YSU ASDS, Statistics, Fall 2019

Lecture 23

Michael Poghosyan

13 Nov 2019

Contents

Last Lecture ReCap

- Give the definition of the $\chi^2(n)$ Distribution.

Last Lecture ReCap

- Give the definition of the $\chi^2(n)$ Distribution.
- Give the $(1-\alpha)$-level CI for σ^2 in the Normal Model.

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **unknown**

Recall that, for the Model $\mathcal{N}(\mu, \sigma^2)$, in the case when μ was unknown, we have obtained the following $(1-\alpha)$-level CI for σ^2:

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **unknown**

Recall that, for the Model $\mathcal{N}(\mu, \sigma^2)$, in the case when μ was unknown, we have obtained the following $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **unknown**

Recall that, for the Model $\mathcal{N}(\mu, \sigma^2)$, in the case when μ was unknown, we have obtained the following $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

This is the same as

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right),$$

where S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}.$$

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
[1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
[8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.

We model this as: say, we weight, by using our scale, something with **exact** weight μ, but we do not know μ.

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.

We model this as: say, we weight, by using our scale, something with **exact** weight μ, but we do not know μ. Our scale will show a number around, very close to μ, let it be, let it be W.

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.

We model this as: say, we weight, by using our scale, something with **exact** weight μ, but we do not know μ. Our scale will show a number around, very close to μ, let it be, let it be W. So we will assume

$$W \sim$$

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.

We model this as: say, we weight, by using our scale, something with **exact** weight μ, but we do not know μ. Our scale will show a number around, very close to μ, let it be, let it be W. So we will assume

$$W \sim \mathcal{N}(\mu, \sigma^2),$$

where σ^2 (or, better, σ) is measuring our scale Precision.

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [8] 3.454406 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.

We model this as: say, we weight, by using our scale, something with **exact** weight μ, but we do not know μ. Our scale will show a number around, very close to μ, let it be, let it be W. So we will assume

$$W \sim \mathcal{N}(\mu, \sigma^2),$$

where σ^2 (or, better, σ) is measuring our scale Precision.

So now, using the above observations (weighting results), we will construct a 90% CI for σ^2.

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

Now, we use **R** to do the rest:

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}},\quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right).$$

Now, we use **R** to do the rest: we want to construct 90% CI, so our $\alpha =$

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}},\quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right).$$

Now, we use **R** to do the rest: we want to construct 90% CI, so our $\alpha = 0.1$.

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

Now, we use **R** to do the rest: we want to construct 90% CI, so our $\alpha = 0.1$. We have 10 observations, so $n = 10$.

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

Now, we use **R** to do the rest: we want to construct 90% CI, so our $\alpha = 0.1$. We have 10 observations, so $n = 10$. We calculate S^2:

```
w <- c(3.449243, 3.450802, 3.453054, 3.448778, 3.452541, 3.
s2 <- var(w)
s2
```

```
## [1] 4.605341e-06
```

Example, Cont'd

The Degrees of Freedom in our case, since we do not know μ, is

Example, Cont'd

The Degrees of Freedom in our case, since we do not know μ, is $n-1=9$, and we calculate the cooresponding quantiles for the $\chi^2(9)$:

```
alpha <- 0.1
lq<-qchisq(alpha/2, df=9); uq<-qchisq(1-alpha/2, df=9)
c(lq,uq)
```

```
## [1]  3.325113 16.918978
```

Example, Cont'd

The Degrees of Freedom in our case, since we do not know μ, is $n - 1 = 9$, and we calculate the cooresponding quantiles for the $\chi^2(9)$:

```
alpha <- 0.1
lq<-qchisq(alpha/2, df=9); uq<-qchisq(1-alpha/2, df=9)
c(lq,uq)
```

```
## [1]  3.325113 16.918978
```

Finally, we calculate our CI endpoints:

```
n <- 10
c((n-1)*s2/uq, (n-1)*s2/lq)
```

```
## [1] 2.449797e-06 1.246516e-05
```

Example, Cont'd

The Degrees of Freedom in our case, since we do not know μ, is $n - 1 = 9$, and we calculate the cooresponding quantiles for the $\chi^2(9)$:

```
alpha <- 0.1
lq<-qchisq(alpha/2, df=9); uq<-qchisq(1-alpha/2, df=9)
c(lq,uq)
```

```
## [1]  3.325113 16.918978
```

Finally, we calculate our CI endpoints:

```
n <- 10
c((n-1)*s2/uq, (n-1)*s2/lq)
```

```
## [1] 2.449797e-06 1.246516e-05
```

Note: The actual value of `sd` I was using was: $sd = 0.002$, so the true value of my σ^2 was

$$\sigma^2 = 4 \cdot 10^{-6}.$$

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \stackrel{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1 - \alpha$-level CI for the unknown parameter σ^2.

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \stackrel{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1 - \alpha$-level CI for the unknown parameter σ^2.

	μ is known	μ is unknown
Piv:		

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1 - \alpha$-level CI for the unknown parameter σ^2.

	μ is known	μ is unknown
Piv:	$\chi^2 = \sum_{k=1}^{n} \left(\frac{X_k - \mu}{\sigma} \right)^2$	$\chi^2 = \sum_{k=1}^{n} \left(\frac{X_k - \overline{X}}{\sigma} \right)^2$
Dis of Piv		

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0,1)$, we want to construct an $1-\alpha$-level CI for the unknown parameter σ^2.

	μ is known	μ is unknown
Piv:	$\chi^2 = \sum_{k=1}^n \left(\frac{X_k - \mu}{\sigma}\right)^2$	$\chi^2 = \sum_{k=1}^n \left(\frac{X_k - \overline{X}}{\sigma}\right)^2$
Dis of Piv	$\chi^2 \sim \chi^2(n)$	$\chi^2 \sim \chi^2(n-1)$
Reg w/$\mathbb{P}$ $1-\alpha$		

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1-\alpha$-level CI for the unknown parameter σ^2.

