LECTURE 2

§6. PRIOR AND POSTERIOR DISTRIBUTIONS

Example 3. A disease occurs with prevalence γ in population and θ indicates that an individual has the disease. Hence

$$P(\theta = 1) = \gamma, \qquad P(\theta = 0) = 1 - \gamma.$$

A diagnostic test gives a result Y, whose distribution function is $F_1(y)$ for a disease individual, and $F_0(y)$ otherwise. The most common type of test declares that a person is diseased if $Y > y_0$, where y_0 is fixed on the basis of past data.

The probability that a person is diseased, given a positive test result, is

$$P\left(\theta = 1/Y > y_0\right) = \frac{\gamma \cdot [1 - F_1(y_0)]}{\gamma \cdot [1 - F_1(y_0)] + (1-\gamma) \cdot [1 - F_0(y_0)]}.$$

This is sometimes called the positive predictive value of test. Its sensitivity and specifity are $1 - F_1(y_0)$ and $F_0(y_0)$.

In more general case, θ can take a finite number of values, labeled $1, 2, ..., k$. We can assign to these values probabilities $p_1, p_2, ..., p_k$ which express our beliefs about θ before we have access to the data. The data y are assumed to be the observed value of a (multidimensional) random variable Y, and $p(y/\theta)$ the density of y given θ (the likelihood function).

Then the conditional probabilities

$$P\left(\theta = j/Y = y\right) = \frac{p_j P(y/\theta = j)}{\sum\limits_{i=1}^{k} p_i P(y/\theta = i)}, \qquad j = 1.2, ..., k,$$

summarize our beliefs about θ after we have observed Y.

The unconditional probabilities $p_1, p_2, ..., p_k$ are called prior probabilities and

$$P\left(\theta = 1/Y = y\right), \quad P\left(\theta = 2/Y = y\right), ..., P\left(\theta = k/Y = y\right)$$

are called posterior probabilities of θ.

When θ can get values continuously on some interval, we can express our beliefs about it with a prior density $p(\theta)$. After we have obtained the data y, our beliefs about θ are contained in the conditional density,

$$p(\theta/y) = \frac{p(\theta) \cdot p(y/\theta)}{\int p(\theta) \cdot p(y/\theta) d\theta}, \tag{6}$$

called posterior density.

Since θ is integrated out in the denominator, it can be considered as a constant with respect to θ. Therefore, the Bayes' formula in (6) is often written as

$$p(\theta/y) \propto p(\theta) \cdot p(y/\theta), \tag{7}$$

which denotes that $p(\theta/y)$ is proportional to $p(\theta) \cdot p(y/\theta)$.