Hilbert space fragmentation in Floquet-driven systems

Ninth-semester presentation

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Out-of-equilibrium quantum dynamics

Equilibrium dynamics

- Equilibrium: ensembles, defined by conserved quantities
- Isolated, time-independent Hamiltonian, H
- Microcanonical description is on an energy shell
- In most ergodic systems $\Rightarrow$ Eigenstate thermalisation hypothesis (ETH) is satisfied
- The microcanonical density matrix is

$$\rho_{\mathrm{mc}} \propto \sum_{E_n \in [E, E+\Delta E]} |n\rangle \langle n|$$

$$\text{Equilibrium ensembles} \xrightarrow{H := H(t)} \text{Out-of-equilibrium dynamics}$$

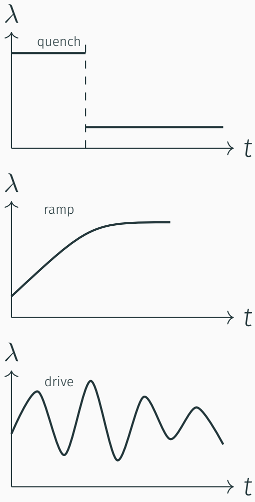
Quenches, ramps, and general drives

Different types of time dependent Hamiltonians are studied in physics:

- Quantum quench: $H_{\mathrm{i}} \to H_{\mathrm{f}}$ at some time
- Ramp / slow quench: control parameter $\lambda(t)$ varies very slowly
- General drive: continuously varying time dependence

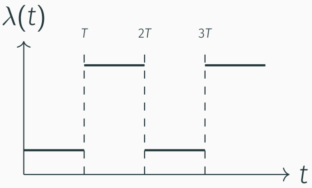
Thus, in general:

$$H(t) = H[\lambda(t)]$$



Periodically driven (Floquet) systems

- Special case: periodic drive
- Hamiltonian repeats with period T
- Focus on evolution over one period
- Stroboscopic times: $t = nT$



$$H(t+T) = H(t)$$

$$U_F = \mathcal{T}\exp\left(-i\int_0^T H(t)\,dt\right)$$

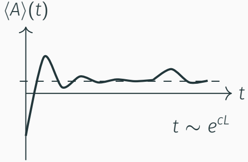
Eigenstate thermalization

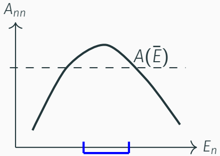
Eigenstate thermalisation hypothesis: why naive time averaging is not enough

- Closed quantum system
- Energy basis: $|\psi(0)\rangle = \sum_m C_m |E_m\rangle$
- Few-body observable A; persistent oscillations
  $\langle A\rangle(t) =$
  $\sum_n p_n A_{nn} + \sum_{m\neq n} c_m^* c_n e^{i(E_m - E_n)t} A_{mn}$
- Cesàro long-time average (diagonal ensemble):

$$\overline{\langle A\rangle} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle A\rangle(t)\, dt = \sum_m |C_m|^2 A_{mm}$$

- Problems: Heisenberg time$\sim e^{cL}$, Initial conditions preserved ;

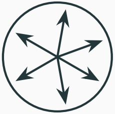




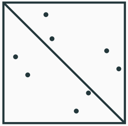
L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, "From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics", *Advances in Physics* (2016).

Eigenstate thermalisation

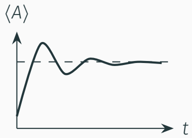
- Chaotic eigenstates take random directions in Hilbert space
- Any local operator, **A** is diagonal in product basis but scrambled in energy basis
- Diagonal elements of **A** vary smoothly versus energy as energy eigenstates are thermal
- Off-diagonals decay exponentially with size and are controlled by temporal fluctuations
- **A** relaxes to plateau given by the ensemble average



random eigenstates



operator in energy basis



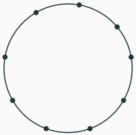
relaxation in time

M. Srednicki, “Chaos and Quantum Thermalization”, Phys. Rev. E 50, 888 (1994)

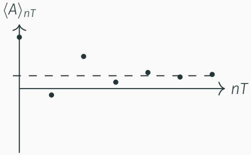
ETH in Floquet heating

- Periodic drive: $H(t+T) = H(t)$
- Floquet unitary:
  $U_F = \mathcal{T} \exp\left(-i \int_0^T H(t)\, dt\right)$
- Effective Floquet Hamiltonian:
  $U_F = e^{-iH_F T}$
- Energy not conserved; only symmetries of U_F (e.g. total S^z)
- ETH for H_F in **global** symmetry sector $\mathcal{S}$: eigenstates $|\phi_\alpha\rangle$ locally random in $\mathcal{S}$
- Diagonal Floquet ETH:

$$\langle \phi_\alpha | A | \phi_\alpha \rangle \approx \frac{1}{\dim \mathcal{S}} \operatorname{Tr}_{\mathcal{S}} A$$



quasienergies in one sector

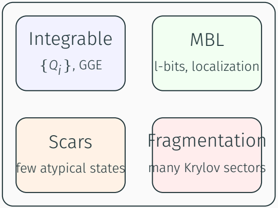


L. D'Alessio and M. Rigol, "Long-time behavior of isolated periodically driven interacting lattice systems", *Physical Review X* (2014).

