DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II **Problem Set 0 - Solutions** **MAR-JUN 23**

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1. (a) Let

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$
$$\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$$

Then,

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\,\mathbf{B} &= \left[(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})\cdot\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\right](B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\
&= \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\
&= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) + \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right) \\
&\qquad + \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)
\end{aligned}$$

(b) The definition of $\hat{\mathbf{r}}$ is

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$$

We have to compute $(\hat{\mathbf{r}}\cdot\nabla)\,\hat{\mathbf{r}}$. Let's first compute the $x-$ component of the quantity.

$$\begin{aligned}
[(\hat{\mathbf{r}}\cdot\nabla)\,\hat{\mathbf{r}}]_x &= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \frac{1}{\sqrt{x^2+y^2+z^2}}\left[x\left(\frac{1}{\sqrt{x^2+y^2+z^2}} + x\left(-\frac{1}{2}\right)\frac{1}{(x^2+y^2+z^2)^{3/2}}2x\right)\right] \\
&\quad + \frac{1}{\sqrt{x^2+y^2+z^2}}\left[y\left(x\left(-\frac{1}{2}\right)\frac{1}{(x^2+y^2+z^2)^{3/2}}2y\right)\right] \\
&\quad + \frac{1}{\sqrt{x^2+y^2+z^2}}\left[z\left(x\left(-\frac{1}{2}\right)\frac{1}{(x^2+y^2+z^2)^{3/2}}2z\right)\right] \\
&= \frac{1}{r}\left[\frac{x}{r} - \frac{1}{r^3}(x^3 + xy^2 + xz^2)\right] \qquad (r = |\mathbf{r}|) \\
&= 0
\end{aligned}$$

Similarly, one can show the other components to be zero. Hence,

$$\boxed{(\hat{\mathbf{r}}\cdot\nabla)\,\hat{\mathbf{r}} = 0}$$

2. Let $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ then

$$\boxed{\nabla \cdot \mathbf{r} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = 1 + 1 + 1 = 3}$$

$$\boxed{\nabla \times \mathbf{r} = \hat{\mathbf{x}}\left(\frac{\partial y}{\partial z} - \frac{\partial z}{\partial y}\right) - \hat{\mathbf{y}}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0}$$

$$\begin{aligned}
\nabla \cdot (\hat{\mathbf{n}} f(r)) &= \nabla\left(\frac{\hat{\mathbf{r}}}{r} f(r)\right) \qquad (r = |\mathbf{r}|) \\
&= \frac{\partial}{\partial x}\left(\frac{x f(r)}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y f(r)}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z f(r)}{r}\right)
\end{aligned}$$

Let's consider the first term

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{x f(r)}{r}\right) &= \frac{f(r)}{r} + x\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial x} \qquad (r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}) \\
&= \frac{f(r)}{r} + \frac{x^2}{r^2}\left(\frac{\partial f(r)}{\partial r} - \frac{f(r)}{r}\right)
\end{aligned}$$

Similarly, one can show

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{y f(r)}{r}\right) &= \frac{f(r)}{r} + \frac{y^2}{r^2}\left(\frac{\partial f(r)}{\partial r} - \frac{f(r)}{r}\right) \\
\frac{\partial}{\partial z}\left(\frac{z f(r)}{r}\right) &= \frac{f(r)}{r} + \frac{z^2}{r^2}\left(\frac{\partial f(r)}{\partial r} - \frac{f(r)}{r}\right)
\end{aligned}$$

Now substituting in

$$\boxed{\nabla \cdot (\hat{\mathbf{n}} f(r)) = \frac{3f(r)}{r} + \frac{\partial f(r)}{\partial r} - \frac{f(r)}{r} = \frac{2f(r)}{r} + \frac{\partial f(r)}{\partial r}}$$

$$\begin{aligned}
\nabla \times (\hat{\mathbf{n}} f(r)) = \hat{\mathbf{x}}&\left(\frac{\partial}{\partial y}\left(\frac{z}{r}f(r)\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}f(r)\right)\right) \\
- \hat{\mathbf{y}}&\left(\frac{\partial}{\partial x}\left(\frac{z}{r}f(r)\right) - \frac{\partial}{\partial z}\left(\frac{x}{r}f(r)\right)\right) \\
+ \hat{\mathbf{z}}&\left(\frac{\partial}{\partial x}\left(\frac{y}{r}f(r)\right) - \frac{\partial}{\partial y}\left(\frac{x}{r}f(r)\right)\right)
\end{aligned}$$

The $\hat{\mathbf{x}}$ component of the desired quantity

$$\frac{\partial}{\partial y}\left(\frac{z}{r}f(r)\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}f(r)\right) = z\left(\frac{1}{r}\frac{\partial f(r)}{\partial r} - \frac{f(r)}{r^2}\right)\frac{y}{r} - y\left(\frac{1}{r}\frac{\partial f(r)}{\partial r} - \frac{f(r)}{r^2}\right)\frac{z}{r} = 0$$

