L E C T U R E   3

Example 4. Let $n = 2$ $(X_1, X_2) = (1, 2)$ be a random sample of size 2 from a distribution with probability density function

$$p(x|\theta) = \binom{x}{3} \theta^x (1-\theta)^{3-x}, \quad x = 0, 1, 2, 3.$$

If the prior density of θ is

$$p(\theta) = \begin{cases} k, & \text{if} \quad 1/2 < \theta < 1 \\ 0, & \text{otherwise} \end{cases}$$

what is the posterior distribution of θ?

Solution. Since $p(\theta)$ is the probability density function of θ, we should get

$$\int_{-\infty}^{\infty} p(\theta)\, d\theta = 1, \quad \text{that is} \quad \int_{1/2}^{1} k\, d\theta = 1.$$

Therefore $k = 2$. The joint density of the sample and the parameter is given by

$$f(x_1, x_2, \theta) = p(x_1|\theta) \cdot p(x_2|\theta) \cdot p(\theta) = \binom{3}{x_1} \theta^{x_1} (1-\theta)^{3-x_1} \cdot \binom{3}{x_2} \theta^{x_2} (1-\theta)^{3-x_2} \cdot 2$$

$$= 2 \binom{3}{x_1} \binom{3}{x_2} \theta^{x_1+x_2} (1-\theta)^{6-x_1-x_2}.$$

Hence,

$$f(1, 2, \theta) = 2 \binom{3}{1} \binom{3}{2} \theta^3 (1-\theta)^3 = 18\, \theta^3 (1-\theta)^3.$$

The marginal distribution of the sample

$$g(1, 2) = \int_{1/2}^{1} f(1, 2, \theta)\, d\theta = 18 \int_{1/2}^{1} \theta^3 (1-\theta)^3\, d\theta = 18 \int_{1/2}^{1} \theta^3 (1 + 3\theta^2 - 3\theta - \theta^3)\, d\theta =$$

$$= 18 \int_{1/2}^{1} (\theta^3 + 3\theta^5 - 3\theta^4 - \theta^6)\, d\theta = \frac{9}{140}.$$

The conditional distribution of the parameter θ given the sample $X_1 = 1$ and $X_2 = 2$ is given by

$$p(\theta | x_1 = 1, x_2 = 2) = \frac{f(1, 2, \theta)}{g(1, 2)} = \frac{18 \cdot \theta^3 (1-\theta)^3}{9/140} = 280\, \theta^3 (1-\theta)^3.$$

Therefore, the posterior distribution of θ is

$$p(\theta | x_1 = 1, x_2 = 2) = \begin{cases} 280 \cdot \theta^3 (1-\theta)^3, & \text{if} \quad 1/2 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}$$

§7. DISTRIBUTION FUNCTIONS

Definition 0. Let $(\Omega, \mathcal{F}, P)$ be a probability space, i. e. Ω is a sample space on which a probability P has been defined. A *random variable* is a function η from Ω to the set of real numbers

$$\eta: \qquad \Omega \longmapsto \mathbb{R}^1,$$

i. e. for every outcome $\omega \in \Omega$ there is a real number, denoted by $\eta(\omega)$, which is called the value of $\eta(\cdot)$ at ω.

We can also give the following definition of distribution function.
The distribution function F of a random variable $\eta(\omega)$ is defined for all real numbers $x \in \mathbb{R}^1$, by the formula

$$F(x) = P(\omega\colon\ \eta(\omega) \le x). \tag{7}$$

In words, $F(x)$ denotes the probability that the random variable $\eta(\omega)$ takes on a value that is less than or equal to x.
Some properties of the distribution function are the following:

Property 0. $0 \le F(x) \le 1$.

Property 1. F is a nondecreasing function, that is, if $x_1 \le x_2$ then $F(x_1) \le F(x_2)$.