Floquet heating is when local observables take infinite-temperature values in $\mathcal{S}$ (they become random)

Violation of eigenstate thermalisation through Hilbert space fragmentation

- ETH is robust but not universal
- Integrability: The presence of a large number of conserved quantities leads to loss of ergodicity and prevents the realisation of long-time thermal steady states.
- MBL: The system becomes non-ergodic due to strong disorder, leading to localisation of all states in its Hilbert space.
- Scars: special nonthermal eigenstates; long-time oscillations
- Fragmentation: local constraints; many disconnected Krylov sectors



D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, “Colloquium: Many-body localization, thermalization, and entanglement”, *Reviews of Modern Physics* (2019); R. Nandkishore and D. A. Huse, “Many-Body Localization and Thermalization in Quantum Statistical Mechanics”, *Annual Review of Condensed Matter Physics* (2015).

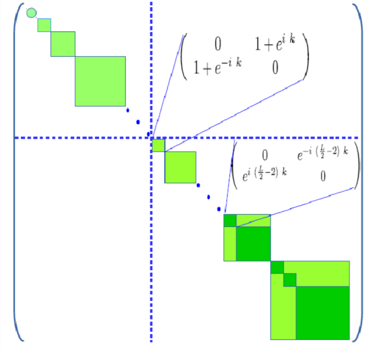
Floquet prethermal fragmentation: restricted exploration of Hilbert space; slowed or prevented heating due to fragmentation in first order Floquet Hamiltonian

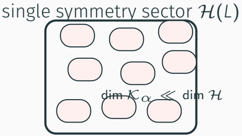
Hilbert space fragmentation (HSF): static picture

- Family of Hilbert spaces $\mathcal{H}(L)$;
  $\dim \mathcal{H}(L) \sim e^{cL}$
- Fragmented operator $X(L)$:

$$\mathcal{H}(L) = \bigoplus_{\alpha=1}^{N_{\text{frag}}(L)} \mathcal{K}_\alpha(L)$$

- Invariance: $X(L)\,\mathcal{K}_\alpha(L) \subseteq \mathcal{K}_\alpha(L)$
- Strong fragmentation:
  $N_{\text{frag}}(L) \sim e^{S_{\text{frag}} L}$
- Typical fragments:
  $\dim \mathcal{K}_\alpha(L) \ll \dim \mathcal{H}(L)$
- Result: eigenstates in tiny islands;
  ETH only inside $\mathcal{K}_\alpha$,
  strong ergodicity breaking

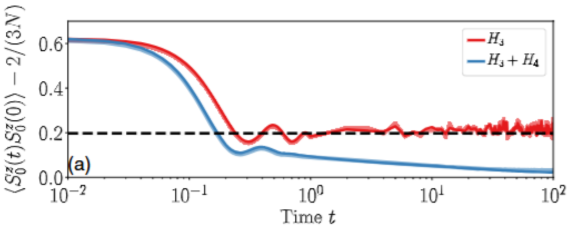




S. Moudgalya and O. I. Motrunich, "Hilbert Space Fragmentation and Commutant Algebras", *Physical Review X* (2022).

Example of HSF in static system

- Spin-1 chain with three-site dipole-conserving Hamiltonian
  $H_3 = -\sum_n \left[S_n^+ (S_{n+1}^-)^2 S_{n+2}^+ + \text{H.c.}\right], H_4 = -\sum_n [S_n^+ S_{n+1}^- S_{n+2}^- S_{n+3}^+ + H.c.]$
- Global conserved quantities: charge and dipole moment $Q = \sum_n S_n^z$, $P = \sum_n n S_n^z$
- Explore autocorrelation: $C_j^z(t) \equiv \langle S_j^z(t) S_j^z(0) \rangle$
- Finite long-time plateau of $C_0^z(t)$ at infinite temperature $\Rightarrow$ persistent local memory, weak ETH violation-strong HSF (H_3).



P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, “Ergodicity Breaking Arising from Hilbert Space Fragmentation in Dipole-Conserving Hamiltonians”, *Physical Review X* (2020).

Hilbert space fragmentation in driven system

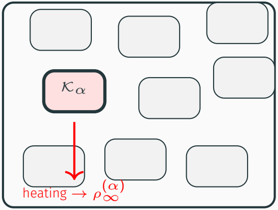
- Fragmentation inherited by the Floquet unitary and projector operator, P_α:

$$U_F = \bigoplus_\alpha U_F^{(\alpha)}, \qquad [U_F, P_\alpha] = 0$$

- Stroboscopic dynamics: $|\psi(0)\rangle \in \mathcal{K}_\alpha \Rightarrow |\psi(nT)\rangle \in \mathcal{K}_\alpha$
- Fragment-restricted infinite-T ensemble:

$$\rho_\infty^{(\alpha)} = \frac{P_\alpha}{\dim \mathcal{K}_\alpha}$$

- ETH inside fragment only; $\overline{\langle A \rangle}_\alpha \approx \frac{1}{\dim \mathcal{K}_\alpha} \mathrm{Tr}(P_\alpha A)$



S. Ghosh, I. Paul, and K. Sengupta, "Prethermal Fragmentation in a Periodically Driven Fermionic Chain", *Physical Review Letters* (2023).