In the same way one can show $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ component also vanishes. Therefore,

$$\boxed{\nabla \times (\hat{\mathbf{n}} f(r)) = 0}$$

3. (a) Given that $\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$,

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) \\
&= 0
\end{aligned}$$

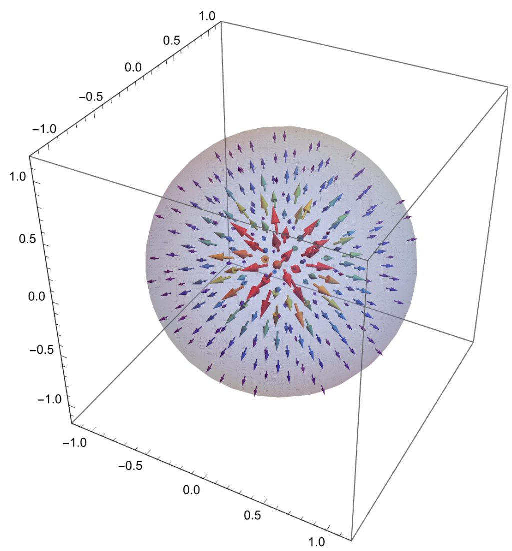

Figure 1: Vector plot of the field $\frac{\hat{\mathbf{r}}}{r^2}$.

Isn't the above result bit surprising? Yes. Because, clearly the vector plot shows the the field is diverging away from origin. In fact, if ever there were a function that ought to have a large positive divergence, this is it. But, the computation shows null divergence. Moreover, you see our field itself is infinite at the origin. Hence, The correct statement is as follows:

$$\boxed{\textit{The divergence of } \mathbf{v} \textit{ is zero at every point except at the origin.}}$$

Then what happens at the origin? That's exactly we will answer in the next part of the question.

(b) Recall the divergence theorem from PH1010. We will consider the volume inside the closed surface at $|\mathbf{r}| = r_0$.

$$\oint \mathbf{v} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{v})\, dV$$

The left hand side surface integral,

$$\begin{aligned}\oint \mathbf{v} \cdot d\mathbf{a} &= \int \left(\frac{1}{r_0^2}\hat{\mathbf{r}}\right) \cdot (r_0^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}) \\ &= \left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right) \\ &= 4\pi\end{aligned}$$

But the right hand side volume integral, $\int (\nabla \cdot \mathbf{v})\, dV$ is zero if we believe our computation in part (a) to be true. Does this imply the divergence theorem is false?

Explanation: The source of the problem arises from the infinite value of the field at the origin. It is obviously true that $\nabla \cdot \mathbf{v} = 0$ at every where except at the origin. Notice in the computation above, the left hand side surface integral of the divergence theorem does not depend on r_0. That means, if we consider the fact that divergence theorem is true, we will always get $\int \nabla \cdot \mathbf{v} dV = 4\pi$, how small the sphere centered at the origin we consider. In other words, we can conclude, the whole contribution is coming from the point $\mathbf{r} = 0$. Therefore, we see that $\nabla \cdot \mathbf{v}$ has the peculiar property that it vanishes everywhere except at one point and yet its integral over any volume containing the point is 4π. What kind of function has such a property? The answer is "Dirac delta function".[1](refer Section 1.5.2 of[1]). Therefore, we finally conclude

$$\boxed{\nabla \cdot \mathbf{v} = 4\pi\delta^3(\mathbf{r})}$$

4. (a)

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta(\mathbf{r} - \mathbf{r}_i)$$

Total charge

$$\begin{aligned}Q &= \int_V \rho(\mathbf{r}) dV \\ &= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} \sum_{i=1}^{N} q_i \delta(\mathbf{r} - \mathbf{r}_i) dxdydz \\ &= \sum_i q_i\end{aligned}$$

[1]Strictly speaking from mathematics point of view Dirac delta is not a function rather a distribution.

(b) Total charge

$$Q = \int \lambda dl = R\int_{\phi=0}^{2\pi} \lambda(\phi)d\phi$$

Let's consider the volume charge distribution as

$$\rho(\mathbf{r}) = A(\phi)\delta\left(\theta - \frac{\pi}{2}\right)\delta(r-R)$$

Because the radius of the circle is R and $\theta = \pi/2$ indicates XY plane. As λ is a function of ϕ, A must be a function of ϕ.

$$\begin{aligned}
Q &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} A(\phi)\delta(\theta-\pi/2)\delta(r-R)r^2\sin\theta dr d\theta d\phi \\
\implies R\int_{\phi=0}^{2\pi}\lambda d\phi &= \int_0^{2\pi} A(\phi)\int_0^{\pi}\delta(\theta-\pi/2)\sin\theta d\theta\int_0^{\infty}\delta(r-R)r^2 dr \\
\implies R\int_{\phi=0}^{2\pi}\lambda d\phi &= R^2\int_0^{2\pi} A(\phi)d\phi
\end{aligned}$$

Hence, $A(\phi) = \frac{\lambda(\phi)}{r}$

$\boxed{\rho(\mathbf{r}) = \frac{\lambda(\phi)}{r}\delta(\theta-\pi/2)\delta(r-R)}$

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics (4th Edition)*. Addison-Wesley, 2013.