Quick Recap:

In the last lecture, we proved our first theorem:

Theorem: For any graph, the reachability relation is an equivalence relation.

We also rephrased Euler's Problem:

which connected graphs have an Eulerian trail?

↓

a graph whose vertex set has only one equivalence class (w.r.t. reachability relation)

↕ same as

a graph that has precisely one connected component.

So, we have two interesting graph properties:

1. Connected or NOT
2. Eulerian or NOT

A connected graph is $\boxed{\text{Eulerian}}$ if it has an Eulerian trail.

↓

Questions:

Do these properties depend on the graph? NO

Do these properties depend on the labels of the vertices and/or edges? NO.

Question: When are two graphs "the same"?

- Clearly, they must have equal number of vertices and equal number of edges.

Back to some set theory: $\boxed{\text{Cardinality}}$ of a Set

same as (size for finite sets)

For a finite set S, $\boxed{\text{cardinality of } S}$, denoted $|S|$, is simply the # of elements in S.

Clearly, $|S| \in \mathbb{N}$.

Let us first focus on finite sets:

Example: Box with blue balls & pink balls.

You want to decide whether

# of blue balls = # of pink balls

but you don't know numbers $(0, 1, 2, \ldots)$.

How would you decide?

Answer: Pair each ~~ed~~ blue ball with a pink ball, and see if any extra blue/pink balls remain. Right?

We can think of this as a function $f : \underset{\text{set of blue balls}}{B} \to \underset{\text{set of pink balls}}{P}$

Let's think about such functions more generally:

Question: If $|P| \geq |B|$, what can we say about the function f?

Answer: For each pink ball $p \in P$, there is $\boxed{\text{at most one}}$ blue ball $b \in B$ such that $f(b) = p$.

$\updownarrow$ same as

Each element in the co-domain is the image of at most one element in the domain.

Such functions are called $\boxed{\text{one-to-one}}$ or $\boxed{\text{injective functions}}$ or $\boxed{\text{injections}}$.

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$\updownarrow$ same as

co-domain of f = range/image of f

A function f is $\boxed{\text{bijective}}$ (or $\boxed{\text{a bijection}}$) if it is injective as well as surjective.

same as

one-to-one as well as onto

Observations for finite sets:

Such finite sets B & P are said to be equicardinal.

B, P : finite sets

$f : B \to P$

① If f is an (1-to-1) injective function then $|P| \geq |B|$.

② If is a (onto) surjective function then $|B| \geq |P|$.

③ Thus: if f is a bijective function then $|B| = |P|$.

We use this last observation to generalize the notion of "equicardinality" to all sets including infinite sets

Two sets (finite or infinite) are $\boxed{\text{equicardinal}}$ if there is a bijection between them; otherwise they are NOT equicardinal.

Philosophical discussion:

In the case of two infinite sets, say B & P,

(1) showing that B & P are equicardinal requires one to construct (or find) a bijection between them

(2) whereas showing that B & P are NOT equicardinal requires one to demonstrate/prove that a bijection (between B & P) does NOT exist!

(2) seems to be much harder than (1), right?

We will return to infinite sets (& their cardinalities) later ..... for now, let's go back to our graph theory discussion .....

Remember?

Question: When are two graphs "the same"?

TIY: Come up with a definition for two simple graphs "being the same". (Hint: use bijections).