Let us now ask the following question:

Question: For $n, k \in \mathbb{N}$ where $k \leq n$,

how many k-element subsets does an n-element set have?

(aka k-subsets) (aka n-set)

In other words: how many different ways of choosing k elements from an n-set?

order does NOT matter.

(I know that you know the answer: $\frac{n!}{k!(n-k)!}$)

How do we prove this using what we already know?

Recall: # of ordered k-tuples of an n-set $= \frac{n!}{(n-k)!}$

(aka k-permutations)

Each k-subset is counted k! times.

(since # of permutations of a k-set $= k!$)

So, we need to divide THIS by k!

This proves the following theorem.

Theorem: For any finite set S with $n \in \mathbb{N}$ elements,

the # of k-subsets (of S) $= \frac{n!}{k!(n-k)!}$.

Notation: $\binom{n}{k}$ READ as "n choose k" denotes the # of k-subsets of an n-set. Thus, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

These numbers $\left\{\binom{n}{k} : n, k \in \mathbb{N},\ k \leq n\right\}$ are called $\boxed{\text{binomial coefficients}}$ (for reasons that will become clear soon) and there are lots of cool identities/formulae involving the binomial coefficients.

Next Goal: To study some binomial coefficient identities and prove them combinatorially.

$\downarrow$

either using bijections or double counting arguments

bijections $\downarrow$ we have already seen some examples

double counting $\downarrow$ we will see an example soon

counting the same thing in two different ways

Let's look at the famous Pascal's triangle:

$\binom{0}{0}$ 1

$\binom{1}{0}$ 1   1 $\binom{1}{1}$ here

$\binom{2}{0}$ 1 $\binom{2}{1}$ 2   1 $\binom{2}{2}$

$\binom{3}{0}$ 1 $\binom{3}{1}$ 3   3 $\binom{3}{2}$ 1 $\binom{3}{3}$

1 $\binom{4}{0}$ 4 $\binom{4}{1}$ 6 $\binom{4}{2}$ 4 $\binom{4}{3}$ 1 $\binom{4}{4}$

$\vdots$

Let us observe that each number is the "sum" of the two numbers above it — one to the left and one to the right.

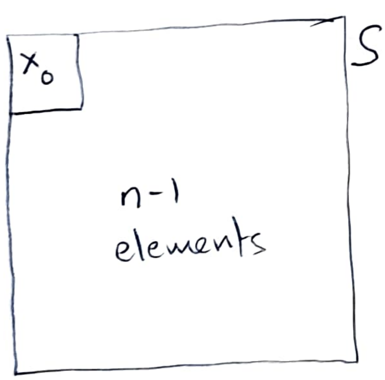
On observing Pascal's triangle, one is tempted to guess that

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \forall\, k, n \in \mathbb{N} \text{ where } 0 < k < n}$$

aka Pascal's Identity/Rule/Formula

Let us prove this combinatorially

LHS is counting # of ways to choose k-elements from an n-element set (say S) [in fact, that is exactly what "n choose k" means]



set with n elements

Let us count this differently.

Fix some element $x \in S$. (can be done since $n > 0$.)

For each choice of k-subset (say T) exactly one of the following holds:

either $x \in T$ — # of ways $= \binom{n-1}{k-1}$

or $x \notin T$

If $x \in T$, we still need to choose k-1 elements from $S-x$

If $x \notin T$, we need to choose all k elements from $S-x$. ↳ # of ways $= \binom{n-1}{k}$

Note that $|S-x| = n-1$.

So, total # of ways to choose k-elements from S $= \binom{n-1}{k} + \binom{n-1}{k-1} = \text{RHS}$.

Thus, LHS $\binom{n}{k}$ AND RHS $\binom{n-1}{k} + \binom{n-1}{k-1}$ are both counting the SAME thing.

Thus, LHS = RHS. This proves Pascal's Identity. ☒

Such a combinatorial proof is called a proof by double counting

Let us observe one more thing from Pascal's triangle.

	SUM OF EACH ROW
1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
⋮	⋮

→ One is tempted to guess:

$$\binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = 2^n \quad \forall\, n \in \mathbb{N}$$

Let's prove this using a double counting argument.

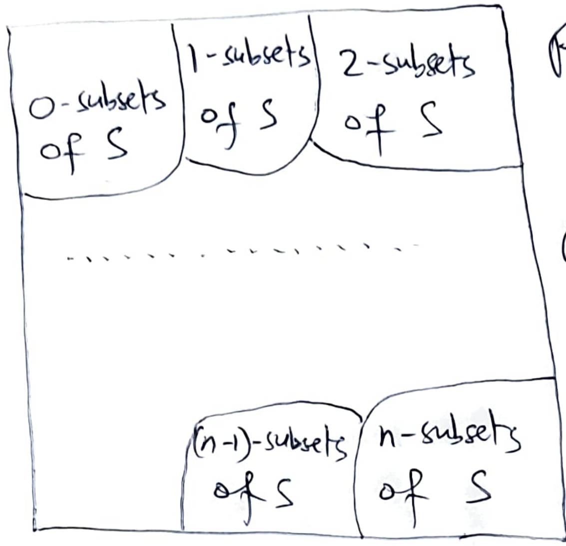
RHS $= 2^n$ is already familiar to us.

It counts the # of subsets of $S = \{1, 2, \ldots, n\}$. (Right?)

→ Let us count THIS in a different manner.

(In particular, $|\mathcal{P}(S)| = 2^n$.)

Idea: Let us partition the power set $\mathcal{P}(S)$ into n parts based on the cardinality of each set.



A partition of $\mathcal{P}(S)$

$\mathcal{P}(S)$: power set of $S := \{1, 2, \ldots, n\}$

collection of all subsets of S

Observe that

$$|\mathcal{P}(S)| = \sum |\mathcal{Q}|$$

$\mathcal{Q}$: some part here

$$= \sum_{k=0}^{n} \binom{n}{k} = \text{RHS}.$$

since # of k-subsets of S is exactly $\binom{n}{k}$

Thus LHS = RHS since they are both counting $|\mathcal{P}(S)|$.

Thus: $\forall n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \blacksquare$$

This is also a special case of the famous

Binomial Theorem: $\forall n \in \mathbb{N}$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

we will prove this using a combinatorial argument as well!

x & y are interchangeable

Let's prove the Binomial Theorem combinatorially:

Binomial Theorem: $\forall n \in \mathbb{N}$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: LHS $= (x+y)^n$

$$= \underbrace{\underbrace{(x+y)}_{\text{bracket 1}}\underbrace{(x+y)}_{\text{bracket 2}}\cdots\cdots\underbrace{(x+y)}_{\text{bracket } n}}_{n \text{ times}}$$

→ Let's think about how multiplication works: finally, we will get a summation of some terms → each term is formed by choosing x from some (say k) of the n brackets and choosing y from the remaining $n-k$ brackets

So, each term in the final summation looks like $x^k y^{n-k}$ for some $k \in \{0,1,\dots,n\}$.

Now, let's ask the following question:

For any $k \in \{0,1,\dots,n\}$, how many terms $x^k y^{n-k}$ will be there in the final summation? This is simply $\binom{n}{k}$. why?

Thus LHS $= \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \text{RHS}.$$ This completes the proof. $\boxtimes$