Lecture 12: Determinant via Multilinearity and Alternating Property

1 Axiomatic Characterization of the Determinant

Theorem 1. *Let F be a field. There exists a unique function*

$$\det : M_{n\times n}(F) \to F$$

satisfying the following properties for all $1 \le i \le n$, all $v_1, \ldots, v_n, v_i' \in F^n$, and all $a \in F$:

1. *(Linearity in each column)*

$$\det(v_1, \ldots, av_i, \ldots, v_n) = a \det(v_1, \ldots, v_i, \ldots, v_n),$$

$$\det(v_1, \ldots, v_i + v_i', \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n).$$

2. *(Alternating property) If $v_i = v_j$ for some $i \neq j$, then*

$$\det(v_1, \ldots, v_n) = 0.$$

3. *(Normalization)*

$$\det(I_n) = \det(e_1, \ldots, e_n) = 1.$$

Remark 1. *We identify $M_{n\times n}(F)$ with $F^n \times \cdots \times F^n$ (n times), by viewing a matrix as a list of its column vectors.*

2 Multilinear and Alternating Maps

Definition 1. *Let V be a vector space over F. A function*

$$H : V^n \to F$$

is called multilinear *if for each $1 \le i \le n$ and fixed $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \in V$, the map*

$$v_i \mapsto H(v_1, \ldots, v_i, \ldots, v_n)$$

is linear.

It is called alternating *if*

$$v_i = v_j \text{ for some } i \neq j \implies H(v_1, \ldots, v_n) = 0.$$

Example 1. *The dot product* $H : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ *defined by*

$$H(a, b) = \sum_{i=1}^{k} a_i b_i$$

is bilinear but not alternating, since $H(v, v) = \sum a_i^2 \neq 0$ *for* $v \neq 0$.

Example 2. *The function* $H : F^2 \times F^2 \to F$ *defined by*

$$H\big((a, b), (c, d)\big) = ad - bc$$

is bilinear and alternating.

3 Swapping Property

Proposition 1. *Let* $H : V^n \to F$ *be multilinear and alternating. Then for any* $1 \leq i < j \leq n$,

$$H(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -H(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).$$

Proof. Since H is alternating,

$$H(v_1, \ldots, v_i + v_j, \ldots, v_i + v_j, \ldots, v_n) = 0.$$

By multilinearity,

$$H(\ldots, v_i, \ldots, v_j, \ldots) + H(\ldots, v_j, \ldots, v_i, \ldots) + H(\ldots, v_i, \ldots, v_i, \ldots) + H(\ldots, v_j, \ldots, v_j, \ldots) = 0.$$

The last two terms vanish by alternation, so

$$H(\ldots, v_i, \ldots, v_j, \ldots) = -H(\ldots, v_j, \ldots, v_i, \ldots).$$

□

4 Expansion Formula for Multilinear Alternating Maps

Let $H : M_{n\times n}(F) \to F$ be multilinear and alternating. Writing the matrix as columns,

$$H \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{i_1, \ldots, i_n} a_{i_1 1} \cdots a_{i_n n} \, H(e_{i_1}, \ldots, e_{i_n}).$$

By alternation, only terms with distinct indices survive. Thus,

$$H(A) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} \, H(e_{\sigma(1)}, \ldots, e_{\sigma(n)}).$$

5 The Symmetric Group and Sign

Recall that S_n is the group of permutations of $\{1, \ldots, n\}$.

Theorem 2. *There exists a unique group homomorphism*

$$\operatorname{sgn} : S_n \to \{\pm 1\}$$

such that $\operatorname{sgn}(\tau) = -1$ *for every transposition* τ. *If* $\sigma = \tau_1 \cdots \tau_m$, *then*

$$\operatorname{sgn}(\sigma) = (-1)^m.$$

If $\sigma = (a_1 \cdots a_k)$ is a k-cycle, then

$$\operatorname{sgn}(\sigma) = (-1)^{k-1}.$$

6 Leibniz Formula for the Determinant

Theorem 3. *The function* $\det : M_{n \times n}(F) \to F$ *defined by*

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

is multilinear, alternating, and satisfies $\det(I_n) = 1$. *Hence it is the unique determinant function.*