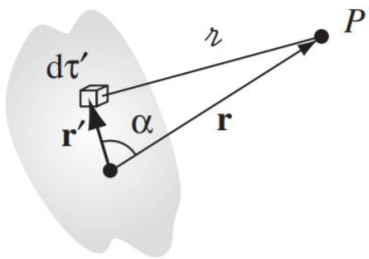
Quadrupole moment



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r\!\!\!/}\rho(\mathbf{r}')\,d\tau'.$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right].$$

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d^3\mathbf{r}'$$

$$r'\cos\theta' = \mathbf{r}'\cdot\hat{\mathbf{r}}$$

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(3\left(\mathbf{r}'\cdot\hat{\mathbf{r}}\right)^2 - \mathbf{r}'\cdot\mathbf{r}'\right)\rho(\mathbf{r}')d^3\mathbf{r}'$$

$$\begin{aligned}\mathbf{r}'\cdot\hat{\mathbf{r}} &= \sum_{i=1}^{3} r'_i\hat{r}_i\\ \mathbf{r}'\cdot\mathbf{r}' &= r'^2\end{aligned}$$

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(3\left(\sum_{i=1}^{3} r'_i\hat{r}_i\right)\left(\sum_{j=1}^{3} r'_j\hat{r}_j\right) - r'^2\right)\rho(\mathbf{r}')d^3\mathbf{r}'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(\sum_{i,j=1}^{3} \hat{r}_i\hat{r}_j 3r'_i r'_j - r'^2\right)\rho(\mathbf{r}')d\tau'$$

Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \sum_{i,j=1}^{3} \hat{r}_i\hat{r}_j\delta_{ij} = 1$$

$$\sum_{i,j=1}^{3} \hat{r}_i\hat{r}_j\delta_{ij} = 1$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(\sum_{i,j=1}^{3} \hat{r}_i\hat{r}_j 3r'_i r'_j - r'^2\right)\rho(\mathbf{r}')d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(\sum_{i,j=1}^{3} \hat{r}_i\hat{r}_j 3r'_i r'_j - r'^2\sum_{i,j=1}^{3} \hat{r}_i\hat{r}_j\delta_{ij}\right)\rho(\mathbf{r}')d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\int\left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}')d\tau'$$

$$V_{\text{quad}} \equiv \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}$$

$$Q_{ij} \equiv \frac{1}{2}\int\left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}')d\tau'$$

$$V_{\text{mon}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}; \quad V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i p_i}{r^2}; \quad V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i\hat{r}_j Q_{ij}}{r^3}; \ \ldots$$

$$Q_{ij} \equiv \frac{1}{2}\int \left(3r'_i r'_j - r'^2 \delta_{ij}\right) \rho(\mathbf{r}') d\tau'$$

For a discrete system of point charges q_l and position

$$\vec{r}_\ell = (r_{x\ell}, r_{y\ell}, r_{z\ell})$$

$$Q_{ij} \equiv \frac{1}{2}\sum_\ell q_\ell \left(3r_{i\ell} r_{j\ell} - \ {r_\ell}^2 \delta_{ij}\right)$$

$$Q_{ij} \equiv \frac{1}{2}\sum_\ell q_\ell \left(3r_{i\ell} r_{j\ell} - \|\vec{r}_\ell\|^2 \delta_{ij}\right)$$

$$Q_{ij} \equiv \frac{1}{2} \sum_{\ell} q_{\ell} \left(3 r_{i\ell} r_{j\ell} - \ r_{\ell}{}^{2} \delta_{ij} \right)$$

$$i, j = \{x, y, z\}$$

$$r_i = r_x = x$$

$$r_j = r_y = y$$

$$Q_{xx} = \frac{1}{2} \sum_{m=1}^{N} q_m (\mathbf{r}') \left(3 x_m x_m - \{x_m^2 + y_m^2 + z_m^2\} \right)$$

$$Q_{xy} = \frac{1}{2} \sum_{m=1}^{N} q_m (\mathbf{r}') (3 x_m y_m)$$

$$= \begin{pmatrix} Q_{xx} & Q_{yz} & Q_{zx} \\ Q_{xy} & Q_{yy} & Q_{zy} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix}$$

$$\mathbf{p} = \alpha \mathbf{E}.$$

only <u>six</u> <u>*independent*</u> components, because $Q_{ij} = Q_{ji}$

i.e. $Q_{xy} = Q_{yx}$
$Q_{xz} = Q_{zx}$
$Q_{yz} = Q_{zy}$

$$Q_{xx} + Q_{yy} + Q_{zz} = 0$$

Q_{ij} is symmetric and traceless

$$\begin{aligned} p_x &= \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z \\ p_y &= \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z \\ p_z &= \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z \end{aligned}$$

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$