Exact bosons though iterative methods

Idea from Girish S. Setlur, revised and rewritten by Rishi Paresh Joshi

$$[c_{\boldsymbol{p},<}, a_{\boldsymbol{p}+\boldsymbol{q}/2}(\boldsymbol{q})] \;=\; n_F(\boldsymbol{p})\; c_{\boldsymbol{p}+\boldsymbol{q},>} \tag{1}$$

$$[c_{\boldsymbol{p},>}, a^{\dagger}_{\boldsymbol{p}-\boldsymbol{q}/2}(\boldsymbol{q})] \;=\; (1-n_F(\boldsymbol{p}))\; c_{\boldsymbol{p}-\boldsymbol{q},<} \tag{2}$$

All other commutators of this type other than the two shown above are zero,

$$[c_{...,>}, a_{..}(..)] = [c_{...,>}, a^{\dagger}_{..}(..)] = [c_{...,<}, a_{..}(..)] = [c_{...,<}, a^{\dagger}_{..}(..)] = 0$$

$a_{\mathbf{k}}(\mathbf{q})$ and $a^{\dagger}_{\mathbf{k}}(\mathbf{q})$ follow the commutation rules of bosonic creation and annihilation operators, ie.

$$[a_{\boldsymbol{k}}(\boldsymbol{q}), a_{\boldsymbol{k'}}(\boldsymbol{q'})] = 0 \; and, \tag{3}$$

$$[a_{\boldsymbol{k}}(\boldsymbol{q}), a^{\dagger}_{\boldsymbol{k'}}(\boldsymbol{q'})] = \delta_{\boldsymbol{k},\boldsymbol{k'}}\delta_{\boldsymbol{q},\boldsymbol{q'}} \tag{4}$$

Iterative solution: 0 order

$$c_{\mathbf{p},<} \;=\; \xi_{\mathbf{p},<}$$

$$c_{\mathbf{p},>} \;=\; \xi_{\mathbf{p},>}$$

Iterative solution: I order

$$c_{\mathbf{p},<} \;=\; \xi_{\mathbf{p},<} - \sum_{\mathbf{q}_1} \xi_{\mathbf{p}+\mathbf{q}_1,>} a^{\dagger}_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1) = I_{I,<}(\mathbf{p})$$

$$c_{\mathbf{p},>} \;=\; \xi_{\mathbf{p},>} + \sum_{\mathbf{q}_1} \xi_{\mathbf{p}-\mathbf{q}_1,<} a_{\mathbf{p}-\mathbf{q}_1/2}(\mathbf{q}_1) = \; I_{I,>}(\mathbf{p})$$

Verification of Ist and 0th order solutions:

We need to substitute the 0th order solution in the RHS of Eq.(1) for $c_{\mathbf{p},<}$ and 1st order solution for $c_{\mathbf{p},<}$ in LHS of Eq.(1) If both the sides are equal, we are done. Similarly, we need to substitute the zeroth-order solution in RHS of Eq. (2), for $c_{\mathbf{p},>}$ and first-order solution for $c_{\mathbf{p},>}$ in LHS of Eq.(2) and show they are equal.

Verifying Eq.(1)

$$[\xi_{\mathbf{p},<} - \sum_{\mathbf{q}_1} \xi_{\mathbf{p}+\mathbf{q}_1,>} a^{\dagger}_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)\;,\; a_{\boldsymbol{p}+\boldsymbol{q}/2}(\boldsymbol{q})]] = \; -\xi_{\mathbf{p}+\mathbf{q}/\mathbf{2},>} \times (-1) = \; c_{\mathbf{p}+\mathbf{q}/\mathbf{2},>}$$

This is from Eq.(4), and it verifies the zeroth and first order iteration, since if $|\boldsymbol{p}| > k_F$, then $c_{\mathbf{p},<}$=0 and the RHS of Eq.(1)=0. This fact is used to verify all iteration solutions that satisfy Eq.(1).

Verifying Eq.(2)

$$[\xi_{\mathbf{p},>} + \sum_{\mathbf{q}_1} \xi_{\mathbf{p}-\mathbf{q}_1,<} a_{\mathbf{p}-\mathbf{q}_1/2}(\mathbf{q}_1)\;,\; a^{\dagger}_{\boldsymbol{p}-\boldsymbol{q}/2}(\boldsymbol{q})] = \; \xi_{\mathbf{p}-\mathbf{q}/\mathbf{2}} = c_{\mathbf{p}-\mathbf{q}/\mathbf{2},<}$$

This is from Eq.(4), and it verifies the zeroth and first order iteration, since if $|\boldsymbol{p}| \le k_F$, then $c_{\boldsymbol{p},>}$=0 and the RHS of Eq.(2)=0. This fact is also used in all the other verification that the iterations satisfy Eq.(2).

