YSU ASDS, Statistics, Fall 2019

Lecture 27

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Last Lecture ReCap

- What are the two Sample Z- or t-Test about?

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- Describe the two Sample t-test.

From the last lecture

Paired t-Test for the Difference of two Normals Means

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[1] The Test will work also in the case when the Differences are nor Normally Distributed, but the Sample Size n is large. We jut need to use the CLT.

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Paired t-Test for the Difference of two Normals Means, Cont'd

Test Statistics: $t = \dfrac{\overline{D} - \mu_0}{S_D/\sqrt{n}}$, where S_D is the Sample Deviation of D.

[2] Or, Asymptotically, $t \approx t(n-1)$ or $t \approx \mathcal{N}(0,1)$, if D_k-s are not Normal, but n is large.

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$\mu_X - \mu_Y \neq \mu_0$	$\lvert t \rvert > t_{n-1,1-\frac{\alpha}{2}}$
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Note: This Test is called the **Paired t-Test**

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Example, cont'd

Here is the code for some hypothetical results from two hypothetical Algorithms:

```
res.algo1 <- rbinom(300, size = 1, prob = 0.4)
res.algo2 <- rbinom(300, size = 1, prob = 0.45)
t.test(res.algo1,res.algo2,paired = T)
```

```
##
##  Paired t-test
##
## data:  res.algo1 and res.algo2
## t = -1.0952, df = 299, p-value = 0.2743
## alternative hypothesis: true difference in means is not
## 95 percent confidence interval:
##  -0.1212004  0.0345337
## sample estimates:
## mean of the differences
##             -0.04333333
```

Likelihood Ratio Test

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where $\mathcal{L}(\theta)$ is our GOF *Likelihood Function*:

$$\mathcal{L}(\theta) = f(X_1|\theta) \cdot f(X_2|\theta) \cdot ... \cdot f(X_n|\theta).$$

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$$\sup_{\theta \in \Theta} \mathcal{L}(\theta) = \mathcal{L}\left(\hat{\theta}^{MLE}\right).$$

And similarly, $\sup_{\theta \in \Theta_0} \mathcal{L}(\theta)$ can be thought as a *restricted MLE for* θ *over* Θ_0.

Neyman-Pearson Lemma

Assume $X_1, ..., X_n \sim \mathcal{F}_\theta$, and we want to Test, at α-level, a Simple Hypothesis vs Simple Hypothesis:

$$\mathcal{H}_0 : \ \theta = \theta_0 \qquad vs \qquad \mathcal{H}_1 : \ \theta = \theta_1.$$

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Let $\mathcal{L}(\theta)$ be the Likelihood Function for $X_1, ..., X_n$, and let

$$LR = \frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\theta_1)}$$

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the significance level is α:

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Then this Test has the highest Power among all other Tests of Significance Level α.

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and the Likelihood Function is:

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Example: LRT

Assume $X_1, X_2, ..., X_n \stackrel{IID}{\sim} \mathcal{N}(\mu, \sigma^2)$, and σ^2 is **known**. We want to construct the LRT, for Significance Level α, for

$$\mathcal{H}_0 : \ \mu = \mu_0 \qquad vs \qquad \mathcal{H}_1 : \ \mu \neq \mu_0.$$

Step 1: We calculate the Likelihood: the PDF is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}},$$

and the Likelihood Function is:

$$\mathcal{L}(\mu) = f(X_1|\mu, \sigma^2) \cdot f(X_2|\mu, \sigma^2) \cdot ... \cdot f(X_n|\mu, \sigma^2)$$

i.e.

$$\mathcal{L}(\mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot e^{-\frac{1}{2\sigma^2} \cdot \sum_{k=1}^{n} (X_k - \mu)^2}$$

Example: LRT, Cont'd

Step 2: Now, we calculate the LR:

$$LR = \frac{\sup_{\mu=\mu_0} \mathcal{L}(\mu)}{\sup_{\mu\in\mathbb{R}} \mathcal{L}(\mu)} =$$

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We can write

$$\sup_{\mu\in\mathbb{R}} \mathcal{L}(\mu) = \mathcal{L}\left(\hat{\mu}^{MLE}\right),$$

since, by the very definition, $\hat{\mu}^{MLE}$ is the global max point of $\mathcal{L}(\mu)$.

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where Z is our Z-Test Statistics. So we have arrived at our GOF Z-Test!

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Log-Likelihood Ratio, its Distribution

Sometimes Statisticians use the Log-Laikelihood Ratio to obtain the LRT: the quantity

$$-2 \cdot \ln LR = -2 \cdot \Big(\ln \big(\sup_{\theta \in \Theta_0} \mathcal{L}(\theta) \big) - \ln \big(\sup_{\theta \in \Theta} \mathcal{L}(\theta) \big) \Big).$$

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And having this Distribution, we can find the Rejection Region using

$$\mathbb{P}(\text{Reject } \mathcal{H}_0 \mid \mathcal{H}_0 \text{ is True}) = \mathbb{P}(-2 \ln LR \geq c') = \alpha.$$

p-Values

3 Methods of Decision Making in Testing

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- Based on the Confidence Interval CI for the Parameter: if θ is our Parameter, (L, U) is a CI of $(1-\alpha)$-level for θ, and our Null is $\mathcal{H}_0: \ \theta = \theta_0$, then we Reject Null if and only if $\theta_0 \notin (L, U)$;

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Note: When doing Tests, say, with `t.test`, **R** is calculating the p-Value, and sometimes also the CI. So, to decide whether to Reject Null or Not, using **R**, you can use the 2nd and 3rd Methods.

