Solving the integral equation of the qs

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1 Fermi-bilinears in terms of $a_{\mathbf{k}}(\mathbf{q})$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,<} \;=\; a^{\dagger}_{\mathbf{p}}(\mathbf{q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,>} \;=\; a_{\mathbf{p}}(-\mathbf{q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,<} \;=\; n_F(\mathbf{p})\;\delta_{\mathbf{q},0} - \sum_{\mathbf{q}_1}\frac{1}{N_>}\;a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>} \;=\; \sum_{\mathbf{q}_1}\frac{1}{N_>}\;a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$[a_{\mathbf{P}}(\mathbf{Q}),N_>] \;=\; a_{\mathbf{P}}(\mathbf{Q})$$

$$[a^{\dagger}_{\mathbf{P}}(\mathbf{Q}),N_>] \;=\; -\,a^{\dagger}_{\mathbf{P}}(\mathbf{Q})$$

$$f(N_>)\;a_{\mathbf{P}}(\mathbf{Q}) = a_{\mathbf{P}}(\mathbf{Q})\;f(N_>-1)$$

$$a^{\dagger}_{\mathbf{P}}(\mathbf{Q})\;f(N_>) = f(N_>-1)\;a^{\dagger}_{\mathbf{P}}(\mathbf{Q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,<} \;=\; a^{\dagger}_{\mathbf{p}}(\mathbf{q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,>} \;=\; a_{\mathbf{p}}(-\mathbf{q})$$

If $\mathbf{q} \neq 0$ then,

$$c_{\mathbf{p}-\mathbf{q}/2,<}c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} \;=\; \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\;\frac{1}{N_>+1}\;a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>} \;=\; \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\;\frac{1}{N_>+1}\;a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

Also we define,

$$c_{\mathbf{p},<}c^{\dagger}_{\mathbf{p},<} \;=\; lim_{\mathbf{q}\to 0}\;c_{\mathbf{p}-\mathbf{q}/2,<}c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}$$

$$c^{\dagger}_{\mathbf{p},>}c_{\mathbf{p},>} \;=\; lim_{\mathbf{q}\to 0}\;c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>}$$

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2 Field operator

Set,

$$e^{iN\theta_{\mathbf{p}}}e^{iN\theta_{\mathbf{p}'}} \;=\; 0$$

and

$$e^{iN\theta_{\mathbf{p}}}e^{-iN\theta_{\mathbf{p}'}} \;=\; \delta_{\mathbf{p},\mathbf{p}'}$$

This means,

$$c_{\mathbf{p},>} = \sum_{\mathbf{q_2}} \; e^{-iN\theta_{\mathbf{p}-\mathbf{q_2}}} \; c_{\mathbf{p}-\mathbf{q}_2,<} \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2,<}}} \; \frac{1}{\sqrt{N_>+1}} \; a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

and

$$c^{\dagger}_{\mathbf{p},<} \;=\; \sqrt{N_>+1} \; \frac{1}{\sqrt{n_{\mathbf{p},<}}} \; c^{\dagger}_{\mathbf{p},<} \; e^{iN\theta_{\mathbf{p}}} \; + \sum_{\mathbf{q_1}} \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1,>}}} \; c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>} \; \frac{1}{\sqrt{N_>+1}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{q_1}}} \; a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)$$

Time evolution of the non-interacting system:
Note that for free fermions,

$$c_{\mathbf{p},s}(t) \;=\; c_{\mathbf{p},s} \; e^{-i\epsilon_p \; t}$$

and

$$a_{\mathbf{k}}(\mathbf{q};t) \;=\; e^{-i\frac{\mathbf{k}.\mathbf{q}}{m} \; t} \; a_{\mathbf{k}}(\mathbf{q})$$

This means,

$$c_{\mathbf{p},>}(t) = \sum_{\mathbf{q_2}} \; e^{-iN\theta_{\mathbf{p}-\mathbf{q_2}}} \; c_{\mathbf{p}-\mathbf{q}_2,<}(t) \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2,<}}} \; \frac{1}{\sqrt{N_>+1}} \; a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2;t)$$

and

$$c^{\dagger}_{\mathbf{p},<}(t) \;=\; \sqrt{N_>+1} \; \frac{1}{\sqrt{n_{\mathbf{p},<}}} \; c^{\dagger}_{\mathbf{p},<}(t) \; e^{iN\theta_{\mathbf{p}}} \; + \sum_{\mathbf{q_1}} \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1,>}}} \; c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>}(t) \; \frac{1}{\sqrt{N_>+1}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{q_1}}} \; a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1;t)$$

