Vector Spaces and Subspaces

1 Vector Spaces

Definition 1.1. *Let F be a field (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$ for prime p). A* vector space *over F is a nonempty set V equipped with two operations:*

$$+ : V \times V \to V, \qquad (v, w) \mapsto v + w,$$

$$\cdot : F \times V \to V, \qquad (a, v) \mapsto av,$$

such that:

1. *$(V, +)$ is an abelian group.*
2. *For all $v, w \in V$ and $a, b \in F$,*

$$a(v + w) = av + aw, \qquad (a + b)v = av + bv,$$

$$a(bv) = (ab)v, \qquad 1v = v.$$

Elements of V are called vectors*, and elements of F are called* scalars*.*

Example 1.2. *The set $F^n = \{(a_1, \ldots, a_n) : a_i \in F\}$ with operations*

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n),$$

$$\alpha(a_1, \ldots, a_n) = (\alpha a_1, \ldots, \alpha a_n),$$

is a vector space over F.

Example 1.3. *Let $F[x]$ be the set of all polynomials with coefficients in F. Then $F[x]$ is a vector space under*

$$(f + g)(x) = f(x) + g(x), \qquad (\alpha f)(x) = \alpha f(x).$$

Example 1.4. *Let V be the set of all functions $f : F \to F$. Then V is a vector space under*

$$(f + g)(x) = f(x) + g(x), \qquad (\alpha f)(x) = \alpha f(x).$$

2 Subspaces

Definition 2.1. *Let V be a vector space over F. A subset $W \subseteq V$ is called a* subspace *of V if:*

1. $W \neq \emptyset$,
2. *for all* $u, v \in W$, $u + v \in W$,
3. *for all* $\alpha \in F$ *and* $u \in W$, $\alpha u \in W$.

Example 2.2. *Let $a_{ij} \in F$. The set*

$$W = \{(x_1, \ldots, x_n) \in F^n : \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0, \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases} \}$$

is a subspace of F^n.

Proof. The zero vector $(0, \ldots, 0)$ satisfies all equations, hence $W \neq \emptyset$. Let $u, v \in W$. Then

$$Au = 0, \quad Av = 0 \implies A(u + v) = Au + Av = 0.$$

Thus $u + v \in W$. For $\alpha \in F$,

$$A(\alpha u) = \alpha Au = 0,$$

so $\alpha u \in W$. Hence W is a subspace. □

Example 2.3. *Let $n \geq 0$ and*

$$F[x]_{\leq n} = \{f(x) \in F[x] : \deg f \leq n\}.$$

Then $F[x]_{\leq n}$ is a subspace of $F[x]$.

Proof. Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n.$$

Then

$$f + g = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \in F[x]_{\leq n},$$

and for $\alpha \in F$,

$$\alpha f = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n \in F[x]_{\leq n}.$$

□

Example 2.4. *Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Then the set of continuous functions (or differentiable functions) is a subspace of V.*

3 Sum and Intersection of Subspaces

Proposition 3.1. *Let U, W be subspaces of a vector space V. Then*

$$U + W = \{u + w : u \in U, w \in W\}$$

is a subspace of V, and

$$U \cap W$$

is also a subspace of V.

Proof. Let $x = u_1 + w_1$ and $y = u_2 + w_2$ in $U + W$. Then

$$x + y = (u_1 + u_2) + (w_1 + w_2) \in U + W,$$

since U, W are subspaces. For $\alpha \in F$,

$$\alpha x = \alpha u_1 + \alpha w_1 \in U + W.$$

Thus $U + W$ is a subspace.

Now let $x, y \in U \cap W$. Then $x, y \in U$ and $x, y \in W$. Hence

$$x + y \in U \cap W, \quad \alpha x \in U \cap W.$$

Therefore $U \cap W$ is a subspace. $\square$