Verifying the equations in Fermions as a non-local particle-hole excitation

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Our goal is to substitute $G_{\mathbf{k},a}$ (split into $G_{\mathbf{k}>}$ and $G_{\mathbf{p}<}$) into the given differential equations,

$$\begin{aligned}-\partial_\lambda\, G_{\mathbf{k}>}(\lambda,\lambda';t,t')\\ &= \sum_{\mathbf{p}} \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})}e^{\lambda+\lambda'}-1}\Big[e^{\lambda+\lambda'}e^{i(t-t')(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}\ G_{\mathbf{p}<}(\lambda,\lambda';t,t')(1-n_F(\mathbf{k}))\\ &\quad\times\Big(\theta(t'-t)+\theta(t-t')e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})}\Big) - G_{\mathbf{k}>}(\lambda,\lambda';t,t')\ n_F(\mathbf{p})\Big]\end{aligned} \tag{1}$$

and

$$\begin{aligned}-\partial_{\lambda'}\ G_{\mathbf{k}<}(\lambda,\lambda';t,t')\\ &= \sum_{\mathbf{p}}\ \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}e^{\lambda+\lambda'}-1}e^{i(t-t')(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}\ G_{\mathbf{p}>}(\lambda,\lambda';t,t')\ n_F(\mathbf{k})\\ &\quad\times\Big(\theta(t-t')+e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}\ \theta(t'-t)\Big) - G_{\mathbf{k}<}(\lambda,\lambda';t,t')\ (1-n_F(\mathbf{p}))\end{aligned} \tag{2}$$

and verify consistency.

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1 Green's Functions we're provided with:

Let's define: $G_{\mathbf{k}>} = \tilde{G}_{\mathbf{k},>}e^{-i\epsilon_{\mathbf{k}}(t-t')}$ and $G_{\mathbf{p}<} = \tilde{G}_{\mathbf{p},<}e^{-i\epsilon_{\mathbf{p}}(t-t')}$.

Their expansion gives:

$$\tilde{G}_{\mathbf{k},>} = e^{\lambda'}\mathrm{sgn}(t-t')\frac{(1-n_F(\mathbf{k}))e^{\theta(t-t')\beta(\epsilon_{\mathbf{k}}-\mu)}}{1+e^{\beta(\epsilon_{\mathbf{k}}-\mu)}e^{\lambda'}} \times \prod_{\mathbf{p},n_F(\mathbf{p})=1} f_1(\lambda) \prod_{\mathbf{p},n_F(\mathbf{p})=0} f_2(\lambda') \tag{3}$$

and,

$$\tilde{G}_{\mathbf{p},<} = \mathrm{sgn}(t-t')\frac{n_F(\mathbf{p})e^{-\theta(t'-t)\beta(\epsilon_{\mathbf{p}}-\mu)}}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{\lambda}} \times \prod_{\mathbf{k},n_F(\mathbf{k})=1} f_1(\lambda) \times \prod_{\mathbf{k},n_F(\mathbf{k})=0} f_2(\lambda') \tag{4}$$

where: $f_1(\lambda) = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}+e^{-\lambda}}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$ and $f_2(\lambda') = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}+1}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$.

We need to substitute these into the differential equation and test for equality. We have cancelled the $e^{-i\epsilon_{\mathbf{k}}(t-t')}$ term in the LHS and RHS. Eq.(1) becomes:

$$\begin{aligned}-\partial_\lambda\, \tilde{G}_{\mathbf{k}>}(\lambda,\lambda';t,t')\\ &= \sum_{\mathbf{p}} \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})}e^{\lambda+\lambda'}-1}\Big[e^{\lambda+\lambda'}\ \tilde{G}_{\mathbf{p}<}(\lambda,\lambda';t,t')(1-n_F(\mathbf{k}))\\ &\quad\times\Big(\theta(t'-t)+\theta(t-t')e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})}\Big) - \tilde{G}_{\mathbf{k}>}(\lambda,\lambda';t,t')\ n_F(\mathbf{p})\Big]\end{aligned} \tag{5}$$

