Definition: Weak Partition [same as partition except that we allow empty parts]

For a set S, a collection of (possibly empty) subsets $(S_1, S_2, \ldots)$ is said to be a $\boxed{\text{weak partition}}$ if:

① $\underbrace{\bigcup S_i = S}_{\text{union of all } S_i\text{s}}$ AND ② $\underbrace{S_i \cap S_j = \emptyset}_{\text{all parts are pairwise-disjoint}}$ $\forall$ distinct i, j

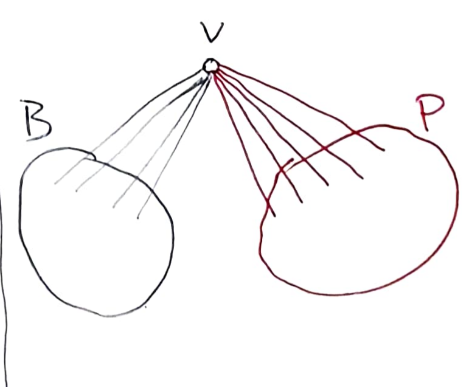
---

Let us prove the FRIENDS & STRANGERS AT A PARTY Theorem again (using different terminology but SAME proof):

Theorem: In any BLUE PINK complete graph on 6 or more vertices, $\exists$ BLUE K_3 OR $\exists$ PINK K_3.

Proof: Let $v \in V(G)$, where G is a BLUE PINK complete graph on 6 or more vertices. We now define a weak partition of $V(G)$ with 3 parts:

illustration:



Part 1: $\{v\}$

Part 2: $\{u \in V(G) - v : uv \text{ is BLUE}\} =: B$

Part 3: $\{u \in V(G) - v : uv \text{ is PINK}\} =: P$

Note that $(\{v\}, B, P)$ is a weak partition of $V(G)$.

Observe that $|B \uplus P| = |V(G) - v| \geq 5$.

Thus $|B| + |P| \geq 5$.

Consequently, either $|B| \geq 3$ or $|P| \geq 3$.

Assume WITHOUT LOSS OF GENERALITY (WLOG/wlog) that $|B| \geq 3$.

Observe that $\forall$ distinct $u, w \in B$, if u & w are joined by a BLUE edge then uvw is a BLUE K_3 and we are DONE.

Now suppose that $\forall$ distinct $u, w \in B$, u & w are joined by a PINK edge.

Consider $x, y, z \in B$ (since $|B| \geq 3$); observe that xyz is a PINK K_3 and we are DONE. ▨

Notation: $\boxed{\text{Disjoint Union}}$ $\uplus$

$B \uplus P$ indicates their union and also indicates that their intersection ($B \cap P$) is EMPTY.

Observe that $|B \uplus P| = |B| + |P|$

This need NOT be true for union (in general), right?

This easy observation is an application of the well-known pigeonhole principle (PHP).

$N = 5$ (edges)

$k = 2$ (colors)

$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{5}{2} \right\rceil = 3$

We used THIS in proof

Theorem: (Pigeonhole Principle — Version 1)

If N objects are placed into k boxes then there IS at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Proof (of PHP – Version 1): By contradiction. Suppose that N objects are placed into k boxes, but there IS NO box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Notation: For any real # x, ceiling of x, denoted by $\lceil x \rceil$ is the smallest integer greater than x.

Thus each box contains at most $\lceil \frac{N}{k} \rceil - 1$ objects.

So, total # of objects

$$\leq k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N; \text{ contradiction!}$$

why? Since $k \geq 0$ and $\lceil \frac{N}{k} \rceil < \frac{N}{k} + 1$

strictly less than

Thus, $\exists$ at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects. $\blacksquare$

We now state another version of the PHP:

Theorem: Let $(S_1, S_2, \ldots, S_t)$ denote a weak partition of a set S with at least $1 + \sum_{i=1}^{t} n_i$ elements where $n_1, n_2, \ldots, n_t \in \mathbb{N}$. Then $\exists\ i \in \{1, 2, \ldots, t\}$ such that S_i has at least n_i elements.

Proof: DIY (using contradiction).

DIY: In the previous proof of FRIENDS & STRANGERS AT A PARTY theorem, we could have used THIS version of PHP. How?

Next Goal: To prove Ramsey's Theorem for graphs

Ramsey's Theorem for Graphs (Version 1): For every $r \in \mathbb{N} - \{0\}$, $\exists$ a smallest positive integer R_r such that any BLUE-PINK complete graph on R_r or more vertices either contains a BLUE K_r or contains a PINK K_r.

[if $r = 3$ then $R_r = 6$]

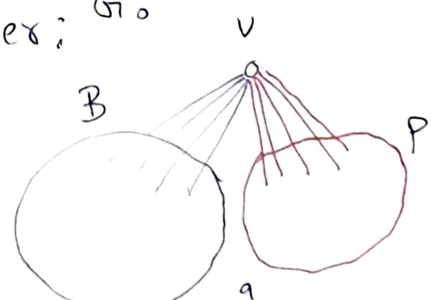
This is the FRIENDS & STRANGERS at a PARTY Theorem!

