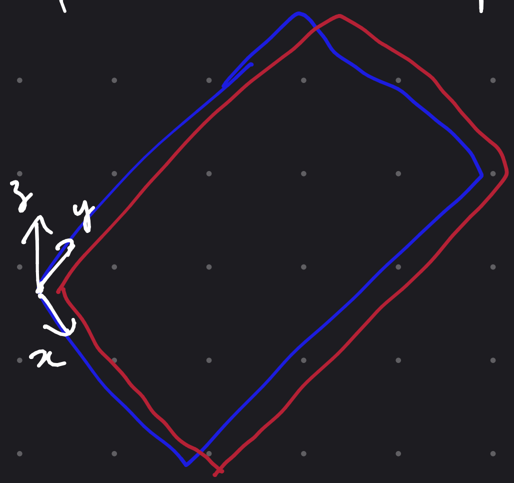
SECTION 6

Quantum spin hall insulator

Quantum spin hall insulator:



$m=0$ No time reversal breaking m term.

$$\hat{H} = \begin{bmatrix} -i\hbar v(\sigma_x \partial_x + \sigma_y \partial_y) & \Delta\delta_0 \\ \Delta\delta_0 & i\hbar v(\sigma_x \partial_x + \sigma_y \partial_y) \end{bmatrix}$$

$$T H T^{-1} = H$$

complex conjugate operator

$$T = \begin{bmatrix} i\sigma_y K & 0 \\ 0 & i\sigma_y K \end{bmatrix}$$

$$\hat{\psi}(\vec{r}) = \begin{bmatrix} \psi_{t\uparrow} \\ \psi_{t\downarrow} \\ \psi_{b\uparrow} \\ \psi_{b\downarrow} \end{bmatrix}$$

For a realistic 2D Chern insulator:

$$\hat{h}_k = \begin{bmatrix} \varepsilon_k + d_{k_z} & d_{k_x} - i d_{k_y} \\ d_{k_x} + i d_{k_y} & \varepsilon_k - d_{k_z} \end{bmatrix}$$

around dirac point: $\vec{k} \approx 0$

$$\vec{d}_{\vec{k}} \simeq [A k_x, -A k_y, M_{\vec{k}}] \qquad A = \hbar v$$

$$M_{\vec{k}} = \Delta + \beta |\vec{k}|^2$$

$$\Omega = \int_{BZ} \frac{dk_x\, dk_y}{2\pi} (\vec{\nabla} \times \vec{A}_{\vec{k}})_z$$

$$= -i \int_{BZ} \frac{dk_x\, dk_y}{2\pi} \sum_{E_i < \mu} \left[\langle \partial_{k_x} \psi_{k,s} | \partial_{k_y} \psi_{k,s} \rangle - \langle \partial_{k_y} \psi_{k,s} | \partial_{k_x} \psi_{k,s} \rangle \right]$$

$$= \frac{1}{4\pi} \int \vec{u}_{\vec{k}} \cdot \left(\frac{\partial \vec{u}_{\vec{k}}}{\partial k_x} \times \frac{\partial \vec{u}_{\vec{k}}}{\partial k_y} \right) dk_x\, dk_y \qquad \vec{u}_{\vec{k}} = \frac{\vec{d}_{\vec{k}}}{|\vec{d}_{\vec{k}}|}$$

Chern no: $$n = \int_0^\infty \frac{2\pi k\, dk}{2\pi} \frac{s}{2k} \frac{\partial u_z}{\partial k}$$ → derived in HW.

