New technique to study many-body systems using non-local operators

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0.1 Acknowledgements

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0.2 Abstract

This report briefly discusses the basics of the cutting-edge formalism developed to study the field of condensed matter in understanding interactions. This report very briefly introduces the past work in the Luttinger liquid model by authors like N. N. Bogolyubov [1], S. Tomonaga [2], J.M. Luttinger [3], D. C. Mattis; E. H. Lieb [4] and F. D. M. Haldane[5]. The main focus is on a new method to study many-body physics, using operators in the Fock space known as Sea-displacement operators[6]. While a lot has been done, some work is still in progress; G. S. Setlur uses this formalism using Sea-displacement operators to calculate the Greens function for a many-body system at finite temperature in many dimensions. This report introduces the creation and annihilation operators in second quantization and defines Fock space. Once we are done with the basics of quantum mechanics, we move towards defining the Sea-displacement operators and some important operators in terms of the Sea-displacement operators. We derive the Fermi Dirac distribution using this formalism. We conclude with the results and future works in this exciting field.

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Chapter 1

History and Introduction

1.1 History

The inspiration behind this study starts with the discovery of the Luttinger liquid model, proposed by S. Tomonaga in 1950[2]. N. N. Bogolyubov inspired it in 1947, while he was trying to explain superfluidity, where he used expressions for bosons in terms of fermions to represent the Hamiltonian as a product of two boson operators[1]. The model showed that second-order interactions between electrons could be modelled as bosonic interactions in a one-dimensional multi-fermionic system under certain constraints. In 1963, J.M. Luttinger reformulated the theory in terms of Bloch sound waves. Through this, an exactly soluble model of a one-dimensional many-fermion system was proposed[3]. Soon after, in 1965, D.C. Mattis and E. H. Liebeld observed that charge density $\rho(\mathbf{p})$ was ipso facto associated with the Fermi-Dirac field. They then used this observation to solve for the and obtain the exact (and nontrivial) energy spectrum, free energy, and dielectric constant. The Luttinger liquid model was also extended to more realistic interactions in one dimension[4]. In 1992, F.D.M. Haldane proposed the of Luttinger's theorem and bosonization of the Fermi surface. This was an important work in this field, using the previously developed ideas and giving a better understanding through bosonization in one dimension[5]. In 1994, A. H. Castro Neto and Eduardo Fradkin bosonized the low-energy excitations of Fermi liquids in d-dimensions in the limit of long wavelengths.

The bosons were a coherent superposition of electron-hole pairs and were related with the displacements of the Fermi surface in some arbitrary direction. They constructed a coherent-state path integral for the "bosonized" theory and showed that it represented histories of the shape of the Fermi surface. The Landau theory of Fermi liquids could be obtained from this formalism in the absence of the nesting of the Fermi surface and singular interactions[7].

1.2 Introduction

We start with the results of the great physicists mentioned above and want to propose a new formalism that can tackle the harder questions like finding the many-particle Green's function for a system with many correlated degrees of freedom. These methods proposed are valuable to progress further research in physics as opposed to using just perturbative methods as they may have more scope in predicting new physical phenomena rather than explaining the current phenomena through perturbative means for taking into consideration interactions in fermionic systems.

For this, G. Setlur in 1995, proposed the idea of Sea-displacement operators in the fermionic subspace. These operators form the basis of this formalism[6].

The definitions for these operators are given in the respective chapter 3, Eq. (3.13),(3.14)(3.15) and (3.16).

In principle, these operators displace a fermion from below the fermi level to above it, hence creating a hole-particle pair. Thus, it forms the heart of our physics.

