YSU ASDS, Statistics, Fall 2019

Lecture 22

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Last Lecture ReCap

- Give the definition of the $(1-\alpha)$-level CI.

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- What is the **R** command to generate 20 random numbers from the *Cauchy*(2) distribution?

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- What is the **R** command to generate 20 random numbers from the *Cauchy*(2) distribution?
- Give a $(1-\alpha)$-level CI for *p* in *Bernoulli*(*p*) Model.

CI by Pivotal Quantity Method

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Then we call $g(X_1, ..., X_n, \theta)$ to be a **Pivot** for our model.

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This will give some inequality of the form $L < \theta < U$, and hence,

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so we can take (L, U) as a CI.

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Note: Usually, we are solving

$$\mathbb{P}\Big(a < g(X_1, X_2, ..., X_n, \theta) < b\Big) = 1 - \alpha.$$

CI for the Mean of $\mathcal{N}(\mu, \sigma^2)$, σ is known, Pivotal Method

Problem: Assume we have a Random Sample

$$X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2).$$

Assume σ^2 is known. Given $\alpha \in (0, 1)$, we want to construct a CI of CL $1 - \alpha$ for μ, using a Pivot.

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But let us consider $\overline{X} - \mu$.

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Clearly,

$$\overline{X} - \mu \sim$$

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The name of the above ratio is Z-**statistics**, and we will meet this again in Hypotheses testing part.

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Because of the symmetry, we need to have that the area of both tails is $\alpha/2$:

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$$b = z_{1-\frac{\alpha}{2}},$$

where $z_{1-\alpha/2}$ is the $1 - \frac{\alpha}{2}$ quantile of the Standard Normal Distribution.

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So we obtained

$$\mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha.$$

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$$\mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha,$$

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and solve for μ:

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Hence, the following interval:

$$\left(\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \ \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

is a $(1 - \alpha)$-level CI for μ.

Example

Example: Assume we want to construct a 95% CI for μ in the $\mathcal{N}(\mu, \sigma^2)$ Model, when σ is given, known.

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Since $1 - \alpha = 0.95$, then $\alpha = 0.05$.

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R gives:

```r
qnorm(0.975)
```

```
## [1] 1.959964
```

so our 95% CI will be

$$\left(\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \;\; \overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right).$$

Example

Example with R: We generate random numbers from $\mathcal{N}(2.31, 4)$ (so here we assume we know the true parameter value of μ).

```
sigma <- 2
n <- 20
smpl <- rnorm(n, mean = 2.31, sd = sigma)
smpl
```

```
##  [1] -0.61517958  0.26141094 -0.47118696  3.42826333  1.
##  [6]  0.06724348  1.18915020  0.70609049  4.60922396  0.
## [11]  3.92192237 -1.71053629  3.07221264  2.21058725  5.
## [16]  4.43852249 -0.54471636  2.27825637  2.97237654 -1.
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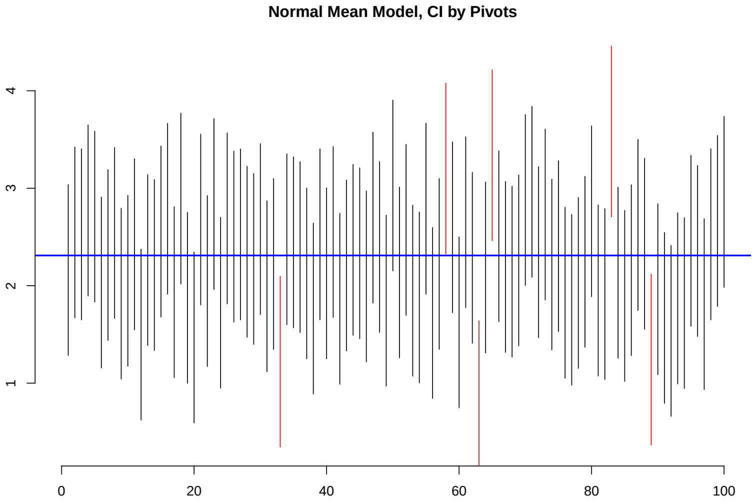
```

Now we construct the 95% CI by the above formula:

```
me <- 1.96* sigma/sqrt(n) #the Margin of Error
c(mean(smpl) - me, mean(smpl) + me)
```

```
## [1] 0.680515 2.433592
```

