CS1200 — Discrete Mathematics for Computer Science — 2023 (Jan-May)

Assignment 1

Release Date: 30/03/2023 **Due Date: 08/04/2023 — 11:00 PM IST**

Collaborator (if any): Aditya Srivastava

Academic Integrity Statement: I, Chandaluru Hema Venkata Raadhes, affirm that I have not given or received any **unauthorized** help (from any source: people, internet, etc.) on this assignment, and that I have written/typed each response on my own, and in my own words.

The marks for each problem (1, 2, 3, 4, 5) are fixed. However, the marks for each subproblem (1a, 1b, 2a, etc.) are tentative — they may be changed during marking if necessary.

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1. [THE RELATION 'IS DISJOINT WITH'] [19]

Given two positive integers n and k (where $k \leq n$), we define the set $U_{n,k}$ as the collection of all k-element subsets of $\{1, 2, \ldots, n\}$. For example, $U_{3,2} := \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ and $U_{3,3} := \{\{1, 2, 3\}\}$.

(a) List all elements of the set $U_{5,2}$. [1]

Response: $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$

(b) Now, we define a relation R (read as: "is disjoint with") on the set $U_{n,k}$. For two (not necessarily distinct) members A and B (of $U_{n,k}$), we say that A is disjoint with B (also written as ARB) if $A \cap B = \emptyset$.

For each of the following, answer "yes" or "no"; if your answer is "yes", explain briefly why; if your answer is "no", give a concrete example to illustrate your point. [6]

(i) Is the relation R reflexive?

Response: No, for it to be reflexive every element $x \in U_{n,k}$ should be related to itself. However $x \cap x \neq \phi$ (for $k > 0$ (k is given as positive integer), x is not ϕ), hence $x \not R\, x$ so R is not reflexive. Example : For $U_{3,2} = \{\{1,2\},\{2,3\},\{1,3\}\}$, $\{1,2\} \cap \{1,2\} \neq \phi$, hence $\{1,2\} \not R \{1,2\}$ and so not reflexive.

(ii) Is the relation R symmetric?

Response: Yes. If $A \cap B = \phi \; then \; B \cap A = \phi$.

(iii) Is the relation R antisymmetric?

Response: No. It is not always anti symmetric . Ex: for $U_{2,1}, \{1\}\ R\ \{2\}\ and\ \{2\}\ R\ \{1\}$. But do note that if k>[n/2] (where [x] represents greatest integer less than or equal to x) then Relation R is antisymmetric as the relation is ϕ.

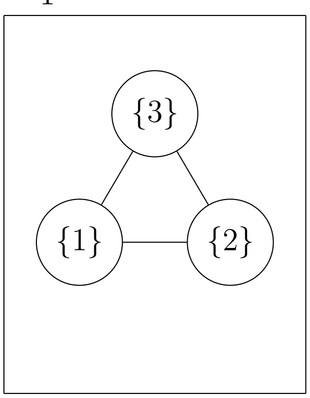
(iv) Is the relation R transitive?

Response: No. Example : for $U_{5,2}$, $\{1,2\}\ R\ \{3,4\}\ and\ \{3,4\}\ R\ \{1,5\}\ however\ \{1,2\} \not R \{1,5\}$.

(c) For each of the following values of n and k, draw the graph $G_{n,k}$ (or digraph $D_{n,k}$) that represents the relation R on the set $U_{n,k}$. (Your drawings should be neat and clean.) [7.5]

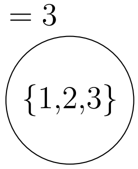
(i) $n = 3$ and $k = 1$

Response:

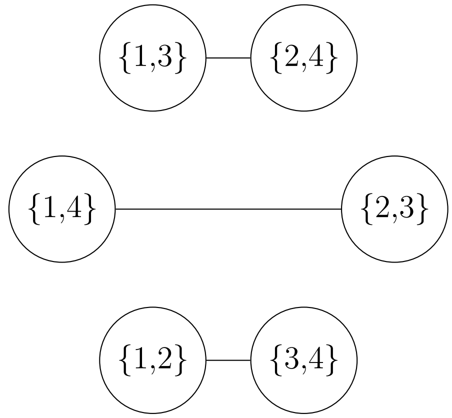


(ii) $n = 3$ and $k = 3$

Response:

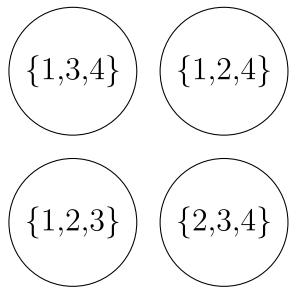


(iii) $n = 4$ and $k = 2$



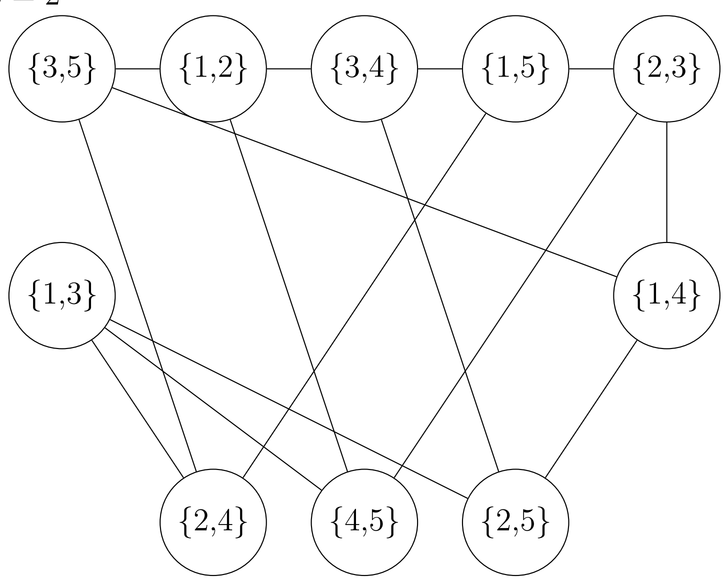
Response:

(iv) $n = 4$ and $k = 3$



Response:

(v) $n = 5$ and $k = 2$



Response:

(d) Given a positive integer n, what does the graph $G_{n,k}$ (or digraph $D_{n,k}$) look like (for the following values of k). (Describe the graph/digraph using words and/or notation covered in lectures.) [4.5]

(i) $k = 1$

Response: Complete Graph with n nodes. Each node contains a set with one element. note that it is complete but does not contain loops.

(ii) $k = n - 1$

Response: Edgeless / Null/Empty Graph with total n nodes . Each node contains set with all elements except one . Exception : When n =2 and k = 1 . There are 2 nodes . E = {{1},{2}}.There is one edge in the graph connecting the 2 nodes.

(iii) $k = n$

Response: Edgeless/Null/Empty Graph with total one node. The node is a set with all elements from 1 to n.

2. [PERFORMING SET OPERATIONS ON RELATIONS, AND INVERSE OF A RELATION] [16]

As we have discussed in lectures, a (binary homogeneous) relation R on a set S may be viewed as a subset of the Cartesian product $S \times S$ (and vice versa).

The goals of this exercise are: (i) to see how different "natural" relations (on the set of integers $\mathbb{Z}$) "relate" with each other (through set operations), and (ii) to define a new operation called the *inverse of a relation.*

For convenience, we will use the notation $R_=$ to denote the "is equal to" $(=)$ relation (as a subset of the universe $\mathbb{Z} \times \mathbb{Z}$). The sets $R_\geq$, $R_\leq$, $R_<$, $R_>$, $R_\neq$, etc. are defined similarly. You may use this convention to define other sets (if required).

(a) Describe (using words or notation) the following sets: (i) $R_< \cup R_>$, (ii) $R_> \cup R_\leq$, (iii) $R_\leq \cap R_\geq$, (iv) $R_> \cap R_\leq$, (v) $R_\leq - R_=$, (vi) $R_\geq - R_>$, (vii) $R_\geq \oplus R_\leq$, (viii) $R_< \oplus R_>$, (ix) $\overline{R_\geq}$ (that is, the complement of $R_\geq$ with respect to the universe $\mathbb{Z} \times \mathbb{Z}$), and (x) $\overline{R_=}$. [10]