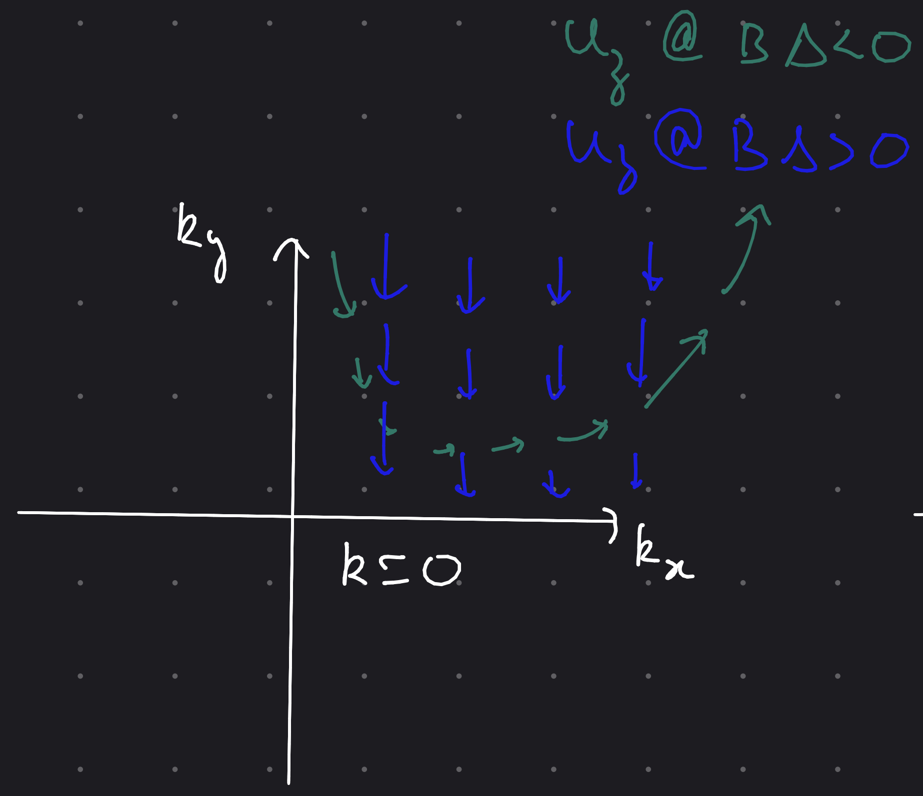
Considering μ is in the band gap, so only valence band is filled.

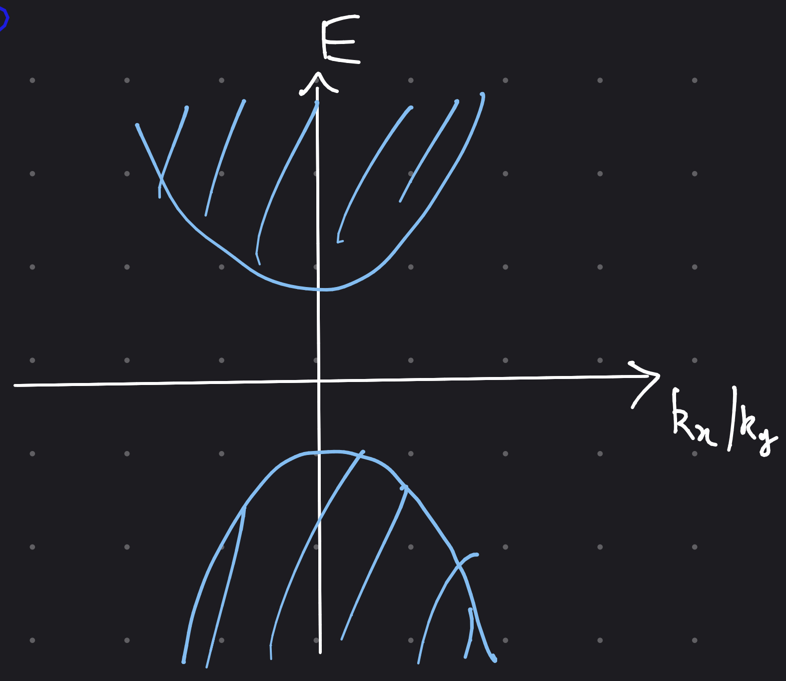
$\Rightarrow s = -1$

$$n = -\left[\frac{u_z(\infty) - u_z(0)}{2} \right] = -\left[\frac{\mathrm{sgn}(\beta) - \mathrm{sgn}(\Delta)}{2} \right]$$

for $B\Delta < 0 \Rightarrow n = -\mathrm{sgn}(B)$ or $\mathrm{sgn}(\Delta)$

for $B\Delta > 0 \Rightarrow n = 0$





● - Topologically non-trivial phase

● - Topologically trivial phase

$$\hat{h}_{\vec{k}} = \sigma_0\left(C + D|\vec{k}|^2\right) + A k_x \sigma_x - A k_y \sigma_y + \left(\Delta + B|\vec{k}|^2\right)\sigma_z$$

Semi-infinite plane: $-\infty \le y \le \infty$, $x \ge 0$

k_y is a good quantum no:

$\Rightarrow$ Eigen energies are related to k_y & there is scattering in x-dir^n.