Example, Simulation



Example, Simulation, Code

```
mu <- 2.31; sigma <- 2
conf.level <- 0.95; a = 1 - conf.level
sample.size <- 20; no.of.intervals <- 100
z <- qnorm(1-a/2) ## our quantile
ME <- z*sigma/sqrt(sample.size) #Margin of Error

plot.new()
plot.window(xlim = c(0,no.of.intervals), ylim = c(mu-2,mu+2))
axis(1); axis(2)
title("Normal Mean Model, CI by Pivots")
for(i in 1:no.of.intervals){
  x <- rnorm(sample.size, mean = mu, sd = sigma)
  lo <- mean(x) - ME; up <- mean(x) + ME
  if(lo > mu || up < mu){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
    segments(c(i), c(lo), c(i), c(up))
  }
}
abline(h = mu, lwd = 2, col = "blue")
```

CI for the Mean of $\mathcal{N}(\mu, \sigma^2)$, σ is **unknown**, PivMe

Problem: Assume we have a Random Sample

$$X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2).$$

Assume σ^2 is **unknown**, which is more realistic.

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Solution: Again we start from $\overline{X}$, having that it is a good Estimator for μ. Again, from the fact that $X_k \sim \mathcal{N}(\mu, \sigma^2)$ are IID, we will have

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Problem: Assume we have a Random Sample

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Assume σ^2 is **unknown**, which is more realistic. Given $\alpha \in (0, 1)$, we want to construct a CI of CL $1 - \alpha$ for μ, using a Pivot.

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We know some good Estimators for σ: let us take, in this case, the following version of Sample SD:

$$S^2 = \frac{\sum_{k=1}^n (X_k - \overline{X})^2}{n-1}, \qquad i.e., \qquad S = \sqrt{\frac{\sum_{k=1}^n (X_k - \overline{X})^2}{n-1}}.$$

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Now, let us replace above σ with S:

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}}.$$

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I know, not sure about you.

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I know, not sure about you. Yet!

t-Distribution

It turns out that the Distribution of above

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is the famous t-Distribution with $n - 1$ degrees of freedom:

[1] See, e.g., https://en.wikipedia.org/wiki/Student's_t-distribution

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is the famous t-Distribution with $n - 1$ degrees of freedom:

Definition: If $X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$ are IID and

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n - 1},$$

then the Distribution of

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is called the **Student's t-Distribution with $n - 1$ degrees of freedom**[1], and is denoted by $t(n - 1)$.

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Student's Paper

Student's Paper

VOLUME VI MARCH, 1908 No. 1

BIOMETRIKA.

THE PROBABLE ERROR OF A MEAN.

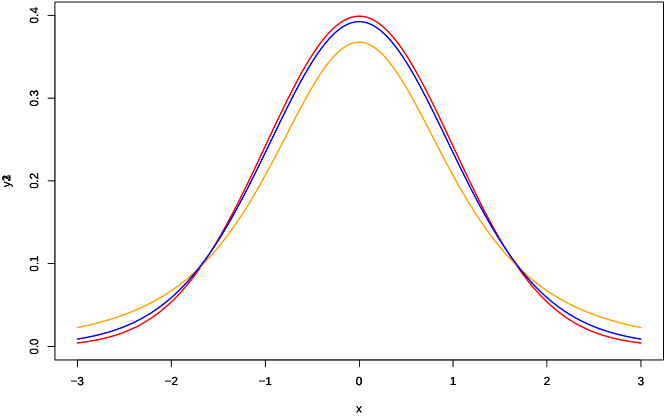
BY STUDENT.

Introduction.

ANY experiment may be regarded as forming an individual of a "population" of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

t-Distribution

```
x <- seq(-3,3, 0.01)
y1 <- dnorm(x); y2 <- dt(x, df = 3); y3 <- dt(x, df = 15)
plot(x,y1, type = "l", col = "red", lwd = 2, ylim = c(0, 0.4))
par(new = T)
plot(x,y2, type = "l", col = "orange", lwd = 2, ylim = c(0, 0.4))
par(new = T)
plot(x,y3, type = "l", col = "blue", lwd = 2, ylim = c(0, 0.4))
```



CI for the Mean of $\mathcal{N}(\mu, \sigma^2)$, σ is **unknown**, PivMe

Back to our Problem, we know that

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1),$$

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Now, for $t \sim t(n-1)$, let us find numbers a and b such that

$$\mathbb{P}(a < t < b) = 1 - \alpha.$$

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Again, because of the symmetry, we need to have that the area of both tails is $\alpha/2$:

$$\mathbb{P}(t \leq -b) = \mathbb{P}(t \geq b) = \frac{\alpha}{2}.$$

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Hence,

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Hence,

$$b = t_{1-\frac{\alpha}{2}}(n-1) = t_{n-1,1-\frac{\alpha}{2}},$$

where $t_{n-1,1-\alpha/2}$ is the $1 - \frac{\alpha}{2}$ quantile of the $t(n-1)$ Distribution.

