A New Result on Parker's Square Sum Conjecture

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Abstract

The square-sum problem asks if there exists a sequence with the integers from 1 to some arbitrary n with the property that each pair of adjacent numbers sums to a square number. Parker's conjecture on the square-sum problem states that such a sequence exists for all $n \geq 25$. We outline a constructive proof that there is a sequence with the square-sum property if $n = 16 \cdot \frac{k(k+1)}{2}$, where k is an integer.

1 Introduction

Consider the sequence:

$$8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9$$

This sequence has the property that every integer from 1 to 15 is used exactly once, and even more remarkably that every adjacent number sums to a square number. This sequence is one of two sequences of the first 15 natural numbers with that property, the only other sequence with these two properties being the reverse of the sequence.

Because $9 + 16 = 25$, we can add 16 to the end of the sequence and preserve the property where adjacent numbers in the sequence sum to a square:

$$8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9, \underline{16}.$$

Similarly, we can add 17 to the beginning of the sequence, as $17 + 8 = 25$:

$$\underline{17}, 8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9, 16.$$

It turns out that we cannot add 18 to either end of the sequence, as $18 + 17 = 35$ and $18 + 16 = 34$, neither of which are squares. This does not prove that it is impossible to construct a sequence of the integers from 1 to 18 with the property that each adjacent number in the sequence sum to a square number. The problem of trying to find these kinds of sequences has come to be known as the **square-sum** problem.

The square-sum problem was first posed by Colombian mathematician Bernardo Recamán Santos. The problem asks us to arrange the set of positive integers from 1 to n in such a way that every pair of adjacent numbers in the arrangement sum to a perfect square, and every number is used exactly once. It can be proven that is impossible for $n < 15$, but the first time it is possible (with the exception of the trivial case of just 1) is when $n = 15$ (which will be shown in section 2.4). Since Santos, it has been found to be possible for more and more n.

In November of 2015, notable mathematician and internet personality Matt Parker released his book titled Things to Make and Do in the Fourth Dimension, in which he talks about the square-sum problem, and remarks that it is known whether or not this can be done for n up to 89. Parker further discussed the problem on the "Numberphile"

YouTube channel[1], where he stated a colleague of his named Charlie Turner had verified for n up to 299, finding solutions for all of them beginning at 25. Because of this, it was conjectured by Parker and others that there is always a solution for any $n \geq 25$.

Obviously the best way to work on the square sum problem is not to check every possible sequence with the first n natural numbers, as by the time we get to $n = 25$, there are already about $1.5 \cdot 10^{25}$ different things to check, which isn't feasible even for the fastest computers.

A better way to complete the problem is to construct a diagram where we put the numbers in dots and draw lines between dots with numbers that sum to squares. The diagram for the numbers 1 to 15 is shown below:

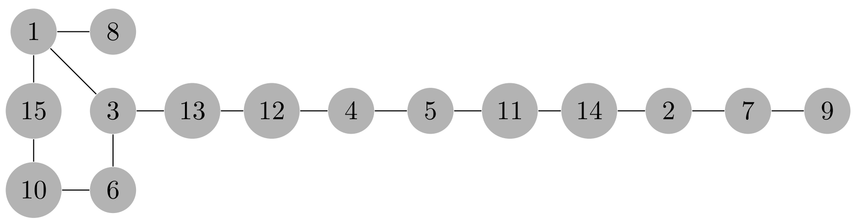

Figure 1: There are lines drawn between the numbers that sum to a perfect square

We can now use this diagram to reconstruct our sequence for $n = 15$:

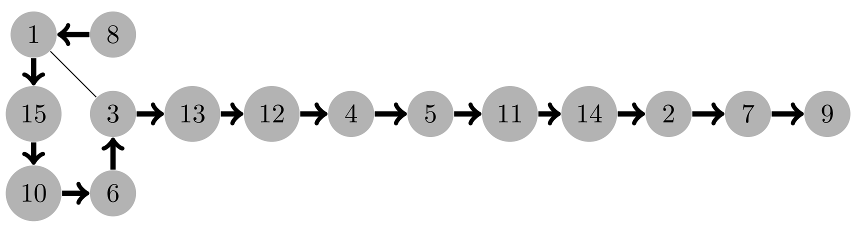

Figure 2: The same diagram as Figure 1, but with the sequence we had earlier highlighted by arrows.

We can now use the same steps as before to construct a similar diagram for the integers 1 to 18, and hopefully find a sequence:

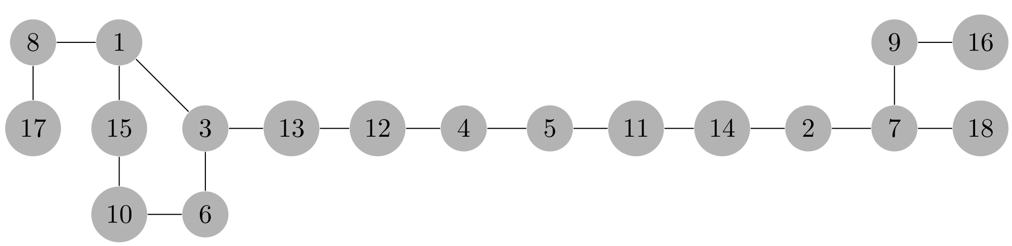

Figure 3: The same kind of diagram as before, but this time with the numbers 1 to 18

In the diagram for 18, we see that 17 only connects to 8. Because of this we know that 17 must be at the beginning of the sequence or the end. The same logic applies to 16 and 18. Because there are 3 numbers and only 2 endpoints of the hypothetical sequence, we cannot construct a sequence for 18.

[1] The two videos can be found at `https://youtu.be/G1m7goLCJDY` and at `https://youtu.be/7_ph5djCCnM`

Creating one of the diagrams and finding a way to visit each of the dots is equivalent to finding one of the special sequences. Mathematicians call the diagrams a **graph**, and the arrow method we used to visit each of the dots on the graph is known as a **Hamiltonian path**.

The diagrams in figures 1, 2, and 3 are all graphs. The dots we put the numbers in are the vertices, and the lines are edges.

Definition 1.1. $\boxed{n}$ *is the graph on* n *vertices labeled with the integers from* 1 *to* n*, and two vertices are connected with an edge if they sum to a square number.*

Example 1.2.

Parker conjectures that there is a Hamiltonian path on $\boxed{n}$ for all $n \geq 25$, figure 4 below shows $\boxed{25}$.

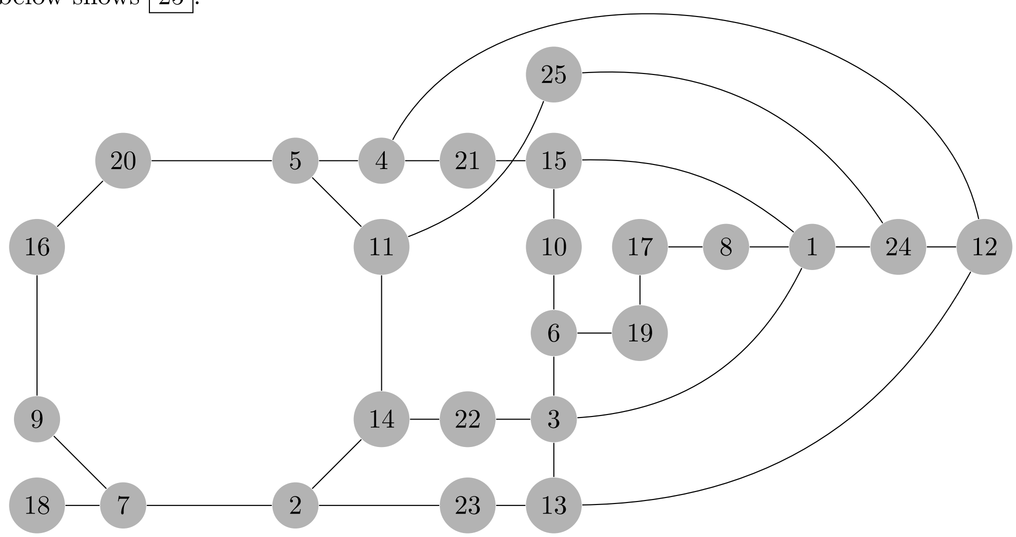

Figure 4: $\boxed{25}$

Remark 1.3. *There are 10 Hamiltonian paths on* $\boxed{25}$*, can you find one or all of them?*

2 Graph Theory

It is known that finding Hamiltonian paths is NP-complete, meaning it is extraordinarily difficult to do on graphs with lots of vertices (assuming $P \neq NP$). This means that it would be the easiest to prove a result for the square-sum if eventually the graph does not have any Hamiltonian paths.

We informally defined graphs and Hamiltonian paths in the introduction, but we will give formal definitions of each of them here.

Definition 2.1. *A* <u>*graph*</u> G *is a finite set* V *of* <u>*vertices*</u>*, and a subset* E *of the set of all pairs or vertices in* V *called* <u>*edges*</u>*.*

Definition 2.2. *A* <u>*walk*</u> *on graph* $G = (V, E)$ *from vertex* $x \in V$ *to vertex* $y \in V$ *is a list* $x, v_1, v_2, \ldots, v_l, y$ *of vertices of* G *such that* $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_l, y\} \in E$*.*

Definition 2.3. *A* <u>*path*</u> *is a walk that does not repeat vertices.*

Definition 2.4. *A* <u>*Hamiltonian path*</u> *(HP) is a path that contains every vertex. If the Hamiltonian path begins and ends on two vertices that share an edge, then it is called a* <u>*Hamiltonian cycle*</u> *(HC).*

2.1 Theorems to Show G Has No Hamiltonian Path

Definition 2.5. *A graph G is called* <u>connected</u> *if for all pairs of distinct vertices in G, there exists a walk between them.*

Theorem 2.6. *If a graph is not connected, then it has no Hamiltonian path.*

Proof. The contrapositive of this theorem is: If a graph has a Hamiltonian path, then it is also connected. This is a much easier proof.

Let $P = v_1, v_2, \ldots, v_n$ be a Hamiltonian path on graph G. By definition P contains all vertices of G. We find a walk from v_i to v_j with 2 cases:

1. For $i < j$, the walk is $v_i, v_{i+1}, v_{i+2}, \ldots, v_{j-1}, v_j$.
2. For $i > j$, the walk is $v_i, v_{i-1}, v_{i-2}, \ldots, v_{j+1}, v_J$.

Therefore if G has a Hamiltonian path, it is connected.

Therefore, by contrapositive, if G is not connected, then G has no Hamiltonian path. ■

Before we go into the next theorem, we must define another property of graphs.

Definition 2.7. *The* <u>degree</u> *of a vertex is the number of edges it is adjacent to.*

Lemma 2.8. *A vertex v of degree 1 must be at the start or end of a Hamiltonian path.*

Proof. Let P be a Hamiltonian path or cycle on graph G. Let vertex v a vertex of degree 1 that is not at the start or end of P. Let vertex w be the vertex v shares an edge with.

Because v is not at the start or end of P, it must be adjacent to two vertices in the sequence. Because v is degree 1, the only vertex it is adjacent to is vertex w, thus w must appear in P twice. Because w appears in P twice, P is not a Hamiltonian path or cycle.

Therefore, by contradiction, a vertex with degree 1 must be at the start or end of a Hamiltonian path. ■

Theorem 2.9. *If a graph has 3 or more vertices of degree 1, then it has no Hamiltonian path.*

Proof. let P be a Hamiltonian path on G, and let vertices v_1, v_2, and v_3 be vertices of degree 1.

Because of lemma 2.8, we know that none of v_1, v_2, or v_3 can be in the middle of P. Without loss of generality, we place v_1 at the beginning of P, and v_2 at the end of P. This leaves v_3 to be in the middle of P.

Therefore, by contradiction, a graph with 3 or more vertices of degree 1 does not have a Hamiltonian path. ■

Theorem 2.10. *If a graph has 1 or more vertices of degree 1, then it has no Hamiltonian cycle.*

Proof. let C be a Hamiltonian cycle on G, and let vertex v be a vertex of degree 1.

Lemma 2.8 tells us that v cannot be in the middle of a path or cycle. Because a cycle is a circular path, every vertex is in the middle of the cycle. If v is in C, then C is not a Hamiltonian cycle.

Therefore, by contradiction, a graph with 1 or more vertices of degree 1 does not have a Hamiltonian cycle. ■

Often there are too many vertices on a graph to analyze it easily. If there is a way to make modifications to a graph which will make it easier to analyze, but will preserve the structure and complexities of the original, it will make our job much easier.

Definition 2.11. *Let G be a graph with at least one vertex of degree 2. Let v be the vertex of degree 2, and let the two vertices it is adjacent to w_1 and w_2, we can remove vertex v from the graph and add a new important edge that connects w_1 to w_2. This process is called smoothing.*

Theorem 2.12. *Finding a Hamiltonian path on a smoothed out graph that uses all of its important edges is equivalent to finding a Hamiltonian path on a graph with no smoothing.*

Proof. Let G be a graph with a Hamiltonian path, and let G' be the resulting graph when we fully smooth G.

In the process of smoothing, we end up removing vertices from G. The fortunately, all of the vertices that are removed are on important edges. If we have a Hamiltonian path on G' that uses all of its important edges, then when we add all of the vertices back into G', because all of the removed vertices were on important edges, and all of the important edges are used in the Hamiltonian path, all of the removed vertices will be on the Hamiltonian path.

Therefore finding a Hamiltonian path on a smoothed graph that uses all of its important edges is equivalent to finding a Hamiltonian path on the same graph without any smoothing. ■

Theorem 2.12 tells us that important edges must be used in a Hamiltonian path on G', or else the corresponding path on G will not be Hamiltonian. We can now use the important edges to create more theorems that we can use to prove that a graph has no Hamiltonian path.

Definition 2.13. *A vertex becomes a terminal vertex if it has some property which will force a Hamiltonian path to have one of its endpoints at that vertex or one of the vertices adjacent to it.*

Example 2.14. *If G is a graph with 2 Hamiltonian paths, and has a vertex v of degree 1. If both of the Hamiltonian paths start at v, but one ends at vertex a, and the other ends at vertex b. If vertex a and vertex b are "far away" from each other, then v is the only terminal vertex because it being degree 1 forces the Hamiltonian paths to start (or end) there. Vertices a and b are not terminal vertices, because they evidently do not force any Hamiltonian paths on G to start or end on them or any vertices adjacent to them.*

The following is a list of some of the criteria for a graph to have a terminal vertex:

1. A vertex is degree 1
2. A vertex is adjacent to 3 important edges

Criterion 1 has been proven in lemma 2.8, but we have yet to prove criterion 2.

Proposition 2.15. *A vertex that is adjacent to 3 important edges is a terminal vertex.*

Proof. A vertex v with 3 important edges will look like figure 3 below. The Hamiltonian path will have to go through v once, and when it does it will take 2 of the important edges, leaving a third left unused. This forces the Hamiltonian path to come back to the vertex to use the important edge. In the final Hamiltonian path, it will end on the last vertex in the final important edge.

Therefore a vertex with 3 important edges is a terminal vertex. ■

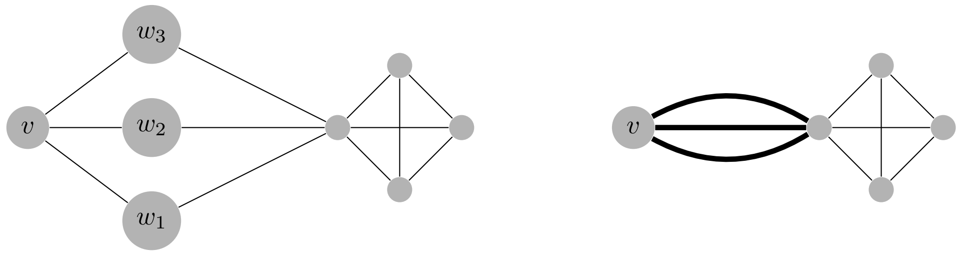

Figure 5: A part of a graph without smoothing (left) and with smoothing (right).

Definition 2.16. *The* <u>*terminal number*</u> *of a graph is a property that counts how many terminal vertices a graph has.*

Proposition 2.17. *A vertex adjacent to 4 important edges is a terminal vertex that adds 2 to a graph's terminal number.*

Proof. The Hamiltonian path must go through the vertex once, and it still can only use 2 of the important edges. This leaves 2 important edges left to take, which means the graph must start and end at two of the vertices adjacent to the vertex.

Therefore a vertex with 4 important edges is a terminal vertex, but because it forces the graph to have both endpoints of a Hamiltonian path be at vertices adjacent to it, it adds 2 to the terminal number of a graph.. ■

Remark 2.18. *Proposition 2.17 generalizes to have the vertex add x to the terminal number of the graph when the vertex is adjacent to $x + 2$ important edges.*

Terminal vertices are made more complicated by "bridges".

Definition 2.19. *An edge in graph G is a* <u>*bridge*</u> *if when it is removed, the graph becomes disconnected.*

Example 2.20. *The dashed edge in figure 6 is a bridge.*

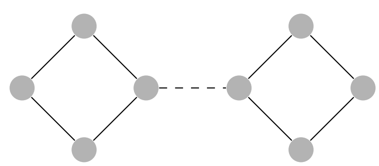

Figure 6: The bridge is the dashed edge

This adds a new way for a vertex to become a terminal vertex: if it is adjacent to 2 important edges as well as a bridge.

Proposition 2.21. *A vertex is a terminal vertex if it is adjacent to a bridge and two important edges.*

Proof. Let G be a graph. Let v be a vertex in G and be adjacent to a bridge and 2 important edges, and let P be a Hamiltonian path on G

Because a Hamiltonian path does not have any sort of intrinsic direction (it can be read backwards and forwards without changing anything about the path), we can have P cross the bridge to get to v without loss of generality. Once P has gotten to v, it has to make a choice between the two important edges. Whichever important edge P does not take after P will be at the end of P. See figure 7.

Therefore a vertex adjacent to a bridge and 2 important edges is a terminal vertex. ■

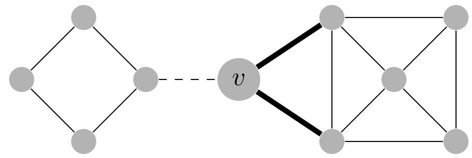

Figure 7: Vertex v is a terminal vertex because it is adjacent to two important edges and a bridge. The important edges are thickened and the bridge is dashed.

It is important to note that if a graph has a bridge, then a Hamiltonian path on that graph must use the bridge, and must visit every vertex on both sides of the bridge. Because we must travel through a vertex that is adjacent to a bridge to travel along the bridge we must visit every vertex on that part of the graph before we go the vertex that is adjacent to the bridge.

