Lecture: Bases and Dimension

1 Bases

Definition 1.1. *Let V be a vector space over a field F and $S \subseteq V$. A set S is called a* basis *of V if it satisfies any (hence all) of the following equivalent conditions:*

1. *S is a minimal spanning set of V.*
2. *S is a maximal linearly independent set.*
3. *Every $v \in V$ can be written uniquely as a finite linear combination of elements of S.*

Proposition 1.2. *If $S \subseteq V$ is a linearly independent spanning set, then S is a basis of V.*

Proof. We prove the contrapositive. Assume that S is not minimal as a spanning set. Then there exists $v \in S$ such that

$$\operatorname{span}(S \setminus \{v\}) = \operatorname{span}(S) = V.$$

Hence v can be written as a linear combination of elements of $S \setminus \{v\}$:

$$v = a_1 v_1 + \cdots + a_n v_n, \quad v_i \in S \setminus \{v\}.$$

Thus

$$a_1 v_1 + \cdots + a_n v_n + (-1)v = 0$$

is a nontrivial linear dependence relation in S, contradicting linear independence. Therefore S is minimal and hence a basis. □

2 Examples

Example 2.1. *Let*

$$W = \{(a, b, c) \in F^3 : a + b + c = 0\}.$$

Then every $w \in W$ can be written as

$$w = (a, b, -a - b) = a(1, 0, -1) + b(0, 1, -1).$$

Hence

$$W = \operatorname{span}\{(1, 0, -1), (0, 1, -1)\}.$$

Moreover, $\{(1, 0, -1), (0, 1, -1)\}$ is linearly independent, so it is a basis of W.

3 Existence of Bases

Theorem 3.1. *Every vector space has a basis.*

Remark 3.2. *Any two bases of the same vector space have the same cardinality.*

4 Finite Spanning Sets Contain a Basis

Theorem 4.1. *Let V be a vector space and let $S \subseteq V$ be a finite set such that* $\text{span}(S) = V$*. Then there exists a subset $B \subseteq S$ such that B is a basis of V.*

Proof. If $S = \varnothing$, then $V = \{0\}$ and $\varnothing$ is a basis. If S is minimal as a spanning set, then S itself is a basis.

Otherwise, S is not minimal, so there exists $v \in S$ such that

$$\text{span}(S \setminus \{v\}) = \text{span}(S) = V.$$

Since $|S \setminus \{v\}| = |S| - 1$, by induction there exists a subset

$$B \subseteq S \setminus \{v\}$$

which is a basis of V. Hence $B \subseteq S$ and B is a basis of V. □

5 Steinitz Replacement Theorem

Lemma 5.1 (Steinitz Replacement Theorem)**.** *Let $S, L \subseteq V$ be finite subsets such that:*

- *S spans V,*
- *L is linearly independent.*

Let $|S| = n$ and $|L| = m$. Then $m \leq n$, and there exists a subset $S' \subseteq S$ with

$$|S'| = n - m$$

such that

$$\text{span}(L \cup S') = V.$$

Proof. We proceed by induction on $m = |L|$.

Base case: $m = 0$. Then $L = \varnothing$. Take $S' = S$. We have $|S'| = n$ and

$$\text{span}(L \cup S') = \text{span}(S) = V.$$

Induction step: Assume the result holds for all linearly independent sets of size m. Let $L = \{v_1, \dots, v_{m+1}\}$ be linearly independent and S a spanning set of size n.

By the induction hypothesis applied to $\{v_1, \dots, v_m\}$, there exists $S_0 \subseteq S$ with $|S_0| = n-m$ such that

$$\text{span}(\{v_1, \dots, v_m\} \cup S_0) = V.$$

Write

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m + b_1 w_1 + \cdots + b_{n-m} w_{n-m},$$

with $w_i \in S_0$.

If all $b_i = 0$, then v_{m+1} is a linear combination of $v_1, \ldots, v_m$, contradicting linear independence. Hence some $b_j \neq 0$. Solving for w_j, we obtain

$$w_j \in \operatorname{span}(\{v_1, \ldots, v_{m+1}\} \cup (S_0 \setminus \{w_j\})).$$

Let

$$S' = S_0 \setminus \{w_j\}.$$

Then $|S'| = (n-m) - 1 = n - (m+1)$ and

$$\operatorname{span}(L \cup S') = V.$$

This completes the induction. □

6 Dimension

Definition 6.1. *If V has a finite basis, the common cardinality of all bases of V is called the* dimension *of V, denoted* $\dim V$*.*

Remark 6.2. *A vector space is called* finite-dimensional *if it has a finite spanning set.*