Similarly, Eq.(2) becomes,

$$\begin{aligned}-\partial_{\lambda'}\ \tilde{G}_{\mathbf{k}<}(\lambda,\lambda';t,t')\\ &= \sum_{\mathbf{p}}\ \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}e^{\lambda+\lambda'}-1}\ \tilde{G}_{\mathbf{p}>}(\lambda,\lambda';t,t')\ n_F(\mathbf{k})\\ &\quad\times\Big(\theta(t-t')+e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}\ \theta(t'-t)\Big) - \tilde{G}_{\mathbf{k}<}(\lambda,\lambda';t,t')\ (1-n_F(\mathbf{p}))\end{aligned} \tag{6}$$

2 Verifying Eq.(1)

If we verify Eq.(5), then it means that we have verified Eq.(1).

2.1 Computing the Left-Hand Side of Eq.(5)

Substituting the given expressions for $\tilde{G}_{\mathbf{k},>}$ and $\tilde{G}_{\mathbf{p},<}$ into the left-hand side (LHS) of the differential equation provided. The LHS is $-\partial_\lambda \tilde{G}_{\mathbf{k},>}(\lambda, \lambda'; t, t')$, which requires us to compute the partial derivative of $\tilde{G}_{\mathbf{k},>}$ with respect to λ.

The expression for $\tilde{G}_{\mathbf{k},>}$ is from Eq.(3):

$$\tilde{G}_{\mathbf{k},>} = e^{\lambda'}\text{sgn}(t-t')\frac{(1-n_F(\mathbf{k}))e^{\theta(t-t')\beta(\epsilon_\mathbf{k}-\mu)}}{1+e^{\beta(\epsilon_\mathbf{k}-\mu)}e^{\lambda'}} \times \prod_{\mathbf{p},n_F(\mathbf{p})=1} f_1(\lambda) \prod_{\mathbf{p},n_F(\mathbf{p})=0} f_2(\lambda')$$

where:
$f_1(\lambda) = \frac{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}$,
$f_2(\lambda') = \frac{e^{-\beta(\epsilon_\mathbf{p}-\mu)}e^{-\lambda'}+1}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}$.

This expression depends on both λ and λ', and we need to compute its partial derivative with respect to λ. Note that λ appears in the product involving $f_1(\lambda)$, while λ' appears in the exponential factor $e^{\lambda'}$ and in the denominator of the fraction as well as in $f_2(\lambda')$.

Since $\tilde{G}_{\mathbf{k},>}$ is a product of terms, we apply the product rule for differentiation. Let us denote the expression as:

$$\tilde{G}_{\mathbf{k},>} = A \cdot B \cdot C$$

where:
$A = e^{\lambda'}\text{sgn}(t-t')\frac{(1-n_F(\mathbf{k}))e^{\theta(t-t')\beta(\epsilon_\mathbf{k}-\mu)}}{1+e^{\beta(\epsilon_\mathbf{k}-\mu)}e^{\lambda'}}$,
$B = \prod_{\mathbf{p},n_F(\mathbf{p})=1} f_1(\lambda)$,
$C = \prod_{\mathbf{p},n_F(\mathbf{p})=0} f_2(\lambda')$.

The partial derivative with respect to λ is:

$$\partial_\lambda \tilde{G}_{\mathbf{k},>} = (\partial_\lambda A) \cdot B \cdot C + A \cdot (\partial_\lambda B) \cdot C + A \cdot B \cdot (\partial_\lambda C)$$