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Example of prethermal HSF in driven system: Model

Hamiltonian given by $H(t) = H_0(t) + H_1$,

$$H_0(t) = V(t) \sum_{j=1..L} \hat{n}_j \hat{n}_{j+1}$$

$$H_1 = \sum_{j=1..L} -J(c_j^\dagger c_{j+1} + H.c.) + \hat{n}_j(V_0 \hat{n}_{j+1} + V_2 \hat{n}_{j+2}),$$

Periodic drive of NN interaction: $V(t) = \begin{cases} -V_1, & t \leq \frac{T}{2} \\ +V_1, & t > \frac{T}{2} \end{cases}$; high-frequency regime, First-order Floquet Hamiltonian $H_F^{(1)}$ from Floquet perturbation theory

$$H_F^{(1)} = \sum_j \hat{n}_j(V_0 \hat{n}_{j+1} + V_2 \hat{n}_{j+2}) - J \sum_j \left[(1 - \hat{A}_j^2) + f(\gamma_0)\,\hat{A}_j^2\right](c_j^\dagger c_{j+1} + \text{h.c.})$$

$$\hat{A}_j = \hat{n}_{j+2} - \hat{n}_{j-1}, \qquad \gamma_0 = \frac{V_1 T}{4\hbar}, \qquad f(\gamma_0) = 0 \text{ at special } \omega_D = \omega_1^*$$

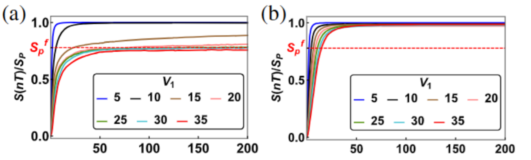
S. Ghosh, I. Paul, and K. Sengupta, “Prethermal Fragmentation in a Periodically Driven Fermionic Chain”, *Physical Review Letters* (2023).

Example of prethermal HSF in driven system: Entanglement entropy

Entanglement entropy and Page value Bipartition chain into $A|B$; reduced state $\rho_A(nT) = \mathrm{Tr}_B\,\rho(nT)$ entanglement entropy is

$$S(nT) = -\mathrm{Tr}_A\left[\rho_A(nT)\ln\rho_A(nT)\right]$$

- Drive frequency near $\omega_D = \omega_1^*$: fragmented $H_F^{(1)}$; initial Fock state confined to fragment f with Page value S_p^f
- Fragmentation only at special driving frequencies $\omega_D = \omega_1^*$



S. Ghosh, I. Paul, and K. Sengupta, "Prethermal Fragmentation in a Periodically Driven Fermionic Chain", *Physical Review Letters* (2023).

D. N. Page, "Average Entropy of a Subsystem", Phys. Rev. Lett. 71, 1291 (1993)

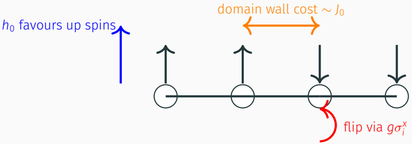
Our model

Our model

- Static Hamiltonian

$$H_{\mathrm{stat}} = -J_0 \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z - h_0 \sum_{i=1}^{L} \sigma_i^z - g \sum_{i=1}^{L} \sigma_i^x, \ |g| \ll |J_0|, |h_0|$$

- $-J_0\sigma_i^z\sigma_{i+1}^z$: ferro / antiferro couplings, domain-wall cost
- $-h_0\sigma_i^z$: longitudinal Zeeman field, biases up / down, breaks TFIM integrability
- $-g\sigma_i^x$: noncommuting transverse field, coherent spin flips
- $h_0 = 0$: integrable TFIM; $h_0 \neq 0$: nonintegrable, chaotic, ETH, ballistic entanglement.



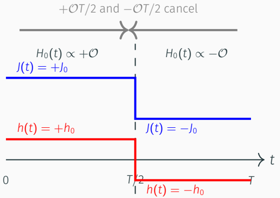
H. Kim and D. A. Huse, "Ballistic Spreading of Entanglement in a Diffusive Nonintegrable System", *Physical Review Letters* (2013).