Note that,

$$c_{\mathbf{p}-\mathbf{q}_2,<}(t) \; .... \; a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2;t) \;=\; e^{-i\epsilon_{\mathbf{p}-\mathbf{q}_2}t} e^{-i\frac{(\mathbf{p}-\mathbf{q}_2/2).\mathbf{q}_2}{m}t} \; c_{\mathbf{p}-\mathbf{q}_2,<} \; .... \; a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

$$= \; e^{-i\frac{(\mathbf{p}-\mathbf{q}_2)^2}{2m}t} e^{-i\frac{(\mathbf{p}-\mathbf{q}_2/2).\mathbf{q}_2}{m}t} \; c_{\mathbf{p}-\mathbf{q}_2,<} \; .... \; a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2) \;=\; e^{-i\epsilon_p t} \; c_{\mathbf{p}-\mathbf{q}_2,<} \; .... \; a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

Similarly,

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>}(t) \; ..... \; a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1;t) \;=\; e^{i\epsilon_p t} \; c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>} \; ..... \; a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)$$

Thus this correspondence gives the correct time evolution. Next we show that this correspondence when back substituted into the equations of the previous section leads to an identity.

3 Back substitution

Set,

$$e^{iN\theta_{\mathbf{p}}}e^{iN\theta_{\mathbf{p}'}} \;=\; 0$$

and

$$e^{iN\theta_{\mathbf{p}}}e^{-iN\theta_{\mathbf{p}'}} \;=\; \delta_{\mathbf{p},\mathbf{p}'}$$

This means,

$$c_{\mathbf{p}-\mathbf{q}/2,>} = \sum_{\mathbf{q_2}} \; e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2-\mathbf{q_2}}} \; c_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<} \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}}} \; \frac{1}{\sqrt{N_>+1}} \; a_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)$$

and

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} \;=\; \sqrt{N_>+1} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}} \; c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}}$$

$$+\sum_{\mathbf{q_1}} \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>}}} \; c^{\dagger}_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>} \; \frac{1}{\sqrt{N_>+1}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{q}/2+\mathbf{q_1}}} \; a_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)$$

$$c^\dagger_{\mathbf{p}+\mathbf{q}/2,>} = \sum_{\mathbf{q}_2} \; a^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2) \; \frac{1}{\sqrt{N_>+1}} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2}}$$

and

$$c_{\mathbf{p}-\mathbf{q}/2,<} \; = \; e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2}} \; c_{\mathbf{p}-\mathbf{q}/2,<} \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2,<}}} \; \sqrt{N_>+1}$$

$$+ \sum_{\mathbf{q}_1} a^\dagger_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) \; e^{iN\theta_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1}} \; \frac{1}{\sqrt{N_>+1}} \; c_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>} \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>}}}$$

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Off-diagonal:

$$c^\dagger_{\mathbf{p}+\mathbf{q}/2,>} c_{\mathbf{p}-\mathbf{q}/2,<} \; = \; \sum_{\mathbf{q}_2} \; a^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2) \; \frac{1}{\sqrt{N_>+1}} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2}} \; c_{\mathbf{p}-\mathbf{q}/2,<}$$

$$= \; \sum_{\mathbf{q}_2} \; a^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2) \; \frac{1}{\sqrt{N_>+1}} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2}} \quad e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2}} \; c_{\mathbf{p}-\mathbf{q}/2,<} \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2,<}}} \; \sqrt{N_>+1}$$

$$= \; a^\dagger_{\mathbf{p}}(\mathbf{q})$$

and

$$c^\dagger_{\mathbf{p}+\mathbf{q}/2,<} c_{\mathbf{p}-\mathbf{q}/2,>} \; = \; \sqrt{N_>+1} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}} \; c_{\mathbf{p}-\mathbf{q}/2,>}$$

$$+ \; \sum_{\mathbf{q}_1} \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>} \frac{1}{\sqrt{N_>+1}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1}} \; a_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) \; c_{\mathbf{p}-\mathbf{q}/2,>}$$

$$= \; \sqrt{N_>+1} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}} \; \sum_{\mathbf{q}_2} \; e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2}} \; c_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<} \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}}} \; \frac{1}{\sqrt{N_>+1}} \; a_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)$$

$$= \; a_{\mathbf{p}}(-\mathbf{q})$$

Diagonal:

$$c^\dagger_{\mathbf{p}+\mathbf{q}/2,>} c_{\mathbf{p}-\mathbf{q}/2,>} \; =$$

$$\sum_{\mathbf{q}_1,\mathbf{q}_2} \; a^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) \; \frac{1}{\sqrt{N_>+1}} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1}}$$

$$e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2}} \; c_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<} \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}}} \; \frac{1}{\sqrt{N_>+1}} \; a_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)$$

$$= \; \sum_{\mathbf{q}_1} \; a^\dagger_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) \; \frac{1}{N_>+1} \; a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

and,

$$c_{\mathbf{p}-\mathbf{q}/2,<} c^\dagger_{\mathbf{p}+\mathbf{q}/2,<} \; =$$

$$e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2}} \; c_{\mathbf{p}-\mathbf{q}/2,<} \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2,<}}} \; \sqrt{N_>+1} \; \sqrt{N_>+1} \; \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2,<} \; e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}}$$

$$+ \sum_{\mathbf{q}_1} a^\dagger_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) \; e^{iN\theta_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1}} \; \frac{1}{\sqrt{N_>+1}} \; c_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>} \; \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>}}}$$

$$\times \; \sum_{\mathbf{q}_2} \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2,>}}} \; c^\dagger_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2,>} \; \frac{1}{\sqrt{N_>+1}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2}} \; a_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2/2}(\mathbf{q}_2)$$

$$= \; \sum_{\mathbf{q}_1} a^\dagger_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) \; \frac{1}{N_>+1} \; a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

Thus we have now verified that the correspondence for the Fermi fields in terms of $a_{\mathbf{k}}(\mathbf{q})$ is fully correct.

4 Integral equation for q

Set ($s = >, <$),

$$\hat{q}_{\mathbf{p},s} \equiv c_{\mathbf{p},s} \frac{1}{\sqrt{n_{\mathbf{p},s}}}$$

so that,

$$\hat{q}_{\mathbf{p},>} \sqrt{n_{\mathbf{p},>}} = \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}} \hat{q}_{\mathbf{p}-\mathbf{q}_2,<} \frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

and

$$\sqrt{n_{\mathbf{p},<}} \hat{q}^{\dagger}_{\mathbf{p},<} = \sqrt{N_> + 1} \hat{q}^{\dagger}_{\mathbf{p},<} e^{iN\theta_{\mathbf{p}}} + \sum_{\mathbf{q}_1} q^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>} \frac{1}{\sqrt{N_> + 1}} e^{-iN\theta_{\mathbf{p}+\mathbf{q}_1}} a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)$$

We now have to solve these,

$$\hat{q}_{\mathbf{p},<} = \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1) e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}} \frac{1}{\sqrt{N_> + 1}} \hat{q}_{\mathbf{p}+\mathbf{q}_1,>} \left(\sqrt{n_{\mathbf{p},<}} - \sqrt{N_> + 1} e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

and

$$\hat{q}_{\mathbf{p},>} \sqrt{n_{\mathbf{p},>}} = \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}} \hat{q}_{\mathbf{p}-\mathbf{q}_2,<} \frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

Hence, to decouple, we first find,

$$\hat{q}_{\mathbf{p}+\mathbf{q}_1,>} = \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}+\mathbf{q}_1-\mathbf{q}_2}} \hat{q}_{\mathbf{p}+\mathbf{q}_1-\mathbf{q}_2,<} \frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}+\mathbf{q}_1-\mathbf{q}_2/2}(\mathbf{q}_2) \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1,>}}}$$

Thus we get the following integral equation:

$$\hat{q}_{\mathbf{p},<} = \sum_{\mathbf{q}_1,\mathbf{q}_2} a^{\dagger}_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1) \frac{e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}} e^{-iN\theta_{\mathbf{p}+\mathbf{q}_1-\mathbf{q}_2}}}{\sqrt{N_> + 1}} \hat{q}_{\mathbf{p}+\mathbf{q}_1-\mathbf{q}_2,<} \frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}+\mathbf{q}_1-\mathbf{q}_2/2}(\mathbf{q}_2) \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1,>}}} \left(\sqrt{n_{\mathbf{p},<}} - \sqrt{N_> + 1} e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

