Midsem assignment:

RISHI PARESH JOSHI

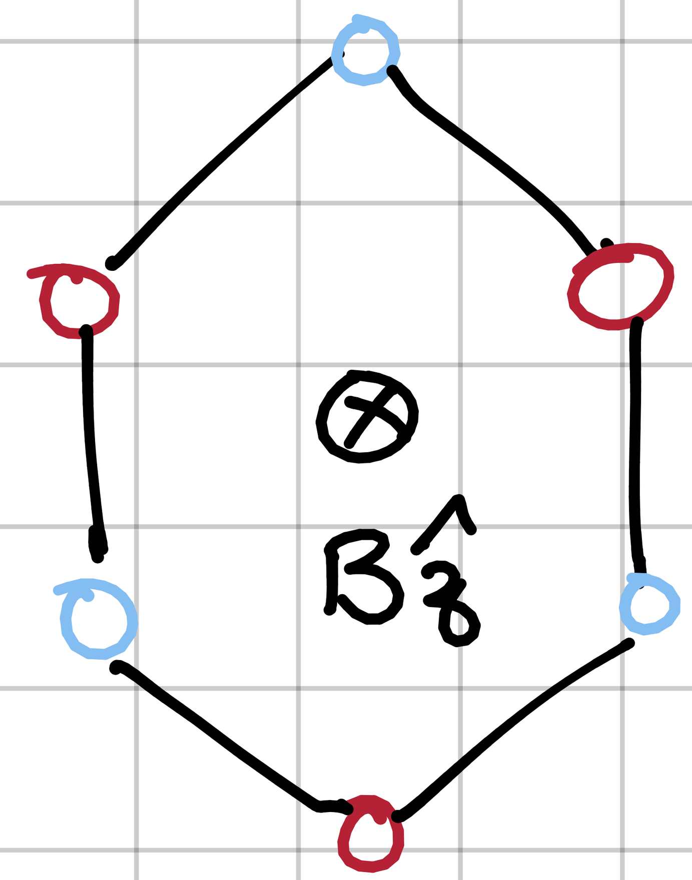
1) QHE in Graphene

2) QHE in 2DEG

QHE – Quantum Hall effect

2DEG – 2 Dimensional electron gas

1)



$$V_{confinement} = M\sigma_z = \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix}$$

How are the Landau levels (LL) modified in presence $V_{confinement} = M\sigma_z$ confinement potential?

Solution:

In presence of magnetic field, $B\hat{z}$, the hamiltonian for the dirac lattice can be written as:

$$H_B = v_F \begin{bmatrix} 0 & \Pi_- \\ \Pi_+ & 0 \end{bmatrix}$$

he standard form deri

where $$\Pi_- = p_x + eA_x - i(p_y + eA_y),$$
$$\Pi_+ = p_x + eA_x - i(p_y + eA_y)$$

or in Landau gauge: $\vec{A} = (-By, 0, 0)$.

On adding the confinement potential, the total hamiltonian is:

$$H_{Total} = H_B + V_{confinement}$$
$$= \begin{bmatrix} M & v_F \hat{\Pi}_- \\ v_F \hat{\Pi}_+ & -M \end{bmatrix}$$

Define ladder operators:

$$\hat{a} = \frac{1}{\sqrt{2\hbar eB}} \hat{\Pi}_-$$

& $$\hat{a}^+ = \frac{1}{\sqrt{2\hbar eB}} \left(p_x - eBy - i\, p_y \right)^+ = \frac{1}{\sqrt{2\hbar eB}} \left(p_x - eBy + i p_y \right)$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar eB}} \hat{\Pi}_+$$

The commutator relations of a & a^+ are:

$$[\hat{a}, \hat{a}^+] = \frac{1}{2\hbar eB} \left[p_x - eBy - ip_y \,,\, p_x - eBy + ip_y \right]$$

$$= \frac{1}{2\hbar eB} \Big(-eByi \underbrace{[y, p_y]}_{i\hbar} + (-i)(-eB) \underbrace{[p_y, y]}_{-i\hbar} \Big)$$

$[\hat{a}, \hat{a}^{+}] = 1 \quad \Rightarrow \hat{a}$ & $\hat{a}^{+}$ are indeed ladder operators.