$$\hat{h}_{\vec{k}} = \underbrace{\varepsilon_k \sigma_0}_{\text{Assymmetric}} + \underbrace{\vec{d}_{\vec{k}} \cdot \vec{\sigma}}_{\text{Symmetric}}$$

$\rightarrow$ hole energy is $-$ve of particle energy

$$\hat{h}_{k_y}(x) = \hat{h}^A_{k_y}(x) + \hat{h}_{k_y}(x)$$

A $\rightarrow$ Assymmetric
S $\rightarrow$ Symmetric

$$\hat{h}^A_{k_y}(x) = \left(C + D k_y^2 - D\partial_x^2\right)\sigma_0$$

$$\hat{h}^S_{k_y}(x) = -iA\partial_x\sigma_x - A k_y \sigma_y + \left(\Delta + B k_y^2 - B\partial_x^2\right)\sigma_z$$

$$\hat{h}^A_{k_y}(x)\,\psi(x) + \hat{h}^S_{k_y}(x)\,\psi(x) = E\,\psi(x)$$

$\psi(x) = 0$ for $x \le 0$

$\psi(x) \to 0$ for $x \to \infty$

$$\int_{-\infty}^{\infty} \hat{h}^A_{k_y}(x)\,\psi(x)\,dx + \int_{-\infty}^{\infty} \hat{h}^S_{k_y}(x)\,\psi(x)\,dx$$

$$= E \int_{-\infty}^{\infty} \psi(x)\,dx$$

B.C:

$\partial_x \psi(x) = 0$ at $x \to \infty$

1st term : $(C + D k_y^2) \int_{-\infty}^{\infty} \psi(x)\,dx$

2nd term : $-A k_y \sigma_y + \Delta + B k_y^2 \sigma_z \int_{-\infty}^{\infty} \psi(x)\,dx$

Now take : $\psi(x) = \phi\, e^{-Kx}$

$$[\,i A K \sigma_x - A k_y \sigma_y - i(\Delta + B k_y^2 - B K^2)\sigma_x \sigma_y\,]\phi = E^S_{k_y}\phi$$

$$\Rightarrow [\,A K \sigma_x - (\Delta + B k_y^2 - B K^2)\sigma_y\,]\,\phi = 0$$

$$-A k_y \sigma \phi = E^S_{k_y}\,\phi$$

Assuming ϕ is eigenstate of σ_y,

$E^S_{k_y} = -A k_y \sigma$, σ is eigenvalue of σ_y

$$K_{1,2} = -\gamma \frac{A}{2B} \pm \sqrt{\left(\frac{A}{2B}\right)^2 + \frac{\Delta}{B} + k_y^2}$$

$\gamma \in \{-1, 1\}$

Since $K_{1,2} < 0$, $\gamma = -\operatorname{sgn}(AB)$

$$K_{1,2} = \left|\frac{A}{2B}\right| \pm \sqrt{\left(\frac{A}{2B}\right)^2 + \frac{\Delta}{B} + k_y^2} > 0$$

for $-k_0 \le k_y \le k_0$, where $k_0 = \sqrt{\frac{-\Delta}{B}}$

$$\psi(x) = \phi\left(A e^{-K_1 x} + B e^{K_2 x}\right)$$

$\psi(x=0) = 0$

$\Rightarrow A = -B$

$$E_{k_y} = C + D k_y^2 + |A| \operatorname{sgn} B \cdot k_y$$

@ $k_y^2 \to 0$, Velocity of edge mode $= \frac{1}{\hbar} \frac{\partial E_{k_y}}{\partial k_y} = \operatorname{sgn} B$

How to make Hamiltonian TRS?

$$\hat{H}_{\vec{k}} = \begin{bmatrix} \hat{h}_k & 0 \\ 0 & \hat{h}^*_{-k} \end{bmatrix}$$

This hamiltonian is $T = i\sigma_y \tau_0 K$ invariant

$$\hat{H}_{\vec{k}} = \varepsilon_k \tau_0 \sigma_0 + \begin{bmatrix} A k_x \tau_x - A k_y \tau_y + M_k \tau_z & 0 \\ 0 & -A k_x \tau_x - A k_y \tau_y + M_k \tau_z \end{bmatrix}^{-1}$$

Make the following transformations:

① $U = \begin{bmatrix} 0 & \tau_z \\ -i\tau_y & 0 \end{bmatrix}$

② $U = \begin{bmatrix} \tau_0 & 0 \\ 0 & \tau_z \end{bmatrix}$

$$\tilde{H}_{\vec{k}} = \varepsilon_k \tau_0 \sigma_0 + \begin{bmatrix} \vec{d}^{+}_{\vec{k}} \cdot \vec{\tau} & 0 \\ 0 & \vec{d}^{-}_{\vec{k}} \cdot \vec{\tau} \end{bmatrix}$$

$$\vec{d}^{\pm}_{\vec{k}} = [A k_x, A k_y, \pm M_{\vec{k}}]$$

Similar as before, calculate $\vec{u}$, then Berry phase, Berry connection & Berry curvature.

