My grades for **End-Sem-Exam**

1a 2

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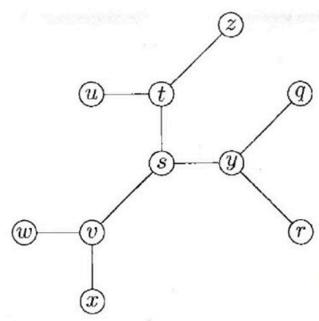
Problem No.	Marks
1	10
2	10
3	10
4	10
5	10
6	10

1. [Leaves and non-leaves in a tree] [10]

a) For the tree shown in the figure below, **write down the partition** of the vertex set into two parts:

- L (set of leaves), and
- N (set of non-leaves) [2]



* Here vertex set is say $V(T)$ where T is the tree
* It can be partitioned into leaves on non leaves
(degree=1) (degree≥2)

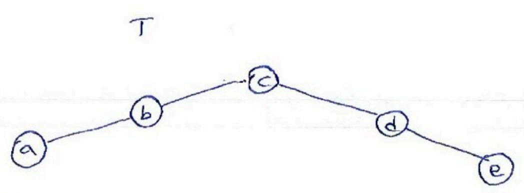
$L = \{q, r, x, w, u, z\}$

$N = \{t, v, y, s\}$

1b 2

b) **Either draw a tree** that has more non-leaves than leaves, or otherwise **prove that** such a tree does **not** exist. [2]

- A tree may have more non leaves than leaves
- Example of such a tree is given below



- The leaves are a, e
  non leaves are b, c, d

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c) **Prove the following.** [6]

"Let T denote a tree (on two or more vertices) that has no vertices of degree two, and let (L, N) denote the partition of the vertex set into leaves and non-leaves respectively. Then $|L| > |N|$."

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[Extra page for Problem 1]

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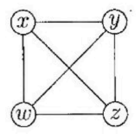
2. [Existence of Blue/Pink Cycle in Blue-Pink Complete Graphs] [10]

a) A drawing of the complete graph K_4 is given below.



Color the edges using two colors — blue and pink — such that there is **no** blue cycle **and** there is **no** pink cycle.

Of course, you don't need to color.
Just write down your sets below. [2]

- B (set of blue edges) := $\{(x,w), (w,z), (z,y)\}$
- P (set of pink edges) := $\{(w,y), (x,z), (x,y)\}$

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b) Prove the following using the pigeonhole principle (PHP).

"Let G denote a blue-pink[1] K_5. Either there exists a blue cycle or there exists a pink cycle (possibly both)."

Write down the version of PHP you are using, and clearly explain how it is applied. [6]

(Hint: You may use the fact that every forest on n vertices has at most $n-1$ edges — without proof.)

- There are 10 edges in K_5.
- Each edge is adjacent to 2 vertical.

Each edge is part of 3 cycles (when colour does not matter)

Why? In fact this is a false statement :)

There are a total of $\binom{5}{3} = 10$ cycles (when colour does not matter)

If you think this is a true statement then you need to prove it :)

Let there be 20 groups, each denotes a cycle either blue or pink, of different choices of 3 vertices

- Fill them with edges from our K_n such that if pink fill in pink cycle with ~~adjacent to the vertices of that edge~~ edge part of that cycle and vice versa.

We fill 60 times as each edge part of 3 of the groups.

We have 10 edges and each edge is a part of 3 groups, then we need to fill only 30 times right? Why 60 times?

o, there is a group with 3 edges by PHP. Hence a 3 cycle.

So you proved, any complete graph contains either a blue K_3 or a pink K_3, which is a wrong statement .

This is wrong :)

PHP say that there exist at least one group with at least 3 edges, not necessarily exactly 3 edges.

[1] In other words, each edge receives exactly one color: blue or pink.

PHP: if n groups and K elements, at least one group of $\lceil \frac{K}{n} \rceil$

not clear

2c 2

c) Deduce the following — using part b.

either has a blue cycle or has a pink cycle (possibly both)." [2]

It means, not proving from beginning, using something (here above part) conclude nicely (not doing more work).

where $n=5$ has either blue cycle or pink cycle

- Say $n>5$.
- Let's inductively assume that for all $n_1 \geq 5$ and $n_1 < n$ the graph K_{n_1} has either blue or pink cycle.
- Take K_n, remove vertex V_n. We get a graph, which is K_{n-1}. (complete graph of pink and blue edges on $n-1$ vertices).

