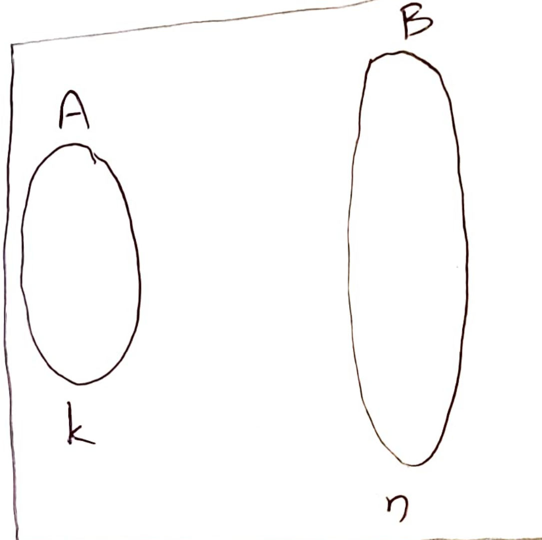
Next goal: Counting the # of permutations of an n-element set.

(Bijections from a set S to itself.)

We will do this by solving/answering a more general question:

Question: How many 1-to-1 (injective) fns are there from a k-element set to an n-element set?

Let A denote a k-element set, and let B denote an n-element set.



Recall (from Module-1): if $\exists$ a 1-to-1 fn from A to B then $|A| \leq |B|$ (that is, $k \leq n$).

Thus, if $k > n$, # of 1-to-1 fns from A to B is ZERO.

Thinking a bit, we may guess that the answer is $n \cdot (n-1) \cdot \ldots \cdot (n-(k-1))$.

Let's prove this using induction.

Theorem: For $k, n \in \mathbb{N}$, the # of 1-to-1 fns from a k-element set to an n-element set =

$$n \cdot (n-1) \cdot \ldots \cdot (n-(k-1)) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

ONLY if $k \leq n$.

Proof: We prove using induction on k.

If $k = 0$, LHS counts the # of 1-to-1 fns from the empty set to some set; there is ONLY one such function: the empty set (thinking of a function as a special case of relations). So, LHS = 1. What about RHS? $\frac{n!}{n!} = 1$.

So, LHS = RHS. Alternatively, $n.(n-1)\cdots(n-k+1) = $ empty product $= 1$ by convention
$(n+1)$

If $k > n$, LHS = 0 and RHS $n.(n-1)\cdots(n-(k-1))$ has a term that equals ZERO (why?); so RHS = 0 = LHS.

Now suppose that $k \leq n$.

(Now suppose that $k \leq n$.)

Now suppose that $k \geq 1$, and assume inductively that the desired conclusion/formula holds for all values smaller than k.

Let A denote a k-element set and let B denote an n-element set. Let $x \in A$ denote any element of A. We have n choices for $f(x)$ (where f is a 1-to-1 fn from A to B) (since $|B| = n$). If we fix any one choice of $f(x)$ then we need to consider the [# of 1-to-1 fns from $\underbrace{A - x}_{k-1 \text{ elements}}$ to $\underbrace{B - f(x)}_{n-1 \text{ elements}}$].

By I.H., this $= (n-1).(n-2)\cdots((n-1)-(k-1)+1)$
$= (n-1).(n-2)\cdots(n-k+1)$

Thus, LHS = # of 1-to-1 fns from A to B $= n * [(n-1).(n-2)\cdots(n-k+1)]$
$= n.(n-1).(n-2)\cdots(n-k+1) = $ RHS.

This completes the proof. ∎

CS1200 Module-3: Counting & Algebraic Structures

Question:

Where have you seen $\frac{n!}{(n-k)!}$ before?

It is the # of ordered k-tuples in an n-element set.

(aka # of k-permutations of $\{1,2,\ldots,n\}$)

For example, ordered 3-tuples of $\{1,2,3,4\}$:

$(1,2,3), (2,1,3), (2,3,1), (1,2,4), (2,4,1), \ldots\ldots$

How many? $\frac{4!}{(4-3)!} = 24$.

In higher mathematics (including this course):

A $\boxed{\text{permutation}}$ is a bijection from a set S to itself.

(aka permutation of S)

Corollary: The # of permutations of an n-element set is $n!$

(put $n=k$ in previous theorem & note that $0! = 1$).

$\boxed{\text{Why is } 0! = 1?}$ $0!$ may be thought of as the empty product $\prod_{x \in \emptyset} x$ and empty product is defined as 1. (why? what about empty sum?)

multiplying an empty set of #s