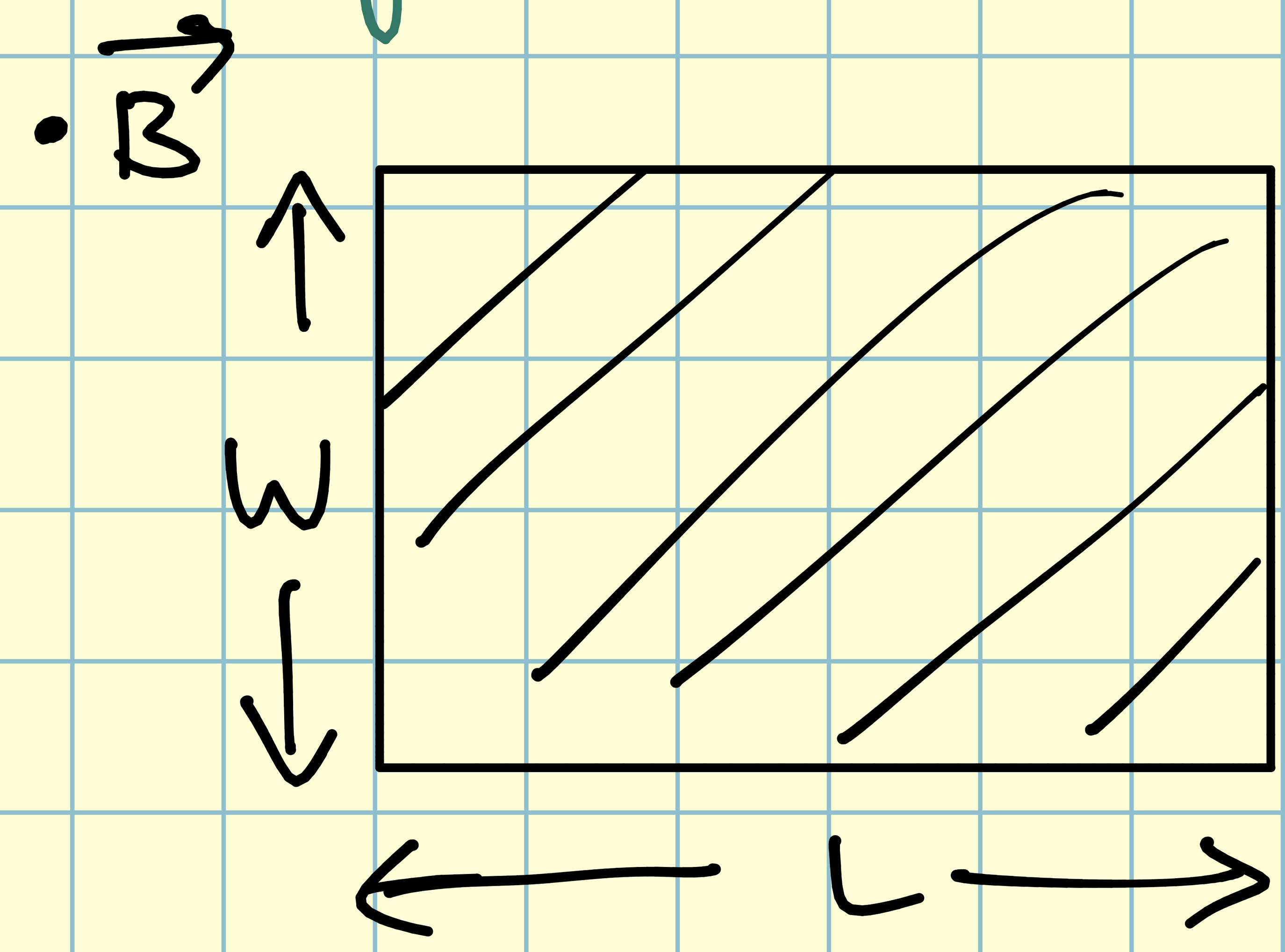
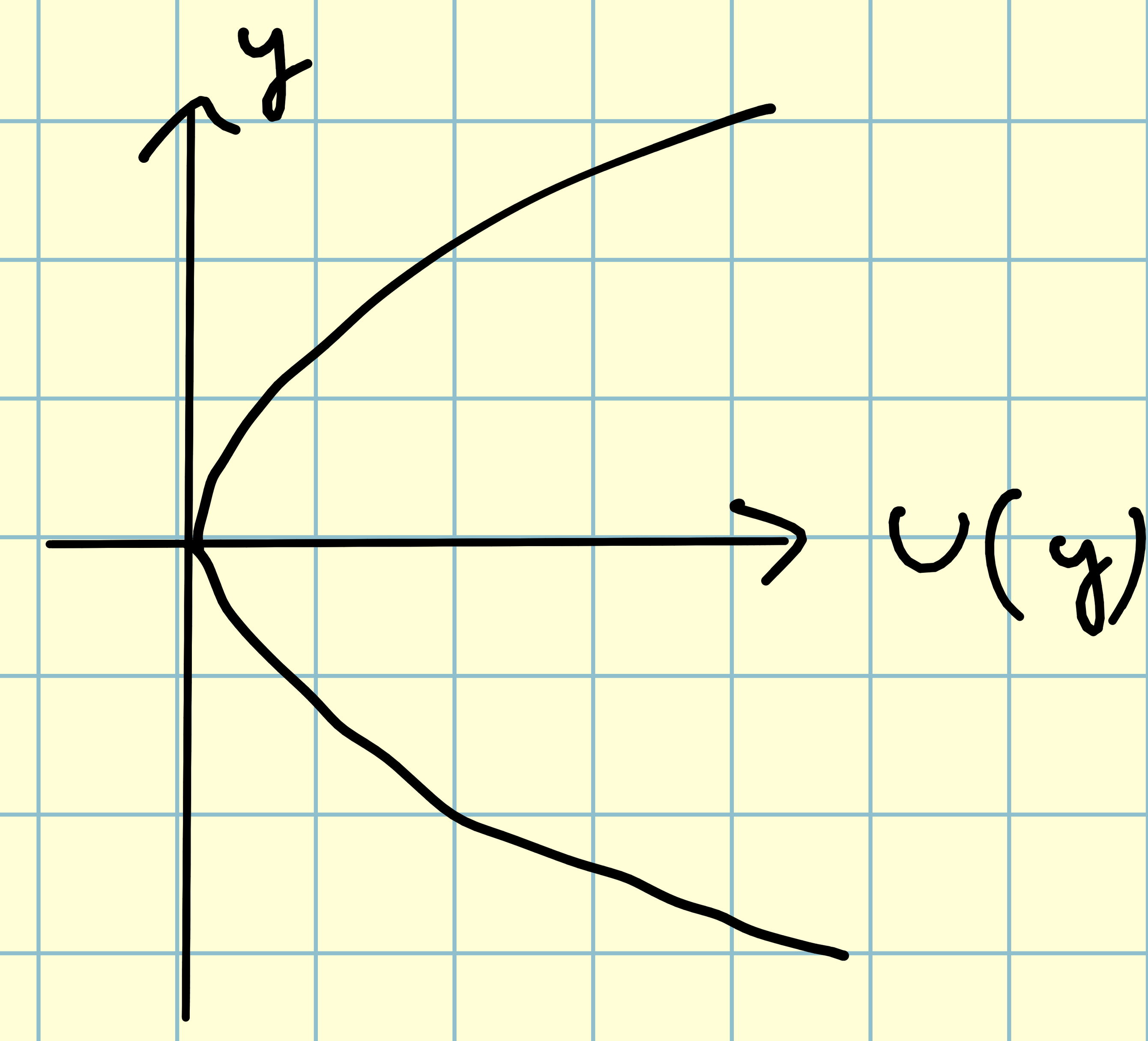
SECTION 10

