Assignment 13

Chiral vs Helical edge mode

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Quantum Hall effect & Quantum spin Hall effect:

Derivation of equations to be used:

$$I_i = \frac{e^2}{h} \sum_{j \neq i} T_{i \leftarrow j} (V_i - V_j) \quad - (1)$$

Landauer-Buttiker formula for current in i^{th} terminal

Adding the conditions for transport of edge states:

Assume:

Probe 'j' is the neighbour of 'i' in the clockwise direction:

$\downarrow$ flows clockwise

$$T^{\downarrow}_{i \leftarrow j} = 0$$

$$T^{\downarrow}_{j \leftarrow i} = 1$$

$\uparrow$ flows counter clockwise

$$T^{\uparrow}_{i \leftarrow j} = 1$$

$$T^{\uparrow}_{j \leftarrow i} = 0$$

$$\boxed{I_i^c = I_i^{\uparrow} + I_i^{\downarrow} = \frac{e^2}{h} \sum_{j\neq i,\sigma} T_{ij}^{\sigma}(V_i - V_j)} \quad -(1)$$

$$I_i^s = \frac{\hbar}{2e}\left(I_i^{\uparrow} - I_i^{\downarrow}\right) = \frac{\hbar}{2e}\left[\frac{e^2}{h}\sum_{j\neq i}\left((T_{ij}^{\uparrow} - T_{ij}^{\downarrow})(V_i - V_j)\right)\right]$$

Defining $\sigma = +1$ for $\uparrow$ & $\sigma = -1$ corresponding to $\downarrow$,

$$\boxed{I_i^s = \frac{e}{4\pi}\sum_{j\neq i,\sigma}\left[\sigma\, T_{ij}^{\sigma}(V_i - V_j)\right]} \quad -(2)$$

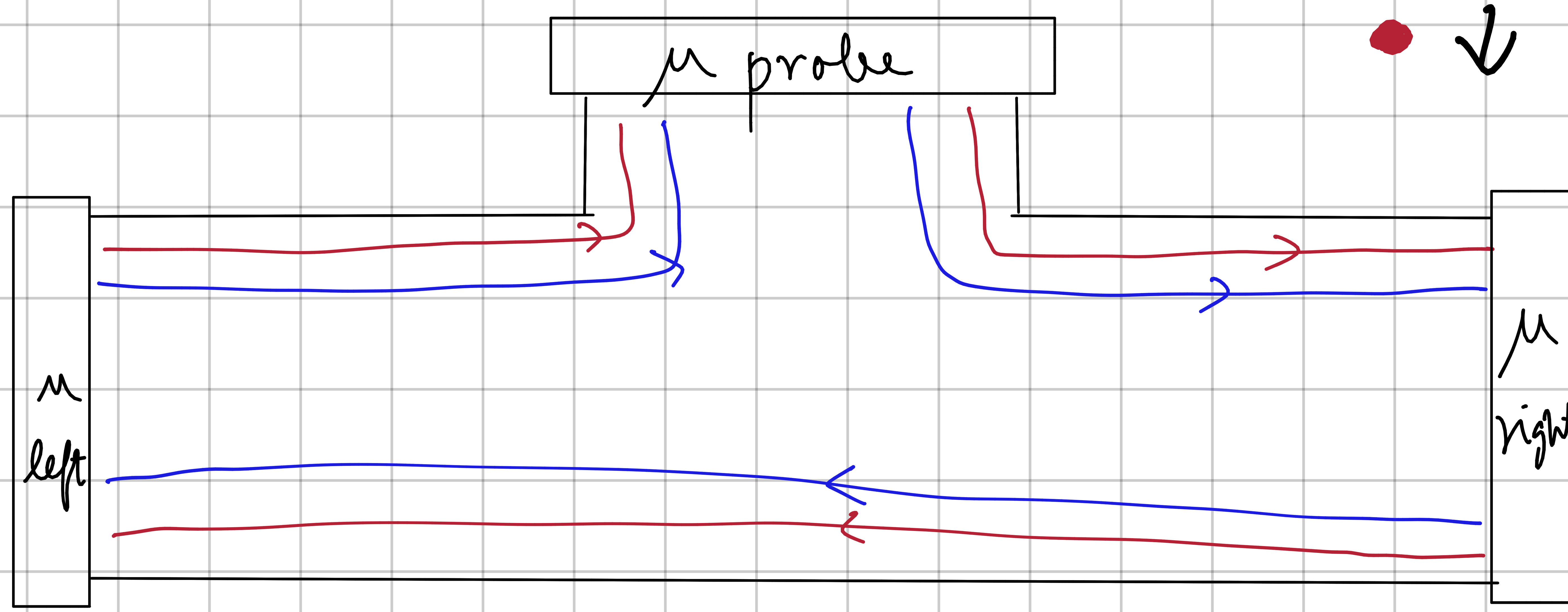
Note: (a) σ doesn't denote spin angular momentum, it denotes helical edge states with different helicity.