Floquet driving protocol in our model

- Time-periodic Hamiltonian $H(t) = H_0(t) + H_1$

$$H_0(t) = -J(t)\sum_i \sigma_i^z \sigma_{i+1}^z - h(t)\sum_i \sigma_i^z, \quad H_1 = -g\sum_i \sigma_i^x$$

- Driving: $J(t), h(t) = \begin{cases} +J_0, +h_0, & 0 \le t \le T/2 \\ -J_0, -h_0, & T/2 < t \le T \end{cases}$
- Diagonal operators $\mathcal{O}_{zz} = \sum_i \sigma_i^z \sigma_{i+1}^z,\ \mathcal{O}_z = \sum_i \sigma_i^z,\ \mathcal{O} = J_0\mathcal{O}_{zz} + h_0\mathcal{O}_z$
- $[H_0(t), H_0(t')] = 0$, $U_0(T, 0) = \mathbb{I}$ (closed micromotion)



Next show results.

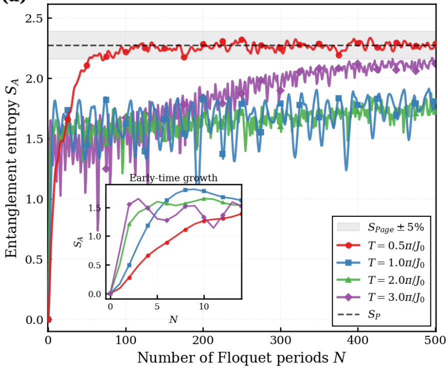
Results

Symmetric-square-wave periodic Hamiltonian, $H(t) = H_0(t) + H_1$

$$H_0(t) = -J(t)\sum_i \sigma_i^z \sigma_{i+1}^z - h(t)\sum_i \sigma_i^z, \quad H_1 = -g\sum_i \sigma_i^x$$

Entanglement dynamics in the Floquet-driven Ising chain

(a)



(b)

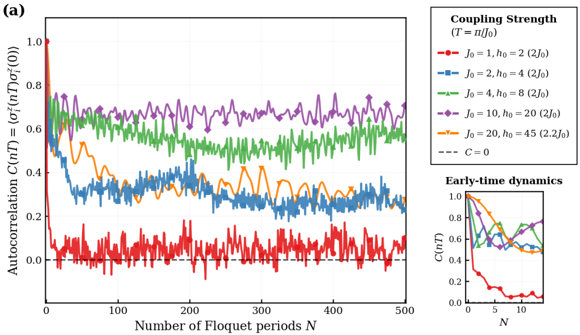
n	T	S_{Page}	S_{final}	S/S_{Page}	$N_{90\%}$
0.5	0.1963	2.2726	2.2830	1.005	42
1.0	0.3927	2.2726	1.8055	0.794	—
2.0	0.7854	2.2726	1.7213	0.757	—
3.0	1.1781	2.2726	2.1175	0.932	308

System Parameters
$L = 8$ sites
$J_0 = 8.0$
$h_0 = 16.0$
$g = 1.0$

Autocorrelation in our model

$$C_j(nT) = \langle \psi_0 | [\sigma_j^z(nT)][\sigma_j^z(0)] | \psi_0 \rangle$$

Coupling strength dependence of autocorrelation in Floquet-driven Ising chain



(b)

J_0	h_0	T	$C(0)$	C_{final}	$\langle \lvert C \rvert \rangle$	σ_C
1	2	3.1416	1.0000	0.1010	0.0544	0.0460
2	4	1.5708	1.0000	0.2275	0.2909	0.0521
4	8	0.7854	1.0000	0.5544	0.5378	0.0452
10	20	0.3142	1.0000	0.7060	0.6659	0.0298
20	45	0.1571	1.0000	0.2762	0.3150	0.0414

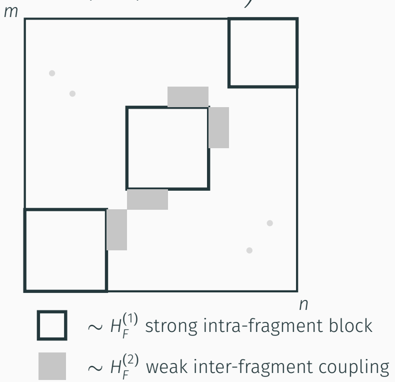
System Parameters
$L = 8$ sites
Site: $i = 4$
$g = 1.0$
Relation: [2, 4, 8, 20, 40, 45]
Period: $T = \pi/J_0$
Ensemble: 1 states

Floquet Hamiltonian and prethermal fragmentation

- Floquet unitary; effective Hamiltonian

$$U_F(T) = \mathcal{T} e^{-i\int_0^T H(t)\,dt}, \quad U_F = e^{-iH_F T}$$

- $(H_F^{(1)})_{nm} = -\langle n| g \sum_{i=1}^{L} \sigma_i^x |m\rangle \left(\delta_{P_{nm},0} + (1-\delta_{P_{nm},0})\,\mathrm{sinc}\!\left(\frac{P_{nm}T}{4\hbar}\right) e^{-\frac{i}{\hbar}P_{nm}\frac{T}{4}} \right)$ constrained single-spin flips; almost block-diagonal in fragmented basis
- Tuning drive period T choose T so sinc-factors remove selected links; prethermal fragments
- Higher orders $H_F^{(2)}, H_F^{(3)}, \ldots$ weak off-diagonal couplings between fragments; slow late-time mixing



