SECTION 4

Applications of Berry curvature & QSHE

Conductance of Berry curvature:

$$\sigma_{xy} = \frac{e^2}{h} \cdot \frac{1}{2\pi} \int_{BZ} \Omega(\vec{k})\, d\vec{k} = \frac{e^2}{h} \cdot C$$

where C is the Chern no:.

PRL (1982) Thouless, Kohmoto, Nightingale, Nijs

Kubo formula:

$$\sigma_{xy} = \frac{i e^2}{A_0 \hbar} \sum_{\epsilon_\alpha \epsilon_\beta < E_F}' \frac{\langle \alpha | \frac{\partial \hat{H}}{\partial k_x} | \beta \rangle \langle \beta | \frac{\partial \hat{H}}{\partial k_y} | \alpha \rangle - \langle \beta \rightarrow \alpha \rangle}{(\epsilon_\alpha - \epsilon_\beta)^2}$$

$$= \frac{i e^2}{2\pi \hbar} \sum' \int d^2\vec{k} \int d^2\vec{r} \left\langle \frac{\partial u^*}{\partial k_x} \frac{\partial u}{\partial k_y} - \frac{\partial u^*}{\partial k_y} \frac{\partial u}{\partial k_x} \right\rangle$$

In 2005 Kane/Mele proposed that Graphene with spin orbit coupling can give rise to QSH effect.

$$\sigma_{ij}^{\Delta} = \frac{e}{\hbar} \int_{BZ} \Omega_{ij}^{\Delta}\, d\vec{k}, \quad \Omega_{ij}^{\Delta} \text{ is calculated}$$

from spin resolved eigenstate of Graphene.

In 2006, Bernevig, Hughes, Zhang predicted HgTe Quantum wells, CdTe Quantum wells can exhibit QSH effect.

In 2007, experiments confirmed QSH effect.

Dirac equation & edge states:

$$\mathcal{H} = -iv\sigma_x \partial_x + M\sigma_z \quad ,$$

$$M = mv^2$$

Eigenenergies : $E_{1,2} = \pm\sqrt{v^2 p_x^2 + M^2}$

Gap @ $p_x = 0$ of $\Delta = |2M|$

Small modification :

$$M(x) = \begin{cases} -m_1 , & \text{if } x<0 \\ m_2 , & \text{if } x \geq 0 \end{cases}$$

Eigenvalue equation :

$$\begin{pmatrix} M(x) & -iv\partial_x \\ -iv\partial_x & -M(x) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

We need to find $x<0$ or $x>0$ solutions:

$x>0$: trial solution $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} e^{-\lambda_+ x}$

$$\begin{pmatrix} m_2 - E & iv\lambda_+ \\ iv\lambda_+ & -m_2 - E \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} = 0$$

$$\begin{vmatrix} m_2 - E & iv\lambda_+ \\ iv\lambda_+ & -m_2 - E \end{vmatrix} = 0 \Rightarrow \lambda_+ = \pm \frac{\sqrt{m_2^2 - E^2}}{v}$$

For $m_2^2 < E^2$, solutions are purely imaginary,
$\Rightarrow$ wavefunction spreads over whole space

$\Rightarrow$ extended / bulk states

For $m_2^2 > E^2$ solutions are real

$\Rightarrow$ wavefunction is localized $\rightarrow$ represents edge states

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} e^{-\lambda_+ x} \Rightarrow \phi_1^+ = \frac{-i v \lambda_+}{m_2 - E} \phi_1^-$$

for $x < 0$: $$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1^- \\ \phi_2^- \end{pmatrix} e^{\lambda_- x}$$

$$\Rightarrow \phi_1^- = \frac{-i v \lambda_-}{m_1 + E} \phi_2^-$$

@ $x = 0$ $$\begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} = \begin{pmatrix} \phi_1^- \\ \phi_2^- \end{pmatrix} \Rightarrow \sqrt{\frac{m_2 + E}{m_2 - E}} = \sqrt{\frac{m_1 - E}{m_1 + E}}$$

