# CS1200 — Discrete Mathematics for Computer Science — 2023 (Jan-May)

## Assignment 3

Release Date: 27/05/2023 **Due Date: 03/06/2023 — 11:00 PM IST**

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**Academic Integrity Statement:** I, Hema Venkata Raadhes Chandaluru, affirm that I have not given or received any **unauthorized** help (from any source: people, internet, etc.) on this assignment, and that I have written/typed each response on my own, and in my own words.

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THE MARKS FOR EACH PROBLEM (1, 2, 3, 4, 5, 6) ARE FIXED. HOWEVER, THE MARKS FOR EACH SUBPROBLEM (1A, 1B, 2A, ETC.) ARE TENTATIVE — THEY MAY BE CHANGED DURING MARKING IF NECESSARY.

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1. [Spot the Bug] [5]

The first time one sees induction (as a proof technique), one often feels that anything(!) can be proved using induction. This is, of course, false.

In this exercise, we will see one such example. Of course, there must be a bug in the proof! Your goal is to find this bug.

**"Theorem" 0.1.** *In any set $S$ of $k$ balls*[1]*, where $k \in \mathbb{N} - \{0\}$, all balls are of the same color.*

*"Proof".* Let $S := \{b_1, b_2, \ldots, b_k\}$ denote a set of $k$ balls. We proceed by induction on $k$ (that is, on the number of balls in $S$).

Observe that if $k = 1$ then the set $S$ contains just one ball; clearly, all balls are of the same color.

Now suppose that $k \geq 2$, and assume the following inductively: "if $T$ is any set with fewer than $k$ balls then all balls (in $T$) are of the same color".

We now define two sets as follows: $Y := S - b_k$ and $Z := S - b_1$. Observe that each of $Y$ and $Z$ is a set comprising $k - 1$ balls. By the induction hypothesis: (i) all balls in $Y$ are of the same color, and likewise, (ii) all balls in $Z$ are of the same color.

Observe that $Y \cap Z$ is nonempty, and let $b$ denote any ball in $Y \cap Z$. In particular, each ball in the set $Y$ has the same color as the ball $b$, and each ball in the set $Z$ has the same color as the ball $b$. This implies that each ball in the set $Y \cup Z$ has the same color as the ball $b$. Observe that $Y \cup Z = S$. Thus, each ball in $S$ has the same color as the ball $b$. It follows that all balls in $S$ are of the same color. □

**Response:** The bug in this proof is the point "observe that $Y \cap Z$ is non empty " . This is because this is not always true for all $k \geq 2$ .
For $k = 2$, if set $S = \{b_1, b_2\}$ and $Y = S - b_2$ and $Z = S - b_1$ . Then $Y \cap Z$ is null set, however in the proof it is stated to be non empty always. This is the bug in the proof.

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[1]We live in a world where each ball is colored using exactly one color. For example, there is no pink ball with blue stripes.

2. [Generalizing Symmetric Difference to any number of sets] [10]

The goal of this exercise is to generalize the "symmetric difference" set operation to more than two sets.

(a) Prove the following lemma using induction:

**Lemma 0.2.** *For $n \in \mathbb{N} - \{0\}$, given a collection of $n$ sets, say $\mathcal{A} := \{A_1, A_2, \ldots, A_n\}$, the set $(\ldots((A_1 \oplus A_2) \oplus A_3) \ldots A_{n-1}) \oplus A_n$ comprises precisely those elements that appear in an odd number of sets in the collection $\mathcal{A}$.*

**Response:** If $A, B$ are two sets , then the symmetric difference of the sets will contain only elements which are in precisely one of the sets. The elements which are in both or none of the two sets will not be in the symmetric difference of the two sets.We will use this without stating below in cases.

Let $S_n = (\ldots((A_1 \oplus A_2) \oplus A_3) \ldots A_{n-1}) \oplus A_n$ , where each $A_i \in \mathcal{A}$.
When n=1 , then the set $S_1$ contains all elements from set $A_1$. Here we can say that $S_1$ contains the elements which are in an odd number of sets of the collection of sets $\mathcal{A}$.As every element in $A_1$ is in an odd number of sets of $\mathcal{A}$ here.
Let us assume inductively that for any collection of sets having less than n sets(where $n \geq 2$) , $S_k$ contains precisely those elements which are in a odd number of those sets.
Now, we have $S_{n-1} \oplus A_n = S_n$.
For each element $a$ (let us consider cases; also please note we will be using the inductive hypothesis below in several places) :
1) $a$ is in even number of sets of $A_1, A_2 ... A_n$ :
-If $a$ is in both $S_{n-1} and A_n$ , then it is not in $S_n$ . Here $a$ is in odd number of sets of $A_1, A_2 ... A_{n-1}$ and $a \in A_n$.
-If $a \notin S_{n-1} and a \notin A_n$ , then it is not in $S_n$ . Here $a$ is in even number of sets of $A_1, A_2 ... A_{n-1}$ and $a \notin A_n$.
In both the above possibilities element $a$ is not in $S_n$. 2)$a$ is in odd number of sets of $A_1, A_2 ... A_n$ :