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and solve for μ:

$$\mathbb{P}\left(\overline{X} - t_{n-1,1-\alpha/2}\cdot\frac{S}{\sqrt{n}} < \mu < \overline{X} + t_{n-1,1-\alpha/2}\cdot\frac{S}{\sqrt{n}}\right) = 1-\alpha.$$

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Hence, the following interval:

$$\left(\overline{X} - t_{n-1,1-\alpha/2}\cdot\frac{S}{\sqrt{n}},\ \ \overline{X} + t_{n-1,1-\alpha/2}\cdot\frac{S}{\sqrt{n}}\right)$$

is a $(1-\alpha)$-level CI for μ.

CI for μ, Normal Model, Notes

Note: To compare:

- If σ is known, $(1-\alpha)$-level CI for μ is
$$\overline{X} \pm z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$
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Note: If we will compare the quantiles of the same level of $\mathcal{N}(0,1)$ with $t(n-1)$, we will see that CIs for the case when σ is unknown are wider than for the case when σ is known. This is intuitive, of course - to compensate the uncertainty in σ, we need to take a wider interval:

```
c(qnorm(0.975), qt(0.975, df = 3), qt(0.975, df = 20))
```

```
## [1] 1.959964 3.182446 2.085963
```

Example

Example: Assume we want to estimate the average time μ our Stat students are spending for a homework. We collect information from 10 students for the last hw, and get the following responses (in hrs):

```
[1] 2.17 1.42 2.13 0.56 1.21 2.22 1.35 2.37 1.47 1.70
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$$X_1, X_2, ..., X_{10} \sim \mathcal{N}(\mu, \sigma^2).$$

We construct a 95% CI for μ, the average time to solve the hw, by the above formula:

Example

```r
n <- 10
cl <- 0.95; a = 1-cl
t <- qt(1-a/2, df = n-1)
smpl<-c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
s <- sd(smpl)
me <-  t*s/sqrt(n) #the Margin of Error
c(mean(smpl) - me, mean(smpl) + me)
```

```
## [1] 1.253748 2.066252
```

Example

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c(mean(smpl) - me, mean(smpl) + me)
```

```
## [1] 1.253748 2.066252
```

Veeery unsatisfactory result, of course! Please spend **more time for your hw, read textbooks**!

Example, cont'd

Later, we will talk about the *t*-**Test**, let me now just do a *t*-Test for the hw solving hours:

```r
smpl <- c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
t.test(smpl)
```

```
##
##  One Sample t-test
##
## data:  smpl
## t = 9.2435, df = 9, p-value = 6.86e-06
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  1.253748 2.066252
## sample estimates:
## mean of x
##      1.66
```

Example, cont'd

We can separate here the CI:

```
smpl <- c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
tst <- t.test(smpl) #Keeping the test result in tst
tst$conf.int
```

```
## [1] 1.253748 2.066252
## attr(,"conf.level")
## [1] 0.95
```

CI for μ, Normal Model, Summary

Let us summarize what we have obtained for this model.

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We have considered 2 cases: when σ^2 was known and unknown. Here we give the summary:

CI for μ, Normal Model, Summary

	σ^2 is known	σ^2 is unknown
Pivot:		

CI for μ, Normal Model, Summary

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Pivot:	$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$	$t = \frac{\overline{X}-\mu}{S/\sqrt{n}}$
Distr. of Pivo:		

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Reg w/prob $1-\alpha$		

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Reg w/prob $1-\alpha$	$-z_{1-\alpha/2} < Z < z_{1-\alpha/2}$	$-t_{n-1,1-\alpha/2} < t < t_{n-1,1-\alpha/2}$
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Distr. of Pivo:	$Z \sim \mathcal{N}(0,1)$	$t \sim t(n-1)$
Reg w/prob $1-\alpha$	$-z_{1-\alpha/2} < Z < z_{1-\alpha/2}$	$-t_{n-1,1-\alpha/2} < t < t_{n-1,1-\alpha/2}$
CI for μ:	$\overline{X} \pm z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$	$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$

