

# **Recursion and Lists**

**Concepts of Programming Languages**  
**Lecture 2**

# Practice Problem

*None today...*

# Outline

- » Cover the **basic expressions** we need to start programming in OCaml, look at some examples
- » Discuss **lists** so we can write actually interesting programs
- » Talk about **tail recursion** and how that affects performance of OCaml programs

# Recall: Functional vs. Imperative

OCaml is a **functional language**. This means a couple things:

- » No state (which means no loops!)
- » We don't think of a program as **describing a procedure**, but as **defining a value using an expression**

# Recall: The Three Components


# Recall: The Three Components

**Syntax:** What a *well-formed* program in your PL?

```
def f():  
    return 3
```



```
define f():  
    3 return
```




# Recall: The Three Components

**Syntax:** What a *well-formed* program in your PL?

```
def f():  
    return 3
```



```
define f():  
    3 return
```




**Type System (Static Semantics):** What is a *valid* program in your PL?

```
x = 2 + 2
```



```
x = 2 + "two"
```



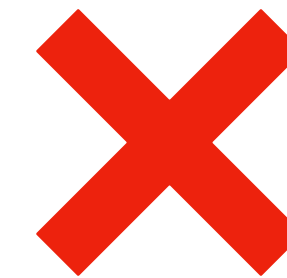
# Recall: The Three Components

**Syntax:** What a *well-formed* program in your PL?

```
def f():  
    return 3
```



```
define f():  
    3 return
```

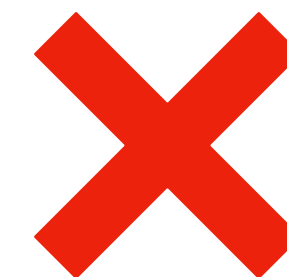


**Type System (Static Semantics):** What is a *valid* program in your PL?

```
x = 2 + 2
```



```
x = 2 + "two"
```



**Semantics (Dynamic Semantics):** What is the *output* of a (valid) program?

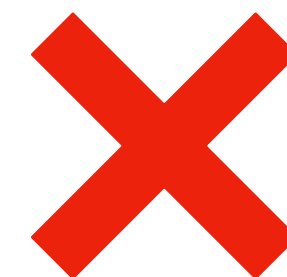
```
>>> 2 + 2
```

```
4
```



```
>>> 2 + 2
```

```
False
```





For every possible expression, we'll  
define the syntax rules, the typing  
rules, and the semantic rules

# **One Last Point: Building Interpreters**

# One Last Point: Building Interpreters

*PL is math* but we still like to *use* PLs. The three components of a PL correspond to the three things we need to *implement* in an **interpreter** of a PL.

# One Last Point: Building Interpreters

*PL is math* but we still like to *use* PLs. The three components of a PL correspond to the three things we need to *implement* in an **interpreter** of a PL.

» **Syntax** is implemented by a **parser**

`parse : string -> expr`

# One Last Point: Building Interpreters

*PL is math* but we still like to *use* PLs. The three components of a PL correspond to the three things we need to *implement* in an **interpreter** of a PL.

» **Syntax** is implemented by a **parser**

```
parse : string -> expr
```

» **Type system** is implemented by a **type checker**

```
type_check : expr -> bool (* valid or not *)
```

# One Last Point: Building Interpreters

*PL is math* but we still like to *use* PLs. The three components of a PL correspond to the three things we need to *implement* in an **interpreter** of a PL.

» **Syntax** is implemented by a **parser**

```
parse : string -> expr
```

» **Type system** is implemented by a **type checker**

```
type_check : expr -> bool (* valid or not *)
```

» **(Dynamic) semantics** is implemented by an **evaluator**

```
eval : expr -> value
```

# Basic Expressions

# Basic Expressions

» Literals

» Let-expressions (local variables)

» If-expressions

» Functions

» Applications



# Basic Expressions

## » Literals

» Let-expressions (local variables)

» If-expressions

» Functions

» Applications

# Primitive Types and Literals

OCaml has a collection of standard literals and types

Type	Literals	Operators
int	0, -2, 13, -023	+, -, *, /, mod
float	3., -1.01	+. , -. , *. , /.
bool	true, false	&&,   , not
char	'b', 'c'	
string	"word", "@*&#"	^

