YSU ASDS, Statistics, Fall 2019

Lecture 20

Michael Poghosyan

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- Maximum Likelihood Estimation

Last Lecture ReCap

- Describe the MLE.

Examples

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$$0, 1, 1, 2, 1, 0, 0, 1, 1$$

from the following Model:

X	0	1	2
$\mathbb{P}(X = x)$	$\frac{\theta}{10}$	$\frac{\theta}{5}$	$1 - \frac{3\theta}{10}$,

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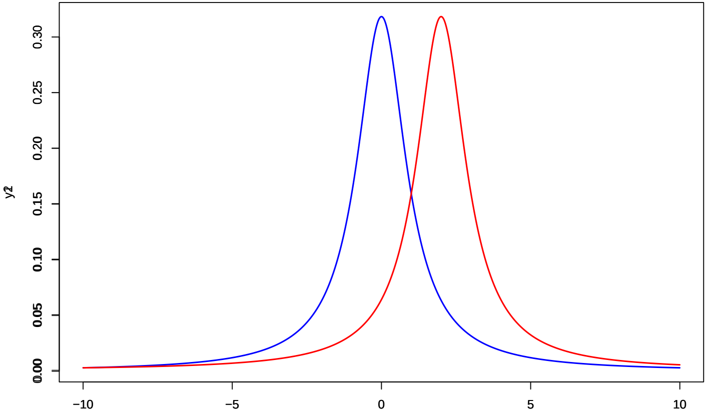
$$f(x|\theta) = \frac{1}{\pi(1 + (x - \theta)^2)}, \qquad x \in \mathbb{R},$$

and $\theta \in \mathbb{R}$ is called the *location parameter*.

PDF of Cauchy(0) and Cauchy(2)

```
x <- seq(from = -10, to = 10, by = 0.01)
y1 <- dcauchy(x); y2 <- dcauchy(x, location = 2);
plot(x, y1, type = "l", lwd = 2, xlim = c(-10,10), col = "blue")
par(new = T)
plot(x, y2, type = "l", lwd = 2, xlim = c(-10,10), col = "red")
```



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Note: It is remarkable, that ML Estimators, under some conditions (if they exist, of course $\ddot\smile$), possess some nice properties. These properties make MLE one of the widely used methods of Estimation.

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So, MLE is **Consistent** and **Asymptotocally Efficient**. And this is why, for large Sample Size n, MLE is the Top 1 Choice, is (almost) unbeatable.

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Note: We will use this later, to construct an (approximate) Confidence Interval for θ and for testing Hypotheses about θ.