# LECTURE 3

Bivariate random vector $(\eta_1, \eta_2)$ has normal distribution if density function has the following form:

$$f(x,y) = c \cdot e^{-Q(x,y)}$$

where $Q(x,y)$ is a positive definite quadratic form.

A quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ik}x_i x_k \qquad \text{(where} \quad a_{ik} = a_{ki})$$

is called positive definite, if it takes on nonnegative values for any $x_1,\ x_2, \cdots, x_n,$ and equal to zero only if $x_1 = \cdots = x_n = 0$. Any positive definite quadratic form by linear transformation is reduced to the following form:

$$\sum_{i=1}^{n} x_i^2. \tag{6.1}$$

In order to a quadratic form to be positive definite it is necessary and sufficient that

$$\Delta_1 > 0 \quad \Delta_2 > 0 \ldots \Delta_k > 0 \ldots \Delta_n > 0$$

where

$$\Delta_k = \begin{bmatrix} a_{11} & \ldots & a_{1k} \\ & \vdots & \\ a_{k1} & \ldots & a_{kk} \end{bmatrix}$$

It is known, that positive definite quadratic form on $x$ and $y$ can be written int the form:

$$Q(x,y) = \frac{(x-a)^2}{2A^2} - r\frac{(x-a)(y-b)}{A \cdot B} + \frac{(y-b)^2}{2B^2},$$

where $A > 0$, $B > 0$ $-1 < r < +1$ are real numbers. If we denote $A = \sigma_1\sqrt{1-r^2}$, $B = \sigma_2\sqrt{1-r^2}$ we obtain

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}} \left[\frac{(x-a)^2}{\sigma_1^2} - 2r\frac{(x-a)(y-b)}{\sigma_1\sigma_2} + \frac{(y-b)^2}{\sigma_2^2}\right]$$

## Rotation of the coordinate axes

It is known that for any $a$, $b$ are real numbers, for any $\sigma_1$, $\sigma_2$ are positive and $-1 < r < 1$ the equation

$$\frac{(x-a)^2}{\sigma_1^2} - 2r\frac{(x-a)(y-b)}{\sigma_1\sigma_2} + \frac{(y-b)^2}{\sigma_2^2} = R^2$$

is an ellipse.

Let us make the following linear transform

$$x = x_1 \cos\alpha - y_1 \sin\alpha + a,$$

$$y = x_1 \sin\alpha + y_1 \cos\alpha + b,$$

where

$$\tan 2\alpha = \frac{2r\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}.$$

Then we obtain (6.1) form.

### Defining Properties of Brownian Motion

There is an increasing interest in the study of systems which vary in time in a random manner. Mathematical models of such systems are known as stochastic processes. We have tried to select topics that are conceptually interesting and that have found fruitful application in various branches of science and technology. A stochastic process can be defined quite generally as any collection of random variables $\eta(t,\omega)$, $t \in T$, defined on a common probability space, where $T$ is a subset of $(-\infty, +\infty)$ and is thought of as the time parameter set. The process is called a *continuous parameter process* if $T$ is an interval having positive length and a *discrete parameter process* if $T$ is a subset of the integers. If the random variables $\eta(t,\omega)$ all take on values from the fixed set $\mathcal{G}$, then $\mathcal{G}$ is called the *state space* of the process.

Since we are interested in non–deterministic systems, we think of $\eta_n(\omega)$, $n \geq 0$, as random variables defined on a common probability space. Little can be said about such random variables unless some additional structure is imposed upon them.

Brownian motion $B(t)$ is a stochastic process with the following properties.

1.(Normal increments) $B(t)-B(s)$ has Normal distribution with mean 0 and variance $t-s$. This implies with $s=0$ that $B(t)-B(0)$ has $N(0,t)$ distribution.

2.(Independence of increments) $B(t)-B(s)$ is independent of the past, that is, of $B_u$, $0 \le u \le s$.

3.(Continuity of paths) $B(t), \quad t \ge 0$ are continuous functions of $t$.

**Example 3.1** Let $B(0)=0$. We calculate $P(B(t) \le 0 \quad for \quad t=2)$ and $P(B(t) \le 0 \quad for \quad t=0,1,2)$. Since $B(2)$ has Normal distribution with mean zero and variance 2, $P(B(t) \le 0 \quad for \quad t=2) = \frac{1}{2}$

Since $B(0)=0$, $P(B(t) \le 0 \quad for \quad t=0,1,2) = P(B(1) \le 0, B(2) \le 0)$. Note that $B(2)$ and $B(1)$ are not independent, therefore this probability cannot be calculated as a product $P(B(1) \le 0)P(B(2) \le 0) = 1/4$. However, using the following decomposition

$$B(2) = B(1) + (B(2) - B(1)),$$

we can reduce calculations to independent variables. Since increments are independent $B(2)-B(1)$ is independent of $B(1)$. By the property of normality of increments of Brownian motion, $B(2)-B(1)$ has the standard Normal distribution. Denote $\hat{B}(1) = (B(2)-B(1))$, than we have

$$\begin{aligned} P(B(1) \le 0, B(2) \le 0) &= P(B(1) \le 0, B(1) + \hat{B}(1) \le 0) \\ &= P(B(1) \le 0, \hat{B}(1) \le -B(1)). \end{aligned}$$

Therefore we obtain

$$P(B(1) \le 0, \hat{B}(1) \le -B(1)) = \int_{-\infty}^{0} P(\hat{B}(1) \le -x) f(x) dx = \int_{-\infty}^{0} \Phi(-x) d\Phi(x),$$

where $\Phi(x)$ and $f(x)$ denote the distribution and the density functions of the standard Normal distribution. Since $\Phi(-x) = 1 - \Phi(x)$ we obtain

$$\int_{-\infty}^{0} (1-\Phi(x)) f(x) dx = \int_{-\infty}^{0} (1-\Phi(x)) d\Phi(x) = \int_{-\infty}^{0} d\Phi(x) - \int_{-\infty}^{0} \Phi(x)\, d\Phi(x) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$