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based on the Test Statistics TS. Assume we already have Observations, and we calculate the value of TS, let us denote that by TS_{obs} (this is just a number). We know that, for a given Significance Level α, we will Reject $\mathcal{H}_0$, iff TS_{obs} will be in the RR.

Now, assume the Distribution of TS, our Test Statistics, **under** $\mathcal{H}_0$, is given like this (I am drawing for Z- or t-Statistics, for Two Tailed Test, the other cases can be considered in a similar way):

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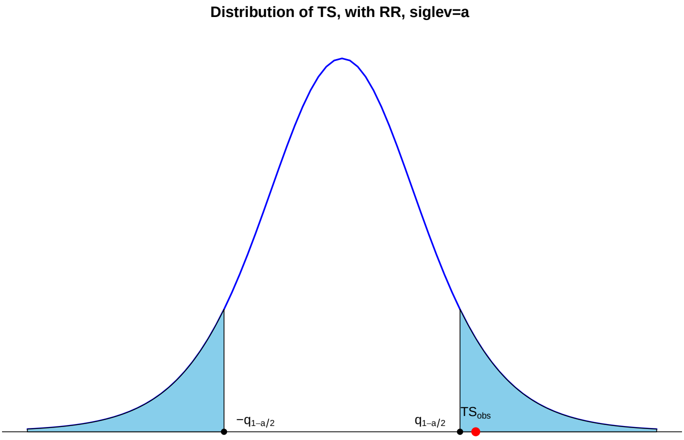

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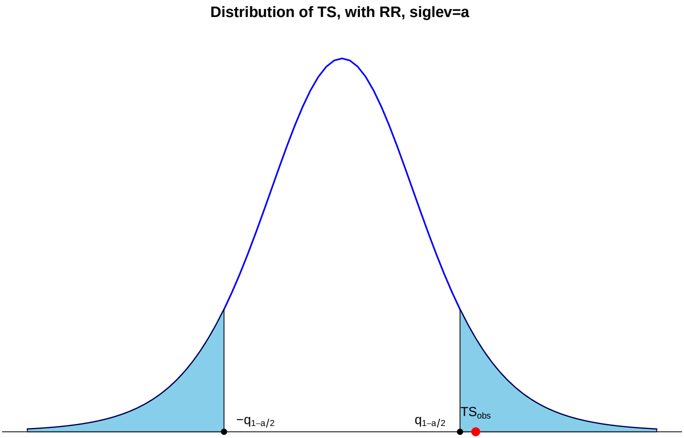

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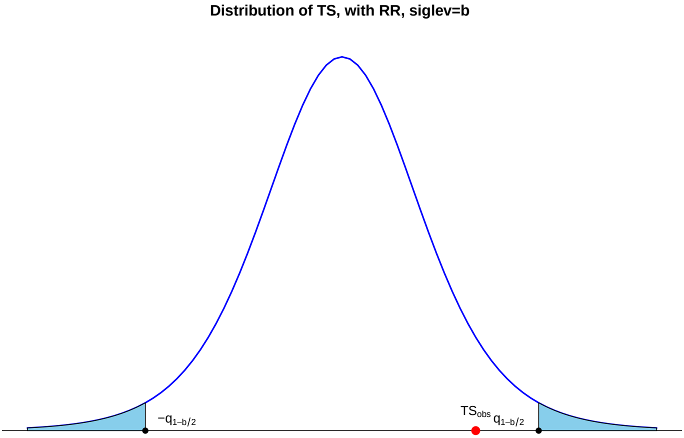

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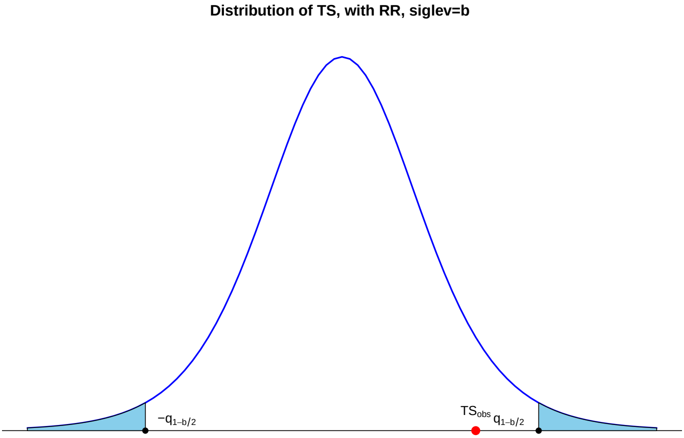

We Do Not Reject $\mathcal{H}_0$ at the level b

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Note: Give here the real line with picture, MP!