Response:
(i) Set of $(a, b) \forall a, b \in \mathbb{Z}$ *and* $a \neq b$.
(ii) $\mathbb{Z} \times \mathbb{Z}$, i.e set of all $(a, b) \forall a, b \in \mathbb{Z}$
(iii) $\{(a, b)|a, b \in \mathbb{Z}, a = b\}$
(iv) ϕ
(v) $\{(a, b)|a, b \in \mathbb{Z}, a < b\}$
(vi) $\{(a, b)|a, b \in \mathbb{Z}, a = b\}$
(vii) $\{(a, b)|a, b \in \mathbb{Z}, a \neq b\}$
(viii) $\{(a, b)|a, b \in \mathbb{Z}, a \neq b\}$
(ix) $\{(a, b)|a, b \in \mathbb{Z}, a < b\}$
(x) $\{(a, b)|a, b \in \mathbb{Z}, a \neq b\}$
Set builder notation used above :
`https://en.wikipedia.org/wiki/Set-builder_notation#:~:text=In%20set%20theory%20and%20its,that%20its%20members%20must%20satisfy.`

(b) For a (binary homogeneous) relation R on a set S, the *inverse of* R, denoted as R^{-1}, is defined as follows: for all (not necessarily distinct) $a, b \in S$, the ordered pair $(a, b) \in R^{-1}$ if $(b, a) \in R$; otherwise, $(a, b) \notin R^{-1}$.

Describe (using words or notation) the following sets: (i) $R_\leq^{-1}$ (read as: $(R_\leq)^{-1}$), (ii) $R_>^{-1}$, (iii) $R_=^{-1}$, and (iv) $R_\neq^{-1}$. [4]

Response:
(i) $\{(a, b)|a, b \in \mathbb{Z}, a \geq b\}$
(ii) $\{(a, b)|a, b \in \mathbb{Z}, a < b\}$
(iii) $\{(a, b)|a, b \in \mathbb{Z}, a = b\}$
(iv) $\{(a, b)|a, b \in \mathbb{Z}, a \neq b\}$

(c) Suppose that D is the digraph that represents a (binary homogeneous) relation R on a set S, how would you easily obtain the digraph (say D^{-1}) that represents the inverse relation R^{-1}? [2]

Response: Simply reverse every arrow in the digraph to obtain the digraph that represents the inverse relation.

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3. [ADDING ELEMENTS TO A RELATION TO OBTAIN A REFLEXIVE OR SYMMETRIC RELATION] [10]

Given a (binary homogeneous) relation R on a set S: (i) if R is not reflexive, can we "add" more elements (to R) to get a reflexive relation (say R^{ref})? (ii) if R is not symmetric, can we add more elements to R to get a symmetric relation (say R^{sym})?

Clearly, the answer is yes (since $S \times S$ is reflexive and symmetric), right? So, we would like to "add" as few elements as possible (in order to obtain/construct the relations R^{ref} and R^{sym}). This is the goal of this exercise.

(a) Define a relation T (that has precisely $|S|$ elements), and use the set union operation to define the reflexive relation R^{ref}.

Response: The relation $T = \{(a, a)|a \in S\}$.
Then $R_{ref} = T \cup R$

(b) Suppose that D is the digraph that represents the relation R, how would you easily obtain the digraph (say D^{ref}) that represents the reflexive relation R^{ref}?

Response: Go to each node in the digraph and : If there is a arc connecting the node to itself leave it as it is. Else add an arc connecting the node to itself.

(c) Suppose that D is the digraph that represents the relation R, how would you easily obtain the digraph or graph (say D^{sym} or G^{sym}) that represents the (symmetric) relation R^{sym}?

Response: For each set of distinct nodes (a,b) check the following : If both arcs (a,b) and (b,a) do not exist move on . If either (a,b) or (b,a) exist then make the other arc so that both exist. If both arcs (a,b) and (b,a) already exist then move on.

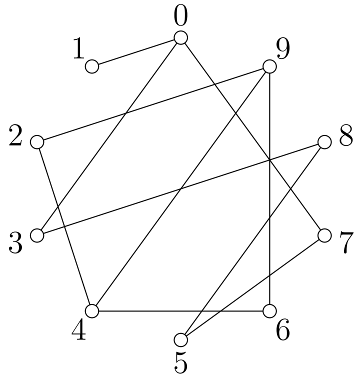
(d) Define R^{sym}. (You may use the operations discussed in Problem 2.)

