LECTURE 5

Example 12. Piles for a building foundation were initially designed for 250-ton capacity each; however, this did not include the effect of high winds that occur only very rarely. On such rare occasions, it is estimated that some of the piles may be subjected to loads as high as 300 tons. In order to assess the safety of the initial design, the engineer in charge wishes to determine the probability of the piles failing under a maximum load of 300 tons. Suppose that from the engineer's experience with this type of piles and the soil condition at the site, he estimated (judgmentally) that the probability p would range from 0.2 to 1.0 with 0.4 as the most likely value; more specifically, p is described by the prior Probability Mass Function (PMF).

$P:$	0.2	0.4	0.6	0.8	1.0
$f(p):$	0.3	0.4	0.15	0.1	0.05

The values of p are discretized at 0.2 intervals to simplify the presentation. On the basis of this prior PMF, the estimated probability of a pile failing at a load of 300 tons would be (by virtue of the total probability theorem):

$$P = 0.2 \cdot 0.3 + 0.4 \cdot 0.4 + 0.6 \cdot 0.15 + 0.8 \cdot 0.1 + 1.0 \cdot 0.05 = 0.44$$

In order to supplement his judgment, the engineer ordered a pile of the same type test-loaded at the site to a maximum load of 300 tons. The outcome of the test shows that the pile failed to carry the maximum load. Based on this single test result, the PMF of p would be revised according to Bayes formula, obtaining the posterior PMF as follows:

$P:$	0.2	0.4	0.6	0.8	1.0
$f(p):$	0.136	0.364	0.204	0.182	0.114

The Bayesian estimate for p therefore is

$$P = 0.2 \cdot 0.136 + 0.4 \cdot 0.364 + 0.6 \cdot 0.204 + 0.8 \cdot 0.182 + 1.0 \cdot 0.114 = 0.55.$$

We see that as a result of the single unsuccessful load test, the probabilities for higher values of p are increased from those of the prior distribution, resulting in a higher estimate for p, namely, $p = 0.55$, whereas the prior estimate was 0.44. Observe that the failure of one test pile does not imply the impossibility of such piles carrying the 300-ton load; instead, the test result merely serves to increase the estimated probability by 0.11 (from 0.44 to 0.55). How the PMF of p changes with increasing number of consecutive test pile failures; the distribution shifts toward $p = 1.0$ as n tends to infinity.

We can show, that if a long sequence of failures is observed, that the corresponding Bayesian estimate for p approaches 1.0 - a result that tends to a classical estimate; in such a case, there is overwhelming amount of observed data to supersede any prior judgment. Ordinarily, however, where observational data are limited, judgment would be important and is reflected properly in the Bayesian estimation process.

Now suppose that each main column is supported on a group of three piles. If the piles carry equal loads and are statistically independent, the probability that none of the piles supporting a column will fail at a total column load of 900 tons (300 tons per pile) can be obtained using Bayes formula. Based on the posterior PMF and denoting X as the number of piles failing, the required probability is:

$$(0.8)^3 \cdot 0.136 + (0.6)^3 \cdot 0.364 + (0.4)^3 \cdot 0.204 + (0.2)^3 \cdot 0.182 = 0.163.$$

Example 13. A traffic engineer is interested in the average rate of accidents ν at an improved road intersection. Suppose that from his previous experience with similar road and traffic conditions, he deduced that the expected accident rate would be between one and three per year, with an average of two, and the prior PMF is

1	2	3
0.3	0.4	0.3

Occurrence of accidents is assumed to be a Poisson process. During the first month after completion of the intersection, one accident occurred.

a) In the light of this observation, revise the estimate of ν;

b) Using the result of part a), determine the probability of no accident in the next six months.