# Basic Expressions

» Literals

» **Let-expressions (local variables)**

» If-expressions

» Functions

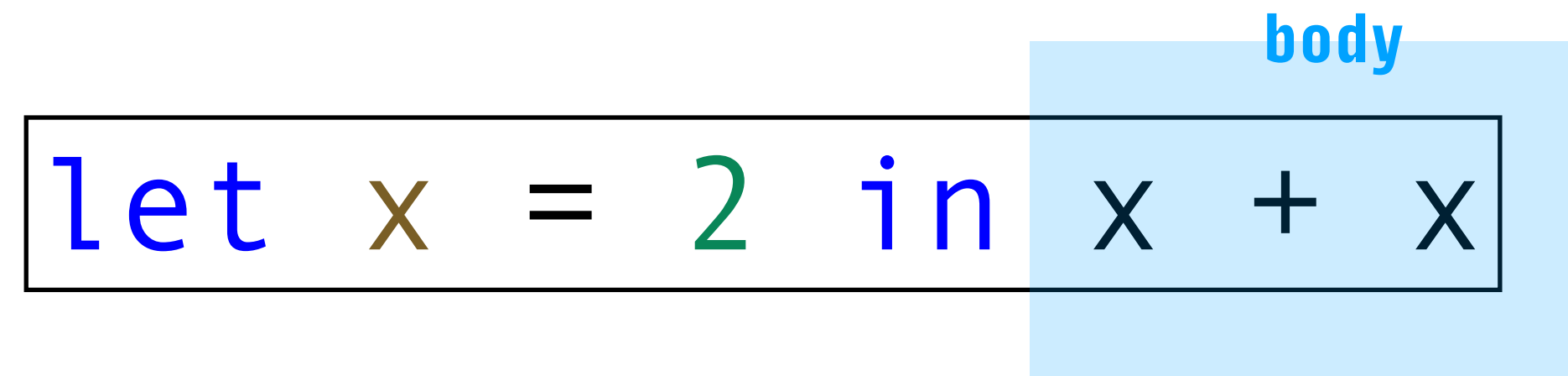
» Applications

# Let-Expressions (Informal)

`let x = 2 in x + x`

body

# Let-Expressions (Informal)

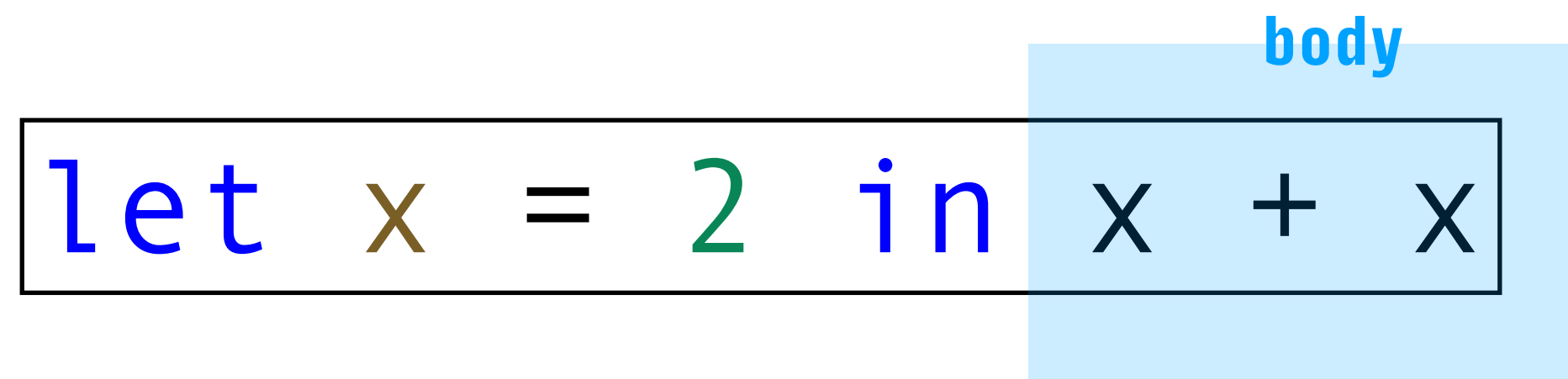


The diagram shows the code `let x = 2 in x + x` enclosed in a black rectangular box. A light blue rectangular area highlights the entire expression, with the word `body` in blue text positioned above the right side of this area. The code itself is color-coded: `let` is blue, `x` is brown, `=` is black, `2` is green, `in` is blue, `x` is brown, `+` is black, and `x` is brown.

```
let x = 2 in x + x
```

**syntax:** `let VARIABLE = EXPRESSION in BODY`

# Let-Expressions (Informal)

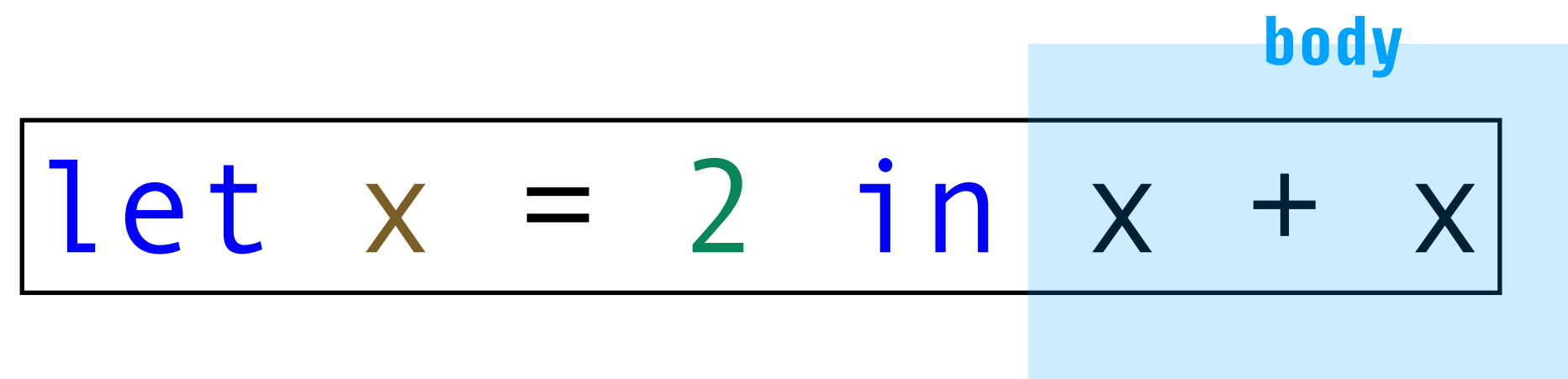


The diagram shows the code `let x = 2 in x + x` enclosed in a black rectangular box. A light blue rectangular box highlights the expression `x + x` on the right side of the `in` keyword. Above the light blue box, the word `body` is written in blue text.

**syntax:** `let VARIABLE = EXPRESSION in BODY`

**typing:** the type is the same as that of BODY *given BODY is well-typed after substituting the VARIABLE in BODY*

# Let-Expressions (Informal)



The diagram shows the expression `let x = 2 in x + x` enclosed in a black rectangular box. The word `let` is blue, `x` is brown, `=` is black, `2` is green, `in` is blue, and `x + x` is black. A light blue rectangular area highlights the entire expression, with the word `body` in blue text positioned above the right side of the box.