We have $n-1 \geq 5$.

So by our inductive assumption K_{n-1} must have either blue or pink cycle.

- Because K_{n-1} is subgraph of K_n. Every edge and vertex of K_{n-1} is present in K_n.

We can say if K_{n-1} has either pink or blue cycle, then K_n also has either pink or blue cycle.

- Hence proved inductively.

3a 6

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3. [Equivalence classes, rings and fields] [10]

a) For a positive integer k, an integer a is said to be *congruent modulo* k to another integer b, denoted by $a \equiv b \pmod{k}$, if $k|(a-b)$ (that is, if k divides $a-b$).

Prove that, for any positive integer k, the relation defined above (on the set of all integers) is an equivalence relation. [4]

For $k \in \{2, 3\}$, **write down** the corresponding equivalence classes — for the relation defined above (on the set of all integers). [2]

→ • K is said to divide n if we have $m \in \mathbb{Z}$ ~~st~~ such that $km = n$

• Above relation is equivalence relation because it is transitive, reflexive and symmetric.

Score 6

(1) Symmetric? Say $k|a-b$, $a, b \in \mathbb{Z}$. Then $k|b-a$ also because $km = a-b$ $m \in \mathbb{Z}$. Hence $k(-m) = b-a$.

Hence if a Related to b, b related to a. $a, b \in \mathbb{Z}$. Which is the definition of symmetric relation

(2) Reflexive? ~~A rela~~ The relation is reflexive because $\forall a \in \mathbb{Z}$ we have a related to itself because $k|a-a$ i.e. k divides 0, we can say $k \times (0) = 0$.

Hence reflexive.

(3) Transitive? Relation is transitive as $\forall a, b, c \in \mathbb{Z}$ if a relates to b and b related to c then a related to c.

If $k|a-b$ and $k|b-c$. $\exists m, n \in \mathbb{Z}$ such that, $km = a-b$, $kn = b-c$.

Hence $k|a-c$ because $k(m+n) = a-c$, $m+n$ is also an integer.

Hence transitive.

• So the relation is an equivalence relation

→ For $k=2$, There are 2 equivalence classes Say A and B.

$A = \{a \in \mathbb{Z} : 2|a\}$

$B = \{b \in \mathbb{Z} : 2|b+1\}$

For $k=3$, there are 3 equivalence classes. Say ~~A, B, C~~ C, D, E.

$C = \{c \in \mathbb{Z} : 3|c\}$

$D = \{d \in \mathbb{Z} : 3|d+1\}$

$E = \{e \in \mathbb{Z} : 3|e+2\}$

9

3b 4

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b) We have discussed in lectures that, for all $k \in \mathbb{N} - \{0, 1\}$, the algebraic structure $(\mathbb{Z}/\mathbb{Z}_k, +, *)$ is a ring.

Recall that a *field* is a commutative ring with the additional property that each element (except for the additive identity) has a multiplicative inverse.

For each of the following values of k, **state whether** $(\mathbb{Z}/\mathbb{Z}_k, +, *)$ is a field or **not**. If it is **not** a field, **explain why** it is **not**. [4]

(i) $k = 2$
(ii) $k = 3$
(iii) $k = 4$
(iv) $k = 6$

~~Multiplicative inverse~~: An element a has multiplicative inverse b if $a*b = b*a = e$. e is the multiplicative identity.

Multiplicative identity: Element e, ~~such that~~ is multiplicative identity for the ring $(S,+,*)$ if $e*a = a*e = a$ $\forall a \in S$.

(i) For $k=2$. We have $\mathbb{Z}/\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

The additive ~~inverse~~ identity is $\bar{0}$.

Each element here except additive inverse has a multiplicative inverse.

The multiplicative identity is $\bar{1}$ by the way. $(\bar{1}*\bar{1}=\bar{1},\ \bar{1}*\bar{0}=\bar{0})$

$\bar{1} * \bar{1} = \bar{1}$.