	μ is known	μ is unknown
Piv:	$\chi^2 = \sum_{k=1}^n \left(\frac{X_k-\mu}{\sigma}\right)^2$	$\chi^2 = \sum_{k=1}^n \left(\frac{X_k-\overline{X}}{\sigma}\right)^2$
Dis of Piv	$\chi^2 \sim \chi^2(n)$	$\chi^2 \sim \chi^2(n-1)$
Reg w/$\mathbb{P}$ $1-\alpha$	$\chi^2_{n,\alpha/2} < \chi^2 < \chi^2_{n,1-\alpha/2}$	$\chi^2_{n-1,\alpha/2} < \chi^2 < \chi^2_{n-1,1-\alpha/2}$

CI for σ^2:

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1-\alpha$-level CI for the unknown parameter σ^2.

	μ is known	μ is unknown
Piv:	$\chi^2 = \sum_{k=1}^n \left(\frac{X_k-\mu}{\sigma}\right)^2$	$\chi^2 = \sum_{k=1}^n \left(\frac{X_k-\overline{X}}{\sigma}\right)^2$
Dis of Piv	$\chi^2 \sim \chi^2(n)$	$\chi^2 \sim \chi^2(n-1)$
Reg w/$\mathbb{P}$ $1-\alpha$	$\chi^2_{n,\alpha/2} < \chi^2 < \chi^2_{n,1-\alpha/2}$	$\chi^2_{n-1,\alpha/2} < \chi^2 < \chi^2_{n-1,1-\alpha/2}$
CI for σ^2:	$\left(\frac{\sum_{k=1}^n (X_k-\mu)^2}{\chi^2_{n,1-\alpha/2}}, \frac{\sum_{k=1}^n (X_k-\mu)^2}{\chi^2_{n,\alpha/2}}\right)$	$\left(\frac{\sum_{k=1}^n (X_k-\overline{X}_n)^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{\sum_{k=1}^n (X_k-\overline{X}_n)^2}{\chi^2_{n-1,\alpha/2}}\right)$

Asymptotic CIs

Asymptotic CIs

Up to now, we have constructed CIs for our important basic Models - say, Bernoulli and Normal. Unfortunately, it is not an easy job to construct CIs for any Parametric Model.

Asymptotic CIs

Up to now, we have constructed CIs for our important basic Models - say, Bernoulli and Normal. Unfortunately, it is not an easy job to construct CIs for any Parametric Model. But fortunately, if the Sample Size is large enough, CLT and Co are helping to construct an *Asymptotic CIs*, covering the unknown Parameter value with high Probability for large Sample Size.

Asymptotic CIs

Up to now, we have constructed CIs for our important basic Models - say, Bernoulli and Normal. Unfortunately, it is not an easy job to construct CIs for any Parametric Model. But fortunately, if the Sample Size is large enough, CLT and Co are helping to construct an *Asymptotic CIs*, covering the unknown Parameter value with high Probability for large Sample Size.

Let us formalize the notion.

Asymptotic CIs

Up to now, we have constructed CIs for our important basic Models - say, Bernoulli and Normal. Unfortunately, it is not an easy job to construct CIs for any Parametric Model. But fortunately, if the Sample Size is large enough, CLT and Co are helping to construct an *Asymptotic CIs*, covering the unknown Parameter value with high Probability for large Sample Size.

Let us formalize the notion.

Again we consider a Parametric Model $\mathcal{F}_\theta$, assuming that we have an (infinite) Random Sample from one of these Distributions:

$$X_1, X_2, ..., X_n, ... \sim \mathcal{F}_\theta.$$

Asymptotic CIs

Up to now, we have constructed CIs for our important basic Models - say, Bernoulli and Normal. Unfortunately, it is not an easy job to construct CIs for any Parametric Model. But fortunately, if the Sample Size is large enough, CLT and Co are helping to construct an *Asymptotic CIs*, covering the unknown Parameter value with high Probability for large Sample Size.

Let us formalize the notion.

Again we consider a Parametric Model $\mathcal{F}_\theta$, assuming that we have an (infinite) Random Sample from one of these Distributions:

$$X_1, X_2, ..., X_n, ... \sim \mathcal{F}_\theta.$$

To construct an Asymptotic CI for θ, we take $\alpha \in (0, 1)$.

Asymptotic CIs

Definition: Assume, for any n, $L_n = L_n(x_1, ..., x_n, \alpha)$, $U_n = U_n(x_1, ..., x_n, \alpha)$ be two functions with $L_n(x_1, ..., x_n, \alpha) \leq U_n(x_1, ..., x_n, \alpha)$ for all $(x_1, ..., x_n, \alpha)$. The sequence of Random Intervals

$$(L_n, U_n) = \big(L_n(X_1, ..., X_n, \alpha);\ U_n(X_1, ..., X_n, \alpha)\big)$$

is called an **Asymptotic Confidence Interval sequence** (or just an Asymptotic Confidence Interval for θ of (Asymptotic) level $1-\alpha$, if for any $\theta \in \Theta$,

$$\liminf_{n \to +\infty} \mathbb{P}(L_n < \theta < U_n) \geq 1 - \alpha.$$

Asymptotic CIs

Definition: Assume, for any n, $L_n = L_n(x_1, ..., x_n, \alpha)$, $U_n = U_n(x_1, ..., x_n, \alpha)$ be two functions with $L_n(x_1, ..., x_n, \alpha) \leq U_n(x_1, ..., x_n, \alpha)$ for all $(x_1, ..., x_n, \alpha)$. The sequence of Random Intervals

$$(L_n, U_n) = \big(L_n(X_1, ..., X_n, \alpha);\ U_n(X_1, ..., X_n, \alpha)\big)$$

is called an **Asymptotic Confidence Interval sequence** (or just an Asymptotic Confidence Interval for θ of (Asymptotic) level $1 - \alpha$, if for any $\theta \in \Theta$,

$$\liminf_{n \to +\infty} \mathbb{P}(L_n < \theta < U_n) \geq 1 - \alpha.$$

Usually, we will have that the limit above exists, so we will use

$$\lim_{n \to +\infty} \mathbb{P}(L_n < \theta < U_n) \geq 1 - \alpha.$$

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$.

