MATH 251 — Lecture 15
Laplace Expansion and Adjugate
(Exam-Optimized A Notes — Annotated Add-ons)

Exam Checklist (Non-negotiable)

You must be able to:

1. State Laplace expansion (row and column).
2. Define minor and cofactor.
3. Explain where the factor $(-1)^{i+j}$ comes from.
4. Define the adjugate matrix correctly.
5. Prove $A\,\mathrm{Adj}(A) = \det(A)I$.
6. Derive $A^{-1} = \dfrac{1}{\det(A)}\mathrm{Adj}(A)$.

Brutal rule: Do not compute large determinants by full expansion.

1 1 Review

Theorem 1 (Theorem 1.)**.** *For $A, B \in M_n(F)$,*

$$\det(AB) = \det(A)\det(B).$$

Proof (Exam Skeleton). Use that determinant is the unique alternating multilinear function of the columns with $\det(I) = 1$. Define $f(B) = \det(AB)$ as a function of the columns of B. Then f is multilinear and alternating in columns of B, and $f(I) = \det(A)$. So by uniqueness, $f(B) = \det(A)\det(B)$, i.e. $\det(AB) = \det(A)\det(B)$.

How to Understand. Think "det = volume scaling". First apply B, then apply A; scale factors multiply.

Exam Tips. This theorem is often used to:

- Prove invertibility: A invertible $\Rightarrow \det(A) \neq 0$ and vice versa.
- Prove $\det(A^{-1}) = 1/\det(A)$ by setting $AB = I$.
- Handle similarity quickly: $\det(QAQ^{-1}) = \det(A)$.

Common Pitfalls.

- Trying to prove this by expanding determinants entry-by-entry (time sink).
- Forgetting the field F matters only for arithmetic, not the proof structure.

Proposition 1 (Proposition 1.). *If Q is invertible,*

$$\det(QAQ^{-1}) = \det(A).$$

Proof (Exam Skeleton).

$$\det(QAQ^{-1}) = \det(Q)\det(A)\det(Q^{-1}) = \det(Q)\det(A)\frac{1}{\det(Q)} = \det(A).$$

How to Understand. Similarity is "change of basis". Determinant is basis-independent, so it must stay the same.
Exam Tips. Whenever you see QAQ^{-1}, immediately replace it with "same determinant, same eigenvalues, same characteristic polynomial" (if allowed in your course context).
Common Pitfalls.

- Writing $\det(Q^{-1}) = \det(Q)^{-1}$ without justifying (justify using $\det(QQ^{-1}) = \det(I) = 1$).
- Confusing QAQ^{-1} with $Q^{-1}AQ$ (both are similar, but be consistent).

2 2 Determinant of a Linear Transformation

Definition 1 (Definition 1.). Fix a basis $\mathcal{B}$ of V:

$$\det(T) = \det([T]_{\mathcal{B}}).$$

Proposition 2 (Proposition 2.). *This is independent of basis.*

Proof (Exam Skeleton). Let $\mathcal{B}$ and $\mathcal{C}$ be two bases, and let Q be the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Then

$$[T]_{\mathcal{C}} = Q[T]_{\mathcal{B}}Q^{-1}.$$

Taking determinants and applying Proposition 1:

$$\det([T]_{\mathcal{C}}) = \det(Q[T]_{\mathcal{B}}Q^{-1}) = \det([T]_{\mathcal{B}}).$$

So $\det(T)$ is well-defined.
How to Understand. $\det(T)$ is an intrinsic property: how T scales n-dimensional volume. Coordinates (bases) should not change it.
Exam Tips. In proofs: "independent of basis" is almost always proven via similarity:

$$\text{same linear map under two bases} \Rightarrow \text{matrices are similar.}$$

Common Pitfalls.

- Forgetting the correct relation: $[T]_{\mathcal{C}} = Q[T]_{\mathcal{B}}Q^{-1}$.
- Claiming independence without referencing similarity or determinant multiplicativity.

3 3 Leibniz Formula

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{\sigma(k),k}.$$

Fix column j and collect terms with $\sigma(j) = i$.
How to Understand. Each permutation σ chooses one entry from each column (and each row exactly once). The sign $\operatorname{sgn}(\sigma)$ enforces alternation (swapping rows flips sign).
Exam Tips. You almost never compute with full Leibniz for $n \geq 4$. You use it to:

- Derive Laplace expansion by grouping terms via $\sigma(j) = i$.
- Explain where $(-1)^{i+j}$ comes from (sign bookkeeping).