**syntax:** `let VARIABLE = EXPRESSION in BODY`

**typing:** the type is the same as that of BODY *given BODY is well-typed after substituting the VARIABLE in BODY*

**semantics:** the is the same as the value of BODY *after substituting the VARIABLE in BODY*

# Example: Ill-Typed Let-Expression

```
let x = 2 in "two" <> x
```

An ill-typed expression will throw a type error when you type it into utop

Note that the body of a let-expression may be ill-typed *depending on the value assigned to its variable*



# A Note on Substitution

let  $x = 2$  in  $x + x$



$2 + 2$

# A Note on Substitution

$$\boxed{\text{let } x = 2 \text{ in } x + x} \longrightarrow \boxed{2 + 2}$$

Formally, we write  $[v/x]e$  to mean "substitute  $v$  for  $x$  in  $e$ ",  
e.g.,  $[3/x](x + x)$  is the same as  $3 + 3$

# A Note on Substitution

$$\boxed{\text{let } x = 2 \text{ in } x + x} \longrightarrow \boxed{2 + 2}$$

Formally, we write  $[v/x]e$  to mean "substitute  $v$  for  $x$  in  $e$ ",  
e.g.,  $[3/x](x + x)$  is the same as  $3 + 3$

Intuitively, substitution is simple: **replace the variable**

# A Note on Substitution

$$\boxed{\text{let } x = 2 \text{ in } x + x} \longrightarrow \boxed{2 + 2}$$

Formally, we write  $[v/x]e$  to mean "substitute  $v$  for  $x$  in  $e$ ",  
e.g.,  $[3/x](x + x)$  is the same as  $3 + 3$

Intuitively, substitution is simple: **replace the variable**

Turns out, this is **very hard** to do correctly, *it's subtle*  
and a source of a lot of mistakes in PL implementations

# Recall: Anatomy of an OCaml Program

```
let x = 3

let y = "string"

(* function definition *)
let square x = x * x

(* recursive function definition *)
let rec f x = if x = 0 then 0 else x + f (x - 1)

(* We can't just print , we assign to wildcard *)
let _ = print_endline("Hello world")
```

An OCaml Program consists of top-level let-expressions, i.e., it is a **collection of named expressions**

# OCaml Programs are Expressions

```
let x = 3 in

let y = "string" in

(* function definition *)
let square x = x * x in

(* recursive function definition *)
let rec f x = if x = 0 then 0 else x + f (x - 1) in

(* We can't just print , we assign to wildcard *)
let _ = print_endline("Hello world") in

()
```

This sequence of top-level let expressions is really shorthand for a **collection of nested local variables**

*(This is a lie, but its a useful one for now)*

# Basic Expressions

» Literals

» Let-expressions (local variables)

» **If-expressions**

» Functions

» Applications

# If-Expressions

```
let abs x = if x > 0 then x else -x
```

Note: OCaml is whitespace agnostic!



# If-Expressions

```
let abs x = if x > 0 then x else -x
```

Note: OCaml is whitespace agnostic!

OCaml has expressions for conditional reasoning

# If-Expressions

```
let abs x = if x > 0 then x else -x
```

Note: OCaml is whitespace agnostic!

OCaml has expressions for conditional reasoning

**Note:** The **else** case is *required* and the **then** and **else** cases must be the *same type* (why?)

# If-Expressions

```
let foo x =  
  if x < 0 then  
    "negative"  
  else if x = 0 then  
    "zero"  
  else  
    "positive"
```

**Answer:** Remember, all we have is expressions. So every if-expression must have a value and a type (and therefore, an **else** case of the same type)

We can do **else if** just by nesting if-expressions! (neat)

# Aside: If-Expressions in Python

```
if x < 0:  
    return -1  
else:  
    return 1
```

if-stmt (Python)

```
return (-1 if x < 0 else 1)
```

if-expr (Python)

If-*statements* in Python are different from if-expressions, but **both are available**

Statements don't have a value, expressions do

# If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

# If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

**Syntax:** if CONDITION then TRUE-CASE else FALSE-CASE

# If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

**Syntax:** if CONDITION then TRUE-CASE else FALSE-CASE

**Typing:** CONDITION must be a Boolean and TRUE-CASE and FALSE-CASE must be the same type. The type is then the same as that of TRUE-CASE and FALSE-CASE

# If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

**Syntax:** if CONDITION then TRUE-CASE else FALSE-CASE

**Typing:** CONDITION must be a Boolean and TRUE-CASE and FALSE-CASE must be the same type. The type is then the same as that of TRUE-CASE and FALSE-CASE

**Semantics:** If CONDITION holds, then we get the TRUE-CASE, otherwise we get the FALSE-CASE



# Basic Expressions

» Literals

» Let-expressions (local variables)

» If-expressions

» **Functions**

» Applications

# Functions

```
let f x y z = x + y + z  
let f (x : int) (y : int) (z : int) : int = x + y + z
```

There are a couple ways of defining functions in OCaml

Note that let-expression can take arguments. ***How should we interpret this? If everything is an expression?***

# Anonymous Functions

```
let f = fun x -> fun y -> fun z -> x + y + z
```

**Answer:** It must be that **functions are expressions as well!**

In OCaml, we can define *anonymous* functions, which are just **functions without names**

You should think of:

```
let f x y z = x + y + z
```

as shorthand for the above

# Aside: Type Annotations

```
let rec fact (n : int) : int =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

# Aside: Type Annotations

```
let rec fact (n : int) : int =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

OCaml has type inference which means we rarely have to *specify* the types of expression in our program

# Aside: Type Annotations

```
let rec fact (n : int) : int =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

OCaml has type inference which means we rarely have to *specify* the types of expression in our program

That said, you **should** include type annotations, especially at the beginning, because they're useful for *documentation* and for *code clarity*

# Aside: Anonymous Functions in Python

