YSU ASDS, Statistics, Fall 2019

Lecture 13

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Probability Reminder

Contents

Last Lecture ReCap

- Give the definition of the convergence in the a.s./ Probability / QM / Distributions sense.

Example

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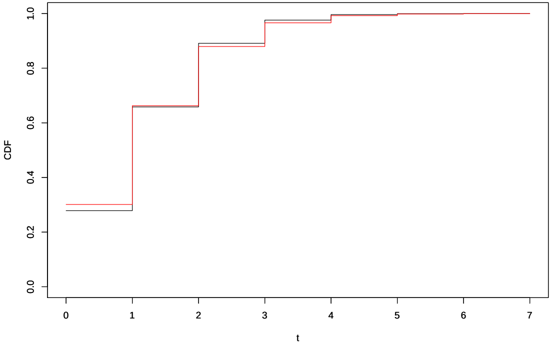
```
lambda <- 1.2; n <- 10; t <- seq(0,7, 0.1)
plot(t,pbinom(t, size = n, prob = lambda/n), type = "s", ylim = c(0,1), ylab = "CDF")
par(new = T)
plot(t, ppois(t, lambda = lambda), type = "s", col = "red", ylim = c(0,1), ylab = "CDF")
```



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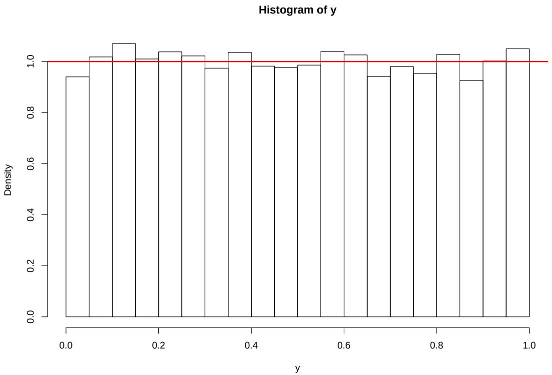
```r
n <- 10000 ## We use Y_n
m <- 10000 ## No. of generated numbers
y <- runif(m, min = 0, max = n)/n
hist(y, freq = F)
abline(h = 1, col = "red", lwd = 2)
```



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Theorem: Assume $X_n \overset{a.s.(P,qm,D)}{\longrightarrow} X$ and $Y_n \overset{a.s.(P,qm,D)}{\longrightarrow} Y$.

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Relationship between Convergence Types

Theorem: (Convergence Relationship Diagram)

$$\begin{tikzcd} X_n \xrightarrow{a.s.} X \arrow[dr] & & \\ & X_n \xrightarrow{\mathbb{P}} X \arrow[r] & X_n \xrightarrow{D} X \\ X_n \xrightarrow{qm} X \arrow[ur] & & \end{tikzcd}$$

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Note: Inverse implications are not always correct. But, say, the following holds: If $X_n \xrightarrow{D} X$ and $X \equiv \textit{constant}$, then $X_n \xrightarrow{P} X$ (X_n and X are defined on the same Probability space).

Limit Theorems

Sequence of IID r.v.

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- X_n-s are independent. Say, in particular,

$$Var(X_1+X_2+...+X_n) = Var(X_1)+Var(X_2)+...+Var(X_n) = n \cdot Var(X_1).$$

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The interpretation of $\mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1)$ and $Var(\overline{X}_n) = \frac{Var(X_1)}{n}$:the values of $\overline{X}_n$ are centered at $\mathbb{E}(X_1)$ and are becoming more and more concentrated around that number as n increases.

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- If $X_k \sim Binom(m, p)$, $k = 1, ..., n$, are Independent, then

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The Weak Law of Large Numbers, WLLN:

If $X_1, X_2, ..., X_n$ are IID, with finite $\mathbb{E}(X_1)$ and Variance $Var(X_1)$, then

$$\overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n} \xrightarrow{\mathbb{P}} \mathbb{E}(X_1), \qquad n \to +\infty,$$

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i.e., for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{X_1 + X_2 + ... + X_n}{n} - \mathbb{E}(X_1)\right| \geq \varepsilon\right) \to 0, \qquad n \to +\infty.$$

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$$\mathbb{P}\left(\left| \frac{X_1 + X_2 + ... + X_n}{n} - \mathbb{E}(X_1) \right| \geq \varepsilon \right) \to 0, \qquad n \to +\infty.$$

Note: This means that for any $\varepsilon > 0$, the chances that $\overline{X}_n$ is far from $\mathbb{E}(X_1)$ more than ε, is very small, if n is large.

