DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II **Problem Set 2 - Solutions** **MAR-JUN 23**

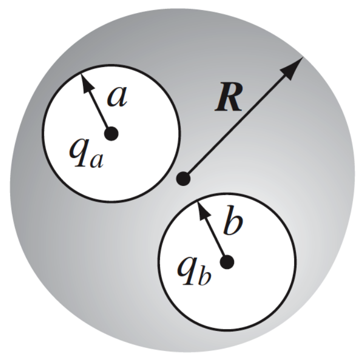

Figure 1:

1. (a) Following the arguments in section (2.5.1) and (2.5.2) of [1]

$$\boxed{\sigma_a = -\frac{q_a}{4\pi a^2}} \qquad \boxed{\sigma_b = -\frac{q_b}{4\pi b^2}} \qquad \boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}}$$

(b) Electric field at a point $\mathbf{r}(> R)$ from the center of the sphere

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{|\mathbf{r}|^3}\mathbf{r}}$$

(c) Field inside the cavity with radius a, and at $\mathbf{r}_a$ from the center of the cavity,

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_a}{|\mathbf{r}_a|^3}\mathbf{r}_a}$$

Field inside the cavity with radius b, and at $\mathbf{r}_b$ from the center of the cavity,

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_b}{|\mathbf{r}_b|^3}\mathbf{r}_b}$$

(d) As both the charges q_a and q_b are in the cavity inside a conductor, the force on each of the charges will be $\boxed{\text{zero}}$.

(e) If another charge q_c was brought near the conductor then, following the arguments in section (2.5.1) and (2.5.2) of [1], the surface charge density of the outer surface σ_R will change (and hence the electric field out side the conductor will also change) to make sure the field inside the conductor is zero. Whereas, the surface charge density of the cavity i.e. σ_a and σ_b will not change, neither the force on each charge.

2. Let the electric fields produced by plate 1 and 2 are $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ respectively. The magnitude can be written as, $|\boldsymbol{E}_1| = |\boldsymbol{E}_2| = \frac{\sigma}{2\epsilon_0}$ (where $\sigma = \frac{Q}{A}$). The electric field between the large conducting plates (region II) is zero and $\frac{\sigma}{\epsilon_0}$ outside (Region I and III). Hence, the electrostatic pressure

$$P = \frac{\sigma^2}{2\epsilon_0}$$

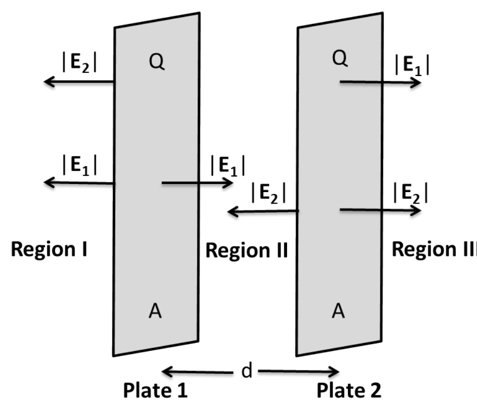

Figure 2:

3.

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right)$$

$$\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right) \qquad (|\mathbf{r}| = r)$$

For the regime $d \ll r$, we neglect the second order term in $\frac{d}{r}$ and the binomial expansion yields,

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

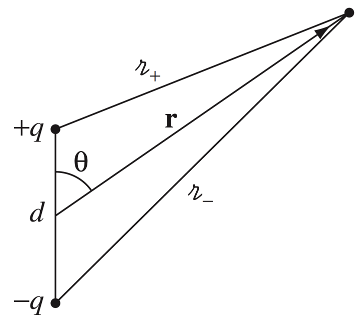

Figure 3:

Therefore,

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

Hence,

$$\boxed{V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}}$$

Once we find the potential of the dipole let's find the electric field.

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla} V \\
&= -\left(\frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}\right) \\
&= -\frac{1}{4\pi\epsilon_0}qd\left(\frac{-2\cos\theta}{r^3}\hat{\mathbf{r}} - \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right) \\
&= \frac{1}{4\pi\epsilon_0 r^3}\Big(3p\cos\theta\hat{\mathbf{r}} - (p\cos\theta\hat{\mathbf{r}} - p\sin\theta\hat{\boldsymbol{\theta}})\Big) \\
&= \boxed{\frac{1}{4\pi\epsilon_0 r^3}\left(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right)}
\end{aligned}$$

[1]

4. Quadrupole moment of a charge distribution, (Problem 3.52 of [1])

$$Q_{ij} = \frac{1}{2}\int \left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}')d\tau'$$

[1] The way we have computed the electric field obviously we have not included the origin. If we consider the origin the right expression would be $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right) - \frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r})}$. We will leave this as an exercise. (*Hint: Problem 3 of Problem Sheet 0*)

where $\mathbf{r}' = \{r'_i\}_{i=1,2,3} = (x', y', z')$ and $|\mathbf{r}'|^2 = r'^2$.

