Lecture: Change of Basis, Decomposition, and Determinant

1 Change-of-Basis Matrices

Let V be a finite-dimensional vector space and let

$$B = \{v_1, \dots, v_n\}, \qquad C = \{w_1, \dots, w_n\}$$

be two bases of V.

Definition 1.1. *The* change-of-basis matrix *from B to C is defined by*

$$[Id]_B^C := \big([v_1]_C \ [v_2]_C \ \cdots \ [v_n]_C\big).$$

It satisfies

$$[Id]_B^C [v]_B = [v]_C \qquad \text{for all } v \in V.$$

Proposition 1.2. *The jth column of $[Id]_B^C$ is the coordinate vector of v_j with respect to C.*

Proposition 1.3.

$$[Id]_B^C [Id]_C^B = I_n = [Id]_C^B [Id]_B^C.$$

Proof. For each j, the jth column of $[Id]_B^C$ is $[v_j]_C$. Applying $[Id]_C^B$ sends this vector to $[v_j]_B = e_j$. Hence $[Id]_C^B [Id]_B^C = I_n$. The other equality follows similarly. □

2 Direct Sum Decomposition and Linear Operators

Example 2.1. *Let*

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + z = 0\}, \qquad U = \operatorname{span}\{(1, 0, 1)\}.$$

Then

$$\mathbb{R}^3 = U \oplus W.$$

Hence every $v \in \mathbb{R}^3$ can be written uniquely as

$$v = u + w, \qquad u \in U,\ w \in W.$$

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(v) = w \quad \text{if } v = u + w \text{ with } u \in U,\ w \in W.$$

Then T is a linear transformation (the projection onto W along U).

Example 2.2. *Since*

$$W = \{(x, y, -x) : x, y \in \mathbb{R}\} = \operatorname{span}\{(1, 0, -1), (0, 1, 0)\},$$

the set

$$B = \{(1, 0, 1), (1, 0, -1), (0, 1, 0)\}$$

is a basis of $\mathbb{R}^3$ *adapted to the decomposition* $\mathbb{R}^3 = U \oplus W$.

3 Matrix Representation and Change of Basis

Let $T : V \to V$ be linear and let B and C be two bases of V.

Theorem 3.1 (Change-of-Basis Formula)**.**

$$[T]_C^C = [Id]_B^C \, [T]_B^B \, [Id]_C^B.$$

Remark 3.2. *This expresses that matrices of the same linear transformation in different bases are* similar *matrices.*

4 Application: Computing High Powers of a Matrix

Example 4.1. *Let*

$$A = \begin{pmatrix} 2 & -2 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 4 \end{pmatrix}.$$

Suppose that for a suitable basis B,

$$D = [L_A]_B^B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Let

$$Q = [Id]_{\mathrm{st}}^B.$$

Then

$$A = QDQ^{-1},$$

and for any integer $n \geq 1$,

$$A^n = QD^nQ^{-1}.$$

Since

$$D^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{pmatrix},$$

this allows efficient computation of A^n.

5 Determinant (Geometric Motivation)

Definition 5.1. *The* determinant *is a function*

$$\det : M_{n\times n}(F) \to F$$

which measures how multiplication by a matrix scales volume.

Remark 5.2. *If* $F = \mathbb{R}$ *and*

$$A = (v_1, \dots, v_n),$$

then $|\det A|$ *is the volume of the parallelepiped*

$$\{a_1v_1 + \cdots + a_nv_n : 0 \le a_i \le 1\}.$$