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$. Here we assume that the Variance $\sigma^2 = Var(X_k)$ is finite, but unknown.

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$. Here we assume that the Variance $\sigma^2 = Var(X_k)$ is finite, but unknown.

So our Problem is to construct an Asymptotic CI of level $1 - \alpha$ for μ.

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$. Here we assume that the Variance $\sigma^2 = Var(X_k)$ is finite, but unknown.

So our Problem is to construct an Asymptotic CI of level $1-\alpha$ for μ.

We consider the following t-Statistics (or, rather, a sequence of Statistics):

$$t_n = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}.$$

where $(S_n)^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X}_n)^2}{n-1}$ is the Sample SD.

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$. Here we assume that the Variance $\sigma^2 = Var(X_k)$ is finite, but unknown.

So our Problem is to construct an Asymptotic CI of level $1-\alpha$ for μ.

We consider the following t-Statistics (or, rather, a sequence of Statistics):

$$t_n = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}.$$

where $(S_n)^2 = \frac{\sum_{k=1}^n (X_k - \overline{X}_n)^2}{n-1}$ is the Sample SD. Unfortunately, this is not a Pivot anymore, in the general case - the Distribution of t_n was $t(n-1)$ **only in the case when** $X_k \sim \mathcal{N}(\mu, \sigma^2)$.

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$. Here we assume that the Variance $\sigma^2 = Var(X_k)$ is finite, but unknown.

So our Problem is to construct an Asymptotic CI of level $1-\alpha$ for μ.

We consider the following t-Statistics (or, rather, a sequence of Statistics):

$$t_n = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}.$$

where $(S_n)^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X}_n)^2}{n-1}$ is the Sample SD. Unfortunately, this is not a Pivot anymore, in the general case - the Distribution of t_n was $t(n-1)$ **only in the case when** $X_k \sim \mathcal{N}(\mu, \sigma^2)$. Fortunately, the Asymptotic Distribution is independent on μ, so t_n is an *Asymptotic Pivot*.

Asymptotic CI for the Mean of General Distribution

Assume we have an observation from a Random Sample $X_1, X_2, ..., X_n, ...$. We want the Estimate, using CIs, the Mean $\mu = \mathbb{E}(X_k)$. Here we assume that the Variance $\sigma^2 = Var(X_k)$ is finite, but unknown.

So our Problem is to construct an Asymptotic CI of level $1-\alpha$ for μ.

We consider the following t-Statistics (or, rather, a sequence of Statistics):

$$t_n = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}.$$

where $(S_n)^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X}_n)^2}{n-1}$ is the Sample SD. Unfortunately, this is not a Pivot anymore, in the general case - the Distribution of t_n was $t(n-1)$ **only in the case when** $X_k \sim \mathcal{N}(\mu, \sigma^2)$. Fortunately, the Asymptotic Distribution is independent on μ, so t_n is an *Asymptotic Pivot*. To see this, we can write t_n as:

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability.

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

So the Asymptotic Distribution of t_n is independent of μ.

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

So the Asymptotic Distribution of t_n is independent of μ.

The rest is standard: first we find numbers a, b such that

$$\mathbb{P}(a < Z < b) = 1 - \alpha, \qquad \textit{where} \quad Z \sim \mathcal{N}(0,1).$$

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0, 1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0, 1).$$

So the Asymptotic Distribution of t_n is independent of μ.

The rest is standard: first we find numbers a, b such that

$$\mathbb{P}(a < Z < b) = 1 - \alpha, \qquad \textit{where} \quad Z \sim \mathcal{N}(0, 1).$$

We can take, as previously, $-a = b = z_{1-\alpha/2}$.

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

So the Asymptotic Distribution of t_n is independent of μ.

The rest is standard: first we find numbers a, b such that

$$\mathbb{P}(a < Z < b) = 1 - \alpha, \qquad \textit{where} \quad Z \sim \mathcal{N}(0,1).$$

We can take, as previously, $-a = b = z_{1-\alpha/2}$. Then

$$\mathbb{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha.$$

Asymptotic CI for the Mean of General Distribution

$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

So the Asymptotic Distribution of t_n is independent of μ.

The rest is standard: first we find numbers a, b such that

$$\mathbb{P}(a < Z < b) = 1 - \alpha, \qquad \textit{where} \quad Z \sim \mathcal{N}(0,1).$$

We can take, as previously, $-a = b = z_{1-\alpha/2}$. Then

$$\mathbb{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha.$$

Since $t_n \xrightarrow{D} \mathcal{N}(0,1)$, then

$$\mathbb{P}(-z_{1-\alpha/2} < t_n < z_{1-\alpha/2}) \to \mathbb{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha.$$

Asymptotic CI for the Mean of General Distribution

We plug the value of t_n here and solve for μ to obtain

$$\mathbb{P}\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}} < \mu < \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right) \to 1 - \alpha$$

Asymptotic CI for the Mean of General Distribution

We plug the value of t_n here and solve for μ to obtain

$$\mathbb{P}\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}} < \mu < \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right) \to 1 - \alpha$$

so the Random Interval (or, rather, the sequence of Intervals)

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}; \quad \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

is a $(1-\alpha)$-level Asymptotic CI for μ.