Proof: We will present two proofs. For $x_1 \le x_2$ the event $\{\omega\colon\ \eta(\omega) \le x_1\}$ is contained in the event $\{\omega\colon\ \eta(\omega) \le x_2\}$ and so cannot have a larger probability (see Property 5 of Probabilities), i. e.

$$P(\omega\colon\ \eta(\omega) \le x_1) \le P(\omega\colon\ \eta(\omega) \le x_2).$$

Therefore, by definition of the distribution function we have $F(x_1) \le F(x_2)$.
Another proof of the property 1 is the following. Let us prove the formula

$$P(\omega\colon\ x_1 < \eta(\omega) \le x_2) = F(x_2) - F(x_1), \qquad \text{for all} \quad x_1 < x_2. \tag{8}$$

This can best be seen by writing the event $\{\omega\colon\ \eta(\omega) \le x_2\}$ as the union of the mutually exclusive events $\{\omega\colon\ \eta(\omega) \le x_1\}$ and $\{\omega\colon\ x_1 < \eta(\omega) \le x_2\}$. That is,

$$\{\omega\colon\ \eta(\omega) \le x_2\} = \{\omega\colon\ \eta(\omega) \le x_1\} \bigcup \{\omega\colon\ x_1 < \eta(\omega) \le x_2\}.$$

and so

$$P(\omega\colon\ \eta(\omega) \leq x_2) = P(\omega\colon\ \eta(\omega) \leq x_1) + P(\omega\colon\ x_1 < \eta(\omega) \leq x_2)$$

which established equation (8). By Axiom 1 the left–hand side of (8) is nonnegative, and therefore $F(x_2) - F(x_1) \geq 0$. The proof is complete.

Property 2. $F(x) \to 1$ as $x \to +\infty$.

Property 3. $F(x) \to 0$ as $x \to -\infty$.

Property 4. $F(x)$ is right continuous. That is, for any x and any decreasing sequence x_n that converges to x,

$$\lim_{n\to\infty} F(x_n) = F(x).$$

Thus, Properties 1 – 4 are necessary conditions for a function $G(x)$ to be a distribution function. However, these properties are also sufficient. This assertion follows from the following theorem which we cite without proof.

Theorem 1 (about Distribution Function). *Let a function $G(x)$, $x \in \mathbb{R}^1$ satisfy the Properties 1 — 4. Then there exist a probability space (Ω, P) and a random variable $\eta(\omega)$ for which distribution function coincides with given function $G(x)$, i. e.*

$$P(\omega\colon \quad \eta(\omega) \leq x) = G(x).$$

Therefore, for giving an example of a random variable we have to cite a function which satisfies the Properties 1 — 4.

We want to stress that in Theorem about distribution function a random variable $\eta(\omega)$ is determined by non–unique way.

Example 5. Let (Ω, P) be a probability space and $P(A) = P(\overline{A}) = 0.5$. We define the following two random variables

$$\eta_1(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in A \\ -1 & \text{if} \quad \omega \notin A \end{cases}, \qquad \eta_2(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in \overline{A} \\ -1 & \text{if} \quad \omega \in A \end{cases}.$$

It is obvious that $\{\omega\colon \quad \eta_1(\omega) \neq \eta_2(\omega)\} = \Omega$. However,

$$F_{\eta_1}(x) = F_{\eta_2}(x) = \begin{cases} 0 & \text{if} \quad x < -1 \\ 0.5 & \text{if} \quad -1 \leq x < 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}.$$

Definition 1. Two random variables $\eta_1(\omega)$ and $\eta_2(\omega)$ are said to be *Identically Distributed* if their distribution functions are equal, that is,

$$F_{\eta_1}(x) = F_{\eta_2}(x) \quad \text{for all} \quad x \in \mathbb{R}^1.$$

§8. EXAMPLES OF DISTRIBUTION FUNCTIONS

Example 6. A random variable $\eta(\omega)$ is said to be *Normally distributed* if its distribution function has the following form

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right) dy, \tag{9}$$

where a and σ are constants, moreover $a \in \mathbb{R}^1$ and $\sigma > 0$.

