$\boxed{\mathbb{Z}: \text{set of integers}}$

Quick Recap: $p, q \in \mathbb{Z}$ and $p \neq 0$.

We say that $\boxed{p \text{ divides } q}$ if there is some $s \in \mathbb{Z}$ such that $q = p \cdot s$.

Many ways to say same thing:

0. p divides q
1. q is divisible by p
2. p is a factor of q
3. p is a divisor of q
4. q is a multiple of p

Defn: A positive integer, greater than 1, is called a $\boxed{\text{prime number}}$ (or just prime) if the only positive divisors of q are 1 & q.

Examples: 2, 3, 5, 7, ..., 17, 19, ..., 29, 31, ...

Defn: A pair of primes p & q (where $p < q$) is called $\boxed{\text{twin primes}}$ if $q - p = 2$.

Theorem: There are infinitely many primes.

→ needs a proof (Module-2)

Twin Prime Conjecture: There are infinitely many twin primes.

→ Alphonse de Polignac 1849 (French mathematician)

↓

NO ONE knows how to prove this (till today)

Another fascinating conjecture about primes:

Goldbach's Conjecture: Every even positive integer, greater than 2, is the sum of two (not necessarily distinct) primes.

A little bit of history: (Rosen 264) (1690-1764) (Rosen 695) (1707-1783)

1742 - letter from Christian Goldbach to Leonhard Euler

Christian Goldbach ↓ conjectured that every odd integer (>5) is sum of 3 primes

Leonhard Euler ↓ replied that if every positive integer (>2) is sum of two primes then $\boxed{\text{this}}$ would be true.

Weak Goldbach Conjecture ↑ proved by Harald Helfgott in 2013

So far, we have seen $\underset{\text{(without proof)}}{\underline{\text{a theorem}}}$ and $\underset{\text{(which may or may NOT be true)}}{\underline{\text{two conjectures}}}$ —

but all of them are about prime numbers, and more generally about natural numbers.

Are there any other interesting theorems/conjectures?

YES! MANY! Are they all about natural numbers? NO.

To discuss other theorems/conjectures (NOT about natural numbers), we need other mathematical creatures...

Sets & Relations

A $\boxed{\text{set}}$ is an unordered collection of objects. → called elements/members of the set

We have discuss a couple of sets so far: $\mathbb{N}$ natural #s

$\mathbb{Z}$ integers

We have also discussed some <u>subsets</u> of $\mathbb{N}$:

$P :=$ set of prime numbers $= \{2, 3, 5, 7, \ldots\}$

$C :=$ set of composite numbers $= \{4, 6, 8, 9, 10, \ldots\}$

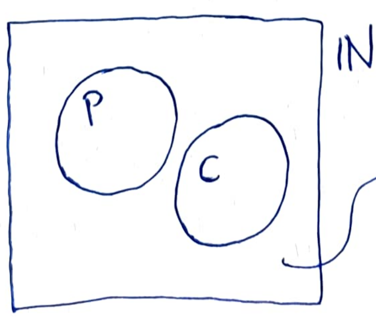
A set A is a subset of a set B if every element of A is an element of B.

Notation: $A \subseteq B$.

Note that $P \subseteq \mathbb{N}$ and $C \subseteq \mathbb{N}$.

Venn Diagram

(John Venn) 1881



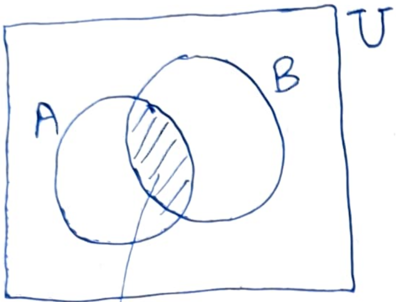
Question: what is here?

Answer: Just 0 & 1.

Observe that P and C have NO common element.

Two sets A and B are $\boxed{\text{disjoint}}$ if they have NO common element.

In general, two sets may have common elements.



U → universal set (aka "the universe") → depends on what you care about

common elements

Example:

$S := \{0, 3, 6, 9, 12, 15, \ldots\}$ → all multiples of 3 (in $\mathbb{N}$)

$T := \{0, 4, 8, 12, 16, 20, 24, \ldots\}$ → all multiples of 4 (in $\mathbb{N}$)

Observe that S & T have common elements — for example: 0, 12, 24, ...