Chern no:

$$n = \sum_{\nu=+,-} \int_{BZ} \frac{dk_x\, dk_y}{2\pi} \left(\vec{\nabla} \times A_{\vec{k},\nu}\right)_z$$

When we are considering contribution from valence band, the u is from:

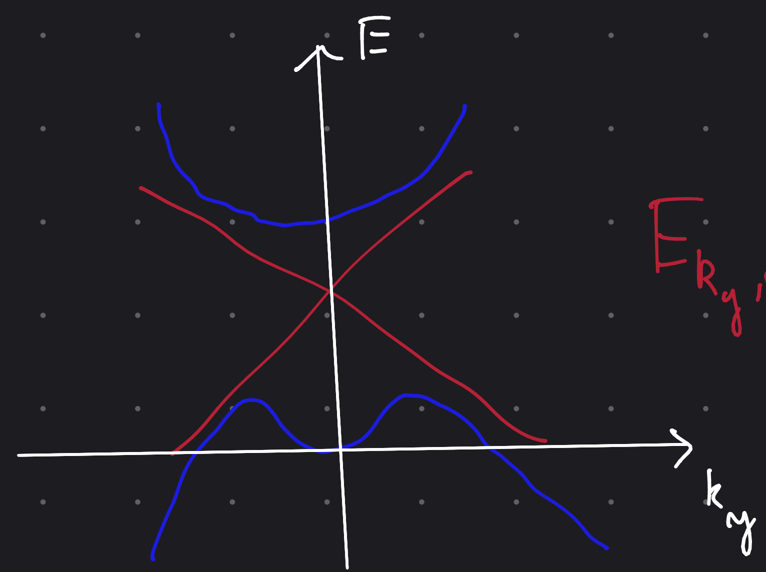
$$u_{z,\nu}(k) = \frac{\nu M_{\vec{k}}}{\sqrt{A^2k^2+M_k^2}}$$

$$\eta = \frac{1}{2}\sum_{\nu=\pm}\int_{\gamma}^{\infty}\frac{\partial u_{z\nu}}{\partial k}\,dk$$

$$= \frac{n_+ + n_-}{2} = 0.$$

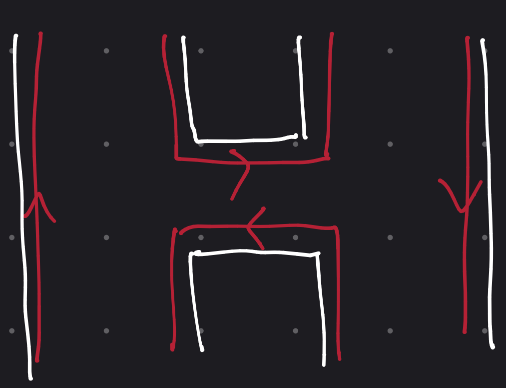
$$n_\pm = \pm\frac{\left(\mathrm{Sgn}(B)-\mathrm{Sgn}(\Delta)\right)}{2}$$

$$Z_2 \text{ invariant} = \frac{n_+-n_-}{2} = \frac{\mathrm{Sgn}(B)-\mathrm{Sgn}(\Delta)}{2}$$



