Ingeniørhøjskolen Århus

Discrete Mathmatics

Hand in 4

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Problems

1 Which of the following sets are well-ordered? (Why/why not?)

a. $S = x \in \mathbb{Q}\ :\ x \geq -10$
b. $S = -2, -1, 0, 1, 2$
c. $S = x \in \mathbb{Q}\ :\ -1 \leq x \leq 1$
d. $S = p\ :\ p\ is\ prime = 2, 3, 5, 7, 9, 11, 13, ...$

First we look at the definition:
A set is well-ordered if every nonempty subset has a least element.
Then we look at the sets:

a. S is not well-ordered set because we can make a subset that doesn't contain a least element. e.g. x > -10.
b. S is a well-ordered set because we can always produce a least element from the subsets of S.
c. S is not a well-ordered set because we can make a subset that doesn't contain a least element. e.g. 0 < x ≤ 1.
d. S is a well-ordered set because we can always produce a least element from the subsets of S.

2 Use mathematical induction to prove that $1 + 5 + 9 + ... + (4n - 3) = 2n^2 - n$ for every positive integer n.

We have $n \in \mathbb{Z}^+$.
We start by establishing the base case:

$$(4n - 3) = 2n^2 - n \Rightarrow (4 * 1 - 3) = 2 * 1^2 - 1 \Rightarrow 1 = 1 \tag{1}$$

$p(k) \Rightarrow p(k + 1)$ Assume $p(k)$
$p(k + 1) \equiv p(k) + (4n - 3) \equiv 2k^2 - k + (4(k + 1) - 3)$

$$2k^2 - k + 4k + 1 \tag{2}$$
$$2k^2 - k + 4k + 1 + 1 - 1 \tag{3}$$
$$2k^2 + 4k + 2 - k - 1 \tag{4}$$
$$2(k^2 + 2k + 1) - (k + 1) \tag{5}$$
$$2(k + 1)^2 - (k + 1) \tag{6}$$

We see that his is our assumption for k + 1.

3 Prove that $2^n > n^3$ for every integer $n \geq 10$

Note: you will need to really work with inequalities. Assume m such that $2^n \leq n^3$
Initial:$2^{10} = 1024 > 1000 = 10^3$
m must be bigger than 10.
$m = k + 1$ where $10 \leq k < m$
$2^k > k^3$
$2^m = 2^{k+1}$
$= 2 * 2^k$
$> 2 * k^3$ because $2^k > k^3$
$= k^3 + k^3$
$\geq k^3 + 10k^2$ because $10 \leq k$
$= k^3 + 3k^2 + 7k^2$
$> k^3 + 3k^2 + 3k + 4k^2$ because $3k^2 > 3k$
$> k^3 + 3k^2 + 3k + 1$ because $4k^2 > 1$
$= (k + 1)^3$
$= m^3$
Which is a contradiction!
This proves that $2^n > n^3$ for every integer $n \geq 10$.

4 Use the method for minimum counterexample to prove that $3|(2^{2n} - 1)$ for every positive integer n.

Base case: $p(1) = 2^{2*1} - 1 = 3$ so True.
Assume: $p(k) = 2^{2k} - 1 = 3m$ where m is an integer.

$p(k + 1) = 2^{2(k+1)} - 1$
$= 2^{2k+2} - 1$
$= 2 * 2 * 2^{2k} - 1$
$= 4 * 2^{2k} - 1$
$= 4 * 2^{2k} - 1 + 4 - 4$
$= 4 * 2^{2k} + 3 - 4$
$= 4 * (2^{2k} - 1) + 3$
$= 4 * (3m) + 3$
$= 3 * (4m + 1)$
Which is divisable by 3.
so p(n) is always divisable by 3 forall $n \in \mathbb{Z}^+$.

5 Use the Strong Principle of Mathmatical Induction to prove the following:

Let $S = i \in \mathbb{Z} : i \geq 2$ and let P be a subset of S with the properties that $2, 3 \in P$ and if $n \in S$, then either $n \in P$ or $n = ab$, where $a, b \in S$. Then every element of S either belongs to P or it can be expressed as a product of elements of P.

Note: read Theorem 11.17, though the proof of 11.17 is not the proof of this question.
Let $LetS = i \in \mathbb{Z} : i \geq 2$ and $P \subseteq S$ and $2, 3 \in P$

Base Case:
n = 2 then $n \in P$.
n = 3 then $n \in P$.
The basecase is trivial.

Next up we assume the strong induction step:
$\forall x \in S, Q(k)$ is true.
for Q(k+1) we make 2 cases:
Case 1:
$(k+1) \in P$ this means we have arrived at the end of our proof.
Case 2:
$(k+1) \notin P$ then $(k+1) = a * b$ where $a, b \in S$
This means that a and b must be larger or equal to 2 (as they are a part of S) and they must be smaller than k + 1. We write:
$k+1 > a \geq 2$ and $k+1 > b \geq 2$.
When substituting a and b into Q(k) we see that Q(a) and Q(b) is in the range of Q(k) which we assumed to be true.
Then a and b either belongs to P or is a product of the elements in P. This leads to: $k+1 = a * b$ is a product of elements in P.
This proves that:
Every element of S either belongs to P or it can be expressed as a product of elements of P.

6 Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 12$, there are non-negative integers a and b such that $n = 3a + 7b$.

Note: this uses generalized strong induction and minimum counterexamples. Given is: $n \geq 12$ and $a, b \in \mathbb{Z}^+$ and $n = 3a + 7b$
We see that p(n): 12 = 3 * 4 + 7 * 0.
But know p(k) does not help us reach k + 1 because our smallest known step is 3.
p(n+3) = 15.
We have to establish the following base cases:
Case 1: $n : 12 = 3 * 4 + 7 * 0$
Case 2: $n : 13 = 3 * 2 + 7 * 1$
Case 3: $n : 14 = 3 * 0 + 7 * 2$

Strong induction step:
It holds for n = 15, because it holds for 12 and we can express it as n+3.
Now we know that we can get every positive integer $\geq$ 15 from our 3 base cases.

Therefore p(n) is true for all $n \geq 12$.