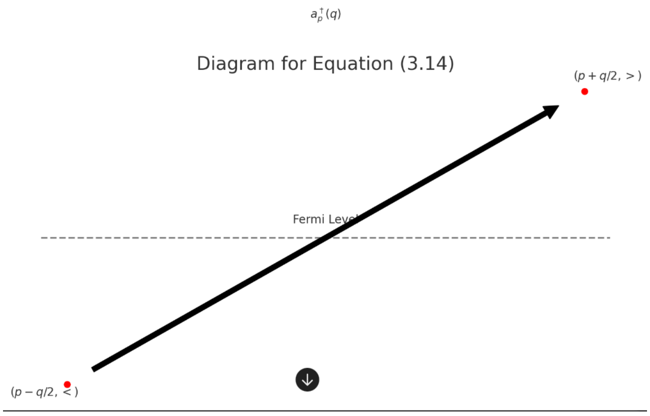

Figure 1.1: Diagram illustrating the process described by Eq. (3.15) The Fermi level is represented by the horizontal dashed line. The lower point represents the state $(p - q/2, <)$, and the upper point represents the state $(p + q/2, >)$. The arrow indicates the action of destroying a particle below the Fermi level and creating one above it, labeled as $a_p^\dagger(q)$.

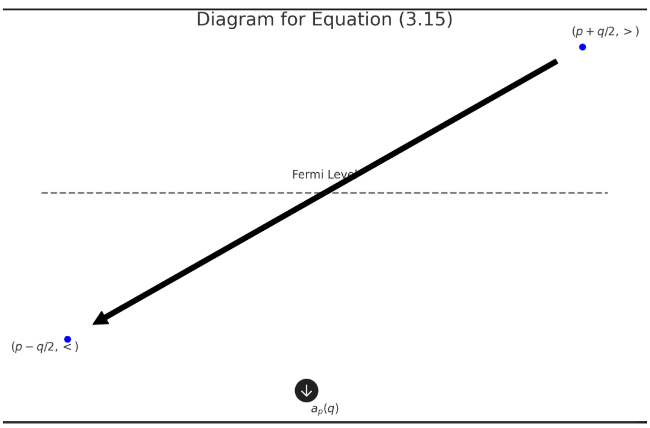

Figure 1.2: Diagram illustrating the process described by Eq. (3.16) The Fermi level is represented by the horizontal dashed line. The lower point represents the state $(p - q/2, <)$, and the upper point represents the state $(p + q/2, >)$. The arrow indicates the action of destroying a particle below the Fermi level and creating one above it, labeled as $a_p(q)$.

Chapter 2

Second quantization: Creation and annihilation operators

2.1 Occupation number representation

The occupation number representation simplifies the description of identical quantum particles by focusing on how many particles occupy each state, avoiding redundancy from labelling. This approach leads to a compact formulation essential for second quantization, a powerful tool in quantum mechanics applicable in various fields, including quantum field theory and statistical mechanics.

Instead of asking which particle is in which level, we ask the question how many particles are there in each level.

By focusing on the number of particles in each state, we achieve a more efficient description. The occupation number representation allows us to describe states without labelling identical particles. This leads to a more compact and manageable representation of particle systems.

