MATH 251 — Final Exam A-Level Review Notes

Change of Basis, Projections, Diagonalization, Determinants

Contents

1 Change of Basis

Final takeaway

Changing basis never changes the vector, only its coordinate representation.

Definition 1.1 (Change of Basis Matrix)**.** *Let V be an n-dimensional vector space and let*

$$B = (v_1, \dots, v_n), \quad C = (w_1, \dots, w_n)$$

be two bases of V. The change of basis matrix from B to C, denoted $[Id]_B^C$, is defined by

$$[Id]_B^C [v]_B = [v]_C \quad \forall v \in V.$$

Proposition 1.2 (Construction Rule)**.** *The j-th column of $[Id]_B^C$ is the coordinate vector of v_j expressed in the basis C. That is, if*

$$v_j = a_{1j} w_1 + \cdots + a_{nj} w_n,$$

then

$$[Id]_B^C = \begin{pmatrix} | & & | \\ [v_1]_C & \cdots & [v_n]_C \\ | & & | \end{pmatrix}.$$

Remark 1.3 (Exam-critical facts)**.**

- $[Id]_B^C$ *converts* B-coordinates into C-coordinates.

- *The reverse change of basis is*
$$[Id]_C^B = ([Id]_B^C)^{-1}.$$
- *Composition rule:*
$$[Id]_B^D = [Id]_C^D [Id]_B^C.$$

Final exam algorithm

1. Identify clearly: "from which basis to which basis".
2. Express each v_j of the *old basis* in terms of the *new basis.*
3. Stack the coordinate vectors as columns.
4. Invert if the direction is reversed.

2 Projections via Direct Sums

Final takeaway

Every projection comes from a direct sum decomposition.

Assume
$$\mathbb{R}^n = U \oplus W.$$
Then every $v \in \mathbb{R}^n$ decomposes uniquely as
$$v = u + w, \quad u \in U,\ w \in W.$$

Definition 2.1 (Projection along U onto W)**.** *Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by*
$$T(v) = w \quad \textit{where } v = u + w.$$

Proposition 2.2 (Fundamental properties)**.** *The projection T satisfies:*
$$T^2 = T, \quad \ker(T) = U, \quad \mathrm{Im}(T) = W.$$

Remark 2.3 (Matrix form trick)**.** *If a basis B is chosen such that*
$$b_1, \dots, b_k \in W, \quad b_{k+1}, \dots, b_n \in U,$$
then
$$[T]_B^B = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 2.4 (Very common trap)**.** *A projection is* not necessarily orthogonal*. Orthogonality requires $U \perp W$, which must be explicitly verified.*

3 Similarity and Diagonalization

Final takeaway

Similarity means the same linear operator under different bases.

Definition 3.1 (Similarity)**.** *Matrices $A, B \in M_n(\mathbb{F})$ are similar if*

$$A = QBQ^{-1}$$

for some invertible Q.

Proposition 3.2 (Change of basis formula)**.** *Let $T : V \to V$ be linear and let B be a basis of V. If*

$$D = [T]_B^B, \quad Q = [Id]_B^{std},$$

then the standard matrix of T is

$$A = QDQ^{-1}.$$

Diagonalization

Definition 3.3 (Diagonalizable matrix)**.** *A matrix $A \in M_n(\mathbb{F})$ is diagonalizable if*

$$A = QDQ^{-1}$$

with D diagonal.

Theorem 3.4 (Diagonalization Criterion)**.** *A is diagonalizable if and only if it has n linearly independent eigenvectors.*

Remark 3.5 (How to kill a diagonalization question)**.**

- *Compute eigenvalues (with algebraic multiplicity).*
- *Compute each eigenspace.*
- *Check:*

$$\sum \dim(E_\lambda) = n.$$

- *If not, **state clearly: "A is not diagonalizable."***

4 Computing High Powers of a Matrix

Theorem 4.1. *If $A = QDQ^{-1}$ with*

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

then for all $k \geq 1$,

$$A^k = QD^kQ^{-1}, \quad D^k = \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k).$$

Remark 4.2 (Mandatory exam step)**.** ***Always check diagonalizability before applying this formula.***

5 Determinants

Final takeaway

The determinant measures volume scaling and detects invertibility.

Definition 5.1 (Axiomatic definition)**.** *The determinant is the unique function* $\det : M_n(\mathbb{F}) \to \mathbb{F}$ *satisfying:*

1. *Multilinearity in columns,*
2. *Alternating property,*
3. $\det(I_n) = 1$.

Theorem 5.2 (Invertibility criterion)**.**

$$\det(A) \neq 0 \iff A \textit{ is invertible.}$$

Moreover, if $\lambda_1, \ldots, \lambda_n$ *are eigenvalues of* A *(with multiplicity),*

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

Theorem 5.3 (Effect of row operations)**.**

- *Swap two rows* $\Rightarrow$ *determinant changes sign.*
- *Multiply a row by* $c \Rightarrow$ *determinant multiplied by* c.
- *Add a multiple of one row to another* $\Rightarrow$ *determinant unchanged.*

Remark 5.4 (Exam strategy)**.** *Reduce to upper triangular form, track all row operations, then multiply diagonal entries.*