How can elliptic curves be used to establish a shared secret over an insecure channel?

Introduction

Internet connections and data go through Internet Service Providers (ISP) which snoop on users' information. [citation needed] An often used method to prevent eavesdropping is through TLS, commonly known as the green padlock next to the address bar or HTTPS, [citation needed] which establishes a secure connection between the user and the website that they are connecting to, such that the ISP only knows which website they have connected to but does not know the content that the user has downloaded or uploaded.

TLS has many different cryptographic techniques to establishing a secure connection. One of which is the Elliptic Curve Diffie-Hellman (ECDH). In this paper, we will examine the mathematical theory underlying the ECDH operation and evaluate its practical application in cybersecurity.

The Discrete Log Problem in $\mathbb{Z}_p^\times$

As a consequence of Fermat's Little Theorem, we can write:

$$a^{p-1} \equiv 1 (\mathrm{mod}\, p)$$
$$a \cdot a^{p-2} \equiv 1 (\mathrm{mod}\, p)$$

For any integer a and prime p. We have found a^{p-2} as a's multiplicative inverse, therefore integers modulo p with multiplication forms a group, as the operation is associative, has an identity element, and every element has an inverse.

Elliptic curves

Let an elliptic curve be denoted by the equation $y^2 = x^3 + Ax + B$ where A and B are constants. Note that the curve is symmetric about the x-axis, since if (x, y) is a point on the curve, $(x, -y)$ is also on the curve.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be distinct points on the curve, where $x_1 \neq x_2$. We can find a new point on the curve by defining a line that goes across the two points, with slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

And line equation

$$y = m(x - x_1) + y_1$$

We substitute this into the equation of the curve:

$$(m(x - x_1) + y_1)^2 = x^3 + Ax + B$$

Expanding and rearranging gives:

$$x^3 - m^2x^2 + (2m^2x_1 - 2y_1m + A)x + 2y_1mx_1 - m^2x_1^2 - y_1 = 0$$

With Vieta's formulas, the sum of roots for the cubic is m^2. We already know two roots of this polynomial as P_1 and P_2 are common points on the curve and the line, so we can find the x coordinate of the third point:

$$x_3 = m^2 - x_1 - x_2$$

In the group law, the y coordinate of the resulting point is flipped: (TODO: explain why)

$$y_3 = -(m(x_3 - x_1) + y_1) = m(x_1 - x_3) - y_1$$

Therefore, we have arrived at $P_3 = (x_3, y_3)$, a third point distinct from P_1 and P_2.

If only one point $P_1 = (x_1, y_1)$ is known, we can use implicit differentiation to find the tangent line:

$$y^2 = x^3 + Ax + B$$
$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + A$$
$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 + A}{2y} = \frac{3x_1^2 + A}{2y_1}$$

With the same line equation $y = m(x - x_1) + y_1$, with the same expanded formula:

$$x^3 - m^2x^2 + ... = 0$$

But this time, x_1 is a repeated root, as a tangent line either touches no other points at all (the case when $y = 0$) or touch one other point.

Therefore, we can find the third point with

$$x_3 = m^2 - 2x_1$$

And

$$y_3 = -(m(x_3 - x_1) + y_1) = m(x_1 - x_3) - y_1$$

Therefore, we can begin to define a group law for points on elliptic curves.

Let $C : y^2 = x^3 + Ax + B$ be the elliptic curve with the set of points that satisfy the given equation. We now show that $C \cup \{\infty\}$ forms a group.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points that are on the curve. Define $P_3 = P_1 + P_2$ to be as follows:

- If $P_1 = P_2 = (x_1, y_1)$, let

$$P_3 = (m^2 - 2x_1, m(x_1 - x_3) - y_1), \text{where } m = \frac{3x_1^2 + A}{2y_1}$$

- If $x_1 = x_2$ but $y_1 \neq y_2$ (N.B. the only case where this happens is $y_1 = -y_2$): let $P_3 = \infty$.
- Otherwise, let

$$P_3 = (m^2 - x_1 - x_2, m(x_1 - x_3) - y_1), \text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Additionally, define $P_1 + \infty = \infty + P_1 = P_1$, as well as $\infty + \infty = \infty$.

Proof that $C \cup \{\infty\}$ forms a group:

1. The operation $+$ is well-defined for any points $P_a + P_b$ where $P_a, P_b \in C \cup \{\infty\}$ as above.
2. ∞ is the identity element, where $P_a + \infty = P_a$ for all $P_a \in C \cup \{\infty\}$.
3. Every element has an inverse: let $P_a = (x, y)$, its inverse is $-P_a = (x, -y)$. TODO show that $-P_a \in C$.
4. The operation is associative, where $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$. This is shown in chapter 2.4 in the book titled “Elliptic Curves: Number Theory and Cryptography”, and the proof gets too long, so we have omitted it here.