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}; \quad \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + z_{1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}}\right)$$

Usually, people use not this one, but the following one:

$$\left(\overline{X}_n - t_{n-1,1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + t_{n-1,1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}}\right)$$

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + z_{1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}}\right)$$

Usually, people use not this one, but the following one:

$$\left(\overline{X}_n - t_{n-1,1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + t_{n-1,1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}}\right)$$

This is because

- the same form of CI was obtained for the Normal Model, when σ^2 was unkown;

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

Usually, people use not this one, but the following one:

$$\left(\overline{X}_n - t_{n-1,1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + t_{n-1,1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

This is because

- the same form of CI was obtained for the Normal Model, when σ^2 was unkown;
- this interval is a little bit larger than the previous one, so it is also an AsympCI for μ of level $1-\alpha$;

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}; \;\; \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

Usually, people use not this one, but the following one:

$$\left(\overline{X}_n - t_{n-1,1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}; \;\; \overline{X}_n + t_{n-1,1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

This is because

- the same form of CI was obtained for the Normal Model, when σ^2 was unkown;
- this interval is a little bit larger than the previous one, so it is also an AsympCI for μ of level $1-\alpha$;
- when $n \geq 30$, these two almost coincide;

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + z_{1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}}\right)$$

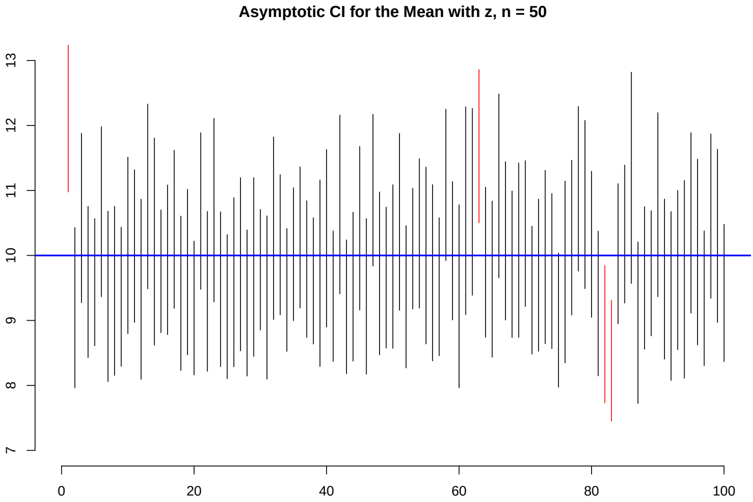
Usually, people use not this one, but the following one:

$$\left(\overline{X}_n - t_{n-1,1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + t_{n-1,1-\alpha/2}\cdot\frac{S_n}{\sqrt{n}}\right)$$

This is because

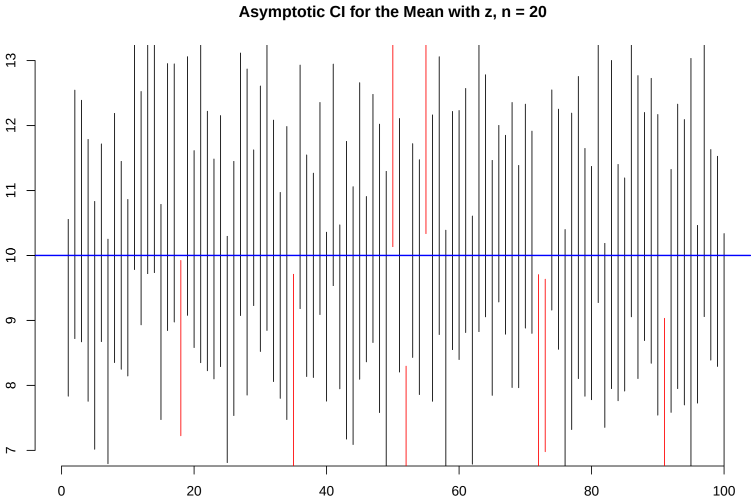
- the same form of CI was obtained for the Normal Model, when σ^2 was unkown;
- this interval is a little bit larger than the previous one, so it is also an AsympCI for μ of level $1-\alpha$;
- when $n \geq 30$, these two almost coincide;
- although in the theory these intervals work for large n, but, in practice, the latter one works also for small n

Example



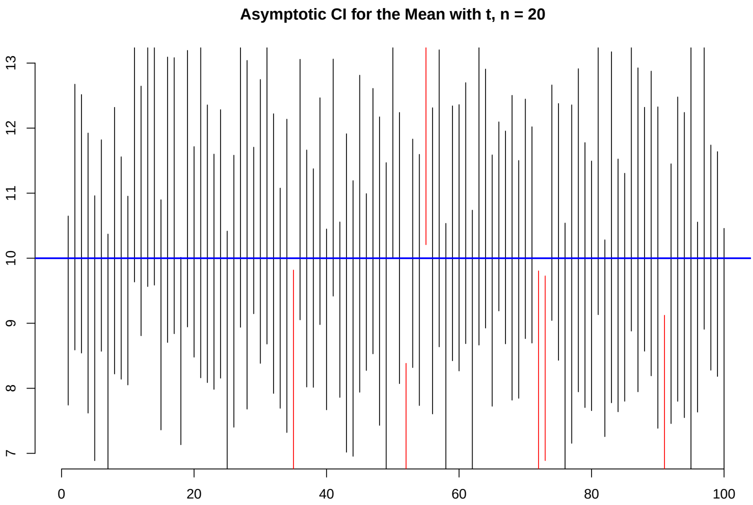
Asymptotic CI for the Mean with z, n = 50
13
12
11
10
9
8
7
0
20
40
60
80
100

Example



Asymptotic CI for the Mean with z, n = 20
7
8
9
10
11
12
13
0
20
40
60
80
100

Example



Asymptotic CI for the Mean with t, n = 20
7
8
9
10
11
12
13
0
20
40
60
80
100

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC.

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC. Scientists have a data about 84 skull sizes of adult Etruscans, giving in average 143.8mm, with a Sample Standard Deviation 5.97mm.

