L E C T U R E 28

APPENDIX: BIVARIATE NORMAL DISTRIBUTION

A very important two–dimensional probability law which is a generalization of the one–dimensional normal probability law is called *Bivariate normal distribution*.

The random variables $\eta_1(\omega)$ and $\eta_2(\omega)$ are said to have a Bivariate normal distribution with parameters $(a_1, a_2, \sigma_1, \sigma_2, \rho)$ if their joint density function is given by

$$\begin{aligned} f(x_1,x_2) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\times \\ &\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-a_1}{\sigma_1}\right)^2 - 2\rho\cdot\frac{(x_1-a_1)(x_2-a_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-a_2}{\sigma_2}\right)^2\right]\right\}. \end{aligned} \tag{A.1}$$

We see that Bivariate normal distribution is determined by 5 parameters. These are a_1, a_2, σ_1, σ_2 and ρ such that $a_1 \in (-\infty,+\infty)$, $a_2 \in (-\infty,+\infty)$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $\rho \in (-1,+1)$.

We know that marginal density function of η_1 is defined by the following formula

$$f_{\eta_1}(x_1) = \int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_2 = \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left(-\frac{(x_1-a_1)^2}{2\sigma_1^2}\right).$$

Similarly, for random variable η_2, we obtain:

$$f_{\eta_2}(x_2) = \int_{-\infty}^{+\infty} f(x_1,x_2)\,dx_1 = \frac{1}{\sigma_2\sqrt{2\pi}}\exp\left(-\frac{(x_2-a_2)^2}{2\sigma_2^2}\right).$$

Therefore, $\eta_1(\omega)$ and $\eta_2(\omega)$ are both normal random variables with respective parameters $\mathcal{N}(a_1,\sigma_1)$ and $\mathcal{N}(a_2,\sigma_2)$.

Thus, the marginal distributions of $\eta_1(\omega)$ and $\eta_2(\omega)$ are both normal, even though the joint distributions for $\rho = 0$, $\rho = 0.3$, $\rho = 0.6$ and $\rho = 0.9$ are quite different from each other.

It is not difficult to verify that

$$\mathrm{Cov}\,(\eta_1,\eta_2) = E\left[(\eta_1-a_1)\cdot(\eta_2-a_2)\right] = \rho\,\sigma_1\sigma_2,$$

i. e. covariance between $\eta_1(\omega)$ and $\eta_2(\omega)$ is equal to $\rho\,\sigma_1\,\sigma_2$ and therefore

$$\rho(\eta_1,\eta_2) = \rho.$$

Thus, we have proved that

$$\eta_1 \quad \text{and} \quad \eta_2 \quad \text{are independent} \quad \text{if and only if the correlation is equal to } 0.$$

Indeed, substituting $\rho = 0$ in (A.1) we obtain

$$f(x_1, x_2) = f_{\eta_1}(x_1) \cdot f_{\eta_2}(x_2).$$

Definition A.1. If $\eta_1(\omega)$ and $\eta_2(\omega)$ have a joint density function $f(x_1, x_2)$, then the conditional density function of $\eta_1(\omega)$, given that $\eta_2(\omega) = x_2$ is defined for all values of x_2 such that $f_{\eta_2}(x_2) \neq 0$, by

$$f_{\eta_1/\eta_2}(x_1|x_2) = \frac{f(x_1, x_2)}{f_{\eta_2}(x_2)}.$$

To motivate this definition, multiply the left–hand side by dx_1 and the right–hand side by $(dx_1 \cdot dx_2)/(dx_2)$ to obtain

$$f_{\eta_1/\eta_2}(x_1|x_2)dx_1 = \frac{f(x_1, x_2)\, dx_1\, dx_2}{f_{\eta_2}(x_2)\, dx_2} \approx \frac{P\{\omega:\ x_1 \leq \eta_1(\omega) \leq x_1 + dx_1, x_2 \leq \eta_2(\omega) \leq x_2 + dx_2\}}{P\{\omega:\ x_2 \leq \eta_2(\omega) \leq x_2 + dx_2\}} =$$
$$= P\{\omega:\ x_1 \leq \eta_1(\omega) \leq x_1 + dx_1 | x_2 \leq \eta_2(\omega) \leq x_2 + dx_2\}.$$

In other words, for small values of dx_1 and dx_2, $f_{\eta_1/\eta_2}(x_1|x_2)$ represents the conditional probability that $\eta_1(\omega)$ is between x_1 and $x_1 + dx_1$, given that $\eta_2(\omega)$ is between x_2 and $x_2 + dx_2$.

One can show that the conditional density of η_1, given that $\eta_2 = x_2$, is the normal density with parameters

$$\mathcal{N}\left(a_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - a_2), \sigma_1^2(1 - \rho^2)\right).$$

Similarly, the conditional density of η_2, given that $\eta_1 = x_1$, is the normal density with parameters

$$\mathcal{N}\left(a_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - a_1), \sigma_2^2(1 - \rho^2)\right).$$

Now we want to prove the formula (36.13).

Firstly, note that the density of bivariate normal distribution takes on a constant values on the ellipses

$$\frac{(x_1 - a_1)^2}{\sigma_1^2} - 2\rho \cdot \frac{(x_1 - a_1)(x_2 - a_2)}{\sigma_1\sigma_2} + \frac{(x_2 - a_2)^2}{\sigma_2^2} = a^2. \tag{A.2}$$

Let's find the probability that point (η_1, η_2) belongs to the ellipse (A.2). We have

$$P(a) = \int\int_{G(a)} f(x_1, x_2)\, dx_1\, dx_2, \tag{A.3}$$

where $G(a)$ is the domain for which ellipse (A.2) is the boundary.

For calculate this integral we consider polar coordinates:

$$x_1 - a_1 = r\,\cos\theta \qquad \text{and} \qquad x_2 - a_2 = r\,\sin\theta.$$

After make the change of variables, integral (A.3) has the form;

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^{2\pi\, a/s} \int_0^{\sqrt{1-\rho^2}} \exp\left(-\frac{r^2\, s^2}{2}\right) r\, dr\, d\theta,$$

where for brevity

$$s^2 = \frac{1}{1-\rho^2}\left[\frac{\cos^2\theta}{\sigma_1^2} - 2\rho\frac{\cos\theta\,\sin\theta}{\sigma_1\,\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right].$$

Integration in r gives:

$$P(a) = \frac{1-\exp\left(-\dfrac{0.5a^2}{1-\rho^2}\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^{2\pi} \frac{d\theta}{s^2}.$$

It is not difficult to integrate in θ, but we can find this integral using probability argument:

$$P(+\infty) = 1 = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^{2\pi} \frac{d\theta}{s^2},$$

from this equality we get:

$$\int_0^{2\pi} \frac{d\theta}{s^2} = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}.$$

and therefore (36.13).

Then the probability that two dimensional vector with density function (36.11) takes the values inside of this ellipse equals to

$$1 - \exp\left(-\frac{0.5a^2}{1-\rho^2}\right). \tag{36.13}$$

Find an ellipse for which the corresponding probability equals $\frac{1}{2}$, that is find a for which

$$1 - \exp\left(-\frac{0.5a^2}{1-\rho^2}\right) = \frac{1}{2}.$$

Therefore

$$a = \sqrt{2\cdot(1-\rho^2)\cdot\ln 2}.$$