete energy and can be written as

$$E_n = nh\nu\ . \tag{3}$$

Now, the average energy is not equal to kT. So, let's compute the average energy

$$\begin{aligned} \langle E \rangle &= \frac{\sum_{n=0}^{\infty} E_n \, e^{\frac{-E_n}{kT}}}{\sum_{n=0}^{\infty} \, e^{\frac{-E_n}{kT}}} = \frac{\sum_{n=0}^{\infty} E_n \, e^{-\beta E_n}}{\sum_{n=0}^{\infty} \, e^{-\beta E_n}} \\ &= \frac{\sum_{n=0}^{\infty} nh\nu \, e^{-\beta nh\nu}}{\sum_{n=0}^{\infty} \, e^{-\beta nh\nu}} \end{aligned} \tag{4}$$

By defining $\beta h\nu = Y$ and $e^{-Y} = X$. We have

$$= h\nu X \frac{(1 + 2X + 3X^2 + \cdots)}{(1 + X + X^2 + X^3 + \cdots)} \tag{5}$$

We have used the identity

$$1 + 2X + 3X^2 + \cdots = \frac{1}{(1-X)^2} \quad \text{and} \quad 1 + X + X^2 + \cdots = \frac{1}{(1-X)}$$

The above equation reduces to

$$= \frac{h\nu}{e^Y - 1} = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} \; . \tag{6}$$

Now, the energy density is

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3 \left(e^{\frac{h\nu}{kT}} - 1 \right)} \; . \tag{7}$$

This is known as Planck's Radiation formula.

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2. **Typical energy of a neutron released from nuclear fission is $3.2 \times 10^{-13} J$. Ignoring the relativistic effect, find out the associated de Broglie wavelength. These neutrons are used to determine the crystal structure where the typical interatomic spacing is $2 \times 10^{-10} m$. How much should you slow down the neutron to make the crystal structure detection possible?**

   The energy of the released neutron is $3.2 \times 10^{-13} J$. The momentum can be computed using $p \approx \sqrt{2mE} = 3.26 \times 10^{-20} \; kg \; ms^{-1}$, the de Broglie wavelength $\lambda = \frac{h}{p} \approx 2.03 \times 10^{-14} \; m$ and velocity $v = \frac{p}{m} = \sqrt{\frac{2E}{m}} \approx 2 \times 10^7 ms^{-1}$.

   The interatomic spacing of crystal is $2 \times 10^{-10} m$. So, the wavelength of neutrons required for crystal structure determination is $\lambda_{cry} = 2 \times 10^{-10} m$. Now we can calculate the momentum as $P_{cry} = \frac{h}{\lambda} \approx 3.31 \times 10^{-24} \; kg \; ms^{-1}$, and the velocity as $v_{cry} \approx 2 \times 10^3 \; ms^{-1}$.

   Therefore the emitted neutrons should have the $\frac{2 \times 10^3}{2 \times 10^7} = 10^{-4}$ times of their original velocity for experiment.

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3. **The size of a given nucleus is $2 \times 10^{-15} m$. How much kinetic energy should an electron possess to become a part of the nucleus?**

   From the uncertainty principle

$$\Delta x \, \Delta p_x \geq \frac{\hbar}{2} \, .$$

   Using this, it is easy to calculate Δp_x as

$$\begin{aligned} \Delta p_x &\geq \frac{\hbar}{2 \, \Delta x} \\ &\geq \frac{1.05 \times 10^{-34}}{2 \times 2 \times 10^{-15}} \\ &\geq 2.62 \times 10^{-20} \, . \end{aligned} \tag{8}$$

   Now the minimum kinetic energy can be computed as

$$\begin{aligned} E &= \frac{p^2}{2 \, m} \\ &\approx \frac{\left(2.62 \times 10^{-20}\right)^2}{2 \times 9.1 \times 10^{-31}} \\ &\approx 3.8 \times 10^{-10} \; J \end{aligned} \tag{9}$$

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4. **Establish the uncertainty principle for a normalized state given by,**

$$\psi(x) = \frac{1}{(\pi \Delta^2)^{1/4}} \exp\left\{\left(-\frac{(x-a)^2}{2\Delta^2}\right)\right\}$$

   This is a Gaussian wave function with Δ quantifying the spread of it.