Response: R^{sym} is union of R and R^{-1} i.e $R^{sym} = R \cup R^{-1}$.

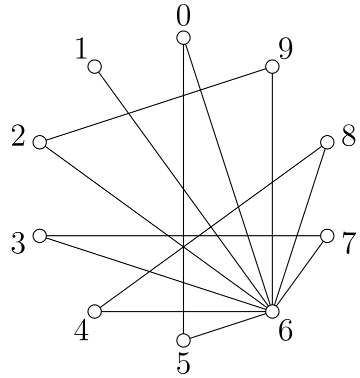
4. [Connected Components in Graphs] [5]

During the lectures, we discussed the reachability relation in graphs, and we used this to define connected components of a graph.

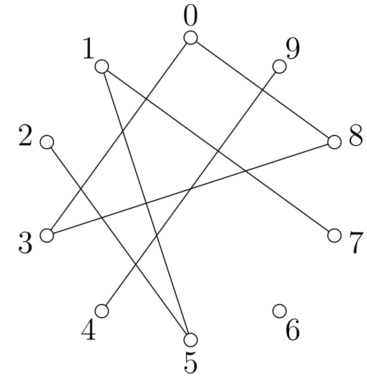
Below, you are given (drawings of) three (simple) graphs — each having vertex set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Edges are drawn as straight line segments joining the two ends.



G_1



G_2

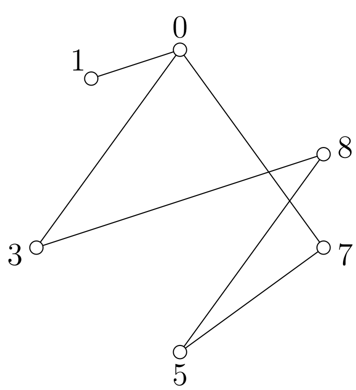


G_3

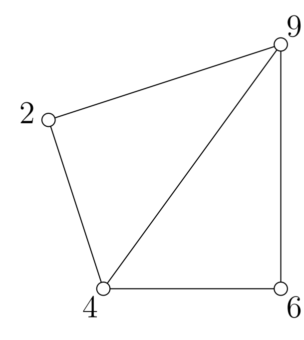
For each of these three graphs, you need to: (i) write the vertex set and edge set of each connected component, and (ii) draw each connected component.

(a) G_1

Response: (i) There are 2 connected components. Let the connected components be A and B.
Vertex Sets:
$V(A) = \{0, 1, 3, 5, 7, 8\}$
$V(B) = \{2, 4, 6, 9\}$
Edge Sets:
$E(A) = \{(0, 1), (0, 3), (0, 7), (3, 8), (5, 7), (5, 8)\}$
$E(B) = \{(2, 4), (2, 9), (4, 6), (4, 9), (6, 9)\}$
(ii)



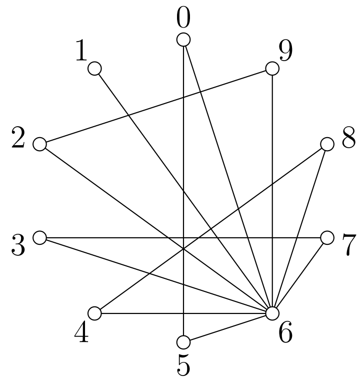
A



B

(b) G_2

Response: (i)There is only one connected component , the graph itself. Vertex Set $V(G_2) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Edge set $E(G_2) = \{(6, 0), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 7), (6, 8), (6, 9), (0, 5), (2, 9), (3, 7), (4, 8), \}$
(ii)



G_2

(c) G_3

Response:

(i)There are 4 connected components. Let the connected components be A,B,C,D .