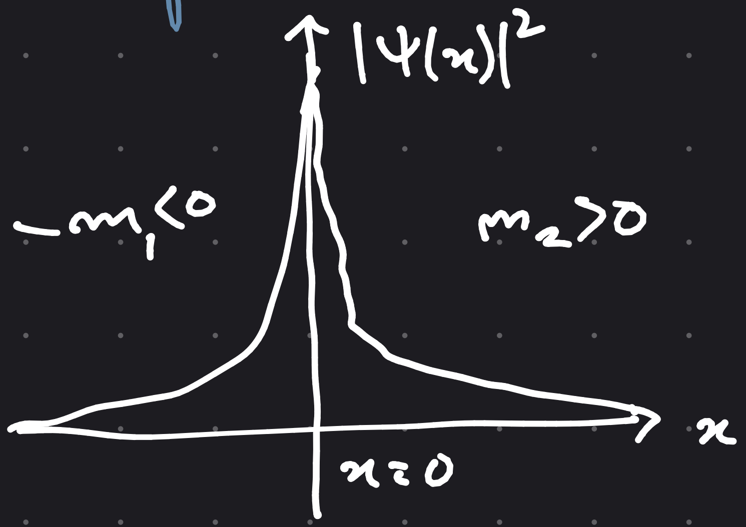
$\Rightarrow$ If $E = 0$ is a solution, we have to have $m(x)$ of opposite signs across $x = 0$.

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-m_2 x/v} \quad \text{(bound state solution)}$$

1. Dirac 1D Hamiltonian admits zero energy bound states localized near the interface.

2. Opposite mass terms may explain existence

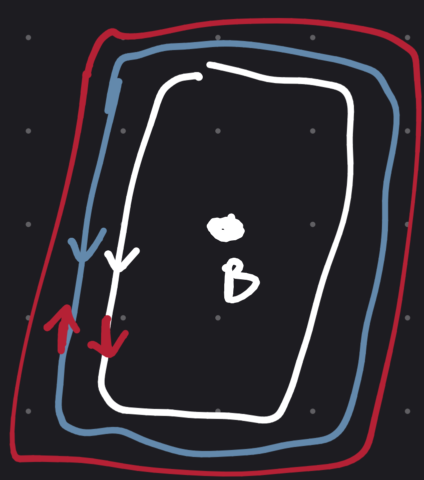
of edge states.



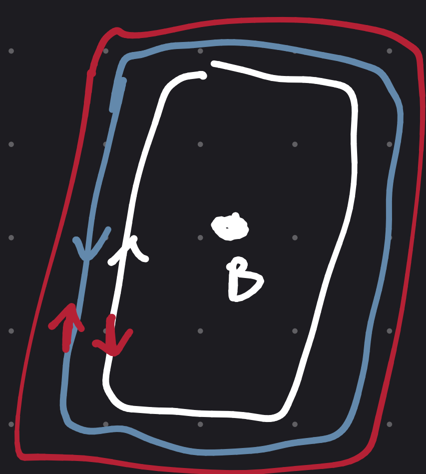
Questions:

1) Topological protection?

2) Spin & momentum are chiral or helical edge modes?



Chiral edge modes



Helical edge modes

1. To go from $H = -iv\partial_x + M\sigma_z$ to

$$H = v\sigma_x p_x + \sigma_z(M + Bp_x^2)$$

$$\sigma_x H\psi = E\sigma_x\psi$$

$$\{v p_x - i\sigma_y(M+Bp_x^2)\}\psi = E\sigma_x\psi$$

for zero energy boundstates occur $E=0$

$$\partial_x\psi = \frac{-i}{v}\left[M + B\partial_x^2\right]\sigma_y\psi$$

Let $\psi = \chi_\eta \, \phi(x)$ with $\phi(0) = 0$,

χ_η is an eigenstate of σ_y with eigenvalue $\eta = \pm 1$

$$\partial_x \phi(x) = -\frac{\eta}{v}\left(M + B\partial_x^2\right)\phi(x)$$

using ansatz:

$$\phi(x) = C e^{-\lambda x}$$

$$-\lambda e^{-\lambda x} = -\frac{\eta}{v}\left(M e^{-\lambda x} + B(-\lambda)^2 e^{-\lambda x}\right)$$

$$\lambda_\pm = \frac{\eta v}{2B} \pm \frac{\eta v \sqrt{1-4MB}}{2B}, \quad \text{I need } \lambda_\pm > 0.$$

To get this:

1. $\eta = \mathrm{sgn}(B)$.
2. $B > 0$ such that 2^{nd} term is always less than 1^{st} term.
3. $v > 0$.

$$\psi_\eta(x) = \frac{C}{\sqrt{2}} \begin{pmatrix} \mathrm{sgn}(B) \\ i \end{pmatrix} \left(e^{-x/\xi_+} - e^{-x/\xi_-}\right)$$

$$\xi_\pm^{-1} = \frac{v}{2|B|}\left(1 \pm \sqrt{1-4MB}\right)$$ is related to penetration depth of edge state into bulk.