Set, $\mathbf{q}_1 - \mathbf{q}_2 = \mathbf{Q}$ and $\mathbf{q}_1 + \mathbf{q}_2 = 2\mathbf{K}$

$$\hat{q}_{\mathbf{p},<} = \sum_{\mathbf{K},\mathbf{Q}} a^{\dagger}_{\mathbf{p}+(\mathbf{K}+\mathbf{Q}/2)/2}(\mathbf{K}+\mathbf{Q}/2) \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}+\mathbf{Q}/2}} e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \hat{q}_{\mathbf{p}+\mathbf{Q},<}$$

$$\frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}+\mathbf{K}+\mathbf{Q}/2-(\mathbf{K}-\mathbf{Q}/2)/2}(\mathbf{K}-\mathbf{Q}/2) \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K}+\mathbf{Q}/2,>}}} \left(\sqrt{n_{\mathbf{p},<}} - \sqrt{N_> + 1} e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

or,

$$\hat{q}_{\mathbf{p},<} = \sum_{\mathbf{K},\mathbf{Q}} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \hat{q}_{\mathbf{p}+\mathbf{Q},<}$$

$$\frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}+\mathbf{K}/2+\mathbf{Q}/2}(\mathbf{K}-\mathbf{Q}) \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \left(\sqrt{n_{\mathbf{p},<}} - \sqrt{N_> + 1} e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

and separating $Q = 0$ and $Q \neq 0$ parts of the sum,

$$\hat{q}_{\mathbf{p},<} = \sum_{\mathbf{K}} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} e^{-iN\theta_{\mathbf{p}}}}{\sqrt{N_> + 1}} \hat{q}_{\mathbf{p},<}$$

$$\frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \left(\sqrt{n_{\mathbf{p},<}} - \sqrt{N_> + 1} e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

$$+ \sum_{\mathbf{K},\mathbf{Q}\neq 0} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \hat{q}_{\mathbf{p}+\mathbf{Q},<}$$

$$\frac{1}{\sqrt{N_> + 1}} a_{\mathbf{p}+\mathbf{K}/2+\mathbf{Q}/2}(\mathbf{K}-\mathbf{Q}) \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \left(\sqrt{n_{\mathbf{p},<}} - \sqrt{N_> + 1} e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

4.1 Try 1

Solve for $\hat{q}_{\mathbf{p},<}$ in terms of $q_{\mathbf{p}+\mathbf{Q},<}$ first...

$$a^\dagger_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K})\ \hat{q}_{\mathbf{p},<}\ =\ c^\dagger_{\mathbf{p}+\mathbf{K},>}c_{\mathbf{p},<}\ c_{\mathbf{p},<}\frac{1}{\sqrt{n_{\mathbf{p},<}}}=0$$

Hence, the first term becomes 0.

$$\hat{q}_{\mathbf{p},<}\ =\ \sum_{\mathbf{K},\mathbf{Q}\neq 0} a^\dagger_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K})\ \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}}\ e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_>+1}}\ \hat{q}_{\mathbf{p}+\mathbf{Q},<}$$

$$\frac{1}{\sqrt{N_>+1}}\ a_{\mathbf{p}+\mathbf{K}/2+\mathbf{Q}/2}(\mathbf{K}-\mathbf{Q})\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}}\ \left(\sqrt{n_{\mathbf{p},<}}\ -\ \sqrt{N_>+1}\ e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

4.2 Try 2

We try to write the θ's explicitly as functions of momentum $\mathbf{p}$. For example, let

$$\theta_{\mathbf{p}}=e^{i\lambda\mathbf{p}},$$

where λ is any constant.

We assume $\hat{q}_{\mathbf{p},s}$ can be written as:

$$\hat{q}_{\mathbf{p},<}=F(\theta(\mathbf{p}))G\times\frac{n_F(\mathbf{p})}{\sqrt{n_{\mathbf{p}}}}$$

and,

$$\hat{q}_{\mathbf{p},>}=F(\theta(\mathbf{p}))G\times\frac{\big(1-n_F(\mathbf{p})\big)}{\sqrt{n_{\mathbf{p}}}}$$

where F is an operator dependent on $\mathbf{p}$, and G is an operator independent of $\mathbf{p}$. Since we need to define the a's in terms of these, we need to find,

$$c_{\mathbf{p},<}=F(\theta(\mathbf{p})G\times n_F(\mathbf{p}),$$
$$c_{\mathbf{p},>}=F(\theta(\mathbf{p})G\times\big(1-n_F(\mathbf{p})\big),$$

$$c^\dagger_{\mathbf{p}-\mathbf{q}_2,<}=G^\dagger F^\dagger(\theta(\mathbf{p}-\mathbf{q}_2))\times n_F(\mathbf{p}-\mathbf{q}_2),$$

and,

$$c^\dagger_{\mathbf{p}+\mathbf{q}_1,>}=G^\dagger F^\dagger(\theta(\mathbf{p}+\mathbf{q}_1))\times\big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big),$$