The Strong LLN

The Strong Law of Large Numbers, SLLN, Kolmogorov

If $X_1, X_2, ..., X_n$ are IID, with finite $\mathbb{E}(|X_1|)$, then

$$\frac{X_1 + X_2 + ... + X_n}{n} \stackrel{a.s.}{\to} \mathbb{E}(X_1), \qquad n \to +\infty,$$

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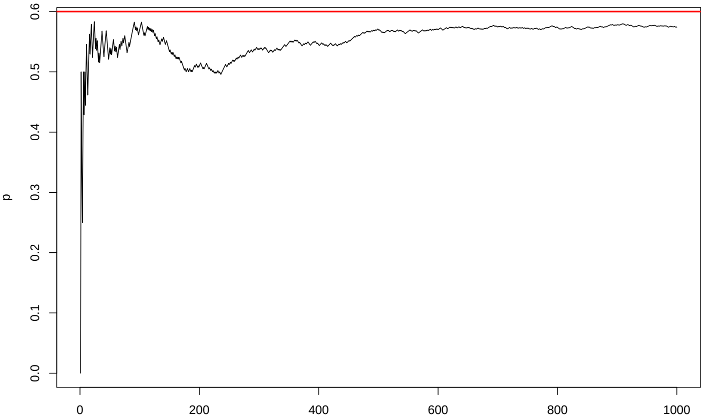
$$\frac{X_1 + X_2 + ... + X_n}{n} \xrightarrow{a.s.} \mathbb{E}(X_1), \qquad n \to +\infty,$$

that is,

$$\mathbb{P}\left(\lim_{n \to +\infty} \frac{X_1 + X_2 + ... + X_n}{n} = \mathbb{E}(X_1)\right) = 1.$$

Visualization of the LLN

```r
n <- 1000; expect <- 0.6
X <- rbinom(n, 1, expect)
S <- cumsum(X); p <- S/(1:n)
plot(p, type = "l")
abline(expect,0, col = "red", lwd = 2)
```



Supplements, LLN

Sometimes we are required to calculate limits of the form:

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To give the general idea of the CLT, let us use the following transform: for a r.v. X, let us denote

$$Standardize(X) = \frac{X - \mathbb{E}(X)}{\sqrt{Var(X)}} = \frac{X - \mathbb{E}(X)}{SD(X)},$$

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Basic Idea of the CLT

The basic idea of the CLT is the following: if we have a sequence of IID r.v. X_n, and we consider their sum S_n or their average $\overline{X}_n$, then

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Easy and beautiful, isn't it?

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Two forms of CLT

Of course, these two forms of the CLT are the same: we have

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Now,

$$\frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma} = \frac{n \cdot (\frac{S_n}{n} - \mu)}{\sqrt{n} \cdot \sigma} = \frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}},$$

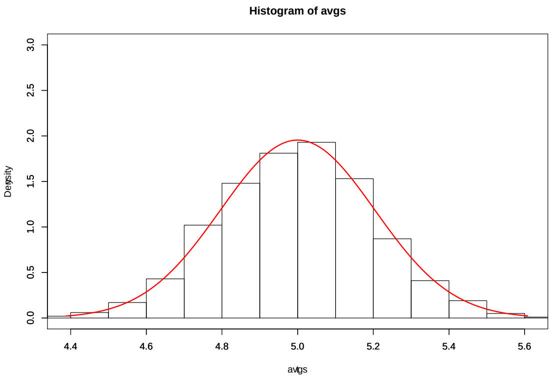
so

$$Standardize(S_n) = Standardize(\overline{X}_n).$$

Hence, the above two versions of CLT are the same, just one is in terms of S_n, the other one is in terms of $\overline{X}_n$.

CLT, Visually

```r
n <- 600 # Sample Size
m <- 1000 # no of Samples
rate <- 0.2
x <- rexp(n*m, rate = rate)
theo.mean <- 1/rate #theoretical mean
theo.sd <- 1/rate #theoretical SD
m <- matrix(x, ncol = m); d <- data.frame(m)
avgs <- sapply(d, mean)
a = theo.mean-3*theo.sd/sqrt(n); b = theo.mean+3*theo.sd/sqrt(n)
hist(avgs, freq = F, xlim = c(a, b), ylim=c(0,3))
par(new = T)
t <- seq(a,b, 0.01)
y <- dnorm(t, mean = theo.mean, sd = theo.sd/sqrt(n))
plot(t,y, type = "l", col="red", lwd = 2, , xlim = c(a,b), ylim=c(0,3))
```



CLT, Visually, v2

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rate <- 0.2
x <- rexp(n*m, rate = rate)
m <- matrix(x, ncol = m); d <- data.frame(m)
avgs <- sapply(d, mean)
qqnorm(avgs, ylab = "Averages"); qqline(avgs)
```