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	σ^2 is known	σ^2 is unknown
Pivot:	$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$	$t = \frac{\overline{X}-\mu}{S/\sqrt{n}}$
Distr. of Pivo:	$Z \sim \mathcal{N}(0,1)$	$t \sim t(n-1)$
Reg w/prob $1-\alpha$	$-z_{1-\alpha/2} < Z < z_{1-\alpha/2}$	$-t_{n-1,1-\alpha/2} < t < t_{n-1,1-\alpha/2}$
CI for μ:	$\overline{X} \pm z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$	$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$

Here S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}.$$

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

Problem: Assume we have a Random Sample

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$$\left(\frac{X_k - \mu}{\sigma}\right)^2.$$

Is this a Pivot?

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Is this a Pivot? Yes, of course. What is the Distirbution of this r.v.? No, of course $\ddot\smile$. Well, we can use our Prob knowledge, but let's keep this to you. The fact is that this Pivot will not give a good result, as we are not using all the information we have.

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So we will use the following as a Pivot:

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$\chi^2(n)$ Distribution

Definition: Assume

$$Z_1, Z_2, ..., Z_n \sim \mathcal{N}(0,1)$$

and Z_k-s are Independent (so they are IID). The Distribution of

$$Y = Z_1^2 + Z_2^2 + ... + Z_n^2$$

is called the **Chi-Squared Distribution with n Degrees of Freedom**, and is denoted by $\chi^2(n)$:

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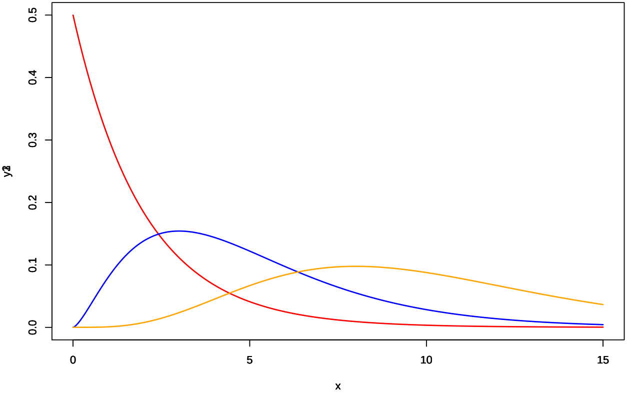
Exercise: What is $\mathbb{E}(Y)$?

$\chi^2(n)$ Distribution, PDF graphs

```
x <- seq(from = 0, to = 15, by = 0.01)
y1<-dchisq(x, df=2); y2<-dchisq(x, df=5); y3<-dchisq(x, df=10)
plot(x,y1,type="l",lwd=2,col="red", ylim=c(0,0.5)); par(new=T)
plot(x,y2,type="l",lwd=2,col="blue", ylim=c(0,0.5)); par(new=T)
plot(x,y3,type="l",lwd=2,col="orange", ylim = c(0,0.5))
```



CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

Back to our CI for σ^2 Problem: We wanted to use the following as a Pivot:

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$$\mathbb{P}(a < Y < b) = 1 - \alpha.$$

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Now, the rest is easy: we obtained

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We plug the value of Y:

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and solve for σ^2:

$$\mathbb{P}\left(\frac{\sum_{k=1}^{n}(X_k - \mu)^2}{\chi^2_{n,1-\frac{\alpha}{2}}} < \sigma^2 < \frac{\sum_{k=1}^{n}(X_k - \mu)^2}{\chi^2_{n,\frac{\alpha}{2}}}\right) = 1 - \alpha$$

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This means that we found a $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{\sum_{k=1}^{n}(X_k - \mu)^2}{\chi^2_{n,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n}(X_k - \mu)^2}{\chi^2_{n,\frac{\alpha}{2}}}\right).$$

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$$\chi^2 = \sum_{k=1}^{n} \left(\frac{X_k - \overline{X}}{\sigma} \right)^2.$$

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **unknown**

Great thing is that this is a Pivot: the Distribution of

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is independent of σ^2, and is:

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Now, doing the same calculations as above, we will arrive at the following $(1-\alpha)$-level CI for σ^2 :

$$\left(\frac{\sum_{k=1}^{n} (X_k - \overline{X})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n} (X_k - \overline{X})^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

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Again, we have obtained the following $(1-\alpha)$-level CI for σ^2 :

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where S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}.$$

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.