Since the $\hat{\Pi}_{+}$ is related to $\hat{a}^{+}$ the raising operator and $\hat{\Pi}_{-}$ is related to $\hat{a}$ the lowering operator,

$\psi_n \propto \begin{bmatrix} |n-1\rangle \\ |n\rangle \end{bmatrix}$ as $\hat{\Pi}_{-}$ lowers $|n\rangle$ in the 2^{nd} row &

$\hat{\Pi}_{+}$ raises $|n-1\rangle$ to $|n\rangle$ in the 1^{st} row.

Let $n \neq 0$,

$$\hat{H}\psi_n = \begin{bmatrix} M & v_F\sqrt{2\hbar eB}\,\hat{a} \\ v_F\sqrt{2\hbar eB}\,\hat{a}^{+} & -M \end{bmatrix} \begin{bmatrix} u\,|n-1\rangle \\ v\,|n\rangle \end{bmatrix}$$

$$= \begin{bmatrix} Mu\,|n-1\rangle + v_F\sqrt{2\hbar eB}\,v\sqrt{n}\,|n-1\rangle \\ v_F\sqrt{2\hbar eB}\,u\sqrt{n}\,|n\rangle - M\,|n\rangle \end{bmatrix} = E_n \begin{bmatrix} u\,|n-1\rangle \\ v\,|n\rangle \end{bmatrix}$$

Thus, subtracting $E\psi_n$ on both sides, we get two equations (for each row):-

$$(M-E_n)u + v_F\sqrt{2\hbar eB}\, v\sqrt{n} = 0 \qquad -(1)$$

$$v_F\sqrt{2\hbar eB}\, u\sqrt{n} - (M+E_n)v = 0 \qquad -(2)$$

In vector form, the equation is:

$$\begin{bmatrix} (M-E_n) & v_F\sqrt{2\hbar eB}\sqrt{n} \\ v_F\sqrt{2\hbar eB}\sqrt{n} & -(M+E_n) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

Hence, for u & v not to be trivial, the determinant must be 0.

$$\Rightarrow -(M-E_n)(M+E_n) - (v_F\sqrt{2\hbar eB}\sqrt{n})^2 = 0$$

$$\Rightarrow \quad E_n^2 = M^2 + 2v_F^2\hbar eBn$$

$$\Rightarrow \quad E_{n,\lambda} = \lambda\sqrt{M^2 + 2nv_F^2\hbar eB} \quad , \quad \lambda = \pm 1$$

For $n=0$ case,

$$H\psi_0 = \begin{pmatrix} M & v_F\sqrt{2\hbar eB}\,\hat{a} \\ v_F\sqrt{2\hbar eB}\,\hat{a}^{\dagger} & -M \end{pmatrix}\begin{pmatrix} 0 \\ |0\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ -M|0\rangle \end{pmatrix} = -M\begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}$$

Thus,

$$E_0 = -M$$

The Landau levels are modified as follows:

For $n \geq 1$ $\quad E_{n,\lambda} = \lambda \sqrt{M^2 + 2n \frac{\hbar^2 v_F^2}{l_B^2}}$, $\lambda = \pm 1$ $\quad \left(l_B = \sqrt{\frac{\hbar}{eB}} \right)$

For $n = 0$ $\quad E_0 = -M.$

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2) Explain the non-violition of Pauli-enclusion principle in 2DEG QHE by solving for the wavefunction in symmetric gauge.

Solution:

In the symmetric gauge, for a uniform magnetic field

$\vec{B} = B\hat{z},$

The magnetic vector potential in symmetric gauge is given by:

$$\vec{A} = \frac{B}{2}(-y, x, 0).$$

The Hamiltonian for an electron in 2DEG in presence of magnetic field is given by:

$$H = \frac{1}{2m}\left[(\hat{p}_x - e\hat{A}_x)^2 + (\hat{p}_y - e\hat{A}_y)^2\right]$$

$$= \frac{1}{2m}\left[\left(\hat{p}_x + \frac{eB}{2}\hat{y}\right)^2 + \left(\hat{p}_y - \frac{eB}{2}\hat{x}\right)^2\right]$$