(b) There is a quantization $\frac{e}{4\pi}$ of spin current which arises because the edge states are top. protected & top. invariant is the $\mathbb{Z}^2$ invariant.

A] Quantum Hall setup:

● $\uparrow$

● $\downarrow$



$$T^{\sigma}_{r\leftarrow l} = T^{\sigma}_{l\leftarrow p} = 0 = T^{\sigma}_{p\leftarrow r}$$

$$T^{\sigma}_{l\leftarrow r} = T^{\sigma}_{p\leftarrow l} = 1 = T^{\sigma}_{r\leftarrow p}$$

left probe $\rightarrow l$

right probe $\rightarrow r$

top probe $\rightarrow p$

$$I^{C}_{l} = \frac{e^2}{h}\sum_{j\neq l,\sigma} T^{\sigma}_{l\leftarrow j}\,(V_i - V_j) \quad (\text{From Eq. } (1))$$

$$= \frac{e}{h}\Big[\cancelto{0}{T^{\uparrow}_{l\leftarrow p}}(\mu_{left}-\mu_{probe}) + \cancelto{0}{T^{\downarrow}_{l\leftarrow p}}(\mu_{left}-\mu_{probe})$$

$$+ \underbrace{T^{\uparrow}_{l\leftarrow r}}_{=1}(\mu_{left}-\mu_{right}) + \underbrace{T^{\downarrow}_{l\leftarrow r}}_{=1}(\mu_{left}-\mu_{right})\Big]$$

$$= \frac{2e}{h}\left(\mu_{left} - \mu_{right}\right) = \frac{2e^2}{h} V \quad \text{(Given)}$$

So, we can take $V_\ell = V$ and $V_r = 0$.

$$I_r^C = \frac{e}{h}\Big[\underbrace{T^{\uparrow}_{r\leftarrow p}}_{=1}\left(\cancel{\mu_{right}}^{\,0} - \mu_{probe}\right) + \underbrace{T^{\downarrow}_{r\leftarrow p}}_{=1}\left(\cancel{\mu_{right}}^{\,0} - \mu_{probe}\right)$$

$$+ \cancel{T^{\uparrow}_{r\leftarrow \ell}}^{\,0}\left(\mu_{right} - \mu_{left}\right) + \cancel{T^{\downarrow}_{r\leftarrow \ell}}^{\,0}\left(\mu_{right} - \mu_{left}\right)\Big]$$

$$= \frac{2e}{h}\left(-\mu_{probe}\right) = -\frac{2e^2}{h} V$$

$$\Rightarrow V_{probe} = -V.$$

$$\Rightarrow I^C_{probe} = \frac{e^2}{h}\Big[T^{\uparrow}_{p\leftarrow \ell}\overbrace{(V - V)}^{\text{cancels}} + T^{\downarrow}_{p\leftarrow \ell}\overbrace{(V - V)}^{\text{cancels}} + \cancel{T^{\uparrow}_{p\leftarrow r}}^{\,0}(V - 0) + \cancel{T^{\downarrow}_{p\leftarrow r}}^{\,0}(V - 0)\Big]$$

$$= \underline{0}$$

$$I^{S}_{probe} = \frac{e}{4\pi} \sum_{j \neq p, \alpha} \alpha \, T^{\alpha}_{p \leftarrow j} (V_p - V_j) \qquad \text{(From Eq. 2)}$$