So it is a ~~ring~~ field

(ii) For $k=3$ We have $\mathbb{Z}/\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

Additive ~~inverse~~ identity $\to \bar{0}$

multiplicative ~~inverse~~ identity $\to \bar{1}$

Score 4

Multiplicative inverse of $\bar{1}$ is $\bar{1}$ $\quad \bar{1}*\bar{1}=\bar{1}$

of $\bar{2}$ is $\bar{2}$ $\quad \bar{2}*\bar{2}=\bar{1}$

So it is a ~~ring~~ field.

Note that the algebraic structure $(\mathbb{Z}/\mathbb{Z}_k,+,*)$ $k\in\mathbb{N}-\{0,1\}$ is also commutative on addition and multiplication as discussed

(iii) $k=4$. $\mathbb{Z}/\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

Here the element $\bar{2}$ does not have multiplicative inverse.

$\bar{2}*\bar{0}=\bar{0}$ $\quad \bar{2}*\bar{3}=\bar{2}$

$\bar{2}*\bar{1}=\bar{2}$

$\bar{2}*\bar{2}=\bar{0}$

10

3EP 0

[Extra page for Problem 3]

- The multiplicative identity is $\bar{1}$, additive identity $\bar{0}$
- Hence it is not a field as $\nexists\, a \in \mathbb{Z}/\mathbb{Z}_4$ such that $\bar{2} * a = \bar{1}$.

(iv) Similarly for $k=6$. $\mathbb{Z}/\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

- Multiplicative identity is $\bar{1}$, additive identity $\bar{0}$
- Here for $\bar{2}$ there is no multiplicative inverse.

$\bar{2} * \bar{0} = \bar{0}$ $\quad$ $\bar{2} * \bar{3} = \bar{0}$
$\bar{2} * \bar{1} = \bar{2}$ $\quad$ $\bar{2} * \bar{4} = \bar{2}$
$\bar{2} * \bar{2} = \bar{4}$ $\quad$ $\bar{2} * \bar{5} = \bar{4}$

- Hence it is not field.

Additive identity : For ring $(S, +, *)$, element x is additive identity $(x \in S)$ when $\forall\, a \in S$ we have ~~a+5=5~~ $a+x = x+a = a$.
In $(\mathbb{Z}/\mathbb{Z}_k, +, *)$ the additive identity is $\bar{0}$.
$k \in \mathbb{N} - \{0, 1\}$

$\longrightarrow$ $(\mathbb{Z}/\mathbb{Z}_2, +, *)$ and $(\mathbb{Z}/\mathbb{Z}_3, +, *)$ are fields

$\longrightarrow$ $(\mathbb{Z}/\mathbb{Z}_4, +, *)$ and $(\mathbb{Z}/\mathbb{Z}_6, +, *)$ are not fields.

4a 3

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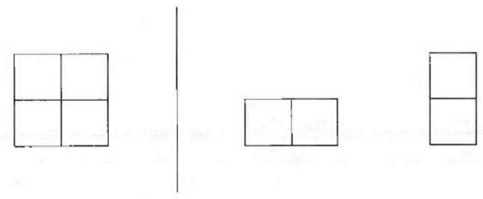
4. [Tilings using Recurrences] [10]

4a) all correct 3

In this problem, our goal is to compute the number of tilings[2] of a $(2 \times n)$-grid for all $n \in \mathbb{N} - \{0\}$ using two types of tiles: box-shaped (2×2) tiles (drawn below on the left) and brick-shaped[3] tiles (drawn below on the right):



a) For convenience, we denote by T_{n-1} the number of tilings of a $(2 \times n)$-grid — using brick-shaped and box-shaped tiles. **Compute** T_0, T_1 & T_2, and **explain your answers** clearly — with drawings (if required). [3]

(1) $T_0 = 1$. (2x1 grid) Can only fill by brick tile.

(2) $T_1 = 3$. (2×2 grid) Say we use box tile, can only use 1 box tile and tiling complete.
Say we use brick tile, (a) brick tiles can be horizontal (both)
(b) Both vertical
Total 3 cases

(3) $T_2 = 5$. Cases: (1) First column contains vertical brick. Then remaining region can be tiled by T_1 cases

box tile, vertical tile impossible here

(2) First column contains part of 1 horizontal brick. Then the other square in first column must also be covered by another ~~brick~~ horizontal brick. Impossible to cover the other square by box-tile or vertical brick.
Leftover region tiling in T_0 ways.

