LECTURE 4

§10. EXAMPLES (continuation)

Definition 2. The mean of the posterior distribution $f(\theta|x_1, x_2, ..., x_n)$ denoted by θ^*, is called Bayes estimate for θ.

Example 10. Using a random sample of size 2, estimate the proportion p of defectives produced by a machine when we assume our prior distribution to be

p	0.1	0.2
$f(p)$	0.6	0.4

Solution: Let X be the number of defectives in our sample. Then the probability distribution for our sample is

$$f(x|p) = \binom{2}{x} p^x (1-p)^{2-x}, \qquad x = 0, 1, 2.$$

From the fact that

$$f(x, p) = f(x|p)\, f(p),$$

we can set up the following table:

$f(x,p)$	0	1	2
0.1	0.486	0.108	0.006
0.2	0.256	0.128	0.016

The marginal distribution for X is then

x	0	1	2
$g(x)$	0.742	0.236	0.022

We obtain the posterior distribution from the formula $f(p|x) = f(x,p)/g(x)$. Hence we have

p	0.1	0.2
$f(p\|x=0)$	0.655	0.345

p	0.1	0.2
$f(p\|x=1)$	0.458	0.542

p	0.1	0.2
$f(p\|x=2)$	0.273	0.727

from which we get

$$p^* = (0.1)(0.655) + (0.2)(0.345) = 0.1345, \quad \text{if} \quad x = 0$$

$$= (0.1)(0.458) + (0.2)(0.542) = 0.1542, \quad \text{if} \quad x = 1$$

$$= (0.1)(0.273) + (0.2)(0.727) = 0.1727, \quad \text{if} \quad x = 2.$$

Example 11. Repeat the previous example using the uniform prior distribution $f(p) = 1$, $0 < p < 1$.

Solution: As before, we find that

$$f(x|p) = \binom{2}{x} p^x (1-p)^{2-x}, \quad x = 0, 1, 2.$$

Now

$$f(x,p) = f(x|p)\, f(p) = \binom{2}{x} p^x (1-p)^{2-x} =$$

$$= (1-p)^2, \quad x = 0, \quad 0 < p < 1$$

$$= 2\,p(1-p), \quad x = 1, \quad 0 < p < 1$$

$$= p^2, \quad x = 2, \quad 0 < p < 1$$

and the marginal distribution for X is obtained by evaluating the integral

$$g(x) = \int_0^1 (1-p)^2\, dp = \frac{1}{3}, \qquad \text{for } x = 0$$

$$= \int_0^1 2\,p\,(1-p)\,dp = \frac{1}{3}, \qquad \text{if } x = 1$$

$$= \int_0^1 p^2\,dp = \frac{1}{3}, \qquad \text{if } x = 2.$$

The posterior distribution is then

$$f(p|x) = \frac{f(x,p)}{g(x)} = 3\binom{2}{x} p^x\,(1-p)^{2-x} =$$

$$= 3(1-p)^2, \quad x = 0, \quad 0 < p < 1$$

$$= 6p(1-p), \quad x = 1, \quad 0 < p < 1$$

$$= 3p^2, \quad x = 2, \quad 0 < p < 1$$

from which we evaluate the point estimate of our parameter to be

$$p^* = 3\int_0^1 p(1-p)^2\,dp = \frac{1}{4}, \qquad \text{if } x = 0$$

$$= 6\int_0^1 p^2(1-p)\,dp = \frac{1}{2}, \qquad \text{if } x = 1$$

$$= 3\int_0^1 p^3\,dp = \frac{3}{4}, \qquad \text{if } x = 2.$$

Comparing these estimates with the values obtained by classical procedures, we see that p^* and $\hat{p}$ are equivalent if $x = 1$, but that $\hat{p} = 0$ for $x = 0$ and $\hat{p} = 1$ for $x = 2$.

A $(1-\alpha)100\%$ Bayesian interval for the parameter θ can be constructed by finding an interval centered at the posterior mean that contains $(1-\alpha)100\%$ of the posterior probability.

Definition 3. The interval $a < \theta < b$ will be called a $(1-\alpha)100\%$ Bayes interval for θ if

$$\int_{\theta^*}^{b} f(\theta|x_1, x_2, ..., x_n)\,d\theta = \int_a^{\theta^*} f(\theta|x_1, x_2, ..., x_n)\,d\theta = \frac{1-\alpha}{2}.$$