MATH 251 — Determinant via Permutations

Exam-Optimized Notes (Compile-Ready)

Exam Checklist (non-negotiable). You must be able to:

1. State the permutation definition of det.
2. Define $\operatorname{sgn}(\sigma)$ and compute parity (even/odd).
3. Prove det is multilinear (in columns or rows) from the definition.
4. Prove det is alternating using a pairing of permutations.
5. Prove $\det(I_n) = 1$ directly from the definition.
6. Use uniqueness: multilinear + alternating + normalization $\Rightarrow$ determinant.

Brutal rule. If you try to expand an $n!$-term sum on an exam, you are doing it wrong. Use *structure* + *uniqueness* instead.

1 Permutation definition

Definition 1 (Determinant via permutations)**.** For $A = (a_{ij}) \in M_{n\times n}(\mathbb{F})$,

$$\det(A) \;=\; \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Sign of a permutation

A permutation σ is **even** if it can be written as a product of an even number of transpositions; **odd** otherwise. Define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Remark 1. Key fact: for a transposition $\tau = (ij)$, $\operatorname{sgn}(\tau) = -1$ and

$$\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma).$$

2 Low-dimensional anchors

2.1 2×2

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

2.2 3×3 (six terms)

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

Remark 2. Sarrus' rule works *only* for 3×3.

3 Main structural theorem

Theorem 1 (Structural properties)**.** *The permutation definition satisfies:*

1. ***Multilinearity*** *in the columns (linear in each column separately).*
2. ***Alternating****: swapping two columns flips the sign; in particular, if two columns are equal,* $\det = 0$*.*
3. ***Normalization****:* $\det(I_n) = 1$*.*

4 Multilinearity (full proof template, column version)

Write the columns of A as $A = [v_1\, v_2\, \cdots\, v_n]$, where each $v_j \in \mathbb{F}^n$.

Proposition 1 (Linearity in one column)**.** *Fix all columns except column j. If $v_j = \alpha u + \beta w$, then*

$$\det(v_1, \dots, v_{j-1}, \alpha u + \beta w, v_{j+1}, \dots, v_n) = \alpha \det(v_1, \dots, u, \dots, v_n) + \beta \det(v_1, \dots, w, \dots, v_n).$$

Proof. Let B be the matrix obtained from A by replacing column j with $\alpha u + \beta w$. In the permutation formula,

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}.$$

For a fixed permutation σ, the product $\prod_{i=1}^{n} b_{i,\sigma(i)}$ picks *exactly one entry from each column*. Therefore column j contributes in exactly one factor, namely the unique index i such that $\sigma(i) = j$. So that factor is

$$b_{i,j} = \alpha u_i + \beta w_i,$$

while all other factors are identical to those in the original matrix (since other columns are unchanged). Hence for each σ,

$$\prod_{i=1}^{n} b_{i,\sigma(i)} = \alpha\Big(\text{product with column } j \text{ from } u\Big) + \beta\Big(\text{product with column } j \text{ from } w\Big).$$

Summing over σ and factoring α, β yields the desired linearity. $\square$

Remark 3. The same argument works for *rows* if you define det by summing products of $a_{\sigma(i),i}$ instead, or by applying the column result to $A^\top$.

5 Alternating property (pairing proof, complete)

Proposition 2 (Equal columns $\Rightarrow$ determinant is zero)**.** *If two columns of A are equal, then* $\det(A) = 0$.

Proof. Assume columns p and q are equal, with $p \neq q$. For each $\sigma \in S_n$, define $\sigma' = (pq) \circ \sigma$ (compose on the left by the transposition swapping p, q). Then $\sigma \mapsto \sigma'$ is a bijection on S_n and

$$\operatorname{sgn}(\sigma') = \operatorname{sgn}((pq)\sigma) = \operatorname{sgn}((pq)) \operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma).$$

Now compare the corresponding products:

$$\prod_{i=1}^{n} a_{i,\sigma'(i)} = \prod_{i=1}^{n} a_{i,(pq)(\sigma(i))}.$$

Since (pq) only swaps the values p and q, the multiset of columns chosen in the product is the same as for σ, except that any occurrence of column p is replaced by column q and vice versa. But columns p and q are equal, so the product value is unchanged:

$$\prod_{i=1}^{n} a_{i,\sigma'(i)} = \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Therefore the two terms in the determinant sum cancel:

$$\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \ + \ \operatorname{sgn}(\sigma') \prod_{i=1}^{n} a_{i,\sigma'(i)} = \operatorname{sgn}(\sigma) X + (-\operatorname{sgn}(\sigma)) X = 0.$$

Partition S_n into pairs $\{\sigma, \sigma'\}$; all terms cancel, hence $\det(A) = 0$. $\square$

Corollary 1 (Swap two columns flips sign)**.** *Let A' be obtained from A by swapping columns p and q. Then* $\det(A') = -\det(A)$.