$\boxed{\text{Notice that } Q_{ij} \text{ is symmetric and traceless by definition.}}$

For the given configuration where, $-q$ is located at $(0, 0, d_1)$ and $+q$ is locate at $(0, 0, -d_2)$,

$$\begin{aligned} Q_{xx} &= \frac{1}{2} \sum_{m=1,2} \left(3x_m^2 - (x_m^2 + y_m^2 + z_m^2)\right) q_m(\mathbf{r}') \\ &= -\frac{q}{2}\left(-d_1^2\right) + \frac{q}{2}\left(-d_2^2\right) \\ &= \frac{q}{2}\left(d_1^2 - d_2^2\right) \end{aligned}$$

Similarly one can obtain,

$$Q_{yy} = \frac{q}{2}\left(d_1^2 - d_2^2\right), \qquad Q_{zz} = -q\left(d_1^2 - d_2^2\right), \qquad Q_{xy} = Q_{yz} = Q_{zx} = 0$$

<u>Case I : $d_1 = d_2$</u>

$$Q_{ij} = 0 \qquad \forall \quad i, j$$

<u>Case II : $d_1 = 1/2, d_2 = 3/2$</u>

$$Q_{xx} = Q_{yy} = -q, \qquad Q_{zz} = 2q, \qquad Q_{xy} = Q_{yz} = Q_{zx} = 0$$

5. (a) The dipole moment

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

Now, if we shift the origin by a constant $\mathbf{r}_0$ then the new dipole moment

$$\tilde{\mathbf{p}} = \int (\mathbf{r}' - \mathbf{r}_0)\rho(\mathbf{r}')d\tau' = \mathbf{p} - \mathbf{r}_0 \int \rho(\mathbf{r}')d\tau' = \mathbf{p} - \mathbf{r}_0 Q$$

Then $Q = 0$ implies $\mathbf{p} = \tilde{\mathbf{p}}$.

$\boxed{\text{If the total charge is zero the dipole moment is independent of the choice of the origin}}$

(b) From part (a)

$$\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{r}_0 Q$$

One can always find a $\mathbf{r}_0 = \mathbf{p}/Q$ for $Q \neq 0$ such that $\tilde{\mathbf{p}}$ vanishes.

(c) For spherically symmetric charge distribution i.e. $\rho = \rho(|\mathbf{r}|)$,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(|\mathbf{r}|)}{|\mathbf{r}|} d\tau$$

The right hand side is nothing but the monopole term. Hence, the higher moments in the multipole expansion must vanish.

6. We will compute the force on the northern hemisphere due to southern hemisphere. Following the argument in the section (2.5.3: *Surface Charge and the Force on a Conductor*) and equation (2.51) of [1], we have the electrostatic pressure $P = \frac{\sigma^2}{2\epsilon_0}$. Hence, the magnitude of the force on an infinitesimal area would be,

$$dF = P da$$

Now, from the symmetry one can conclude that the net force would be along $\hat{\mathbf{z}}$. Therefore,

$$\begin{aligned}
\mathbf{F} &= \hat{\mathbf{z}} \int \frac{\sigma^2}{2\epsilon_0} da \cos\theta \\
&= \hat{\mathbf{z}} \frac{\sigma^2}{2\epsilon_0} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \cos\theta R^2 \sin\theta d\theta d\phi \\
&= \hat{\mathbf{z}} \frac{\sigma^2}{2\epsilon_0} 2\pi R^2 \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta d\theta \\
&= \hat{\mathbf{z}} \frac{\sigma^2}{2\epsilon_0} 2\pi R^2 \times \frac{1}{2} \\
&= \boxed{\frac{Q^2}{32\pi R^2 \epsilon_0} \hat{\mathbf{z}}} \qquad (\because \sigma = \frac{Q}{4\pi R^2})
\end{aligned}$$

Alternative solution: One can obtain the same answer by calculating the field ($\boldsymbol{E}$) created by the southern (northern) hemisphere on the surface of the northern (southern) hemisphere. The pressure is $\sigma \boldsymbol{E}$.

7. We will follow the exact same method we used in problem 4.

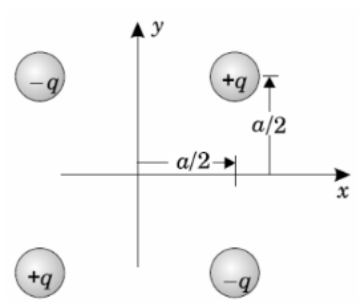

Figure 4:

$$
\begin{aligned}
Q_{xx} &= \frac{1}{2}\sum_{m=1}^{4}\left(3x_m^2 - (x_m^2 + y_m^2 + z_m^2)\right) q_m(\mathbf{r'}) \\
&= \frac{q}{2}\left(3\left(\frac{a}{2}\right)^2 - 2\left(\frac{a}{2}\right)^2\right) - \frac{q}{2}\left(3\left(\frac{-a}{2}\right)^2 - 2\left(\frac{a}{2}\right)^2\right) \\
&+ \frac{q}{2}\left(3\left(\frac{-a}{2}\right)^2 - 2\left(\frac{a}{2}\right)^2\right) - \frac{q}{2}\left(3\left(\frac{a}{2}\right)^2 - 2\left(\frac{a}{2}\right)^2\right) \\
&= 0
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Q_{yy} &= \frac{1}{2}\sum_{m=1}^{4}\left(3y_m^2 - (x_m^2 + y_m^2 + z_m^2)\right) q_m(\mathbf{r'}) = 0 \\
Q_{zz} &= \frac{1}{2}\sum_{m=1}^{4}\left(3z_m^2 - (x_m^2 + y_m^2 + z_m^2)\right) q_m(\mathbf{r'}) = 0 \quad \text{one can readily see } Q_{zz} \text{ as } -(Q_{xx} + Q_{yy}) \\
Q_{xy} &= \frac{1}{2}\sum_{m=1}^{4}\left(3x_m y_m\right) q_m(\mathbf{r'}) = 4 \times \frac{1}{2} \times \frac{3qa^2}{4} = \frac{3}{2}qa^2 \\
Q_{yz} &= \frac{1}{2}\sum_{m=1}^{4}\left(3y_m z_m\right) q_m(\mathbf{r'}) = 0 \qquad (z_m = 0 \quad \forall \quad m) \\
Q_{zx} &= \frac{1}{2}\sum_{m=1}^{4}\left(3z_m x_m\right) q_m(\mathbf{r'}) = 0 \qquad (z_m = 0 \quad \forall \quad m)
\end{aligned}
$$

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.