-If $a \in S_{n-1} and a \notin A_n$ , then it is in $S_n$ . Here $a$ is in odd number of sets of $A_1, A_2 ... A_{n-1}$ and $a \notin A_n$ .
-If $a \notin S_{n-1} and a \in A_n$ , then it is in $S_n$ . Here $a$ is in even number of sets of $A_1, A_2 ... A_{n-1}$ and $a \in A_n$.
In both the above possibilities $a$ is in $S_n$.

Hence if the Lemma holds for all collections of less than n sets, then it holds for collections of n sets also. Hence , Inductively proved. □

(b) Given a collection of $n$ sets, say $\mathcal{A}$, where $n \in \mathbb{N} - \{0\}$, explain briefly why the following expression (that is, a way of writing) is unambiguous, and explain what it means.

$$\bigoplus_{A \in \mathcal{A}} A$$

**Response:** This expression is a way of generalising the symmetric difference of sets. It represents set which comprises precisely those elements that appear in an odd number of sets in the collection $\mathcal{A}$. This representation is unambiguous because whichever order we take symmetric difference of sets we will reach the same final result.

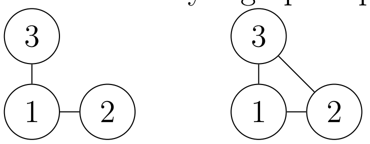
3. [Non-leaves in a Tree] [12]

See if you can come up with a tree where each vertex is a leaf. How many such trees are you able to construct (up to isomorphism)? Not many, I suppose. The goal of this exercise is to prove that this is not a coincidence, and more!

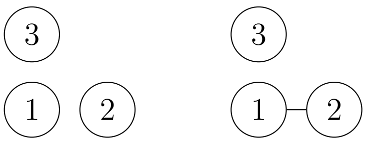
(a) Prove, using induction, that every tree on three or more vertices, has at least one non-leaf.[2]

**Response:** Any connected graph of one and two vertices clearly are trees.

There are only 4 graphs upto isomorphism for 3 vertices out of which only one is a tree.



$G_1$ $G_2$



$G_3$ $G_4$

Here the only one which is a tree is $G_1$ and it clearly has a vertex which is not a leaf (specifically vertex 1).

Take the number of vertices of a tree to be n.
{Base case}For any tree of n=3 we have that it must contain a vertex which is not a leaf as shown above.
How do we prove that trees with more vertices also have such vertices. Let us use induction.

Take a tree $G$ with $n > 3$ and assume inductively that " every tree with less than n vertices has a non-leaf vertex".

We have proved previously in that every tree has atleast one leaf , and that the graph we get after removal of a leaf is also a tree.

Let us take a leaf $x$ in tree $G$ and remove it , $T = G - x$. Removal of this leaf removes the vertex and the edge associated with it. $T$ has n-1 vertices , so it must have a vertex which is not a leaf by our assumption.
T is a subgraph of G clearly. Every vertice and every edge of T is in G,
By our assumption we can say that T has atleast one non leaf vertex. Let us take this vertex , lets say $a$. As every edge in tree T is present in Tree G, we can safely say that the degree of $a$ in G is greater than or equal to the degree of $a$ in T . As $a$ is not a leaf in T it has degree$\geq$ 2 in tree T. Hence it also has degree $\geq$ 2 in G.
So we can say that G has a non leaf vertex.
If all trees of less than n vertices have a vertex which is not a leaf then all trees of n vertices also have a vertex which is not a leaf.
Hence proved inductively. □

(b) Now, let's ask ourselves: which trees have exactly one non-leaf? Describe an infinite family of (nonisomorphic) trees, say $\mathcal{T}$, with the following properties:

- For each $n \in \mathbb{N} - \{0, 1, 2\}$, the family $\mathcal{T}$ contains precisely one tree on $n$ vertices.