Hence, now $a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$ becomes,

$$a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)=c^\dagger_{\mathbf{p}-\mathbf{q}_2,<}c_{\mathbf{p},>}$$
$$a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)=G^\dagger F^\dagger((\theta(\mathbf{p}-\mathbf{q}_2))n_F(\mathbf{p}-\mathbf{q}_2)\times F((\theta(\mathbf{p}))G\big(1-n_F(\mathbf{p})\big)$$
$$=\big(1-n_F(\mathbf{p})\big)n_F(\mathbf{p}-\mathbf{q}_2)G^\dagger F^\dagger((\theta(\mathbf{p}-\mathbf{q}_2)))F((\theta(\mathbf{p}))G$$

and $a^\dagger_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)$ is,

$$a^\dagger_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)=c^\dagger_{\mathbf{p}+\mathbf{q}_1,>}c_{\mathbf{p},<}$$
$$a^\dagger_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)=G^\dagger F^\dagger((\theta(\mathbf{p}+\mathbf{q}_1))\big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)\times F((\theta(\mathbf{p}))Gn_F(\mathbf{p})$$
$$=\big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)n_F(\mathbf{p})G^\dagger F^\dagger((\theta(\mathbf{p}+\mathbf{q}_1))F((\theta(\mathbf{p}))G$$

The last operators we need are the implicit, $\hat{q}_{\mathbf{p}-\mathbf{q}_2,<}$

$$\hat{q}_{\mathbf{p}-\mathbf{q}_2,<}=F(\theta(\mathbf{p}-\mathbf{q}_2))G\times\frac{n_F(\mathbf{p}-\mathbf{q}_2)}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2}}}$$

and $\hat{q}_{\mathbf{p}+\mathbf{q}_1,>}$

$$\hat{q}_{\mathbf{p}+\mathbf{q}_1,>} = F(\theta(\mathbf{p}+\mathbf{q}_1))G \times \frac{\big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1}}}$$

We can now try to satisfy these equations:

$$\hat{q}_{\mathbf{p},>}\ \sqrt{n_{\mathbf{p},>}} \ = \ \sum_{\mathbf{q}_2} \ e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ \hat{q}_{\mathbf{p}-\mathbf{q}_2,<}\ \frac{1}{\sqrt{N_>+1}}\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

and,

$$\hat{q}_{\mathbf{p},<} \ = \ \sum_{\mathbf{q}_1} a^\dagger_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)\ e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}}\ \frac{1}{\sqrt{N_>+1}}\ \hat{q}_{\mathbf{p}+\mathbf{q}_1,>}\ \Big(\sqrt{n_{\mathbf{p},<}} \ - \ \sqrt{N_>+1}\ e^{-iN\theta_\mathbf{p}}\Big)^{-1}$$

The equation for $\hat{q}_{\mathbf{p},>}\ \sqrt{n_{\mathbf{p},>}}$,

$$\hat{q}_{\mathbf{p},>}\ \sqrt{n_{\mathbf{p},>}} \ = \ \sum_{\mathbf{q}_2} \ e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ \hat{q}_{\mathbf{p}-\mathbf{q}_2,<}\ \frac{1}{\sqrt{N_>+1}}\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

becomes,

$$F(\theta(\mathbf{p}))G \times \frac{\big(1-n_F(\mathbf{p})\big)}{\sqrt{n_\mathbf{p}}}\ \sqrt{n_{\mathbf{p},>}} \ = \ \sum_{\mathbf{q}_2} \big(\ e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ F(\theta(\mathbf{p}-\mathbf{q}_2))G \times \frac{n_F(\mathbf{p}-\mathbf{q}_2)}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2}}}$$
$$\frac{1}{\sqrt{N_>+1}}\ \big(1-n_F(\mathbf{p})\big)n_F(\mathbf{p}-\mathbf{q}_2)G^\dagger F^\dagger((\theta(\mathbf{p}-\mathbf{q}_2)))\big)F((\theta(\mathbf{p}))G$$

Cancelling $\sqrt{n_{\mathbf{p},>}}$ in the LHS and the term $\big(1-n_F(\mathbf{p})\big)$ (since we are interested in $|\mathbf{p}|>k_F$, else the equation trivially holds), in LHS and RHS, gives,

$$F(\theta(\mathbf{p}))G = \ \sum_{\mathbf{q}_2} \big(\ e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ F(\theta(\mathbf{p}-\mathbf{q}_2))G \times \frac{n_F(\mathbf{p}-\mathbf{q}_2)}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2}}}$$
$$\frac{1}{\sqrt{N_>+1}}\big)n_F(\mathbf{p}-\mathbf{q}_2)G^\dagger F^\dagger((\theta(\mathbf{p}-\mathbf{q}_2)))\big)F((\theta(\mathbf{p}))G$$