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC. Scientists have a data about 84 skull sizes of adult Etruscans, giving in average 143.8mm, with a Sample Standard Deviation 5.97mm.

Scientists believe Etruscans were Italians.

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC. Scientists have a data about 84 skull sizes of adult Etruscans, giving in average 143.8mm, with a Sample Standard Deviation 5.97mm.

Scientists believe Etruscans were Italians. They believe that the skull size is not changing much through time.

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC. Scientists have a data about 84 skull sizes of adult Etruscans, giving in average 143.8mm, with a Sample Standard Deviation 5.97mm.

Scientists believe Etruscans were Italians. They believe that the skull size is not changing much through time. Nowadays, adult Italians skull size is in average 132.4mm.

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC. Scientists have a data about 84 skull sizes of adult Etruscans, giving in average 143.8mm, with a Sample Standard Deviation 5.97mm.

Scientists believe Etruscans were Italians. They believe that the skull size is not changing much through time. Nowadays, adult Italians skull size is in average 132.4mm.

We want to construct a 95% AsymptoCI to see if it is supporting scientists hypothesis.

Example

Example: There was an ancient civilization called Etruscan Civilization, living in the area corresponding to Italy, around 8th century BC. Scientists have a data about 84 skull sizes of adult Etruscans, giving in average 143.8mm, with a Sample Standard Deviation 5.97mm.

Scientists believe Etruscans were Italians. They believe that the skull size is not changing much through time. Nowadays, adult Italians skull size is in average 132.4mm.

We want to construct a 95% AsymptoCI to see if it is supporting scientists hypothesis.

We model our problem like this: we assume the skull sizes of Italians are coming from some Distribution with some Mean μ and Variance σ^2, σ^2 is unknown.

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, ..., X_{84}.$$

where X_k is the skull size of k-th Etruscan person.

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, ..., X_{84}.$$

where X_k is the skull size of k-th Etruscan person. We have that $n = 84$, $\overline{X} = 143.8$ and $S = 5.97$.

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, ..., X_{84}.$$

where X_k is the skull size of k-th Etruscan person. We have that $n = 84$, $\overline{X} = 143.8$ and $S = 5.97$.

Then, we know how to construct an Asympto 95% CI for μ:

$$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, ..., X_{84}.$$

where X_k is the skull size of k-th Etruscan person. We have that $n = 84$, $\overline{X} = 143.8$ and $S = 5.97$.

Then, we know how to construct an Asympto 95% CI for μ:

$$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

giving us

```
x.bar <- 143.8; s <- 5.97; n <- 84
a <- 0.05; t <- qt(1-a/2, df = n-1)
me <- t*s/sqrt(n)
c(x.bar - me, x.bar +me)
```

```
## [1] 142.5044 145.0956
```

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, ..., X_{84}.$$

where X_k is the skull size of k-th Etruscan person. We have that $n = 84$, $\overline{X} = 143.8$ and $S = 5.97$.

Then, we know how to construct an Asympto 95% CI for μ:

$$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

giving us

```
x.bar <- 143.8; s <- 5.97; n <- 84
a <- 0.05; t <- qt(1-a/2, df = n-1)
me <- t*s/sqrt(n)
c(x.bar - me, x.bar +me)
```

```
## [1] 142.5044 145.0956
```

We know the true value of μ, it is $\mu = 132.4$.

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, ..., X_{84}.$$

where X_k is the skull size of k-th Etruscan person. We have that $n = 84$, $\overline{X} = 143.8$ and $S = 5.97$.

Then, we know how to construct an Asympto 95% CI for μ:

$$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

giving us

```
x.bar <- 143.8; s <- 5.97; n <- 84
a <- 0.05; t <- qt(1-a/2, df = n-1)
me <- t*s/sqrt(n)
c(x.bar - me, x.bar +me)
```

```
## [1] 142.5044 145.0956
```

We know the true value of μ, it is $\mu = 132.4$. So we can deduce that ...

Example, Cont'd

If we believe that Etruscans are Italians, then we have a Sample from that Distrib:

$$X_1, X_2, \ldots, X_{84}.$$

where X_k is the skull size of k-th Etruscan person. We have that $n = 84$, $\overline{X} = 143.8$ and $S = 5.97$.

Then, we know how to construct an Asympto 95% CI for μ:

$$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

giving us

```
x.bar <- 143.8; s <- 5.97; n <- 84
a <- 0.05; t <- qt(1-a/2, df = n-1)
me <- t*s/sqrt(n)
c(x.bar - me, x.bar +me)
```

```
## [1] 142.5044 145.0956
```

We know the true value of μ, it is $\mu = 132.4$. So we can deduce that ... they were looong skul Italians $\ddot{\smile}$

Asymptotic CI for the general case, using MLE

Using the Asymptotic Normality of the MLE, we can obtain a $(1 - \alpha)$-level Asymptotic CI for the general case.

Asymptotic CI for the general case, using MLE

Using the Asymptotic Normality of the MLE, we can obtain a $(1-\alpha)$-level Asymptotic CI for the general case.

Assume we have a (regular) Parametric Model $\mathcal{F}_\theta$, and a Random Sample from it

$$X_1, X_2, ..., X_n, ... \sim \mathcal{F}_\theta.$$

Asymptotic CI for the general case, using MLE

Using the Asymptotic Normality of the MLE, we can obtain a $(1-\alpha)$-level Asymptotic CI for the general case.

Assume we have a (regular) Parametric Model $\mathcal{F}_\theta$, and a Random Sample from it

$$X_1, X_2, ..., X_n, ... \sim \mathcal{F}_\theta.$$

We want to construct a $(1-\alpha)$-level Asymptotic CI for θ.

Asymptotic CI for the general case, using MLE

Using the Asymptotic Normality of the MLE, we can obtain a $(1-\alpha)$-level Asymptotic CI for the general case.