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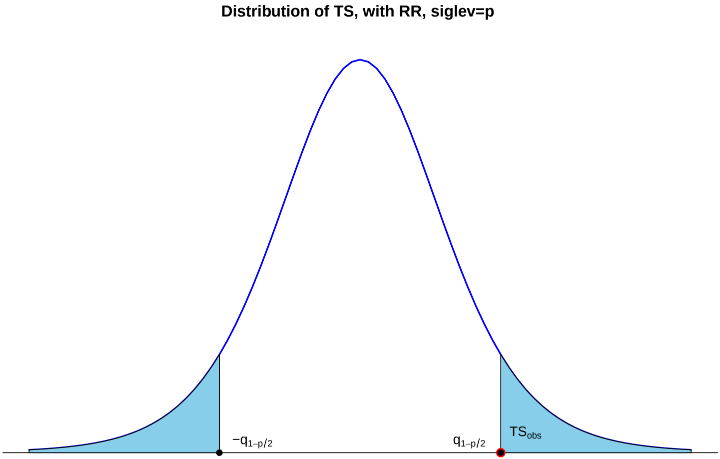
Note: Give here the real line with picture, MP!

Now, we denote $p = \alpha^*$ and call it the *p*-**Value of the Test:**

$$p\text{-Value} = p = \inf\{\alpha : \text{ we Reject } \mathcal{H}_0 \text{ at level } \alpha\}.$$

p-Values



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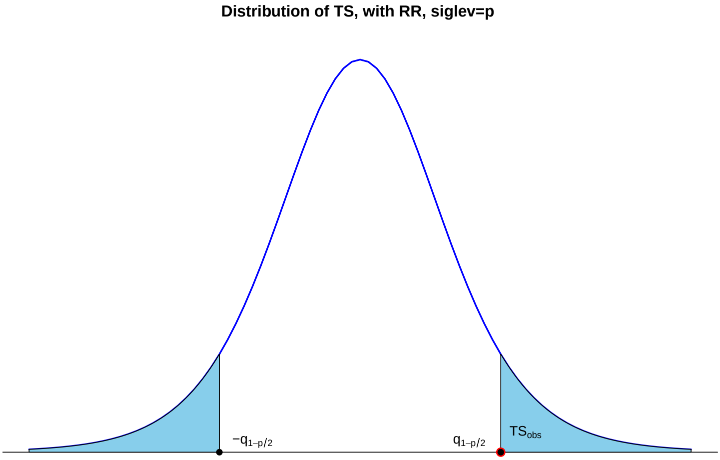

p-Value, the inf value of α at which we Reject $\mathcal{H}_0$

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It is clear from the Figure above that

$$p\text{-Value} = p = \mathbb{P}(|TS| > TS_{obs} \mid \mathcal{H}_0 \text{ is True}).$$

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And, if p-Value is small, then TS_{obs} is very unprobable, very unbelievable, under $\mathcal{H}_0$, so we safely Reject $\mathcal{H}_0$.

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To Remember:

- If p-Value$< \alpha$, then we Reject $\mathcal{H}_0$
- If p-Value$\geq \alpha$, then we Fail to Reject $\mathcal{H}_0$

R Code for the Graphics

```
df <- 8;
x <- seq(-4,4,0.1); y <- dt(x, df = df)
plot.new()
plot.window(xlim = c(-4, 4), ylim = c(-0.05,0.4))
plot(x,y, type="l",col="blue",lwd=2,xaxt="n",yaxt="n",
     bty="n",xlab="",ylab="")
abline(h=0)
title("Distribution of TS, with RR, siglev=a ")
qpoint <- 1.5; tspoint <- 1.7
cord.x <- c(qpoint,seq(qpoint,4,0.01),4)
cord.y <- c(0,dt(seq(qpoint,4,0.01), df=df),0)
polygon(cord.x,cord.y,col='skyblue')
points(c(qpoint), c(0), pch=20, cex=1.4)
text(c(qpoint-0.38),c(0.01),labels=expression("q"[1-a/2]))
cord.x1 <- c(-4,seq(-4,-qpoint,0.01),-qpoint)
cord.y1 <- c(0,dt(seq(-4,-qpoint,0.01), df=df),0)
polygon(cord.x1,cord.y1,col='skyblue')
points(c(-qpoint), c(0), pch=20, cex=1.4)
text(c(-qpoint+0.4),c(0.01),labels=expression("-q"[1-a/2]))
points(c(tspoint), c(0), col="red", pch=19, cex=1.4)
text(c(tspoint), c(0.02), labels = expression("TS"[obs]))
```

Example

Example: Assume we have an observation from $\mathcal{N}(\mu, 3^2)$, and we want to Test

$$\mathcal{H}_0 : \ \mu = 1.2 \qquad \textit{vs} \qquad \mathcal{H}_1 : \ \mu \neq 1.2.$$

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and the value of p-Value is ⌣̈

```
1-(pnorm(1.72,mean=0,sd=1)-pnorm(-1.72,mean=0,sd=1))
```

```
## [1] 0.08543244
```