   With the given above wave-function, the expectation values are

   (a) Expectation of $\hat{x}$:

$$\begin{aligned} \langle \hat{x} \rangle &= \int_{-\infty}^{\infty} \psi^* x \psi dx \\ &= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} x \; \exp\left\{\left(-\frac{(x-a)^2}{\Delta^2}\right)\right\} dx \\ &= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} (z+a) \; \exp\left\{\left(-\frac{z^2}{\Delta^2}\right)\right\} dz \\ &= a \, . \end{aligned} \tag{10}$$

   (b) Expectation of $\hat{x}^2$:

$$\begin{aligned} \left\langle \hat{x}^2 \right\rangle &= \int_{-\infty}^{\infty} \psi^* x^2 \psi dx \\ &= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} x^2 \; \exp\left\{\left(-\frac{(x-a)^2}{\Delta^2}\right)\right\} dx \\ &= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} (z^2 + 2az + a^2) \; \exp\left\{\left(-\frac{z^2}{\Delta^2}\right)\right\} dz \\ &= a^2 + \frac{\Delta^2}{2} \, . \end{aligned} \tag{11}$$

(c) Expectation of $\hat{p}$:

$$\begin{aligned}
\langle \hat{p} \rangle &= \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{d}{dx}\right) \psi dx \\
&= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} \exp\left\{\left(-\frac{(x-a)^2}{2\Delta^2}\right)\right\} \left(-i\hbar \frac{d}{dx}\right) \exp\left\{\left(-\frac{(x-a)^2}{2\Delta^2}\right)\right\} dx \\
&= 0 \, . \qquad (12)
\end{aligned}$$

(d) Expectation of $\hat{p}^2$:

$$\begin{aligned}
\left\langle \hat{p}^2 \right\rangle &= \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{d}{dx}\right)^2 \psi dx \\
&= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} \exp\left\{\left(-\frac{(x-a)^2}{2\Delta^2}\right)\right\} \left(-i\hbar \frac{d}{dx}\right)^2 \exp\left\{\left(-\frac{(x-a)^2}{2\Delta^2}\right)\right\} dx \\
&= \frac{\hbar^2}{2\Delta^2} \, . \qquad (13)
\end{aligned}$$

Now, $\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = \frac{\Delta}{\sqrt{2}}$ and $\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = \frac{\hbar}{\sqrt{2}\Delta}$.
It is easy to verify that

$$\Delta x \Delta p = \frac{\hbar}{2}$$

Note : Here, we have used the formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{0}^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_{0}^{\infty} e^{-ax^2+bx} dx = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}$$

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5. **A particle is moving in one dimension, has ground state wavefunction given by $\psi_0 = Ae^{-\frac{\alpha}{2\hbar}x^2}$ (where α is some real constant and A is the normalization constant) belonging to the energy eigenvalue $E_0 = \frac{\hbar\alpha}{2m}$ (where m is the mass). Determine the potential in which the particle moves.**

   As already discussed in class about the Schrödinger equation, i.e.,

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\,\psi(x)$$

Given the ground state wavefunction as well as the belonging energy eigenvalue as

$$\psi_0(x) = Ae^{-\frac{\alpha}{2\hbar}x^2}$$
$$E_0 = \frac{\hbar\alpha}{2m} \ .$$

It is easy to verify that

$$\frac{d^2\psi_0(x)}{dx^2} = \frac{-\alpha}{\hbar}\psi_0(x) + \frac{\alpha^2 x^2}{\hbar^2}\psi_0(x) \ .$$

Now, using the Schrödinger equation, we have

$$\begin{aligned} V(x)\ \psi_0(x) &= E_0\ \psi_0(x) + \frac{\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2} \\ &= \frac{\hbar\alpha}{2m}\psi_0(x) + \frac{\hbar^2}{2m}\left(\frac{-\alpha}{\hbar}\psi_0(x) + \frac{\alpha^2 x^2}{\hbar^2}\psi_0(x)\right) \\ &= \frac{\alpha^2 x^2}{2m}\psi_0(x) \ . \end{aligned} \tag{14}$$

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6. **An electron having de Broglie wavelength** $2 \times 10^{-12} m$**. Calculate its**
   (a) **Kinetic Energy.**
   (b) **Phase velocity.**
   (c) **Group Velocity.**

From the de Broglie wavelength relation, we have

$$p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}}{2 \times 10^{-12}} = 3.31 \times 10^{-22}\ kg\ ms^{-1} \ .$$

As we know, the rest energy of electrons, i.e., $E_0 = 8.187 \times 10^{-14}$ joules. Using the relation for kinetic energy(K.E.) as $K.E. = E - E_0 = \sqrt{E_0^2 + p^2c^2} - E_0 = 4.678 \times 10^{-17} J.$

Now, we have to compute the electron's velocity; for this, we will use

$$E = \frac{E_0}{\sqrt{1 - \frac{v^2}{c^2}}} \implies v = c\sqrt{1 - \frac{E_0^2}{E^2}} \approx 0.771c\ \ ms^{-1} \ . \tag{15}$$

With the help of this

(a) Phase velocity : $v_p = \frac{c^2}{v} = 1.30c\ \ ms^{-1}$.
(b) Group velocity : $v_g = v = 0.771c\ \ ms^{-1}$

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7. **Do you think $\frac{1}{3}\cos\left(\frac{\pi}{6}x\right)$ is a valid normalized eigenfunction for an infinite potential well? Appropriately prove your answer. If yes, sketch the potential well and the eigenfunction.**

Yes, this is a valid wave function.

