MATH 251 — Lecture 12 (Exam-Optimized A Notes)
Determinant via Multilinearity and Alternation

0. Exam Checklist (non-negotiable)

You must be able to:

1. State the determinant axioms (multilinearity, alternation, normalization) *precisely*.
2. Prove the **swap-sign property** from multilinearity + alternation (no black-box).
3. Expand a multilinear map in the standard basis and justify why **only permutations survive**.
4. Define $\operatorname{sgn}(\sigma)$ and compute $H(e_{\sigma(1)}, \ldots, e_{\sigma(n)})$.
5. Derive the **Leibniz formula** and prove it satisfies the axioms.
6. Use **uniqueness** to show $H = \det$ without expanding an $n!$-term sum.

Brutal rule: If your write-up is missing *any* of: swap-sign proof, permutation reduction, sgn-evaluation on permuted basis, uniqueness argument, then it is **not** A-level exam notes for this lecture.

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1 Axiomatic characterization

1.1 Matrices as column tuples

Fix a field $\mathbb{F}$ and $n \in \mathbb{N}$. Identify a matrix $A \in M_{n\times n}(\mathbb{F})$ with its ordered list of columns:

$$A = (v_1, \ldots, v_n), \qquad v_j \in \mathbb{F}^n.$$

Thus a function $H : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is equivalently a function $H : (\mathbb{F}^n)^n \to \mathbb{F}$.

Definition 1.1 (Multilinear and alternating)**.** Let V be an $\mathbb{F}$-vector space. A map $H : V^n \to \mathbb{F}$ is

- **multilinear** if for each slot i, fixing all other inputs, the map

$$v_i \longmapsto H(v_1, \ldots, v_i, \ldots, v_n)$$

  is linear in v_i;

- **alternating** if whenever $v_i = v_j$ for some $i \neq j$, we have $H(v_1, \ldots, v_n) = 0$.

Remark 1.2 (Common mistake). Multilinear does *not* mean "linear in all variables at once". It means linear in each slot separately (holding the others fixed).

1.2 Determinant axioms

Theorem 1.3 (Axioms for the determinant)**.** *There exists a unique function* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *such that:*

1. ***(Linearity in each column)*** *For all* $a \in \mathbb{F}$ *and all column vectors* $v_i, v_i' \in \mathbb{F}^n$*,*

$$\det(v_1, \dots, av_i, \dots, v_n) = a\, \det(v_1, \dots, v_i, \dots, v_n),$$

$$\det(v_1, \dots, v_i + v_i', \dots, v_n) = \det(v_1, \dots, v_i, \dots, v_n) + \det(v_1, \dots, v_i', \dots, v_n).$$

2. ***(Alternating)*** *If* $v_i = v_j$ *for some* $i \neq j$*, then* $\det(v_1, \dots, v_n) = 0$*.*

3. ***(Normalization)*** $\det(I_n) = \det(e_1, \dots, e_n) = 1$*.*

Remark 1.4 (Why normalization matters). Multilinearity + alternation alone do not give uniqueness: if H works, then cH works for any $c \in \mathbb{F}$. Normalization forces $c = 1$.

2 Warm-up sanity checks (useful anchors)

Example 2.1 (Dot product: bilinear but not alternating)**.** $H : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, $H(a, b) = \sum_{i=1}^k a_i b_i$ is bilinear, but not alternating since $H(v, v) = \sum_i v_i^2 \neq 0$ for $v \neq 0$.

Example 2.2 (The 2×2 area form)**.** $H : (\mathbb{F}^2)^2 \to \mathbb{F}$ defined by

$$H\big((a, b), (c, d)\big) = ad - bc$$

is bilinear and alternating. It is exactly det on 2×2 matrices.

3 Swap-sign property (MUST know proof)

Proposition 3.1 (Swapping two columns flips the sign)**.** *Let* $H : V^n \to \mathbb{F}$ *be multilinear and alternating. Then for any* $1 \leq i < j \leq n$*,*

$$H(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\,H(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

Proof. Because H is alternating, if two slots are equal then the value is 0. In particular,

$$0 = H(\dots, v_i + v_j, \dots, v_i + v_j, \dots).$$

Expand by multilinearity in the i-th and j-th slots:

$$\begin{aligned} 0 = {} & H(\dots, v_i, \dots, v_j, \dots) + H(\dots, v_j, \dots, v_i, \dots) \\ & + H(\dots, v_i, \dots, v_i, \dots) + H(\dots, v_j, \dots, v_j, \dots). \end{aligned}$$

The last two terms vanish by alternation (two equal inputs), hence

$$H(\dots, v_i, \dots, v_j, \dots) = -H(\dots, v_j, \dots, v_i, \dots).$$

□

Remark 3.2 (Auto-corollaries you should use without thinking). (1) Two equal columns $\Rightarrow H = 0$ (this is alternation). (2) Two swaps $\Rightarrow$ sign returns (consistent with $(-1)^2 = 1$).