```
lambda x: x + 1
```

Python

```
fun x -> x + 1
```

OCaml

$$\lambda x. x + 1$$

Math

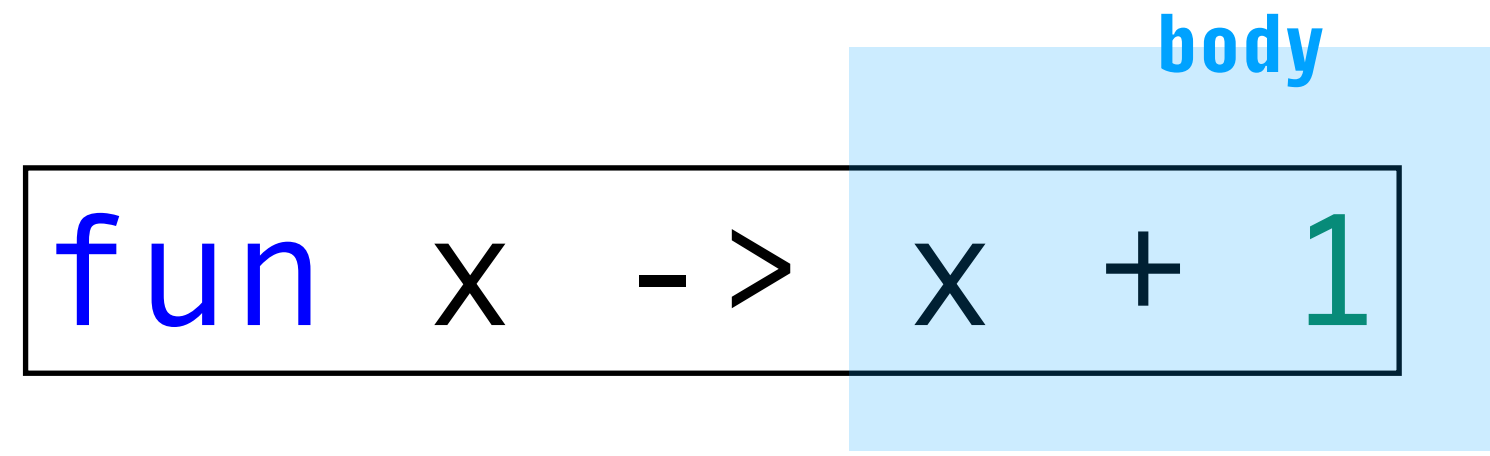
There are also anonymous function in Python!

They're called *lambdas*, based on the **lambda calculus**, a mathematical formulation of a functional PL that dates back to the 1930s, invented by **Alonzo Church**

*(You'll find a lot of functional ideas hidden in languages like Python)*



# Functions (Informal)



```
fun x -> x + 1
```

**Syntax:** `fun VAR-NAME -> EXPR`

**Typing:** the type of a function is **`T1 -> T2`** where **`T1`** is the type of the input and **`T2`** is the type of the output

**Semantics:** A function will evaluate to special *function value* (printed as `<fun>` by utop)



# Important: Curried Functions

```
let f = fun x -> fun y -> fun z -> x + y + z
```

The only kind of function we have is *single argument*

This seems restrictive, but ultimately it doesn't affect us at all

We can *simulate* multi-argument functions with nested functions. This is called **Currying** after Haskell Curry

# Important: Curried Functions

```
let f = fun x -> fun y -> fun z -> x + y + z
```

We should think of the above function as something which takes an input and returns **another function**

In other words, we *partially apply* the function

# Basic Expressions

» Literals

» Let-expressions (local variables)

» If-expressions

» Functions

» **Applications**

# Application

```
(fun x -> fun y -> x + y + 1) 3 2
```

Application is done by *juxtaposition* which means we put the arguments next to the function

Application is *left-associative*, which means we pass arguments from left to right

# Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

# Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

**Syntax:** FUNCTION-EXPR ARG-EXPR

# Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

**Syntax:** FUNCTION-EXPR ARG-EXPR

**Typing:** If FUNCTION-EXPR is of type  $T1 \rightarrow T2$ ,  
and ARG-EXPR is of type  $T1$ , then the type is  $T2$

# Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

**Syntax:** FUNCTION-EXPR ARG-EXPR

**Typing:** If FUNCTION-EXPR is of type  $T1 \rightarrow T2$ ,  
and ARG-EXPR is of type  $T1$ , then the type is  $T2$

**Semantics:** Substitute the value of ARG-EXPR into  
the body of FUNCTION-EXPR and evaluate that



# Application (Example)

# Application (Example)

```
(fun x -> fun y -> x + y + 1) 3 2
```

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`((fun x -> (fun y -> x + y + 1)) 3) 2`

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`( (fun x -> (fun y -> x + y + 1)) 3 ) 2`

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`( (fun x -> (fun y -> x + y + 1)) 3 ) 2`  
`(fun y -> 3 + y + 1) 2`

*evaluates to*

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`( (fun x -> (fun y -> x + y + 1)) 3 ) 2`

*evaluates to*

`(fun y -> 3 + y + 1) 2`

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`( (fun x -> (fun y -> x + y + 1)) 3 ) 2`

*evaluates to*

`(fun y -> 3 + y + 1) 2`

*evaluates to*

`3 + 2 + 1`



# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`((fun x -> (fun y -> x + y + 1)) 3) 2`

*evaluates to*

`(fun y -> 3 + y + 1) 2`

*evaluates to*

`3 + 2 + 1`

*is the same as*

`(3 + 2) + 1`

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`((fun x -> (fun y -> x + y + 1)) 3) 2`

*evaluates to*

`(fun y -> 3 + y + 1) 2`

*evaluates to*

`3 + 2 + 1`

*is the same as*

`(3 + 2) + 1`

*evaluates to*

`5 + 1`

# Application (Example)

`(fun x -> fun y -> x + y + 1) 3 2`

*is the same as*

`(fun x -> (fun y -> x + y + 1)) 3 2`

*is the same as*

`((fun x -> (fun y -> x + y + 1)) 3) 2`

*evaluates to*

`(fun y -> 3 + y + 1) 2`

*evaluates to*

`3 + 2 + 1`

*is the same as*

`(3 + 2) + 1`

*evaluates to*

`5 + 1`

*evaluates to*

`6`

# demo

(extended example)

# Lists

# What is a list?