(3) First column contains part of box tile.
Remaining can be tiled in T_0 ways.

→ Total $T_1 + T_0 + T_0 = 5$.

Good!

[2] As per examples discussed in lec[illegible]ns: "no two tiles should overlap" and "each square of the grid should be covered by a tile".

[3] These may be placed horizontally as (1×2)-tiles, or vertically as (2×1)-tiles.

4b 3.5

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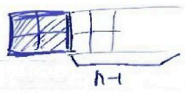
4b) Recurrence relation correct **3.5**

b) Write down a recurrence for T_n, and **explain** clearly why this recurrence is correct. [3.5]

• Say A_n is the set of all tilings of $2\times(n+1)$ grid by the tiles stated above.
$T_n = |A_n|$.

• Partition A_n into B_n, C_n, D_n. [Say $n \geq 2$ here]

(1) B_n: All tilings such that both squares of first column are covered by box tiles only



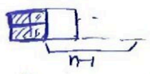
Observe that, there are T_{n-2} ways to tile remaining region
$|B_n| = T_{n-2}$

(2) C_n: All tilings such that exactly one square in first column is covered by ~~horizontal~~ box tile.

Observe this is impossible.
$|C_n| = 0$

(3) D_n: All tilings such that exactly zero of the squares in first column are covered by box tiles. So they must only be covered by brick tiles.

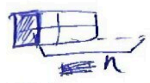
Subcase 1: ~~The~~ Both first column ~~has~~ ~~is~~ squares are covered by ~~by precisely~~ horizontal brick tiles



This subcase has T_{n-2} ways to tile.

Subcase 2: Exactly one square of first column covered by horizontal brick tile. Observe this is impossible as the other square would need to be covered by vertical brick tile which is not possible without overlap. 0 ways in this subcase.

Subcase 3: None of the first column squares are covered by ~~precisely 0~~ horizontal bricktile. So they must be covered by vertical brick tile.



There are T_{n-1} ways to tile remaining region.

$* \; |D_n| = T_{n-2} + T_{n-1}$

• As B_n, C_n, D_n partition A_n we have $|A_n| = |B_n| + |C_n| + |D_n|$
$T_n = 2T_{n-2} + T_{n-1}$.

13

4c 3.5

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c) **Solve** the recurrence obtained in part (b) — using the initial conditions from part (a) — and **write down** a closed form formula for T_n. **Explain the steps** followed clearly. [3.5]

4c) Correct 3.5

→ The recurrence relation is $T_n = T_{n-1} + 2T_{n-2}$.

$T_n - T_{n-1} - 2T_{n-2} = 0.$

→ The characteristic equation is $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

Roots of this equation are $x_1 = 2,\ x_2 = -1$

→ When a characteristic equation of a linear homogenous recurrence relation of degree 2 has 2 distinct roots then:

The closed form of the recurrence $T_n = c_1 T_{n-1} + c_2 T_{n-2}$ can we written as $T_n = \alpha_1 x_1^n + \alpha_2 x_2^n$ where α_1, α_2 are determined by the initial conditions.

→ Here we have $T_0 = 1$, $T_1 = 3$.

$T_n = \alpha_1 2^n + \alpha_2 (-1)^n$

(1) $T_0 = \alpha_1 (2)^0 + \alpha_2 (-1)^0 = \alpha_1 + \alpha_2 = 1$ --- (1)

(2) $T_1 = \alpha_1 (2)^1 + \alpha_2 (-1)^1 = 2\alpha_1 - \alpha_2 = 3$ — (2)

Add (1) and (2) we have:- $3\alpha_1 = 4 \Rightarrow \alpha_1 = 4/3$

Substition of this in (1) again gives, $4/3 + \alpha_2 = 1$

$\Rightarrow \alpha_2 = -1/3$.