QUANTUM HALL EFFECT

(FULLY QUANTUM)

Fully Quantum Picture of QHE:





$L > W$, $\vec{B}$ in z-dir^n

① There are confined electrons in y dir^

$U(y) \neq 0$

@ $\vec{B} = 0$,

$$\left[E_s + \frac{p_x^2}{2m^*} + \frac{p_y^2}{2m^*} + \frac{1}{2} m^* \omega_0^2 y^2\right] \psi(x,y) = E\, \psi(x,y)$$

Here, E_s is the subband energy, i.e. E_{2a} assuming only a single subband is occupied in z dir^n:

$$E_{21} = E_s$$

Assuming a plane wave solution:

$$\psi(x,y) = e^{ikx}\chi(y)$$

$$\left[E_s + \frac{\hbar^2 k^2}{2m^*} + \frac{p_y^2}{2m} + \frac{1}{2} m\omega_0^2 y^2\right] e^{ikx}\chi(y) = E\, e^{ikx}\chi(y)$$

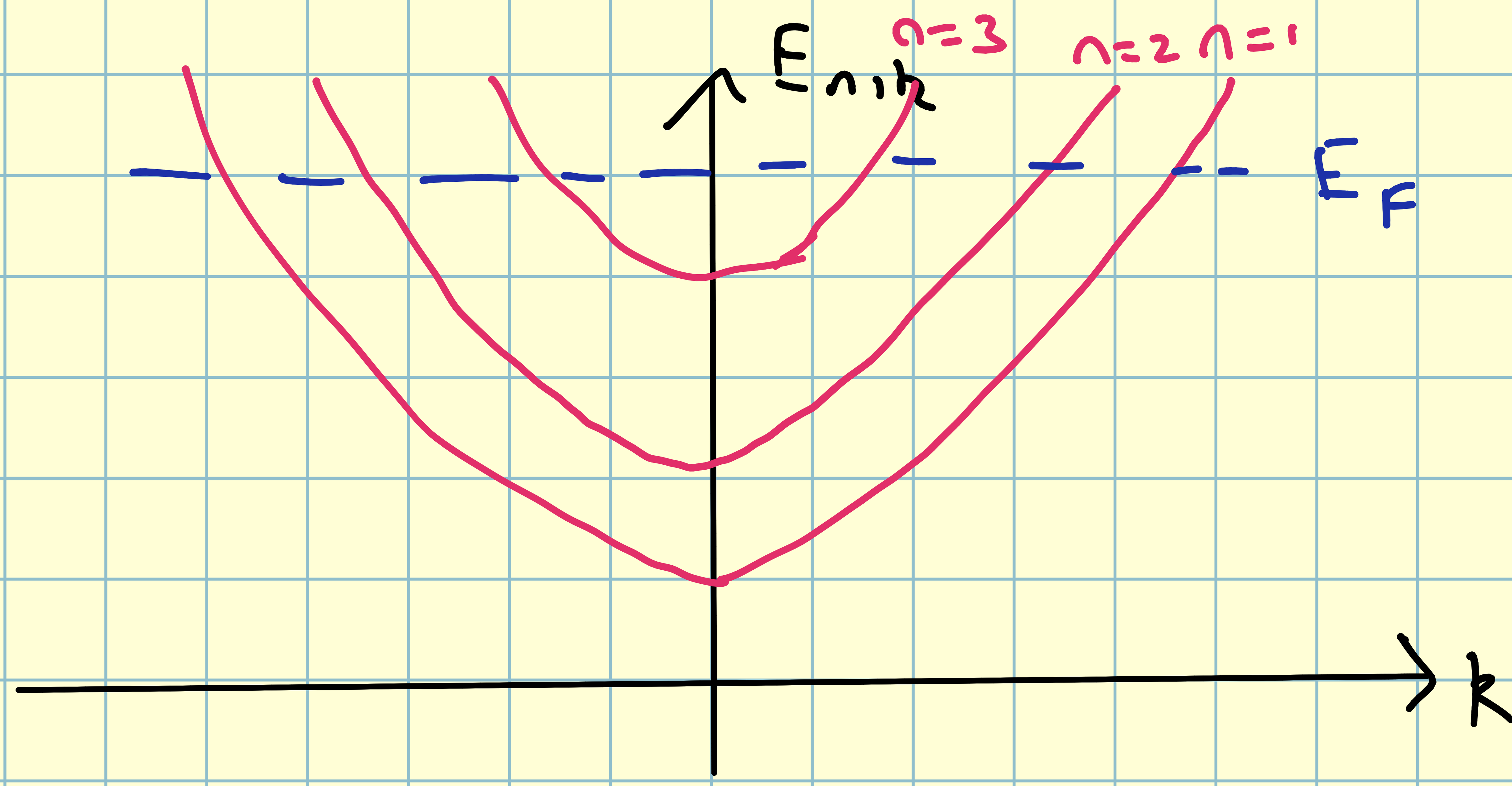
$$\left[\frac{p_y^2}{2m^*} + \frac{1}{2} m\omega_0^2 y^2\right]\chi(y) = \varepsilon\,\chi(y)$$

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$$

$$E_{n,k} = E_s + \frac{\hbar^2 k^2}{2m^*} + \left(n + \frac{1}{2}\right)\hbar\omega_0$$

