MATH 251 — Determinants (Exam-Optimized A Notes)

0. Exam Checklist (Non-negotiable)

You must be able to:

1. State how determinant behaves under row/column operations.
2. Compute determinants using triangular reduction.
3. Use $\det(AB) = \det(A)\det(B)$ without expanding.
4. Detect instantly when $\det = 0$.
5. Prove invertible $\iff$ $\det \neq 0$ using multiplicativity.
6. Explain determinant as volume scaling.

Brutal rule: If you expand a 5×5 determinant directly on an exam, you are losing time.

1. Transpose

Definition 1. *For $A \in M_n(F)$, the transpose A^T swaps rows and columns.*

Theorem 1.

$$\det(A^T) = \det(A)$$

Proof. Using the Leibniz formula:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Since $(A^T)_{i,j} = a_{j,i}$,

$$\det(A^T) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

Let $\tau = \sigma^{-1}$. Then $\operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma)$, hence

$$\det(A^T) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} a_{i,\tau(i)} = \det(A).$$

$\square$

2. Triangular Matrices

Theorem 2. *If A is triangular, then*

$$\det(A) = \prod_{i=1}^{n} a_{ii}.$$

Proof. In the Leibniz formula, any permutation other than the identity selects at least one entry below (or above) the diagonal, which is zero.

Thus only the identity permutation contributes:

$$\det(A) = \prod_{i=1}^{n} a_{ii}.$$

□

3. Determinant and Row/Column Operations

3.1 Core Rules

Operation	Effect on determinant
Swap rows	Multiply by -1
Multiply row by c	Multiply by c
Row + multiple	No change

Same for columns.

3.2 Immediate Consequences

Proposition 1. *Two equal rows* $\Rightarrow \det = 0$.

Proof. Swapping the two equal rows multiplies the determinant by -1, but the matrix does not change.

Thus $\det(A) = -\det(A)$, so $\det(A) = 0$. □

Proposition 2. *One zero row* $\Rightarrow \det = 0$.

Proof. Determinant is multilinear in rows. If one row is zero, every term in the expansion contains a factor 0. □

4. Computation Strategy (Exam Algorithm)

Standard workflow:

1. Use row operations to create zeros.
2. Reach triangular matrix.
3. Track swaps and scalings.
4. Multiply diagonal.

Priority order:

$$\text{Triangular} > \text{Row reduction} > \text{Cofactor expansion}$$

5. Multiplicativity

Theorem 3.

$$\det(AB) = \det(A)\det(B)$$

Proof. Fix A and define $f(B) = \det(AB)$.
Then f is:

- multilinear in columns of B
- alternating

Also $f(I) = \det(A)$.
By uniqueness of determinant as the alternating multilinear function normalized by $\det(I) = 1$,

$$f(B) = \det(A)\det(B).$$

Hence $\det(AB) = \det(A)\det(B)$. □

Exam Usage:

- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(A^k) = (\det A)^k$

6. Invertibility Criterion

Theorem 4. *Equivalent:*

1. *A invertible*
2. $\det(A) \neq 0$
3. *Columns form a basis*
4. *Rank $= n$*

Proof. (1)⇒(2): If $AB = I$, then

$$\det(A)\det(B) = 1 \Rightarrow \det(A) \neq 0.$$

(2)⇒(3): If determinant were zero, columns would be linearly dependent.
(3)⇒(4): Basis ⇒ rank n.
(4)⇒(1): Full rank square matrix is invertible.

□

Proof Strategy (exam):
If $AB = I$ then

$$\det(A)\det(B) = 1 \Rightarrow \det(A) \neq 0.$$

7. Instant Zero Tests (High Yield)

- Two proportional rows
- Linear dependence
- Zero on triangular diagonal
- One column is combination of others

8. Common Mistakes (Seen Every Midterm)

- Forgetting swap changes sign
- Thinking row addition changes determinant
- Believing $\det(A + B) = \det(A) + \det(B)$ (False)

9. Conceptual Meaning

Theorem 5. *Determinant measures volume scaling and orientation of the linear map $T(x) = Ax$.*

Proof. A linear map sends the unit cube to a parallelepiped.
The determinant equals its signed volume.
If $\det(A) = 0$, volume collapses to lower dimension. If negative, orientation reverses. □

One-Page Compression (Read Before Exam)

- Triangular $\Rightarrow$ multiply diagonal.
- Swap $\Rightarrow$ sign change.
- Row addition $\Rightarrow$ no change.
- $\det(AB) = \det(A)\det(B)$.
- $\det = 0 \iff$ not invertible.
- Goal in computation: create zeros first.