Bulk P_{nm} values for states connected by $\sum_{i=1}^{L} \sigma_i^x$

Take $J_0 = h_0/2$

flip $i = 2$

$$\downarrow \quad \uparrow \quad \downarrow \quad \xrightarrow{\sigma_2^x} \quad \downarrow \quad \downarrow \quad \downarrow$$

Figure 1: $P_{nm} = 0$

flip $i = 2$

$$\uparrow \quad \uparrow \quad \downarrow \quad \xrightarrow{\sigma_2^x} \quad \uparrow \quad \downarrow \quad \downarrow$$

Figure 2: $P_{nm} = -4J_0$

flip $i = 2$

$$\uparrow \quad \uparrow \quad \uparrow \quad \xrightarrow{\sigma_2^x} \quad \uparrow \quad \downarrow \quad \uparrow$$

Figure 3: $P_{nm} = -8J_0$

Edge P_{nm} values for states connected by H_1

For open boundary conditions, additional edge processes occur:

flip edge $i = 1$

$$\uparrow \quad \downarrow \quad \downarrow \quad \xrightarrow{\sigma_1^x} \quad \downarrow \quad \downarrow \quad \downarrow$$

Figure 4: $P_{nm} = -2J_0$

flip edge $i = 1$

$$\uparrow \quad \uparrow \quad \downarrow \quad \xrightarrow{\sigma_1^x} \quad \downarrow \quad \uparrow \quad \downarrow$$

Figure 5: $P_{nm} = -6J_0$

From the different driving, we can fix all spin flips other than $P_{nm} = 0$. In this case we see that blocks of $\uparrow$ in states do not change. This is our local conserved quantity.

Time period of drive and allowed transitions in periodic boundary conditions

$$(H_F^{(1)})_{nm} = -\langle n| g \sum_{i=1}^{L} \sigma_i^x |m\rangle \left(\delta_{P_{nm},0} + (1-\delta_{P_{nm},0}) \,\mathrm{sinc}\!\left(\frac{P_{nm}T}{4\hbar}\right) e^{-\frac{i}{\hbar}P_{nm}\frac{T}{4}} \right)$$

Table 1: First-order (single flip) connectivity for PBC at $h_0 = 2J_0$. Parentheses indicate $PT/4$.

T	$P = 0$	$P = 4J_0$	$P = 8J_0$
$\frac{\pi}{2J_0}$	on (0)	on $(\frac{\pi}{2})$	off (π)
$\frac{\pi}{J_0}$	on (0)	off (π)	off (2π)

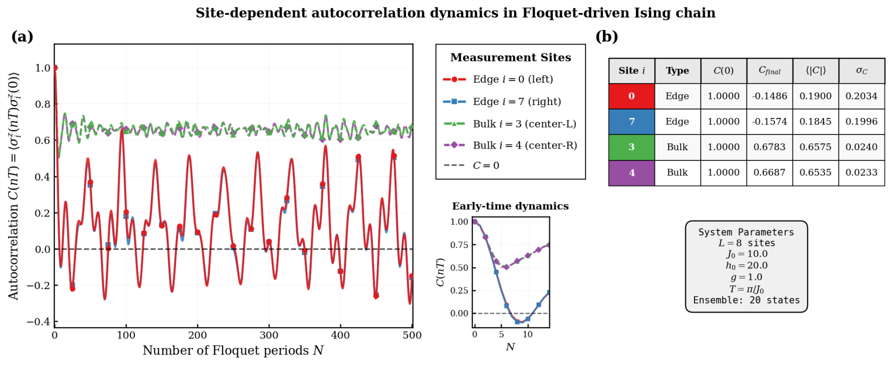
Time period of drive and allowed transitions in open boundary conditions

$$(H_F^{(1)})_{nm} = -\langle n| g \sum_{i=1}^{L} \sigma_i^x |m\rangle \left(\delta_{P_{nm},0} + (1-\delta_{P_{nm},0})\,\mathrm{sinc}\!\left(\frac{P_{nm}T}{4\hbar}\right) e^{-\frac{i}{\hbar}P_{nm}\frac{T}{4}} \right)$$

Table 2: First-order (single flip) connectivity for OBC at $h_0 = 2J_0$ (including edge flips). Parentheses indicate $PT/4$.