Now that we have all of the information relevant to terminal vertices, we can finally use them to find other ways to prove a graph does not have a Hamiltonian path or cycle.

Theorem 2.22. *If a graph has a terminal number of 3 or more, then it does not have a Hamiltonian path.*

Proof. Let G be a graph with a Hamiltonian path and terminal number 3.

When a graph has terminal number 3, there are 3 places where a Hamiltonian path must end. Because a path only has 2 endpoints, the pigeonhole principle tells us that one of the places where the Hamiltonian path must end (given by the terminal number) is not at the end of a Hamiltonian path.

Therefore, by contradiction, a graph with terminal number 3 has no Hamiltonian path ∎

We can adapt this proof for Hamiltonian cycles

Theorem 2.23. *If a graph has any terminal vertices, then it does not have a Hamiltonian cycle.*

Proof. Let G be a graph with a Hamiltonian cycle and at least 1 terminal vertex.

Because a Hamiltonian cycle does not have any endpoints, there is nowhere for the terminal vertex to be in the cycle.

Therefore, by contradiction, a graph with any terminal vertices does not have a Hamiltonian cycle. ∎

Theorem 2.24. *If a graph has a bridge and two terminal vertices on the same side of the bridge, then the graph has no Hamiltonian path.*

Proof. Let G be a graph with a bridge and two terminal vertices (v_1 and v_2) on the same side of the bridge. Let G have a Hamiltonian path called P. Let b_1 and b_2 be the two vertices adjacent to the bridge.

Without loss of generality we let P start at v_1. In order to visit vertices on the other side of the bridge, we must cross the bridge, visiting both b_1 and b_2. Because v_2 is a terminal vertex, we must have P end there. Because there is a bridge, the only way to go to the other side of the graph and end P at v_2 is to cross the bridge. When we do this we have visited both b_1 and b_2 twice, thus P is not a Hamiltonian path.

Therefore, by contradiction, a graph with two terminal vertices on the same side of a bridge does not have a Hamiltonian path. ∎

2.2 Applying the Theorems

Now we have all of the theorems we need to start exploring the graphs for $\boxed{2}$ to $\boxed{24}$ (Note that we will not be doing $\boxed{1}$ because that is trivial.).

Theorem 2.25. $\boxed{n}$ *is not connected for* $2 \leq n \leq 13$.

Proof. Because there is not a walk from vertex 2 to vertex 1 in any of $\boxed{2}$ to $\boxed{13}$, none of the graphs are connected, therefore by theorem 2.6, none of $\boxed{2}$ through $\boxed{13}$ have a Hamiltonian path. See the pictures in Figure 12 on the next page.

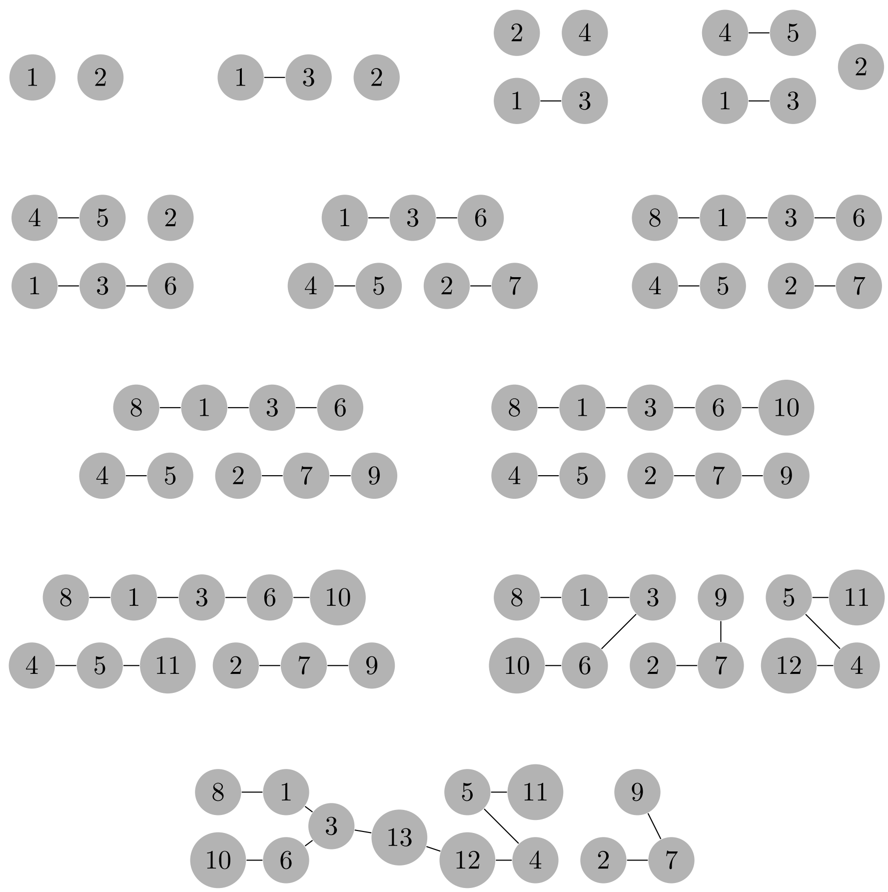

Figure 8: $\boxed{2}$ to $\boxed{13}$

■

Theorem 2.26. $\boxed{14}$ *does not have a Hamiltonian path.*

Proof. $\boxed{14}$ has 3 vertices of degree 1 (see figure 2 below), therefore by theorem 2.9, $\boxed{14}$ does not have a Hamiltonian path.

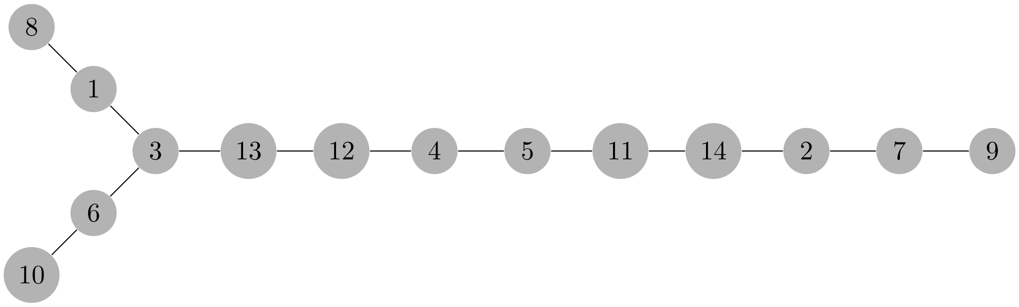

Figure 9: $\boxed{14}$

■

Theorem 2.27. $\boxed{15}$, $\boxed{16}$, *and* $\boxed{17}$ *all have a Hamiltonian path, as shown in the Introduction.*

Proof. See figure 3 below.

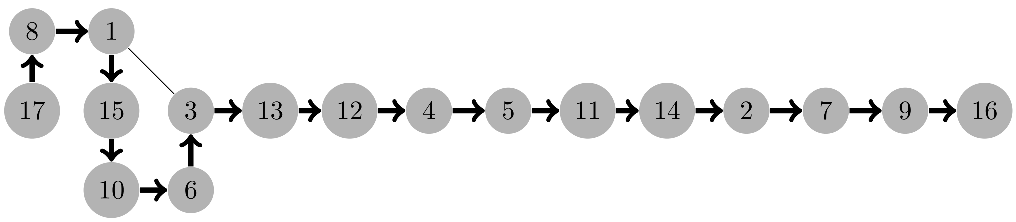

Figure 10: $\boxed{17}$ and its Hamiltonian path. For the Hamiltonian path for $\boxed{16}$, ignore vertex 17. For the Hamiltonian path on $\boxed{15}$, ignore vertex 16 and vertex 17

■

Theorem 2.28. *None of* $\boxed{18}$ *through* $\boxed{22}$ *have Hamiltonian paths.*

Proof. $\boxed{18}$ has 3 vertices of degree 1 (vertices 16, 17, and 18), therefore by theorem 2.9, $\boxed{18}$ does not have a Hamiltonian path. See figure 4 below.

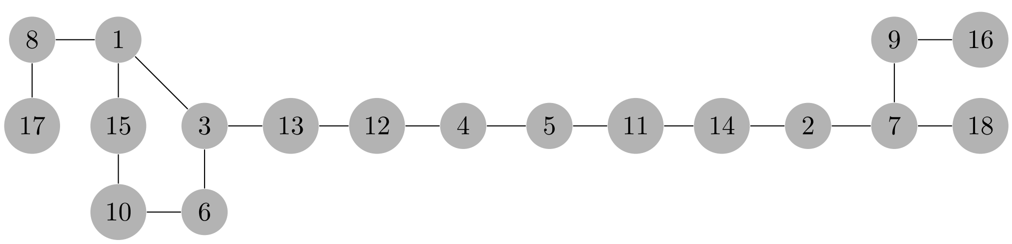

Figure 11: $\boxed{18}$

$\boxed{19}$ has two terminal vertices (vertices 16 and 18, because they are both degree 1) on the same side of a bridge (the edge connecting vertex 2 to vertex 7). Therefore by theorem 2.24, $\boxed{19}$ does not have a Hamiltonian path. See figure 5 below.

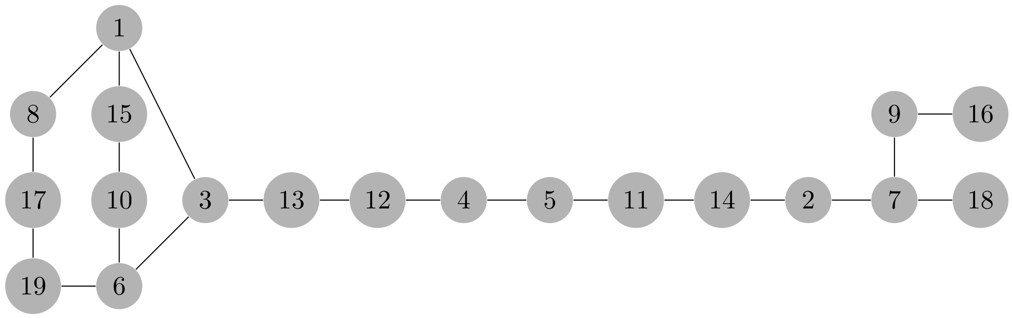

Figure 12: $\boxed{19}$

In $\boxed{20}$, vertex 18 is a terminal vertex because it is degree 1. The edge connecting vertices 18 and 7 is a bridge. Vertex 7 is also adjacent 2 chains of degree 2 vertices. Thus by proposition 2.21, vertex 7 is a terminal vertex. The edge between vertex 4 and vertex 5 is a bridge. Thus there are two vertices on the same side of a bridge and therefore, by theorem 2.24, $\boxed{20}$ does not have a Hamiltonian path. See figure 6 below.

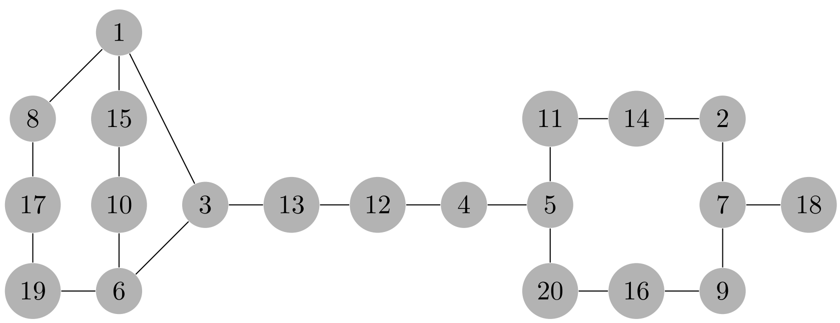

Figure 13: $\boxed{20}$

In $\boxed{21}$, vertex 18 is a terminal vertex because it is degree 1. The edge connecting vertices 18 and 7 is a bridge. Vertex 7 is also adjacent 2 chains of degree 2 vertices. Thus by proposition 2.21, vertex 7 is a terminal vertex. The edge between vertex 4 and vertex 5 is a bridge. Thus there are two vertices on the same side of a bridge and therefore, by theorem 2.24, $\boxed{21}$ does not have a Hamiltonian path. See figure 7 below.

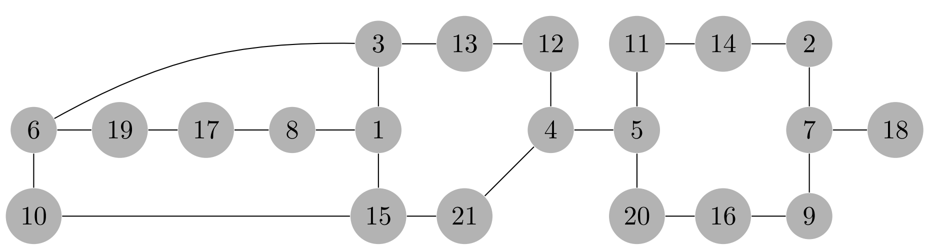

Figure 14: $\boxed{21}$

In $\boxed{22}$, vertex 18 is a terminal vertex because it is degree 1. The edge connecting vertices 18 and 7 is a bridge. Vertex 7 is also adjacent 2 chains of degree 2 vertices. Thus by proposition 2.21, vertex 7 is a terminal vertex. Vertex 14 is adjacent to 3 chains of degree 2 vertices. Thus by Proposition 2.15, 14 is a terminal vertex. Because $\boxed{22}$ has 3 terminal vertices, its terminal number is 3, and therefore by theorem 2.22 it does not have a Hamiltonian path.

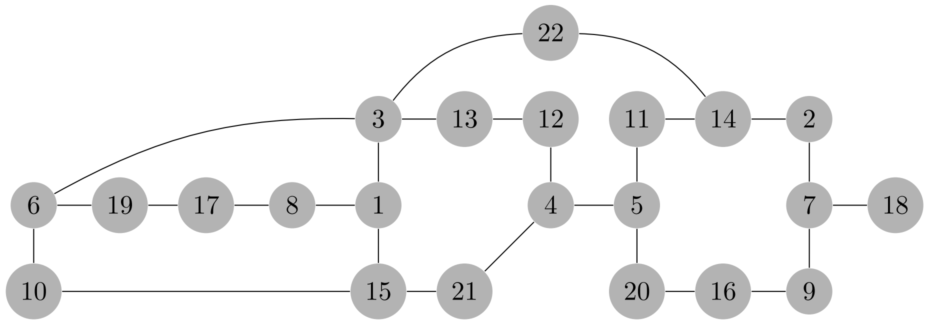

Figure 15: $\boxed{22}$

■

Theorem 2.29. $\boxed{23}$ *has a Hamiltonian path*

Proof. See figure 11 for the Hamiltonian path.

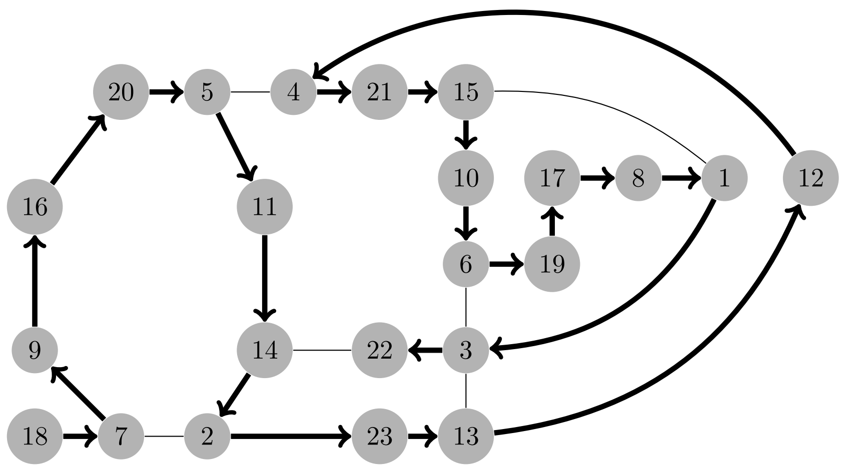

Figure 16: $\boxed{23}$ and its Hamiltonian path

■

Theorem 2.30. $\boxed{24}$ *does not have a Hamiltonian path.*

We omit the proof as it cannot be done without significant case work (as far as we know).

2.3 Degrees of Vertices

One of the easiest ways to prove that a graph has no Hamiltonian path is to use theorem 2.9; which states that a graph does not have a Hamiltonian path if it has 3 vertices of degree 1. Because of this we would like to have a formula for the degree of any vertex v in graph $\boxed{n}$. Let us do a couple examples before we prove a formula.

Example 2.31. *Vertex* 7 *in* $\boxed{50}$ *is degree 5. The vertices it connects to are 2* ($7+2=9$)*, 9* ($7+9=16$)*, 18* ($7+18=25$)*, 29* ($7+29=36$)*, and 42* ($7+42=49$).

Example 2.32. *Vertex* 8 *in* $\boxed{50}$ *is degree 4. The vertices it connects to are 1* $(8+1=9)$*, 17* $(8+17=25)$*, 28* $(8+28=36)$*, and 41* $(8+41=49)$*.*

From these two examples, we find that we are counting squares in a certain range. But we also notice that the degree of vertex 8 is one less than the degree of vertex 7. When we look at the squares, we see that vertex 7 has an edge with the vertex it sums to 16 with, but vertex 8 does not. This is because $8+8=16$, so vertex 8 would have to have an edge go from itself to itself, which we don't allow for in $\boxed{n}$.

Now we will find and prove a formula for the degree of vertex v in $\boxed{n}$

Lemma 2.33. *The number of integer squares in the interval* $[1,x]$ *is* $\lfloor\sqrt{x}\rfloor$*.*

Proof. let α^2 be in the interval: $[1,x]$. Therefore $\alpha \in [1, \sqrt{x}\ ]$. The number of integers in the interval $[1,\sqrt{x}]$ is $\lfloor\sqrt{x}\rfloor$. Therefore there are $\lfloor\sqrt{x}\rfloor$ perfect squares less than or equal to x. ∎

Definition 2.34. *Let* ϵ_v *be a function such that*

$$\epsilon_v = \begin{cases} 1 & 2v \text{ is a square} \\ 0 & \text{else} \end{cases}$$

Remark 2.35. *A closed-form formula for* ϵ_v *is* $\epsilon_v = \lfloor\sqrt{2v}\rfloor - \lceil\sqrt{2v}\rceil + 1$

Theorem 2.36. *The degree of vertex* v *in graph* $\boxed{n}$ *is* $\lfloor\sqrt{n+v}\rfloor - \lfloor\sqrt{v}\rfloor - \epsilon_v$

Proof. The largest valued vertex that vertex v can be connected to is vertex n. Therefore the sum $n+v$ is the largest sum that vertex v could possibly have in $\boxed{n}$. Lemma 2.33 tells us that there are $\lfloor\sqrt{n+v}\rfloor$ square numbers less than or equal to $n+v$. This gives us $\lfloor\sqrt{n+v}\rfloor$ as an upper bound for the degree of vertex v.