The velocity of electrons:

$$v_{n,k} = \frac{1}{\hbar}\frac{dE}{dk} = \frac{\hbar k}{m^*}$$



② Unconfined electrons in non-zero $\vec{B}$-field

$$\left[E_s + \frac{1}{2m^*}\left(i\hbar\vec{\nabla} + eA\right)^2\right]\psi(x,y) = E\,\psi(x,y)$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\frac{1}{2m^*}\left(i\hbar\vec{\nabla} + eA\right)^2\psi(x,y) = \frac{1}{2m^*}\left[-\hbar^2\nabla^2\psi + i\hbar\vec{\nabla}(e\vec{A})\psi + e\vec{A}\cdot i\hbar\vec{\nabla}\psi + e^2A^2\psi\right]$$

$$= E'\psi$$

$$\frac{1}{2m^*}\left[-\hbar^2\nabla^2\psi + i\hbar\,\vec{\nabla}(e\vec{A})\psi + e\vec{A}\cdot i\hbar\vec{\nabla}(\psi) + e^2A^2\psi\right] = E'\psi$$

$$\Rightarrow \frac{1}{2m^*}\left[-\hbar^2\nabla^2\psi + \left(i\hbar e\frac{\partial A_x}{\partial x} + i\hbar e\frac{\partial A_y}{\partial y}\right)\psi + 2i\hbar e\left(A_x\frac{\partial \psi}{\partial x} + A_y\frac{\partial \psi}{\partial y}\right)\right.$$

$$\left. + e^2(A_x^2 + A_y^2)\psi\right] = E'\psi$$

Gauge invariance:-

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

Since,

$$\vec{\nabla}\times\vec{\nabla}\chi = 0,$$ where χ is a scalar:

$$\vec{B} = B\hat{z} \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

Cases arise may be:

1. $\vec{A} = (-yB,\ 0,\ 0)$
2. $\vec{A} = (0,\ xB,\ 0)$

} Landau gauge

3. $\vec{A} = \left(-\frac{1}{2}yB,\ \frac{1}{2}xB,\ 0\right) \rightarrow$ Symmetric gauge

In the first gauge

$$A_x = -yB\ ,\ A_y = 0,$$

$$\left[E_s + \frac{1}{2m^*}\left(-\hbar^2\nabla^2\psi + i\hbar eB\,\cancelto{0}{\frac{\partial y}{\partial x}} - 2i\hbar e\, yB\frac{\partial \psi}{\partial x} + e^2 y^2 B^2\psi\right.\right] = E'\psi$$

Consider the solution :

$$\psi(x,y) = \frac{1}{\sqrt{L}} e^{ikx} \chi(y)$$

$$\Big[E_s e^{ikx} \chi(y) + \frac{\hbar^2 k^2}{2m^*} \chi(y) e^{ikx} - \frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial y^2} \big(\chi(y)\big) e^{ikx}$$

$$- \frac{2i\hbar}{2m^*} eBy \cdot ik \cdot \chi(y) e^{ikx} + \frac{e^2 B^2 y^2}{2m^*} e^{ikx} \chi(y) \Big]$$

$$= E e^{ikx} \cdot \chi(y)$$

$$\Big[E_s - \frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial y^2} + \frac{1}{2m^*} (eBy + \hbar k)^2 \Big] \chi(y) = E \chi(y)$$

$$\left[E_s - \frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial y^2} + \frac{e^2B^2}{2m^*}\left(y + \frac{\hbar k}{eB}\right)^2\right]\chi(y) = E\,\chi(y)$$

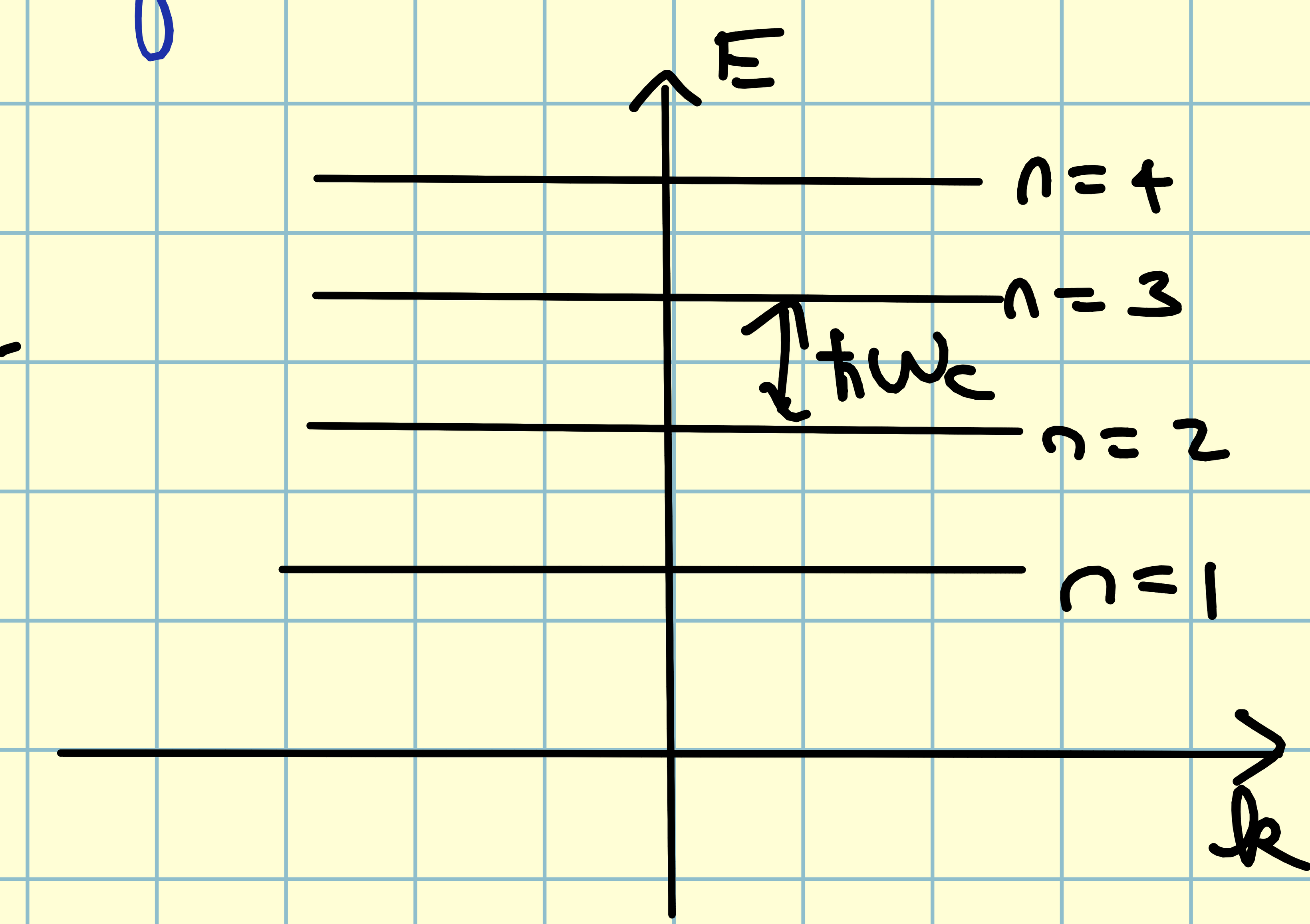
$$\left[E_s - \frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m^*\omega_c^2\left(y + y_k\right)^2\right]\chi(y) = E\,\chi(y)$$

$$y_k = \frac{\hbar k}{eB} \quad , \quad \omega_c = \frac{eB}{m^*}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c + E_s$$