$S \cap T := \{0, 12, 24, \ldots\}$

Definition

A, B: sets

The $\boxed{\text{intersection of A and B}}$, denoted by $A \cap B$, is the set that contains those elements which are members of both A and B.

where two (or more) things meet

Question: What does it mean for two sets to be disjoint?

If A & B are disjoint, what can we say about their intersection?

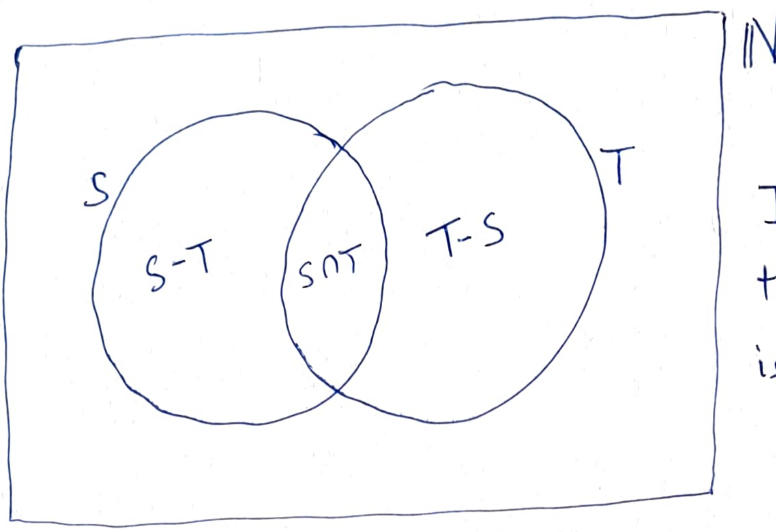
Answer: It should be "empty", right?

Notation: $\emptyset$ OR $\{\}$: EMPTY SET / NULL SET

Two sets A & B are $\boxed{\text{disjoint}}$ if $A \cap B = \emptyset$.

Back to our example: $S := \{0,3,6,9,12,15,\dots\}$ → all multiples of 3 (in $\mathbb{N}$)

$T := \{0,4,8,12,16,20,24,\dots\}$ → all multiples of 4 (in $\mathbb{N}$)



In our example, the universe (U) is $\mathbb{N}$.

Question:
What are some other sets that we can observe in the above Venn Diagram?

Answers: ① Multiples of 3 that are NOT multiples of 4 (in $\mathbb{N}$)
② Multiples of 4 that are NOT multiples of 3 (in $\mathbb{N}$)
③ Natural numbers that are either multiples of 3 OR are multiples of 4 (OR of both)

Question: How should we denote all of these sets?

Intuitively it makes sense to use: ① $S-T$ → $S \setminus T$
② $T-S$ → $T \setminus S$
(many books use: $S \setminus T$, $T \setminus S$)

③ $S+T$? Makes sense definitely, but we will NOT use this. Instead $\boxed{S \cup T}$.

Definition

A, B: sets

The $\boxed{\text{union of } A \text{ and } B}$ (act of joining together), denoted by $A \cup B$, is the set that contains those elements which are members of A OR members of B (or members of both).

Question: Have we missed ~~each~~ any other observable sets in the example Venn Diagram?

Answer: YES

① Natural numbers that are NOT multiples of 3

② Natural numbers that are NOT multiples of 4

Question: How should we denote these sets?

Answer: We can use set difference, right?

① ~~S~~ $\mathbb{N} - S$

② $\mathbb{N} - T$

Let's write down some more definitions: (eyes rolling emoji)

Set Difference

→ I find this confusing!

A, B : sets

As per Rosen: $\boxed{\text{difference of } A \text{ and } B}$, denoted by $A - B$ (or $A \setminus B$), is the set containing those elements that are in A but NOT in B.

I will generally say "A minus B".

Complement of a set (~~with~~ with respect to a given universe)

w.r.t.

U : universe

A : some set (subset of U)

The $\boxed{\text{complement of } A}$ (w.r.t. U), denoted by $\bar{A}$, is the set $U - A$.

~~Eg~~ why complement?

one of the meanings: counterpart (one of two mutually completing parts)