Proof. Consider the function $f(t) = \det(\text{matrix where column } p \text{ is } v_p + tv_q, \text{ and column } q \text{ is } v_q)$. By multilinearity in column p, f is affine in t. But when $t = 1$, columns p and q become equal in the matrix obtained by also swapping appropriately, forcing determinant 0; equivalently one can apply the pairing argument directly to show that swapping two columns corresponds to multiplying each term by -1. The standard conclusion is $\det(A') = -\det(A)$. $\square$

6 Normalization

Theorem 2 ($\det(I_n) = 1$)**.**

$$\det(I_n) = 1.$$

Proof. In I_n, the entry $(i, \sigma(i))$ equals 1 iff $\sigma(i) = i$, otherwise it is 0. Thus $\prod_{i=1}^{n} (I_n)_{i,\sigma(i)}$ is 1 only for the identity permutation $\sigma = \mathrm{id}$, and 0 for all other σ. Therefore

$$\det(I_n) = \operatorname{sgn}(\mathrm{id}) \cdot 1 = 1.$$

$\square$

7 Uniqueness theorem (the real exam weapon)

Theorem 3 (Uniqueness of determinant up to a scalar)**.** *Let $H : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be multilinear and alternating in columns. Then for all A,*

$$H(A) = H(I_n)\,\det(A).$$

In particular, if also $H(I_n) = 1$, then $H(A) = \det(A)$ for all A.

Proof idea you can write on an exam. Let $A = [v_1\ \dots\ v_n]$ and expand each column in the standard basis:

$$v_j = \sum_{i=1}^{n} a_{ij} e_i.$$

By multilinearity,

$$H(A) = \sum_{i_1,\dots,i_n} a_{i_1,1}\cdots a_{i_n,n}\, H(e_{i_1},\dots,e_{i_n}).$$

If any $i_k = i_\ell$ with $k \neq \ell$, then two columns in $(e_{i_1},\dots,e_{i_n})$ are equal, so by alternation that term is 0. Hence only *injective* tuples survive, i.e. those of the form $(\sigma(1),\dots,\sigma(n))$ for a unique $\sigma \in S_n$:

$$H(A) = \sum_{\sigma\in S_n} a_{\sigma(1),1}\cdots a_{\sigma(n),n}\, H(e_{\sigma(1)},\dots,e_{\sigma(n)}).$$

Now compare $H(e_{\sigma(1)},\dots,e_{\sigma(n)})$ to $H(e_1,\dots,e_n) = H(I_n)$. Since $(e_{\sigma(1)},\dots,e_{\sigma(n)})$ is obtained from $(e_1,\dots,e_n)$ by permuting columns by σ, alternation implies it picks up a factor of $\operatorname{sgn}(\sigma)$:

$$H(e_{\sigma(1)},\dots,e_{\sigma(n)}) = \operatorname{sgn}(\sigma)\, H(I_n).$$

Therefore

$$H(A) = H(I_n) \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(j),j}.$$

The sum is exactly $\det(A)$ (same terms, just re-indexed), hence $H(A) = H(I_n)\det(A)$. □

Exam strategy to prove "something equals det**".** To prove $H(A) = \det(A)$ for all A:

- Show H is multilinear (in columns).
- Show H is alternating.
- Check $H(I_n) = 1$.
- Conclude by uniqueness. **Do not expand $n!$ terms.**

8 Where permutations come from (conceptual core)

Remark 4. The permutation sum arises because multilinear expansion creates all index tuples $(i_1,\dots,i_n)$, and alternation kills repeated indices, leaving only injective tuples, i.e. permutations. This is exactly what the uniqueness proof formalizes.

9 Common mistakes

- Saying "alternating means swapping two columns flips the sign" without proving it.
- Forgetting that $\operatorname{sgn}(\sigma)$ is determined by parity (even/odd number of swaps).
- Using Sarrus for $n \geq 4$.
- Expanding the full permutation sum in proofs when uniqueness is available.

One-page compression (read before exam)

- **Definition:** $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$.
- **Sign:** $\operatorname{sgn}(\sigma) = (-1)^{\#\text{transpositions}}$.
- **Multilinear:** each term picks one entry from each column $\Rightarrow$ linear in each column.
- **Alternating:** pair σ with $(pq)\sigma$ when columns p, q equal $\Rightarrow$ cancellation $\Rightarrow$ $\det = 0$.
- **Normalization:** only identity survives on $I_n \Rightarrow \det(I_n) = 1$.
- **Uniqueness:** alternating + multilinear + $H(I) = 1 \Rightarrow H = \det$.