Because $\boxed{n}$ does not contain non-positive integers, there is no possible way for vertex v to sum to a square number that is less than or equal to itself. Lemma 2.33 tells us that there are $\lfloor\sqrt{v}\rfloor$ square numbers less than or equal to itself. We must subtract this from the rest of the formula.

Because $\boxed{n}$ is a simple graph, it does not allow for loops- edges that go from a vertex to itself. This means numbers which are exactly half of a square number will have one less edge than we would expect. This means we must subtract 1 from the formula whenever v is half of a square number, which is accomplished if we add ϵ_v to the formula.

Putting all of this together gives the desired formula of $\lfloor\sqrt{n+v}\rfloor - \lfloor\sqrt{v}\rfloor - \epsilon_v$ ∎

Corollary 2.37. *The degree of vertex* n *in* $\boxed{n}$ *is* $\lceil\sqrt{2n}\rceil - \lfloor\sqrt{n}\rfloor - 1$

Proof. We use Theorem 2.36 to find that the formula for the degree of vertex n in $\boxed{n}$ is $\lfloor\sqrt{n+n}\rfloor - \lfloor\sqrt{n}\rfloor - \epsilon_n$. We can use the equation in remark 2.35 to simplify the formula.

$$\begin{aligned} \text{degree}(n) &= \lfloor\sqrt{n+n}\rfloor - \lfloor\sqrt{n}\rfloor - \epsilon_n \\ &= \lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor + (\lfloor\sqrt{2n}\rfloor - \lceil\sqrt{2n}\rceil + 1) \\ &= \lceil\sqrt{2n}\rceil - \lfloor\sqrt{n}\rfloor - 1 \end{aligned}$$

∎

Remark 2.38. *Theorem 2.36 tells us that the degree of any fixed vertex goes to infinity as n goes to infinity. Therefore we can never use theorem 2.9 to prove that* $\boxed{n}$ *has no Hamiltonian path. Indeed for* $n \geq 31$ *there are no vertices of degree 1, and for* $n \geq 71$ *there are no vertices of degree 2.*

2.4 The Connectedness of $\boxed{n}$

In order to prove that $\boxed{n}$ is connected for $n \geq 14$, we must first prove the following lemma:

Lemma 2.39. *The degree of vertex n in $\boxed{n}$ is greater than 0 for $n \geq 5$*

Proof. The degree of vertex n is proven in Corollary 2.37 to be $\lceil\sqrt{2n}\rceil - \lfloor\sqrt{n}\rfloor - 1$. To find when the degree of vertex n is less than 1, we will solve the following inequality:

$$\begin{aligned} \text{degree}(n) &\leq 0 \\ \lceil\sqrt{2n}\rceil - \lfloor\sqrt{n}\rfloor - 1 &\leq 0 \end{aligned}$$

Because we can't directly do the algebra we want to floor and ceiling functions, we will have to make the following substitutions:

$$\begin{aligned} \theta_1 &= \lceil\sqrt{2n}\rceil - \sqrt{2n} \\ \lceil\sqrt{2n}\rceil &= \sqrt{2n} + \theta_1 \end{aligned}$$

$$\begin{aligned} \theta_2 &= \sqrt{n} - \lfloor\sqrt{n}\rfloor \\ \lfloor\sqrt{n}\rfloor &= \sqrt{n} - \theta_2 \end{aligned}$$

Note that $\theta_1, \theta_2 \in [0, 1)$. Now we substitute these back into our original inequality:

$$\begin{aligned} \lceil\sqrt{2n}\rceil - \lfloor\sqrt{n}\rfloor - 1 &\leq 0 \\ \sqrt{2n} + \theta_1 - \sqrt{n} + \theta_2 - 1 &\leq 0 \\ \sqrt{2n} - \sqrt{n} &\leq 1 - \theta_1 - \theta_2 \end{aligned}$$

To make this easier to work with, we will substitute $\theta_3 = 1 - \theta_1 - \theta_2$. This means that $\theta_3 \in (-1, 1]$, and $(\theta_3)^2 \in [0, 1]$. Finally we note that $3 - s\sqrt{2} > 0$, so when we divide by it later, we don't have to worry about the direction of the inequality.

$$\begin{aligned} \sqrt{2n} - \sqrt{n} &\leq 1 - \theta_1 - \theta_2 \\ \sqrt{2n} - \sqrt{n} &\leq \theta_3 \\ 2n + n - 2\sqrt{2n}\sqrt{n} &\leq (\theta_3)^2 \\ 3n - 2\sqrt{2n^2} &\leq (\theta_3)^2 \\ n(3 - 2\sqrt{2}) &\leq (\theta_3)^2 \\ n &\leq \frac{(\theta_3)^2}{3 - 2\sqrt{2}} \end{aligned}$$

The largest value for n for which this inequality is true is when $\theta_3 = 1$, which occurs when $n = \frac{1}{3-2\sqrt{2}} \approx 5.8$. This means that for $n \geq 6$, the degree of vertex n is at least 1, because it is an integer greater than zero. For the lemma, we need to prove this for $n \geq 5$. We do this by substituting $n = 5$ into the formula for the degree of n and finding that it is 1:

$$\begin{aligned} \text{degree}(n) &= \lceil\sqrt{2n}\rceil - \lfloor\sqrt{n}\rfloor - 1 \\ \text{degree}(5) &= \lceil\sqrt{2 \cdot 5}\rceil - \lfloor\sqrt{5}\rfloor - 1 \\ \text{degree}(5) &= 4 - 2 - 1 \\ \text{degree}(5) &= 1 \end{aligned}$$

Therefore the degree of vertex n is greater than or equal to 1 for $n \geq 5$. ∎

Theorem 2.40. $\boxed{n}$ *is connected for* $n \geq 14$.

Proof. We will use induction to prove that $\boxed{n}$ is connected for $n \geq 14$. Figure 2 above shows that $\boxed{14}$ is connected, making that our base case.

If $\boxed{n}$ is connected, then there is a walk between every pair of disjoint edges. When we add vertex $n+1$ to $\boxed{n}$ to make it $\boxed{n+1}$, lemma 2.39 tells us that it is degree at least 1 if $n \geq 5$. Because vertex $n+1$ is degree at least 1, it shares an edge with an edge with at least one vertex that already exists in $\boxed{n}$. Let w be a vertex that vertex $n+1$ shares an edge with.

To prove that $\boxed{n+1}$ is connected we must prove that there is a walk between every two pairs of distinct vertices. Because $\boxed{n}$ is connected there is already a walk between every pair of distinct vertices that doesn't include vertex $n+1$. This means that there are walks from every vertex in $\boxed{\text{n}}$ to vertex w. We can add vertex $n+1$ to the end of all of those walks to make walks from vertex $n+1$ to every other vertex in $\boxed{n+1}$.

Therefore, by induction, $\boxed{n}$ is connected for $n \geq 14$ ∎

3 Main Result

3.1 Introduction

Consider the following table

72	49	95	74
70	51	93	76
68	53	91	78
66	55	89	80
64	57	87	82
62	59	85	84
60	61	83	86
58	63	81	88
56	65	79	90
54	67	77	92
52	69	75	94
50	71	73	96
48			

Table 1: An interesting table

This table has some very special properties. It contains every integer in the interval $[48, 96]$. Each time the row number increases by one, the number in each column either increases or decreases by 2. If the table is read left to right and top to bottom, we find that each number in the table sums to a square number with its neighbors in a repeating pattern. The neighbors in columns 1 and 2 sum to $121 = 11^2$, the neighbors in columns 2 and 3 sum to $144 = 12^2$, the neighbors in columns 3 and 4 sum to $169 = 13^2$, and the neighbors in columns 4 and 1 sum to $144 = 12^2$.

These properties allow us to read a string of numbers with the intended property of the square-sum problem, but with some of the lower numbers missing. If we decide to create a graph like that of $\boxed{n}$, but only with the numbers on the table, we get something rather interesting:

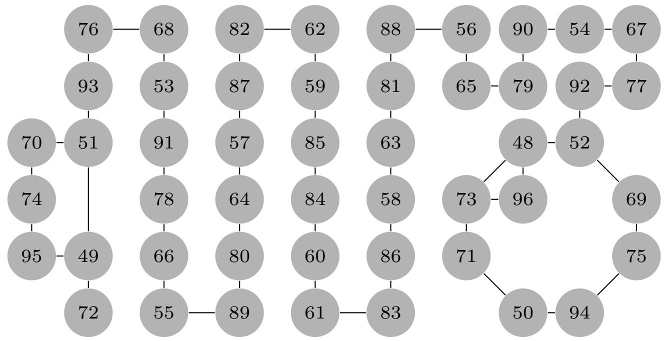

Figure 17: The numbers from 1 in the form of a graph with all of the desired edges

On the graph, there is an extremely long chain of degree 2 vertices (34 vertices long). This restricts the amount of freedom we have for a Hamiltonian path on such a graph, making it much easier to find one. If we could somehow string arbitrarily many of these tables/graphs together, we could create an arbitrarily long Hamiltonian path with the square-sum property.

It turns out that this is possible to do, but the way to do it is not immediately clear, and requires almost twice as many tables/graphs as what one would intuit.

3.2 Single Table Exploration

3.2.1 The Interval of a Table

Definition 3.1. *We will refer to the entry in row i and column j in the table with middle square m as $\underline{T_{i,j}^m}$. We will refer to the table with the middle square m as T^m.*

Definition 3.2. *We will refer to the path read from T^m as $\underline{T^m}$.*

The vertices in a table are depended upon the squares used in them. Those squares will always be $(m-1)^2$ for columns 1 and 2, m^2 for columns 2 and 3, $(m+1)^2$ for columns 3 and 4, and m^2 again for columns 4 and 1.

Theorem 3.3. *Each time we increase the row number of a column, the vertices in columns 2 and 4 increase by 2, and the vertices in columns 1 and 3 decrease by 2*

Proof. If we start at vertex $T_{0,1}^m$ which sums with vertex $T_{0,2}^m$ to be $(m-1)^2$, then $T_{0,2}^m = m^2 - T_{0,1}^m$. Then $T_{0,3}^m = (m+1)^2 - T_{0,2}^m$, $T_{0,4}^m = m^2 - T_{0,3}^m$, and $T_{1,1}^m = (m-1)^2 - T_{0,4}^m$. When we substitute all of these equations into one another, we get $T_{1,1}^m = (m-1)^2 - m^2 + (m+1)^2 - m^2 + T_{0,1}^m$. We will now simplify this simplify this equation to find out what happens to column 1 when we increase the row number by 1:

$$\begin{aligned} T_{1,1}^m &= (m-1)^2 - m^2 + (m+1)^2 - m^2 + T_{0,1}^m \\ &= m^2 - 2m + 1 - m^2 + m^2 + 2m + 1 - m^2 + T_{0,1}^m \\ &= m^2 \cdot (1 - 1 + 1 - 1) + m \cdot (2 - 2) + 2 + T_{0,1}^m \\ &= T_{0,1}^m + 2 \end{aligned}$$

Note that if we were to do the same for $T_{0,3}^m$ and $T_{1,3}^m$, we would get the same result, as we would be multiplying by -1 twice. Now we do the same process to column 2 to find out what happens when we increase the row number:

$$
\begin{aligned}
T^m_{1,2} &= m^2 - (m+1)^2 + m^2 - (m-1)^2 + T^m_{0,2} \\
&= m^2 - m^2 - 2m - 1 + m^2 - m^2 + 2m - 1 + T^m_{0,2} \\
&= m^2 \cdot (1 - 1 + 1 - 1) + m \cdot (2 - 2) - 2 + T^m_{0,2} \\
&= T^m_{0,2} - 2
\end{aligned}
$$

Note that if we were to do the same for $T^m_{0,4}$ and $T^m_{1,4}$, we would get the same result, as we would be multiplying by -1 twice.

Therefore columns 1 and 3 decrease by 2 when we increase the row number, and columns 2 and 4 increase by 2 when we increase the row number. ■

Lemma 3.4. *Integers that are half of a square number are even.*

Proof. let k be an integer, and let $h = \frac{k^2}{2}$.

If k is odd, then k^2 is odd, which would make h not an integer, so we don't care. If k is even, then there exists some integer l such that $k = 2l$. If we substitute this into our equation for h, we get:

$$
\begin{aligned}
h &= \frac{k^2}{2} \\
&= \frac{(2l)^2}{2} \\
&= \frac{4l^2}{2} \\
&= 2 \cdot l^2
\end{aligned}
$$

Therefore, because l^2 is an integer, and $h = 2 \cdot l^2$, h is even. Therefore integers that are half of a square number are even. ■

Theorem 3.5. *Columns 1 and 4 consist only of even numbers, and columns 2 and 3 consist only of odd numbers.*

Proof. First we must determine the entries in the 0th row of each column. As a rule we will always start with $T^m_{0,1} = \frac{m^2}{2}$. Then we have $T^m_{0,2} = (m-1)^2 - T^m_{0,1}$, $T^m_{0,3} = m^2 - T^m_{0,2}$, and $T^m_{0,4} = (m+1)^2 - T^m_{0,3}$.

Now we will substitute all of our equations and solve for each $T^m_{0,j}$.

$$\begin{aligned}
T^m_{0,1} &= \frac{m^2}{2} \\
\\
T^m_{0,2} &= (m-1)^2 - T^m_{0,1} \\
&= n^2 - 2m + 1 - \frac{n^2}{2} \\
&= \frac{m^2}{2} - 2m + 1 \\
\\
T^m_{0,3} &= m^2 - T^m_{0,2} \\
&= m^2 - \frac{m^2}{2} + 2m - 1 \\
&= \frac{m^2}{2} + 2m - 1 \\
\\
T^m_{0,4} &= (m+1)^2 - T^m_{0,3} \\
&= m^2 + 2m + 1 - \frac{m^2}{2} - 2m + 1 \\
&= \frac{m^2}{2} + 2
\end{aligned}$$

Because m is even by definition, $T^m_{0,1}$ is even by lemma 3.4. We will now find the parity of each $T^m_{0,j}$:

$$\begin{aligned}
T^m_{0,1} &\equiv \text{even} \\
\\
T^m_{0,2} &= \frac{m^2}{2} - 2m + 1 \\
&\equiv \text{even} - \text{even} + 1 \\
&\equiv \text{odd} \\
\\
T^m_{0,3} &= \frac{m^2}{2} + 2m - 1 \\
&\equiv \text{even} + \text{even} - 1 \\
&\equiv \text{odd} \\
\\
T^m_{0,4} &= \frac{m^2}{2} + 2 \\
&\equiv \text{even} + 2 \\
&\equiv \text{even}
\end{aligned}$$

When you add 2 to a number, it doesn't change its parity. Theorem 3.3 tells us that we only ever add 2 to a number in between columns. Therefore each column is comprised only of numbers of a single parity. We know the parity of all of the numbers in the 0th row, therefore the parity of each column is the same as the parity of their entries in the 0th row.

Therefore columns 1 and 4 are comprised only of even numbers, and columns 2 and 3 are comprised only of odd numbers. ■

Now that we can construct the tables much faster, the question arises: What interval of values will be in the table and path?

Theorem 3.6. *The table contains every integer in the interval* $[\frac{m^2}{2} - 2m,\ \frac{m^2}{2} + 2m]$.

Proof. We use the 0th row and theorem 3.3 to create the generalized table (note that r will represent the row number):

1	2	3	4
$\frac{m^2}{2}$	$\frac{m^2}{2} - 2m + 1$	$\frac{m^2}{2} + 2m - 1$	$\frac{m^2}{2} + 2$
$\frac{m^2}{2} - 2$	$\frac{m^2}{2} - 2m + 1 + 2$	$\frac{m^2}{2} + 2m - 1 - 2$	$\frac{m^2}{2} + 2 + 2$
$\frac{m^2}{2} - 4$	$\frac{m^2}{2} - 2m + 1 + 4$	$\frac{m^2}{2} + 2m - 1 - 4$	$\frac{m^2}{2} + 2 + 4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{m^2}{2} - 2r$	$\frac{m^2}{2} - 2m + 1 + 2r$	$\frac{m^2}{2} + 2m - 1 - 2r$	$\frac{m^2}{2} + 2 + 2r$

Because we want the table to represent a Hamiltonian path, we will end the table on the entry before we get a repeated value. Theorem 3.5 tells us that this can only happen between columns 1 and 4 or 2 and 3 due to parity. We also see in the generalized graph, that columns 1 and 4 are diverging as the row number increase, whereas columns 2 and 3 are converging as the row number increases. This gives us two possibilities: the repeated value will be in column 2 or in column 3.

If the table ends with an entry in column 2, then the repeated value will be in columns 2 and 3, in the same row. Using this we can set up the following equation to find when this will be the case, noting that m is even and r is an integer:

$$\begin{aligned} \frac{m^2}{2} + 2m - 1 - 2r &= \frac{m^2}{2} - 2m + 1 + 2r \\ 4m &= 2 + 4r \\ 2m &= 1 + 2r \\ 2 \cdot \text{even} &\equiv 1 + 2 \cdot \text{integer} \\ \text{even} &\equiv 1 + \text{even} \\ \text{even} &\equiv \text{odd} \end{aligned}$$

Thus, because we have a contradiction, the repeated value cannot be in the same row. Because the only other option for the repeated value is to have it be $T^m_{i,3}$ and $T^m_{i+1,2}$.

We set solve the following equation to find which row the repeated value will be in:

$$\begin{aligned} T^m_{r+1,2} &= T^m_{r,3} \\ \frac{m^2}{2} - 2m + 1 + 2(r+1) &= \frac{m^2}{2} + 2m - 1 - 2r \\ 2 + 2(r+1) + 2r &= 4m \\ 4r &= 4m - 4 \\ r &= m - 1 \end{aligned}$$

Thus the repeated value appears at $T^m_{m,2}$.

