MATH 251 — Determinants (Exam-Optimized A Notes)

0. Exam Checklist (Non-negotiable)

You must be able to:

1. State how determinant behaves under row/column operations.
2. Compute determinants using triangular reduction.
3. Use $\det(AB) = \det(A)\det(B)$ without expanding.
4. Detect instantly when $\det = 0$.
5. Prove invertible $\iff$ $\det \neq 0$ using multiplicativity.
6. Explain determinant as volume scaling.

Brutal rule:
If you expand a 5×5 determinant directly on an exam, you are losing time.
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1 1. Transpose

Definition 1. *For $A \in M_{m\times n}(\mathbb{F})$, the transpose A^T swaps rows and columns.*

Theorem 1.

$$\det(A^T) = \det(A)$$

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2 2. Triangular Matrices

Theorem 2. *If A is triangular, then*

$$\det(A) = \prod_{i=1}^{n} a_{ii}.$$

Exam usage:
Always try to reach triangular form first.
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3 3. Determinant and Row/Column Operations

3.1 Core Rules

Operation	Effect on determinant
Swap rows	Multiply by -1
Multiply row by c	Multiply by c
Row + multiple	No change

Same for columns.
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3.2 Immediate Consequences

Proposition 1. *Two equal rows* $\Rightarrow \det = 0$.

Proposition 2. *One zero row* $\Rightarrow \det = 0$.

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4 4. Computation Strategy (Exam Algorithm)

Standard workflow:

1. Use row operations to create zeros.
2. Reach triangular matrix.
3. Track swaps and scalings.
4. Multiply diagonal.

Priority order:

$$\text{Triangular} \; > \; \text{Row reduction} \; > \; \text{Cofactor expansion}$$

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5 5. Multiplicativity

Theorem 3.
$$\det(AB) = \det(A)\det(B)$$

Exam Usage

- $\det(A^{-1}) = 1/\det(A)$
- $\det(A^k) = (\det A)^k$

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6 6. Invertibility Criterion

Theorem 4. *Equivalent:*

1. *A invertible*
2. $\det(A) \neq 0$
3. *Columns form a basis*
4. *Rank* $= n$

Proof Strategy (exam):
If $AB = I$ then

$$\det(A)\det(B) = 1 \Rightarrow \det(A) \neq 0.$$

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7 7. Instant Zero Tests (High Yield)

- Two proportional rows
- Linear dependence
- Zero on triangular diagonal
- One column is combination of others

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8 8. Common Mistakes (Seen Every Midterm)

- Forgetting swap changes sign
- Thinking row addition changes determinant
- Believing $\det(A + B) = \det(A) + \det(B)$ (False)

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9 9. Conceptual Meaning

Determinant measures:

- Volume scaling
- Orientation
- Invertibility of linear map

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One-Page Compression (Read Before Exam)

- Triangular $\Rightarrow$ multiply diagonal.
- Swap $\Rightarrow$ sign change.
- Row addition $\Rightarrow$ no change.
- $\det(AB) = \det(A)\det(B)$.
- $\det = 0 \iff$ not invertible.
- Goal in computation: create zeros first.