Lecture: Dimension and Bases

1 Uniqueness of Dimension

Theorem 1.1. *Let V be a vector space with a finite basis. Then any two bases of V have the same number of elements.*

Proof. Let S_1 and S_2 be two bases of V with $|S_1| = n < \infty$. Since S_1 is a basis, it is linearly independent. Hence any finite subset of S_2 is linearly independent.

Applying the replacement lemma to the linearly independent set S_2 and the spanning set S_1, we obtain

$$|S_2| \leq |S_1| = n.$$

By symmetry, exchanging the roles of S_1 and S_2 yields

$$|S_1| \leq |S_2|.$$

Therefore $|S_1| = |S_2|$. $\square$

Remark 1.2. *This theorem justifies the definition of the dimension of a finite-dimensional vector space.*

2 Characterization of Bases

Proposition 2.1. *Let V be a finite-dimensional vector space with $\dim V = n$, and let $S \subseteq V$ with $|S| = n$. Then the following are equivalent:*

1. *S spans V.*
2. *S is linearly independent.*
3. *S is a basis of V.*

Proof. By definition of a basis, (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Let B be a basis of V with $|B| = n$. If S is linearly independent and $|S| = |B|$, applying the replacement lemma to the linearly independent set S and the spanning set B shows that S spans V. Hence S is a basis.

(1) $\Rightarrow$ (3): Assume that S spans V. If S were not minimal as a spanning set, then there would exist a proper subset $S' \subset S$ with $|S'| < n$ that still spans V. This contradicts the replacement lemma. Hence S is minimal and therefore linearly independent. Thus S is a basis. $\square$

Remark 2.2. *A basis is precisely a set which is both linearly independent and spanning.*

3 Subspaces and Dimension

Proposition 3.1. *Let V be a finite-dimensional vector space and let $W \subseteq V$ be a subspace. Then W is finite-dimensional and*

$$\dim W \leq \dim V,$$

with equality if and only if $W = V$.

Proof. Let S be a basis of V and let L be a basis of W. Applying the replacement lemma to the linearly independent set L and the spanning set S, we obtain

$$|L| \leq |S| = \dim V.$$

Hence $\dim W \leq \dim V$.

If $\dim W = \dim V$, then $|L| = |S|$. By the previous proposition, L is a basis of V. Thus $W = \operatorname{span}(L) = V$. □

4 Infinite-Dimensional Counterexample

Remark 4.1. *The previous proposition is not true for infinite-dimensional vector spaces.*

Example 4.2. *Let*

$$W = \{f(x) \in F[x] : f(0) = 0\}.$$

Then W is a proper subspace of $F[x]$, but

$$\{x^n : n \geq 1\}$$

is a basis of W and

$$\{x^n : n \geq 0\}$$

is a basis of $F[x]$. Thus $W \neq F[x]$, but $\dim W = \dim F[x]$.

5 Extending Linearly Independent Sets

Proposition 5.1. *Let V be a finite-dimensional vector space and let $L \subseteq V$ be linearly independent. Then L can be extended to a basis of V.*

Proof. Let S be any basis of V. Apply the replacement lemma to the linearly independent set L and the spanning set S. Then there exists a subset $S' \subseteq S$ such that

$$B = L \cup S'$$

spans V and

$$|B| = |L| + |S'| = \dim V.$$

By the previous proposition, B is a basis of V. □

6 Example: Lagrange Polynomials

Example 6.1. *Let* $a_1, \ldots, a_n \in F$ *be distinct. For* $1 \le i \le n$*, define*

$$f_i(x) = \prod_{\substack{1 \le j \le n \\ j \ne i}} \frac{x - a_j}{a_i - a_j}.$$

Then

$$f_i(a_j) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

Hence $\{f_1, \ldots, f_n\}$ *is linearly independent and therefore a basis of* $F[x]_{\le n-1}$.