→ The closed form ~~expression~~ equation for ~~T_n is~~ recurrence is,

$T_n = (4/3)(2)^n - (1/3)(-1)^n$

$= \dfrac{2^{n+2}}{3} + \dfrac{(-1)^{n+1}}{3} = \dfrac{(2^{n+2} + (-1)^{n+1})}{3}$

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[Extra page for Problem 4]

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5a 5

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5. [Proofs using Induction] [10]

a) **Prove the following statement using induction.** [5]

"You have infinitely many 3 rupees notes as well as 4 rupees notes. For any integer $n \geq 6$, you can pay precisely n rupees."

→ Induction parameter n

→ Observe that for $n=6$, we can pay by choosing ~~₹~~ 2, 3 rupee notes
$n=7$, we can pay via 1, 3 rupee note and 1, 4 rupee note
$n=8$, we can pay via 2, 4 rupee notes

→ For a certain $n \geq 9$
Assume inductively that we can pay ~~any~~ precisely n_1 rupees ($n_1 \geq 6$) and $n_1 < n$ ~~be~~.

→ Now we must prove can pay n rupees also.
Observe that it is possible to pay $n-3$ rupees by our inductive assumption. (and note $n-3 \geq 6$)
So if we pay $(n-3)$ rupees and then pay the remaining by a 3 rupee note we are done as we have paid n rupees.

5/5 5

Hence inductively proved.

5b **5**

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b) **Prove the following statement using induction.** [5]

"For all $n \in \mathbb{N} - \{0\}$, every orientation D_n, of the complete graph K_n, contains a Hamiltonian dipath."

(An *orientation of* K_n is any digraph that is obtained from the complete graph K_n by replacing each edge uv by an arc: either (u,v) or (v,u). A dipath is *Hamiltonian* if it contains each vertex.)

→ Let us prove by induction with induction parameter n. (number of vertices of D_n).

→ Observe that for a digraph D_n ~~which is~~ where $n=1$.
We have a Hamiltonian dipath. This ~~path~~ contains no ~~edges~~. arcs.
We can denote it just as v_1 where v_1 is the one vertex of D_1.
(Base case)

→ Say $n>1$ now.

→ Assume inductively that for every $n_1 < n$, $n_1 \in \mathbb{N} - \{0\}$ we have a hamiltonian dipath for any D_{n_1}.

→ We must prove that any D_n contains a hamiltonian dipath.
Take one vertex of this digraph D_n. Remove it.
Observe that the resulting ~~digg~~ digraph is an orientation of K_{n-1}.
Because there is exactly one arc between every pair of distinct vertices.
Say this ~~dig~~ digraph is D_{n-1}.

By our assumption, there is a hamiltonian dipath in D_{n-1}.
[note $D_n - v_n = D_{n-1}$, where v_n is removed vertex]

- Say this hamiltonian dipath is $v_1 a_1 v_2 a_2 \ldots v_{n-2} a_{n-2} v_{n-1}$
- ~~Observe~~, *If there is an arc from v_n to v_1 in D_n then we are done
$v_n a v_1 a_1 v_2 \ldots v_{n-2} a_{n-2} v_{n-1}$ would be a hamiltonian dipath of D_n

*If there is an arc from v_{n-1} to v_n in D_n, similarly we are DONE

*If both these are not true, i.e there is arc from v_1 to v_n in D_n and arc from v_n to v_{n-1} in D_n
(as there is ~~atte~~ exactly one arc between distinct vertices)

5/5 **5**

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[Extra page for Problem 5]

Let i be the largest index, ($\le i \le n-1$) such that there is an arc from V_i to V_n. —(condition 1.0)

Note that this is possible as we already know in this case that there is arc from V_1 to V_n.

Then we can say that there is arc from V_n to V_{i+1} for sure. This is because $i<n-1$, hence $i+1<n$. This means that vertex V_{i+1} is part of D_{n-1} indeed. If there was arc from V_{i+1} to V_n then there would be a contradiction as $i+1>i$ and condition 1.0 violated. So arc exists from V_n to V_{i+1}.

Then we can say hamiltonian dipath exists in D_n and it is

$V_1\, a_1 \ldots V_i\, a_0\, V_n\, a_p\, V_{i+1} \ldots V_{n-1}$

Where a_0 denotes arc from V_i to V_n and a_p denotes arc from V_n to V_{i+1}.

- We have shown D_n contains hamiltonian dipath if D_{n-1} contains hamiltonian dipath.