[2]In a tree, a *non-leaf* is any vertex that is not a leaf.

- Each member of $\mathcal{T}$ is a tree that has exactly one non-leaf.

**Response:** Let us define the such a tree for every n by specifying the edge and vertex sets. I will not prove it is a tree as it is not asked and it is clear as well.

For a every $n \in \mathbb{N} - \{0, 1, 2\}$ , let us define exactly one tree $T_n$ of n vertices and $T_n \in \mathcal{T}$
Which is :
Tree $T_n$ :
$V(T_n) = \{1, 2...n\}$
$E(T_n) = \{(1, 2), (1, 3)...(1, n)\}$

(c) Prove, using induction, the following:

**Theorem 0.3.** *Every tree on three or more vertices has at least one non-leaf; furthermore, it has exactly one non-leaf if and only if it belongs to the family $\mathcal{T}$.*

**Response:** (1) If a graph belongs to the set $\mathcal{T}$ then it has precisely one vertex which is not a leaf : The vertex 1 is not a leaf because it clearly has degree greater than or equal to 2. The total vertices in the tree apart from vertex 1 is greater than or equal to 2 because each tree in the set has atleast 3 vertices in total. So the vertex 1 is not a leaf.
Every other vertex in set {2,3,...n} is a leaf , because each of them has degree equal to one . (The only vertex each of them is adjacent to is vertex 1).

(2) Now we must also prove the converse. If a tree has precisely one non leaf it must belong to the set $\mathcal{T}$.
Let us take a tree of n vertices. How can we prove that this tree belongs to the set $\mathcal{T}$.
By part (a) we can take that such as tree would have atleast 3 vertices , $n \geq 3$.
For $n = 3$ we have the only possible tree is $G_1$ in part (a) and it is clearly part of $\mathcal{T}$. $V(G_1) = \{1, 2, 3\}, E(G_1) = \{(1, 2), (1, 3)\}$. This is the tree of 3 vertices in $\mathcal{T}$.
Take a tree $T$ of $n \geq 4$ vertices.
Take the one non leaf vertex as 1. And the leaves are vertices $2, 3...n$.
Remove the vertex $n$ from this tree. This vertex is a leaf. By quiz2 problem we know that T-n is also a tree. Call this tree as $T_1$. When we remove a vertex from a graph the degree of every vertex must either stay the same or decrease. So the number of non leaves can only decrease but not increase.
We also know that every tree that has atleast 3 vertices must have atleast one non leaf. Hence $T_1$ must have precisely one non leaf because $T$ has one non leaf , and $T_1$ must have less that or equal to number of non leaves as $T$ and also $T_1$ has atleast 3 vertices ($T_1$ has n-1 vertices , and $n-1 \geq 3$).
Now by our assumption $T_1 \in \mathcal{T}$.
$V(T_1) = \{1, \ldots n-1\}$
$E(T_1) = \{(1, 2), \ldots (1, n-1)\}$
Now if take the tree $T$ it must have every edge and vertice that $T_1$ has. Also the vertex $n$ is a leaf so it has degree one. So it must be connected to the vertex 1. This is true because if it were connected to any other vertex(note all the other vertices except 1 are leaves) , those vertices would have degree more than 1 and would be a leaf would result in a contradiction.
So for tree $T$ :
$V(T) = \{1, \ldots n\}$
$E(T) = \{(1, 2) \ldots (1, n)\}$
Clearly this is the tree of n vertices in $\mathcal{T}$.
So if a tree of atleast 3 vertices has precisely one non leaf then it belongs to $\mathcal{T}$

We have proved both sides of the implications and The first point of Theorum 0.3 has been proved in part (a) hence Theorum 0.3 holds true. □

4. [Ordering the elements of a Finite Poset nicely] [10]

The goal of this exercise is to show that, for any finite poset, one can order its elements "nicely" as stated below.

**Theorem 0.4.** *Given any finite poset $(S, \preccurlyeq)$ with $n := |S|$ elements, there exists an ordering of the members of $S$ — say $a_1, a_2, \ldots, a_n$ — such that, for all $i, j \in \{1, 2, \ldots, n\}$, if $a_i \preccurlyeq a_j$ then $i \leq j$.*

Prove the above theorem using induction.

**Response:** Take a POSET with n number of elements.

If n=1 then certainly there is such an ordering as specified because the order would just be $a_1$ where it is the only element of the POSET.$a_1 \preccurlyeq a_1 and 1 \leq 1$.