Therefore the full generalized table is:

1	2	3	4
$\frac{m^2}{2}$	$\frac{m^2}{2} - 2m + 1$	$\frac{m^2}{2} + 2m - 1$	$\frac{m^2}{2} + 2$
$\frac{m^2}{2} - 2$	$\frac{m^2}{2} - 2m + 1 + 2$	$\frac{m^2}{2} + 2m - 1 - 2$	$\frac{m^2}{2} + 2 + 2$
$\frac{m^2}{2} - 4$	$\frac{m^2}{2} - 2m + 1 + 4$	$\frac{m^2}{2} + 2m - 1 - 4$	$\frac{m^2}{2} + 2 + 4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{m^2}{2} - 2(m-1)$	$\frac{m^2}{2} - 2m + 1 + 2(m-1)$	$\frac{m^2}{2} + 2m - 1 - 2(m-1)$	$\frac{m^2}{2} + 2 + 2(m-1)$
$\frac{m^2}{2} - 2(m)$			

The interval of a given table T^m is $[\frac{m^2}{2}-2m, \frac{m^2}{2}+2m]$. We can use the updated table to find possible values for the upper and lower bounds of the interval. The lowest value in the table is $\frac{m^2}{2}-2m$, the last entry in column 1. The highest value in the table is not as obvious, but there are only two possibilities: the first entry in column 3 or the last entry in column 4. We find that the entry in column 4 is the highest by simplifying it:

$$\begin{aligned}
\frac{m^2}{2}+2+2(m-1) &> \frac{m^2}{2}+2m-1 \\
\frac{m^2}{2}+2+2m-2 &> \frac{m^2}{2}+2m-1 \\
\frac{m^2}{2}+2m &> \frac{m^2}{2}+2m-1
\end{aligned}$$

Thus we know that the interval is $[\frac{m^2}{2}-2m,\ \frac{m^2}{2}+2m]$. Now all that's left is to prove that every integer within the interval's range appears in the table.

Every number in the range is used in the tables, and there are no duplicates.

There are m rows of 4 columns, and one extra entry in column 1. This means that there are $4m+1$ entries in the table. The interval covers $\frac{m^2}{2}+2m-(\frac{m^2}{2}-2m)+1 = 4m+1$ integers. By the pigeonhole principle, proving that every integer is in the table is equivalent to proving that there are no duplicates in the table. We will prove that there is no duplicates.

There will be no duplicates in any column, as when the row number changes, the entry in the column always changes.

There will be no duplicates between the following pairs of columns due to parity: (1,2), (1,3), (2,4), (3,4).

Therefore we only need to check for duplicates between the following pairs of columns: (1,4) and (2,3).

We defined the table to have as many rows as possible without having a duplicate between columns 2 and 3, so by definition, the table does not contain any duplicates between columns 2 and 3.

Columns 1 and 4 start 2 away from each other and then diverge, so it is impossible to have duplicates between columns 1 and 4.

Because there cannot be duplicates between any pair of columns or in a single column, there are no duplicates in the entire graph.

Therefore, by the pigeonhole principle the interval contains every integer in its bounds with no duplicates. ■

3.2.2 The Triangle Property

Proposition 3.7. *There is a triangle in $\underline{T^m}$ that involves the three vertices on an end of T^m.*

Proof. Consider the table T^m. The last vertex in T^m is $T^m_{m,1}$ and equals $\frac{m^2}{2}-2m$. The second to last vertex ($T^m_{m-1,4}$) is $\frac{m^2}{2}+2m$. The third to last vertex ($T^m_{m-1,3}$) is $\frac{m^2}{2}+1$. By the established relationship between T^m and $\underline{T^m}$. The third to last vertex connects to the second to last vertex, and the second to last vertex connects to the last vertex.

If the sum of the value of the third to last vertex with the last vertex is a perfect square, then there is a connection between them in $\boxed{\text{n}}$ that can be considered to be a part of $\underline{T^m}$.

$$
\begin{aligned}
& \tfrac{m^2}{2} - 2m + \tfrac{m^2}{2} + 1 \\
= & \quad m^2 - 2m + 1 \\
= & \quad (m1)(m-1)
\end{aligned}
$$

Because the set of integers is closed under subtraction and multiplication, the sum of the third to last and last vertices is a perfect square. Therefore, there is a triangle of the three vertices on an end of $\square\square_m$. ■

3.2.3 The Pentagon Property

Proposition 3.8. *There is a pentagon involving five of the six first vertices on the end of T^m that does not involve the triangle described in section 3.2.2.*

Proof. We must show that the second vertex T^m and the sixth vertex in $\underline{T^m}$ sum to a perfect square, because this implies that they are connected in $\boxed{\text{n}}$ and that that connection can then be considered a part of $\underline{T^m}$.

By Table 3 in section 3.3.2, the second vertex ($T^m_{0,2}$) is $\frac{m^2}{2} - 2m + 1$ and the sixth vertex ($T^m_{1,2}$) is $\frac{m^2}{2} - 2m + 3$.

$$
\begin{aligned}
& \tfrac{m^2}{2} - 2n + 1 + \tfrac{m^2}{2} - 2m + 3 \\
= & \quad m^2 - 4m + 4 \\
= & \quad (m-2)(m-2)
\end{aligned}
$$

Because the set of integers is closed under subtraction and multiplication, the sum of the second and sixth vertices is a perfect square. Therefore, there is a pentagon involving five of the six first vertices on the end of T^m that does not involve the triangle described in section 3.2.2.

$\Rightarrow$ the sum of x_2 and x_6 is a perfect square, and therefore x_2 and x_6 are connected in $\boxed{\text{n}}$. ■

3.2.4 The Nonagon Property

Proposition 3.9. *There is a nonagon involving the last nine vertices on the end of T^m that involves the triangle as defined in 3.2.2.*

Proof. We must show that the final vertex in T^m and the ninth to last vertex in $\underline{T^m}$ sum to a perfect square, because this implies that they are connected in $\boxed{\text{n}}$ and that that connection can then be considered a part of T^m.

By Table 3 in section 3.3.2, the ninth to last vertex is $\frac{m^2}{2} - 2m + 4$, and the last vertex is $\frac{m^2}{2} - 2m$.

$$
\begin{aligned}
& \tfrac{m^2}{2} - 2m + 4 + \tfrac{m^2}{2} - 2m \\
= & \quad m^2 - 4m + 4 \\
= & \quad (m-2)(m-2)
\end{aligned}
$$

Because the set of integers is closed under subtraction and multiplication, the sum of the ninth to last and last vertices is a perfect square. Therefore, there is a nonagon of the three vertices on the end of T^m that has a triangle as described in section 3.2.2. ■

Remark 3.10. *The edge between the last and ninth to last vertices in T^m will not be used in the path, as the squares used as edges are only $(m-1)^2$, m^2, and $(m+1)^2$, not $(m-2)^2$.*

Remark 3.11. *Because three of the vertices in the nonagon form a triangle, and two of the edges in the triangle are used in the nonagon, the vertex that, in the nonagon, is between the other two members of the triangle can be excluded from the cycle to form an octagon.*

3.3 Properties of Multiple Tables

3.3.1 Tables 4 Apart

Tail-Tail

Examining two tables with middle square values that have a difference of four can be very useful. We are able to prove that the final vertex in a table m (the lone number in row m + 1) will connect to the final vertex in the table for m + 4. The proof is as follows:

Proof. As shown above, the final vertex in a table T^m, can be found using the formula $\frac{m^2}{2} - 2m$. It follows that the formula for the final vertex in a table m + 4 follows the formula $\frac{(m+4)^2}{2} - 2(m+4)$.

$$\begin{aligned} SomeSquare &= (\frac{m^2}{2} - 2n) + (\frac{(m+4)^2}{2} - 2(m+4)) \\ &= \frac{m^2}{2} - 2m + \frac{m^2 + 8m + 16}{2} - 2m - 8 \\ &= \frac{m^2}{2} - 2m + \frac{m^2}{2} + 4m + 8 - 2m - 8 \\ &= m^2 \end{aligned} \tag{1}$$

■

Note you can also read the table backwards, so to speak. Beginning at the end and ending at the beginning. Along with the fact that you use every number in each table exactly once, that is sufficient to prove there is a Hamiltonian path from the beginning of the table T^m to the beginning of the table for m + 4 by a tail to tail connection.

Head-head Failure

Because we know that two tables that is four apart contain each number exactly once and form Hamiltonian paths on their own. It would be nice to find a way to attach the head of a table for m to the head of a table for $m + 4$, but we have deemed that difficult enough to stop trying. The tables would have to contain more than 2000 vertices each. That is where we stopped testing. If this could work, however, we could use head-head, tail-tail, head-head and so on back and forth infinitely many times.

3.3.2 Tables Two Apart

Repeated Section

There are some interesting properties when examining two tables with middle squares that are only two apart. Among them is what we are calling, "the repeated section." Consider the tables for $m = 4$ and 6 respectively:

m = 4:

8	1	15	10
6	3	13	12
4	5	11	14
2	7	9	16
(0)			

m = 6:

18	7	29	20
16	9	27	22
14	11	25	24
12	13	23	26
10	15	21	28
8	17	19	30
6			

Notice that the final two columns for the table m = 4 are repeated in the first two columns for m = 6. Or more accurately, the same numbers occur, however, they are flipped upside down and backwards. This is an expansion of corollary 3.24.

Large Contiguous Intersections
Let m be an even integer ≥ 4, and consider the table of values in the interval $[\frac{m^2}{2} - 2m, \frac{m^2}{2} + 2m]$ as the table was defined previously. (Note, m must be even.) See T^m (Table 2). Then simplify algebraically. See T^m again (Table 3). Then find the T^{m+2}. See T^{m+2} (Table 4). Simplify algebraically. See T^{m+2} again (Table 5).

Proposition 3.12. *For all tables (m even, $m \geq 4$), columns 3 and 4 in T^m appear in reverse-row-order within columns 2 and 1 (respectively) in T^{m+2}.*

That is to say:

$$\{T^m_{k,p} | 0 \leq k \leq m-1 \wedge p = 3,4\} = \{T^{m+2}_{l,q} | m \geq l \geq 1 \wedge q = 2,1\}$$

Which is to say:

$$\begin{aligned} & \{T^m_{0,3}, T^m_{0,4}, T^m_{1,3}, T^m_{1,4}, \ldots, T^m_{m-1,3}, T^m_{m-1,4}\} \\ = \quad & \{T^{m+2}_{m,2}, T^{m+2}_{m,1}, T^{m+2}_{m-1,2}, T^{m+2}_{m-1,1}, \ldots, T^{m+2}_{2,2}, T^{m+2}_{2,1}, T^{m+2}_{1,2}, T^{m+2}_{1,1}\} \end{aligned}$$

row	1	2	3	4
0	$\frac{m^2}{2}$	$\frac{m^2}{2} - 2m + 1$	$\frac{m^2}{2} + 2m - 1$	$\frac{m^2}{2} + 2$
1	$\frac{m^2}{2} - 2$	$\frac{m^2}{2} - 2m + 1 + 2$	$\frac{m^2}{2} + 2m - 1 - 2$	$\frac{m^2}{2} + 2 + 2$
2	$\frac{m^2}{2} - 4$	$\frac{m^2}{2} - 2m + 1 + 4$	$\frac{m^2}{2} + 2m - 1 - 4$	$\frac{m^2}{2} + 2 + 4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m - 2	$\frac{m^2}{2} - 2(m-2)$	$\frac{m^2}{2} - 2m + 1 + 2(m-2)$	$\frac{m^2}{2} + 2m - 1 - 2(m-2)$	$\frac{m^2}{2} + 2 + 2(m-2)$
m - 1	$\frac{m^2}{2} - 2(m-1)$	$\frac{m^2}{2} - 2m + 1 + 2(m-1)$	$\frac{m^2}{2} + 2m - 1 - 2(m-1)$	$\frac{m^2}{2} + 2 + 2(m-1)$
m	$\frac{m^2}{2} - 2(m)$			

Table 2: T^m

row	1	2	3	4
0	$\frac{m^2}{2}$	$\frac{m^2}{2} - 2m + 1$	$\frac{m^2}{2} + 2m - 1$	$\frac{m^2}{2} + 2$
1	$\frac{m^2}{2} - 2$	$\frac{m^2}{2} - 2m + 3$	$\frac{m^2}{2} + 2m - 3$	$\frac{m^2}{2} + 4$
2	$\frac{m^2}{2} - 4$	$\frac{m^2}{2} - 2m + 5$	$\frac{m^2}{2} + 2m - 5$	$\frac{m^2}{2} + 6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m-2	$\frac{m^2}{2} - 2m + 4$	$\frac{m^2}{2} - 3$	$\frac{m^2}{2} + 3$	$\frac{m^2}{2} + 2m - 2$
m-1	$\frac{m^2}{2} - 2m + 2$	$\frac{m^2}{2} - 1$	$\frac{m^2}{2} + 1$	$\frac{m^2}{2} + 2m$
m	$\frac{m^2}{2} - 2m$			

Table 3: Simplified T^m

3.3.3 Formula for row, column in tables 2 apart

Similarly to how there is a relationship between the position in a table for m and the number that occupies it, there is a relationship between what number can appear in a

row	1	2	3	4
0	$\frac{(m+2)^2}{2}$	$\frac{(m+2)^2}{2} - 2(m+2) + 1$	$\frac{(m+2)^2}{2} + 2(m+2) - 1$	$\frac{(m+2)^2}{2} + 2$
1	$\frac{(m+2)^2}{2} - 2$	$\frac{(m+2)^2}{2} - 2(m+2) + 3$	$\frac{(m+2)^2}{2} + 2(m+2) - 3$	$\frac{(m+2)^2}{2} + 4$
2	$\frac{(m+2)^2}{2} - 4$	$\frac{(m+2)^2}{2} - 2(m+2) + 5$	$\frac{(m+2)^2}{2} + 2(m+2) - 5$	$\frac{(m+2)^2}{2} + 6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m-3	$\frac{(m+2)^2}{2} - 2(m+2) + 6$	$\frac{(m+2)^2}{2} - 5$	$\frac{(m+2)^2}{2} + 5$	$\frac{(m+2)^2}{2} + 2(m+2) - 4$
m-2	$\frac{(m+2)^2}{2} - 2(m+2) + 4$	$\frac{(m+2)^2}{2} - 3$	$\frac{(m+2)^2}{2} + 3$	$\frac{(m+2)^2}{2} + 2(m+2) - 2$
m-1	$\frac{(m+2)^2}{2} - 2(m+2) + 2$	$\frac{(m+2)^2}{2} - 1$	$\frac{(m+2)^2}{2} + 1$	$\frac{(m+2)^2}{2} + 2(m+2)$
m	$\frac{(m+2)^2}{2} - 2(m+2)$			

Table 4: This is the next table: T^{m+2}

row	1	2	3	4
0	$\frac{m^2}{2} + 2m + 2$	$\frac{m^2}{2} - 1$	$\frac{m^2}{2} + 4m + 5$	$\frac{m^2}{2} + 2m + 4$
1	$\frac{m^2}{2} + 2m$	$\frac{m^2}{2} + 1$	$\frac{m^2}{2} + 4m + 3$	$\frac{m^2}{2} + 2m + 6$
2	$\frac{m^2}{2} + 2m - 2$	$\frac{m^2}{2} + 3$	$\frac{m^2}{2} + 4m + 1$	$\frac{m^2}{2} + 2m + 8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m-3	$\frac{m^2}{2} + 4$	$\frac{m^2}{2} + 2m - 3$	$\frac{m^2}{2} + 2m + 7$	$\frac{m^2}{2} + 4m + 2$
m-2	$\frac{m^2}{2} + 2$	$\frac{m^2}{2} + 2m - 1$	$\frac{m^2}{2} + 2m + 5$	$\frac{m^2}{2} + 4m + 4$
m-1	$\frac{m^2}{2} + 0$	$\frac{m^2}{2} + 2m + 1$	$\frac{m^2}{2} + 2m + 3$	$\frac{m^2}{2} + 4m + 6$
m	$\frac{m^2}{2} - 2$			

Table 5: Simplified T^{m+2}

particular row or column when examining two tables whose m values have a difference of two. We will refer to the relationship of tables with m values with a difference of two as two consecutive tables. The formulas are shown on the next page. Note that the final number in row r is unlikely to connect to the first number in row $r+1$. In this case, we are defining $a = m$ and $b = m + 2$. The tables show a relationship between the position in a combination of tables and the number that occupies it.

1	2	3	4	5	6
-	-	-	-	$\frac{b^2}{2}+2b-2(0)$	$\frac{b^2}{2}+2b-2(b-1)+1$
-	-	-	$\frac{b^2}{2}-2b+2(0)$	$\frac{b^2}{2}+2b-2(1)$	$\frac{b^2}{2}+2b-2(b-2)+1$
-	-	$\frac{b^2}{2}-2b+2(b-1)+1$	$\frac{b^2}{2}-2b+2(1)$	$\frac{b^2}{2}+2b-2(2)$	$\frac{b^2}{2}+2b-2(b-3)+1$
$\frac{a^2}{2}-2a+2(a-0)$	$\frac{a^2}{2}-2a+2(0)+1$	$\frac{b^2}{2}-2b+2(b-2)+1$	$\frac{b^2}{2}-2b+2(2)$	$\frac{b^2}{2}+2b-2(3)$	$\frac{b^2}{2}+2b-2(b-4)+1$
$\frac{a^2}{2}-2a+2(a-1)$	$\frac{a^2}{2}-2a+2(1)+1$	$\frac{b^2}{2}-2b+2(b-3)+1$	$\frac{b^2}{2}-2b+2(3)$	$\frac{b^2}{2}+2b-2(4)$	$\frac{b^2}{2}+2b-2(b-5)+1$
$\frac{a^2}{2}-2a+2(a-2)$	$\frac{a^2}{2}-2a+2(2)+1$	$\frac{b^2}{2}-2b+2(b-4)+1$	$\frac{b^2}{2}-2b+2(4)$	$\frac{b^2}{2}+2b-2(5)$	$\frac{b^2}{2}+2b-2(b-6)+1$
...	...	...	...	...	...
$\frac{a^2}{2}-2a+2(a-(a-1)$	$\frac{a^2}{2}-2a+2(a-1)+1$	$\frac{b^2}{2}-2b+2(b-(b-1))+1$	$\frac{b^2}{2}-2b+2(b-1)$	$\frac{b^2}{2}+2b-2(b)$	$\frac{b^2}{2}+2b-2(b-(b+1))+1$
$(\frac{a^2}{2}-2a+2(a-(a))$	-	$(\frac{b^2}{2}-2b+2(b-b)+1)$	$\frac{b^2}{2}-2b+2(b)$	$(\frac{b^2}{2}+2b-2(b))$	$\frac{b^2}{2}+2b-2(b-b)+1$

Here is the same table, while substituting 4 for a and 6 for b and solving:

1	2	3	4	5	6
-	-	-	-	30	19
-	-	17	(8)	28	21
8	1	15	10	26	23
6	3	13	12	24	25
4	5	11	14	22	27
2	7	9	16	20	29
(0)	-	(7)	18	(18)	31

Other Intersections

The table for m, where m is some integer > 4, will be referred to as T^m, with rows i, 0 through m and columns j, 1 through 4. Each cell is therefore $T^m_{i,j}$. Consider valid m values to be even integers greater than or equal to four.

