Recall:

Theorem: Let $G = (V,E)$ be a conn. graph.

TFAE (the following are equivalent):

1. G has an Eulerian tour.
2. Each vtx. of G has even degree.
3. A directed graph D can be obtained from G such that $d^{in}(v) = d^{out}(v)$ for each $v \in V$ in D.
4. $E(G)$ can be partitioned into cycles.

$\updownarrow$ same as

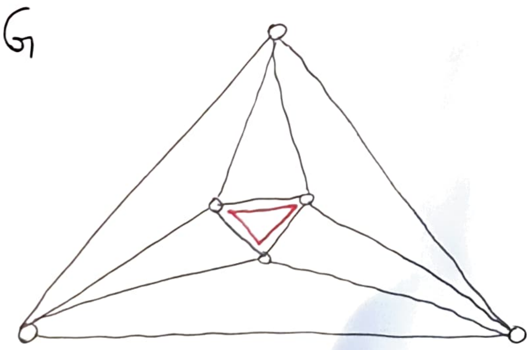
G admits (has) a cycle partition.

We have ONLY proved ④ $\Rightarrow$ ② in lectures (so far).

TODAY, we will discuss ② $\Rightarrow$ ④. Proof later.

First some examples.

G



Our goal is to get a cycle partition.

First, let us get a cycle.

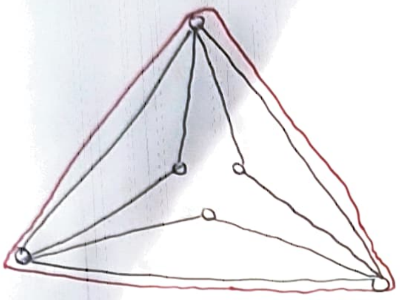
Recall Theorem: If each vertex (of a graph G) has degree ≥ 2 then G has a cycle.

Each vtx has degree ≥ 2.

So there is a cycle. Consider the cycle C_1 shown in RED color.

Let us remove the (edges of) cycle C_1.

$G - E(C_1)$:



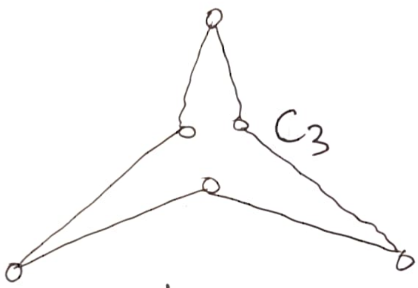
Once again, each vtx. has degree ≥ 2.

(Lucky coincidence? YES)

Consider the cycle C_2 shown in RED.

Let us remove (edges of) C_2.

$(G - E(C_1)) - E(C_2)$:



Once again, each vtx. has degree ≥ 2.

(Lucky coincidence? YES)

Now the graph $(G - E(C_1)) - E(C_2)$ itself is a cycle, say C_3.

So, $\{C_1, C_2, C_3\}$ is a cycle partition.

We are on our way to discover a recursive procedure (aka recursive algorithm) to find a cycle partition of any given graph whose each vtx. has even degree.

Repeatedly find and remove (edges of) cycles until graph has NO edges.

Why CAN this be DONE? This needs more thought & careful reasoning.

Just because it works on one example, does NOT mean it should always work.

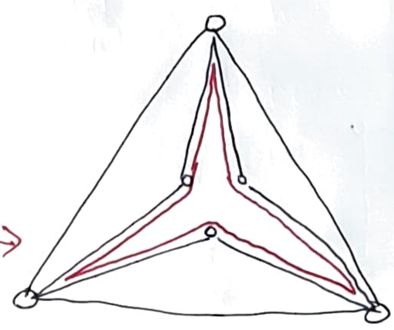
Question: After we remove any cycle (that is, edges of a cycle), how do we know that we will be able to find another cycle in the "remaining graph" (unless it is empty graph)?