As the wavefunction is in the form of cosine, the potential is an infinite symmetric potential of length L. And the normalization constant is $\sqrt{\frac{2}{L}}$.

As already discussed in class about symmetric potential

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, & \text{if } n = \text{even} \\ \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}, & \text{if } n = \text{odd} \end{cases} \tag{16}$$

By comparing with the above wavefunction, one can conclude $L = 18$ units and $n = 3$. So, the above wavefunction corresponds to the quantum number n as odd value.

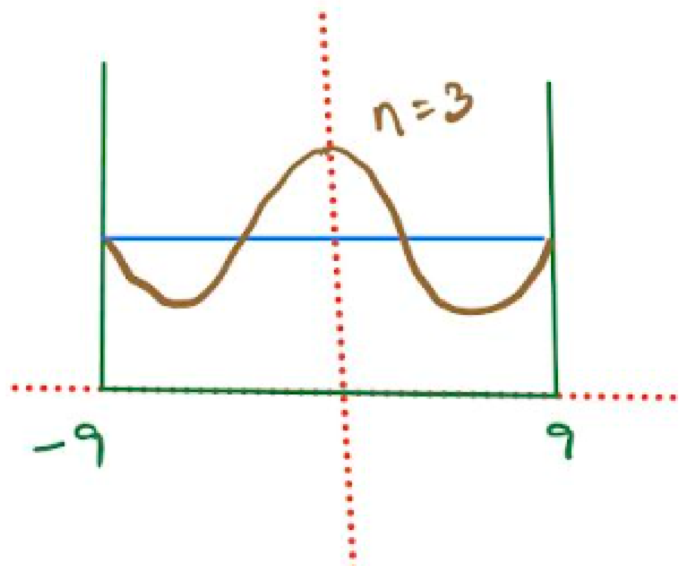

Figure 1: Sketch of potential well and the wavefunction. Inside the well (i.e., $-9 < x < 9$), the potential is zero, and outside the well (i.e., $x \leq -9$ and $x \geq 9$), it is infinite.

Suggested questions:

8. **By considering a cubical cavity of volume L^3, show that the density of standing waves in the frequency range ν and $\nu + d\nu$ is**

$$G(\nu)d\nu = \frac{8\pi\nu^2}{c^2}d\nu$$

To explain the blackbody spectrum, Rayleigh and Jeans proposed a classical calculation by considering the blackbody as a radiation-filled cavity at temperature T. As the walls of the blackbody are assumed to be a perfect reflector, there must be a standing EM wave in radiation. In order for a node to occur at each wall, the path length between the walls (L) in any direction should be an integral multiple of half wavelength ($\lambda/2$), i.e.,

$$n_i(\lambda/2) = L \tag{17}$$

where, $i \in \{x, y, z\}$. From the above equation, one can write,

$$n_i = \frac{2L}{\lambda} = 1, 2, 3, \cdots \tag{18}$$

Now for standing waves inside a cubic cavity, we can write,

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{2L}{\lambda}\right)^2 = n^2, \tag{19}$$

where, $n_i = 0, 1, 2, \cdots$ and all n_is are not zero simultaneously. Now we want to calculate the number of standing waves $g(\lambda)d\lambda$ within the wavelength range λ to $\lambda + d\lambda$ as a function of frequency. To do that, we first consider a spherical shell of radius n and thickness dn. As we are interested in positive values only, we consider only one octant of the shell. As the wave is polarized in two perpendicular planes, we multiply this number by a factor 2. Finally, the number of standing waves can be written as,

$$g(n)dn = 2\left(\frac{1}{8}\right)4\pi n^2 dn = \pi n^2 dn \tag{20}$$

From Eq. 19 and Eq. 20, we can write,

$$g(\nu)d\nu = \pi\left(\frac{2L\nu}{c}\right)^2 \frac{2L}{c}d\nu = \frac{8\pi L^3}{c^3}\nu^2 d\nu \tag{21}$$

Therefore the density of standing waves in a cavity can be written as,

$$G(\nu)d\nu = \frac{1}{L^3}g(\nu)d\nu = \frac{8\pi}{c^3}\nu^2 d\nu \tag{22}$$