Iterative solution: II order

$$c_{\mathbf{p},<} \;=\; \xi_{\mathbf{p},<} - \sum_{\mathbf{q}} \xi_{\mathbf{p+q},>} a^\dagger_{\mathbf{p+q}/2}(\mathbf{q}) - \sum_{\mathbf{q},\mathbf{q}_1} \xi_{\mathbf{p+q-q}_1,<} a^\dagger_{\mathbf{p+q}/2}(\mathbf{q}) a_{\mathbf{p+q-q}_1/2}(\mathbf{q}_1) = \; I_{II,<}(\mathbf{p})$$

and

$$c_{\mathbf{p},>} \;=\; \xi_{\mathbf{p},>} + \sum_{\mathbf{q}} \xi_{\mathbf{p-q},<} a_{\mathbf{p-q}/2}(\mathbf{q}) - \sum_{\mathbf{q},\mathbf{q}_1} \xi_{\mathbf{p-q+q}_1,>} a^\dagger_{\mathbf{p-q+q}_1/2}(\mathbf{q}_1) a_{\mathbf{p-q}/2}(\mathbf{q}) = \; I_{II,>}(\mathbf{p})$$

Verification of IInd and Ist order solutions:
Verifying Eq.(1)

$$[\xi_{\mathbf{p},<} - \sum_{\mathbf{q}'} \xi_{\mathbf{p+q}',>} a^\dagger_{\mathbf{p+q}'/2}(\mathbf{q}') - \sum_{\mathbf{q}',\mathbf{q}_1} \xi_{\mathbf{p+q}'-\mathbf{q}_1,<} a^\dagger_{\mathbf{p+q}'/2}(\mathbf{q}') a_{\mathbf{p+q}'-\mathbf{q}_1/2}(\mathbf{q}_1),\; a_{\boldsymbol{p}+\boldsymbol{q}/2}(\boldsymbol{q})] = \; I_{I,>}(\mathbf{p}+\mathbf{q})$$

Breaking the commutator into different sums, the commutator with $\xi_{\mathbf{p},<} = 0$, and from the previous order iteration, the commutator with the second term is just $\xi_{\mathbf{p+q},>}$, We need to find the commutator with the third term. Taking the summation outside the commutator with the third term,

$$-\sum_{\mathbf{q}',\mathbf{q}_1} \xi_{\mathbf{p+q}'-\mathbf{q}_1,<} [a^\dagger_{\mathbf{p+q}'/2}(\mathbf{q}') a_{\mathbf{p+q}'-\mathbf{q}_1/2}(\mathbf{q}_1),\; a_{\boldsymbol{p}+\boldsymbol{q}/2}(\boldsymbol{q})] = -(-\sum_{\mathbf{q}_1} \xi_{\mathbf{p+q-q}_1,<} a_{\mathbf{p+q-q}_1/2}(\mathbf{q}_1))]$$

Hence, by adding the third term, we get $I_{I,>}(\mathbf{p}+\mathbf{q})$. How do I know this is the most general solution, why can't there be $a \cdot a^\dagger$ convolution?

Verifying Eq.(2)

$$[\xi_{\mathbf{p},>} + \sum_{\mathbf{q}'} \xi_{\mathbf{p-q}',<} a_{\mathbf{p-q}'/2}(\mathbf{q}') - \sum_{\mathbf{q}',\mathbf{q}_1} \xi_{\mathbf{p-q}'+\mathbf{q}_1,>} a^\dagger_{\mathbf{p-q}'+\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p-q}'/2}(\mathbf{q}'),\; a^\dagger_{\boldsymbol{p}-\boldsymbol{q}/2}(\boldsymbol{q})] = I_{I,<}(\mathbf{p}-\mathbf{q})$$

Looking at the second and third term, since we are taking a commutator with $a^\dagger_{\boldsymbol{p}-\boldsymbol{q}/2}(\boldsymbol{q})$, hence terms with $-\mathbf{q}'$ become $-\mathbf{q}$. Thus, we will get $\xi_{\mathbf{p-q},<}$ from the second term. The third term gives us $-\sum_{\mathbf{q}_1} \xi_{\mathbf{p-q+q}_1,>} a^\dagger_{\mathbf{p-q+q}_1/2}(\mathbf{q}_1)$

Iterative solution: III order

$$\begin{aligned} c_{\mathbf{p},<} \;=\; & \xi_{\mathbf{p},<} - \sum_{\mathbf{q}_1} \xi_{\mathbf{p+q}_1,>} a^\dagger_{\mathbf{p+q}_1/2}(\mathbf{q}_1) - \sum_{\mathbf{q}_2,\mathbf{q}_1} \xi_{\mathbf{p+q}_2-\mathbf{q}_1,<} a^\dagger_{\mathbf{p+q}_2/2}(\mathbf{q}_2) a_{\mathbf{p+q}_2-\mathbf{q}_1/2}(\mathbf{q}_1) \\ & + \frac{1}{2} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \xi_{\mathbf{p+q}_3-\mathbf{q}_2+\mathbf{q}_1,>}\; a^\dagger_{\mathbf{p+q}_3-\mathbf{q}_2+\mathbf{q}_1/2}(\mathbf{q}_1) a^\dagger_{\mathbf{p+q}_3/2}(\mathbf{q}_3) a_{\mathbf{p+q}_3-\mathbf{q}_2/2}(\mathbf{q}_2) \end{aligned}$$