4 Multilinear expansion and why permutations appear

4.1 Full multilinear expansion

Write each column in the standard basis $(e_1, \dots, e_n)$:

$$v_j = \sum_{i=1}^{n} a_{ij}\, e_i \qquad (j = 1, \dots, n).$$

Here $A = (a_{ij})$ is the matrix with entries a_{ij} (column index is j).

Lemma 4.1 (Full multilinear expansion)**.** *If $H : (\mathbb{F}^n)^n \to \mathbb{F}$ is multilinear, then*

$$H(v_1, \dots, v_n) = \sum_{i_1,\dots,i_n \in \{1,\dots,n\}} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}\, H(e_{i_1}, e_{i_2}, \dots, e_{i_n}).$$

Proof. Expand in slot 1 using linearity: $v_1 = \sum_{i_1} a_{i_1 1} e_{i_1}$. Then expand slot 2, etc., through slot n. Each expansion pulls out the corresponding scalar, giving the stated sum. □

4.2 Alternation kills repeated indices

Proposition 4.2 (Only permutations survive)**.** *If H is multilinear and alternating, then every term in Lemma 4.1 with $i_p = i_q$ for some $p \neq q$ vanishes. Hence*

$$H(A) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}\, H(e_{\sigma(1)}, \dots, e_{\sigma(n)}).$$

Proof. If $i_p = i_q$, then $e_{i_p} = e_{i_q}$, so two input columns to H are equal. Alternation forces that term to be 0. Thus only n-tuples $(i_1, \dots, i_n)$ with all entries distinct survive; such an n-tuple is exactly a permutation of $(1, \dots, n)$, i.e. $(i_1, \dots, i_n) = (\sigma(1), \dots, \sigma(n))$ for some $\sigma \in S_n$. □

Remark 4.3 (Common mistake). Do *not* claim "alternating means swapping changes sign" without proof. Swap-sign is a *derived* property (Proposition 3.1) using multilinearity + alternation.

5 The sign map and H on permuted basis columns

5.1 Transpositions and parity

Definition 5.1 (Transposition)**.** A transposition is a permutation that swaps two elements (and fixes the rest), denoted $(i\,j)$.

Theorem 5.2 (Sign homomorphism)**.** *There exists a unique group homomorphism* $\operatorname{sgn} : S_n \to \{\pm 1\}$ *such that* $\operatorname{sgn}(\tau) = -1$ *for every transposition τ. If $\sigma = \tau_1 \cdots \tau_m$ is a product of transpositions, then* $\operatorname{sgn}(\sigma) = (-1)^m$*. Moreover, if $\sigma = (a_1 a_2 \cdots a_k)$ is a k-cycle, then* $\operatorname{sgn}(\sigma) = (-1)^{k-1}$.

Remark 5.3 (What you actually need on an exam). Two facts suffice: (i) each swap contributes a factor -1; (ii) $\operatorname{sgn}(\sigma)$ depends only on whether σ is even/odd (well-defined).

5.2 Evaluating H on permuted basis columns

Let $H : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be multilinear and alternating, and set

$$c := H(e_1, \dots, e_n) = H(I_n) \in \mathbb{F}.$$

Proposition 5.4. *For all $\sigma \in S_n$,*

$$H(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \operatorname{sgn}(\sigma)\, c.$$

Proof. Write σ as a product of transpositions: $\sigma = \tau_1 \cdots \tau_m$. Applying τ_r corresponds to swapping two columns, which flips sign by Proposition 3.1. Each transposition multiplies the value by -1, so after m swaps:

$$H(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (-1)^m H(e_1, \dots, e_n).$$

By Theorem 5.2, $(-1)^m = \operatorname{sgn}(\sigma)$, giving the result. □

6 Structure theorem, Leibniz formula, and uniqueness

6.1 Structure theorem (general multilinear alternating maps)

Theorem 6.1 (Structure theorem)**.** *If $H : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is multilinear and alternating, then*

$$H(A) = c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}, \quad \text{where } c = H(I_n).$$