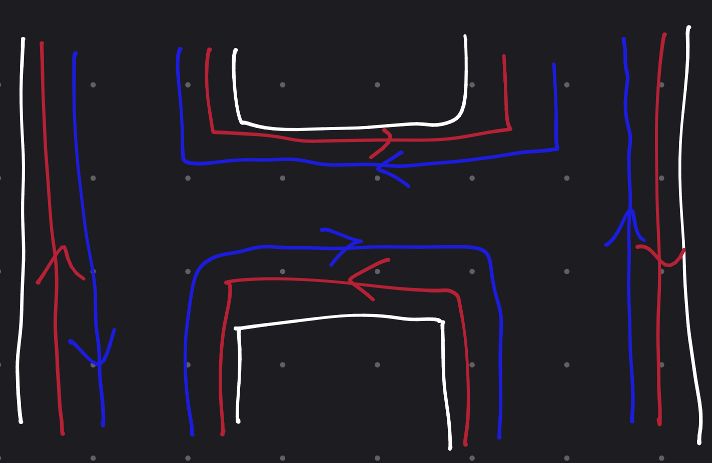
$$E_{k_y,\sigma} = C+Dk_y^2+|A|\,\mathrm{sgn}B\,k_y\sigma$$

(a) Chiral QH:



$T_{i+1,i}=1$

(b) helical SQH



$T_{i+1,i} = 1 = T_{i,i+1}$

$$I_i = \frac{e^2}{h}\sum_{j=1}^{N}\left(T_{ji}V_i - T_{ij}V_j\right)$$

$$R_{14,14} = \frac{h}{e^2}$$

$$R_{14,23} = 0$$

$$R_{14,14} = \frac{V_1 - V_4}{I_1} = \frac{3h}{4e^2}$$

$$R_{14,23} = \frac{h}{4e^2}$$

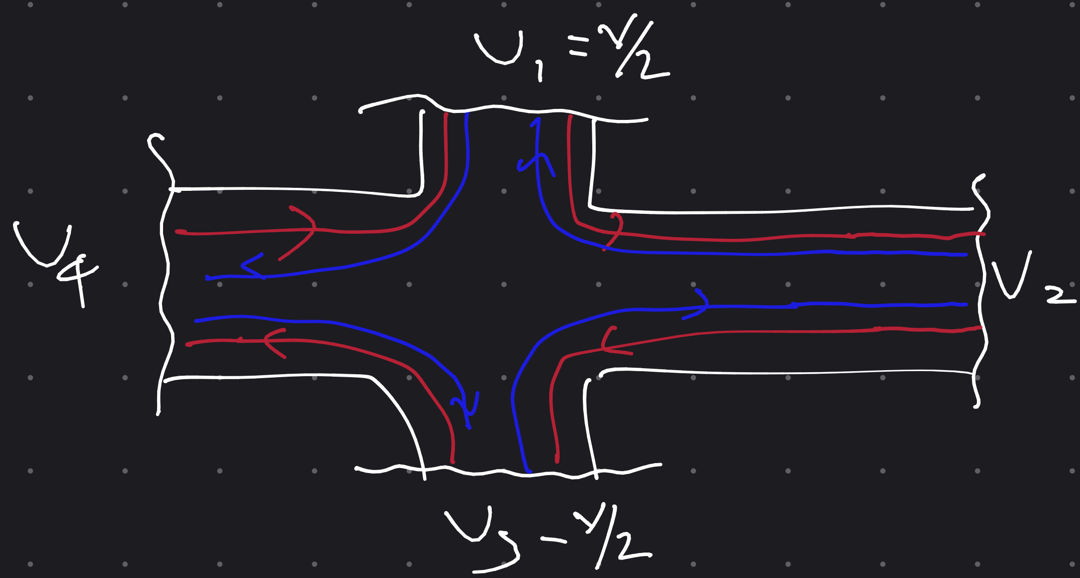
Spin based Landauer Buttiker formalism

$$I_i^{sp} = I_i^{\uparrow} - I_i^{\downarrow}$$

$$I_i^{sp} = \frac{\hbar}{2e}\,\frac{e^2}{h} \sum_{j\neq i,\sigma} \sigma \left[T_{ji}^{\sigma} V_i - T_{ij}^{\sigma} V_j\right]$$

$$I_i^{ch} = I_i^{\uparrow} + I_i^{\downarrow} = \frac{e^2}{h} \sum_{j\neq i,\sigma} \left(T_{ij}^{\sigma} V_i - T_{ij}^{\sigma} V_j\right)$$

Ex.



All are current probes.

$V_2 = V_4 = 0 \qquad V_1 = -V_3 = \frac{V}{2}$

$I_i^{sp} \neq 0 \qquad I_i^{ch} = 0 \rightarrow$ Pure spin current.

In the above setup, find spin current & charge current.

$$I_2^{ch} = 0$$

$$I_2^{sp} = \frac{\hbar}{2e}\,\frac{e^2}{h}\cdot V = \frac{e}{4\pi} V$$