Assume we have a (regular) Parametric Model $\mathcal{F}_\theta$, and a Random Sample from it

$$X_1, X_2, ..., X_n, ... \sim \mathcal{F}_\theta.$$

We want to construct a $(1-\alpha)$-level Asymptotic CI for θ.

Let $\hat{\theta}_n^{MLE}$ be the ML Estimator for θ.

Asymptotic CI for the general case, using MLE

Using the Asymptotic Normality of the MLE, we can obtain a $(1-\alpha)$-level Asymptotic CI for the general case.

Assume we have a (regular) Parametric Model $\mathcal{F}_\theta$, and a Random Sample from it

$$X_1, X_2, ..., X_n, ... \sim \mathcal{F}_\theta.$$

We want to construct a $(1-\alpha)$-level Asymptotic CI for θ.

Let $\hat{\theta}_n^{MLE}$ be the ML Estimator for θ. Then we know that

$$\frac{\hat{\theta}_n^{MLE} - \theta}{\sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\theta}_n^{MLE}\right)}}} \xrightarrow{D} \mathcal{N}(0,1)$$

Asymptotic CI for the general case, using MLE

This means that

$$\mathbb{P}\left(-z_{1-\alpha/2} < \frac{\hat{\theta}_n^{MLE} - \theta}{\sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\theta}_n^{MLE}\right)}}} < z_{1-\alpha/2}\right) \to 1 - \alpha$$

Asymptotic CI for the general case, using MLE

This means that

$$\mathbb{P}\left(-z_{1-\alpha/2} < \frac{\hat{\theta}_n^{MLE} - \theta}{\sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\theta}_n^{MLE}\right)}}} < z_{1-\alpha/2}\right) \to 1 - \alpha$$

giving us a $(1-\alpha)$-level Asymptotic CI for θ:

Asymptotic CI for the general case, using MLE

This means that

$$\mathbb{P}\left(-z_{1-\alpha/2} < \frac{\hat{\theta}_n^{MLE} - \theta}{\sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\theta}_n^{MLE}\right)}}} < z_{1-\alpha/2}\right) \to 1 - \alpha$$

giving us a $(1-\alpha)$-level Asymptotic CI for θ:

$$\left(\hat{\theta}_n^{MLE} - z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\theta}_n^{MLE}\right)}};\;\; \hat{\theta}_n^{MLE} + z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\theta}_n^{MLE}\right)}}\right).$$

Asymptotic CI for the Bernoulli Model

We have already constructed a CI for p in the $Bernoulli(p)$ Model. Here we want to construct an Asymptotic CI for p, which usually gives smaller length interval.

Asymptotic CI for the Bernoulli Model

We have already constructed a CI for p in the *Bernoulli*(p) Model. Here we want to construct an Asymptotic CI for p, which usually gives smaller length interval.

The Problem is the following: we have a Random Sample

$$X_1, X_2, ..., X_n, ... \sim Bernoulli(p),$$

and $\alpha \in (0, 1)$.

Asymptotic CI for the Bernoulli Model

We have already constructed a CI for p in the *Bernoulli*(p) Model. Here we want to construct an Asymptotic CI for p, which usually gives smaller length interval.

The Problem is the following: we have a Random Sample

$$X_1, X_2, ..., X_n, ... \sim \mathit{Bernoulli}(p),$$

and $\alpha \in (0, 1)$. Our aim is to construct a $(1-\alpha)$-level Asymptotic CI for p.

Asymptotic CI for the Bernoulli Model

We have already constructed a CI for p in the *Bernoulli*(p) Model. Here we want to construct an Asymptotic CI for p, which usually gives smaller length interval.

The Problem is the following: we have a Random Sample

$$X_1, X_2, ..., X_n, ... \sim Bernoulli(p),$$

and $\alpha \in (0, 1)$. Our aim is to construct a $(1-\alpha)$-level Asymptotic CI for p.

We know that $\hat{p}_n^{MLE} =$

Asymptotic CI for the Bernoulli Model

We have already constructed a CI for p in the *Bernoulli*(p) Model. Here we want to construct an Asymptotic CI for p, which usually gives smaller length interval.

The Problem is the following: we have a Random Sample

$$X_1, X_2, ..., X_n, ... \sim Bernoulli(p),$$

and $\alpha \in (0,1)$. Our aim is to construct a $(1-\alpha)$-level Asymptotic CI for p.

We know that $\hat{p}_n^{MLE} = \overline{X}_n$. Also, we have calculated the Fisher Information for the Bernoulli case: $\mathcal{I}(p) =$

Asymptotic CI for the Bernoulli Model

We have already constructed a CI for p in the $Bernoulli(p)$ Model. Here we want to construct an Asymptotic CI for p, which usually gives smaller length interval.

The Problem is the following: we have a Random Sample

$$X_1, X_2, ..., X_n, ... \sim Bernoulli(p),$$

and $\alpha \in (0,1)$. Our aim is to construct a $(1-\alpha)$-level Asymptotic CI for p.

We know that $\hat{p}_n^{MLE} = \overline{X}_n$. Also, we have calculated the Fisher Information for the Bernoulli case: $\mathcal{I}(p) = \dfrac{1}{p(1-p)}$.