Term 1: $\partial_\lambda A$

A does not depend on λ (since λ appears only in $f_1(\lambda)$, which is in B), so:

$$\partial_\lambda A = 0$$

Term 2: $\partial_\lambda C$

$C = \prod_{\mathbf{p},n_F(\mathbf{p})=0} f_2(\lambda')$ depends only on λ', not λ, so:

$$\partial_\lambda C = 0$$

Term 3: $\partial_\lambda B$

$B = \prod_{\mathbf{p},n_F(\mathbf{p})=1} f_1(\lambda)$. For a product, the derivative is:

$$\partial_\lambda B = \sum_{\mathbf{p},n_F(\mathbf{p})=1} (\partial_\lambda f_1(\lambda)) \prod_{\mathbf{q}\neq\mathbf{p},n_F(\mathbf{q})=1} f_1(\lambda)$$

Now, we compute $\partial_\lambda f_1(\lambda)$:

$$f_1(\lambda) = \frac{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}$$

The denominator is constant with respect to λ, so:

$$\partial_\lambda f_1(\lambda) = \frac{\partial_\lambda(e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda})}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}} = \frac{0+(-e^{-\lambda})}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}} = -\frac{e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}$$

Thus:

$$\partial_\lambda B = \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(-\frac{e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}\right) \prod_{\mathbf{q}\neq\mathbf{p}, n_F(\mathbf{q})=1} f_1(\lambda)$$

Therefore:

$$\partial_\lambda \tilde{G}_{\mathbf{k},>} = A \cdot \left(\sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(-\frac{e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}\right) \prod_{\mathbf{q}\neq\mathbf{p}, n_F(\mathbf{q})=1} f_1(\lambda)\right) \cdot C$$

$$-\partial_\lambda \tilde{G}_{\mathbf{k},>} = -A \cdot C \cdot \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(-\frac{e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}\right) \prod_{\mathbf{q}\neq\mathbf{p}, n_F(\mathbf{q})=1} f_1(\lambda)$$

$$= A \cdot C \cdot \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \frac{e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}} \prod_{\mathbf{q}\neq\mathbf{p}, n_F(\mathbf{q})=1} f_1(\lambda)$$

Multiplying and dividing by $f_1(\lambda) = \frac{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}{1+e^{-\beta(\epsilon_\mathbf{p}-\mu)}}$ to make the product over all $\mathbf{q}$,

$$-\partial_\lambda \tilde{G}_{\mathbf{k},>} = A \cdot C \cdot \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(\frac{e^{-\lambda}}{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}\right) \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda)$$

Substitutinge A and C:

$$-\partial_\lambda \tilde{G}_{\mathbf{k},>} = \left(e^{\lambda'} \mathrm{sgn}(t-t') \frac{(1-n_F(\mathbf{k}))e^{\theta(t-t')\beta(\epsilon_\mathbf{k}-\mu)}}{1+e^{\beta(\epsilon_\mathbf{k}-\mu)}e^{\lambda'}}\right) \cdot \left(\prod_{\mathbf{p}, n_F(\mathbf{p})=0} f_2(\lambda')\right) \cdot \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(\frac{e^{-\lambda}}{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}\right) \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda)$$

Substitution into the LHS

The LHS of the differential equation is $-\partial_\lambda \tilde{G}_{\mathbf{k},>}(\lambda, \lambda'; t, t')$, so the above result is the substituted form. The expression is as follows:as follows:

$$-\partial_\lambda \tilde{G}_{\mathbf{k},>} = \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(\frac{e^{-\lambda}}{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}\right) e^{\lambda'} \mathrm{sgn}(t-t') \frac{(1-n_F(\mathbf{k}))e^{\theta(t-t')\beta(\epsilon_\mathbf{k}-\mu)}}{1+e^{\beta(\epsilon_\mathbf{k}-\mu)}e^{\lambda'}} \cdot \prod_{\mathbf{q}, n_F(\mathbf{q})=0} f_2(\lambda') \cdot \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda)$$

$$= \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \left(\frac{e^{-\lambda}}{e^{-\beta(\epsilon_\mathbf{p}-\mu)}+e^{-\lambda}}\right) \tilde{G}_{\mathbf{k},>}$$