```
let _ = 1 :: 2 :: 3 :: []  
let _ = 1 :: (2 :: (3 :: []))  
let _ = [1; 2; 3]
```

A list is an ordered *variable-length homogeneous* collection of data

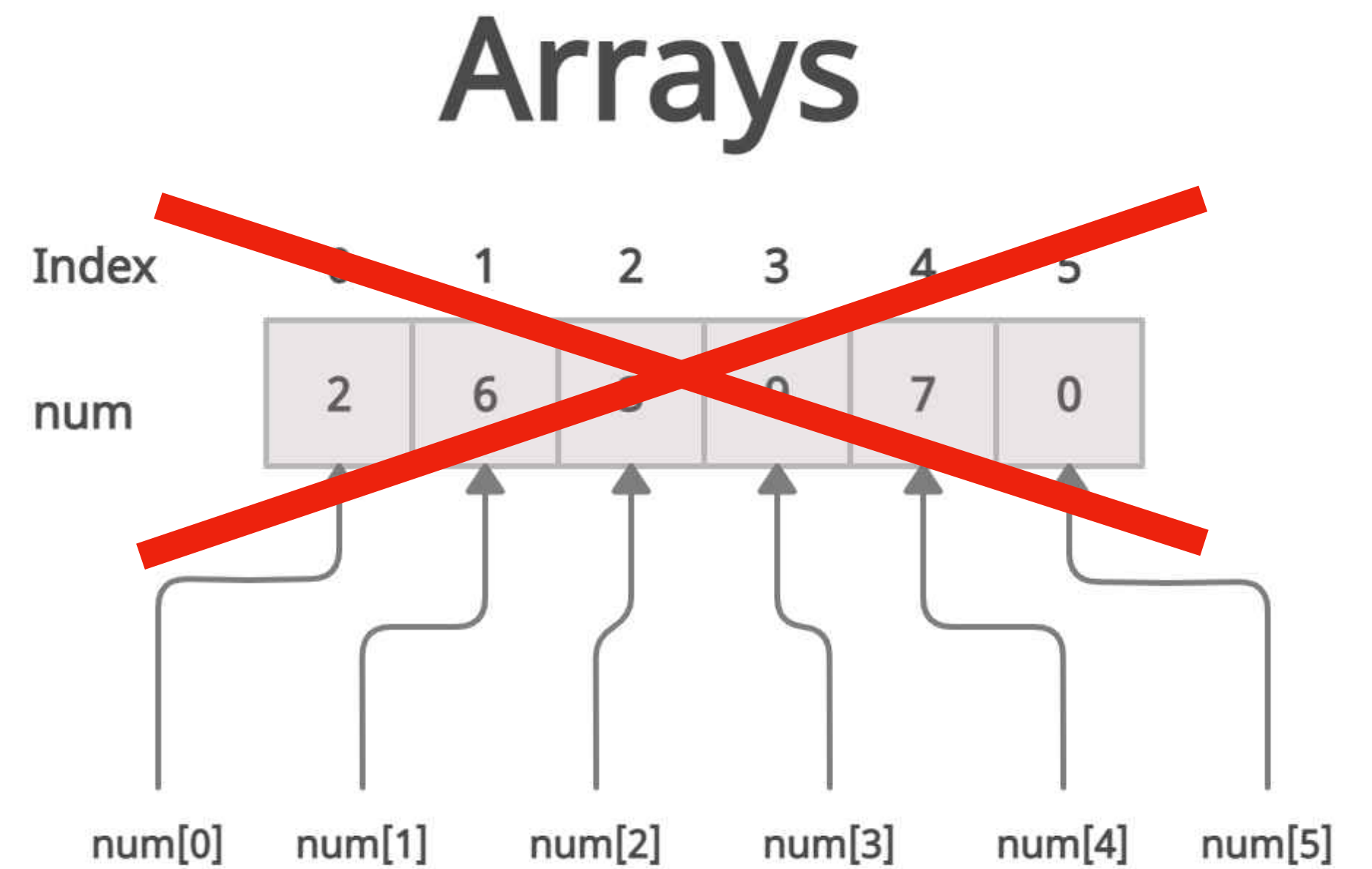
Many important operations on data can be represented as operations on lists (e.g., updating all users in a database)

# What is a list not?

A list is *not* an array. We don't have constant-time indexing

A list is *not* mutable. **No data structures in FP are mutable**

(You should think of a list structurally as more like a linked list, sort of)



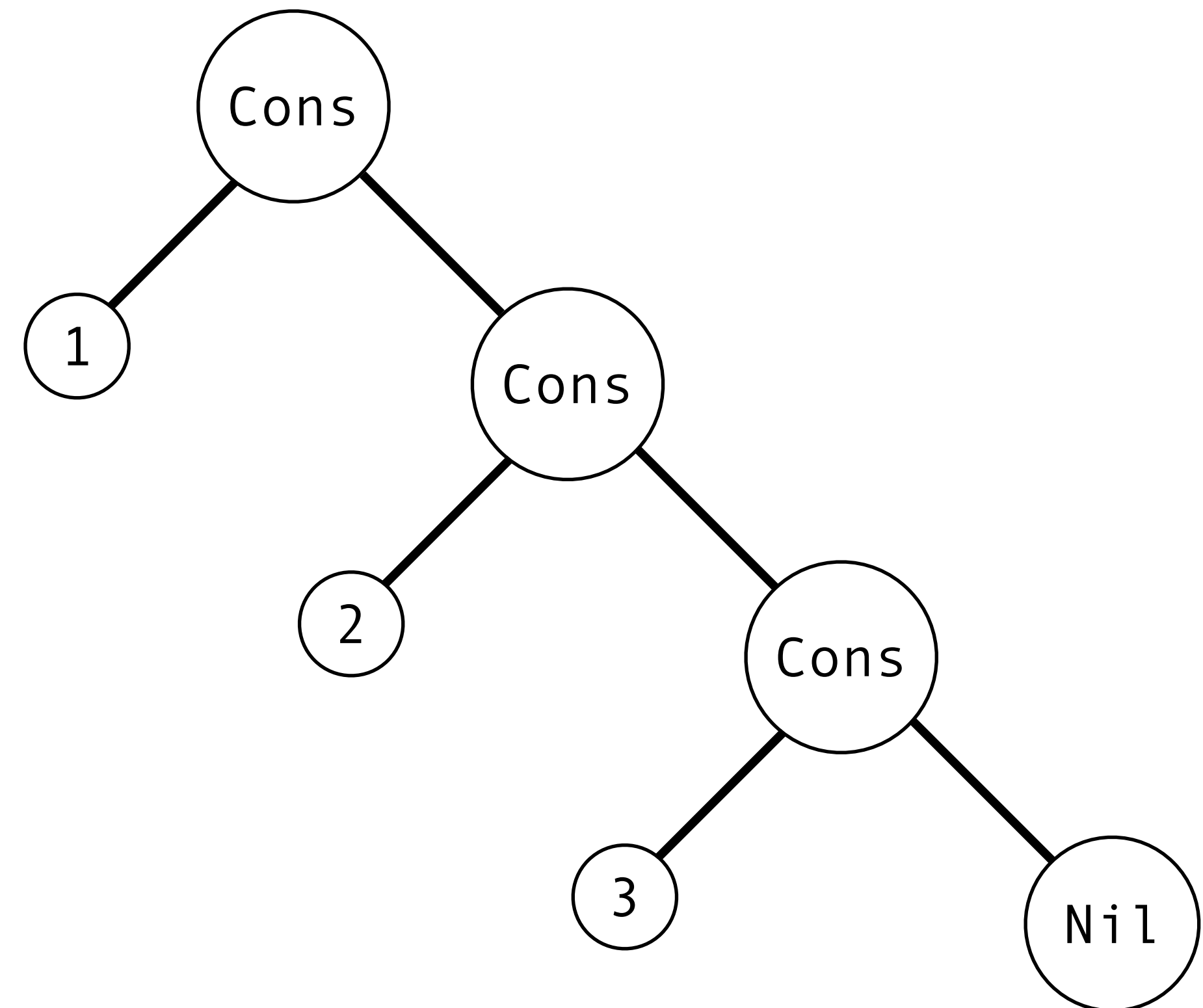
# The Picture

We can think of the list

`1 :: 2 :: 3 :: []`

as a leaning tree with data  
a leaves

(this will generalize to  
other *algebraic* data types)





# List Syntax (Informally)