Common Pitfalls.

- Mixing the order in the product: here it is *column-indexed* by k.
- Forgetting $\sigma(k)$ is a row index.

4 4 Minors and Cofactors

Definition 2 (Definition 2. Minor:)**.** A_{ij} = matrix deleting row i and column j.

How to Understand. In many texts, the *submatrix* is A_{ij}, and the *minor* is $\det(A_{ij})$. Your note uses "Minor" to mean the submatrix object; that is fine *as long as you are consistent*, and you always take $\det(A_{ij})$ when needed.
Common Pitfalls.

- Biggest grading trap: writing "minor" when the marker expects $\det(A_{ij})$.
- Confusing A_{ij} (a matrix of size $(n-1) \times (n-1)$) with its determinant (a scalar).

Definition 3 (Definition 3. Cofactor:)**.**

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

How to Understand. Cofactor = signed minor. The factor $(-1)^{i+j}$ is the parity of how many swaps are needed to move row i and column j into the "top-left" position in a Laplace-style argument.
Exam Tips. Memorize the checkerboard sign pattern:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Common Pitfalls.

- Dropping the sign (common).
- Writing $(-1)^{i-j}$ (wrong).

5 5 Key Lemma (Permutation Restriction)

Lemma 1 (Lemma 1.)**.**

$$\sum_{\sigma(j)=i} \operatorname{sgn}(\sigma) \prod_{k \neq j} a_{\sigma(k),k} = (-1)^{i+j} \det(A_{ij}).$$

Proof (Exam Skeleton). Start from Leibniz and restrict to permutations with $\sigma(j) = i$. Such permutations pick entry a_{ij} from column j. Removing row i and column j leaves a bijection between:

- permutations in S_n with $\sigma(j) = i$, and
- permutations in S_{n-1} of the remaining indices.

Under this bijection, the product over $k \neq j$ becomes exactly a Leibniz product for the submatrix A_{ij}. The sign changes by $(-1)^{i+j}$ (moving row i and column j into position contributes parity $i + j$). Hence the restricted sum equals $(-1)^{i+j} \det(A_{ij})$.
How to Understand. This lemma is the *engine* behind Laplace expansion:

Group Leibniz terms by which row is chosen in a fixed column.

Exam Tips. If asked "where does $(-1)^{i+j}$ come from?", the safe answer is:

- restriction $\sigma(j) = i$ reduces to $(n-1)$-permutations, and
- parity shifts by moving the chosen row/column into place.

You do *not* need to fully formalize the bijection unless the exam demands it; present the skeleton cleanly.
Common Pitfalls.

- Treating this lemma as "magic" and not mentioning restriction/bijection.
- Losing track of what indices are fixed (j fixed; i varies).

6 6 Laplace Expansion

Theorem 2 (Theorem 2 (Laplace). Column expansion:)**.**

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

If $l \neq j$:

$$\sum_{i=1}^{n} a_{ij} C_{il} = 0.$$

Row expansion:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

If $l \neq i$:

$$\sum_{j=1}^{n} a_{ij} C_{lj} = 0.$$

Proof (Exam Skeleton). (1) Column expansion: start from Leibniz formula and group terms by $\sigma(j) = i$:

$$\det(A) = \sum_{i=1}^{n} a_{ij} \left(\sum_{\sigma(j)=i} \operatorname{sgn}(\sigma) \prod_{k \neq j} a_{\sigma(k),k} \right).$$

Apply Lemma 1 to the inner sum to get $\det(A) = \sum_i a_{ij} C_{ij}$.

(2) Off-diagonal (column): form a matrix B by replacing column l of A with column j (so B has two identical columns). Then $\det(B) = 0$ (alternating property). Expanding $\det(B)$ along column l gives exactly $\sum_i a_{ij} C_{il}$, hence it is 0.

(3) Row expansion and its off-diagonal identity are analogous (apply the same argument to A^T, or repeat with rows).

How to Understand. Laplace is simply:

determinant is linear in one chosen row/column, with cofactors as coefficients.

The "zero-sum" identities say: *if you plug the wrong column into cofactors, alternation forces 0.*

Exam Tips. Computations:

- Always expand along the row/column with most zeros.
- If no zeros: do row operations first (keeping track of determinant rules).