Asymptotic CI for the Bernoulli Model

The above obtained CI is:

$$\left(\hat{p}_n^{MLE} - z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{p}_n^{MLE}\right)}};\ \ \hat{p}_n^{MLE} + z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{p}_n^{MLE}\right)}}\right).$$

Asymptotic CI for the Bernoulli Model

The above obtained CI is:

$$\left(\hat{p}_n^{MLE} - z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{p}_n^{MLE}\right)}};\ \ \hat{p}_n^{MLE} + z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{p}_n^{MLE}\right)}}\right).$$

By plugging the values for our case, we'll obtain the following Asymptotic CI of level $(1-\alpha)$ for p:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}};\ \ \overline{X}_n + z_{1-\alpha/2} \cdot \sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}}\right).$$

Asymptotic CI for the Bernoulli Model

The above obtained CI is:

$$\left(\hat{p}_n^{MLE} - z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{p}_n^{MLE}\right)}};\;\; \hat{p}_n^{MLE} + z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{p}_n^{MLE}\right)}}\right).$$

By plugging the values for our case, we'll obtain the following Asymptotic CI of level $(1-\alpha)$ for p:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}};\;\; \overline{X}_n + z_{1-\alpha/2} \cdot \sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}}\right).$$

People use this if $np > 5$ and $n(1-p) > 5$.

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Taking an $\alpha \in (0,1)$, we want to construct a $(1-\alpha)$-level Asymptotic CI for λ.

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Taking an $\alpha \in (0, 1)$, we want to construct a $(1 - \alpha)$-level Asymptotic CI for λ.

We know that $\hat{\lambda}_n^{MLE} =$

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Taking an $\alpha \in (0, 1)$, we want to construct a $(1-\alpha)$-level Asymptotic CI for λ.

We know that $\hat{\lambda}_n^{MLE} = \overline{X}_n$, and also, $\mathcal{I}(\lambda) =$

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Taking an $\alpha \in (0,1)$, we want to construct a $(1-\alpha)$-level Asymptotic CI for λ.

We know that $\hat{\lambda}_n^{MLE} = \overline{X}_n$, and also, $\mathcal{I}(\lambda) = \dfrac{1}{\lambda}$.

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Taking an $\alpha \in (0,1)$, we want to construct a $(1-\alpha)$-level Asymptotic CI for λ.

We know that $\hat{\lambda}_n^{MLE} = \overline{X}_n$, and also, $\mathcal{I}(\lambda) = \dfrac{1}{\lambda}$. Then, using the formula above:

$$\left(\hat{\lambda}_n^{MLE} - z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\lambda}_n^{MLE}\right)}};\ \ \hat{\lambda}_n^{MLE} + z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\lambda}_n^{MLE}\right)}}\right),$$

Example, Asymptotic CI for Poisson Model

Example: We consider the Poisson Model $Pois(\lambda)$, and a Random Sample from it:

$$X_1, X_2, ..., X_n, ... \sim Pois(\lambda).$$

Taking an $\alpha \in (0,1)$, we want to construct a $(1-\alpha)$-level Asymptotic CI for λ.

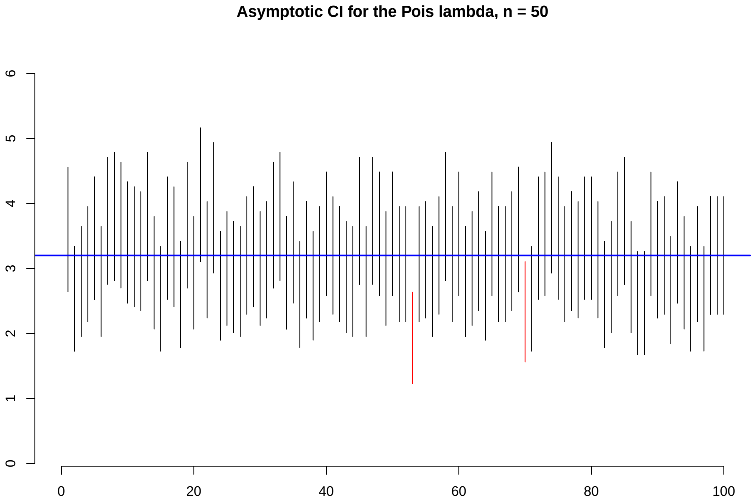
We know that $\hat{\lambda}_n^{MLE} = \overline{X}_n$, and also, $\mathcal{I}(\lambda) = \dfrac{1}{\lambda}$. Then, using the formula above:

$$\left(\hat{\lambda}_n^{MLE} - z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\lambda}_n^{MLE}\right)}};\;\; \hat{\lambda}_n^{MLE} + z_{1-\alpha/2} \cdot \sqrt{\frac{1}{n \cdot \mathcal{I}\left(\hat{\lambda}_n^{MLE}\right)}}\right),$$

we'll obtain the following Asymptotic CI of level $(1-\alpha)$ for λ:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \sqrt{\frac{\overline{X}_n}{n}};\;\; \overline{X}_n + z_{1-\alpha/2} \cdot \sqrt{\frac{\overline{X}_n}{n}}\right).$$

Example, in **R**



Asymptotic CI for the Pois lambda, n = 50
0
1
2
3
4
5
6
0
20
40
60
80
100

Example, in **R**, Code

```
lambda <- 3.2
conf.level <- 0.95; a = 1 - conf.level
sample.size <- 15; no.of.intervals <- 100
z <- qnorm(1-a/2)

plot.new()
plot.window(xlim=c(0,no.of.intervals),ylim=c(lambda-3,lambda+3))
axis(1); axis(2)
title("Asymptotic CI for the Pois lambda, n = 50")
for(i in 1:no.of.intervals){
  x <- rpois(sample.size, lambda = lambda)
  ME <- z*sqrt(mean(x)/sample.size) #Margin of Error
  lo <- mean(x) - ME;  up <- mean(x) + ME
  if(lo > lambda || up < lambda){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
    segments(c(i), c(lo), c(i), c(up))
  }
}
abline(h = lambda, lwd = 2, col = "blue")
```

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and $Y_1, Y_2, ..., Y_m$, that can be also of different sizes.

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and Y_1, Y_2, ..., Y_m, that can be also of different sizes. We assume that

- X_k-s are IID,

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and $Y_1, Y_2, ..., Y_m$, that can be also of different sizes. We assume that

- X_k-s are IID,
- Y_k-s are also IID,

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and Y_1, Y_2, ..., Y_m, that can be also of different sizes. We assume that

- X_k-s are IID,
- Y_k-s are also IID,
- X_k-s and Y_j-s are Independent.