NO edges
ONLY vertices

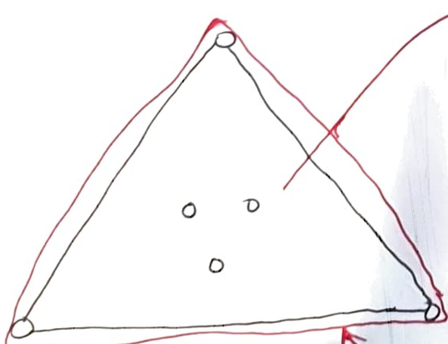
IF in the "remaining graph", each vtx. has degree ≥ 2,
then $\exists$ a cycle (by other theorem).

In fact, this need NOT be TRUE.
Remember: "Lucky coincidence" ?

Let's go back to $G - E(C_1)$:
What if we choose C_3 shown in RED ?



Then $(G - E(C_1)) - E(C_3)$ is:



Now, we have vertices of degree 0.

But what if we choose a component where each vtx. has degree ≥ 2, and then apply the other theorem?

Also, we have NOT yet discussed one important observation.

Lemma: G: graph where each vtx. has even degree.
C: cycle of G.
H := G − E(C).
Then each vtx. of H has even degree.

Proof: DIY

Now, let us go back to our recursive algorithm:

Given a graph G where each vtx. has even degree:
repeatedly ~~remove edges~~ find a cycle and remove edges of the cycle until NO edges are remaining.

Each time we remove edges of a cycle, we get a graph with fewer edges and again each vtx. has even degree.

due to THIS lemma

In fact, we only care to show existence of cycle. For this apply other theorem to any component that has ≥ 1 edge.

G: graph
If each vtx. has degree ≥ 2 then G has a cycle.

But why do we care about algorithms in THIS course?

$\boxed{\text{We don't.}}$

However, it is $\boxed{\text{NOT}}$ possible to discuss induction without recursion. They are 2 sides of the SAME COIN.

Now, we will see/discuss the INDUCTION viewpoint.

The "essence" of induction (without formalism):

GOAL: To prove something about a class of mathematical objects.

IDEA/PLAN: To get/construct systematically a "smaller" object in the SAME class.

Assume that the statement to be proved is TRUE for the "smaller" object, and show that it is TRUE for the "bigger" object.

IMPORTANT CHECK: The statement needs to be proved for the "smallest" objects in our class.

Example:

$\mathcal{E}$: set of (conn.) graphs where each vtx has even degree

To prove: Each member of $\mathcal{E}$ admits a cycle partition.

Lemma: Let $G \in \mathcal{E}$. If we remove the edges of any cycle C from G we get a graph H in $\mathcal{E}$ with fewer edges.

Assume that H has a cycle partition, say $\mathcal{C}$. Then $\mathcal{C} \cup \{C\}$ is a cycle partition of G.

(continued)

What are the "smallest" objects?
The objects from which you can NOT get smaller objects in the SAME class in a meaningful way.

We will NOW write down a complete proof using induction:

⋮

before that, let us ask ourselves a question:

Question: Do we really need our graphs to be connected?

Answer: NO.

We will in fact prove for all graphs where each vtx. has even degree.

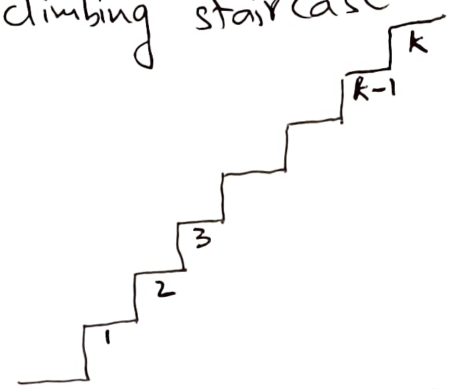
Example continued:

The empty graphs (ONLY vertices; NO edges) are the graphs with fewest edges in $\mathcal{E}$.

For each such graph, $\emptyset$ is a cycle partition.

ANALOGY:

climbing staircase



step 0 (base of staircase)

How do you teach a child to climb a staircase?

① You teach the child to reach step 0 → the base of the staircase. Base case

② You teach the child how to go from step k-1 to step k. Induction Step