Lecture: Hom Spaces and Isomorphisms

1 The Vector Space $\mathrm{Hom}(V, W)$**Hom(V,W)**

Definition 1.1. *Let V and W be vector spaces over a field F. Define*

$$\mathrm{Hom}(V, W) = \{\, T : V \to W \mid T \textit{ is linear} \,\}.$$

Proposition 1.2. $\mathrm{Hom}(V, W)$ *is a vector space over F with operations*

$$(T_1 + T_2)(v) = T_1(v) + T_2(v), \qquad (aT)(v) = aT(v),$$

for all $T_1, T_2 \in \mathrm{Hom}(V, W)$, $a \in F$, and $v \in V$.

Proof. For $T_1, T_2 \in \mathrm{Hom}(V, W)$ and $v, w \in V$,

$$(T_1 + T_2)(v + w) = T_1(v + w) + T_2(v + w) = T_1(v) + T_1(w) + T_2(v) + T_2(w),$$

so $T_1 + T_2$ is linear. Similarly, for $a \in F$,

$$(aT)(v + w) = aT(v + w) = aT(v) + aT(w),$$

so aT is linear. The zero map $0(v) = 0_W$ is linear and $(-T)(v) = -T(v)$ is the additive inverse. Thus $\mathrm{Hom}(V, W)$ is a vector space. □

Remark 1.3. *If $V = W$, then* $\mathrm{Hom}(V, V)$ *together with composition of maps is a ring.*

2 Isomorphisms

Definition 2.1. *Let V and W be vector spaces over F. A bijective linear map*

$$T : V \to W$$

is called an isomorphism*. If such a map exists, we write*

$$V \cong W$$

and say that V and W are isomorphic.

Proposition 2.2. *If $T : V \to W$ is an isomorphism, then its inverse*

$$T^{-1} : W \to V$$

is also linear.

Proof. From elementary algebra,

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2).$$

Let $w \in W$ and $a \in F$. Write $w = T(v)$ for some $v \in V$. Then

$$T^{-1}(aw) = T^{-1}(aT(v)) = T^{-1}(T(av)) = av = aT^{-1}(w).$$

Hence T^{-1} is linear. □

Remark 2.3. *Isomorphism is an equivalence relation on vector spaces.*

3 Isomorphisms and Dimension

Proposition 3.1. *Let V and W be isomorphic vector spaces over F. Then V is finite-dimensional if and only if W is finite-dimensional. In this case,*

$$\dim V = \dim W.$$

Proof. Let $T : V \to W$ be an isomorphism and assume V is finite-dimensional. Let $\{v_1, \dots, v_n\}$ be a basis of V.

Spanning: For any $w \in W$, since T is surjective, there exists $v \in V$ with $T(v) = w$. Write

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Then

$$w = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n),$$

so $\{T(v_1), \dots, T(v_n)\}$ spans W.

Linear independence: If

$$a_1 T(v_1) + \cdots + a_n T(v_n) = 0,$$

then

$$T(a_1 v_1 + \cdots + a_n v_n) = 0.$$

Since T is injective,

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

hence $a_1 = \cdots = a_n = 0$.

Thus $\{T(v_1), \dots, T(v_n)\}$ is a basis of W, and $\dim W = \dim V$. □

4 Coordinate Isomorphism

Theorem 4.1. *Let V be a finite-dimensional vector space over F and let*

$$B = \{v_1, \dots, v_n\}$$

be a basis of V. Define

$$T : V \to F^n, \qquad T(v) = [v]_B.$$

Then T is an isomorphism. In particular,

$$V \cong F^n.$$

Proof. Let

$$v = a_1 v_1 + \cdots + a_n v_n, \qquad w = b_1 v_1 + \cdots + b_n v_n.$$

Then

$$T(v + w) = [v + w]_B = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} = [v]_B + [w]_B = T(v) + T(w),$$

and for $a \in F$,

$$T(av) = [av]_B = \begin{pmatrix} aa_1 \\ \vdots \\ aa_n \end{pmatrix} = a[v]_B = aT(v).$$

Thus T is linear.

Injectivity follows from $[v]_B = 0 \Rightarrow v = 0$, and surjectivity since every vector in F^n is the coordinate vector of some $v \in V$. Hence T is an isomorphism. □

5 Classification of Finite-Dimensional Vector Spaces

Corollary 5.1. *Let V and W be finite-dimensional vector spaces over F. Then*

$$V \cong W \quad \iff \quad \dim V = \dim W.$$

Proof. ($\Rightarrow$) follows from the previous proposition. ($\Leftarrow$) If $\dim V = \dim W = n$, then

$$V \cong F^n \cong W.$$

□