Deep Learning

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YSU, Krisp

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Outline

Back-Propagation

Question: How to calculate the derivative of the function $\sin x^2$?

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Theorem 1

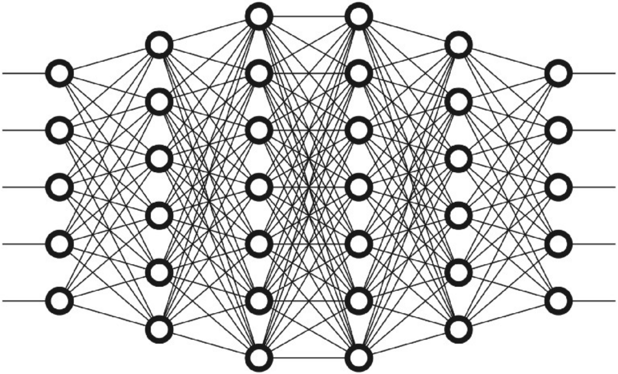
Given n functions $f_1, \dots, f_n$ with the composite function

$$f = f_1 \circ (f_2 \circ \cdots (f_{n-1} \circ f_n)),$$

if each function f_i is differentiable at its immediate input, then the composite function is also differentiable by the repeated application of Chain Rule, where the derivative is

$$\frac{df}{dx} = \frac{df_1}{df_2}\frac{df_2}{df_3} \cdots \frac{df_n}{dx}.$$

Back-Propagation



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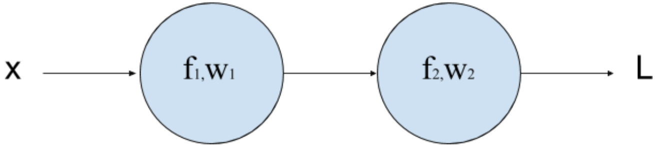

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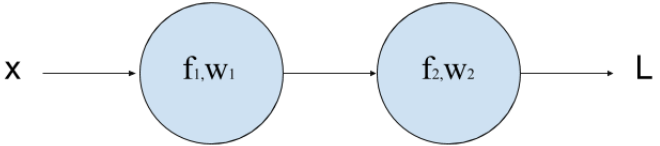

In this case we have the following function

$$L(w_1, w_2) = (f_2(w_2 f_1(w_1 x)) - y)^2$$

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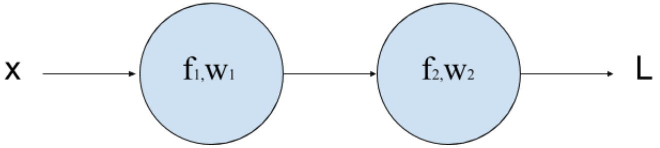

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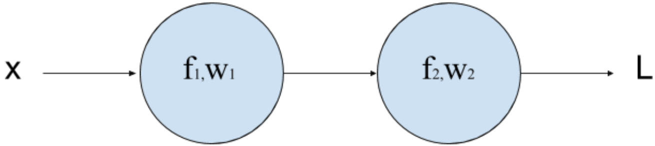

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$$L\left(w_1, w_2\right) = \left(f_2\left(w_2 f_1\left(w_1 x\right)\right) - y\right)^2$$

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$$\frac{\partial L}{\partial w_2} = \frac{\partial L}{\partial f_2} \frac{\partial f_2}{\partial\left(w_2 f_1\right)} \frac{\partial\left(w_2 f_1\right)}{\partial w_2},$$

$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial f_2} \frac{\partial f_2}{\partial\left(w_2 f_1\right)} \frac{\partial\left(w_2 f_1\right)}{\partial f_1} \frac{\partial\left(f_1\right)}{\partial\left(w_1 x\right)} \frac{\partial\left(w_1 x\right)}{\partial w_1}.$$

Outline

Data Normalization

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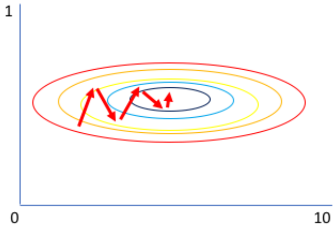
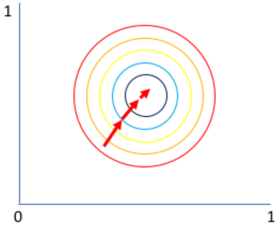
Why normalize?



