L E C T U R E 14

Definition 2. The mean of the posterior distribution $f(\theta|x_1, x_2, ..., x_n)$ denoted by θ^*, is called Bayes estimate for θ.

Example 17.6.(See Example 11.) Using a random sample of size 2, estimate the proportion p of defectives produced by a machine when we assume our prior density of p is

$$p(\theta) = \begin{cases} k, & \text{if} \quad 1/2 < \theta < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution: Since for any density function (therefore also and in this example)

$$\int p(\theta)\, d\theta = 1$$

Hence $k = 2$.

As before, we find that

$$f(x|p) = \binom{2}{x} p^x (1-p)^{2-x}, \quad x = 0, 1, 2.$$

Now

$$f(x, p) = f(x|p)\, f(p) =$$

$$= 2(1-p)^2, \quad x = 0, \quad \frac{1}{2} < p < 1$$

$$= 4\, p(1-p), \quad x = 1, \quad \frac{1}{2} < p < 1$$

$$= 2p^2, \quad x = 2, \quad \frac{1}{2} < p < 1$$

and the marginal distribution for X is obtained by evaluating the integral

$$g(x) = \int_{1/2}^{1} 2(1-p)^2\, dp = \frac{1}{12}, \qquad \text{for } x = 0$$

$$= \int_{1/2}^{1} 4\, p\, (1-p)\, dp = \frac{1}{3}, \qquad \text{if } x = 1$$

$$= \int_{1/2}^{1} 2p^2\, dp = \frac{7}{12}, \qquad \text{if } x = 2.$$

The posterior distribution is then

$$f(p|x) = \frac{f(x,p)}{g(x)} =$$

$$= 24(1-p)^2, \quad x=0, \quad \frac{1}{2} < p < 1$$

$$= 12p(1-p), \quad x=1, \quad \frac{1}{2} < p < 1$$

$$= \frac{24}{7}p^2, \quad x=2, \quad \frac{1}{2} < p < 1$$

from which we evaluate the point estimate of our parameter to be

$$p^* = 24\int_{1/2}^{1} p(1-p)^2\,dp = \frac{5}{8} = 0.625, \qquad \text{if } x=0$$

$$= 12\int_{1/2}^{1} p^2(1-p)\,dp = \frac{11}{16} = 0.6875, \qquad \text{if } x=1$$

$$= \frac{24}{7}\int_{1/2}^{1} p^3\,dp = \frac{45}{56}, \qquad \text{if } x=2.$$

Example 17.7 (see Example 14). An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a standard deviation of 100 hours and mean μ. The firm belives, that μ is surely between 770 and 830 hours and it is felt that a more realistic Bayesian approach would be to assume the prior distribution

$$f(\mu) = \begin{cases} \frac{1}{60}, & \text{if } \quad 770 < \mu < 830 \\ 0, & \text{otherwise.} \end{cases}$$

If a random sample of 25 bulbs gives an average life of 780 hours, find the posterior distribution

$$p(\mu|x_1, x_2, ..., x_{25}).$$

Solution: Multiplying the density of our sample

$$f(x_1, x_2, ..., x_{25}|\mu) = \frac{1}{(2\pi)^{25/2}\cdot 100^{25}}\exp\left[-\frac{1}{2}\sum_{i=1}^{25}\left(\frac{x_i-\mu}{100}\right)^2\right], \quad -\infty < x_i < \infty,\ i=1,2,...,25$$

by our prior

$$f(\mu) = \begin{cases} \frac{1}{60}, & \text{if } \quad 770 < \mu < 830 \\ 0, & \text{otherwise.} \end{cases}$$

we obtain the joint density of the random sample and μ. That is,

$$f(x_1, x_2, ..., x_{25}, \mu) = \begin{cases} \frac{1}{60}\left(\frac{1}{\sqrt{2\pi}100}\right)^{25} \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{25}\left(\frac{x_i-\mu}{100}\right)^2\right\}\right] & \text{if} \quad 770 < \mu < 830 \\ 0, & \text{otherwise.} \end{cases}$$