Proof. By Proposition 4.2,

$$H(A) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}\, H(e_{\sigma(1)}, \dots, e_{\sigma(n)}).$$

Use Proposition 5.4 to replace $H(e_{\sigma(1)}, \dots, e_{\sigma(n)})$ by $\operatorname{sgn}(\sigma)c$ and factor out c. □

6.2 Leibniz formula

Definition 6.2 (Leibniz formula)**.** Define $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ by

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

Theorem 6.3 (Leibniz determinant satisfies the axioms)**.** *The function* det *in Definition 6.2 is multilinear, alternating, and* $\det(I_n) = 1$*.*

Proof. **Multilinear in each column.** Fix all columns except column j. In the Leibniz sum, each term contains exactly one factor from column j, namely $a_{\sigma(j)j}$. Hence the whole expression is linear in the entries of column j, i.e. linear in that column vector.

Alternating. If two columns p and q are equal, pair each permutation σ with $\sigma \circ (p\,q)$. The paired terms have opposite signs (since $(p\,q)$ is a transposition) but equal products (columns p and q are identical), so they cancel. Therefore $\det(A) = 0$.

Normalization. For I_n, we have $a_{ij} = \delta_{ij}$. The product $a_{\sigma(1)1} \cdots a_{\sigma(n)n}$ equals 1 iff $\sigma = \mathrm{id}$ and 0 otherwise. Thus $\det(I_n) = \operatorname{sgn}(\mathrm{id}) \cdot 1 = 1$. □

6.3 Uniqueness (the exam strategy)

Theorem 6.4 (Uniqueness of the determinant)**.** *If $H : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is multilinear, alternating, and satisfies $H(I_n) = 1$, then $H = \det$.*

Proof. By the structure theorem (Theorem 6.1),

$$H(A) = c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}, \quad c = H(I_n).$$

If $H(I_n) = 1$, then $c = 1$, so $H(A) = \det(A)$ for all A. □

Remark 6.5 (Exam-grade takeaway)*.* To prove two functions equal det, **do not** expand a huge Leibniz sum. Prove:

$$\text{multilinear} + \text{alternating} + \text{correct normalization} \implies H = \det$$

by Theorem 6.4. This is the intended efficient method.

7 Fast consequences (column operations from axioms)

Let $A = (v_1, \dots, v_n)$.

Proposition 7.1 (Two equal columns $\Rightarrow \det = 0$)*.* *If A has two equal columns, then* $\det(A) = 0$.

Proposition 7.2 (Scaling a column scales the determinant)**.** *If A' is obtained by replacing one column v_j with av_j for $a \in \mathbb{F}$, then* $\det(A') = a \det(A)$.

Proposition 7.3 (Add a multiple of one column to another: determinant unchanged)**.** *If A' is obtained from A by replacing column j with $v_j + av_i$ for some $i \neq j$, then* $\det(A') = \det(A)$.

Proof. By multilinearity in column j,

$$\det(\dots, v_j + av_i, \dots) = \det(\dots, v_j, \dots) + a \det(\dots, v_i, \dots).$$

The second term has two identical columns (column i and column j are both v_i), hence it is 0 by alternation. Therefore $\det(A') = \det(A)$. □

Remark 7.4 (Common mistake)*.* Do not memorize "row/column operations" as isolated rules. For this lecture, every rule above is a one-line consequence of the axioms.

8 Mini drill (self-test in 5 minutes)

1. Let $H : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be multilinear and alternating with $H(I_n) = 2$. Express $H(A)$ in terms of $\det(A)$.
2. Use Proposition 3.1 to show: if a column is 0, then $\det(A) = 0$.
3. For $n = 3$, write the Leibniz formula explicitly (6 terms) and mark which permutations are even/odd.

One-page compression (read right before the exam)

- **Axioms:** multilinear in each column + alternating + $\det(I_n) = 1$.
- **Swap-sign:** $0 = H(\dots, v_i + v_j, \dots, v_i + v_j, \dots)$ then expand $\Rightarrow$ swap flips sign.
- **Permutations appear:** full multilinear expansion; repeats die by alternation.
- **Basis evaluation:** $H(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \operatorname{sgn}(\sigma) H(I_n)$.
- **Structure:** any multilinear alternating H equals $H(I_n)\cdot$Leibniz-sum.
- **Uniqueness trick:** multilinear + alternating + $H(I_n) = 1 \Rightarrow H = \det$.