Gradient of larger parameter dominates the update



Both parameters can be updated in equal proportions

Standard Normalization

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Denote $\mu^j = \frac{1}{n}\sum_{i=1}^n x_i^j$, $\sigma^j = \sqrt{\frac{1}{n}\sum_{i=1}^n \left(x_i^j - \mu^j\right)^2}$

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Our new data will be $(z_i, y_i)_{i=1}^n$, $z_i \in \mathbb{R}^k$, $y_i \in \mathbb{R}^m$.

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  Answer: **No**

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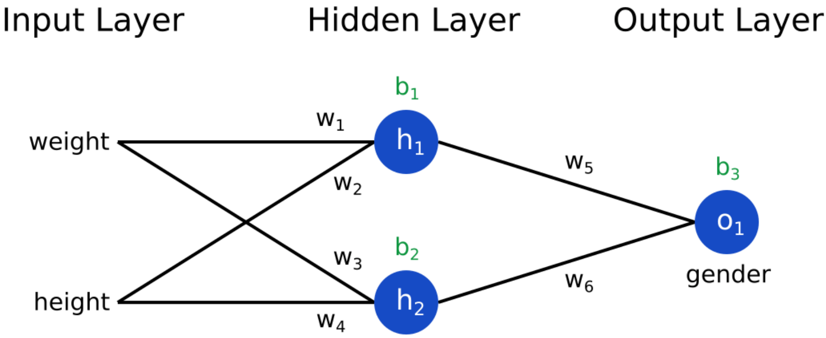

Random Initialization

$$h_1 = f\left(w_1 x_1 + w_2 x_2 + b_1\right), \quad h_2 = f\left(w_3 x_1 + w_4 x_2 + b_2\right)$$

$$o_1 = g\left(w_5 h_1 + w_6 h_2 + b_3\right)$$

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Let $w_1^0 = w_3^0 = c_1$, $w_2^0 = w_4^0 = c_2$, $w_5^0 = w_6^0 = c_3$ and $b_1^0 = b_2^0 = c_4$.

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During the optimization we need to do the following steps

$$w_i^1 = w_i^0 - \alpha \frac{\partial L}{\partial w_i}\left(w^0\right), i = 1, \dots, 6,$$

where $w^0 = \left(w_1^0, \dots, w_6^0\right)$.

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$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial o_1} \cdot g'\left(w_5 h_1 + w_6 h_2 + b_3\right) \cdot w_5 \cdot f'\left(w_1 x_1 + w_2 x_2 + b_1\right) \cdot x_1$$

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So we see that $w_1^1 = w_3^1 = const$, $w_2^1 = w_4^1 = const$.

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So we see that $w_1^1 = w_3^1 = const$, $w_2^1 = w_4^1 = const$. In the same way we can prove that $b_1^1 = b_2^1 = const$.

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- If the weights in a network start too small, then the signal shrinks as it passes through each layer until it's too tiny to be useful.
- If the weights in a network start too large, then the signal grows as it passes through each layer until it's too massive to be useful.

What's Xavier initialization?

Initializing the weights in your network by drawing them from a distribution with zero mean and a specific variance

$$\mathrm{Var}\left(W\right) = \frac{1}{n_{in}},$$

where W is the initialization distribution for the neuron in question, and n_{in} is the number of neurons feeding into it. The distribution used is typically Gaussian or uniform.

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where W is the initialization distribution for the neuron in question, and n_{in} is the number of neurons feeding into it. The distribution used is typically Gaussian or uniform. Glorot & Bengio's paper originally recommended using

$$\text{Var}(W) = \frac{2}{n_{in} + n_{out}},$$

where n_{out} is the number of neurons the result is fed to.

