Lecture: Coordinates, Interpolation, and Linear Transformations

1 Lagrange Interpolation

Let $a_0, \dots, a_n \in F$ be distinct scalars. Define for $0 \leq i \leq n$:

$$f_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - a_j}{a_i - a_j}.$$

Proposition 1.1. *For all $0 \leq i, j \leq n$,*

$$f_i(a_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Theorem 1.2 (Lagrange Interpolation Formula)**.** *For any polynomial $g(x) \in F[x]_{\leq n}$,*

$$g(x) = \sum_{i=0}^{n} g(a_i) f_i(x).$$

Conversely, for any $b_0, \dots, b_n \in F$, the polynomial

$$g(x) = \sum_{i=0}^{n} b_i f_i(x)$$

is the unique polynomial of degree $\leq n$ such that $g(a_i) = b_i$ for all i.

Example 1.3. *Find $g(x) \in \mathbb{R}[x]_{\leq 2}$ such that*

$$g(1) = 5, \quad g(2) = 8, \quad g(3) = -4.$$

Using the Lagrange basis $\{f_0, f_1, f_2\}$, we obtain

$$g(x) = 8f_1(x) + 5f_0(x) - 4f_2(x) = -3x^2 + 6x + 5.$$

2 Coordinates with Respect to a Basis

Let V be a vector space over F with finite basis

$$B = \{v_1, \ldots, v_n\}.$$

Then every $v \in V$ can be written uniquely as

$$v = a_1 v_1 + \cdots + a_n v_n, \qquad a_i \in F.$$

Definition 2.1. *With the notation above, the column vector*

$$[v]_B := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

is called the coordinate vector of v with respect to B.

Remark 2.2. *1. Coordinates depend on the ordering of the basis B.*

2. We usually write coordinate vectors as column vectors when working with matrices.

Example 2.3. *Let $V = \mathbb{R}[x]_{\leq 2}$ and $B = \{1, x, x^2\}$. For $p(x) = -3x^2 + 6x + 5$, we have*

$$[p]_B = \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}.$$

If instead we use the basis

$$C = \left\{ \tfrac{1}{2}(x^2 - 5x + 6),\ -(x^2 - 4x + 3),\ \tfrac{1}{2}(x^2 - 3x + 2) \right\},$$

then

$$[p]_C = \begin{pmatrix} 8 \\ 5 \\ -4 \end{pmatrix}.$$

3 Linear Transformations

Definition 3.1. *Let V and W be vector spaces over F. A function $T : V \to W$ is called a* linear transformation *if for all $v_1, v_2 \in V$ and $a \in F$,*

$$T(v_1 + v_2) = T(v_1) + T(v_2), \qquad T(av) = aT(v).$$

Remark 3.2. *For any $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in V$,*

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

Definition 3.3. *The* kernel *and* image *of T are defined by*

$$\ker(T) = \{v \in V : T(v) = 0\}, \qquad \operatorname{im}(T) = \{T(v) : v \in V\}.$$

Proposition 3.4. *Let $T : V \to W$ be linear. Then $\ker(T)$ is a subspace of V and $\operatorname{im}(T)$ is a subspace of W.*

Proof. We already know $\ker(T)$ and $\operatorname{im}(T)$ are subgroups. It suffices to check closure under scalar multiplication.

If $v \in \ker(T)$ and $a \in F$, then

$$T(av) = aT(v) = a \cdot 0 = 0,$$

so $av \in \ker(T)$.

If $w \in \operatorname{im}(T)$, then $w = T(u)$ for some $u \in V$. For $a \in F$,

$$aw = aT(u) = T(au) \in \operatorname{im}(T).$$

□

Remark 3.5. *T is injective if and only if $\ker(T) = \{0\}$.*

4 Examples of Linear Transformations

Example 4.1 (Zero Map)**.** *The map $T : V \to W$ defined by $T(v) = 0$ for all $v \in V$ is linear. Here $\ker(T) = V$ and $\operatorname{im}(T) = \{0\}$.*

Example 4.2 (Identity Map)**.** *The map $T : V \to V$ defined by $T(v) = v$ is linear.*

Example 4.3 (Derivative Operator)**.** *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by*

$$T(f(x)) = f'(x).$$

Then T is linear. Moreover,

$$\ker(T) = \mathbb{R} \quad (\textit{constant polynomials}), \qquad \operatorname{im}(T) = \mathbb{R}[x].$$

If instead $T : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n}$ is defined by

$$T(f) = f',$$

then

$$\ker(T) = \mathbb{R}, \qquad \operatorname{im}(T) = \mathbb{R}[x]_{\leq n-1}.$$