Lets us inductively assume that "All POSETS with less than or equal to n-1 elements can be ordered in such a way as specified in the theorum 0.4 ."

If $n \geq 2$ , then take one element , say $x$.
Remove this element from the set S of POSET $(S, \preccurlyeq)$.
Can we say that the set S-x on the relation $\preccurlyeq$ is also a poset. Yes we can. It is reflexive because every element of S-x is related with itself as the relation was reflexive with set S also.
As $\preccurlyeq$ is antisymmetric on S for every $a, b \in S$ atmost one of $a \preccurlyeq b$ , $b \preccurlyeq a$ holds. So for every $a, b \in S - x$ atmost one of $a \preccurlyeq b$ , $b \preccurlyeq a$ holds. So , $\preccurlyeq$ is antisymmetric on S-x.
As $\preccurlyeq$ is transitive on S; for every $a, b, c \in S$ if $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$. So for every $a, b, c \in S - x$ if $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.Hence $\preccurlyeq$ is transitive on S-x

$(S - x, \preccurlyeq)$ is also a poset , and has n-1 elements. By our assumption we can order the elements of S-x as $a_1, a_2, \ldots, a_{n-1}$–$O_1$ such that, for all $i, j \in \{1, 2, \ldots, n-1\}$, if $a_i \preccurlyeq a_j$ then $i \leq j$.

By antisymmetric property we have that for every $a \in S - x$, either $a$ is not comparable to x, or exactly one of $a \preccurlyeq x$ and $x \preccurlyeq a$ holds.
Let $S_1$ denote the set of elements in $a \in S$ such that $a \preccurlyeq x$. And $S_2$ denote the elements $a \in S$ such that $x \preccurlyeq a$.
Claim : The order satisfying the theorum 0.4 for poset $(S, \preccurlyeq)$ is $a_1, \ldots, x, \ldots a_{n-1}$ where $a_1, a_2 \ldots a_{n-1}$ –$O_2$ is the order for poset $(S - x, \preccurlyeq)$ and all elements of $S_1$ are located before x in the order and all elements of $S_2$ are located after x.
This order is possible because every element in $b \in S_2$ is present after every element in $a \in S_1$ in order $O_1$ because $a \preccurlyeq b$ (This is true by transtive property $a \preccurlyeq x, x \preccurlyeq b => a \preccurlyeq b$). Every $b \in S_2$ must be present after every $a \in S_1$ in order $O_1$. Take the element of $y \in S_1$ which is the most forward in the order for poset $(S, \preccurlyeq)$. Then we can place x right in front of y. All elements of $S_2$ were in front of y in order $O_1$ so they are also in front of x in order $O_2$.
In case $S_1$ is empty then we can place x in a position which is before every element of $S_2$ in $O_1$ , simple position would just be to place x such that it comes at first position in $O_2$.
Now we are sure that such an order is possible , but how are we so sure that the order suffices the conditions?
Proof of claim : This order satisfies the conditions because:
(1)We know that for every $a_i$ and $a_j$ (elements of S-x) if $a_i \preccurlyeq a_j$ then $i \leq j$. ( As order is assumed to

be possible for S-x).
(2)We also know that for every element $a \in S_1$ , $a \preccurlyeq x$. As all elements of $S_1$ are before x in the order , the indices are such that indice of $a <$ indice of x. Similarly we can show for elements of $S_2$ that they have index more than that of x as they in front of x in the order .
As the statement holds for all element pairs , we can say that the theorum holds true. Hence proved. $\square$

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5. [Immediate Predecessor Relation of a Finite Poset] [11]

Recall that, for a poset $(S, \preccurlyeq)$, we defined the *immediate predecessor relation* as follows. For $x, y \in S$, we say that $x$ is an *immediate predecessor of* $y$, denoted by $x \lhd y$, if: (i) $x \prec y$ and (ii) there is no $z \in S$ such that $x \prec z \prec y$.

We discussed in lectures (without proof) that the entire partial order ($\preccurlyeq$) can be reconstructed from the immediate predecessor relation ($\lhd$) — in case $S$ is finite. The goal of this exercise is to prove this.

(a) Prove the following lemma using induction.