Vertex Sets :

$V(A) = \{0, 3, 8\}$

$V(B) = \{1, 2, 5, 7\}$

$V(C) = \{4, 9\}$

$V(D) = \{6\}$

Edge Sets:

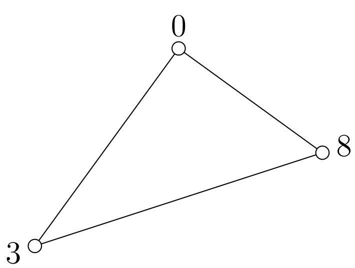
$E(A) = \{(0, 3), (0, 8), (3, 8)\}$

$E(B) = \{(2, 5), (5, 1), (1, 7)\}$

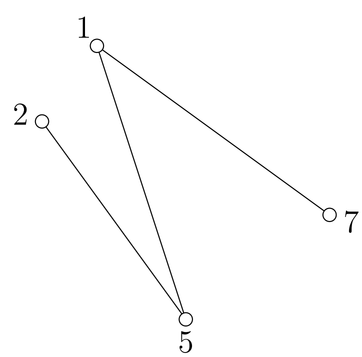
$E(C) = \{(4, 9)\}$

$E(D) = \phi$

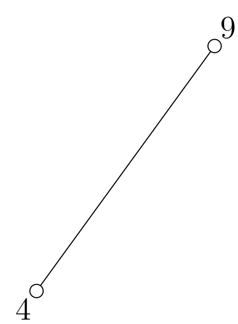
(ii)



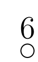
A



B



C



D

5. [REACHABILITY IN DIGRAPHS] [10]

During the lectures, we have defined and discussed the reachability relation in graphs, and we have proved various properties of this relation.

The goal of this exercise is to define and discuss the reachability relation in digraphs.

A *directed walk* in a digraph $D := (V, A)$ is any sequence of vertices and arcs — say $v_1 e_1 v_2 e_2 \ldots v_{k-1} e_{k-1} v_k$ — where each v_i is a vertex — and each e_j is an arc with tail v_j and head v_{j+1}. For such a directed walk, we say that v_1 is the *start vertex*, and v_k is the *end vertex*.

Given a digraph $D := (V, A)$, for two (not necessarily distinct) vertices $u, w \in V$, we say that w *is reachable from* u (or that u *is able to reach* w), denoted by uRw, if there is a directed walk Q (in D) with u as the start vertex, and w as the end vertex.

(a) Given any digraph D, is the relation R reflexive? If yes, explain why; if no, give a concrete example to illustrate your point.

Response: Yes , we can walk from every vertex to itself . If vertex is v_n, then the walk would just be defined as " v_n ".

(b) Given any digraph D, is the relation R symmetric? If yes, explain why; if no, give a concrete example to illustrate your point, and describe using graph-theoretical language the relation R^{sym} (as defined in Problem 3 part (d)).

Response: No, the digraph is not necessarily symmetric. Example : $D := (V, A)$ *where* $V = \{1, 2, 3\}$ *and* $A = \{(1, 2), (2, 3)\}$. Here 1 R 3 but 3 $\not R$ 1 . Just because we can walk from a to b doesnt mean we can walk from b to a in a directed graph as all the edges have a directon .

Description of R^{sym} :

R^{sym} here is the relation of all node pairs (nodes not necessarily distinct) when the first node is reachable from the second or second is reachable from the first or both nodes are reachable from each other .
OR
R^{sym} is relation of all node pairs which have a directed walk between them . Direction of the walk is irrelevant.
OR
Given the underlying undirected graph of D:=(V,A), lets say G:=(V,E), then for two (not necessarily distinct) vertices $u, v \in V$,if u is reachable from v(there is a walk between them) then $u\ R^{sym}\ v$.

(c) Given any digraph D, is the relation R antisymmetric? If yes, explain why; if no, give a concrete example to illustrate your point.

Response: No, the digraph is not necessarily antisymmetric. Example : $D := (V, A)$ *where* $V = \{1, 2, 3\}$ *and* $A = \{(1, 2), (2, 3), (3, 1)\}$. Here 1 R 3 and 3 R 1 . The digraph here is not antisymmetric as both 1 R 3 and 3 R 1. If we can walk from a to b then we may be able to walk from b to a also if required directed edges exist.

(d) Given any digraph D, is the relation R transitive? If yes, explain why; if no, give a concrete example to illustrate your point.

Response: Yes , the relation R is transitive because if we can walk from vertex a to b and we can walk from b to c then we can walk from a to c by combining the walks. i.e. If $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$, which is the definition of transitivity.