Linear Independence and Bases

1 Linear Independence

Definition 1.1. *Let V be a vector space over a field F and $S \subseteq V$. We say that S is* linearly dependent *if there exist distinct vectors $v_1, \ldots, v_n \in S$ and scalars $a_1, \ldots, a_n \in F$, not all zero, such that*

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

If no such relation exists, then S is linearly independent*.*

Remark 1.2.

1. *$\varnothing$ is linearly independent.*
2. *If $0 \in S$, then S is linearly dependent.*
3. *S is linearly independent iff*

$$a_1 v_1 + \cdots + a_n v_n = 0 \ \Rightarrow\ a_1 = \cdots = a_n = 0.$$

Examples

Example 1.3. *Let $S = \{v\}$. Then S is linearly independent if and only if $v \neq 0$.*

Example 1.4. *Let $S = \{v, w\}$. Then S is linearly independent if and only if neither vector is a scalar multiple of the other.*

Example 1.5. *In F^n, the standard vectors $e_1, \ldots, e_n$ form a linearly independent set.*

Example 1.6. *In $F[x]$, the set $\{x^n : n \in \mathbb{N}\}$ is linearly independent.*

Example 1.7. *In the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$, the set*

$$\{e^x, \sin x, \cos x\}$$

is linearly independent.

Proof. Suppose

$$ae^x + b\sin x + c\cos x = 0 \quad \text{for all } x \in \mathbb{R}.$$

If $a \neq 0$, then e^x is unbounded while $\sin x$ and $\cos x$ are bounded, a contradiction. Hence $a = 0$. Thus $b\sin x + c\cos x = 0$ for all x. Plugging in $x = 0$ gives $c = 0$, and plugging in $x = \frac{\pi}{2}$ gives $b = 0$. Therefore $a = b = c = 0$, and the set is linearly independent. □

2 Maximal Linearly Independent Sets

Definition 2.1. *A linearly independent set $S \subseteq V$ is called* maximal linearly independent *if for every $v \notin S$, the set $S \cup \{v\}$ is linearly dependent.*

Example 2.2. *In F^n, the set $\{e_1, \ldots, e_n\}$ is a maximal linearly independent set.*

3 Bases

Theorem 3.1 (Characterization of Bases)**.** *Let V be a vector space over F and $S \subseteq V$. The following are equivalent:*

1. *S is a minimal spanning set of V.*
2. *S is a maximal linearly independent set.*
3. *Every vector $v \in V$ can be written uniquely as*

$$v = a_1 v_1 + \cdots + a_n v_n, \quad v_i \in S.$$

Definition 3.2. *A* basis *of V is a set $S \subseteq V$ satisfying the equivalent conditions above.*