T	$P=0$	$P=2J_0$ (edge)	$P=4J_0$	$P=6J_0$ (edge)	$P=8J_0$	Summary
$\frac{\pi}{2J_0}$	on (0)	on ($\frac{\pi}{4}$)	on ($\frac{\pi}{2}$)	on ($\frac{3\pi}{4}$)	off (π)	only $\lvert P\rvert = 8J_0$ killed
$\frac{\pi}{J_0}$	on (0)	on ($\frac{\pi}{2}$)	off (π)	on ($\frac{3\pi}{2}$)	off (2π)	kills $\lvert P\rvert = 4, 8$
$\frac{2\pi}{J_0}$	on (0)	off (π)	off (2π)	off (3π)	off (4π)	all nonzero P killed

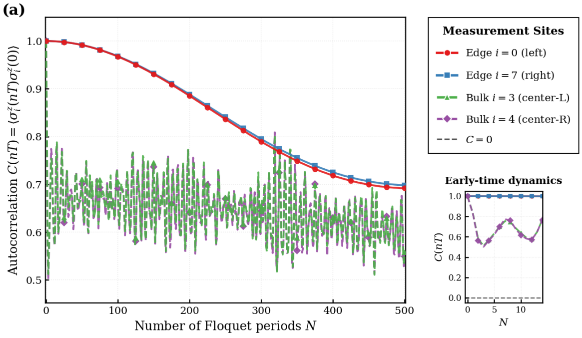
The time period is $T = \pi/J_0$.



Site i	Type	$C(0)$	C_{final}	$\langle \lvert C \rvert \rangle$	σ_C
0	Edge	1.0000	-0.1486	0.1900	0.2034
7	Edge	1.0000	-0.1574	0.1845	0.1996
3	Bulk	1.0000	0.6783	0.6575	0.0240
4	Bulk	1.0000	0.6687	0.6535	0.0233

The time period is $T = 2\pi/J_0$.

Site-dependent autocorrelation dynamics in Floquet-driven Ising chain



(b)

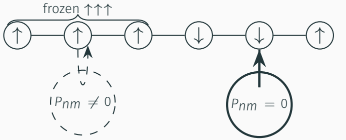
Site i	Type	$C(0)$	C_{final}	$\langle \lvert C \rvert \rangle$	σ_C
0	Edge	1.0000	0.6920	0.8045	0.0896
7	Edge	1.0000	0.6982	0.8093	0.0875
3	Bulk	1.0000	0.5577	0.6395	0.0518
4	Bulk	1.0000	0.5497	0.6370	0.0535

System Parameters
$L = 8$ sites
$J_0 = 10.0$
$h_0 = 20.0$
$g = 1.0$
$T = 2\pi/J_0$
Ensemble: 20 states

Conclusion

Summary: first-order Floquet selection and fragmentation

- Drive, basis:
  TLFIM, symmetric sign flip; $h_0 = 2J_0$;
  diagonal $\mathcal{O}$ in σ^z basis
- Floquet selection: $P_{nm} = 0$ flips
- Matrix elements σ_i^x follow bulk three-site rule; extra edge flips violate it at $T = \pi/J_0$
- Conserved quantities: $I_i = n_i^\uparrow n_{i+1}^\uparrow$;
- Fragmentation, heating:
  frozen ↑↑ domains; heating confined within each fragment



Resonant (solid) vs blocked (dashed) flips

$$(H_F^{(1)})_{nm} = (H_1)_{nm}\,\mathrm{sinc}\Big(\frac{P_{nm}T}{4}\Big)e^{-iP_{nm}T/4}, \quad H_1 = -g\sum_i \sigma_i^x$$

$$I_i = n_i^\uparrow n_{i+1}^\uparrow, \quad n_i^\uparrow = \tfrac{1}{2}(1+\sigma_i^z)$$

Future directions

- Graph-theoretic analysis of H_F connectivity vs T, g/J_0, L; invariant components, sector dimensions
- Entanglement saturation vs fragment Page values; entanglement-based confirmation of fragmentation
- Symmetry structure and edge modes of Floquet spectrum; bulk–edge contrast in autocorrelation plateaus
- Higher-order Floquet corrections, larger L; distinguish true fragmentation vs prethermal plateaus; robustness in thermodynamic limit
- Search for explicit emergent conserved quantities enabling Floquet-engineered constrained dynamics

Questions?

ETH backup slides

Random matrix theory picture: operator matrix elements

- Simple basis $\{|i\rangle\}_{i=1}^{\mathcal{D}}$: $\hat{O}$ diagonal

$$\hat{O} = \sum_i O_i \, |i\rangle\langle i|$$

- Chaotic eigenstates of $\hat{H}$:

$$\psi_i^m := \langle i|E_m\rangle \,, \quad |E_m\rangle = \sum_i \psi_i^m \, |i\rangle$$

- Operator in energy basis:

$$O_{mn} = \langle E_m|\hat{O}|E_n\rangle = \sum_i O_i (\psi_i^m)^* \psi_i^n$$

- RMT eigenvectors: random orthogonal

$$\overline{(\psi_i^m)^* \psi_j^n} = \frac{1}{\mathcal{D}} \delta_{mn} \delta_{ij}$$

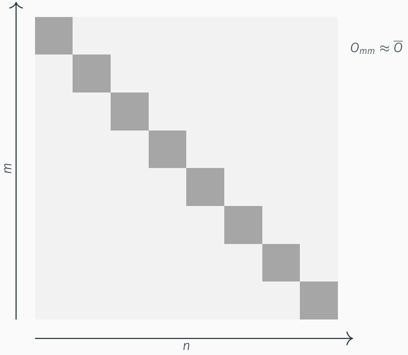


Figure 6: Diagonal-dominated operator matrix in chaotic energy basis; RMT cartoon of ETH structure.