**Lemma 0.5.** *Let $(S, \preccurlyeq)$ denote a finite poset, and let $x, y \in S$ such that: (i) $x \prec y$ and (ii) there exist at most $n \in \mathbb{N}$ elements $z \in S$ satisfying $x \prec z \prec y$. Then either $x \lhd y$, or there exist $z_1, z_2, \ldots, z_k \in S$ such that $x \lhd z_1 \lhd \cdots \lhd z_k \lhd y$.*

**Response:** Let us define an increasing sequence of k elements between two elements x,y to be $z_1, z_2 ... z_k$ such that $x \prec z_1 \prec z_2 ... z_k \prec y$ .And longest increasing sequence would be one which has largest value of its length , k .
If for x,y ($x \prec y$)we have longest increasing sequence between them with $k = 0$ that means that $x \lhd y$ because if there were atleast one $z$ such that $x \prec z \prec y$ then longest increasing subsequence must be having $k \geq 1$, but here $k = 0$ so no such $z$ exists.

If longest increasing sequence between x,y has length $k \geq 1$, Let us inductively assume that for a POSET$(S, \preccurlyeq)$,for every pair of elements a,b ($a \prec b$)with longest increasing sequence of length $l$ between a,b (sequence $z_1, ... z_l$), where $l < k$ , we will have $a \lhd z_1 \lhd ... z_l \lhd b$.

We can write $x \prec z_1 \prec .... z_k \prec y$, where $z_1 ... z_k$ is the longest increasing sequence between x,y of length k .
Take elements $x, z_k$ . Observe that a longest increasing sequence between these two elements must be $z_1, z_2 ... z_{k-1}$ which is of length k-1. If there were a longer sequence then there would also be a longer sequence for x,y. Say we have for some $c > k - 1$, that $x \prec w_1 \prec ... w_c \prec z_k$. Then we can write $x \prec w_1 \prec ... w_c \prec z_k \prec y$ by transitivity of the relation. Here x,y has increasing sequence of length c+1 between them which is more than k which is a contradiction .
Hence a longest increasing sequence of $x, z_k$ is $z_1, z_2 ... z_{k-1}$ . By our inductive assumption we can say that $x \lhd z_1 \lhd ... z_{k-1} \lhd z_k$.
Now we shall claim that not only is $z_k \prec y$ but also $z_k \lhd y$. How can we say this? If this was not the case there would exist some $w$ such that $z_k \prec w \prec y$, and then we would have that $x \prec z_1 \prec z_2 ... z_k \prec w \prec y$. Then $z_1, z_2 ... z_k, w$ would be an increasing sequence between x,y. But this would result in a contradiction as this is of length k+1 and we assumed longest increasing sequence between x,y was of length k.
Hence we have $z_k \lhd y$. So we have $x \lhd z_1 \lhd ... z_{k-1} \lhd z_k$ and we have that $z_k \lhd y$.
Hence we have $x \lhd z_1 \lhd ... z_{k-1} \lhd z_k \lhd y$.
Hence if $x \prec y$ then either $x \lhd y$ or there exist $z_1, z_2, \ldots, z_k \in S$ such that $x \lhd z_1 \lhd \cdots \lhd z_k \lhd y$.

Hence proved. □

(b) Use Lemma 0.5 to deduce the following theorem.

**Theorem 0.6.** *Let $(S, \preccurlyeq)$ denote a finite poset, and let $\lhd$ denote the corresponding immediate predecessor relation. Then for any two elements $x, y \in S$: $x \prec y$ if and only if either $x \lhd y$, or there exist elements $z_1, z_2, \ldots, z_k \in S$ such that $x \lhd z_1 \lhd \cdots \lhd z_k \lhd y$.*

**Response:** In Lemma 0.5 we proved that : Let $(S, \preccurlyeq)$ denote a finite poset, and let $\lhd$ denote the corresponding immediate predecessor relation. Then for any two elements $x, y \in S$: if $x \prec y$ then $x \lhd y$, or there exist elements $z_1, z_2, \ldots, z_k \in S$ such that $x \lhd z_1 \lhd \cdots \lhd z_k \lhd y$.
If we prove the converse of this also , then Theorum 0.6 will have been proven.
(1) If $x \lhd y$ then $x \prec y$. As this is stated in the start of this question. (2) If $z_1, z_2, \ldots, z_k \in S$

such that $x \lhd z_1 \lhd \cdots \lhd z_k \lhd y$, then we can say $x \prec z_1 \prec \cdots \prec z_k \prec y$ and then by transitivity property of relation we can say $x \prec y$. Hence the converse is proved.

As Lemma 0.5 and its converse are proved Theorum 0.6 is proved.

$\square$

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6. [A COUNTRY WITH DIRECT ONE-WAY ROAD JOINING EACH PAIR OF CITIES] [12]

In this exercise, we will prove the following surprising fact using induction.