2.2 Computing the Right-Hand Side of Eq.(5)

The RHS of Eq.(5) is given by:

$$\sum_{\mathbf{p}} \frac{1}{e^{\beta(\epsilon_\mathbf{k}-\epsilon_\mathbf{p})}e^{\lambda+\lambda'}-1} \left[e^{\lambda+\lambda'}\ \tilde{G}_{\mathbf{p}<}(\lambda, \lambda'; t, t')(1-n_F(\mathbf{k}))\ \times \left(\theta(t'-t)+\theta(t-t')e^{\beta(\epsilon_\mathbf{k}-\epsilon_\mathbf{p})}\right) - \tilde{G}_{\mathbf{k}>}(\lambda, \lambda'; t, t')\ n_F(\mathbf{p})\right]$$

This means that the sum is implicitly over $n_F(\mathbf{p}) = 1$.

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \frac{1}{e^{\beta(\epsilon_\mathbf{k}-\epsilon_\mathbf{p})}e^{\lambda+\lambda'}-1} \left[e^{\lambda+\lambda'}\ \tilde{G}_{\mathbf{p}<}(\lambda, \lambda'; t, t')(1-n_F(\mathbf{k}))\ \times \left(\theta(t'-t)+\theta(t-t')e^{\beta(\epsilon_\mathbf{k}-\epsilon_\mathbf{p})}\right) - \tilde{G}_{\mathbf{k}>}(\lambda, \lambda'; t, t')\ \right]$$

We summarize a couple of things we noticed:

- $(1-n_F(\mathbf{k})) = 1$ and $n_F(\mathbf{p}) = 1$ are in both LHS and RHS. Hence, we need to check only when these conditions are met.
- We get a summation over p in LHS after taking the derivative, and we had a summation over p in RHS before. If these match term by term, our result is verified. If not we will have to struggle.

We take the cases,

2.2.1 Case 1: t' is greater than t

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} e^{\lambda+\lambda'} - 1} \Big[e^{\lambda+\lambda'} \ \tilde{G}_{\mathbf{p}<}(\lambda, \lambda'; t, t')(1 - n_F(\mathbf{k})) \ - \tilde{G}_{\mathbf{k}>}(\lambda, \lambda'; t, t') \Big]$$

$$= \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \frac{(1 - n_F(\mathbf{k}))}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} e^{\lambda+\lambda'} - 1} \Big[- \frac{e^{\lambda+\lambda'} \ e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda}} + \frac{e^{\lambda'}}{1 + e^{\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda'}} \Big] \times \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda) \times \prod_{\mathbf{q}, n_F(\mathbf{q})=0} f_2(\lambda')$$

Using mathematical, we find that,

$$\frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} e^{\lambda+\lambda'} - 1} \Big[- \frac{e^{\lambda+\lambda'} \ e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda}} + \frac{e^{\lambda'}}{1 + e^{\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda'}} \Big]$$
$$= - \left(\frac{e^{-\lambda}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} + e^{-\lambda}} \right) \times \frac{e^{\lambda'}}{1 + e^{\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda'}}$$

which is on the LHS when $t' > t$. Thus, the summation matches term by term.

Interesting question: Would we require this term-by-term matching condition, and can it also be related to creating local equations that may be simpler to solve?

2.2.2 Case 2: t' is lesser than t

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \frac{(1 - n_F(\mathbf{k}))}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} e^{\lambda+\lambda'} - 1} \Big[e^{\lambda+\lambda'} \ e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} \tilde{G}_{\mathbf{p}<}(\lambda, \lambda'; t, t') - \tilde{G}_{\mathbf{k}>}(\lambda, \lambda'; t, t') \Big]$$

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=1} \frac{(1 - n_F(\mathbf{k}))}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} e^{\lambda+\lambda'} - 1} \left[\frac{e^{\lambda+\lambda'} \ e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})}}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda}} - \frac{e^{\lambda'} e^{\beta(\epsilon_{\mathbf{k}}-\mu)}}{1 + e^{\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda'}} \right] \times \prod_{\mathbf{k}, n_F(\mathbf{k})=1} f_1(\lambda) \times \prod_{\mathbf{k}, n_F(\mathbf{k})=0} f_2(\lambda')$$

