LECTURE 23

When the process starts from state i, the probability that the process will ever reenter state i is

$$F_i = \sum_{n=0}^{\infty} f_i(n) = \lim_{N\to\infty} \sum_{n=0}^{N} f_i(n).$$

State i is said to be ***recurrent*** if $F_i = 1$, and ***transient***, if $F_i < 1$. Therefore, a state i is recurrent if and only if, after the process starts from state i, the probability of its returning to state i after some finite length of time is one.

Suppose that the process starts in state i and i is recurrent. Hence, with probability 1, the process will eventually reenter state i. However, by the definition of a Markov chain, it follows that the process will be starting over again when it reenters state i and, therefore, state i will eventually be visited again. Continual repetition of this argument leads to the conclusion that if state i is recurrent then, starting in state i, the process will reenter state i infinitely often.

On the other hand, suppose that state i is transient. Hence, each time the process enters state i there will be a positive probability, namely $1 - F_i$, that it will never again enter that state. Therefore, starting in state i, the probability that the process will be in state i for exactly n time period equals $F_i^{n-1} \cdot (1 - F_i)$, $n \geq 1$. In other words, if state i is transient then, starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean $1/(1 - F_i)$.

Consider a transient state i. Then the probability that a process starting from state i returns to state i at least once is $F_i < 1$. Because of the Markov property, the probability that the process returns to state i at least twice is $(F_i)^2$, and, repeating the argument, we see that the probability that the process returns to i at least k times is $(F_i)^k$ for $k = 1, 2, \ldots$.

For a recurrent state i, $p_{i,i}(n) > 0$ for some $n \geq 1$. Define the period of state i, denoted by d_i, as the greatest common divisor of the set of positive integers n such that $p_{i,i}(n) > 0$. A recurrent state i is said to be aperiodic if its period $d_i = 1$, and ***periodic***, if $d_i > 1$.

Example 96. Consider a Markov chain consisting of the four states 1, 2, 3, 4, and having a transition probability matrix

$$\P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The classes of this Markov chain are $\{1,2\}$, $\{3\}$ and $\{4\}$. Note that while state 1 (or 2) is accessible from state 3, the reverse is not true. Since state 4 is an absorbing state, that is, $P_{4,4} = 1$, no other state is accessible from it.

Thus, for irreducible Markov chain every state can be reached from every other state in a finite number of steps. In other words, for all $i, j \in \mathcal{G}$, there is an integer $n \geq 1$ such that $p_{i,j}(n) > 0$.

It follows that state i is recurrent if and only if, starting in state i, the expected number of time periods that the process is in state i is infinite. Letting

$$A_n = \begin{cases} 1 & \text{if} \quad \eta_n(\omega) = i, \\ 0 & \text{if} \quad \eta_n(\omega) \neq i \end{cases}$$

we have that $\sum\limits_{n=0}^{\infty} A_n$ represents the number of periods that the process is in state i. Also

$$E\left[\sum_{n=0}^{\infty} A_n/\eta_0 = i\right] = \sum_{n=0}^{\infty} E\left[A_n/\eta_0 = i\right] == \sum_{n=0}^{\infty} P\{\eta_n = i/\eta_0 = i\} = \sum_{n=0}^{\infty} P_{ii}(n).$$

We have proved the following theorem.

Theorem 28. State i is recurrent if and only if

$$\sum_{n=0}^{\infty} P_{ii}(n) = +\infty.$$

Equivalently, state i is transient if and only if

$$\sum_{n=0}^{\infty} P_{ii}(n) < +\infty.$$

Theorem 29 (Solidarity). All states of an irreducible Markov chain are of the same type:

1) If one state of an irreducible Markov chain is periodic, then all states are periodic and have the same period.

2) If one state is recurrent, then so are all states.

Example 97. Consider a two–state Markov chain with $p_{1,2} = 0$ and $p_{2,1} = 1$, that is

$$¶ = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

In this case the state 2 is transient and state 1 is absorbing. The chain is not irreducible, but the limiting state probabilities exist (since $¶(n) = ¶$) and are given by $p_1 = 1$ and $p_2 = 0$. This says that eventually the chain will remain in state 1 (after at most one transition).

§35. LIMITING DISTRIBUTIONS

For a Markov chain with a countably infinite state space $\mathcal{G}$, computation of $P_{i,j}(n)$ poses problems.

For a large number of Markov chains it turns out that $P_{i,j}(n)$ converges, as $n \to \infty$, to a value p_j that depends only on j. That is, for large values of n, the probability of being in state j after n transitions is approximately equal to p_j no matter what the initial state was. It can be shown that for a finite element state space $\mathcal{G}$, a sufficient condition for a Markov chain to possess this property is that for some $n > 0$,

$$P_{i,j}(n) > 0 \quad \text{for all} \quad i, j \in \mathcal{G}. \tag{120}$$

Markov chains that satisfy (120) are said to be ergodic. Since Theorem 27 yields

$$P_{i,j}(n+1) = \sum_{k \in \mathcal{G}} P_{i,k}(n) \cdot P_{k,j}$$

it follows, by letting $n \to \infty$, that for ergodic chains

$$p_j = \sum_{k \in \mathcal{G}} p_k \cdot P_{k,j}. \tag{121}$$

Furthermore, since $1 = \sum_{j \in \mathcal{G}} P_{i,j}(n)$, we also obtain, by letting $n \to \infty$,

$$\sum_{j \in \mathcal{G}} p_j = 1. \tag{122}$$