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It easy to see that for all $i = 1, \ldots, n$ we have

$$\operatorname{Var}(W_iX_i) = E(X_i)^2 \operatorname{Var}(W_i) + E(W_i)^2 \operatorname{Var}(X_i) + \operatorname{Var}(W_i)\operatorname{Var}(X_i)$$

$$= \operatorname{Var}(W_i)\operatorname{Var}(X_i) = \operatorname{Var}(W_1)\operatorname{Var}(X_1)$$

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$$\mathrm{Var}\left(Y\right) = \mathrm{Var}\left(W_1X_1 + W_2X_2 + \ldots + W_nX_n\right) = n\mathrm{Var}\left(W_1\right)\mathrm{Var}\left(X_1\right).$$

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Glorot Bengio's formula needs a tiny bit more work. If you go through the same steps for the backpropagated signal, you find that you need

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to keep the variance of the input gradient the output gradient the same. These two constraints can only be satisfied simultaneously if $n_{in} = n_{out}$, so as a compromise, Glorot & Bengio take the average of the two:

$$\mathrm{Var}\,(W_1) = \frac{2}{n_{in} + n_{out}}$$

Case of the activation function

Note that this was about a linear neuron, but it works also for tanh and sigmoid functions. For rectifying nonlinearities this is not true and there is a paper where authors suggest using

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Outline

Dropout

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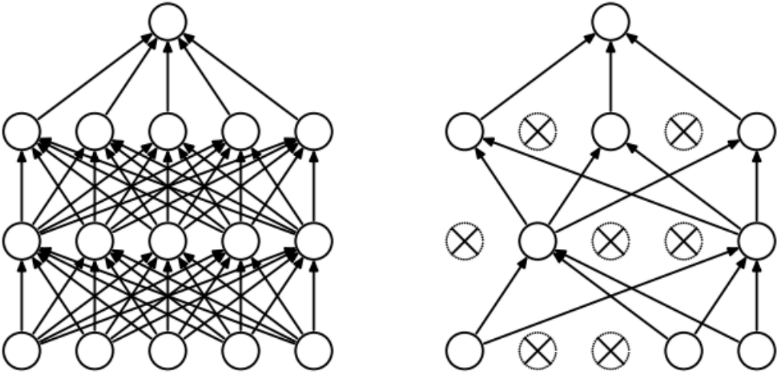
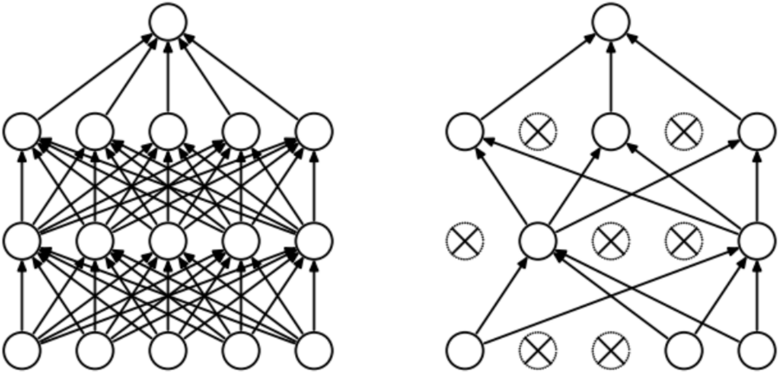


(a) Standard Neural Net (b) After applying dropout.

Dropout



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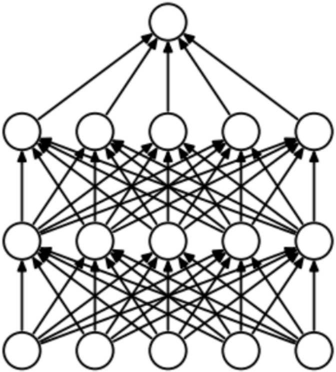
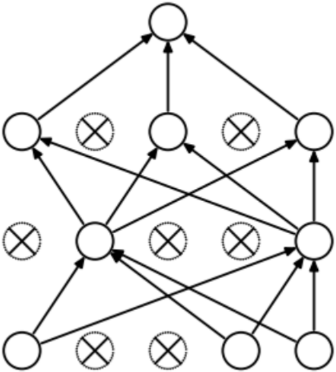
What to do during the inference?

Dropout



(a) Standard Neural Net



(b) After applying dropout.

What to do during the inference?

Answer: Scale units by $\frac{1}{1 - rate}$ during the training and set rate=1 during the inference.