In order to show the correctness of Example 6 we have to verify that the function in the right–hand side of (9) satisfies the Properties 1 — 4. The correctness of Example 6 can be found in the Appendix of the present lecture.

The normal distribution plays a central role in probability and statistics. This distribution is also called the Gaussian distribution after Carl Friedrich Gauss, who proposed it as a model for measurement errors.

Example 7. A random variable is said to be *Uniformly distributed* on the interval (a, b) if its distribution function is given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \le a \\ \dfrac{x-a}{b-a} & \text{if} \quad a \le x \le b \\ 1 & \text{if} \quad x \ge b \end{cases}. \tag{10}$$

It is obvious that the function (10) satisfies all Properties 1 — 4.

Example 8. A random variable is said to be *Exponentially distributed* with parameter $\lambda > 0$ if its distribution function is given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ 1 - e^{-\lambda x} & \text{if} \quad x \ge 0 \end{cases}. \tag{11}$$

It is obvious that the function (11) satisfies all Properties 1 — 4.

Like the Poisson distribution, the exponential distribution depends on the only parameter.

Example 9. If $\eta(\omega) \equiv c$ then corresponding distribution function has the form

$$F(x) = \begin{cases} 0 & \text{if} \quad x < c \\ 1 & \text{if} \quad x \geq c \end{cases}. \tag{12}$$

Consider the experiment of flipping a symmetrical coin once. The two possible outcomes are "heads" (outcome ω_1) and "tails" (outcome ω_2), that is, $\Omega = \{\omega_1, \omega_2\}$. Suppose $\eta(\omega)$ is defined by putting $\eta(\omega_1) = 1$ and $\eta(\omega_2) = -1$. We may think of it as earning of the player who receives or loses a dollar according as the outcome is heads or tails. Corresponding distribution function has the form

$$F(x) = \begin{cases} 0 & \text{if} \quad x < -1 \\ 1/2 & \text{if} \quad -1 \leq x < 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}.$$

Lemma 1. *Let $F(x)$ be a distribution function of a random variable $\eta(\omega)$. Then for any real number x we have*

$$P\{\omega: \quad \eta(\omega) = x\} = F(x) - F(x-0), \tag{13}$$

where $F(x-0)$ is the left–hand limit at x.

Therefore, for continuous distribution function (as in Examples 6 — 8) we have

$$P(\omega: \quad \eta(\omega) = x) = 0 \qquad \text{for any} \quad x \in \mathbb{R}^1. \tag{14}$$

§9. CONTINUOUS RANDOM VARIABLES

We say that $\eta(\omega)$ is an *absolutely continuous* random variable if there exists a function $f(x)$ defined for all real numbers and the distribution function $F(x)$ of the random variable $\eta(\omega)$ is represented in the form

$$F(x) = \int_{-\infty}^{x} f(y)\, dy. \tag{14}$$

The function f is called the *Density function* of $\eta(\omega)$.

A function $f(x)$ must have certain properties in order to be a density function. Since $F(x) \to 1$ as $x \to +\infty$ we obtain

Property 1.

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1. \tag{15}$$

Property 2. $f(x)$ is a nonnegative function.

Proof: Differentiating both sides of (14) yields

$$f(x) = \frac{d}{dx} F(x). \tag{16}$$

That is, the density is the derivative of the distribution function. We know that the first derivative of a nondecreasing function is always nonnegative. Therefore the proof is complete as $F(x)$ is nondecreasing.

Remarkably that these two properties are also sufficient for a function $g(x)$ be a density function.

Theorem 2 (About Density Function). *Let a function* $g(x)$, $x \in \mathbb{R}^1$ *satisfy (15) and, in addition, satisfies the condition*

$$g(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^1.$$

Then there exist a probability space (Ω, P) *and an absolutely continuous random variable* $\eta(\omega)$ *for which density function coincides with given function* $g(x)$.