How does one prove such a theorem?

Let's try a strategy similar (combined with induction) to our proof of FRIENDS & STRANGERS at a PARTY Theorem

we will consider a complete graph G on some "big" number of vertices, and we will look at the whole graph from the point of view of 1 vertex — say v — and consider a weak partition $(\{v\}, B, P)$ as we did earlier:

G:



If $G[B]$ has a BLUE K_{r-1} we can combine THIS K_{r-1} with v to get BLUE K_r; however, vertex v does NOT help with PINK color in $G[B]$! This suggests we should prove a stronger theorem.

(Version 2)

Ramsey's Theorem for Graphs: For every $b, p \in \mathbb{N} - \{0\}$ $\exists$ a smallest positive integer $R_{b,p}$ such that any BLUE-PINK complete graph on $R_{b,p}$ or more vertices either contains a BLUE K_b or contains a PINK K_p.

This seems better for our strategy and implies previous version

why? put $b = p = r$ and $R_r = R_{b,p}$

Another point: We do NOT need to show existence (or find formula for) smallest such positive integer $R_{b,p}$

we simply need to show existence of some positive integer

why?

If some positive integer exists then smallest positive integer also exists (Right?)

we will actually show that

$$\binom{b+p-2}{b-1} = \binom{b+p-2}{p-1}$$

works, and to make it work we will use Pascal's Identity.

We will prove the following theorem:

Ramsey's Theorem for Graphs (Version 3): Let $b, p \in \mathbb{N} - \{0\}$.

Every BLUE-PINK complete graph on $\binom{b+p-2}{b-1} = \binom{b+p-2}{p-1}$ or more vertices either contains a BLUE K_b or contains a PINK K_p.

$\binom{n}{k} = \binom{n}{n-k}$. we will use this many times.

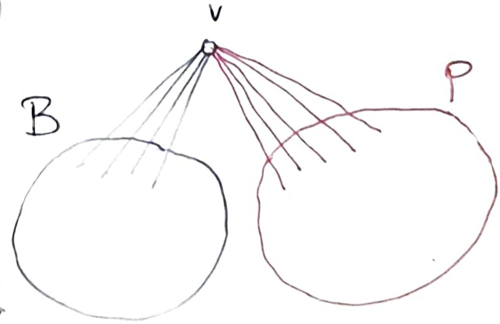
Proof: We proceed by induction on $b+p$.

Base case: (DIY) Either $b=1$ or $p=1$.

Induction Step: Now suppose that $b \geq 2$ and $p \geq 2$ and [assume inductively that the desired conclusion holds for all $b', p' \in \mathbb{N} - \{0\}$ whenever $b'+p' < b+p$.] → Induction Hypothesis

Consider a ~~graph~~ BLUE-PINK complete graph on $\binom{b+p-2}{b-1} = \binom{b+p-2}{p-1}$ or more vertices, say G, and let $v \in V(G)$.

We define a weak partition of $V(G)$ as follows: see figure →



Part 1: $\{v\}$

Part 2: $B := \{u \in V(G) - v : uv \text{ is BLUE}\}$

Part 3: $P := \{u \in V(G) - v : uv \text{ is PINK}\}$

We will find Pascal's Identity useful: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

By Pascal's Identity: $\binom{b+p-2}{b-1} = \binom{b+p-3}{b-1} + \binom{b+p-3}{b-2} = \binom{b+p-3}{b-2} + \binom{b+p-3}{p-2}$

Pigeonhole Principle Used Here

Claim: Either $|B| \geq \binom{b+p-3}{b-2}$

or $|P| \geq \binom{b+p-3}{p-2}$.

alternatively, one may directly use PHP.

Proof: Suppose NOT.

Then $|B| \leq \binom{b+p-3}{b-2} - 1$

and $|P| \leq \binom{b+p-3}{p-2} - 1$

using this

So $|V(G)| = |\{v\} \cup B \cup P| = |\{v\}| + |B| + |P|$

$\leq 1 + \left(\binom{b+p-3}{b-2} - 1\right) + \left(\binom{b+p-3}{p-2} - 1\right)$

$= \binom{b+p-3}{b-2} + \binom{b+p-3}{p-2} - 1 < \binom{b+p-2}{b-1}$ $\square$

Case 2:

$|P| \geq \binom{b+p-3}{p-2}$.

DIY.

Follow the same ideas/arguments.

Case 1: $|B| \geq \binom{b+p-3}{b-2} = \binom{(b-1)+p-2}{(b-1)-1}$.

Let $b' := b-1$ & $p' := p$. Observe that $b'+p' < b+p$.

By Induction Hypothesis, $G[B]$ either contains BLUE K_{b-1} or contains PINK K_p.

If $G[B]$ contains PINK K_p then G contains PINK K_p; DONE.

If $G[B]$ contains BLUE K_{b-1}, say H, then $G[V(H) \cup \{v\}]$ is BLUE K_b in G; DONE.