We established the identity

$$\sum_{i=1}^{n}(x_i - \mu)^2 = \sum_{i=1}^{n}(x_i - \overline{x})^2 + n(\overline{x} - \mu)^2,$$

which enables us to write

$$f(x_1, x_2, ..., x_{25}, \mu) = \begin{cases} A \exp\left[-\frac{1}{2}\left(\frac{\mu-780}{20}\right)^2\right] & \text{if} \quad 770 < \mu < 830 \\ 0, & \text{otherwise.} \end{cases}$$

where A is a function of the sample values. The marginal distribution of the sample is then

$$g(x_1, x_2, ..., x_{25}) = A\sqrt{2\pi}\,20 \int_{770}^{830} \frac{1}{\sqrt{2\pi}20}\, e^{-(1/2)[(\mu-780)/20]^2}\, d\mu =$$

$$= 20\,A\,\sqrt{2\pi}\left[\Phi\left(\frac{830-780}{20}\right) - \Phi\left(\frac{770-780}{20}\right)\right] =$$

$$= 20\,A\,\sqrt{2\pi}\,[\Phi(2.5) - \Phi(-0.5)] = 20\,A\,\sqrt{2\pi}\,[\Phi(2.5) + \Phi(0.5) - 1] =$$

$$= 20\,A\,\sqrt{2\pi}[0.9938 + 0.6915 - 1] = 20\,A\,\sqrt{2\pi}0.6853 = 13.706\,A\,\sqrt{2\pi}.$$

and the posterior distribution is

$$f(\mu|x_1, x_2, ..., x_{25}) = \frac{f(x_1, x_2, ..., x_{25}, \mu)}{g(x_1, x_2, ..., x_{25})} =$$

$$= \begin{cases} \dfrac{1}{13.706\,\sqrt{2\,\pi}} \exp\left[-\dfrac{1}{2}\left(\dfrac{\mu-780}{20}\right)^2\right] & \text{if} \quad 770 < \mu < 830 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. The interval $a < \theta < b$ will be called a $(1-\alpha)100\%$ Bayes interval for θ if

$$\int_{\theta^*}^{b} f(\theta|x_1, x_2, ..., x_n)\, d\theta = \int_{a}^{\theta^*} f(\theta|x_1, x_2, ..., x_n)\, d\theta = \frac{1-\alpha}{2}.$$

Appendix. Mode of a probability distribution.

A mode of a continuous probability distribution is a value at which the probability density function attains its maximum value.

The mode of the set of data values is the value that appears most often. If $\eta(\omega)$ is a discrete random variable, the mode is the value x (that is, $\eta = x$) at which the probability mass function takes its maximum value. In other words, it is the value that is most likely to be sampled.

The mode is not necessary unique to a given discrete distribution, since the probability mass function make take the same maximum value at several points x_1, x_2, etc.

When the probability density function of a continuous distribution has multiple local maxima it is common to refer to all the local maxima as modes of the distribution. Such a continuous distribution is called multimodal (as opposite to unimodal). A mode of a continuous probability distribution is often considered to be any value x at which its probability density function has a locally maximal value, so any peak is a mode.

Example 17.8. Sample $(1, 2, 2, 3, 4, 7, 9)$ has sample mean $\hat{X} = \frac{\sum_{i=1}^{7} X_i}{7} = 4$, sample median $\tilde{X} = 3$ and sample mode equal 2.

Example 17.9. η has the following discrete distribution:

$$\begin{array}{rcccc} \eta: & 1 & 2 & 3 & 4 \\ p: & 0.15 & 0.3 & 0.25 & 0.3 \end{array}$$

Then we see that the mode is not unique. In fact, we have two modes 2 and 4.

Example 17.10. Let η have a density function:

$$f(x) = \begin{cases} \frac{3}{4}x^2(2-x) & \text{if} \quad 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the mode.

Proof: We have to find the first derivative of $f(x)$:

$$f'(x) = 2x(2-x) - x^2 = -3x^2 + 4x.$$

To calculate the pick of $f(x)$ we have to solve the equation $f'(x) = 0$. We obtain $-3x^2+4x = 0$. Thus the roots are $x_1 = \frac{4}{3}$ and $x_2 = 0$. The second rood $x = 0$ does not belong to the interval $[0, 2]$. The second derivative $f''(x) = -6x + 4$ and $f''(4/3) = -4 < 0$. Therefore mode is $\frac{4}{3}$