Define: $\hat{\Pi}_x = \hat{p}_x + \frac{eB}{2}\hat{y}$ $\quad \Pi_y = \hat{p}_y - \frac{eB}{2}\hat{x}$

$$l_B = \sqrt{\frac{\hbar}{eB}} \qquad \omega_c = \frac{eB}{m}$$

and the operators $\hat{a}$ & $\hat{a}^+$:

$$\hat{a} = \frac{l_B}{\sqrt{2}\hbar}\left[\hat{p}_x + \frac{eB}{2}\hat{y} - i\left(\hat{p}_y - \frac{eB}{2}\hat{x}\right)\right] = \frac{l_B}{\sqrt{2}\hbar}\left(\hat{\Pi}_x - i\hat{\Pi}_y\right)$$

$$\hat{a}^+ = \frac{l_B}{\sqrt{2}\hbar}\left[\hat{p}_x + \frac{eB}{2}\hat{y} + i\left(\hat{p}_y - \frac{eB}{2}\hat{x}\right)\right] = \frac{l_B}{\sqrt{2}\hbar}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right)$$

and the 'centre guiding' operators:

$$\hat{b} = \frac{l_B}{\sqrt{2}\hbar}\left[\hat{p}_x - \frac{eB}{2}\hat{y} - i\left(\hat{p}_y + \frac{eB}{2}\hat{x}\right)\right] = \frac{l_B}{\sqrt{2}\hbar}\left[\hat{Q}_x - i\hat{Q}_y\right]$$

$$\hat{b}^+ = \frac{l_B}{\sqrt{2}\hbar}\left[\hat{p}_x - \frac{eB}{2}\hat{y} + i\left(\hat{p}_y + \frac{eB}{2}\hat{x}\right)\right] = \frac{l_B}{\sqrt{2}\hbar}\left[\hat{Q}_x + i\hat{Q}_y\right]$$

the commutation relation are given by:

$$[\hat{a}, \hat{a}^+] = \left[\frac{l_B}{\sqrt{2}\hbar}(\hat{\Pi}_x + i\hat{\Pi}_y), \frac{l_B}{\sqrt{2}\hbar}(\hat{\Pi}_x - i\hat{\Pi}_y)\right]$$

$$= \frac{l_B^2}{2\hbar^2}\left[\hat{\Pi}_x + i\hat{\Pi}_y, \hat{\Pi}_x - i\hat{\Pi}_y\right]$$

$$= \left(\frac{l_B^2}{\cancel{2}\hbar^2}\right)(-\cancel{2}i)\left[\hat{\Pi}_x, \hat{\Pi}_y\right] = -i\left[\frac{eB}{2}[\hat{p}_x, \hat{x}] + \frac{eB}{2}[\hat{y}, \hat{p}_y]\right] \times \frac{l_B^2}{\hbar^2}$$

$$= (-i)(i\cancel{\hbar}\,\cancel{eB})\frac{\cancel{\hbar}}{\cancel{eB}\,\cancel{\hbar^2}} = \boxed{1}$$

$$[\hat{b}, \hat{b}^+] = \frac{l_B^2}{2\hbar^2}\left[\hat{Q}_x - i\hat{Q}_y, \hat{Q}_x + i\hat{Q}_y\right] = \frac{l_B^2}{2\hbar^2}\, 2i\left[\hat{Q}_x, \hat{Q}_y\right]$$

$$= \frac{l_B^2}{\hbar^2}\, i\left(\frac{eB}{2}[p_x, x] - \frac{eB}{2}[y, p_y]\right)$$

$$= \frac{l_B^2}{\hbar^2}\, i\,(-i\hbar e B) = 1$$

$$[\hat{a}, \hat{b}] = \frac{l_B^2}{2\hbar^2}\left(\hat{\Pi}_x + i\hat{\Pi}_y, \hat{Q}_x - i\hat{Q}_y\right)$$