→ Hence we can say $\forall n \in N_{\ge 0}$, D_n contains hamiltonian dipath by induction.

Hence proved.

6a 3

6. [Combinatorial proofs] [10]

a) **Prove that** the following sets are equicardinal. [3]

$S := \{j \in \mathbb{N} : 5|j\}$ and $T := \{k \in \mathbb{N} : 7|k\}$.

- Take the set of natural numbers $\mathbb{N}$.
  Define a function $f: \mathbb{N} \to S$ where $f(x) = 5x$ $x \in \mathbb{N}$
  Define a function $g: S \to \mathbb{N}$ where $g(x) = x/5$ $x \in S$.

Observe: $f(g(s)) = s$ $\forall s \in S$
$g(f(n)) = n$ $\forall n \in \mathbb{N}$

Proof: Note that every element of set S is divisible by the number 5. (and ≥ 0)
That means that number divided by 5 is also ~~even~~ a natural number.

$*$ $g(s) = s/5$; $f(g(s)) = 5(s/5) = s$
$*$ $f(n) = 5n$; $g(f(n)) = \frac{(5n)}{5} = n$ Hence, proved.

- This means that f is onto and one-one, why?
  (1) If f ~~was~~ were not onto then $\exists s_1 \in S$ such that $\nexists n_1 \in \mathbb{N}$ such that $f(n_1) = s_1$.
  That would imply $f(g(s)) = s$ cannot be true for all $s \in S$ as it cannot be true for such s_1.
  Hence f must be onto.
  (2) If f were not one-one, we would have $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$ and $f(n_1) = f(n_2)$.
  If that were the case then, $g(f(n_1)) = g(f(n_2))$.
  Which would imply $n_1 = n_2$; as we have proved $g(f(n)) = n$ $\forall n \in \mathbb{N}$. Which is a contradiction.
  Hence f is one-one.
- Hence f is a bijection. One-one and onto.
  We can say that $|\mathbb{N}| = |S|$ as if there exists bijection from one set to another they are equicardinal.
- Without loss of generality we can say $|\mathbb{N}| = |T|$. Similar proof, just replacement of 5 with 7 everywhere in function definitions.

Hence $|\mathbb{N}| = |S| = |T|$

6b 3.5

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b) Let $n \in \mathbb{N} - \{0\}$. We will compute the cardinalities of two sets, namely A_n and B_n (defined below). [3.5]

(i) Recall that an S-*string* is a string formed by elements in the set S. For example, if $S := \{0, 1, 2\}$ then $0, 012, 02122201021$ are some S-strings.

Let A_n denote the set of all S-strings of length n — where $S := \{0, 1, 2\}$. **Compute** $|A_n|$ and **explain** briefly why your answer is correct.

(ii) Let $Z := \{1, 2, \ldots, n\}$. We define the set B_n as follows. $B_n := \{(X, Y) : X \subseteq Z, Y \subseteq Z, X \subseteq Y\}$.

Prove the following.

$$|B_n| = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}$$

write your claim before you begin base case.

(i) Base case: If $n=1$, then we have 3 possibilities for S-strings. 1 or 2 or 0

If $n \geq 2$,
Assume inductively that for any $n_1 < n$ there are 3^{n_1} ways to make S-string.

Inductive step: The last element of S-string of length n has 3 possibilities.
The first $n-1$ elements will have the same number of possibilities as an S-string of $(n-1)$ length.
3^{n-1} possibilities. (By inductive assumption)
So length n S-string has $3 \cdot 3^{n-1}$ possibilities, 3^n.

$|A_n| = 3^n$

(ii) We have $Z := \{1, 2 \ldots n\}$ and $Y \subseteq Z$ and $X \subseteq Y$. (note $X \subseteq Z$ is obvious from this)

- The set X can have anywhere from 0 to n elements. The number of ways to choose $0 \leq i \leq n$ elements from n elements is $\binom{n}{i}$. The number of ways to pick X set.
- Observe that all these i elements MUST be in Y also, as X is subset of Y. [X is subset of Y if $\forall x \in X$, $x \in Y$ also]

Hence Y contains every element that is in X and ~~some~~ may contain some of the $n-i$ other elements also.