Using Mathematica, we find that:

$$\frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})} e^{\lambda+\lambda'} - 1} \left[\frac{e^{\lambda+\lambda'} \ e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{p}})}}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda}} - \frac{e^{\lambda'} e^{\beta(\epsilon_{\mathbf{k}}-\mu)}}{1 + e^{\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda'}} \right]$$
$$= \left(\frac{e^{-\lambda}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} + e^{-\lambda}} \right) \times \frac{e^{\lambda'} e^{\beta(\epsilon_{\mathbf{k}}-\mu)}}{1 + e^{\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda'}}$$

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3 Verifying Eq.(2)

If we verify Eq.(6), then it means that we have verified Eq.(2)

3.1 Computing the Left-Hand side of Eq.(6)

Substituting the given expressions for $\tilde{G}_{\mathbf{p},>}$ and $\tilde{G}_{\mathbf{k},<}$ into the left-hand side (LHS) of the differential equation provided. The LHS is $-\partial_{\lambda'} \tilde{G}_{\mathbf{k},<}(\lambda, \lambda'; t, t')$, which requires us to compute the partial derivative of $\tilde{G}_{\mathbf{k},<}$ with respect to λ. The expression for $\tilde{G}_{\mathbf{k},<}$is from Eq.(4):

$$\tilde{G}_{\mathbf{k},<} = \mathrm{sgn}(t - t') \frac{n_F(\mathbf{k}) e^{-\theta(t'-t)\beta(\epsilon_{\mathbf{k}}-\mu)}}{1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda}} \times \prod_{\mathbf{p}, n_F(\mathbf{p})=1} f_1(\lambda) \times \prod_{\mathbf{p}, n_F(\mathbf{p})=0} f_2(\lambda')$$

where:

$f_1(\lambda) = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} + e^{-\lambda}}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$,

$f_2(\lambda') = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'} + 1}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$. To compute the partial derivative $-\partial_{\lambda'} \tilde{G}_{\mathbf{k},<}(\lambda, \lambda'; t, t')$, we begin by carefully analyzing the given expression for $\tilde{G}_{\mathbf{k},<}$. The expression is a product of three distinct factors, and we must determine how each depends on the variable λ' to apply the product rule for differentiation correctly. Let us denote the expression as:

$$\tilde{G}_{\mathbf{k},<} = A \times B \times C,$$

where:
1. $A = \text{sgn}(t-t')\frac{n_F(\mathbf{k})e^{-\theta(t'-t)\beta(\epsilon_{\mathbf{k}}-\mu)}}{1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}e^{\lambda}}$,
2. $B = \prod_{\mathbf{p},n_F(\mathbf{p})=1} f_1(\lambda)$,
3. $C = \prod_{\mathbf{p},n_F(\mathbf{p})=0} f_2(\lambda')$,

with the functions defined as: $f_1(\lambda) = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}+e^{-\lambda}}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$,
$f_2(\lambda') = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}+1}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$.

The goal is to compute $-\frac{\partial}{\partial\lambda'}\tilde{G}_{\mathbf{k},<}$, where $\tilde{G}_{\mathbf{k},<}$ depends on both λ and λ', as well as t and t'. Since this is a partial derivative with respect to λ', we treat λ, t, and t' as constants.

Factor A: This term depends on λ, t, t', and various parameters (β, $\epsilon_{\mathbf{k}}$, μ, etc.), but it contains no λ'. Thus, $\frac{\partial A}{\partial\lambda'} = 0$.

Factor B: The product involves $f_1(\lambda)$, which depends only on λ and not on λ'. Hence, $\frac{\partial B}{\partial\lambda'} = 0$.

Factor C: This is a product over $\mathbf{p}$ where $n_F(\mathbf{k}) = 0$, and each $f_2(\lambda')$ explicitly depends on λ'. Therefore, the derivative will act solely on this factor.