Summary of case 2:

Unconfined electrons in finite B field with V=0.



$$E_n = E_s + \left(n + \frac{1}{2}\right)\hbar\omega_c \quad ; \; n = 0, 1, 2, \ldots$$

$$\omega_c = \frac{eB}{m^*}$$

$$\chi_{n,k}(y) = u_n(q + q_k) \quad \text{with}$$

$$q = \sqrt{\frac{m\omega_c}{\hbar}}\, y \quad , \quad q_k = \sqrt{\frac{m\omega_c}{\hbar}}\, y_k$$

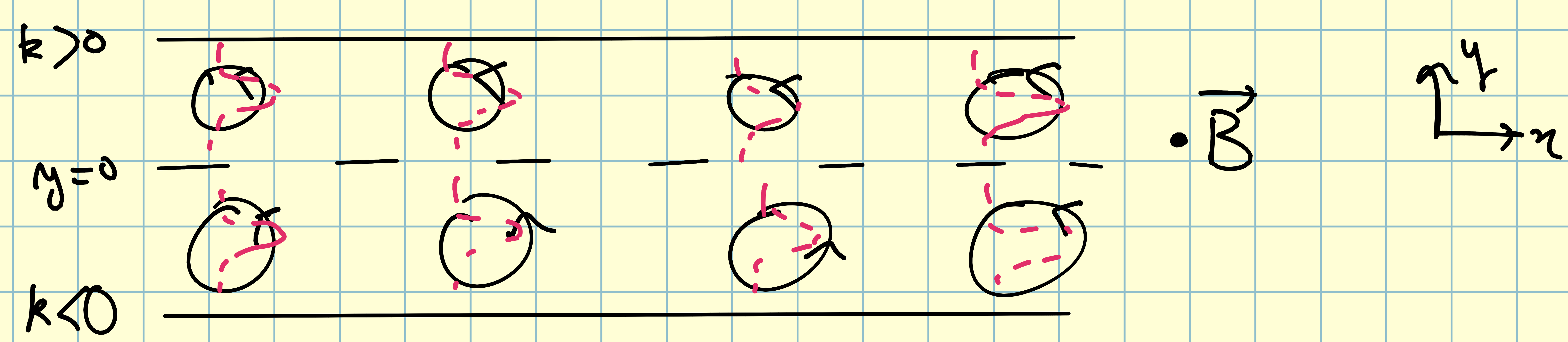
$$v_{n,k} = \frac{1}{\hbar}\frac{dE_n}{dk} = 0$$

Area Occupied by an electronic state:

$$\delta S = \frac{\Phi_0}{g_s B}$$

Radial extent of a particular state:

$$r = \sqrt{\delta S} = \sqrt{\frac{\Phi_0}{g_s B}}$$



Case 3:-

Confined electrons ($U \neq 0$) and finite magnetic field $B \neq 0$

$$\vec{A} = -\hat{x} B y$$

Schrodinger eqn with plane wave solution:

$$\psi(x,y) = \frac{1}{\sqrt{L}} e^{ikx} \chi(y)$$

$$\left[E_s - \frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial y^2} + \frac{\hbar^2}{2m^*} \left(\frac{eBy}{\hbar} + k \right)^2 + \frac{1}{2} m^* \omega_0^2 y^2 \right] \chi(y) = E \chi(y)$$

$$\frac{\hbar^2}{2m^*} \cdot \frac{1}{l_B^2} = \frac{\hbar^2}{2m^*} \frac{eB}{\hbar} = \frac{\hbar eB}{2m^*} = \frac{\hbar \omega_c}{2} \qquad l_B = \frac{\hbar}{eB}$$

Divide the eqn by $\frac{\hbar\omega_c}{2}$

$$\left[\frac{2E_s}{\hbar\omega_c} - l_B^2\frac{\partial^2}{\partial y^2} + \Big(\underbrace{l_B\frac{eBy}{\hbar}}_{y/l_B} + l_B k\Big)^2 + \frac{m^*\omega_0^2 y^2}{\hbar\omega_c}\right]\chi(y) = \frac{2E}{\hbar\omega_c}\chi(y)$$

$$\left[\frac{2E_s}{\hbar\omega_c} - l_B^2\frac{\partial^2}{\partial y^2} + \left(\frac{y}{l_B} + l_B k\right)^2 + \frac{m^{*2}\omega_0^2 y^2 l_B^2}{\hbar^2}\right]\chi(y) = \frac{2E}{\hbar\omega_c}\chi(y)$$

Let $\quad K = k\,l_B, \qquad \eta = \frac{y}{l_B}, \qquad \frac{m^{*2}\omega_0^2 y^2 l_B^2}{\hbar^2} = \frac{m^{*2}\omega_0^2\eta^2 l_B^4}{\hbar^2} = \frac{m^{*2}\omega_0^2\eta^2}{e^2B^2}$

$$= \frac{\omega_0^2}{\omega_c^2}\eta^2$$

$$\left[\frac{2E_s}{\hbar\omega_c} - \frac{\partial^2}{\partial\eta^2} + (\eta+K)^2 + \alpha^2\eta^2\right]\chi(\eta) = \frac{2E}{\hbar\omega_c}\chi(\eta)$$

$$(\eta+K)^2+\alpha^2\eta^2 = \eta^2+K^2+2\eta K+\alpha^2\eta^2$$

$$= (1+\alpha^2)\left(\eta+\frac{K}{1+\alpha^2}\right)^2 + K^2 - \frac{K^2}{1+\alpha^2}$$

$$= (1+\alpha^2)\left(\eta+\frac{K}{1+\alpha^2}\right)^2 + \frac{\alpha^2 K^2}{1+\alpha^2}$$

$$\left[-\frac{d^2}{d\eta^2} + (1+\alpha^2)(\eta+\eta_c)^2\right]\chi(\eta) = \left[\frac{2E}{\hbar\omega_c} - \frac{\alpha^2K^2}{1+\alpha^2} - \frac{2E_s}{\hbar\omega_c}\right]\chi(\eta)$$

This is a shifted harmonic oscillator

$$\left[-\frac{d^2}{d\eta^2} + (1+\alpha^2)(\eta+\eta_c)^2\right]\chi(\eta) = \left[\frac{2E}{\hbar\omega_c} - \frac{\alpha^2k^2}{1+\alpha^2} - \frac{2E_s}{\hbar\omega_c}\right]\chi(\eta)$$

$$E_n = \frac{\alpha^2 K^2}{1+\alpha^2} \cdot \frac{\hbar\omega_c}{2} + E_s + \left(n+\frac{1}{2}\right)\hbar\Omega$$

$$\Omega^2 = \omega_c^2 + \omega_0^2$$

$$K^2 = k^2 l_B^2$$

$$\alpha^2 = \frac{\omega_0^2}{\omega_c^2}$$

$$\frac{\alpha^2 K^2}{1+\alpha^2} = \frac{\omega_0^2}{\omega_0^2+\omega_c^2} \cdot k^2 \left(\frac{\hbar}{eB}\right)^2 \times \frac{\omega_c^2}{\omega_c^2} = \frac{\hbar^2 k^2}{2m^*} \times \left(\frac{\omega_0}{\Omega}\right)^2$$

The Landau levels are no longer constant and independent of 'k' but are dependent on 'n'.

