Simple random phase approximation (SRPA) vs. Generalised random phase approximation (GRPA)

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1 SRPA versus GRPA - I

Defn: $a_{\mathbf{k}}(\mathbf{q}) = c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<} c_{\mathbf{k}+\mathbf{q}/2,>}$, $n_{\mathbf{p}} = c^{\dagger}_{\mathbf{p}} c_{\mathbf{p}}$.

This means (these are exact),

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')] = 0$$

and

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q}) \, (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2})$$

and

$$[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})] \;=\; n_F(\mathbf{p}) \, c_{\mathbf{p}+\mathbf{q},>} \, \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

$$[c_{\mathbf{p},>}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \;=\; (1 - n_F(\mathbf{p})) \, c_{\mathbf{p}-\mathbf{q},<} \, \delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

and

$$[c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \;=\; [c_{\mathbf{p},>}, a_{\mathbf{k}}(\mathbf{q})] \;=\; 0$$

Now the big question is this, should we say (these are approximate),

SRPA: $[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'} \; n_F(\mathbf{k} - \mathbf{q}/2)(1 - n_F(\mathbf{k} + \mathbf{q}/2))$

OR

GRPA: $[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'} \; n_F(\mathbf{k} - \mathbf{q}/2)(1 - n_F(\mathbf{k} + \mathbf{q}/2))(n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2})$.

Suppose we select the simple **SRPA**.

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; [c_{\mathbf{p},<}, \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'} \; n_F(\mathbf{k} - \mathbf{q}/2)(1 - n_F(\mathbf{k} + \mathbf{q}/2))] \;=\; 0$$

On the other hand,

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; [[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})], a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] + [a_{\mathbf{k}}(\mathbf{q}), [c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]]$$

this means,

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; [n_F(\mathbf{p}) \, c_{\mathbf{p}+\mathbf{q},>} \, \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] + 0$$
$$= n_F(\mathbf{p}) \; \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2} \, (1 - n_F(\mathbf{p}+\mathbf{q})) \, c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<} \, \delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}$$

Thus we have reached a contradiction.
Suppose we select the **GRPA**.

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'} \; n_F(\mathbf{k} - \mathbf{q}/2)(1 - n_F(\mathbf{k} + \mathbf{q}/2)) \, [c_{\mathbf{p},<}, n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}] \;=$$
$$= \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'} \; n_F(\mathbf{k} - \mathbf{q}/2)(1 - n_F(\mathbf{k} + \mathbf{q}/2)) \, n_F(\mathbf{p}) \, (\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2}) \, c_{\mathbf{p}}$$

On the other hand,

$$[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]] = [[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] + [a_{\mathbf{k}}(\mathbf{q}),[c_{\mathbf{p},<},a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]]$$

this means,

$$[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]] = [n_F(\mathbf{p})\,c_{\mathbf{p}+\mathbf{q},>}\,\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2},a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] + 0$$

$$= n_F(\mathbf{p})\;\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}\,(1 - n_F(\mathbf{p}+\mathbf{q}))\,c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<}\,\delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}$$

This is not contradiction at least for $\mathbf{k} = \mathbf{k}'$ and $\mathbf{q} = \mathbf{q}'$.

2 SRPA versus GRPA - II

$$[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})] = n_F(\mathbf{p})\,c_{\mathbf{p}+\mathbf{q},>}\,\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

$$[c_{\mathbf{p},>},a_{\mathbf{k}}^{\dagger}(\mathbf{q})] = (1 - n_F(\mathbf{p}))\,c_{\mathbf{p}-\mathbf{q},<}\,\delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

Take a further commutator with $a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')$ and $a_{\mathbf{k}'}(\mathbf{q}')$,

$$[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] = n_F(\mathbf{p})\,[c_{\mathbf{p}+\mathbf{q},>},a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]\,\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

$$[[c_{\mathbf{p},>},a_{\mathbf{k}}^{\dagger}(\mathbf{q})],a_{\mathbf{k}'}(\mathbf{q}')] = (1 - n_F(\mathbf{p}))\,[c_{\mathbf{p}-\mathbf{q},<},a_{\mathbf{k}'}(\mathbf{q}')]\,\delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

In **SRPA**,

$$[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] = [c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]] + [[c_{\mathbf{p},<},a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')],a_{\mathbf{k}}(\mathbf{q})]$$

$$= [c_{\mathbf{p},<},\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))] + 0 = 0$$

whereas,

$$n_F(\mathbf{p})\,[c_{\mathbf{p}+\mathbf{q},>},a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]\,\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2} = n_F(\mathbf{p})\,(1 - n_F(\mathbf{p}+\mathbf{q}))\,c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<}\,\delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}\,\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

which is a contradiction. Whereas in **GRPA**,

$$[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] = [c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]] + [[c_{\mathbf{p},<},a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')],a_{\mathbf{k}}(\mathbf{q})]$$

$$= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\,(\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\,c_{\mathbf{p},<}$$

These two are equal when $\mathbf{q}' = \mathbf{q}$ (**GRPA**). Also,

$$[[c_{\mathbf{p},>},a_{\mathbf{k}}^{\dagger}(\mathbf{q})],a_{\mathbf{k}'}(\mathbf{q}')] = -n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\,[c_{\mathbf{p},>},n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}]$$

$$= -n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\,(\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\,c_{\mathbf{p},>}$$

and

$$(1 - n_F(\mathbf{p}))\,[c_{\mathbf{p}-\mathbf{q},<},a_{\mathbf{k}'}(\mathbf{q}')]\,\delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2} = (1 - n_F(\mathbf{p}))\,n_F(\mathbf{p}-\mathbf{q})\,c_{\mathbf{p}-\mathbf{q}+\mathbf{q}',>}\,\delta_{\mathbf{k}',\mathbf{p}-\mathbf{q}+\mathbf{q}'/2}\,\delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

These two are equal when $\mathbf{q} = \mathbf{q}'$ (**GRPA**).

3 Exact bosons

The goal now is to find exact bosons $b_{\mathbf{k}}(\mathbf{q})$ such that,

$$[b_{\mathbf{k}}(\mathbf{q}),b_{\mathbf{k}'}(\mathbf{q}')] = 0; \quad [b_{\mathbf{k}}(\mathbf{q}),b_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] = n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\,\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}$$

whereas the starting point is **GRPA**,

$$[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}(\mathbf{q}')] = 0; \quad [a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}^{\dagger}(\mathbf{q}')] = n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\,(n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2})\,\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}$$

and

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q}) \left(\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}\right)$$

Note that,

$$n_{\mathbf{p}} \neq n_F(\mathbf{p}) + \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) - \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})$$

whereas,

$$n_{\mathbf{p}} = n_F(\mathbf{p}) + \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) - \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})$$

and

$$(N_>)^2 = \sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q})$$

Set,

$$a_{\mathbf{k}}(\mathbf{q}) = \Gamma(b^{\dagger}b; \mathbf{k}, \mathbf{q}) b_{\mathbf{k}}(\mathbf{q})$$

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')] = 0; \quad [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2)) \left(n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}\right) \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'}$$

and

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q}) \left(\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}\right)$$

$$n_{\mathbf{p}} = n_F(\mathbf{p}) + \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) - \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})$$

and

$$(N_>)^2 = \sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q})$$