IIT M-CS1200 : Discrete Math (Mar - Jul 2023) Release Date: 15-06-2023

Tutorial No: 6 Tutorial Date: 20-06-2023

1. [DERANGEMENTS]

   Seven students attend a party, leaving their hats at the door. At the end of the party, they hastily grab a hat on their way out.

   (a) How many different ways could this happen so that exactly *two* students leave with their own hat?

   (b) How many different ways could this happen so that exactly *three* students leave with their hat?

2. [COMBINATORIAL PROOFS]

   (a) Give a double counting proof to show that, for all $k, n \in \mathbb{N}$, where $2 < k < n$, the following holds:

   $$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

   (b) Give a combinatorial proof to show that, for all $k, i, n \in \mathbb{N}$, where $0 < i < k < n$, the following holds:

   $$\binom{n}{k} = \sum_{j=0}^{i} \binom{i}{j}\binom{n-i}{k-j}$$

3. [RECURRENCE RELATIONS]

   An $S$-string is a string formed by elements in the set $S$. For example, if $S = \{0, 1, 2\}$ then

   $$0, 02, 0000, 0021221, 2220, 1102$$

   are some of the examples of $S$-strings. Let $S = \{0, 1\}$.

   For each of the following problems, find a recurrence relation and initial conditions for the number of $S$-strings of length $n$ that:

   (a) do NOT have two consecutive $0$'s.

   (b) do NOT have three consecutive $0$'s.

   (c) that have two consecutive $0$'s.

   (d) that have three consecutive $0$'s.

   Solve all the parts above with $S = \{0, 1, 2\}$.

4. [RECURRENCE RELATIONS]

   Establish a recurrence relation and determine the initial conditions for calculating the number of ways to pay a bill of $n$ rupees, where $n \in \mathbb{N} - \{0\}$, using coins with denominations of $1, 2, 5$ and $10$?

   Note that the order of coins used to pay the bill matter, i.e., we can pay a bill of $3$ rupees in three ways: $1 + 2$, $2 + 1$ and $1 + 1 + 1$.

5. [COMBINATORIAL PROOFS AND RECURRENCE RELATIONS]

   For $N \in \mathbb{N} - \{0\}$, consider the following sets of tilings:

- Let $T_1(N)$ be the set of tilings of $(2 \times N)$-grid using brick-shaped $(2 \times 1)$ tiles
- Let $T_2(N)$ be the set of tilings of $(1 \times N)$-grid using brick-shaped $(2 \times 1)$ and box-shaped $(1 \times 1)$ tiles

(a) Show that there exists a Bijection between sets $T_1$ and $T_2$.

(b) Establish a recurrence relation and determine the initial conditions for calculating the number of tilings in both cases.

(c) Did you observe anything?

6. [PHP ON INTEGER SEQUENCES]

   Let $A \subset \{1, 2, \ldots 2n\}$ be a collection of $n+1$ unique integers for some positive integer $n$. Prove the following:

   (a) $A$ contains two integers such that one divides the other.

   (b) $A$ contains two integers that are co-primes.

7. [PHP]

   A student has a worksheet of $30$ problems and works over it over a span of $20$ days, solving at least one problem every day. Show that there exists a stretch of consecutive days where the student solves exactly $9$ questions.

8. [PHP ON INTEGER SEQUENCES]

   Let $X = \{x_1, x_2.., x_7\}$ be a set of positive integers each less than or equal to $10$. Let $Y = \{y_1, y_2.., y_{10}\}$ be a set of positive integers each less than or equal to $7$. Prove that there exist non empty subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that

$$\sum_{x \in X'} x = \sum_{y \in Y'} y$$