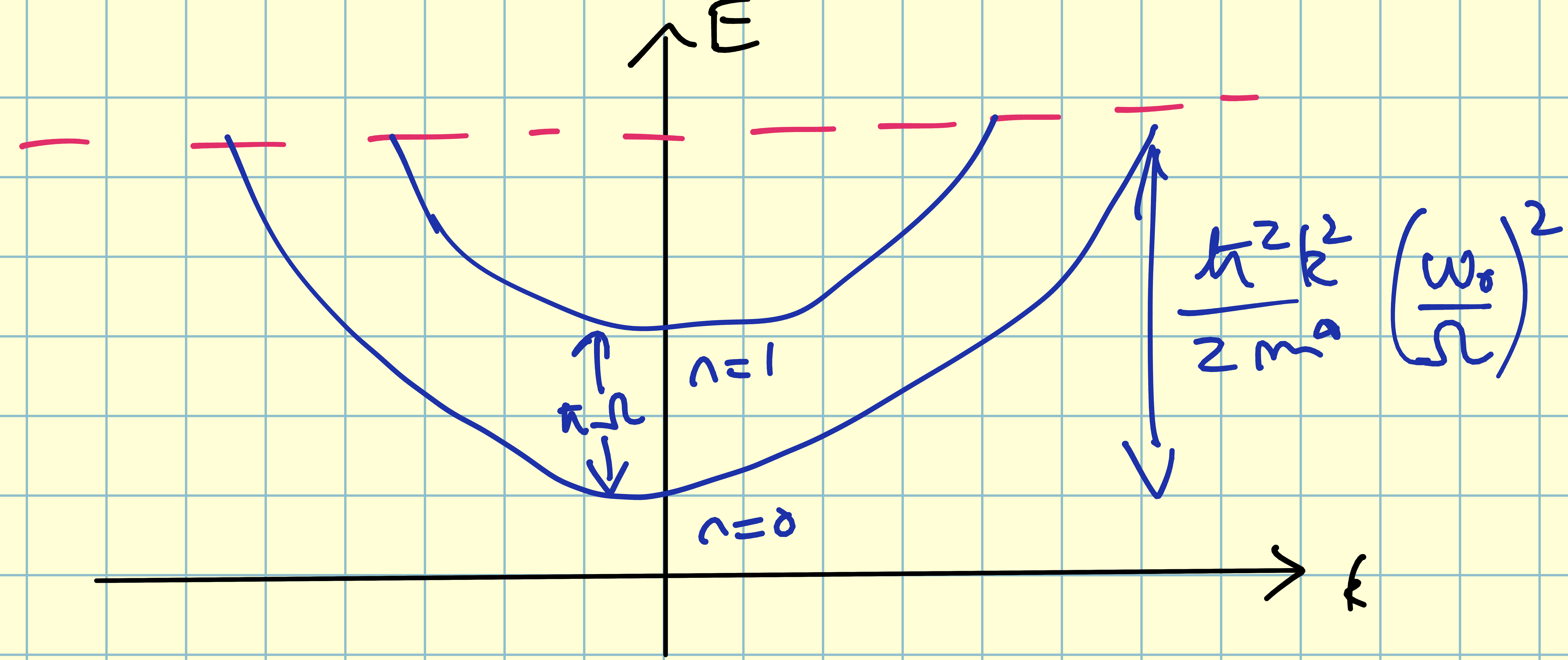
$$v_{n,k} = \frac{1}{\hbar}\frac{dE_{n,k}}{dk} = \frac{\hbar k}{m^*}\frac{\omega_0^2}{\Omega^2}$$

Eigenfunctions :

$$\psi_{n,k} = e^{ikx}\chi_n(\eta+\eta_c) = e^{ikx}\chi_n\left(\frac{y}{l_B}+\frac{y_{n,k}}{l_B}\right)$$

$$\eta_c = \frac{K}{1+\alpha^2} = \frac{k l_B \omega_c^2}{\omega_0^2+\omega_c^2} = \frac{y_{n,k}}{l_B}$$

$$y_{n,k} = \frac{k l_B^2 \omega_c^2}{\Omega^2} = k L_B^2 \qquad L_B^2 = \frac{l_B^2\omega_c^2}{\Omega^2}$$



The eigenfunctions are centred at $Y_{n,k} \sim k_n(E_F)\cdot l_B^2$

$$Y_{n,k} = \pm \sqrt{\frac{2m^*\Omega^2}{\hbar^2\omega_0^2}\left(E_F - E_s\left(n+\frac{1}{2}\right)\hbar\Omega\right.} \cdot l_B^2$$

If $k>0$ then $Y_{n,k}>0$

& $Y_{0k} > Y_{1k} > Y_{2k} > \dots$

if $k<0$, then $Y_{n,k}<0$

& $Y_{0,-k} > y_{1,-k} > Y_{2,-k} \dots$

$$\chi_n\left(\frac{y+y_{n,k}}{l_B}\right) \sim e^{\left(-\frac{1}{2}\frac{(y+y_{n,k})^2}{l_B^2}\right)}$$

$\Rightarrow$ States are localized in y dir^n.

In a realistic potential:

$$V(y) = \begin{cases} 0 & \text{, for } |y| < b \quad \text{(bulk)} \\ \frac{1}{2} m^* \omega_0^2 (|y| - b)^2 & \text{, for } |y| > b \end{cases}$$