$$= \frac{e}{4\pi} \Big[(+1)\, T^{\uparrow}_{p \leftarrow \ell} (V_p - V) + (-1)\, T^{\downarrow}_{p \leftarrow \ell} (V_p - V)$$

cancels out as $T^{\uparrow}_{p \leftarrow \ell} = T^{\downarrow}_{p \leftarrow \ell} = 1$

$$(+1)\, \cancelto{0}{T^{\uparrow}_{p \leftarrow r}} (0 - V_p) + (-1)\, \cancelto{0}{T^{\downarrow}_{p \leftarrow r}} (0 - V_p) \Big]$$

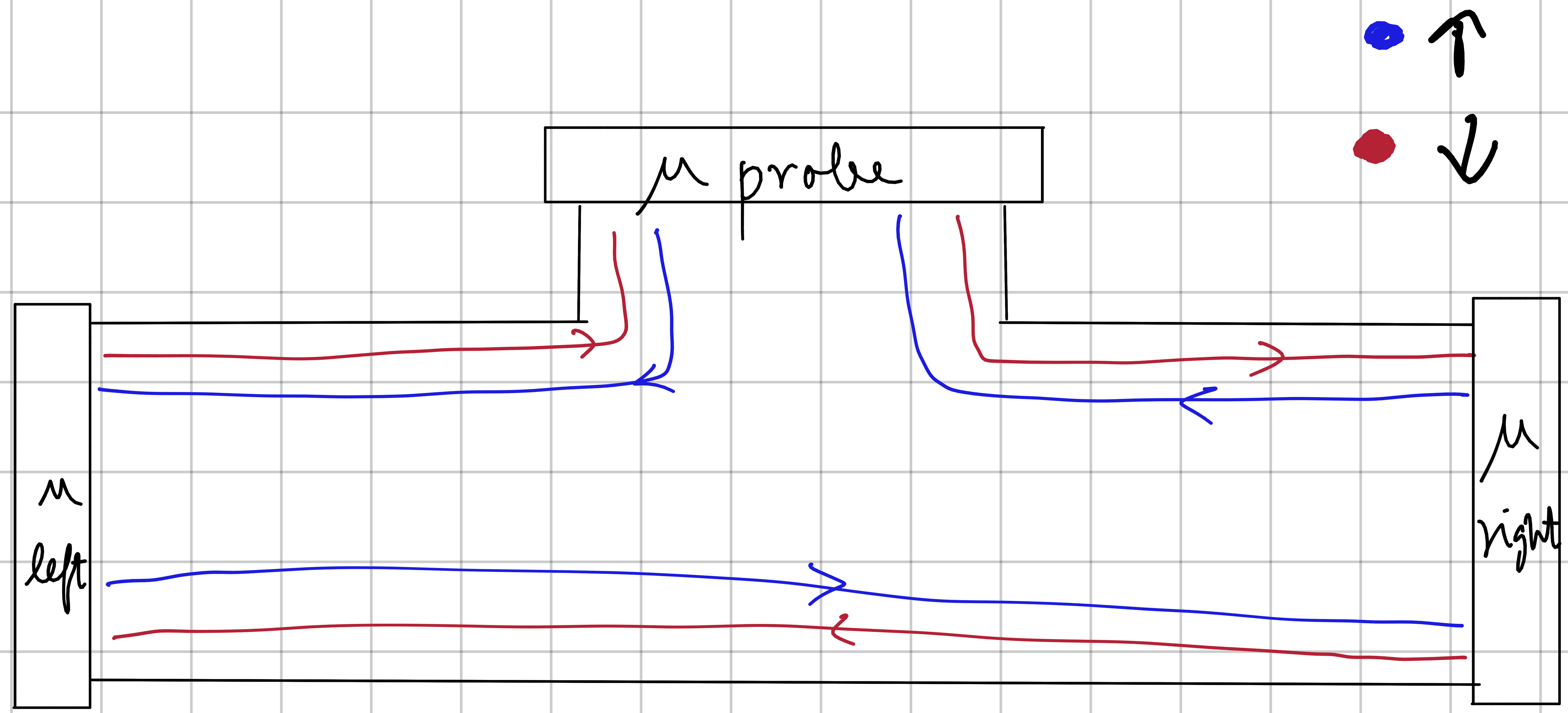
$$= \underline{\underline{0}}.$$

B] Quantum Spin Hall setup:

$$T^{\uparrow}_{r \leftarrow \ell} = T^{\uparrow}_{p \leftarrow r} = 1 = T^{\uparrow}_{\ell \leftarrow p} \qquad T^{\uparrow}_{\ell \leftarrow r} = T^{\uparrow}_{r \leftarrow p} = 0 = T^{\uparrow}_{p \leftarrow \ell}$$

$$T^{\downarrow}_{r \leftarrow \ell} = T^{\downarrow}_{p \leftarrow r} = 0 = T^{\downarrow}_{\ell \leftarrow p} \qquad T^{\downarrow}_{\ell \leftarrow r} = T^{\downarrow}_{r \leftarrow p} = 1 = T^{\downarrow}_{p \leftarrow \ell}$$

left probe → ℓ
right probe → r
top probe → p



$$I_\ell^c = \frac{e^2}{h}\Big[\cancelto{0}{T^{\uparrow}_{\ell\leftarrow r}}(V_\ell - V_r) + \underbrace{T^{\downarrow}_{\ell\leftarrow r}}_{=1}(V_\ell - V_r) + \underbrace{T^{\uparrow}_{\ell\leftarrow p}}_{=1}(V_\ell - V_p) + \cancelto{0}{T^{\downarrow}_{\ell\leftarrow p}}(V_\ell - V_p)\Big] = \frac{e^2}{h}\left[2V_\ell - (V_r + V_p)\right]$$

(From Eq. (1))

We can take $V_r = 0$ as reference:

$$I_\ell^c = \frac{e^2}{h}\left[2V_\ell - V_p\right] = \frac{3}{2}\frac{e^2}{h}V$$

$$I_r^C = \frac{e^2}{h}\Big[\underbrace{T^{\uparrow}_{r\leftarrow l}}_{=1}(V_r - V_l) + \cancelto{0}{T^{\downarrow}_{r\leftarrow l}}(V_r - V_l)$$
$$+\cancelto{0}{T^{\uparrow}_{r\leftarrow p}}(V_r - V_p) + \underbrace{T^{\downarrow}_{r\leftarrow p}}_{=1}(V_r - V_p)\Big]$$

$$= \frac{e^2}{h}\left[-V_l - V_p\right] = -\frac{3}{2}\frac{e^2}{h}V$$

$$\Rightarrow \quad 2V_l - V_p = 3/2\, V$$
$$-V_l - V_p = -3/2\, V$$

$$\Rightarrow \quad 3V_l = 3V \quad \Rightarrow \quad \boxed{V_l = V} \qquad \boxed{V_p = V/2}$$

Cancel

$$I_p^C = \frac{e^2}{h}\Big[\cancelto{0}{T^{\uparrow}_{p\leftarrow l}}(-V/2) + \underbrace{T^{\downarrow}_{p\leftarrow l}}_{=1}(-V/2)$$
$$+\underbrace{T^{\uparrow}_{p\leftarrow r}}_{=1}(V/2) + \cancelto{0}{T^{\downarrow}_{p\leftarrow r}}(V/2)\Big] = 0$$

$$I_p^S = \frac{e}{4\pi}\left[\cancel{T^{\uparrow}_{p\leftarrow l}}^{\to 0}(V_p - V_l) + (-)T^{\downarrow}_{p\leftarrow l}(V_p - V_l) \right.$$
$$\left. T^{\uparrow}_{p\leftarrow r}(V_p - 0) + (-)\cancel{T^{\downarrow}_{p\leftarrow r}}^{\to 0}(V_p - 0) \right] \quad (\text{From Eq. } 2)$$

$$\boxed{I_p^S = \frac{e^2}{4\pi}(V) = \frac{e}{4\pi}\left(\mu_{left} - \mu_{right}\right)}$$
