

Vector Spaces: Direct Sums, Span, and Linear Independence

1 Direct Sum of Vector Spaces

Definition 1.1. Let U and W be vector spaces over the same field F . Define

$$U \oplus W := \{(u, w) : u \in U, w \in W\}.$$

With operations

$$\begin{aligned}(u, w) + (u', w') &= (u + u', w + w'), \\ \alpha(u, w) &= (\alpha u, \alpha w),\end{aligned}$$

the set $U \oplus W$ is a vector space over F , called the (external) direct sum of U and W .

Remark 1.2. 1. As abelian groups, $U \oplus W = U \times W$.

2. Inductively, we define $V_1 \oplus \cdots \oplus V_n$. For example,

$$F^n = F \oplus \cdots \oplus F \quad (n \text{ times}).$$

3. For infinite collections, there is a distinction between \bigoplus and \prod .

Definition 1.3. Let V be a vector space and U, W be subspaces of V . We say that V is an internal direct sum of U and W , written

$$V = U \oplus W,$$

if

$$V = U + W \quad \text{and} \quad U \cap W = \{0\}.$$

2 Span

Definition 2.1. Let $S \subseteq V$. The span of S , denoted $\text{span}(S)$, is the set of all finite linear combinations of elements of S :

$$\text{span}(S) = \left\{ \sum_{i=1}^n a_i v_i : n \geq 1, v_i \in S, a_i \in F \right\}.$$

By convention,

$$\text{span}(\emptyset) = \{0\}.$$

Proposition 2.2. *The span can equivalently be described as*

$$\text{span}(S) = \bigcap \{ W \subseteq V : W \text{ is a subspace and } S \subseteq W \}.$$

Proof. Every subspace containing S must contain all linear combinations of elements of S , hence contains $\text{span}(S)$. Conversely, $\text{span}(S)$ itself is a subspace containing S . Thus it is the smallest subspace containing S . \square

Proposition 2.3. *$\text{span}(S)$ is a subspace of V .*

Proof. If $S = \emptyset$, then $\text{span}(S) = \{0\}$, which is a subspace. Assume $S \neq \emptyset$. Let

$$x = \sum_{i=1}^m a_i v_i, \quad y = \sum_{j=1}^n b_j w_j,$$

with $v_i, w_j \in S$. Then

$$x + y = \sum_{k=1}^{\max(m,n)} (a_k + b_k) u_k \in \text{span}(S),$$

and for $\alpha \in F$,

$$\alpha x = \sum_{i=1}^m (\alpha a_i) v_i \in \text{span}(S).$$

Hence $\text{span}(S)$ is a subspace. \square

3 Spanning Sets

Definition 3.1. *Let V be a vector space and $S \subseteq V$. We say S spans V if*

$$\text{span}(S) = V.$$

If no proper subset of S spans V , then S is called a minimal spanning set.

4 Linear Independence

Definition 4.1. *A subset $S \subseteq V$ is linearly dependent if there exist distinct vectors $v_1, \dots, v_n \in S$ and scalars $a_1, \dots, a_n \in F$, not all zero, such that*

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

If no such relation exists, S is linearly independent.

Remark 4.2. 1. \emptyset is linearly independent.

2. If $0 \in S$, then S is linearly dependent.

3. S is linearly independent iff

$$a_1 v_1 + \cdots + a_n v_n = 0 \Rightarrow a_1 = \cdots = a_n = 0.$$