Remark 3.13. ***Portal Method*** *- A way to find Hamiltonian paths on $\bigcirc_n$ using many T^m's. The noun portal denotes an entry that is in one of the large contiguous intersections of two tables. The verb portal is used to indicate that the next entry to be read into the HP$\boxed{n}$ after the portal is the next vertex in the the other table the portal appears in. Each portal vertex is only read into the HP$\boxed{n}$ once.*

Proposition 3.14. *There exists a portal between $T^m_{m-1,2}$ and $T^{m+2}_{1,2}$ for all valid m.*

Proof. By Table 3 in section 3.3.2, we observe that $T^m_{m-1,2} = \frac{m^2}{2} - 1$. We observe that $T^{m+2}_{1,2}$ also equals $\frac{m^2}{2} - 1$. Therefore, there exists a portal between $T^m_{m-1,2}$ and $T^{m+2}_{1,2}$ for all valid m. ■

Proposition 3.15. *There exists a portal between $T^{m+2}_{m-1,1}$ and $T^m_{0,1}$ for all valid m.*

Proof. By Table 3 in section 3.3.2, we observe that $T^{m+2}_{m-1,1} = \frac{m^2}{2}$. We also observe that $T^m_{0,1}$ also equals $\frac{m^2}{2}$. Therefore, there exists a portal between $T^{m+2}_{m-1,1}$ and $T^m_{0,1}$ for all valid m. ■

Proposition 3.16. *There exists a portal between $T^m_{1,1}$ and $T^{m+2}_{m,1}$.*

Proof. By Table 3 in section 3.3.2, we observe that $T^m_{1,1} = \frac{m^2}{2} - 2$. We also observe that $T^{m+2}_{m,1}$ also equals $\frac{m^2}{2} - 2$. Therefore, there exists a portal between $T^m_{1,1}$ and $T^{m+2}_{m,1}$. ■

Proposition 3.17. *There exists a portal between $T^m_{m-1,4}$ and $T^{m+4}_{m,1}$.*

Proof. By Table 3 in section 3.3.2, we observe that $T^m_{m-1,4} = \frac{m^2}{2} + 2m$. Table 3 implies that $T^{m+4}_{m+4,1} = \frac{(m+4)^2}{2} - 2(m-8) = \frac{m^2}{2} + 2m$. Therefore, $T^m_{m-1,4} = T^{m+4}_{m+4,1}$. Therefore, there exists a portal between $T^m_{m-1,4}$ and $T^{m+4}_{m+4,1}$. ■

Remark 3.18. *Proposition 3.17 means that two tables, T^m and T^{m+4}, and their two corresponding paths, $\underline{T^m}$ and $\underline{T^{m+4}}$ can be strung together to create a path that covers all the vertices in the interval [$\frac{m^2}{2} - 2n, \frac{(m+4)^2}{2} + 2(m+4)$].*

3.3.4 Expansion of Formula for Row, Column in Tables 2 Apart

Refer back to the table with formulas to find a number in a particular row and column, we've found that you can actually expand to include an arbitrary amount of tables. For our purposes, we will again use letters to denote even integers, beginning with $a = 4$ and ending with an arbitrary z. Building from left to right, we will look at as though we are examining the right square. In other words, when looking at column 3, we will denote its formula in relation to $b = 6$. It gets tricky when figuring out how many rows are in each column. Typically, the first column for a particular even square h has h rows. The right, or second, column has $h + 1$ rows. However, the left, or first, column for the even numbers a and c do not fit this description. Theirs are exactly opposite.

Proposition 3.19. *For letters $L = a, c, e,$ ...*
the "right square," "left column" is
$\frac{L^2}{2} - 2L + 2(L - r)$ *for* $r \leq L + 1$ *and*
the "right square," "right column" is
$\frac{L^2}{2} - 2L + 2(r - 1) + 1$ *for* $r \leq L + 1$.

Proposition 3.20. *For letters $L = b, d, f, \ldots$*
the "right square," "left column" is
$\frac{L^2}{2} - 2L + 2(L - r) + 1$ *for* $r \leq L$ *and*
the "right square," "right column" is
$\frac{L^2}{2} - 2L + 2(r - 1)$ *for* $r \leq L + 1$.

3.3.5 m Values

Proposition 3.21. Corollary of ?? *All m values must be even.*

Proof. Assume m is odd. Let q be $\in \mathbb{N}^+$. We find the lowest vertex in the interval of $\underline{T^m}$.

$$\begin{aligned}
\Rightarrow \quad & m = 2q + 1 \\
\Rightarrow \quad & \frac{(2q+1)^2}{2} - 2(2q + 1) \\
= \quad & \frac{4q^2+4q+1}{2} - 4q - 2 \\
= \quad & \frac{4q^2-4q-3}{2} \\
& \Rightarrow\Leftarrow
\end{aligned}$$

Because the integers are not closed under addition, the lowest vertex in the interval of $\underline{T^m}$ when m is odd is not an integer.

Therefore m must be even. ∎

Proposition 3.22. Corollary of ?? *The smallest m value for $\underline{T^m}$ is 4.*

Proof. Assume that the smallest m value is 2. Let m = 2. All vertices must be > 0. We find the lowest vertex of the interval of $\underline{T^m}$.

$$\begin{aligned}
& \frac{m^2}{2} - 2m \\
\Rightarrow \quad & \frac{4}{2} - 2(2) \\
= \quad & 2 - 4 \\
= \quad & -2 \\
& \Rightarrow\Leftarrow
\end{aligned}$$

Therefore, the smallest m value for $\underline{T^m}$ is 4.

∎

3.3.6 Triangle numbers

The triangle numbers are given by $1 + 2 + \ldots + m = \frac{m(m+1)}{2}$ for $m \geq 0$.

Proposition 3.23. *The upper bound of the interval of vertices covered by a set of tables is always 16 times a triangle number.*

Proof. By the section **??** and definition **??**, the largest table in a set of tables has an m value that is divisible by 4. The largest vertex covered by a set of tables is the upper bound for the largest table. The upper bound is $\frac{m^2}{2} + 2m$ Therefore, the largest vertex in a set of tables is $\frac{(4p)^2}{2} + 2(4p)$, where p is a positive integer such that $m = 4p$. Let q be an integer ≥ 0.

$$\begin{aligned}
& \frac{(4p)^2}{2} + 2(4p) && (2) \\
= \quad & \frac{16p^2}{2} + 8p \\
= \quad & 8p^2 + 8p \\
= \quad & 8p(p + 1) && (3)
\end{aligned}$$

$$\begin{aligned} & 16\frac{q(q+1)}{2} \\ = \quad & 8q(q+1) \end{aligned} \tag{4}$$

Equation (1) can be written in the form of equation (2).

$$8p(p+1) = 8q(q+1) \tag{5}$$

Equation (2) equals equation (3), see equation (4). Therefore, the largest vertex covered by a set of tables is 16 times a triangle number. ■

Corollary 3.24. *Define consecutive tables to mean two tables whose m values have a difference of two. Then the numbers in the sequence $a(x) = 2x^2 - 2$, where x is a positive integer, are the only numbers to appear in three distinct, consecutive tables. All other numbers appear in exactly two consecutive tables.*

3.3.7 Requirements for the set of tables for $\boxed{\mathbf{n}}$.

We establish definitions.

Definition 3.25. *Tables T^{m_1}, T^{m_2}, are consecutive, with T^{m_2} following T^{m_1}, iff $m_2 = m_1 + 2$.*

Definition 3.26. *Two T^ms are considered neighbors if they are consecutive.*

Definition 3.27. $\underline{\boxed{\boxed{n}}}$ *is the set that contains all the consecutive tables from T^4 through T^M (inclusive of T^4 and T^M), where M is divisible by four. By extension,* $\boxed{\boxed{n}}$ *contains vertices 1 through n (inclusive of 1 and n), where $n = \frac{m_{max}^2}{2} + 2m_{max}$ (see section* **??***).*

Proposition 3.28. *The cardinality of* $\boxed{\boxed{n}}$ *is odd for all $n \in \mathbb{N}^+$.*

Proof. base case: Because the set containing only T^4 fulfills all of the requirements in definition 3.27, and T^4 is the smallest T^m, and $\frac{4^2}{2} + 2 * 4 = 16$, the smallest $\boxed{\boxed{\mathrm{n}}}$ is $\boxed{\boxed{16}}$. The cardinality of $\boxed{\boxed{16}}$ is 1.

inductive hypothesis: $\exists k \in \mathbb{N}^+$ such that $\boxed{\boxed{\frac{k^2}{2} + 2k}}$ has an odd cardinality.

inductive step: Assume the inductive hypothesis is true.

Beginning with the set $\boxed{\boxed{\mathrm{k}}}$, let its cardinality ($c$) be $2m + 1$, $m \in \mathbb{N}$. Note that $\mathbb{N}$ is closed under addition and multiplication. Let $l = m + 1$.

$$\begin{aligned} c &= 2m + 1 \\ \Rightarrow c + 2 &= 2m + 1 + 2 \\ \Rightarrow c + 2 &= 2m + 2 + 1 \\ \Rightarrow c + 2 &= 2(m + 1) + 1 \\ \Rightarrow c + 2 &= 2l + 1 \end{aligned}$$

Therefore, the cardinality of $\boxed{\boxed{\mathrm{n}}}$ is odd for all $n \in \mathbb{N}^+$. ■

3.3.8 Description of How to Obtain $\boxed{\boxed{\mathbf{n}}}$ from several T^ms

The following is a description of how $\boxed{\boxed{\text{n}}}$ can be read to create a HP$\boxed{\text{n}}$. We begin with definitions.

Definition 3.29. *The noun vertex indicates an entry in a T^m, a node in a graph, or the numerical value that corresponds with either.*

Definition 3.30. *A plain vertex is a vertex that appears in only one $T^m \in \boxed{\boxed{n}}$.*

Definition 3.31. *For some $T^m_{i,j}$, if $j = 4$, the subsequent vertex with respect to T^m is the vertex $T^m_{i+1,1}$. If $i = m$, the subsequent vertex is not defined. Otherwise, the subsequent vertex is the vertex $T^m_{i,j+1}$.*

Definition 3.32. *A series of vertices is consecutive with respect to some T^m iff each vertex in the series is followed in the series by its subsequent vertex in T^m.*

Definition 3.33. *The verb read is used to mean the following. Copy the value(s) of a vertex or a series of consecutive vertices from the indicated table, and paste them into the the HP$\boxed{n}$.*

Definition 3.34. *The noun portal denotes an entry that is in one of the large contiguous intersections of two tables. The verb portal is used to indicate that the next entry to be read after the portal is the next vertex in the the other table the portal appears in.*

Definition 3.35. *The noun wormhole denotes an entry in a table that is duplicated in another table, where their duplication is not accounted for by a large contiguous intersection. The verb wormhole is used to indicate that the next entry to be read after the wormhole is the entry that follows the wormhole in the other table the wormhole appears in.*

Always begin reading in the upper left hand corner of the largest table in $\boxed{\boxed{\text{n}}}$. Read left to right, top to bottom, like English text. Portals and wormholes can be used to move from reading one table to reading another table. Only some of the portals in the large contiguous intersections (LCIs, see section 3.3.2 for definition) of the T^ms are used to move between tables when reading $\boxed{\boxed{\text{n}}}$ into a HP$\boxed{\text{n}}$. The group of portals in the LCI that are used is called the usable subgroup of the LCI, or usLCI. The vertices that are part of the LCI and not part of the usLCI are treated, when reading, as though they were plain vertices. Similarly, only some of the wormholes are used to move between tables when reading $\boxed{\boxed{\text{n}}}$ into a HP$\boxed{\text{n}}$. The wormholes that are not used to move between tables are treated, when reading, as though they were plain vertices.

For each $T^m_{i,j}$, read it into the path, then determine the next vertex to be read into the path using the following algorithm.

1. Let $\mathbb{M} = \{m | T^m \in \boxed{\boxed{\text{n}}}\}$.
2. Let M be the largest member of $\mathbb{M}$.
3. Let $\mathbb{W}^m$ be the set of all the wormholes in T^m.
4. Let $\mathbb{W}^m_u$ be the set of wormholes in T^m that are used in constructing the HP$\boxed{\text{n}}$.
5. Determine if $T^m_{i,j}$ is in a usLCI.
   (a) If $m = M$, then $usLCI = \{T^m_{k,p} | 2 \leq k \leq m - 3 \wedge p \in \{1, 2\}\}$.
   (b) If $m = 4$, then $usLCI = \{T^m_{k,p} | 0 \leq k \leq 3 \wedge p \in \{3, 4\}\}$.

(c) If $m \neq M$, and $m \neq 4$, and m is divisible by 4, then there are two usLCIs: $usLCI = \{T^m_{k,p} | 2 \leq k \leq m-3 \wedge p = 1,2\}$, and $usLCI = \{T^m_{l,q} | 0 \leq l \leq m-1 \wedge q = 3,4\}$.

(d) If m is not divisible by 4, then there are two usLCIs: $usLCI = \{T^m_{k,p} | 2 \leq k \leq m-2 \wedge p = 1,2\}$, where and and $usLCI = \{T^m_{l,q} | 1 \leq l \leq m-2 \wedge q = 3,4\}$.

6. If $T^m_{i,j}$ is in a usLCI, then find its column number, j.

(a) If $j = 1$, then the next vertex to be read into the HP is the subsequent vertex of $T^{m-2}_{m-2-r,5-c}$ in T^{m-2}.

(b) If $j = 2$, then the next vertex in the HP is its subsequent vertex in T^m.

(c) If $j = 3$, then the next vertex in the HP is the subsequent vertex of $T^{m+2}_{m-r,5-c}$ in T^{m+2}.

(d) If $j = 4$, then the next vertex in the HP is its subsequent vertex in T^m.

7. If $T^m_{i,j}$ is not in a usLCI, determine if $T^m_{i,j}$ is a used wormhole.

(a) If $m = M$, then $\mathbb{W}^M_u = \{T^M_{M,1}\}$.

(b) If $m = 4$, then $\mathbb{W}^4_u = \{T^4_{0,1}\}$.

(c) If $m \neq M$ and $m \neq 4$ and m is divisible by 4, then $\mathbb{W}^m_u = \{T^m_{0,1}, T^m_{m,1}\}$.

(d) If m is not divisible by 4, then $\mathbb{W}^m_u = \{T^m_{m-1,1}\}$.

8. If $T^m_{i,j}$ is a used wormhole, then the same entry appears in $T^{m_w}_{r,s}$. The next vertex in HP $\boxed{\text{n}}$ is the subsequent vertex of $T^{m_w}_{r,s}$ in T^{m_w}.

9. If $T^m_{i,j}$ is not in a usLCI and not a used wormhole, then the next vertex in the HP is the subsequent vertex of $T^m_{i,j}$ in T^m.

10. Repeat steps 5 through 10 with the vertex just added to the HP $\boxed{\text{n}}$.

Proposition 3.36. *Columns 1 and 2 of T^m contain all of the natural numbers in [].* *Columns 3 and 4 of T^m contain all of the natural numbers in [].*

Proof. Let r denote the row number of an entry in T^m.

It follows directly from the $0th$ row of T^m and Theorem 3.3 that column 1 of T^m is comprised (in ascending row order) of the sequence

$$\{\frac{m^2}{2}, \frac{m^2}{2} - 2, \frac{m^2}{2} - 4, \frac{m^2}{2} - 6, \ldots, \frac{m^2}{2} - 2m\}.$$

Consider $[\frac{m^2}{2} - 2m, \frac{m^2}{2}]$. It is clear that the set of entries in column 1 is a strict subset of this interval. Because of Theorem 3.5, we know that the set of entries in column 1 contains all of the even natural numbers in the interval.

It follows directly from the $0th$ row of T^m and Theorem 3.3 that column 2 of T^m is comprised (in ascending row order) of the sequence

$$\{\frac{m^2}{2} - 2m + 1, \frac{m^2}{2} - 2m + 3, \frac{m^2}{2} - 2m + 5, \frac{m^2}{2} - 2m+, , \ldots, \frac{m^2}{2} - 2m + 1 + 2(m-1)\}.$$

Consider $[\frac{m^2}{2} - 2m, \frac{m^2}{2}]$. It is clear that the set of entries in column 1 is a strict subset of this interval. Because of Theorem 3.5, we know that the set of entries in column 1 contains all of the odd natural numbers in the interval.

Because the natural numbers in $[\frac{m^2}{2} - 2m, \frac{m^2}{2}]$ are only even and odd numbers, columns 1 and 2 contain, together, all of the natural numbers in $[\frac{m^2}{2} - 2m, \frac{m^2}{2}]$.

Because T^m has four columns and contains all of the natural numbers in $[\frac{m^2}{2} - 2m, \frac{m^2}{2} + 2m]$, and because columns 1 and 2 contain all of the natural numbers in $[\frac{m^2}{2} - 2m, \frac{m^2}{2}]$, columns 3 and 4 must contain all of the natural numbers in $[\frac{m^2}{2} + 1, \frac{m^2}{2} + 2m]$. ■

Proposition 3.37. *Column 3 in T^m occurs in reverse-row-order in column 2 in T^{m+2}. Column 4 in T^m occurs in reverse-row-order in column 1 in T^{m+2}.*

Proof.

Definition 3.38. *The reverse of a finite, ordered set of m discrete elements, $\{a_1, a_2, a_3, a_4, \ldots, a_{m-1}, a_m\}$, is $\{a_m, a_{m-1}, \ldots, a_4, a_3, a_2, a_1\}$.*

Let $\mathbb{C}_x^m$ be the set of items in column x of table T^m, where the items are ordered by their row number, from lowest row number (0) to highest row number (m). Let the reverse of $\mathbb{C}_x^m$ be $\mathbb{C}_x^{-m}$.