Proof questions:

- Column expansion proof = "group Leibniz terms".
- Zero-sum proof = "make two columns equal $\Rightarrow$ det = 0".

Common Pitfalls.

- Mixing which index is fixed (in column expansion j fixed, sum over i).
- Forgetting the zero-sum identity is *off-diagonal* and must mention "two equal columns".
- Using C_{li} instead of C_{il} (index order matters).

7 7 Adjugate Matrix

Definition 4 (Definition 4.)**.**

$$\operatorname{Adj}(A) = (C_{ji})$$

(cofactor matrix transpose).

How to Understand. Adjugate is defined so that the Laplace identities become matrix multiplication identities. The transpose is *not optional*: it is exactly what aligns indices in $(A \operatorname{Adj}(A))_{ij}$.

Exam Tips. If you are unsure: write the (i, j) entry of $A \operatorname{Adj}(A)$ and see what cofactor index you need:

$$(A \operatorname{Adj}(A))_{ij} = \sum_k a_{ik} (\operatorname{Adj}(A))_{kj} = \sum_k a_{ik} C_{jk}.$$

This forces $\operatorname{Adj}(A)_{kj} = C_{jk}$, i.e. transpose.

Common Pitfalls.

- Your own note already flagged the #1 mistake: using A_{ij} instead of A_{ji} (equivalently forgetting transpose).
- Confusing cofactor matrix vs adjugate.

Example 2×2 (as in your note)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathrm{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exam Tips. For 2×2, this is the fastest route; for $n \geq 3$, do not compute adjugate explicitly unless asked.

8 8 Fundamental Identity

Theorem 3 (Theorem 3.)**.**

$$A\,\mathrm{Adj}(A) = \mathrm{Adj}(A)\,A = \det(A)I_n.$$

Proof (Exam Skeleton). Do it entrywise. For the (i, j) entry:

$$(A\,\mathrm{Adj}(A))_{ij} = \sum_{k=1}^{n} a_{ik} C_{jk}.$$

If $i = j$, this is exactly the Laplace row expansion of $\det(A)$ along row i, so it equals $\det(A)$. If $i \neq j$, this is the Laplace *off-diagonal* zero-sum identity (row version), so it equals 0. Hence $A\,\mathrm{Adj}(A) = \det(A)I_n$. The identity $\mathrm{Adj}(A)\,A = \det(A)I_n$ follows similarly (or by applying the first identity to A^T and transposing).
How to Understand. This theorem says: *cofactors are designed to behave like "inverse times determinant"*. Adjugate is the "almost inverse".
Exam Tips. If the exam asks to prove it, the safest template is:

1. Write (i, j) entry of the product.
2. Split into cases $i = j$ vs $i \neq j$.
3. Invoke Laplace expansion + zero-sum identity.

Do **not** try to prove by abstract reasoning; entrywise is short and marker-friendly.
Common Pitfalls.

- Forgetting to separate diagonal/off-diagonal.
- Using the wrong cofactor index: it must be C_{jk} (not C_{kj}).

Corollary 1 (Corollary 1.)**.** *If* $\det(A) \neq 0$,

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A).$$

Proof (Exam Skeleton). From $A\,\mathrm{Adj}(A) = \det(A)I_n$, multiply both sides by $1/\det(A)$. Then

$$A \left(\frac{1}{\det(A)} \mathrm{Adj}(A) \right) = I_n,$$

so the bracket is A^{-1}.
How to Understand. Adjugate gives an explicit inverse formula, but it is mainly theoretical for large n.
Exam Tips.

- Always state the condition $\det(A) \neq 0$ before writing the inverse formula.
- For computations: use row-reduction for inverse; adjugate is too slow for $n \geq 3$ unless matrix has many zeros/special form.

Common Pitfalls.

- Writing inverse exists without stating $\det(A) \neq 0$.
- Treating the formula as a computational method for big matrices (time trap).

One-Page Compression (Before Exam)

- Minor $\to$ Cofactor $(-1)^{i+j}$
- Laplace = determinant as linear combination of cofactors
- Off-diagonal Laplace sums are zero
- Adjugate = transpose of cofactor matrix
- $A\,\mathrm{Adj}(A) = \det(A)I$
- Inverse formula follows immediately

Common Mistakes

- Using A_{ij} instead of A_{ji} inside Adj
- Forgetting $(-1)^{i+j}$
- Missing the zero-sum Laplace identity
- Mixing row vs column expansion indices