LECTURE 22

Example 33.2. (A random walk). Consider a Markov chain whose state space consists of the integers $\{0, \pm 1, \pm 2, ...\}$ and have transition probabilities given by

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 0, \pm 1, \pm 2, ...,$$

where $0 < p < 1$.

In other words, at each transition the particle moves with probability p one unit to the right and with probability $1-p$ one unit to the left. One colorful interpretation of this process is that it represents the wanderings of a drunken man as he walks along a straight line. Another is that it represents the winnings of a gambler who on each play of the game either wins or loses one dollar.

Since all states clearly communicate it follows from Theorem 28 that they are either all transient or all recurrent. So let us consider state 0 and attempt to determine if $\sum_{n=0}^{\infty} P_{00}(n)$ is finite or infinite. Obviously, we have

$$P_{00}(2n+1) = 0, \qquad n = 1, 2, ...$$

On the other hand, we have

$$P_{00}(2n) = \binom{2n}{n} p^n \, (1-p)^n.$$

Using now Stirling's formula

$$n! = n^{n+1/2} e^{-n} \sqrt{2\,\pi}$$

we obtain

$$P_{00}(2n) \approx \frac{(4p(1-p))^n}{\sqrt{\pi\, n}}.$$

It is obvious that $4p(1-p) \leq 1$ with equality holding if and only if $p = 1/2$. Hence

$$\sum_{n=0}^{\infty} P_{00}(n) = +\infty$$

if and only if $p = 1/2$. Therefore from Theorem 28, the one-dimensional random walk is recurrent if and only if $p = 1/2$. Intuitively, if $p \neq 1/2$ there is positive probability that a particle initially at the origin will drift to $+\infty$ if $p > 1/2$ (to $-\infty$ if $p < 1/2$) without ever returning to the origin.

§34. CLASSIFICATION OF STATES

State j is said to be *accessible* from state i if $P_{i,j}(n) > 0$ for some $n \geq 0$. Note that this implies that state j is accessible from state i if and only if, starting in i, it is possible that the process will ever enter state j. This is true since if j is not accessible from i, then

$$P(\text{ever enter } j \,/\text{start in } i\,) = P(\bigcup_{n=0}^{\infty}\{\eta_n(\omega) = j \,/\eta_0(\omega) = i\,\}) \leq$$
$$\leq \sum_{n=0}^{\infty} P(\eta_n(\omega) = j \,/\eta_0(\omega) = i\,) = \sum_{n=0}^{\infty} P_{i,j}(n) = 0.$$

Two states i and j that are accessible to each other are said to be *communicate*. Note that any state communicates with itself since, by definition,

$$P_{i,i}(0) = P(\eta_0(\omega) = i \,/\eta_0(\omega) = i\,) = 1.$$

The relation of communication satisfies the following three properties:

1) State i communicates with state i for all $i \in \mathcal{G}$;

2) If state i communicates with state j, then state j communicates with state i;

3) If state i communicates with state j, and state j communicates with state k, then state i communicates with state k.

Properties 1) and 2) follow immediately from the definition of communication. To prove 3) suppose that i communicates with j, and j communicates with k. Thus, there exist integers n and m such that $P_{i,j}(n) > 0$ and $P_{j,k}(m) > 0$. Now by (116), we have that

$$P_{i,k}(n+m) = \sum_{l \in \mathcal{G}} P_{i,l}(n) P_{l,k}(m) \geq P_{i,j}(n) P_{j,k}(m).$$

Hence, state k is accessible from state i. Similarly, we can show that state i is accessible from state k. Hence, states i and k communicates.

Two states that communicate are said to be in the same *class*. It is an easy consequence of 1), 2), and 3) that any two classes of states are either identical or disjoint. Therefore, the concept of communication is an equivalent relation. In other words, the concept of communication divides the state space up into a number of separate classes. The Markov chain is said to be *irreducible* if there is only one class, that is, if all states communicate with each other.

Consider an arbitrary, but fixed state i. We define, for each integer $n \geq 1$,

$$f_i(n) = P\{\eta_n = i, \eta_k \neq i \quad \text{for any} \quad k = 1, 2, ..., n-1 \,/\eta_0 = i\,\}.$$

In other words, $f_i(n)$ is the probability that, starting from state i, the first return to state i occurs at the nth transition. Clearly, $f_i(1) = P_{ii}$ and $f_i(n)$ may be calculated recursively according to

$$P_{ii}(n) = \sum_{k=0}^{n} f_i(k)\, P_{ii}(n-k), \qquad n \geq 1 \tag{119}$$

where we define $f_i(0) = 0$ for all i. Equation (119) is derived by decomposing the event from which $P_{ii}(n)$ is computed according to the time of the first returns to state i. Indeed, consider all the possible realizations of the process for which $\eta_0 = i$, $\eta_n = i$ and the first return to state i occurs at the kth transition. Call this event B_k. The events B_k, $k = 1, 2, ..., n$ are clearly mutually exclusive. The probability of the event that the first return is at the kth transition is by definition $f_i(k)$. In the remaining $n-k$ transitions, we are dealing only with those realizations for which $\eta_n = i$. Using the Markov property, we have

$$P(B_k) = P(\text{first return is at } k\text{th transition} \,/\eta_0 = i\,)\, P(\eta_n = i\,/\eta_k = i\,) = f_i(k)\, P_{ii}(n-k)$$

(recall that $P_{ii}(0) = 1$). Hence

$$P(\eta_n = i\,/\eta_0 = i\,) = \sum_{k=1}^{n} P(B_k) = \sum_{k=1}^{n} f_i(k)\, P_{ii}(n-k) = \sum_{k=0}^{n} f_i(k)\, P_{ii}(n-k),$$

since by definition $f_i(0) = 0$.