Bulk modes: $\quad E_n = \left(n+\frac{1}{2}\right)\hbar\omega_c \qquad v_n = 0$

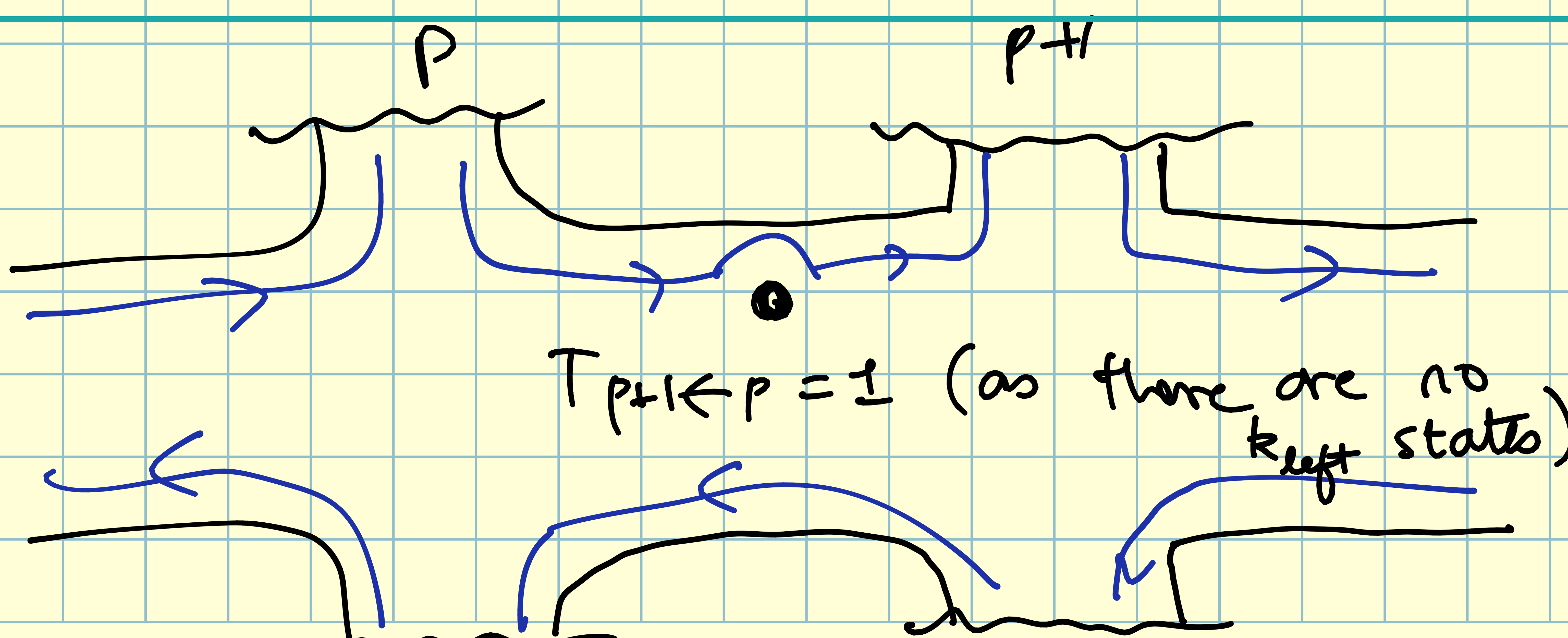
Edge modes: $\quad E_n = E_s + \left(n+\frac{1}{2}\right)\hbar\Omega + \frac{\hbar^2 k^2}{2 m_B}$

$$m_B = m^* \frac{\Omega^2}{\omega_0^2} \; ; \; v_{n,k} = \frac{\hbar k}{m_B}$$

$$\langle y_n \rangle - \langle y_{n-1} \rangle = \text{const} \times \Omega \left(1 - \frac{1}{2} \cdot \frac{E_F - E_s - (n+\frac{1}{2})\hbar\Omega}{E_F - E_S - (n-\frac{1}{2})\hbar\Omega} \right) \cdot L_B^2$$

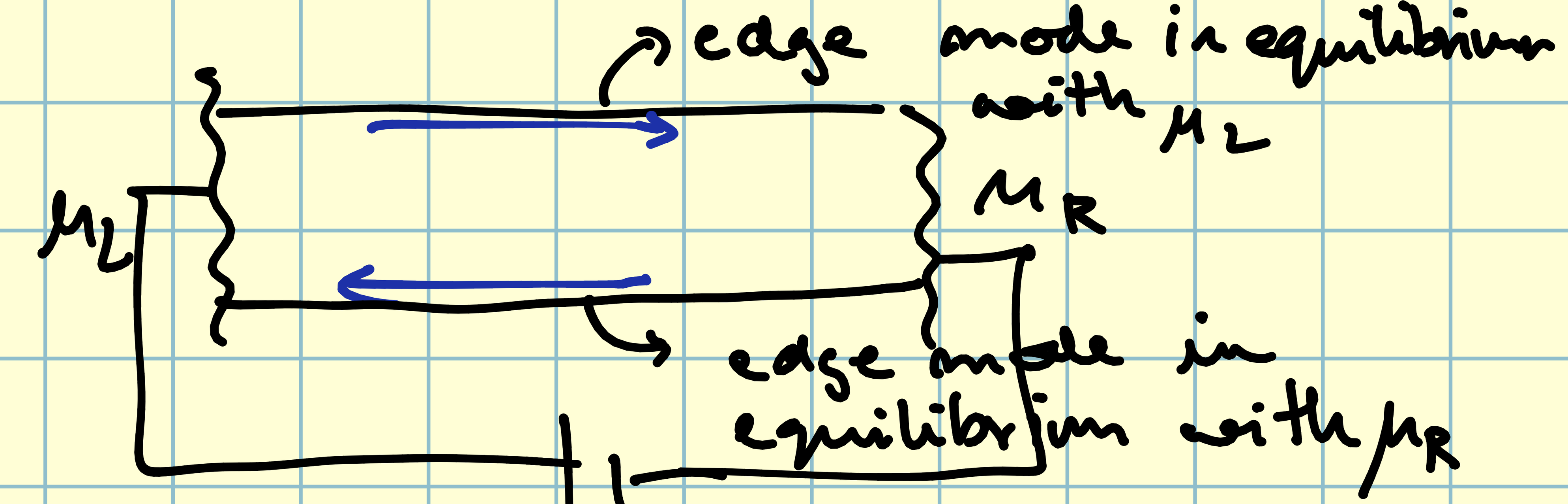
$$\Omega \cdot L_B = \frac{l_B^2 \omega_c^2}{\Omega^2} \rightarrow \text{independent of } B.$$

$$\langle y_{n,k} \rangle - \langle y_{n,-k} \rangle = 2 \text{ const.} \times \Omega \cdot l_B^2 \sim B$$



Edge mode transport is robust to impurity scattering.

The flow of current is independent of scatterers (upto a high extent). This is called chiral transport.