Proof: Let us define a function

$$G(x) = \int_{-\infty}^{x} g(y)\, dy.$$

It is not difficult to verify that $G(x)$ satisfies all conditions 1 — 4 for distribution function. Therefore by Theorem 1 about distribution function, there exists a random variable $\eta(\omega)$ for which distribution function coincides with $G(x)$. By definition of density function, $g(x)$ is a density function of the random variable $\eta(\omega)$. The proof is complete.

Therefore, for giving an example of an absolutely continuous random variable we have to cite a *non-negative* function which satisfies (15).

The normally distributed random variable (see Example 6) is absolutely continuous and its density function has the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right), \tag{17}$$

where a and σ are constant, moreover $a \in \mathbb{R}^1$ and $\sigma > 0$.

The uniformly distributed random variable over the interval (a, b) (see Example 7) is absolutely continuous and its density function has the form

$$f(x) = \begin{cases} 0 & \text{if } x \notin (a, b) \\ \dfrac{1}{b-a} & \text{if } a \leq x \leq b \end{cases}. \tag{18}$$

It is obvious that the function (18) satisfies (15).

The exponentially distributed random variable with parameter $\lambda > 0$ (see Example 8) is absolutely continuous and its density function has the form

$$f(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ \lambda\, e^{-\lambda x} & \text{if} \quad x > 0 \end{cases}. \tag{19}$$

It is obvious that the function (19) satisfies (15).

We obtain from (8) that

$$\begin{aligned} &P(\omega\colon\ a \le \eta(\omega) \le b) = P(\omega\colon\ a \le \eta(\omega) < b) = \\ &= P(\omega\colon\ a < \eta(\omega) \le b) = P(\omega\colon\ a < \eta(\omega) < b) = \int_a^b f(x)\, dx. \end{aligned} \tag{20}$$

As the distribution function of an absolutely continuous random variable is continuous at all points thus

$$P(\omega\colon\ \eta(\omega) = x) = 0$$

for any fixed x.

Therefore this equation states that the probability that an absolutely continuous random variable will assume any fixed value is zero.

A somewhat more intuitive interpretation of the density function may be obtained from (18). If $\eta(\omega)$ is an absolutely continuous random variable having density function $f(x)$, then for small dx

$$P(\omega\colon\ x \le \eta(\omega) \le x + dx) = f(x)\, dx + o(dx).$$

In general case (for any distribution function) we have the following formulae:

$$P(\omega\colon\ a \le \eta(\omega) \le b) = F(b) - F(a-0),$$

$$P(\omega\colon\ a \le \eta(\omega) < b) = F(b-0) - F(a-0),$$

$$P(\omega\colon\ a < \eta(\omega) \le b) = F(b) - F(a),$$

$$P(\omega\colon\ a < \eta(\omega) < b) = F(b-0) - F(a),$$

where $F(a-0)$ is the left-limit of $F(x)$ at point a.

APPENDIX 3.

Proof of Lemma 1: **Let us prove the following equation**

$$P(\omega: \quad \eta(\omega) < x) = F(x-0), \tag{21}$$

i. e. we want to compute the probability that $\eta(\omega)$ is less than x.
It is not difficult to verify that

$$\{\omega: \quad \eta(\omega) < x\} = \bigcup_{n=1}^{\infty} A_n,$$

where $A_n = \left\{\omega: \quad \eta(\omega) \leq x - \dfrac{1}{n}\right\}$.
A_n is an increasing sequence and therefore, tends to the event $\{\omega: \quad \eta(\omega) < x\}$. Thus

$$A_n \uparrow \{\omega: \quad \eta(\omega) < x\}.$$

By a property of Probability we get

$$P(A_n) \uparrow P(\omega: \quad \eta(\omega) < x).$$

Hence (21) is proved.
As

$$P(\omega: \quad \eta(\omega) = x) = P(\omega: \quad \eta(\omega) \leq x) - P(\omega: \quad \eta(\omega) < x) = F(x) - F(x-0)$$

the assertion of the Lemma follows from the equation (21). The proof is complete.