```
let _ = 1 :: 2 :: 3 :: []  
let _ = 1 :: (2 :: (3 :: []))  
let _ = [1; 2; 3]
```

# List Syntax (Informally)

```
let _ = 1 :: 2 :: 3 :: []  
let _ = 1 :: (2 :: (3 :: []))  
let _ = [1; 2; 3]
```

**[]** stands for the empty list (a.k.a. **nil**), the list with no elements

# List Syntax (Informally)

```
let _ = 1 :: 2 :: 3 :: []  
let _ = 1 :: (2 :: (3 :: []))  
let _ = [1; 2; 3]
```

`[]` stands for the empty list (a.k.a. `nil`), the list with no elements

`x :: xs` stands for the list `xs` with `x` prepended to it. The symbol `::` is pronounced "cons" and is a *right associative* operator

# List Syntax (Informally)

```
let _ = 1 :: 2 :: 3 :: []  
let _ = 1 :: (2 :: (3 :: []))  
let _ = [1; 2; 3]
```

`[]` stands for the empty list (a.k.a. `nil`), the list with no elements

`x :: xs` stands for the list `xs` with `x` prepended to it. The symbol `::` is pronounced "cons" and is a *right associative* operator

`[x1; x2; ...; xn]` is a list literal. It's shorthand for a list of a known length

# Example

*Construct a function **generate** which, given integers  $n$ , returns a list consisting of the first  $n$  positive integers*

# Lists Semantics (Informally)

**$[2 + 3; 4 * 12; 2 - 1] \Downarrow [5; 48; 1]$**

We evaluate the list  $[e_1; e_2; \dots; e_k]$  by evaluating each element of the list (from right to left)

# Destructing Lists

```
match l with  
| [] -> (* something *)  
| x :: xs -> (* something else *)  
| ... (* other patterns??? *)
```

As with any type in OCaml, we can use **pattern matching** to destruct lists

With pattern matching, we describe the value we want based on the *shape* of the list we're matching on

# Example

*Implement the function **double** where **double l** is the same as the list **l** but with every element doubled*



# Lists are Immutable

```
val remove_all_negatives : int list -> int list
```

# Lists are Immutable

```
val remove_all_negatives : int list -> int list
```

All data structures in FP are immutable

# Lists are Immutable

```
val remove_all_negatives : int list -> int list
```

All data structures in FP are immutable

If we want to "update" a list, we have to produce an *entirely new* list

# Lists are Immutable

