YSU ASDS, Statistics, Fall 2019

Lecture 28

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Contents

- Intro To The Linear Regression

Last Lecture ReCap

- Give two alternative definitions of the p-Value of a Test.

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- Give two alternative definitions of the p-Value of a Test.
- Using p-Values, in which case we Reject Null?

Linear Regression

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From the Statistical Learning Perspective

Intro to Linear Regression

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to get some information about θ, in particular,

- to find a good Point Estimator and Estimate;
- to find a CI for θ of given CL;
- to Test a Hypothesis about θ, say, is it likely that $\theta = 3.1415$ or not.

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Recall that, in the Descriptive Statistics part, we considered two Datasets, and defined the Sample Covariance and Correlation Coefficients, to measure the Linear Relationship between that Datasets. That was defined for two **Numerical Dataset**, without any assumptions behind the Process generating that Datasets. Now, if we assume that that Datasets are coming from some Distribution, we are at the stage of doing a Statistical Inference, Statistical Analysis.

Elements of the Supervised Learning

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We will assume $\mathcal{X} \subset \mathbb{R}^d$ (or, maybe, in other d-Dim Space), and a typical element $\mathbf{x}$ of $\mathcal{X}$ will have the form

$$\mathbf{x} = (x_1, x_2, ..., x_d)$$

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We will assume also that $\mathcal{Y} \subset \mathbb{R}$, and we will call the elements of $\mathcal{Y}$ to be the **Labels**.

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Problem: Given a Feature vector $\mathbf{x}$, other than $\mathbf{x}_k$, predict its Label y.

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- Numerical/Quantitative, if $x_k \in \mathbb{R}$

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Regression Problems:

- $\mathcal{Y} = \mathbb{R}$ - **1D Regression**

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We will assume that the observation $(\mathbf{x}, y) \in \mathcal{X} \times \mathcal{Y}$ is a realization of the pair of r.v.s $(\mathbf{X}, Y)$ that is coming from some unknown Distribution $\mathcal{F}$:

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The general idea/Problem is, having Data, to infer $\mathcal{F}$.

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$$(\mathbf{X}_1, Y_1), (\mathbf{X}_2, Y_2), ..., (\mathbf{X}_n, Y_n)$$

construct a "good" Prediction Function

$$g : \mathcal{X} \to \mathcal{Y},$$

that will predict the Label of $\mathbf{X}$.

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To assess goodness of the Predictor g, we take a **Loss** function. We will call any function of the form

$$\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$$

a **Loss** function, and we will assume that:

$$\ell(y_1, y_2) \geq 0, \qquad \forall y_1, y_2 \in \mathcal{Y}, \qquad \textit{and} \qquad \ell(y, y) = 0.$$

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- For The Binary Classification:

$$\ell(y_1, y_2) = \begin{cases} 1, & y_1 \neq y_2 \\ 0, & y_1 = y_2 \end{cases}$$

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We want to have this Loss as small as possible. Well, under our setting, this will be a R.V., so we define the **Average Loss** or the **Risk** of the Predictor g to be

$$Risk(g) = \mathbb{E}\Big(\ell(Y, g(\mathbf{X}))\Big).$$

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Now, we can state our Problem of finding a good Predictor: Find g minimizing the Risk, i.e., find

$$g^* \in \underset{g}{argmin}\, Risk(g).$$

Example

Toy Example: Assume $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{0, 1\}$, and we have the Joint Distribution of (X, Y):

$Y \setminus X$	1	2	3
0	0.1	0.2	0.1
1	0.2	0.1	0.3

Assume

$$g(x) = \begin{cases} 0, & \textit{if } x \textit{ is even} \\ 1, & \textit{otherwise} \end{cases}$$

and $\ell(y_1, y_2) = |y_1 - y_2|$. Calculate the $Risk(g)$.

Solution: OTB

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Usually, we assume that g comes from a Parametric Family of functions, which we call a Predictive Model:

$$g \in \mathcal{G} = \{g(\mathbf{x}|\theta),\ \theta \in \Theta\}, \qquad \textit{where} \quad g(\mathbf{x}|\theta) : \mathcal{X} \to \mathcal{Y}.$$

and Θ is some Parameter Set (1D or more).

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- In the general Regression/Classification Problems, we can have $g(\mathbf{x}|\theta)$ to be a Neural Network, where $\mathbf{x}$ is our input, θ is the vector of all NN weights, and $g(\mathbf{x}|\theta)$ is the output of the NN.

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and we want to find $g^* \in \mathcal{G}$ such that

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If $\mathcal{G}$ coincides with the set of all measurable functions, thet g^*, if exists, is called the **Bayes Predictor**.

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If $\mathcal{G}$ coincides with the set of all measurable functions, thet g^*, if exists, is called the **Bayes Predictor**. And if g^* is a Bayes Predictor, then its Risk, $Risk(g^*)$, is called the **Bayes Risk**.

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So, instead of trying to minimize $Risk(g)$, we can try to minimize

$$ERM(g) = \frac{1}{n} \cdot \sum_{k=1}^{n} \ell(Y_k, g(\mathbf{X}_k)),$$

which is called the **Empirical Risk Measure of** g.

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Example: Now, assume $(X, Y) \sim \mathit{Unif}(D)$, where D is the triangle with vertices at $(-1, 0)$, $(0, 1)$ and $(1, 0)$. What is $Y|X = x$?

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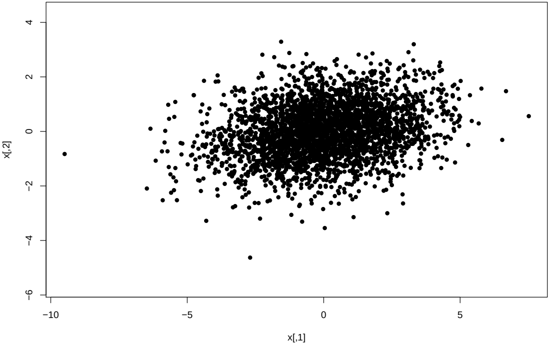
$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 4 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$



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