From the noted theorems and corollaries, we know the following about T^m:

1. The table has four columns, numbered 1, 2, 3, and 4 with the parities even, odd, odd, and even, respectively. (Theorem 3.5)
2. The table contains all of the natural numbers $[\frac{m^2}{2} - 2m, \frac{m^2}{2} + 2m]$. (Theorem 3.6)
3. Columns 1 and 2 contain the vertices $[\frac{m^2}{2} - 2m, \frac{m^2}{2}]$. (Corollary 3.36.)
4. Columns 3 and 4 contain the vertices $[\frac{m^2}{2} + 1, \frac{m^2}{2} + 2m]$. (Corollary 3.36.)
5. $\mathbb{C}_1^m = \{v|\frac{m^2}{2} \geq v \geq \frac{m^2}{2} - 2m \wedge v \equiv \mathit{even}\}$, (elements are ordered by magnitude from greatest to least). (Theorem 3.3)
6. $\mathbb{C}_2^m = \{v|\frac{m^2}{2} - 2m \leq v \leq \frac{m^2}{2} \wedge v \equiv \mathit{odd}\}$ (elements are ordered by magnitude from least to greatest). (Theorem 3.3)
7. $\mathbb{C}_3^m = \{v|\frac{m^2}{2} + 2m \geq v \geq \frac{m^2}{2} + 1 \wedge v \equiv \mathit{odd}\}$ (elements are ordered by magnitude from greatest to least). (Theorem 3.3)
8. $\mathbb{C}_4^m = \{v|\frac{m^2}{2} + 1 \leq v \leq \frac{m^2}{2} + 2m \wedge v \equiv \mathit{even}\}$ (elements are ordered by magnitude from least to greatest). (Theorem 3.3)

First, find $\mathbb{C}_x^{m+2}$ for $x = 1, 2$. Take the ordering of elements by magnitude to be as noted above for each x value.

$$\begin{aligned}
\mathbb{C}_1^{m+2} &= \{v|\frac{(m+2)^2}{2} \geq v \geq \frac{(m+2)^2}{2} - 2(m+2) \wedge v \equiv \mathit{even}\} \\
\mathbb{C}_2^{m+2} &= \{v|\frac{(m+2)^2}{2} - 2(m+2) \leq v \leq \frac{(m+2)^2}{2} \wedge v \equiv \mathit{odd}\}
\end{aligned}$$

The above simplify to the following, preserving the order of the elements.

$$\mathbb{C}_1^{m+2} = \{v|\frac{m^2}{2} + 2m + 2 \geq v \geq \frac{m^2}{2} - 2 \wedge v \equiv \mathit{even}\} \tag{6}$$

$$\mathbb{C}_2^{m+2} = \{v|\frac{m^2}{2} - 2 \leq v \leq \frac{m^2}{2} + 2m + 2 \wedge v \equiv \mathit{odd}\} \tag{7}$$

We also know the following. The order of elements in $\mathbb{C}_3^{-m}$ (8) is opposite the order of elements in $\mathbb{C}_3^m$ (7.). The order of elements in $\mathbb{C}_4^{-m}$ (9) is opposite the order of elements in $\mathbb{C}_4^m$ (8.).

$$\mathbb{C}_3^{-m} = \{v|\frac{m^2}{2} + 1 \leq v \leq \frac{m^2}{2} + 2m \wedge v \equiv \mathit{odd}\} \tag{8}$$

$$\mathbb{C}_4^{-m} = \{v|\frac{m^2}{2} + 2m \geq v \geq \frac{m^2}{2} + 1 \wedge v \equiv \mathit{even}\} \tag{9}$$

It is clear that $\mathbb{C}_3^{-m}$ (8) is a strict subset of $\mathbb{C}_2^{m+2}$ (7), and that each of these sets has the property that its elements are ordered by magnitude from least to greatest. It is also clear that $\mathbb{C}_4^{-m}$ (9) is a strict subset of $\mathbb{C}_1^{m+2}$ (6), and that each of these sets has the property that its elements are ordered by magnitude from greatest to least. Therefore, column 3 of T^m occurs in reverse-row-order in column 2 in T^{m+2}, and column 4 in T^m occurs in reverse-row-order in column 1 in T^{m+2}. ■

Remark 3.39. *The tabular reason for why the method of stringing multiple consecutive tables together does not apply to sets of even numbers of T^ms (tables) is that T^4 must be read from top to bottom, as described in section 3.3.8. If T^4 is read from bottom to top, as B is (see section 3.3.8), then T^6 is read from top to bottom, so 16 comes before 9 in its traversal. Using the portal method, the 16 in T^6 is followed by the 16 in T^4, then by the 0 in T^4. The 0 in T^4 is a dead end (because it only appears in T^4) and we know that 9 and all the vertices to the right of/below 16 in T^6 have not yet been reached.*

Remark 3.40. *A HP on honeycomb graph when honeycomb graph is constructed from an even number of T^ms has been found by computer for every n tested. We have not found a pattern in the path that is true for all the honeycomb graphs tested, so we have no generalization to propose. We believe, however, that honeycomb graphs constructed from an even number of T^ms has a path for many n values.*

4 Proving that the Method Hits Every Number Without Duplicates

Before we can prove that the method actually does use every number without any duplicates, we must convert the table method into one that involves graph theory.

To convert the table method into a graph method, we follow the following 6 steps:

1. convert each individual table into a graph with no edges other than the ones that what we read in the path.
2. "Stack" the new graphs on top of one another, with the smallest table at the bottom and the largest table at the top (it should resemble something of an inverted pyramid).
3. Move each graph to the left or right so that the repeated vertices in each graphs line up with the graphs above and below it (this will require tables that aren't a multiple of 4 to be backwards).
4. Connect each of the vertices that are the same value with dashed edges. This represents them technically being the same vertex, but shown in multiple places to help represent the structure of the graph.
5. Remove each vertical dashed line by condensing the two visual representations of that vertex into one vertex.
6. The resulting graph is what we will call the "honeycomb graph".

Example 4.1.

We will use the steps on the tables $m = 4, 6, 8$ on the next 2 pages

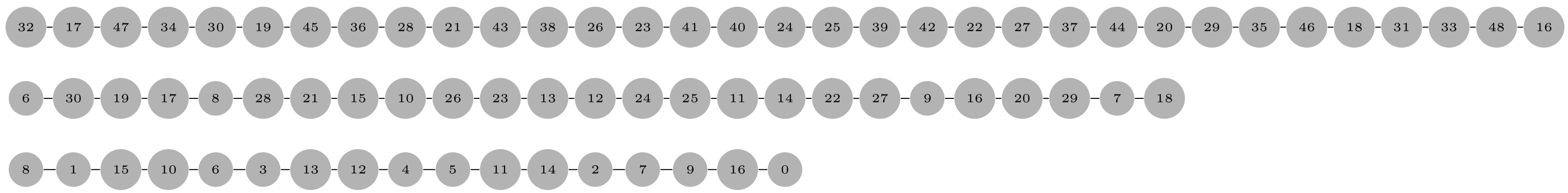

Figure 18: steps 1, 2, and 3

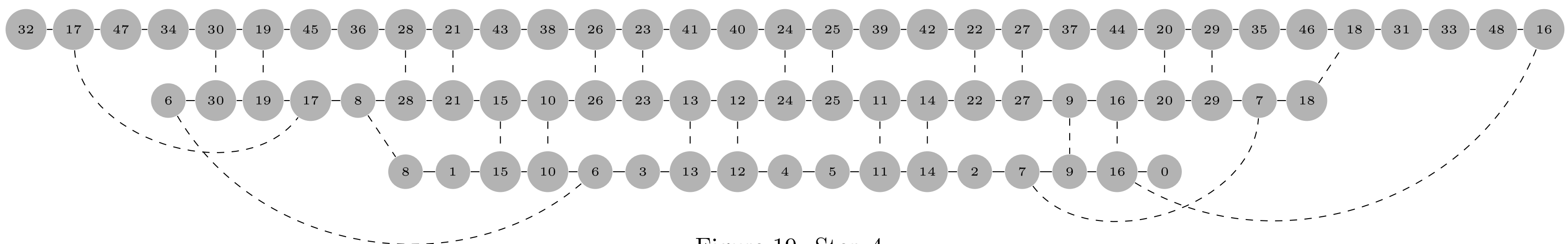

Figure 19: Step 4

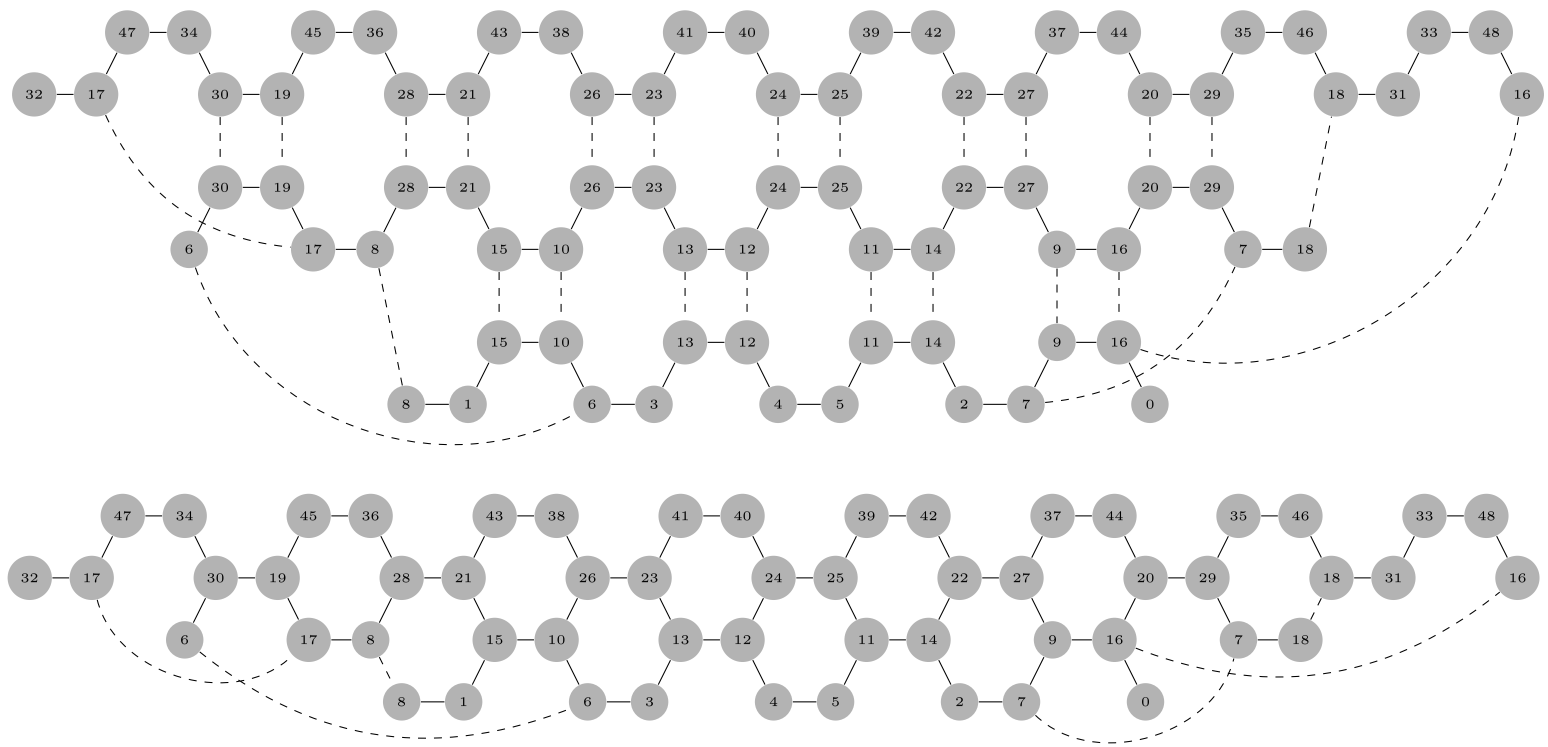

Figure 20: Step 5 done in two substeps. We will refer to the graph that isn't quite in honeycomb form the transitional graph.

Before we apply the table method to the corresponding honeycomb graph, we will relate each of the necessary parts of the table method to the honeycomb graph and transitional graph.

1. **Table -** In the transitional graph, a table is 2 layers of vertices connected by horizontal edges, but the two layers must be connected by diagonal edges. In the honeycomb graph, it is any two layers of vertices connected by horizontal edges.

2. **Large Contiguous Intersection -** In both the transitional graph and the honeycomb graph, the large contiguous intersection is a single layer of vertices connected by horizontal edges.

3. **Portal -** In the transitional graph a portal is represented by a vertical dashed edge. In the honeycomb graph there isn't really a good representation of portals, as the "portals" have been eliminated to make the graph easier to read, but they can still be conceptualized as staying at the same vertex and focusing on different 'tables' on the graph.

4. **Wormhole -** In the transitional graph a wormhole is the curved dashed edge between two different tables between a vertex that is shown in the two 'tables', and is a member of one of the following sequences: $a_1(n) = 2n^2 - 1$ and $a_2(n) = 2n^2 - 2$..In the honeycomb graph a wormhole is a curved dashed edge between different visual representations of the same vertex in two layers if the two vertices are in one of the following sequences: $a_1(n) = 2n^2 - 1$ and $a_2(n) = 2n^2 - 2$.

Below is a visual representation of the table method being used on a transitional graph and its corresponding honeycomb graph.

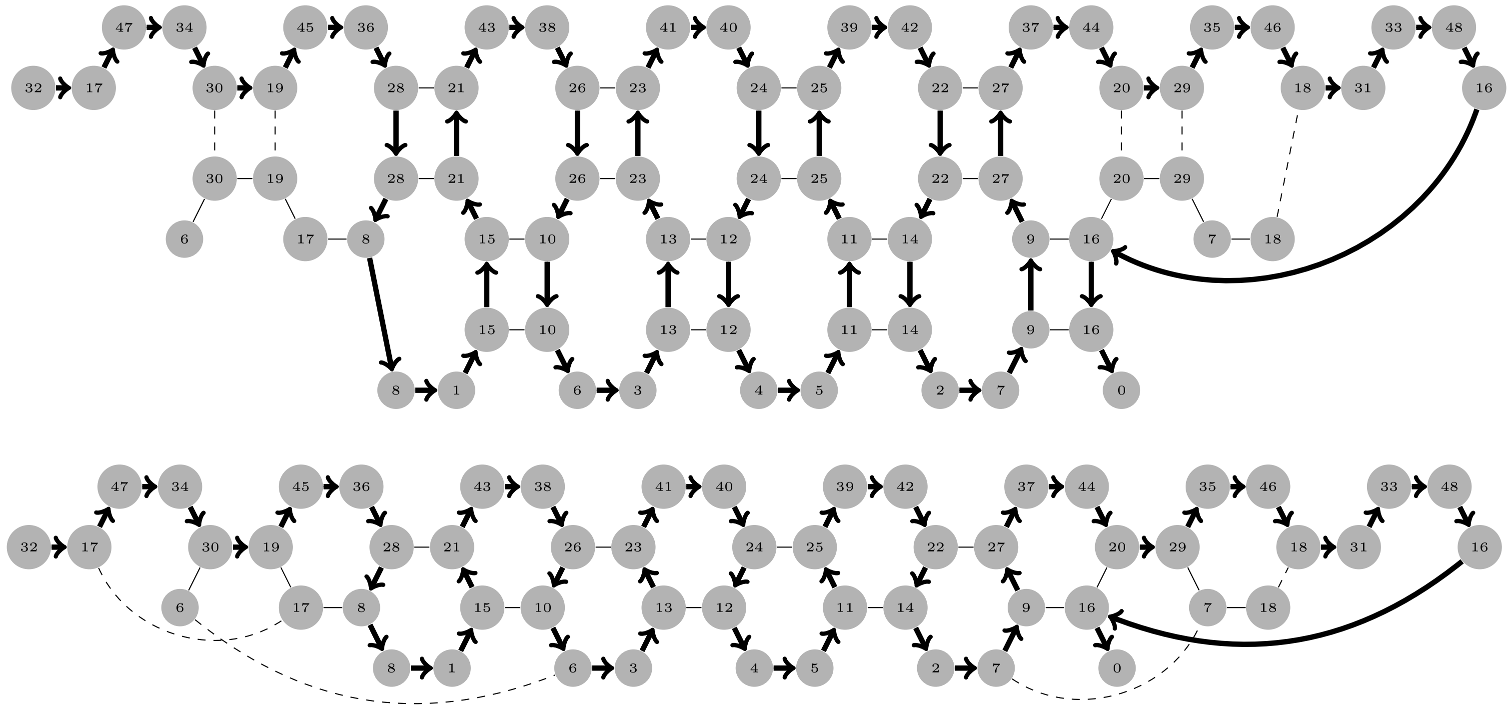

Figure 21: The table method on both the transitional graph and the honeycomb graph

The table method will always result in the zig-zag like pattern for the Hamiltonian path on the honeycomb graph, with it ending with a descent through the 16 times triangular numbers. We can now use this information to prove that the table method uses every number without any duplicates.

We would like to prove that the honeycomb graph has a repeating structure to it, and that the path that the table method provides will always be constructed in a way that will always be Hamiltonian.

Definition 4.2. *A* <u>*row*</u> *on the honeycomb graph is a row of horizontal edges which all represent sums to a square number. We label the rows by what square the edges represent.*

Note that because of the way we define tables and the honeycomb graph, rows can only be labeled with odd numbers.

Definition 4.3. *If we call α the square the edges in the row sum to, then the row begins with vertex $\frac{(\alpha+1)^2}{2}$ and ends with vertex $\frac{(\alpha-1)^2}{2}-2$.*

Example 4.4. *Row 3 contains the following vertices in this order:* $8, 1, 6, 3, 4, 5, 2, 7, 0$*, with edges between vertices* $\{8, 1\}, \{6, 3\}, \{4, 5\}, \{2, 7\}$*. Vertex* 0 *is not part of an edge in row 3.*

Lemma 4.5. *Row α contains every integer in the interval $[\frac{\alpha^2-3}{2}-\alpha, \frac{\alpha^2+1}{2}+\alpha]$.*

Proof. We get row α from columns 1 and 2 of table $m = \alpha + 1$. The generalized form of the table tells us that column 1 contains every even integer in the interval $[\frac{m^2}{2}-2m, \frac{m^2}{2}]$ and column 2 contains every odd number in the interval $[\frac{m^2}{2}-2m+1, \frac{m^2}{2}-1]$. When we combine the two intervals, we find that the resulting interval contains every integer in the interval $[\frac{m^2}{2}-2m, \frac{m^2}{2}]$.