$$\left.\begin{array}{l} [\hat{\Pi}_x, \hat{Q}_x] = 0\,, \quad [\hat{\Pi}_x, \hat{Q}_y] = \frac{eB}{2}[\hat{p}_x, \hat{x}] + \frac{eB}{2}[\hat{y}, \hat{p}_y] = 0 \\ [\hat{\Pi}_y, \hat{Q}_x] = -\frac{eB}{2}[\hat{p}_y, y] - \frac{eB}{2}[\hat{x}, \hat{p}_x] = 0\,, \quad [\hat{\Pi}_y, \hat{\Pi}_y] = 0. \end{array}\right\} \Pi, Q \text{ com.}$$

$$\Rightarrow [\hat{a}, \hat{b}] = 0$$

$$\Rightarrow [\hat{a}, \hat{b}^+] = \frac{l_B^2}{2\hbar}[\hat{\Pi}_x + i\hat{\Pi}_y, \hat{Q}_x + i\hat{Q}_y] = 0.$$

This means that these are raising & lowering operators for different quantum numbers.

Also from $\hat{\Pi}$ & $\hat{Q}$ commutators,

$$[\hat{b}, \hat{H}] = 0.$$

The action of $\hat{a}$ & $\hat{a}^{+}$ on states:

$$\hat{a}|n,m\rangle = \sqrt{n}\,|n-1,m\rangle \quad , \quad \hat{a}^{+}|n,m\rangle = \sqrt{n+1}\,|n+1,m\rangle$$

The action of $\hat{b}$ & $\hat{b}^{+}$ on states:

$$\hat{b}|n,m\rangle = \sqrt{m}\,|n,m-1\rangle \quad , \quad \hat{b}^{+}|n,m\rangle = \sqrt{m+1}\,|n,m+1\rangle$$

$$\hat{a}|0,0\rangle = 0 = \hat{b}|0,0\rangle$$

The wavefunctions are found by solving for

$\langle z,\bar{z}|n,m\rangle = \psi_{n,m}(z,\bar{z})$ where,

$z = x+iy$ & $\bar{z} = x-iy$.

* Mark O. Goerbig's notation is confusing so I am using mine.

Finding the lowest landau level:

$$\Pi_x = -i\hbar\partial_x + \frac{eB}{2}y \quad , \quad \Pi_y = -i\hbar\partial_y - \frac{eB}{2}x$$

$$\hat{a}|0,0\rangle = 0$$

$$\frac{l_B}{\sqrt{2}\hbar}\left(-i\hbar\partial_x + \frac{eB}{2}y - i\left(-i\hbar\partial_y - \frac{eB}{2}x\right)\right)\psi_{0,0}(z,\bar{z}) = 0$$

Since $\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$ $\left(\because \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right)$

$$\Rightarrow \frac{l_B}{\sqrt{2}\hbar}\left(-i\hbar\left(\partial_z + \partial_{\bar{z}}\right) - i\hbar\left(\partial_z - \partial_{\bar{z}}\right) - i\frac{eB}{2}(x - iy)\right)\psi_{0,0}(z,\bar{z}) = 0$$

$$\Rightarrow \frac{l_B}{\sqrt{2}\hbar}\left(-2i\hbar\partial_z - i\frac{\hbar}{2l_B^2}\bar{z}\right)\psi_{0,0}(z,\bar{z}) = 0$$

$$\Rightarrow \left(\frac{i}{2\sqrt{2}l_B}\bar{z} + \sqrt{2}\,i\,l_B\partial_z\right)\psi_{0,0}(z,\bar{z}) = 0$$

$$\Rightarrow -i\sqrt{2}\left(l_B\partial_z + \frac{\bar{z}}{4l_B}\right)\psi_{0,0}(z,\bar{z}) = 0$$

$$\Rightarrow \partial_z\psi_{0,0}(z,\bar{z}) = \frac{\bar{z}}{4l_B^2}\psi_{0,0}(z,\bar{z})$$

$$\Rightarrow \quad \ln(\psi_{0,0}) = \frac{-1}{4 l_B^2} \bar{z}\left[\int dz + f(\bar{z})\right]$$

$$\ln(\psi_{0,0}) = \frac{-|z|^2}{4 l_B^2} + \frac{\bar{z} f(\bar{z})}{4 l_B^2}$$

$$\Rightarrow \quad \psi_{0,0}(z, \bar{z}) = e^{\frac{-|z|^2}{4 l_B^2}} g(\bar{z})$$

Since $\psi_{0,0}$ is analytical, $g(\bar{z})$ is analytical. Since $g := g(\bar{z})$, it is anti-analytical. Hence, $g(\bar{z})$ is a contant.