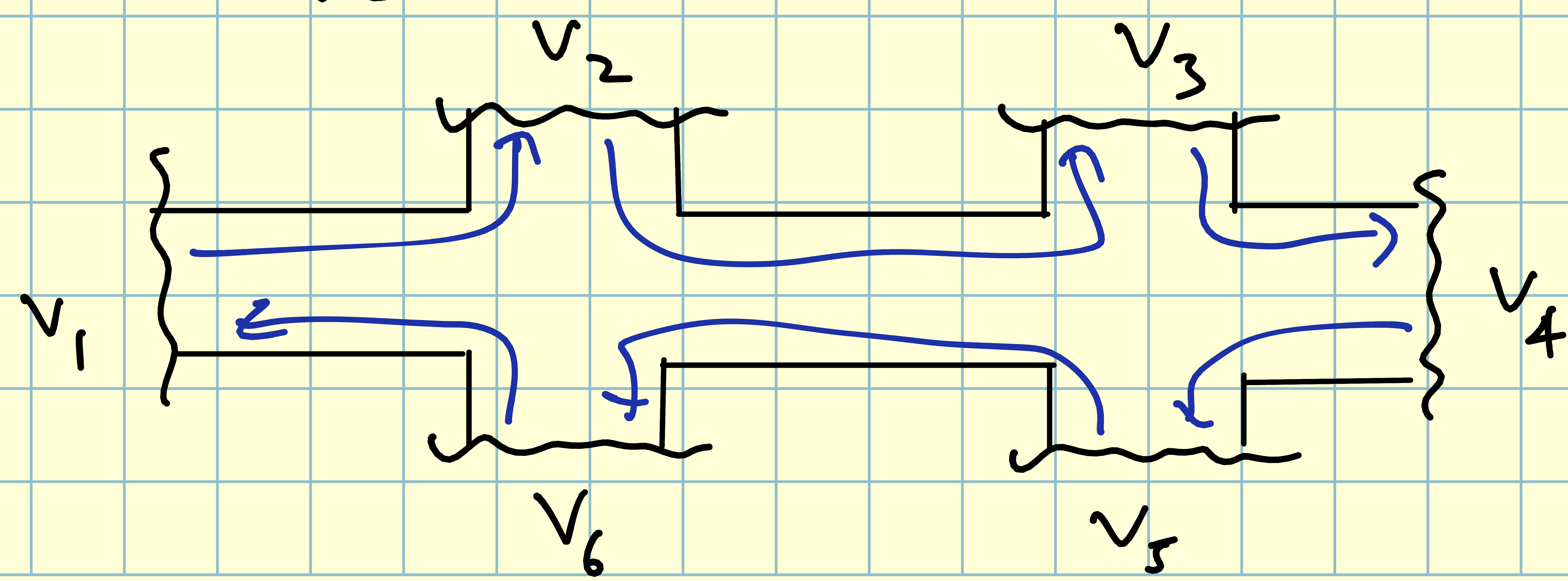


$$I = \frac{2(-e)}{h}\sum_{n=1}^{M}\int_{\mu_R}^{\mu_L} dE = \frac{2(-e)}{h}\cdot M(\mu_L - \mu_R)$$

$$I = \frac{2e^2}{h}\cdot MV$$

Ex.

(1)



2, 3, 5, 6 are voltage probes.

1, 4 are current probes.

'N' edge modes below E_F

$$I_p = \sum_{\substack{q \\ q \neq p}} G_{pq} (V_p - V_q)$$

$$G_{21} = \frac{2e^2}{h} N = G_{32} = G_{43} = G_{54} = G_{65} = G_{16} = \overline{N}$$

The rest of G_{pq}'s are 0.

