YSU ASDS, Statistics, Fall 2019

Lecture 19

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02 Nov 2019

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- Give the definition of the Fisher Information.

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- Give the CR LB.

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- Give the CR LB.
- Give a method to check if an Estimator is MVUE.
- Describe the MoM.

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So assume $\theta = (\theta_1, \theta_2)$. Then we need 2 Equations to find θ_1 and θ_2. The MoM says: first try to solve the following system:

$$\begin{cases} \textbf{1-st order Theoretical Moment} = \textbf{1-st order Empirical Moment} \\ \textbf{2-nd order Theoretical Moment} = \textbf{2-nd order Empirical Moment} \end{cases}$$

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Example

Example: Find the MoM Estimator for (μ, σ^2) in the $\mathcal{N}(\mu, \sigma^2)$ Model.

Solution: OTB

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Example: Find the MoM Estimator for (a, b) in the $Unif\,[a, b]$ Model.

Solution: OTB

Example

Example: Let us do an experiment in **R**, concerning the last example:

```r
a <- 2.5; b <- 3.24
x <- runif(10, min = a, max = b)
x.bar <- mean(x)
z <- sqrt(3* (mean(x^2) - x.bar^2))
a.hat.MoM<- x.bar - z
b.hat.MoM <- x.bar + z
c(a.hat.MoM, b.hat.MoM)
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Of course, we can just take $\hat{a} = X_{(1)}$ and $\hat{b} = X_{(n)}$:

```
c(min(x), max(x))
```

```
## [1] 2.513637 3.031960
```

Notes

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Note: Sometimes we are interested not in the Estimate of the Parameter itself, but in the Estimate for some function of that Parameter. Say, we want to Estimate $h(\theta)$, where θ is our unknown Parameter. Then one of the approaches is the Plug-in Principle: find an Estimator $\hat{\theta}$ for θ, say, using the MoM, and then plug that in h, to obtain $h(\hat{\theta})$ as an Estimator for $h(\theta)$.

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the solution of which we denote by $\hat{\theta}_n$. This gives

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Now, by the WLLN, $\overline{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}(X_1) = e(\theta)$, so

$$\hat{\theta}_n = e^{-1}(\overline{X}_n) \xrightarrow{\mathbb{P}} e^{-1}(e(\theta)) = \theta.$$

The Maximum Likelihood Method

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Ok, let's do some calculations.

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$$\text{If } p = 0.2, \text{ then} \quad \mathbb{P}(HHHHHHH) = (0.2)^7 \approx 1.28e{-}05 = 1.28 \cdot 10^{-5}$$

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So our guess was to **select the value of p giving the highest likelihood to our outcome**.

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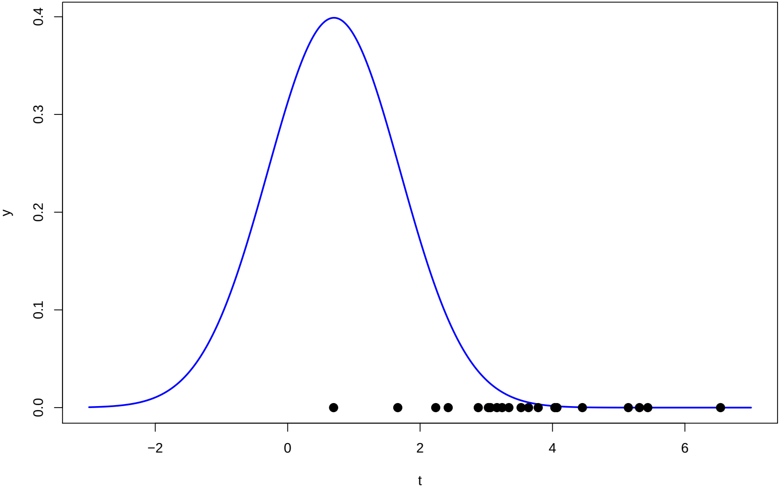
Idea of Maximum Likelihood Estimation: We choose that value of our parameter, under which **our Observation is the most Probable**.

Idea of the MLE

Guess which Distribution makes our Data most Probable:



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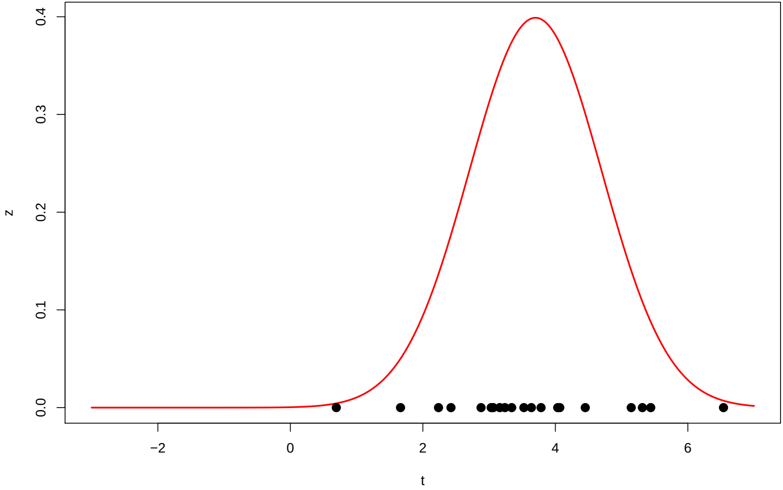

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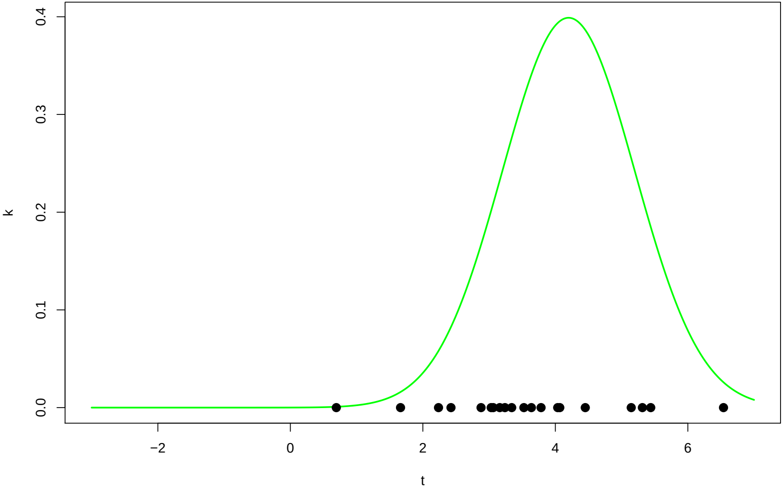

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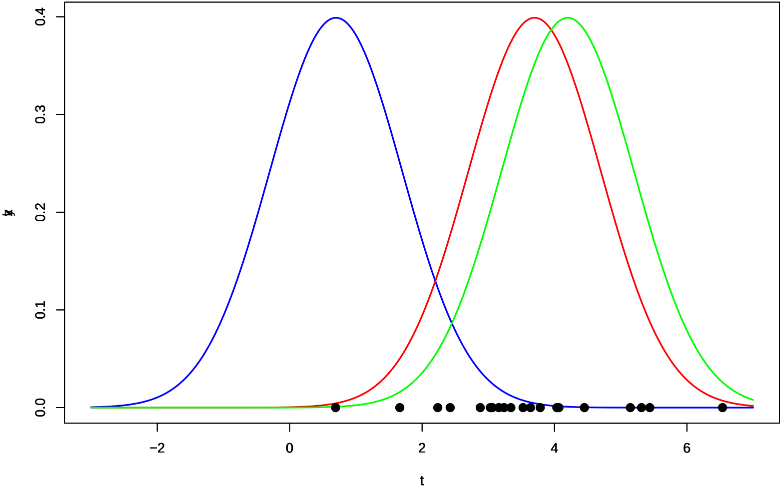

Problem Statement Again

Again, assume we have an Observation $x: \ x_1, ..., x_n$, from one of the Distributions of Parametric Family $\mathcal{F}_\theta$, with the PD(M)F $f(x|\theta)$, $\theta \in \Theta$.

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We, instead of our Observation, take a Random Sample

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to *generalize*, to have our method work also for *unseen Data*, to get a result for all possible Observations, i.e., to construct an **Estimator** for θ.

And the Maximum Likelihood Method is saying: **choose that value of θ, under which it is most likely to get $X_1, X_2, ..., X_n$.**

Likelihood

Definition: The **Likelihood Function** for the above Model and Random Sample is the Joint PD(M)F of $X_1, ..., X_n$, **considered as a function of the parameter** θ, and **calculated at the Random Sample**, i.e., it is given by[1]

$$\mathcal{L}(\theta) = \mathcal{L}_n(X_1, ..., X_n|\theta) = f(X_1|\theta)\cdot f(X_2|\theta)\cdot ...\cdot f(X_n|\theta), \qquad \theta \in \Theta.$$

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The **Log-Likelihood Function** is the function

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Also we define the **Negative Log-Likelihood Function** to be

$$-\ell(\theta) = -\ln \mathcal{L}(\theta).$$

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Now, the Maximum Likelihood Method suggests **to find a point that makes our Likelihood Maximal**:

Maximum Likelihood Method

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And in the case if we have an Observation $x: \ x_1, x_2, ..., x_n$ from the above Model (from one of the Distributions of that Model), the **Maximum Likelihood Estimate** (again **MLE**) of the parameter θ is the value of $\hat{\theta}^{MLE}$ on our Observation.

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i.e., the points of maximum of $\mathcal{L}(\theta)$ and $\ln \mathcal{L}(\theta)$ coincide. And, in the rest, we will find the Max points of the **Log-Likelihd** function.

Calc + Optim Refresher

Here it is desirable to have a slide about how to find the maximum points of a function $\ell(\theta)$ for $\theta \in \Theta$, considering:

- 1D Case
- *n*-D Case
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I know that you can fill this slide, so I am keeping it to you*.

* In fact, I realized that one slide will not be enough, and was lazy to prepare them ⌣̈

Examples

Example: Find the MLE for p in the *Bernoulli*(p) Model.

Solution: OTB

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Example: Find the MLE Estimator for λ in the $Exp(\lambda)$ Model.

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