```
val remove_all_negatives : int list -> int list
```

All data structures in FP are immutable

If we want to "update" a list, we have to produce an *entirely new* list

In reality the data is not literally duplicated, there are optimizations which allow for shared data

# Practice Problem

*Implement the function*

***remove\_all\_negatives : int list -> int list***

*where remove\_all\_negative l is the same as the list l but with all negative numbers removed*

# Deep Pattern Matching

```
match <expr> with
| [] -> <expr>
| [h1; h2] -> <expr>
| h1::h2::t -> <expr>
| h::t -> <expr>
| .....
```

Pattern matching is very general. We can match on more complex patterns than just empty and nonempty

# Example

*Implement the function*

***delete\_every\_other : int list -> int list***

*such that **delete\_every\_other** **l** is the first, third, fifth,..., and so on elements of **l***

# A Note on Polymorphism

```
let rec length l =  
  match l with  
  | [] -> 0  
  | x :: xs -> 1 + length xs
```

What is the type of the length function?

Does this function depend on the values in the list?



# The List Type

`[1;2;3]`

`int list`

`["1";"2";"3"]`

`string list`

`[[1;1];[2;2];[3;3]]`

`int list list`

# The List Type

`[1;2;3]`

`int list`

`["1";"2";"3"]`

`string list`

`[[1;1];[2;2];[3;3]]`

`int list list`

The `list` type is an example of a **parametrized** type

# The List Type

`[1;2;3]`

`int list`

`["1";"2";"3"]`

`string list`

`[[1;1];[2;2];[3;3]]`

`int list list`

The `list` type is an example of a **parametrized** type

A function on lists is *polymorphic* (with respect to the list parameter) if it can be apply to a list parametrized by *any* type

# The List Type

`[1;2;3]`

`int list`

`["1";"2";"3"]`

`string list`

`[[1;1];[2;2];[3;3]]`

`int list list`

The `list` type is an example of a **parametrized** type

A function on lists is *polymorphic* (with respect to the `list` parameter) if it can be apply to a list parametrized by *any* type

For this, we need *type parameters* to stand for *any* type:

# The List Type

`[1;2;3]`

`int list`

`["1";"2";"3"]`

`string list`

`[[1;1];[2;2];[3;3]]`

`int list list`

The `list` type is an example of a **parametrized** type

A function on lists is *polymorphic* (with respect to the list parameter) if it can be apply to a list parametrized by *any* type

For this, we need *type parameters* to stand for *any* type:

`'a, 'b, 'c, ...`

# Not all functions can be polymorphic

```
let rec sum l =  
  match l with  
  | [] -> 0  
  | x :: xs -> x + sum xs
```

# Not all functions can be polymorphic

```
let rec sum l =  
  match l with  
  | [] -> 0  
  | x :: xs -> x + sum xs
```

Can this function be applied to a list parametrized by any type?

# Not all functions can be polymorphic

```
let rec sum l =  
  match l with  
  | [] -> 0  
  | x :: xs -> x + sum xs
```

Can this function be applied to a list parametrized by any type?

**Answer:** No, it can only be applied to **int lists**



# Not all functions can be polymorphic

```
let rec sum l =  
  match l with  
  | [] -> 0  
  | x :: xs -> x + sum xs
```

Can this function be applied to a list parametrized by any type?

**Answer:** No, it can only be applied to **int lists**

OCaml's type inference is good at "guessing" when functions are polymorphic

# Practice Problem

*Implement the function*

***reverse : 'a list -> 'a list***

*such that **reverse** l is the same as l but in reverse order*

# Tail Recursion

demo

(even the wrong way)

# Tail Recursion

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

not tail recursive

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

tail recursive

A recursive function is **tail recursive** if it does not perform any computations on the result of a recursive call

# **Why do we care?**

# Why do we care?

Recursive functions are *expensive* with respect to the call-stack

# Why do we care?

Recursive functions are *expensive* with respect to the call-stack

**Tail-call elimination** is an optimization implemented by OCaml's compiler which *reuses* stack frames



# Why do we care?

Recursive functions are *expensive* with respect to the call-stack

**Tail-call elimination** is an optimization implemented by OCaml's compiler which *reuses* stack frames

*Tail-recursive functions "behave iteratively"*

# The Picture

fact 5

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

fact 1

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

fact 1

fact 0

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

fact 1

fact 0

$\Rightarrow$  1



# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

fact 1

$\Rightarrow 1 * 1 = 1$

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

$\Rightarrow 2 * 1 = 2$

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

$\Rightarrow 3 * 2 = 6$

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

fact 4

$\Rightarrow 4 * 6 = 24$

# The Picture

fact 5

$\Rightarrow 5 * 24 = 120$

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

# The Picture

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

fact 5

$\Rightarrow 5 * 24 = 120$

fact 4

$\Rightarrow 4 * 6 = 24$

fact 3

$\Rightarrow 3 * 2 = 6$

fact 2

$\Rightarrow 2 * 1 = 2$

fact 1

$\Rightarrow 1 * 1 = 1$

fact 0

$\Rightarrow 1$

**1 frame per  
recursive call**

# The Picture

loop 1 5

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4



# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

fact 60 2

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

fact 60 2

fact 120 1

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

fact 60 2

fact 120 1

fact 120 0

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

fact 60 2

fact 120 1

fact 120 0

⇒ **120**

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

fact 60 2

fact 120 1  
⇒ **120**

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3

fact 60 2

⇒ **120**

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

loop 20 3  
⇒ **120**



# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

loop 5 4

⇒ 120

# The Picture

loop 1 5 ⇒ <b>120</b>
--------------------------

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

# The Picture

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5  
⇒ 120

loop 5 4  
⇒ 120

loop 20 3  
⇒ 120

fact 60 2  
⇒ 120

fact 120 1  
⇒ 120

fact 120 0  
⇒ 120

1 frame per  
recursive call

BUT THE VALUE  
DOESN'T CHANGE  
ON IT'S WAY UP  
THE CALL STACK

# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 1 5

# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 5 4

# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 20 3

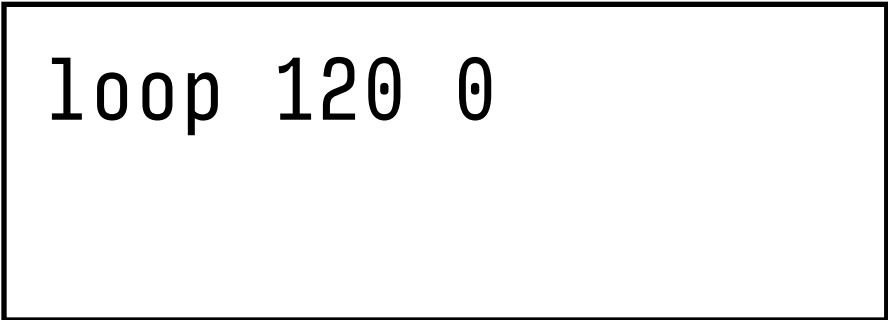
# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 120 1

# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```



loop 120 0



# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 120 0
⇒ 120

# The Picture (Optimized)

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

loop 120 0 ⇒ 120
---------------------

**1 frame  
for every  
recursive  
call**

# Tail Position