References: Srednicki [1]

Random Matrix Theory picture: fluctuations and ETH-like structure

- Averages:

$$\overline{O_{mm}} = \overline{O}, \quad \overline{O_{mn}} = 0 \ (m \neq n)$$

- Off-diagonal variance

$$\overline{|O_{mn}|^2} = \frac{1}{\mathcal{D}} \overline{O^2}, \quad m \neq n$$

- ETH-like structure

$$O_{mn} \approx \overline{O}\,\delta_{mn} + \sqrt{\frac{\overline{O^2}}{\mathcal{D}}}\, R_{mn}$$

- Scaling: $|O_{mn}| \sim \mathcal{D}^{-1/2} \sim e^{-S/2}$

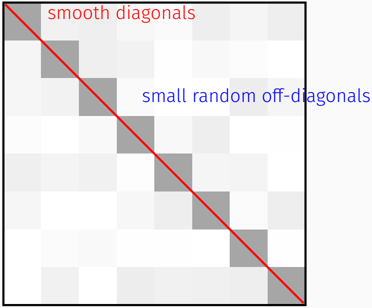


Figure 7: Operator matrix in chaotic basis: almost-constant diagonal band plus $\mathcal{D}^{-1/2}$-scaled random off-diagonals; random-matrix ETH cartoon .

References: Srednicki [1]

Backup

- Fluctuations around plateau:

$$\delta A(t) = \langle A\rangle(t) - \overline{\langle A\rangle} = \sum_{m\neq n} C_m^* C_n e^{i(E_m - E_n)t} A_{mn}$$

- Time-averaged variance:

$$\overline{\delta A^2} = \sum_{m\neq n} |C_m|^2 |C_n|^2 |A_{mn}|^2$$

- ETH off-diagonals: $|A_{mn}|^2 \sim e^{-S(\bar{E})}$
- Narrow shell; bounded f_A, R_{mn}; sum of weights ≤ 1
- Result: $\overline{\delta A^2} \lesssim e^{-S(\bar{E})} \sim e^{-cL}$; fluctuations exponentially small

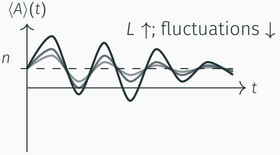


Figure 8: Off-diagonal ETH $\rightarrow$ exponentially small temporal fluctuations; larger systems look more “thermal” in time. D’Alessio et al. (2016) [2].

ETH ansatz: energy and locality

- Nonintegrable, “quantum chaotic” lattice system
- ETH ansatz for few-body A:

$$A_{mn} = A(\bar{E})\,\delta_{mn} + e^{-S(\bar{E})/2} f_A(\bar{E},\omega)\,R_{mn}$$

- Mean energy $\bar{E} = (E_m + E_n)/2$; frequency $\omega = E_n - E_m$
- $A(\bar{E})$: smooth microcanonical value
- $S(\bar{E})$: thermodynamic entropy $\sim s(\bar{e})L$
- R_{mn}: random, zero mean, unit variance

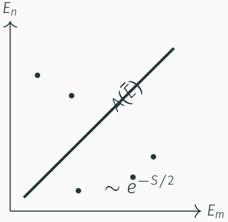


Figure 9: ETH structure: smooth diagonals, entropy-suppressed random off-diagonals.

References: Deutsch (1991) [3], Srednicki (1994) [1], D’Alessio et al. (2016) [2]

ETH violation backup

Entanglement entropy as ETH diagnostic

Backup

- Bipartition: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$; $\dim \mathcal{H}_A = d_A$, $\dim \mathcal{H}_B = d_B$
- Eigenstate reduced state: $\rho_A^{(\alpha)} = \mathrm{Tr}_B \, |E_\alpha\rangle\langle E_\alpha|$
- Entanglement entropy: $S_A(\alpha) = -\,\mathrm{Tr}_A \left[\rho_A^{(\alpha)} \log \rho_A^{(\alpha)}\right]$
- ETH expectation: $\rho_A^{(\alpha)} \approx \rho_A^{\mathrm{th}}(e)$, $S_A(\alpha) \approx S_A^{\mathrm{th}}(e) \simeq s(e)\,|A|$
- Random / Page behaviour: $S_A^{\mathrm{Page}} \simeq \log d_A - \dfrac{d_A}{2d_B}$
- Diagnostic: finite fraction with $S_A(\alpha) \leq S_A^{\mathrm{th}}(e) - \delta|A|$; one-to-one map of entropy, $\Rightarrow$ there is finite trace distance $\rho_A^{(\alpha)} \neq \rho_A^{\mathrm{th}}(e)$; ETH breakdown

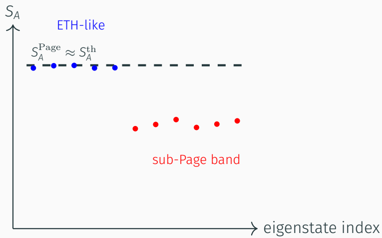


Figure 10: Eigenstate entanglement entropy; Page / thermal line versus sub-Page band.

