LECTURE 9

Square matrices are best understood in terms of quantities called eigenvalues and eigenvectors.

Definition 9.1. Let $\mathbf{A}$ be a $k \times k$ square matrix and $\mathbf{I}$ be the $k \times k$ identity matrix. Then the scalars λ_1, λ_2,..., λ_k satisfying the polynomial equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

are called the eigenvalues of a matrix $\mathbf{A}$.

For example, let

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}.$$

Then

$$|\mathbf{A} - \lambda \mathbf{I}| = \left| \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| =$$

$$= \left| \begin{bmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix} \right| = (1-\lambda)(3-\lambda) = 0$$

implies that there are two roots, $\lambda_1 = 1$ and $\lambda_2 = 3$. The eigenvalues of $\mathbf{A}$ are 3 and 1. Let

$$\begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}.$$

Then the equation

$$|\mathbf{A} - \lambda \mathbf{I}| = \left| \begin{bmatrix} 13-\lambda & -4 & 2 \\ -4 & 13-\lambda & -2 \\ 2 & -2 & 10-\lambda \end{bmatrix} \right| = -\lambda^3 + 36\lambda^2 - 405\lambda + 1458 = 0$$

has three roots: $\lambda_1 = 9$, $\lambda_2 = 9$, and $\lambda_3 = 18$; that is 9, 9, and 18 are the eigenvalues of $\mathbf{A}$.

Definition 9.2. Let $\mathbf{A}$ be a square matrix of dimension $k \times k$ and let λ be an eigenvalue of $\mathbf{A}$. If $\mathbf{x}$ is a nonzero vector of dimension $k \times 1$ $(\mathbf{x} \neq \underbrace{(0, ..., 0)}_{k\,times})$ such that

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}$$

then $\mathbf{x}$ is said to be an eigenvector of the matrix $\mathbf{A}$ associated with the eigenvalue λ.

An equivalent condition for λ to be a solution of the eigenvalue-eigenvector equation is $|\mathbf{A} - \lambda \mathbf{I}| = 0$. This follows because the statement that $\mathbf{Ax} = \lambda \mathbf{x}$ for some λ and $\mathbf{x} \neq 0$ implies that

$$0 = (\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = x_1 \text{Column}_1 (\mathbf{A} - \lambda \mathbf{I}) + x_2 \text{Column}_2 (\mathbf{A} - \lambda \mathbf{I}) + ... + x_k \text{Column}_k (\mathbf{A} - \lambda \mathbf{I}).$$

That is, the columns of $\mathbf{A} - \lambda \mathbf{I}$ are linearly dependent so,

$$|\mathbf{A} - \lambda \mathbf{I}| = 0,$$

as asserted. Following Definition 9.1 we have shown that the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}.$$

are $\lambda_1 = 1$ and $\lambda_2 = 3$. The eigenvectors associated with these eigenvalues can be determined by solving the following equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From the first expression, we obtain

$$x_1 = x_1$$

$$x_1 + 3x_2 = x_2$$

or

$$x_1 = -2x_2$$

There many solutions for x_1 and x_2.

Setting $x_2 = 1$ (arbitrarily) gives $x_1 = -2$, and hence,

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue 1. From the second expression,

$$x_1 = 3x_1$$

$$x_1 + 3x_2 = 3x_2$$

implies that $x_1 = 0$ and $x_2 = 1$ (arbitrarily), and hence, we get

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue 3. It is usual practice to determine an eigenvector so that it has length unity. That is, if $\mathbf{Ax} = \lambda\mathbf{x}$, we take $\mathbf{e} = \frac{\mathbf{x}}{\sqrt{\mathbf{x}'\mathbf{x}}}$ as the eigenvector corresponding to λ. For example, the eigenvector for $\lambda_1 = 1$ is $\mathbf{e}' = \left[\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$.

Definition 9.3. A quadratic form $Q(\mathbf{x})$ in the k variables x_1, x_2,..., x_k is $Q(\mathbf{x}) = \mathbf{x}'\mathbf{Ax}$, where $\mathbf{x}' = [x_1, x_2, ..., x_k]$ and $\mathbf{A}$ is a $k \times k$ symmetric matrix. Note that a quadratic form can be written as

$$Q(\mathbf{x}) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} x_i x_j.$$

Fore example,

$$Q(\mathbf{x}) = [x_1, x_2] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1x_2 + x_2^2.$$

$$Q(\mathbf{x}) = [x_1, x_2, x_3] \begin{bmatrix} 1 & 3 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + 6x_1x_2 - x_2^2 - 4x_2x_3 + 2x_3^2.$$

symmetric square matrix can be reconstructed from its eigenvalues and eigenvectors. The particular expression reveals the relative importance of each pair according to the relative size of the eigenvalue and the direction of the eigenvector.

Theorem 9.1. The spectral decomposition. Let $\mathbf{A}$ be a $k \times k$ symmetric matrix. Then $\mathbf{A}$ can be expressed in terms of its k eigenvalue-eigenvector pairs $(\lambda_i, \mathbf{e}_i)$ as

$$\mathbf{A} = \sum_{i=1}^{k} \lambda_i \mathbf{e}_i \mathbf{e}_i'.$$

For example, let

$$A = \begin{bmatrix} 2.2 & 0.4 \\ 0.4 & 2.8 \end{bmatrix}$$

Then

$$|A-\lambda I| = \lambda^2 - 5\lambda + 6.16 - 0.16 = (\lambda-3)(\lambda-2)$$

so A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$. The corresponding eigenvectors are $\mathbf{e}_1' = \left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right]$ and $\mathbf{e}_2' = \left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right]$, respectively. Consequently,

$$A = \begin{bmatrix} 2.2 & 0.4 \\ 0.4 & 2.8 \end{bmatrix} = 3 \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right] + 2 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} \left[\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right] =$$

$$\begin{bmatrix} 0.6 & 1.2 \\ 1.2 & 2.4 \end{bmatrix} + \begin{bmatrix} 1.6 & -0.8 \\ -0.8 & 0.4 \end{bmatrix}$$

The ideas that lead to the spectral decomposition can be extended to provide a decomposition for a rectangular, rather than a square matrix. If $\mathbf{A}$ is a rectangular matrix, then the vectors in the expansion of $\mathbf{A}$ are the eigenvectors of the square matrices $\mathbf{AA}'$ and $\mathbf{A}'\mathbf{A}$.

Theorem 9.2. Singular-Value decomposition. Let $\mathbf{A}$ be an $m \times k$ matrix of real numbers. Then there exist an $m \times m$ orthogonal matrix $\mathbf{U}$ and a $k \times k$ orthogonal matrix $\mathbf{V}$ such that

$$\mathbf{A} = \mathbf{U}\Lambda\mathbf{V}'$$

where the $m \times k$ matrix Λ has (i,i) entry $\lambda_i \geq 0$ for $i = 1, 2, ..., \min(m,k)$ and the other entries are zero. The positive constants λ_i are called the singular values of $\mathbf{A}$.