```
let rec fact n =  
  if n <= 0  
  then 1  
  else n * fact (n - 1)
```

not tail recursive

computation after the recursive call

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

tail recursive

Tail-call optimizations apply to functions whose recursive calls are in **tail position**

**Intuition:** A call is in tail position if there is no computation *after* the recursive call

# Aside: Tail Position More Formally

`let rec f x1 x2 ... xk = e`

A recursive call `f e1 e2 ... ek` is in tail position in `e` if:

- » it does not appear in `e`, or `e` is the recursive call itself
- » `e = if e1 then e2 else e3` and the call does not appear in `e1` and it is in tail position in `e2` and `e3`
- » `e` is a **match-expression** and the call is in tail position in every branch, and does not appear in the matched expression
- » `e = let x = e1 in e2` and the call does not appear in the `e1` and it is in tail position in `e2`

# Aside: Tail Position More Formally

`let rec f x1 x2 ... xk = e`

A recursive call `f e1 e2 ... ek`<sup>\*</sup> is in tail position in `e` if:

- » it does not appear in `e`, or `e` is the recursive call itself
- » `e = if e1 then e2 else e3` and the call does not appear in `e1` and it is in tail position in `e2` and `e3`
- » `e` is a **match-expression** and the call is in tail position in every branch, and does not appear in the matched expression
- » `e = let x = e1 in e2` and the call does not appear in the `e1` and it is in tail position in `e2`

<sup>\*</sup> `f` cannot appear in `e1 ... ek`

# Tail Recursion and Lists

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

# Tail Recursion and Lists

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

We need to take care with tail-recursion and lists

# Tail Recursion and Lists

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

We need to take care with tail-recursion and lists

*Does the above program concatenate two lists?*



```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

append [1;2;3] [4;5;6]

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

`loop [1;2;3] [4;5;6]`

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

```
match [1;2;3] with  
| [] -> [4;5;6]  
| x :: xs -> loop xs (x :: [4;5;6])
```

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

```
match 1 :: [2;3] with  
| [] -> [4;5;6]  
| x :: xs -> loop xs (x :: [4;5;6])
```

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

loop [2;3] (1 :: [4;5;6])

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

loop [2;3] [1;4;5;6]

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

```
match [2;3] with  
| [] -> [1;4;5;6]  
| x :: xs -> loop xs (x :: [1;4;5;6])
```

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

match 2 :: [3] with

| [] -> [1;4;5;6]

| x :: xs -> loop xs (x :: [1;4;5;6])



```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

loop [3] (2 :: [1;4;5;6])

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

loop [3] [2;1;4;5;6]

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

```
match [3] with  
| [] -> [2;1;4;5;6]  
| x :: xs -> loop xs (x :: [2;1;4;5;6])
```

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

```
match 3 :: [] with  
| [] -> [2;1;4;5;6]  
| x :: xs -> loop xs (x :: [2;1;4;5;6])
```

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

loop [] (3 :: [2;1;4;5;6])

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

loop [] [3;2;1;4;5;6]

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

```
match [] with  
| [] -> [3;2;1;4;5;6]  
| x :: xs -> loop xs (x :: [3;2;1;4;5;6])
```

```
let append l r =  
  let rec loop acc l =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop xs (x :: acc)  
  in loop l r
```

[3;2;1;4;5;6]

***whoops!***



# Tail Recursion and Lists

```
let append l r =  
  let rec loop l acc =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop (x :: acc) xs  
  in loop l r
```

We need to take care with tail-recursion and lists

*Does the above program concatenate two lists?*

# Tail Recursion and Lists

```
let append l r =  
  let rec loop l acc =  
    match l with  
    | [] -> acc  
    | x :: xs -> loop (x :: acc) xs  
  in loop l r  
      should be (List.rev l)
```

We need to take care with tail-recursion and lists

*Does the above program concatenate two lists?*

# Accumulators

```
let fact n =  
  let rec loop acc n =  
    if n <= 0  
    then acc  
    else loop (n * acc) (n - 1)  
  in loop 1 n
```

*Our accumulator pattern is almost always tail recursive*

# Code Example

*Implement the function*

***reverse : 'a list -> 'a list***

*The implementation must be tail recursive*

# Summary

OCaml is a language of **expressions**, everything is an expression

**Lists** are used to process collections of homogeneous data

We can use **tail-recursion** to make our implementations more memory efficient, but we have to be careful when working with lists