References:
Page 1993 [4];
Foong & Kanno 1994 [5];
Ghosh–Paul–Sengupta 2023 [6].

Infinite-temperature autocorrelations as ETH diagnostic

- Static H; local A; Hilbert dimension D
- Infinite-T autocorrelation: $C_A(t) = \frac{1}{D}\,\mathrm{Tr}[A(t)A(0)]$, $A(t) = e^{iHt}Ae^{-iHt}$
- Long-time limit (nondegenerate gaps): $C_A^\infty = \frac{1}{D}\sum_n |A_{nn}|^2$
- ETH scaling (traceless, charge-orthogonal A): $|A_{nn}|^2 \sim e^{-S(E)}$; $C_A^\infty \sim e^{-cL} \to 0$
- Mazur projection onto conserved $\{Q_k\}$: $A_\parallel = \sum_{k,\ell}(A, Q_k)[C^{-1}]_{k\ell}Q_\ell$, $C_A^\infty \geq D_A = \frac{1}{D}\,\mathrm{Tr}\left(A_\parallel^\dagger A_\parallel\right)$
- Fragmentation: conserved projectors P_α; finite $A_\parallel$; nonvanishing C_A^∞ plateau

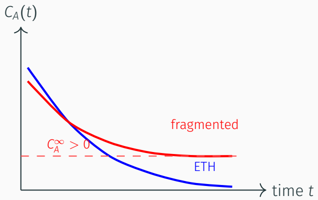


Figure 11: Infinite-T autocorrelations; ETH decay versus nonzero plateau from conserved components.

References:
Mazur 1969 [7];
Nandkishore & Huse 2015 [8];
Sala et al. 2020 [9];
Ghosh–Paul–Sengupta 2023 [6].

Spin 1/2 chain backup

Spin-$\frac{1}{2}$ Ising chain: microscopic playground

- Static Hamiltonian

$$H_{\text{stat}} = -J_0 \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z - h_0 \sum_{i=1}^{L} \sigma_i^z - g \sum_{i=1}^{L} \sigma_i^x, \ |g| \ll |J_0|, |h_0|$$

- Minimal quantum lattice model for magnetism and quantum phase transitions
- Sites $i = 1, \ldots, L$; local space $\mathcal{H}_i \cong \mathbb{C}^2$, Total space $\mathcal{H} = \bigotimes_{i=1}^{L} \mathcal{H}_i$, $\dim \mathcal{H} = 2^L$
- Computational basis: σ^z eigenstates $\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$, Product states $|s_1 \cdots s_L\rangle$ as classical spin configurations σ_i^z: local magnetisation; σ_i^x: single-spin flip, quantum fluctuations

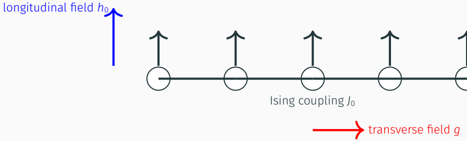


Figure 12: Spin-$\frac{1}{2}$ chain with nearest-neighbour Ising coupling and longitudinal / transverse fields.

Floquet perturbation theory

Floquet perturbation theory and first-order $H_F^{(1)}$

Interaction picture; first-order H_F

- Split Hamiltonian:

$$H(t) = H_0(t) + V$$

- Interaction picture:

$$U^I(t,0) = U_0(0,t)\,U(t,0)$$

$$V_I(t) = U_0(0,t)\,V\,U_0(t,0)$$

$$i\hbar\,\partial_t U^I(t,0) = V_I(t)\,U^I(t,0)$$

- One-period Floquet operator:

$$U(T,0) = U_0(T,0)\,U^I(T,0)$$

- If $U_0(T,0) = \mathbb{I}$:

$$U(T,0) = U^I(T,0) = e^{-\frac{i}{\hbar}H_F T}$$

- First-order Floquet Hamiltonian:

$$H_F^{(1)} = \frac{1}{T}\int_0^T dt\, V_I(t)$$

Application: Ising drive

- Symmetric square wave $J(t), h(t)$:

$$[H_0(t), H_0(t')] = 0, \quad U_0(T,0) = \mathbb{I}$$

- $\mathcal{O}$–eigenbasis:
  $\mathcal{O}\,|m\rangle = P_m\,|m\rangle\,, \quad P_{nm} = P_n - P_m$
- First-order matrix element:
  $(H_F^{(1)})_{nm} = -\left\langle n \middle| g\sum_{i=1}^{L}\sigma_i^x \middle| m\right\rangle$
  $\left[\delta_{P_{nm},0} + (1-\delta_{P_{nm},0})\,\mathrm{sinc}\left(\frac{P_{nm}T}{4\hbar}\right)e^{-\frac{i}{\hbar}P_{nm}\frac{T}{4}}\right].$
- Find T so that *sinc* term vanishes and take $J_0 = h_0/2$