Now we put the interval in terms of α using the equation $m = \alpha + 1$:

$$\begin{aligned}
[\frac{m^2}{2}-2m, \frac{m^2}{2}] &= [\frac{(\alpha+1)^2}{2}-2(\alpha+1), \frac{(\alpha+1)^2}{2}] \\
&= [\frac{\alpha^2+2\alpha+1}{2}-2\alpha-2, \frac{\alpha^2+2\alpha+1}{2}] \\
&= [\frac{\alpha^2-3}{2}-\alpha, \frac{\alpha^2+1}{2}+\alpha]
\end{aligned}$$

■

Lemma 4.6. *Row α and row $\alpha+2$ both contain the following three numbers: $\frac{\alpha^2+1}{2}+\alpha-2$, $\frac{\alpha^2+1}{2}+\alpha-1$, and $\frac{\alpha^2+1}{2}+\alpha$*

Proof. We use the interval in theorem 4.5 to find where the intervals for rows α and $\alpha+2$ overlap. The interval for row α is $[\frac{\alpha^2-3}{2}-\alpha, \frac{\alpha^2+1}{2}+\alpha]$. To find the interval for row $\alpha+2$, we substitute $\alpha+2$ into the interval for α

$$\begin{aligned}
[\frac{\alpha^2-3}{2}-\alpha, \frac{\alpha^2+1}{2}+\alpha] &= [\frac{(\alpha+2)^2-3}{2}-(\alpha+2), \frac{(\alpha+2)^2+1}{2}+(\alpha+2)] \\
&= [\frac{\alpha^2+4\alpha+4-3}{2}-\alpha+2, \frac{\alpha^2+4\alpha+4+1}{2}+\alpha+2] \\
&= [\frac{\alpha^2+1}{2}+\alpha-2, \frac{\alpha^2+9}{2}+3\alpha]
\end{aligned}$$

The intervals overlap on exactly 3 integers: $\frac{\alpha^2+1}{2}+\alpha-2$, $\frac{\alpha^2+1}{2}+\alpha-1$, and $\frac{\alpha^2+1}{2}+\alpha$, Therefore they are in both rows ■

Lemma 4.7. *Row $\alpha + 2$ overhangs the beginning of row α by three numbers, and overhangs the end by 2 numbers.*

Proof. Because we get the rows from tables, we will use tables m and $m + 2$ to get where the overhang starts and stops.

Row α comes comes from columns 1 and 2 of T^m, where $m = \alpha + 1$. This means we can use this, and the large contiguous intersections to find where the overhang will start and end.

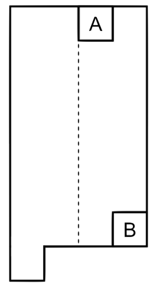

Figure 22: Table m. The entirety of row α is on the left of the dashed line, and some of row $\alpha + 2$ is on the right of the dashed line. The boxes with the "A" and "B" in them represent entries that will be the last entries in row $\alpha + 2$ that do not overhang row α.

In Table T^{m+2}, we can use entries A and B to find how much of row $\alpha+2$ hangs over the side of row α on each side. Because the shared entries in tables T^m and T^{m+2} get "rotated", entry A will be near the bottom of T^{m+2}, and entry B will be near the top. Therefore every entry in columns 1 and 2 below entry A will overhang the start of row α, and every entry in columns 1 and 2 above entry B will overhang the end of row α. Proposition 3.37 allows us to find where A and B are in table T^{m+2}

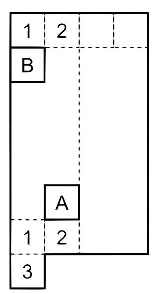

Figure 23: Table $m + 2$. The numbers in the dashed boxes are counting how many entries in row $\alpha + 2$ overhang row α

Therefore row $\alpha + 2$ overhangs the beginning of row α by 3 entries, and it overhangs the end of row α by 2 entries. ■

Theorem 4.8. *The honeycomb graph of arbitrarily many tables has a repeating structure on the left and right sides of the graph.*

Proof. Whenever we add a new table to the graph, we add exactly 1 row. Because a set of tables is only valid if there are an odd number of them, we add 2 rows to the honeycomb graph to make it valid. The fact that we have to add two tables allows us to bypass the fact that that each row is added in an alternating forwards-backwards fashion.

Because all of the rows have the same structure, if where the duplicates are in the rows stay consistent, and where the rows are added is consistent, then the honeycomb graph will have a repeating structure.

Lemma 4.5 tells us that the range of each row scales linearly, and lemma 4.7 tells us that every row is placed in the same spot relative to the row below it. Lemma 4.6 tells us that the every row has the same types of duplicates in the same spot.

Therefore, the graph has a repeating structure. ■

This is the arbitrarily large honeycomb graph in 5 sections: the left- and right-most parts of the graph, the center of the graph, and the repeating structure on the left and the right of the graph.

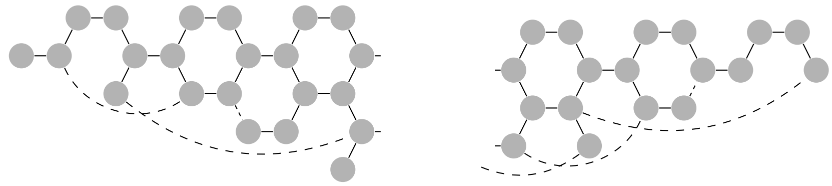

Figure 24: The left-most and right-most part of the arbitrarily large honeycomb graph

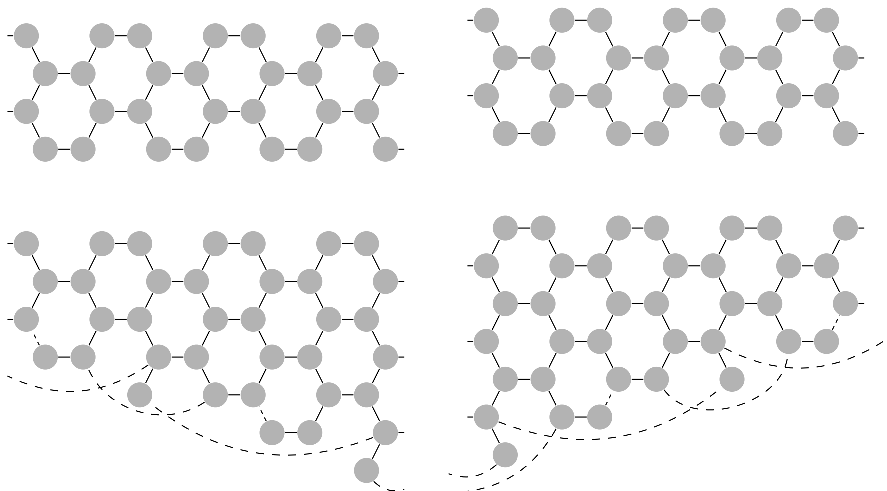

Figure 25: The repeating structure on the left and right of the graph

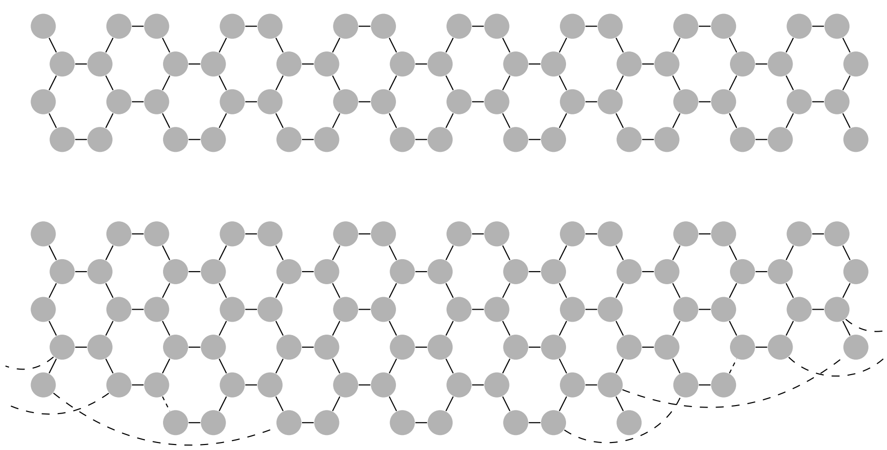

Figure 26: The center of the graph

Now that we have a picture of the arbitrarily large honeycomb graph, we can use it to prove that the path given by the table method is indeed Hamiltonian.

Theorem 4.9. *The path given by the table method is Hamiltonian.*

Proof. Recall that the dashed lines represent a what looks like two vertices actually secretly being the same vertex. The path will be Hamiltonian if it used every vertex once. For vertices that aren't in two places at once (they don't have a dashed line attached to them) this is easy. For vertices that are in two places at once, one of two conditions must be met:

1. Only one of the visual representations of the vertex is used.
2. Both of the visual representations of the vertex are used, and the path travels along the dashed line that joins the two visual representations.

Figures 27 through 29 show us that this happens

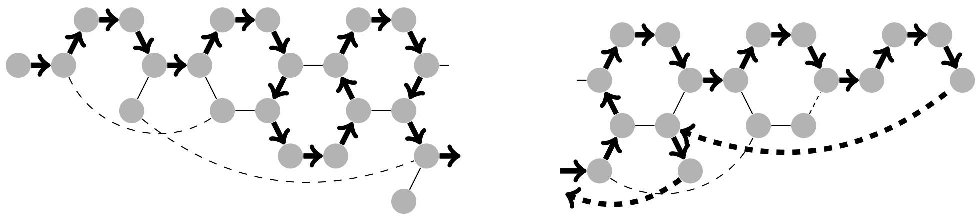

Figure 27: The left-most and right-most part of the arbitrarily large honeycomb graph and the Hamiltonian path derived from the table method.

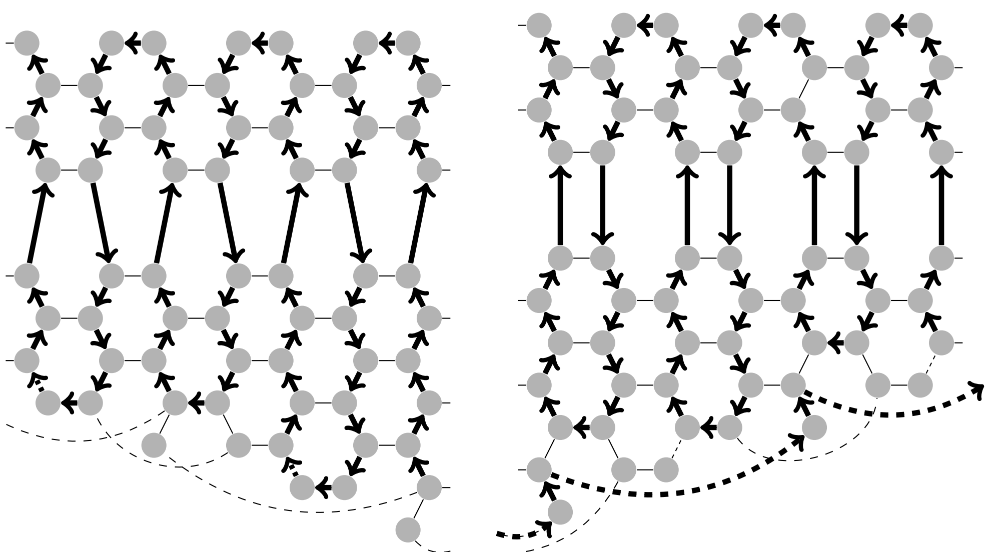

Figure 28: The repeating structure on the left and right of the graph

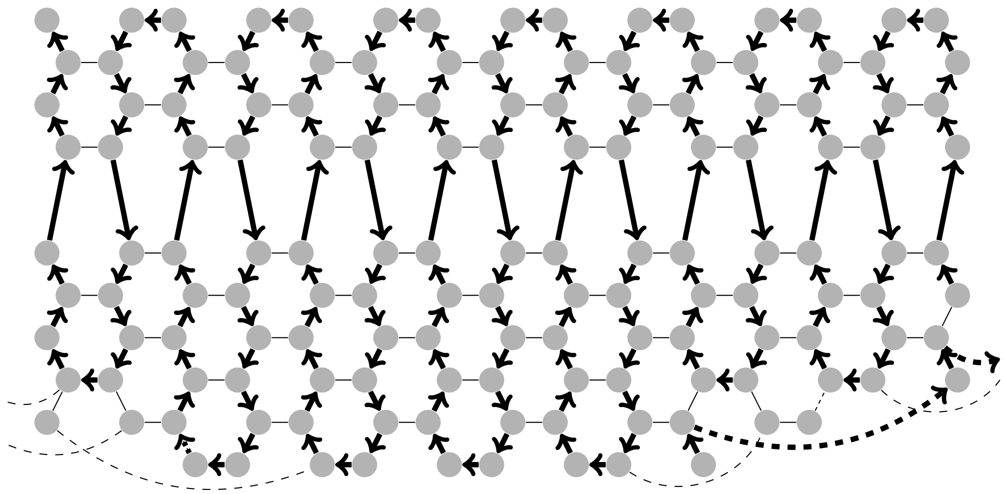

Figure 29: The center of the graph

On all of the sections of the graph, all of the vertices that aren't in two place at once are used exactly once, and all of the vertices that are in two places at once are only visited once, or are both visited using the dashed line that represents them being the same vertex.

Therefore the path given by the table method is Hamiltonian.

■

5 Algebraic Method to Generate the Paths

5.1 Instructions for the Algebraic Method

These instructions are very similar to code, and are written that way to help put this method into code.

Choose a number m which is divisible by 4.

Start at 0, then go up the sequence $a(n) = 16 \cdot \frac{n(n+1)}{2}$ (the triangular numbers times 16) until you reach $\frac{m^2}{2} + 2m$.

> **Definition: "Swoop" -** A swoop of height x means you take the most recent number in the string (we call this the "current number"), and subtract it from $(m+1)^2$ to get the new current number (every time we get a new current number we add it to the end of the string).
>
> Then we take the $2x-1$ greatest square numbers less than or equal to m^2. To get the next current numbers, we subtract the current number from each of these, going from greatest to least.
>
> Next we subtract the newest current number from $(m-2x+1)^2$ to get the next current number.
>
> Finally we once again subtract the current number from the $2x-1$ greatest squares less than or equal to m^2, this time going from least to greatest. Sometimes on the very first subtraction in this one, you will get a number twice in a row. That is because that number is half of the square we are looking at. When this happens just write down one of them.
>
> The swoop is now complete.

To find the path, do swoops of the given heights in the following order: $1, 1, 3, 3, 5, 5, \ldots, \frac{m}{2} - 3, \frac{m}{3} - 3, \frac{m}{2} - 1, \frac{m}{2} - 1, \frac{m}{2} - 1, \frac{m}{2} - 1, \frac{m}{2} - 3, \frac{m}{2} - 3, \ldots, 5, 5, 3, 3, 1, 1$.

The path is now complete.

5.2 Python Code

This is some python code to generate the path in the form of a list.

```
def swoop(swoop_height, n, swoop_start, path):
    swoop_current = swoop_start
    swoop_current = (n + 1)**2 - swoop_current
    path.append(swoop_current)
    swoop_height_counter = 1
    while swoop_height_counter <= swoop_height:
        swoop_current = (n - 2*(swoop_height_counter - 1))**2 - swoop_current
        path.append(swoop_current)
        swoop_height_counter += 1
    swoop_current = (n - 2*swoop_height + 1)**2 - swoop_current
    path.append(swoop_current)
    swoop_height_counter = 1
    while swoop_height_counter <= swoop_height:
        swoop_current = (n - 2*(swoop_height - swoop_height_counter))**2 - swoop_current
        if swoop_current not in path:
            path.append(swoop_current)
        swoop_height_counter += 1
    swoop_start = swoop_current
    return(swoop_start)

def Generate_series(n):
    if not (n % 4 == 0):
        print("n must be divisible by 4 in order for this method to work")
    else:
        path = []
        path.append(0)
        order = n/4
        counter_order = 1
        while counter_order <= order:
            triangular = ((counter_order)*(counter_order + 1)) // 2
            path.append(16*triangular)
            counter_order += 1
        swoop_start = 16*triangular
        swoop_max_height = (n // 2) - 1
        swoop_counter = 1
        while swoop_counter <= swoop_max_height:
            swoop_start = swoop(swoop_counter, n, swoop_start, path)
            swoop_start = swoop(swoop_counter, n, swoop_start, path)
            swoop_counter += 2
        swoop_counter += -2
        while swoop_counter >= 1:
            swoop_start = swoop(swoop_counter, n, swoop_start, path)
            swoop_start = swoop(swoop_counter, n, swoop_start, path)
            swoop_counter += -2
        return(path)
```

6 Closing Remarks

Of all the methods we covered, the honeycomb/tabular method proved the most fruitful for our purposes, even if it may not have seemed the most promising at first. The limitations of our result are clear. We were not able to prove that the square-sum problem for all $n \geq 25$. What we were able to do, however, is provide a constructive proof for infinitely many n. There are other methods that we will discuss in the appendix that we believe will be more likely to solve the square sum problem for every n. All of the possible proofs we discuss are existence proofs, unlike our constructive proof.

Appendix A Alternative conjectures

A.1 Hamiltonian Cycles

Definition A.1. *A Hamiltonian Cycle is a Hamiltonian path where the endpoint points are connected.*

Although much of what has been discussed previously has been related to Hamiltonian paths and rather than Hamiltonian cycles, a significant portion of our work has looked at the cycles. Note that a Hamiltonian cycle is a stronger case of a Hamiltonian path.

Conjecture A.2. *There exists a Hamiltonian cycle for all $n \geq 32$.*

We have checked conjecture A.2 for all n up to 22,000. After that, it seemed likely our conjecture was true and we began exploring methods to prove it. There were methods that seemed very likely to work, but we did not finish them. Proving there exists a Hamiltonian cycle for n is sufficient to prove there exists a Hamiltonian path for n because a Hamiltonian cycle implies there is at least one Hamiltonian path. Some of those methods we deemed worth further investigation will be discussed below.

Some necessary but not sufficient criteria we have noted for a Hamiltonian cycle are the following:

1) It must not have vertices of degree 1 or less 2) If a vertex has degree two, it must be adjacent to both of its connections 3) There can neither be a vertex nor an edge such that if it is removed, there is no Hamiltonian path 4) It must use every vertex exactly once

A.2 Hamiltonian Connectedness

Definition A.3. *A graph is Hamiltonian connected if there exists a Hamiltonian path between any two vertices.*

Conjecture A.4. *In a graph n where two vertices are connected if they sum to a perfect square, the graph is Hamiltonian connected for all $n \geq 32$.*

This conjecture has been verified using sage code for all n up to $n = 126$.

A.3 Various Shapes

Definition A.5. *A group of three numbers form a triangle if they are each connected in a graph.*

Definition A.6. *A group of four numbers form a rectangle if they form a Hamiltonian cycle among those four numbers.*

Definition A.7. *A group of k numbers form a shape of k vertices if they form a cycle of length k.*

The following two conjectures and succeeding remark could help to prove our previous two conjectures.

Conjecture A.8. *There exists a pentagon such that n is the highest number for all $n \geq 71$.*

Conjecture A.9. *There exists a rectangle such that n is the highest number for all $n \geq 189$.*

Remark A.10. *It is likely you will use A.9 to explain that there are more rectangles than vertices for a given graph, which will show importance in the Investigate Further section below.*

Definition A.11. *The diameter of a graph is the maximum eccentricity of any vertex in the graph. That is, it is the greatest distance between any pair of vertices.*

Conjecture A.12. *The diameter of $\boxed{n}$ is 4 for all $n \geq 329$*

Conjecture A.13. *If $\boxed{\infty}$ is $\boxed{n}$ with infinitely many vertices, then the diameter of $\boxed{\infty}$ is 3.*

Appendix B Important Code

All of our code can be operated using using Sage notebook. Some may require additional packages.