Solution:

a) Let A be the event that an accident occurred in one month. The posterior probabilities

$$P(\nu = 1/A) = \frac{e^{-1/12} \cdot 1/12 \cdot 0.3}{e^{-1/12} \cdot 1/12 \cdot 0.3 + e^{-1/6} \cdot 1/6 \cdot 0.4 + e^{-1/4} \cdot 1/4 \cdot 0.3} = 0.166$$

Similarly,

$$P(\nu = 2/A) = 0.411$$

$$P(\nu = 3/A) = 0.423.$$

Hence the updated value of ν is

$$\nu = 0.166 \cdot 1 + 0.411 \cdot 2 + 0.423 \cdot 3 = 2.26$$

accidents per year.

b) Let B the event of no accidents in the next six months. Then By total probability formula is

$$P(B) = e^{-1/2}\, 0.166 + e^{-1}\, 0.411 + e^{-3/2}\, 0.423 = 0.346$$

Example 14. An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a standard deviation of 100 hours. Prior experience leads us to assume μ to be a value of a normal random variable M with a mean equal to 800 hours and a standard deviation of 10 hours. If a random sample of 25 bulbs have an average life of 780 hours, find a 95% Bayes interval for μ.

Solution. Multiplying the density of our sample

$$f(x_1, x_2, ..., x_{25}|\mu) = \frac{1}{(2\pi)^{25/2} \cdot 100^{25}} \exp\left[-\left(\frac{1}{2}\right)\sum_{i=1}^{25}\left(\frac{x_i - \mu}{100}\right)^2\right], \quad -\infty < x_i < \infty,\ i = 1, 2, ..., 25$$

by our prior

$$f(\mu) = \frac{1}{\sqrt{2\pi} \cdot 10}\, e^{-(1/2)[(\mu-800)/10]^2}, \quad -\infty < \mu < \infty,$$

we obtain the joint density of the random sample and M. That is,

$$f(x_1, x_2, ..., x_{25}, \mu) = \frac{1}{(2\pi)^{13} \cdot 10^{51}} \exp\left[-\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{25}\left(\frac{x_i - \mu}{100}\right)^2 + \left(\frac{\mu - 800}{10}\right)^2\right\}\right].$$

We established the identity

$$\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}(x_i-\overline{x})^2+n(\overline{x}-\mu)^2,$$

which enables us to write

$$f(x_1,x_2,...,x_{25},\mu)=\frac{1}{(2\pi)^{13}\cdot 10^{51}}\,\exp\left[-\left(\frac{1}{2}\right)\sum_{i=1}^{25}\left(\frac{x_i-780}{100}\right)^2\right]\times$$
$$\times e^{-(1/2)\{25[(780-\mu)/100]^2+[(\mu-800)/10]^2\}}.$$

Completing the square in the second exponent, we have

$$25\left(\frac{780-\mu}{100}\right)^2+\left(\frac{\mu-800}{10}\right)^2=\frac{\mu^2-1592\mu+633,680}{80}=\frac{(\mu-796)^2+64}{80}.$$

The joint density of the sample and M can now be written

$$f(x_1,x_2,...,x_{25},\mu)=K\,e^{-(1/2)[(\mu-796)/\sqrt{80}]^2},$$

where K is a function of the sample values. The marginal distribution of the sample is then

$$g(x_1,x_2,...,x_{25})=K\sqrt{2\pi}\sqrt{80}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{80}}\,e^{-(1/2)[(\mu-796)/\sqrt{80}]^2}\,d\mu=K\,\sqrt{2\pi}\sqrt{80},$$

and the posterior distribution is

$$f(\mu|x_1,x_2,...,x_{25})=\frac{f(x_1,x_2,...,x_{25},\mu)}{g(x_1,x_2,...,x_{25})}=\frac{1}{\sqrt{2\pi}\sqrt{80}}\,e^{-(1/2)[(\mu-796)/\sqrt{80}]^2},\quad -\infty<\mu<\infty,$$

which is normal with mean $\mu^*=796$ and standard deviation $\sigma^*=\sqrt{80}$. The 95% Bayes interval for μ is then given by

$$\mu^*-1.96\sigma^*<\mu<\mu^*+1.96\sigma^*.$$

That is,

$$796-1.96\sqrt{80}<\mu<796+1.96\sqrt{80}$$

or

$$778.5<\mu<813.5$$

Ignoring the prior information about μ and comparing this result with that given by the classical 95% confidence interval

$$780-(1.96)\frac{100}{5}<\mu<780+(1.96)\frac{100}{5}$$

or

$$740.8<\mu<819.2$$

we notice that the Bayes interval is shorter than the classical confidence interval.