$$I_1 = \overline{N} (V_1 - V_6)$$

$$I_2 = 0 = \bar{N}(V_2 - V_1) \Rightarrow V_2 = V_1$$

$$I_3 = 0 \Rightarrow V_3 = V_2$$

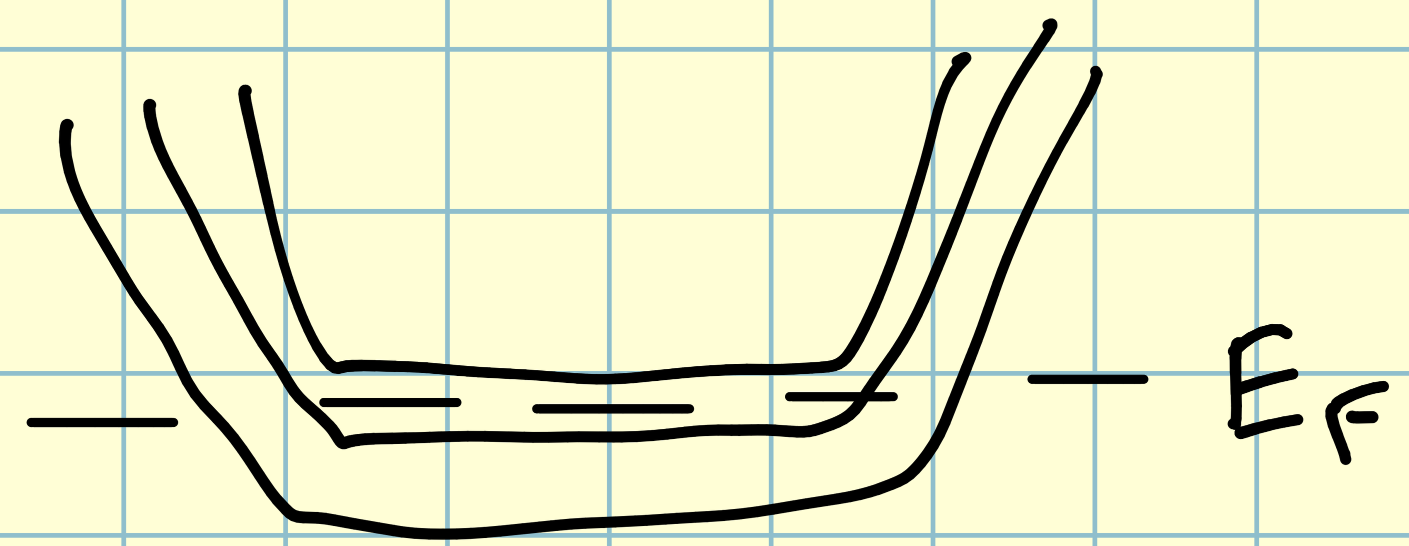
$$I_4 = \bar{N}(V_4 - V_3)$$

$$I_5 = 0 \Rightarrow V_5 = V_4$$

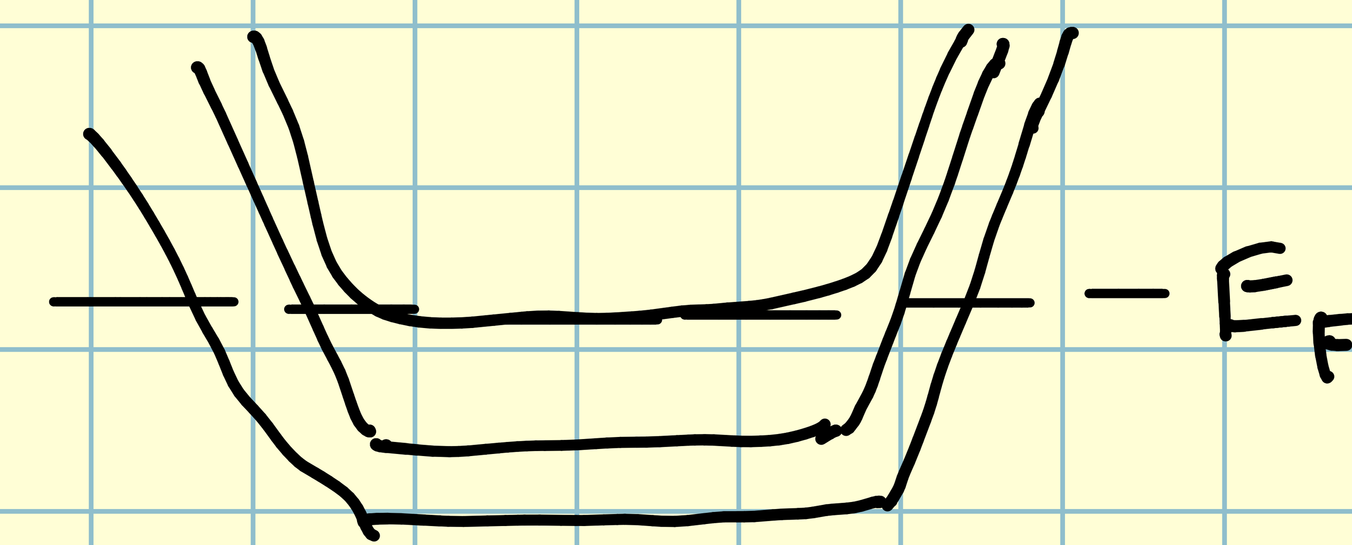
$$I_6 = 0 \Rightarrow V_6 = V_5$$

$$R_H = \frac{V_2 - V_6}{I_1} = \frac{V_1 - V_6}{\bar{N}(V_1 - V_6)} = \frac{h}{2e^2 \bar{N}}$$

$$R_2 = \frac{V_2 - V_3}{I_1} = 0$$

② 

Case - 1



Case - 2

$$\left(N+\frac{1}{2}\right)\hbar\omega_c < E_F < \left(N+\frac{3}{2}\right)\hbar\omega_c$$

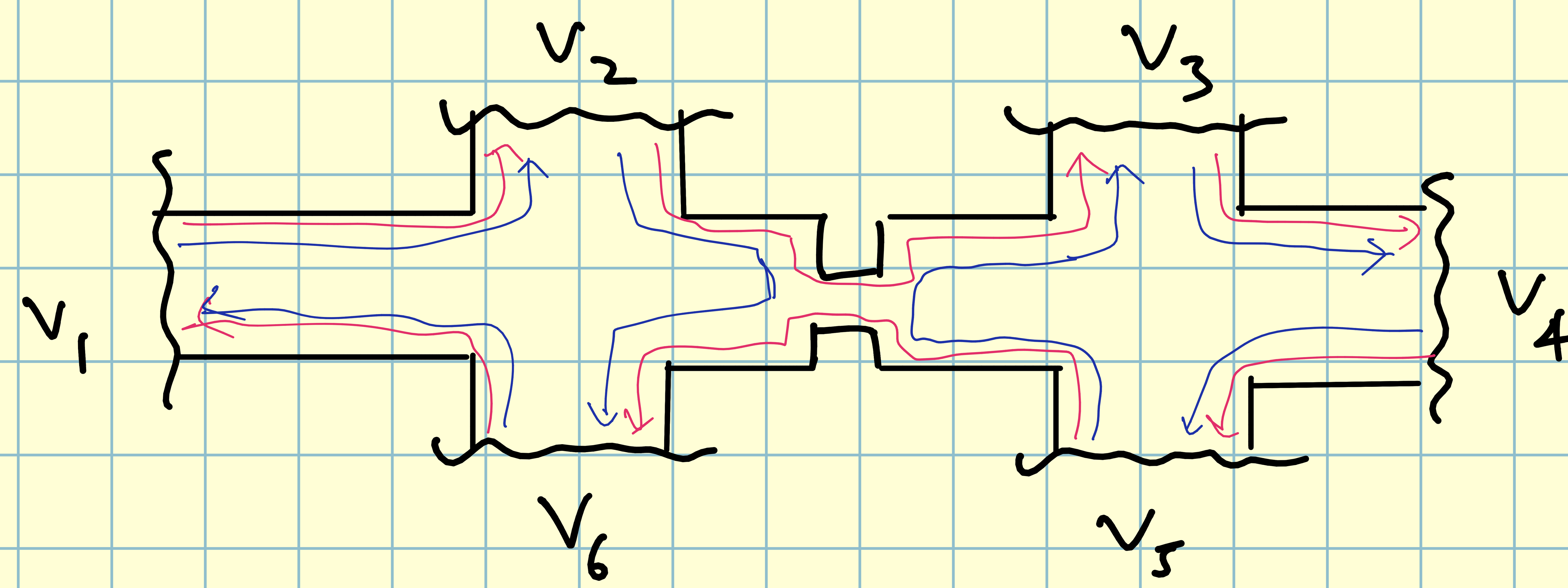
$$N \sim \left[\frac{m^* E_F}{\hbar e B}\right] = \text{no. of edge modes}$$

when 'B' increases $N \to N-1$

when 'B' decreases $N \to N+1$

G_H
$1/B$
R_L
$1/B$

Now let us simulate the backscattering



N edge modes below E_F

M edge modes can transmit

$N-M$ backscatter.

$$I_1 = \bar{N}\,(V_1 - V_6)$$

$$I_2 = 0 \Rightarrow V_2 = V_1$$

$$I_3 = \bar{M}\,(V_3 - V_2) + (\bar{N} - \bar{M})(V_3 - V_5) = 0$$

$$I_4 = \bar{N}\,(V_4 - V_3)$$

$$I_5 = 0 \Rightarrow V_5 = V_4$$

$$I_6 = \bar{M}\,(V_6 - V_5) + (\bar{N} - \bar{M})(V_6 - V_2) = 0$$

Taking $V_1 = V$

$$V_3 = \left(\frac{\bar{N}}{\bar{M}} - 1\right)\left(V_3 - V_4\right) + V$$

$$\left(2 - \frac{\bar{N}}{\bar{M}}\right) V_3 = \left(1 - \frac{\bar{N}}{\bar{M}}\right) V_4 + V$$

$$\Rightarrow R_H = \frac{1}{\bar{N}}$$

$$R_L = \frac{V_2 - V_3}{I_1} = \frac{1}{\bar{M}} - \frac{1}{\bar{N}}$$

Difference b/w edge modes & ballistic transport :-

Edge modes	Ballistic modes
1) No backscattering	Always backscattering
2) Probability of transmission via an edge mode = 1	Quantum : sum over amplitudes Classical : sum probabilities
3) Length of sample: $L > l_i, l_e$	Quantum : $\lambda_F \approx L < l_i, l_e$ Classical : $\lambda_F \ll L < l_i, l_e$