Since A and B are independent of λ', the partial derivative of the product simplifies to:

$$\frac{\partial}{\partial\lambda'}(A \cdot B \cdot C) = A \cdot B \cdot \frac{\partial C}{\partial\lambda'},$$

where A and B are treated as constants with respect to λ'. Thus:

$$-\frac{\partial}{\partial\lambda'}\tilde{G}_{\mathbf{k},<} = -A \cdot B \cdot \frac{\partial C}{\partial\lambda'}.$$

The factor C is a product:

$$C = \prod_{\mathbf{p},n_F(\mathbf{p})=0} f_2(\lambda'),$$

where $f_2(\lambda') = \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}+1}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}$. For a product of functions $\prod_i g_i(x)$, the derivative is:

$$\frac{\partial}{\partial x}\prod_i g_i(x) = \left(\prod_i g_i(x)\right)\sum_i \frac{g_i'(x)}{g_i(x)}.$$

Here, $g_i = f_2(\lambda')$ for each $\mathbf{p}$ in the set where $n_F(\mathbf{k}) = 0$. We need to compute the logarithmic derivative $\frac{f_2'(\lambda')}{f_2(\lambda')}$.

Let the numerator of $f_2(\lambda')$ be $N = e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'} + 1$ and the denominator be $D = 1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}$, so $f_2(\lambda') = \frac{N}{D}$. Since D is constant with respect to λ':

$$\frac{\partial f_2}{\partial\lambda'} = \frac{1}{D}\frac{\partial N}{\partial\lambda'}.$$

Now, compute:

$$\frac{\partial N}{\partial\lambda'} = \frac{\partial}{\partial\lambda'}\left(e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'} + 1\right) = e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} \cdot (-e^{-\lambda'}) = -e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'},$$

since the derivative of $e^{-\lambda'} = -e^{-\lambda'}$ and the constant term 1 has derivative zero. Thus:

$$\frac{\partial f_2}{\partial\lambda'} = \frac{-e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}}{1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}.$$

Logarithmic Derivative:

$$\frac{\frac{\partial f_2}{\partial\lambda'}}{f_2(\lambda')} = \frac{-\frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}}{\frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}+1}{1+e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}} = -\frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}e^{-\lambda'} + 1}.$$

For the product C:

$$\frac{\partial C}{\partial \lambda'} = C \sum_{\mathbf{p}, n_F(\mathbf{k})=0} \frac{\frac{\partial f_2}{\partial \lambda'}}{f_2(\lambda')} = C \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \left(- \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'} + 1} \right).$$

Substituting back gives us:

$$-\frac{\partial}{\partial \lambda'} \tilde{G}_{\mathbf{k},<} = -A \cdot B \cdot \frac{\partial C}{\partial \lambda'} = -A \cdot B \cdot C \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \left(- \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'} + 1} \right).$$

The two negatives on the RHS of the above equation cancel, yielding:

$$-\frac{\partial}{\partial \lambda'} \tilde{G}_{\mathbf{k},<} = (A \cdot B \cdot C) \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'} + 1}.$$

Since $A \cdot B \cdot C = \tilde{G}_{\mathbf{k},<}$, the final result is:

$$-\frac{\partial}{\partial \lambda'} \tilde{G}_{\mathbf{k},<}(\lambda, \lambda'; t, t') = \tilde{G}_{\mathbf{k},<}(\lambda, \lambda'; t, t') \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'}}{e^{-\beta(\epsilon_{\mathbf{p}}-\mu)} e^{-\lambda'} + 1}.$$

3.2 Computing the Right-Hand Side of Eq.(6)

The RHS of Eq.(6) is given by:

$$\sum_{\mathbf{p}} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})} e^{\lambda+\lambda'} - 1} \left[\tilde{G}_{\mathbf{p}>}(\lambda, \lambda^{'}; t, t')\ n_F(\mathbf{k}) \times \left(\theta(t - t^{'}) + e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}\ \theta(t^{'} - t) \right) - \tilde{G}_{\mathbf{k}<}(\lambda, \lambda^{'}; t, t')\ (1 - n_F(\mathbf{p})) \right]$$

