Recall the question we asked yesterday: (I mean, last lecture) $(n \in \mathbb{N} - \{0\})$.

Is it possible to tile a $(2^n \times 2^n)$-GRID (EXCEPT for any ARBITRARY square) using L-shaped tiles?

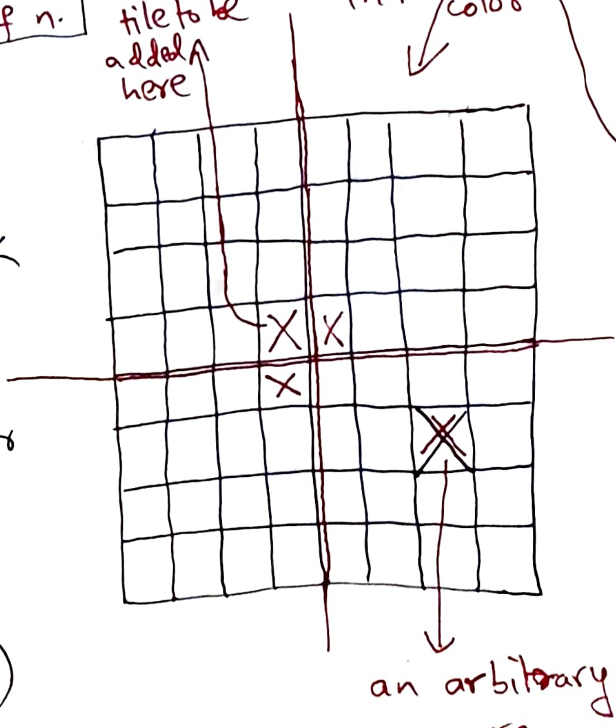
↓

Think of this as a "puzzle" you need to solve (for some "big" value of n) and suppose that you know how to solve the same puzzle (for "smaller" values of n).

THIS IS THE ESSENCE OF INDUCTION

and suppose that the answer is YES for "smaller" values of n.

↓ for example:



IDEA:
Let us break THIS grid into 4 smaller grids

↓

(each $2^{n-1} \times 2^{n-1}$)

IDEA continued:
And let us make a "smart" choice of an (arbitrary ;-)) square in each of these smaller grids.

Now take a tiling of each smaller grid minus the chosen square & combine ALL of these, add a tile. DONE.

Let us write down a complete proof using induction.

Theorem: Let $n \in \mathbb{N} - \{0\}$ and let L denote a $(2^n \times 2^n)$-grid. Then L minus any arbitrary square, say s, can be tiled using L-shaped tiles.

Sorry, I flipped it by mistake

Proof: We will prove using induction on $\boxed{n.}$ INDUCTION PARAMETER

BASE CASE

If $n=1$ then L is a 2×2 grid and removing any arbitrary square leaves an L-shaped grid which can be clearly tiled using one L-shaped tile. DONE.

INDUCTION STEP BEGINS HERE

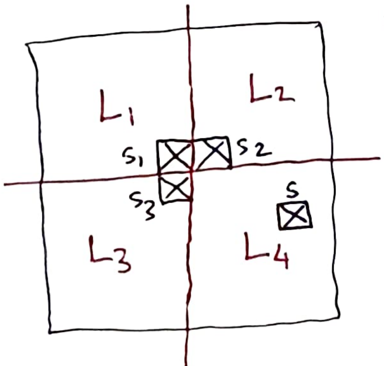
Now suppose that $n \geq 2$ AND assume inductively that,

INDUCTION HYPOTHESIS

$\forall k < n, k \in \mathbb{N} - \{0\}$, a $(2^k \times 2^k)$-grid MINUS an arbitrary square can be tiled using L-shaped tiles.

We break the $(2^n \times 2^n)$-grid L into FOUR $(2^{n-1} \times 2^{n-1})$-grids as follows:

Let us call these "smaller" grids L_1, L_2, L_3, L_4.



Let s denote the arbitrary square in L. Assume without loss of generality that s lies in L_4.

means that this is a safe assumption

Why?

Let s_i denote the square in L_i that is adjacent with 2 of the remaining 3 grids $\forall i \in \{1,2,3\}$.

By I.H., $L_i - s_i$ can be tiled using L-shaped tiles $\forall i \in \{1,2,3\}$.

By I.H., $L_4 - s$ can be tiled using L-shaped tiles.

Combine these 4 tilings and ADD one more tile covering s_1, s_2 & s_3. We have a tiling of $L - s$. This completes proof. ∎

A few comments about the previous proof:

① One can easily write code to solve the "puzzle" recursively.

tiling a $(2^n \times 2^n)$-grid minus any arbitrary square

Recursion & Induction, in this sense, are two sides of the SAME COIN.

② Suppose that I had asked you the following question instead:

- A more specific question: Is it possible to tile a $(2^n \times 2^n)$-grid EXCEPT square $(2,2)$ using L-shaped tiles?

2nd row & 2nd column

How would one answer this specific question? (if you did NOT know the answer to the general question)

(arbitrary square version)

It is NOT clear!

Interestingly, and somewhat ironically/surprisingly, it is easier to answer the more general question.

To put it differently, some/many times, it is easier to prove "stronger" theorems using induction than to prove "weaker" theorems. ("stronger" theorem implies "weaker" theorem.)

This phenomenon is referred to as "using a stronger induction hypothesis".

TIY: Prove the following theorem:
Theorem: The sum of first n odd numbers is a perfect square.