Consider a country with $n \geq 1$ cities — where there is a direct[3] one-way road joining each pair of cities. There exists a city that can be reached from every other city either directly, or indirectly via exactly one city.

(a) Write down a theorem statement that captures the fact stated in the above paragraph — in the language/terminology of digraphs. (You may introduce new definitions and/or notation — if required — for the sake of convenience. Such definitions and/or notation, if any, should be clearly stated before the theorem statement.)

**Response:** Let us define a relation $R$ on the $V(D)$ where D is a digraph. For any two vertices $a, b \in V(D), a\ R\ b$ if arc (a,b) is part of the digraph or arcs (a,c) and (c,b) are part of the digraph for some vertex $c \in V(D) - \{a, b\}$. (or both are true)

**Theorum:** There is a digraph $D_n$ of n vertices which is an orientation of a complete graph. (i.e between every pair of vertices there exist precisely one arc , ex: for vertices a,b precisely one arc of (a,b) and (b,a) exists but not both)
In this digraph there exist atleast one vertex in the vertex set of the digraph ($a \in V(D_n)$) such that $\forall x \in V(D_n) - a$ we have $x\ R\ a$ .

(b) Prove the theorem you have stated in part (a) using induction. (Marks will be awarded only if part (a) is correct.)

**Response:** Notation : For a digraph D , Define hop parameter of a vertice $a \in V(D)$ to be the number of vertices $x \in V(D) - a$ such that $x\ R\ a$.

Let $D_n$ be a digraph of n vertices which is an orientation of $k_n$ complete graph.
For a digraph of n=1, we have that the single vertex $a$. As $V(D_1) - a = \phi$, this vertex satisfies the theorum. For all vertices in $x \in \phi$ , we have that $x\ R\ a$.

Assume inductively that the theorum holds for all digraphs that have less than or equal to n-1 vertices. Take a vertex x in the the digraph $D_n$ and remove it. We will get another digraph , let us denote this with $D_{n-1}$. Clearly $D_{n-1}$ is an orientation of the complete graph $k_{n-1}$.
By our assumption we have that $D_{n-1}$ satisfies the theorum , there exist $a$ such that $x\ R\ a$ for all $x \in V(D_{n-1}) - a$.
$y \in S_1$ if and only if $y \in V(D_{n-1}) - a$ and arc (y,a) exists.
$y \in S_2$ if and only if $y \notin S_1$ ,$y \in V(D_{n-1}) - a$ and arcs (y,c) and (c,a) exist for some $c \in S$. (note that obviously $c \in S_1$ because arc (c,a) exists.)

Observe $S_1 \cup S_2 \cup a = V(D_{n-1})$. Every element in vertex set of $V(D_{n-1})$except for $a$ will be in one of the sets $S_1$ or $S_2$ because for all vertices in $y \in V(D_{n-1}) - a$ we have (y,a) arc exists or (y,c) and (c,a) exists (or both) . Also see $S_1 \cap S_2 = \phi$ as elements of $S_1$ will not be in $S_2$ by definition.

Now we get to the fun part. Lets talk about the vertex x in $D_n$.
Cases :
(1)If the arc (x,a) exists then we are done . Because then $x\ R\ a$ and $\forall y \in V(D_{n-1}) - a, y\ R\ a$ by assumption. As $V(D_n) = V(D_{n-1}) \cup \{x\}$, we have for all elements $y \in V(D_n) - a$ we have $y\ R\ a$. So we are done.
(2) Moving forward , if arc (x,c) exists for some $c \in S_1$ then also we are done. Because (c,a) arc also exists. So $x\ R\ a$. So similarly as previous case here too every element of $V(D_n) - a$ is related

[3]Here, 'direct' means that it does <u>not</u> go through any other city

to $a$.
(3) If both above cases are not true , then $\forall y \in S_1$ arc (y,x) exists and arc (a,x) exists.
So ,here :
- $a \; R \; x$.
- $\forall y \in S_1, y \; R \; x$
- for $y \in S_2$ we have arcs (y,c) and (c,x) exist where $c \in S_1$. So $y \; R \; x$

As clearly , $S_1 \cup S_2 \cup a = V(D_n) - x$ all elements of $V(D_n) - x$ are related to x here. So we are done.

So, if theorum true for all digraphs which are orientations of complete graphs with less than or equal to n-1 vertices then the theorum is true for all digraphs which are orientations of complete graph of n vertices.
Hence proved inductively.

□