This means that the sum is implicitly over $n_F(\mathbf{p}) = 0$

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})} e^{\lambda+\lambda'} - 1} \left[\tilde{G}_{\mathbf{p}>}(\lambda, \lambda^{'}; t, t')\ n_F(\mathbf{k}) \times \left(\theta(t - t^{'}) + e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}\ \theta(t^{'} - t) \right) - \tilde{G}_{\mathbf{k}<}(\lambda, \lambda^{'}; t, t') \right]$$

Using,

$$\tilde{G}_{\mathbf{p},>} = e^{\lambda'} \mathrm{sgn}(t - t') \frac{(1 - n_F(\mathbf{p})) e^{\theta(t-t')\beta(\epsilon_{\mathbf{p}}-\mu)}}{1 + e^{\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda'}} \times \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda) \prod_{\mathbf{q}, n_F(\mathbf{q})=0} f_2(\lambda')$$

and

$$\tilde{G}_{\mathbf{k},<} = \mathrm{sgn}(t - t') \frac{n_F(\mathbf{k}) e^{-\theta(t'-t)\beta(\epsilon_{\mathbf{k}}-\mu)}}{1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda}} \times \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda) \times \prod_{\mathbf{q}, n_F(\mathbf{q})=0} f_2(\lambda')$$

where:

$f_1(\lambda) = \frac{e^{-\beta(\epsilon_{\mathbf{q}}-\mu)} + e^{-\lambda}}{1 + e^{-\beta(\epsilon_{\mathbf{q}}-\mu)}}$,

$f_2(\lambda') = \frac{e^{-\beta(\epsilon_{\mathbf{q}}-\mu)} e^{-\lambda'} + 1}{1 + e^{-\beta(\epsilon_{\mathbf{q}}-\mu)}}$

3.2.1 Case 1: t' is greater than t

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})} e^{\lambda+\lambda'} - 1} \left[\frac{-e^{\lambda'} e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}}{1 + e^{\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda'}}\ n_F(\mathbf{k}) + \frac{n_F(\mathbf{k}) e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}}{1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda}} \right] \times \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda) \prod_{\mathbf{q}, n_F(\mathbf{q})=0} f_2(\lambda')$$

Or,

$$RHS = \tilde{G}_{\mathbf{k},<} \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})} e^{\lambda+\lambda'} - 1} \left[\frac{-e^{\lambda'} e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})} (1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)} e^{\lambda})}{(1 + e^{\beta(\epsilon_{\mathbf{p}}-\mu)} e^{\lambda'}) e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}} + 1 \right]$$

This is equal to the LHS, when simplified on Mathematica.

3.2.2 Case 2: t' is lesser than t

$$RHS = \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}e^{\lambda+\lambda'}-1}\left[e^{\lambda'}\frac{(1-n_F(\mathbf{p}))e^{\beta(\epsilon_{\mathbf{p}}-\mu)}}{1+e^{\beta(\epsilon_{\mathbf{p}}-\mu)}e^{\lambda'}}n_F(\mathbf{k}) - \frac{n_F(\mathbf{k})}{1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}e^{\lambda}}\right] \times \prod_{\mathbf{q}, n_F(\mathbf{q})=1} f_1(\lambda) \times \prod_{\mathbf{q}, n_F(\mathbf{q})=0} f_2(\lambda')$$

$$RHS = \tilde{G}_{\mathbf{k},<} \sum_{\mathbf{p}, n_F(\mathbf{p})=0} \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}})}e^{\lambda+\lambda'}-1}\left[\frac{e^{\lambda'}e^{\beta(\epsilon_{\mathbf{p}}-\mu)}(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}e^{\lambda})}{1+e^{\beta(\epsilon_{\mathbf{p}}-\mu)}e^{\lambda'}} - 1\right]$$

This, too, is equal to the LHS when simplified on Mathematica.