$\psi_{0,0}(z, \bar{z})$ is analytical ~~as~~ it must be differentiable at every point.

$$\Rightarrow \quad \psi_{0,0}(z,\bar{z}) = C\, e^{-\frac{|z|^2}{4\ell_B^2}}$$

Normalizing,

$$\int |\psi_{0,0}|^2 d^2r = |C|^2 \int_0^{2\pi} d\phi \int_0^{\infty} r\,dr \left(e^{-\frac{r^2}{4\ell_B^2}}\right)^2$$

$$= 2\pi |C|^2 \ell_B^2 = 1$$

$$\Rightarrow |C| = \frac{1}{\sqrt{2\pi \ell_B^2}}$$

$$\Rightarrow \boxed{\psi_{0,0} = \frac{1}{\sqrt{2\pi\ell_B^2}}\, e^{-\frac{|z|^2}{4\ell_B^2}}}$$

For $n=0$ $m>0$, we apply $b^{\dagger}$ on $\psi_{0,0}(z,\bar{z})$.

$$\hat{b}^{\dagger} \rightarrow \frac{l_B}{\sqrt{2}\hbar}\left(-i\hbar\partial_x - \frac{\hbar}{2l_B^2}y + i\left[-i\hbar\partial_y + \frac{\hbar}{2l_B^2}x\right]\right)$$

$$= \frac{l_B}{\sqrt{2}\hbar}\left(-i\hbar\left(\partial_z+\partial_{\bar{z}}\right) + i\hbar\left(\partial_z - \partial_{\bar{z}}\right) + \frac{i\hbar}{2l_B^2}(x+iy)\right) = \frac{l_B}{\sqrt{2}\hbar}\left(-i\hbar\partial_{\bar{z}} + \frac{i\hbar}{2l_B^2}z\right)$$

$$\Rightarrow i\sqrt{2}\left(\frac{z}{4l_B} - l_B\,\partial_{\bar{z}}\right)\psi_{0,0} = \psi_{0,1}$$

$$\Rightarrow \boxed{\left[i\sqrt{2}\left(\frac{z}{4l_B} - l_B\,\partial_{\bar{z}}\right)\right]^m \frac{1}{\sqrt{2\pi l_B^2}}\, e^{-\frac{|z|^2}{4l_B^2}} = \psi_{0,m}}$$

For higher landau levels,

$$\hat{a}^{+} = \frac{l_B}{\sqrt{2}\hbar}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right) \to i\sqrt{2}\left(l_B \partial_{\bar{z}} + \frac{z}{4l_B}\right)$$

$$\left(\hat{a}^{+}\right)^{n}\left(\hat{b}^{+}\right)^{n}|0,0\rangle$$

$$\downarrow$$

$$\Psi_{n,m}(z,\bar{z}) = \left[i\sqrt{2}\left(l_B \partial_{\bar{z}} + \frac{z}{4l_B}\right)\right]^{n}\left[i\sqrt{2}\left(\frac{z}{4l_B} - l_B \partial_{\bar{z}}\right)\right]^{m} \frac{1}{\sqrt{2\pi l_B^2}} e^{\frac{-|z|^2}{4l_B^2}}$$

Using special functions, the solution can be written as:

$$\Psi_{n,m} = \sqrt{\frac{n!}{(m+n)!}} \times \frac{1}{\sqrt{2\pi l_B^2}} \left(\frac{z}{l_B}\right)^{m} L_n^{m}\left(\frac{|z|^2}{2l_B^2}\right) e^{\frac{-|z|}{4l_B^2}}$$

where L_m^n is the Laguerre polynomial.

The pauli exclusion principle is naturally accounted for by different m for same n. There are many m for the same n and contributes to the peaks in the density of states at each landau level, n.