We can further simplify this to (by cancelling $F(\theta(\mathbf{p}))G$ from LHS and RHS),

$$I = \ \sum_{\mathbf{q}_2} n_F(\mathbf{p}-\mathbf{q}_2)\ e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ F(\theta(\mathbf{p}-\mathbf{q}_2))G \times \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2}}}\ \frac{1}{\sqrt{N_>+1}}G^\dagger F^\dagger((\theta(\mathbf{p}-\mathbf{q}_2))) \tag{1}$$

The equation for $\hat{q}_{\mathbf{p},<}$

$$\hat{q}_{\mathbf{p},<}\ \ \Big(\sqrt{n_{\mathbf{p},<}} \ - \ \sqrt{N_>+1}\ e^{-iN\theta_\mathbf{p}}\Big) = \ \sum_{\mathbf{q}_1} \ a^\dagger_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)\ e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}}\ \frac{1}{\sqrt{N_>+1}}\ \hat{q}_{\mathbf{p}+\mathbf{q}_1,>}$$

becomes,

$$F(\theta(\mathbf{p}))G\frac{n_F(\mathbf{p})}{\sqrt{n_\mathbf{p}}}\ \ \Big(\sqrt{n_{\mathbf{p},<}} \ - \ \sqrt{N_>+1}\ e^{-iN\theta_\mathbf{p}}\Big) =$$
$$\sum_{\mathbf{q}_1}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}}\ \big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)n_F(\mathbf{p})G^\dagger F^\dagger((\theta(\mathbf{p}+\mathbf{q}_1))F((\theta(\mathbf{p}))G$$
$$\frac{1}{\sqrt{N_>+1}}F(\theta(\mathbf{p}+\mathbf{q}_1))G\frac{\big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1}}}.$$

We can further simplify, using $[n_F(\mathbf{p})F((\theta(\mathbf{p}))G,\ \frac{1}{\sqrt{N_>+1}}] = 0$, and $[n_F(\mathbf{p})F((\theta(\mathbf{p}))G, F(\theta(\mathbf{p}+\mathbf{q}_1))G\frac{\big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1}}}] = 0$, since for $\mathbf{q}_1=0$, we get 0.

We can hence cancel $n_F(\mathbf{p})F((\theta(\mathbf{p}))G$ from LHS and RHS, we get,

$$\left(I \ - \ \frac{\sqrt{N_>+1}\ e^{-iN\theta_\mathbf{p}}}{\sqrt{n_\mathbf{p}}}\ \right) = \ \sum_{\mathbf{q}_1}\ \big(1-n_F(\mathbf{p}+\mathbf{q}_1)\big)\ e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}}G^\dagger F^\dagger((\theta(\mathbf{p}+\mathbf{q}_1))\ \frac{1}{\sqrt{N_>+1}}F(\theta(\mathbf{p}+\mathbf{q}_1))G\frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1}}}. \tag{2}$$

We need to solve, for $F(\theta(\mathbf{p}))$ and G using Eq. (2)

$$\left(I \; - \; \frac{\sqrt{N_> + 1}\; e^{-iN\theta_{\mathbf{p}}}}{\sqrt{n_{\mathbf{p}}}}\right) = \sum_{\mathbf{q}_1} \big(1 - n_F(\mathbf{p} + \mathbf{q}_1)\big)\; e^{iN\theta_{\mathbf{p}+\mathbf{q}_1}} G^\dagger F^\dagger((\theta(\mathbf{p} + \mathbf{q}_1))$$

$$\times \frac{1}{\sqrt{N_> + 1}} F(\theta(\mathbf{p} + \mathbf{q}_1)) G \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1}}}$$

and Eq.(1)

$$I = \sum_{\mathbf{q}_2} n_F(\mathbf{p} - \mathbf{q}_2)\; e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\; F(\theta(\mathbf{p} - \mathbf{q}_2)) G \times \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2}}}\; \frac{1}{\sqrt{N_> + 1}} G^\dagger F^\dagger((\theta(\mathbf{p} - \mathbf{q}_2))).$$

I don't know how to proceed after substituting for I using Eq.(1) in Eq.(2)...