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6c 3.5

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c) For $n \in \mathbb{N} - \{0\}$, **establish a bijection** between the sets A_n and B_n (defined in part (b)), and **deduce** the following identity. [3.5]

$$\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} = 3^n$$

(You need to prove that your bijection is indeed a bijection!)

- let us define a function $p: \{1,2 .. n\} \to \{0,1,2\}$.
  For a particular S-string if i^{th} position is 0 $p(i) = 0$

Do you mean ith element is not present?

- Let us define $f: A_n \to B_n$ such that for a particular $a \in A_n$
  we have $f(a) = (X, Y)$. $X = \{c \in \mathbb{Z} : p(c) = 2\}$
  $Y = \{d \in \mathbb{Z} : p(c) = 1 \text{ or } p(c) = 2\}$

  $g: B_n \to A_n$ such that $a \in A_n$.
  $g((X,Y)) = a$, i^{th} element of a is 0 if i is absent in X, Y
  i^{th} element of a is 1 if it is present in Y, not in X
  i^{th} element of a is 2 if it is present in both X, Y.

- ~~If~~ $g(f(a)) = a \ \forall a \in A_n$. Why? [Say $f(a) = (X, Y)$ by the way]
  ~~For~~ * If i^{th} element of a is 0, ~~then say $f(a) = (X,Y)$~~
  $i \notin X$, $i \notin Y$. Hence $g(f(a))$ also has i^{th} element 0, by definition of g.
  * If i^{th} element of a is 1, ~~then say $f(a) = (X,Y)$~~
  $i \notin X$, $i \in Y$. Hence $g(f(a))$ also has i^{th} element 1.

~~* If i^{th}~~

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[Extra page for Problem 6]

- For every of these $(n-i)$ elements they ~~may~~ can be in Y or they cannot (2 possibilities).

  So the number of ways to pick arbitrary number of these $(n-i)$ elements is 2^{n-i}.

- ~~The total number of~~ The total number of ways to choose X with i elements and Y with given conditions is hence

  $\binom{n}{i} \cdot 2^{n-i}$

- The total ways to chose X, Y with given condition is hence

  $\sum_{i=0}^{n} \binom{n}{i} \cdot 2^{n-i}$ because i can range from 0 to n as stated by us previously.

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[Extra page: may be used for **any** problem]

*If i^{th} element of a is 2, ~~then say $f(a)=(x,y)$~~ $i\in X, i\in Y$. And hence $g(f(a))$ has i^{th} element 2.

As this is true for all possibilities of each position of a. [i^{th} element of a is $c\in\{0,1,2\}$ then i^{th} element of $g(f(a))$ is c]

$g(f(a))=a \;\forall a\in A_n$

• $f(g(b))=b \;\forall b\in B_n$. Why? Say $b=(X,Y)$. $f(g(b))=(X_1,Y_1)$

*If i element is such that $i\in X, i\in Y$ then i^{th} element of $g(b)$ S-string is 2. And then $i\in X_1, i\in Y_1$ by definition.

*If i element is such that $i\notin X, i\in Y$ then i^{th} element of $g(b)$ S-string is 1. And then $i\notin X_1, i\in Y_1$.

*If i element is such that $i\notin X, i\notin Y$. Then i^{th} element of $g(b)$ S-string is 0. And then $i\notin X_1, i\notin Y_1$.

*i element cannot be such that $i\in X, i\notin Y$ because that would contradict $X\subseteq Y$.

As this covers all possibilities of all elements $i\in\{1,..n\}$, we can safely say,

$f(g(b))=b \;\forall b\in B_n$ as we have gotten that for each $i\in\{1,..n\}$ then ~~$i\in X$ iff $i\in X_1$~~ $i\in X$ iff $i\in X_1$, $i\in Y$ iff $i\in Y_1$.

*We have proved in 6(a) that when $f(g(b))=b \;\forall b\in B_n$ $g(f(a))=a \;\forall a\in A_n$

Then there is a bijection from A_n to B_n. And so $|A_n|=|B_n|$.

$\Rightarrow \sum_{i}^{n}\binom{n}{i}2^{n-i}=3^n$ // by previous parts.

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[Extra page: may be used for any problem]

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