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and Y_1, Y_2, ..., Y_m, that can be also of different sizes. We assume that

- X_k-s are IID,
- Y_k-s are also IID,
- X_k-s and Y_j-s are Independent.

The general Problem is the following: assume the Mean of the Distribution behind X_k-s is μ_X, and the Mean of the Distribution behind Y_k-s is μ_Y.

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and Y_1, Y_2, ..., Y_m, that can be also of different sizes. We assume that

- X_k-s are IID,
- Y_k-s are also IID,
- X_k-s and Y_j-s are Independent.

The general Problem is the following: assume the Mean of the Distribution behind X_k-s is μ_X, and the Mean of the Distribution behind Y_k-s is μ_Y. We do not know μ_X and μ_Y, and, in fact, we are not too much interested in μ_X and μ_Y, rather, we are interested in the difference between the Means:

$$\mu_X - \mu_Y.$$

CI for the Difference between the Means of Two Samples

Assume We have two Samples: $X_1, X_2, .., X_n$ and $Y_1, Y_2, ..., Y_m$, that can be also of different sizes. We assume that

- X_k-s are IID,
- Y_k-s are also IID,
- X_k-s and Y_j-s are Independent.

The general Problem is the following: assume the Mean of the Distribution behind X_k-s is μ_X, and the Mean of the Distribution behind Y_k-s is μ_Y. We do not know μ_X and μ_Y, and, in fact, we are not too much interested in μ_X and μ_Y, rather, we are interested in the difference between the Means:

$$\mu_X - \mu_Y.$$

So we want to construct a $(1-\alpha)$-level CI for this difference, given α.

CI for the Difference between the Means of Two Samples

Here we need to consider different cases.

CI for the Difference between the Means of Two Samples

Here we need to consider different cases.

Case 1: X_k, Y_k are Normal, with known Variances

In this case we assume

$$X_1, X_2, \ldots, X_n \overset{IID}{\sim} \mathcal{N}(\mu_X, \sigma_X^2) \qquad \textit{and} \qquad Y_1, Y_2, \ldots, Y_m \overset{IID}{\sim} \mathcal{N}(\mu_Y, \sigma_Y^2)$$

and σ_X^2 and σ_Y^2 are **known**. Also we have that X_k-s and Y_j-s are Independent.

CI for the Difference between the Means of Two Samples

Here we need to consider different cases.

Case 1: X_k, Y_k are Normal, with known Variances

In this case we assume

$$X_1, X_2, \ldots, X_n \overset{IID}{\sim} \mathcal{N}(\mu_X, \sigma_X^2) \qquad \textit{and} \qquad Y_1, Y_2, \ldots, Y_m \overset{IID}{\sim} \mathcal{N}(\mu_Y, \sigma_Y^2)$$

and σ_X^2 and σ_Y^2 are **known**. Also we have that X_k-s and Y_j-s are Independent.

It is easy to find a good Estimator for $\mu_X - \mu_Y$: we can just take

CI for the Difference between the Means of Two Samples

Here we need to consider different cases.

Case 1: X_k, Y_k are Normal, with known Variances

In this case we assume

$$X_1, X_2, ..., X_n \stackrel{IID}{\sim} \mathcal{N}(\mu_X, \sigma_X^2) \qquad \textit{and} \qquad Y_1, Y_2, ..., Y_m \stackrel{IID}{\sim} \mathcal{N}(\mu_Y, \sigma_Y^2)$$

and σ_X^2 and σ_Y^2 are **known**. Also we have that X_k-s and Y_j-s are Independent.
It is easy to find a good Estimator for $\mu_X - \mu_Y$: we can just take $\overline{X} - \overline{Y}$.

CI for the Difference between the Means of Two Samples

Here we need to consider different cases.

Case 1: X_k, Y_k are Normal, with known Variances

In this case we assume

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu_X, \sigma_X^2) \qquad \textit{and} \qquad Y_1, Y_2, ..., Y_m \overset{IID}{\sim} \mathcal{N}(\mu_Y, \sigma_Y^2)$$

and σ_X^2 and σ_Y^2 are **known**. Also we have that X_k-s and Y_j-s are Independent.

It is easy to find a good Estimator for $\mu_X - \mu_Y$: we can just take $\overline{X} - \overline{Y}$. So we will base our construction of CI on this, by finding a Pivot using $\overline{X} - \overline{Y}$.

Case 1: X_k, Y_k are Normal, with known Variances

It is not so hard to see that

$$\overline{X} - \overline{Y} \sim$$

Case 1: X_k, Y_k are Normal, with known Variances

It is not so hard to see that

$$\overline{X} - \overline{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

Case 1: X_k, Y_k are Normal, with known Variances

It is not so hard to see that

$$\overline{X} - \overline{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

Now, we can obtain a Pivot: we will just consider

Case 1: X_k, Y_k are Normal, with known Variances

It is not so hard to see that

$$\overline{X} - \overline{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

Now, we can obtain a Pivot: we will just consider

$$Z = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim$$

Case 1: X_k, Y_k are Normal, with known Variances

It is not so hard to see that

$$\overline{X} - \overline{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

Now, we can obtain a Pivot: we will just consider

$$Z = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim \mathcal{N}(0, 1).$$

Case 1: X_k, Y_k are Normal, with known Variances

It is not so hard to see that

$$\overline{X} - \overline{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

Now, we can obtain a Pivot: we will just consider

$$Z = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim \mathcal{N}(0, 1).$$

Then, using this Pivot, we will obtain the following $(1-\alpha)$-level CI for $\mu_X - \mu_Y$:

$$(\overline{X} - \overline{Y}) \pm z_{1-\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}.$$