and

$$\begin{aligned} c_{\mathbf{p},>} \;=\; & \xi_{\mathbf{p},>} + \sum_{\mathbf{q}_1} \xi_{\mathbf{p-q}_1,<} a_{\mathbf{p-q}_1/2}(\mathbf{q}_1) - \sum_{\mathbf{q}_2,\mathbf{q}_1} \xi_{\mathbf{p-q}_2+\mathbf{q}_1,>} a^\dagger_{\mathbf{p-q}_2+\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p-q}_2/2}(\mathbf{q}_2) \\ & - \frac{1}{2} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \xi_{\mathbf{p-q}_3+\mathbf{q}_2-\mathbf{q}_1,<}\; a^\dagger_{\mathbf{p-q}_3+\mathbf{q}_2/2}(\mathbf{q}_2) a_{\mathbf{p-q}_3+\mathbf{q}_2-\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p-q}_3/2}(\mathbf{q}_3) \end{aligned}$$

Verification of IIIrd and IInd order solutions:

Iterative solution: IV order

$$
\begin{aligned}
c_{\mathbf{p},<} &= \xi_{\mathbf{p},<} - \sum_{\mathbf{q}_1} \xi_{\mathbf{p}+\mathbf{q}_1,>} a^\dagger_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1) - \sum_{\mathbf{q}_2,\mathbf{q}_1} \xi_{\mathbf{p}+\mathbf{q}_2-\mathbf{q}_1,<} a^\dagger_{\mathbf{p}+\mathbf{q}_2/2}(\mathbf{q}_2) a_{\mathbf{p}+\mathbf{q}_2-\mathbf{q}_1/2}(\mathbf{q}_1) \\
&\quad + \frac{1}{2} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \xi_{\mathbf{p}+\mathbf{q}_3-\mathbf{q}_2+\mathbf{q}_1,>}\ a^\dagger_{\mathbf{p}+\mathbf{q}_3-\mathbf{q}_2+\mathbf{q}_1/2}(\mathbf{q}_1) a^\dagger_{\mathbf{p}+\mathbf{q}_3/2}(\mathbf{q}_3) a_{\mathbf{p}+\mathbf{q}_3-\mathbf{q}_2/2}(\mathbf{q}_2) \\
&+ \frac{1}{4} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4} \xi_{\mathbf{p}+\mathbf{q}_4-\mathbf{q}_3+\mathbf{q}_2-\mathbf{q}_1,<}\ a^\dagger_{\mathbf{p}+\mathbf{q}_4/2}(\mathbf{q}_4) a^\dagger_{\mathbf{p}+\mathbf{q}_4-\mathbf{q}_3+\mathbf{q}_2/2}(\mathbf{q}_2) a_{\mathbf{p}+\mathbf{q}_4-\mathbf{q}_3+\mathbf{q}_2-\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}+\mathbf{q}_4-\mathbf{q}_3/2}(\mathbf{q}_3)
\end{aligned}
$$

and

$$
\begin{aligned}
c_{\mathbf{p},>} &= \xi_{\mathbf{p},>} + \sum_{\mathbf{q}_1} \xi_{\mathbf{p}-\mathbf{q}_1,<} a_{\mathbf{p}-\mathbf{q}_1/2}(\mathbf{q}_1) - \sum_{\mathbf{q}_2,\mathbf{q}_1} \xi_{\mathbf{p}-\mathbf{q}_2+\mathbf{q}_1,>} a^\dagger_{\mathbf{p}-\mathbf{q}_2+\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2) \\
&\quad - \frac{1}{2} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \xi_{\mathbf{p}-\mathbf{q}_3+\mathbf{q}_2-\mathbf{q}_1,<}\ a^\dagger_{\mathbf{p}-\mathbf{q}_3+\mathbf{q}_2/2}(\mathbf{q}_2) a_{\mathbf{p}-\mathbf{q}_3+\mathbf{q}_2-\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}-\mathbf{q}_3/2}(\mathbf{q}_3) \\
&+ \frac{1}{4} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4} \xi_{\mathbf{p}-\mathbf{q}_4+\mathbf{q}_3-\mathbf{q}_2+\mathbf{q}_1,>}\ a^\dagger_{\mathbf{p}-\mathbf{q}_4+\mathbf{q}_3-\mathbf{q}_2+\mathbf{q}_1/2}(\mathbf{q}_1) a^\dagger_{\mathbf{p}-\mathbf{q}_4+\mathbf{q}_3/2}(\mathbf{q}_3) a_{\mathbf{p}-\mathbf{q}_4+\mathbf{q}_3-\mathbf{q}_2/2}(\mathbf{q}_2) a_{\mathbf{p}-\mathbf{q}_4/2}(\mathbf{q}_4)
\end{aligned}
$$