B.1 Hamiltonian Paths/Cylces

```
# this cell imports
from sage.graphs.generic_graph_pyx import find_hamiltonian as fh
import math
import copy

## To find a Hamiltonian path ##
# set the n value
n=32

# this makes a dictionary into a graph
G=Graph(graph_alg(n))

# if a Hamiltonian path exists, this finds a Hamiltonian path in the graph
# Sucessive runs of this cell can return different paths
a,b=fh(G,find_path=true) # This is for a path, change to find_path=false for cycles
print b
# plot(G)


### Less than ideal find_all_paths functions. Does not find all paths yet ###
def find_all_paths(n, graph, start, end, path=[]):
    path = path + [start]
    if start == end and len(path) == n:
        return path
    if start not in graph:
        return None
    for node in graph[start]:
        if node not in path:
            newpath = find_all_paths(n, graph, node, end, path)
            if newpath: return newpath
    return None

def find_ham_cycles(n, v):
    sv = find_starting_point(n)
    graph = graph_alg(n)
    if len(sv) == 1:  ##for n=18 to n=32
        start = sv[0]
        for i in range(1, n + 1):
            end = i
            print('first loop', find_all_paths(n, graph, start, end))
    elif len(sv) == 2:  ##for n=15 to n=17
        start = sv[0]
        end = sv[1]
    elif len(sv) > 2: ##for n<=14 and n=18
        print("BAD!!!")
    else:
        # for i in range(1, n + 1):
        # start = i
        # start = 1
        # for j in range(1, n + 1):
```

```
        # end = j
        start = v
        for value in graph[v]:
            end = value
            print('second loop', find_all_paths(n, graph, start, end))
```

B.2 Hamiltonian Connectedness

```
""" Gives T/F as to whether graph is Hamiltonian connected """
def f_1(n):
    G = graph_alg(n)
    cycles = []
    for v in G:
        for w in G:
            if w != v and [v,w] not in cycles and [w,v] not in cycles:
                for v_neighbor in G[v]:
                    H = copy.deepcopy(G)
                    H[v] = [v_neighbor, w]
                    for w_neighbor in G[w]:
                        H[w] = [w_neighbor, v]
                        a, b = fh(Graph(H), find_path=false)
                        if a and [v,w] not in cycles and [w,v] not in cycles:
                            cycles.append([v, w])
            else:  # if w==v
                pass  # then just go on to the next w
    return len(cycles) == binom(n, 2)
```

B.3 Degrees

```
### count_degrees function. This creates a dictionary of the degrees for each vertex ###
def count_degrees(n):
    square_list = s_l(n)
    #cube_list = cube_list(n)
    graph = graph_alg(n)
    deg_dict = {}
    for n1 in range(1, n + 1):
        deg_dict[n1] = len(graph[n1])
    return deg_dict

## This was used in tandem with the following two functions ##
def c_degrees(n, mn):
    graph = graph_alg(n)
    deg_dict = {}
    for n1 in range(1, n + 1):
        deg_dict[n1] = len(graph[n1])
    if mn:
        m = min(deg_dict.items(), key=lambda x: x[1])
        return m
    else: return deg_dict
### Average Degrees for a given n ###
def avg_degrees(n):
    sum = 0
    deg_dict = c_degrees(n, False)
    for i in range(len(deg_dict)):
        sum += deg_dict[i + 1]
    avg = sum / n
    return avg

### Minimum degree for a given n, there were some nice properties following from this ###
    deg_dict = c_degrees(n, False)
    m = min(deg_dict.items(), key=lambda x: x[1])
    return m

### find_degrees function. This uses the formula we discussed in a previous section ###
```

```
def find_degrees(n, v):
    n_degs = math.floor((n+v)**.5)-math.floor((v**.5))+math.ceil(math.ceil((2*v)**.5)-(2*v)**.5)-1
    return num_degrees

### find_starting_point function. This makes a list of vertices with degree one ###
def find_starting_point(n):
    starting_vertices = []
    degs = count_degrees(n)
    for i in range(n):
        if degs[i + 1] == 1:
            starting_vertices.append(i + 1)
    return starting_vertices
```

B.4 Sum list Functions

```
## This function makes a list of square numbers ##
def s_l(n):
    square_list = []
    for i in range(2, int(math.ceil((2*n)**.5))):
        square_list.append(i**2)
    return square_list

## Generalized version of s_l. p is for power, ex: p=2 in square_list ##
def sum_list(n, p):
    s_list = []
    for i in range(2, math.ceil((2*n)**(1/p))):
        s_list.append(i**p)
    return s_list

## Prime list ##
def prime_list(n):
    primes = [1]
    for possiblePrime in range(2, 2*n):
        isPrime = True
        for num in range(2, possiblePrime):
            if possiblePrime % = 0:
                isPrime = False
        if isPrime:
            primes.append(possiblePrime)
    return primes
```

B.5 Graph Algorithm

```
## This function creates a dictionary of connections that will be used to make a graph ##
def graph_alg(n):
    set_list = s_l(n)
    #set_list = c_l(n)
    #set_list = prime_list(n)
    d = {}
    for n1 in range(1, n + 1):
        l = []
        for item in set_list:
            if item - n1 >0 and item - n1 <=n:
                if n1 != item - n1:
                    l.append(item - n1)
        d[n1] = l
    return d

### Updated graph_alg function with a specified range ###
def ranged_graph_alg(v_min, v_max):
    set_list = s_l(v_max)
    #set_list = c_l(n)
    #set_list = prime_list(n)
    d = {}
    for n1 in range(v_min, v_max + 1):
        l = []
        for item in set_list:
            if item - n1 > v_min and item - n1 <= v_max:
                if n1 != item - n1 and item >= v_min:
                    l.append(item - n1)
        d[n1] = l
```

```
    return d
```

B.6 Diameter

Definition B.1.

```
### This will find the diameter of a graph ###
def find_diameter(n, n_max):
    with open("text_file_diameter", mode="a") as out_file_1: # receives n values and the diameter of the graph
        while n < n_max:
            H=Graph(graph_alg(n))
            DM = H.diameter()
            out_file_1.write("n=" + str(n) + ' ' + "diameter=" + str(DM) + '\n')
            n+=1
    out_file_1.close
```

B.7 Shapes

```
### Find triangles. For if you want the triangles printed ###
## This is a ##
def find_triangles(n):
    graph = graph_alg(n)
    used_dict = {}
    trials = 1
    counter = 0
    nums_unused = []
    for i in range(1, n + 1):
        nums_unused.append(i)
    while trials <= n:
        for connection in graph[trials]:
            for node in graph[connection]:
                if node != connection and node != trials:
                    tup = (trials, connection, node)
                    if node in graph[trials] and tup not in used_dict:
                        print(tup)
                        if trials in nums_unused:
                            nums_unused.remove(trials)
                        if connection in nums_unused:
                            nums_unused.remove(connection)
                        if node in nums_unused:
                            nums_unused.remove(node)
                        used_dict[(trials, connection, node)] = (trials, connection, node)
                        used_dict[(connection, trials, node)] = (connection, trials, node)
                        used_dict[(trials, node, connection)] = (trials, node, connection)
                        used_dict[(connection, node, trials)] = (connection, node, trials)
                        used_dict[(node, connection, trials)] = (node, connection, trials)
                        used_dict[(node, trials, connection)] = (node, trials, connection)
                        counter += 1
        trials += 1
    print('Total triangles:', counter)
    print('Number unused:', len(nums_unused), '. Min unused in triangles:', nums_unused[0])

## This is if you just want how many there are ##
def fast_n_lazy_find_triangles(n):
    graph = graph_alg(n)
    trials = 1
    counter = 0
    nums_unused = []
    for i in range(1, n + 1):
        nums_unused.append(i)
    while trials <= n:
        for connection in graph[trials]:
            for node in graph[connection]:
                if node in graph[trials] and node != connection and node != trials:
                    #print(trials, node, connection)
                    if trials in nums_unused:
                        nums_unused.remove(trials)
                    if connection in nums_unused:
                        nums_unused.remove(connection)
                    if node in nums_unused:
```

```
                    nums_unused.remove(node)
                counter += 1
        trials += 1
    print('Total triangles:', int(counter / 6))
    print('Number unused:', len(nums_unused), '. Min unused in triangles:', nums_unused[0])

""" this function finds all pentagons less than n """
def find_less_than_pentagons_1(n):  # see circled diagram page 20 in my notebook for variable names v-v4
    """ returns dictionary with integer keys to list of tuple of int values
    where each key is an n value and each tuple is a pentagon with the key as its largest value """
    set_list = s_l(n)
    graph = graph_alg(n)
    dp = {}  # dictionary of pentagons
    for v in graph:
        for v1 in graph[v]:
            if v1 < v:
                for v2 in graph[v1]:
                    if v2 < v:
                        for v3 in graph[v2]:
                            if v3 < v:
                                if v3 != v1 and v3 != v2:
                                    for v4 in graph[v3]:
                                        if v4 < v:
                                            if v4 != v1 and v4 != v2 and v4 != v3:
                                                if v4 in graph[v]:
                                                    dp[v] = [(v, v1, v2, v3, v4)]
                                        else: break
                            else: break
                    else: break
            else: break
    return dp

""" find rectangles """
def find_less_than_rectangles(n):
    """ returns dictionary with integer keys to a list of tuple int values where each key is an n value
    and each tuple is a rectangle with the key as its largest value"""
    set_list = s_l(n)
    graph = graph_alg(n)
    dr = {}
    for v in graph:

        for v1 in graph[v]:
            if v1 >= v:
                break
            else:

                for v2 in graph[v1]:
                    if v2 >= v:
                        break
                    else:

                        for v3 in graph[v2]:
                            if v3 >= v:
                                break
                            elif v3 < v and v3 != v1:  # implies v3 != v2 and v3 != v

                                if v3 in graph[v]:
                                    dr[v] = [(v, v1, v2, v3)]
    return dr


""" this function prints output of the pentagon finding function """
def run_the_pent(n):
    print()
    dp = find_less_than_pentagons_1(n)
    print("finding pentagons for n =", n)
    print(dp)
    no_pents_list = []
    one_pent_list = []
    mult_pents_list = []
    for key in range(1, n+1):
```

```
        if key not in dp and key >= 14:
            no_pents_list.append((key, "no pentagon"))
        elif key in dp and len(dp[key]) > 1:
            mult_pents_list.append((key, "%d pentagons" %len(dp[key])))
        elif key in dp and len(dp[key]) == 1:
            one_pent_list.append((key))
        else:
            raise Exception("else case failure")
    print("no pentagons list")
    for tup in no_pents_list:
        print(tup)
    print('-'*15)
    print("multiple pentagons list")
    for tup in mult_pents_list:
        print(tup)
    print('-'*15)
```

B.8 Other useful functions

```
### fixes lists for the disjoint edges function ###
def get_rid_of_none(mylist):
    yourlist = []
    while mylist:
        sublist = mylist.pop()
        new_sublist = []
        for (a, b, c) in sublist:
            new_sublist += [(a, b)]
        yourlist.append(new_sublist)
    return yourlist

### List all disjoint edges ###
def disjoint_edges(n, n_max):
    from sage.graphs.independent_sets import IndependentSets
    with open("text_file_disjoint_edges", mode="a") as out_file_1: # receives all pairs of disjoint edges
        while n < n_max:
            H=Graph(graph_alg(n))
            L=H.line_graph()
            is2 = list((x for x in IndependentSets(L) if len(x) == 2))
            l=len(is2)
            out_file_1.write("n="+str(n)+' '+"amount="+str(l)+'\n\n'+str(get_rid_of_none(is2))+'\n\n\n')
            n+=1
    out_file_1.close

### Test if the generalized petersen graph is a subgraph. Helps find Hamiltonian connectedness ###
def petersen_subgraph(n, n_max):
    with open("text_file_petersen_subgraph", mode="a") as out_file_1:
    #this receives n values and whether the graph has a generalized Petersen graph as a 'large' subgraph ##
        while n < n_max:
            H=Graph(graph_alg(n))
            GP=graphs.GeneralizedPetersenGraph(n/2, 3)
            T = GP.is_subgraph(H)
            out_file_1.write("n=" + str(n) + ' ' + "GPG subgraph? " + str(T) + '\n')
            if T:
                print("n=" + str(n) + ' ' + "GPG subgraph? " + str(T))
            n+=4
    out_file_1.close


### Finds largest amount of vertices you can remove. Starting from vertex 1 and going up. ###
## Has a probelm with degree one vertices as well ##
def find_max_vert(n, n_max):
    with open("text_file_maximum_vertices_maximum.txt", mode="a") as out_file_1:
        with open("text_file_maximum_vertices_info.txt", mode="a") as out_file_2:
            with open("text_file_maximum_vertices_paths.txt", mode="a") as out_file_3:
                def find_v(m, v):
                    H=Graph(ranged_graph_alg(v, m))
                    tf, hp = fh(H,find_path=true)
                    if tf:
                        out_file_2.write(str(tf) + " for v= " + str(v-1) + '\n')
                        out_file_3.write("v=" + str(v) + ' ' + str(hp) + '\n')
                        find_v(m, v+1)
```

```
                if not tf:
                    out_file_2.write("FALSE FOR v=" + str(v-1) + " !!!!")
                    out_file_1.write("n=" + str(m) + " maximum vertex removed is " + str(v-2) + '\n')
            while n < n_max:
                out_file_2.write("n=" + str(n) + '\n\n')
                out_file_3.write("n=" + str(n) + '\n\n')
                find_v(n, 1)
                out_file_2.write('\n\n\n')
                out_file_3.write('\n\n\n')
                print("n=" + str(n) + " done")
                n+=1

### Finding Independence Number ###
def write_independence_number(n, n_max):
    from sage.graphs.independent_sets import IndependentSets
    with open("text_file_independence_number", mode="a") as out_file_1: #contains independence number of graph
        while n < n_max:
            H=Graph(graph_alg(n))
            Im = IndependentSets(H, maximal = True)
            number_of = [0] * H.order()
            for x in Im: number_of[len(x)] += 1
            for i in range(len(number_of)):
                if number_of[i] != 0:
                    independence_number = i
            out_file_1.write("n=" + str(n) + ' ' + 'independence number: ' + str(independence_number) + '\n')
            n+=1
        out_file_1.close
```

Appendix C Investigate Further

There were many avenues we discovered along the way that did not apply to the work already stated, but we believe will eventually take someone in a promising direction. Below we will discuss some of those.

C.1 Shapes

As mentioned earlier, there are occasionally shapes that show up in various graphs. The tables above even contain triangles, pentagons, and nonagons. However, shapes are much more common when we do not restrict the graph to fewer vertices. For instance, the graph for $n = 50$ contains four triangles, the smallest of which is the 6, 19, 30 triangle. We have spent some time exploring these shapes, and it turns out they can be very useful when trying to find a Hamiltonian cycle for $n + 1$ given there is a Hamiltonian cycle for n.

Consider a Hamiltonian cycle for n containing two adjacent vertices a and b such that both a and b can be connected to $n + 1$. Then we could make a cycle for $n + 1$ by putting the vertex $n + 1$ between a and b and keeping the rest of the cycle the same. This would be using a triangle property between a, b, and $n + 1$ This example is below.

Cycle for n: ......z, a, b, c ......
Cycle for $n + 1$ ...... z, a, $n + 1$ b, c ......

This is what we refer to as a level one shift. A level two shift would utilize a pentagon, a level three would utilize a heptagon, and so on.

You can often rearrange a cycle within itself using rectangles, hexagons, octagons, etc, which also often tend to appear in our square sum graph. We refer to those rearrangements as rectangular switches, hexagonal shuffles, etc.

We believe a proof that there always exists a rectangle that can set up a level two shift (utilizing a pentagon) is a good first step. After that, you would likely need to prove it always leads to, "the right cycle" for n in order to set up a level shift. Then all that would need to be done is a proof that there always exists a pentagon (or some shape) such that n is the highest number. That would be sufficient to prove that there exists a Hamiltonian cycle for all n greater than or equal to the point when both of those conjectures are true.

According to our code results, we believe there is always a pentagon such that n is the highest number beginning at $n = 71$. We also believe there always exists a rectangle such that n is the highest number at $n = 189$. That would also imply there are more rectangles than vertices beginning at $n = 220$, showing there would have to be, "the right rectangle" to set up a level shift by manipulating the cycle for n.

We have created a google slides presentation that explains the different shifts and shuffles more thoroughly. We believe this could one day lead to a solid proof. The link is below.

`https://docs.google.com/presentation/d/1z2Ajz1RS4Wld-x7xKsd_pTNMHtioiHZW1DKQKWwBtWc/edit?usp=sharing`

Here is the output of our code examining the various shapes and their frequencies in the square sum graph:

`https://docs.google.com/spreadsheets/d/19fs5FyqRiMNwdTyk4RCHK5CzDhlfPMtKBGzbCG6wpjk/edit?usp=sharing`

Appendix D Possible Proof Methods for the Future

There are many ways we believe a proof of Parker's Conjecture might come from. Here are some of them organized into categories:

D.1 Proofs by Contradiction

1. There exists a vertex $g \in \boxed{\text{n}}$ such that $g \notin P$, where P is the longest path in $\boxed{\text{n}}$.
2. There exists some n such that $\boxed{\text{n}}$ has a hamiltonian path/cycle, but $\boxed{\text{n+1}}$ does not.
3. No smallest counterexample

D.2 Possible Inductive Hypotheses

1. Hamiltonian path (starts at 25)
2. Hamiltonian cycle (starts at 32)
3. Hamiltonian connected (starts at 32)
4. every edge used in at least one Hamiltonian path (starts at 27?)
5. Take an arbitrarily long Hamiltonian path (like the one found here), and remove vertices

D.3 Possible Direct Proofs

1. There is always a level add in
2. it is always possible to shuffle to a HP/HC that does have a level add in
3. if we add "red" edges to a graph of $\boxed{\text{n}}$ such that they connect every two vertices that vertex $n + 1$ would have been adjacent to, then prove there is always a Hamiltonian path that uses a "red" edge

D.4 Possible Proofs by Corollary

1. The amount of Hamiltonian paths a graph has approaches some non-zero constant or proportion would prove that there is always a Hamiltonian path.
2. The number of vertices that can be removed where $\boxed{\text{n}}$ (minus the removed vertices) still has a Hamiltonian path approaches some non-zero constant or proportion would prove there is always a Hamiltonian path when no vertices are removed.