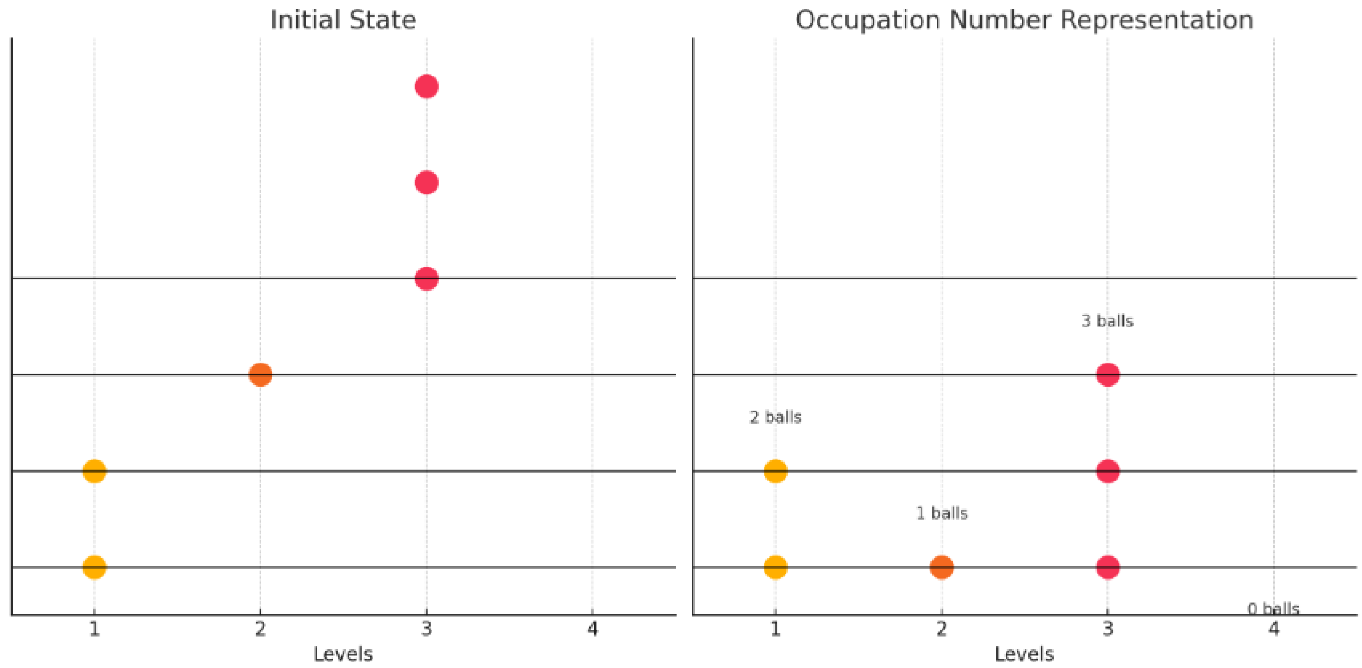

Figure 2.1: Initial state and occupation number representation of a 6-Ball System. The left diagram shows the initial state of the balls across four levels. The right diagram represents the occupation numbers, indicating the number of balls at each level.

2.1.1 Classification of Identical particles under occupation number representation

Identical particles can be classified as bosons or fermions, each requiring different mathematical treatments. Symmetric states describe bosons, while fermions are described by antisymmetric states. The symmetrization and antisymmetrization processes are essential for building appropriate bases for identical particle systems. These operations help in obtaining valid states in quantum mechanics.

In terms of equations for more clarity, let's consider a system of N identical particles who's state space $\mathcal{V}_N$ is the tensor product of the single particle state spaces V_i.

$$\mathcal{V}_N = V_1 \otimes V_2 \otimes \cdots \otimes V_N$$

The symmetric operator is given by,

$$\hat{S}^{\dagger} = \frac{1}{N!}\sum_{\alpha}\hat{P}_{\alpha},$$

and the anti-symmetric operator is given by,

$$\hat{S}^{-} = \frac{1}{N!}\sum_{\alpha}\eta_{\alpha}\hat{P}_{\alpha},$$

where $\hat{P}_{\alpha}$ is the permutation operator and η_{α} is given by,

$$\eta_{\alpha} = \begin{cases} +1 & \textit{even } \hat{P}_{\alpha} \\ -1 & \textit{odd } \hat{P}_{\alpha}. \end{cases}$$

The state space $\mathcal{V}_N$ consists of subspace $\mathcal{V}_{N+}$ and $\mathcal{V}_{N-}$ which represent Bosonic and Fermionic subspace respectively.

$$\mathcal{V}_N \implies \begin{cases} \text{Bosons:} & \mathcal{V}_{N+} \\ \text{Fermions:} & \mathcal{V}_{N-} \end{cases}$$

The basis states in $\mathcal{V}_N$ be given by:

$$\{|u_{i_1}\rangle \otimes |u_{j_2}\rangle \otimes \cdots \otimes |u_{p_N}\rangle\} \in \mathcal{V}_N, \tag{2.1}$$

where i represents the energy level and $1, 2, 3 \cdots, N$ represent the particle number. Each state in $\mathcal{V}_N$ is a linear combination of the basis states in $\mathcal{V}_N$.

$$|\psi\rangle \in \mathcal{V}_N \implies |\psi\rangle = \sum_{i_1 j_2 \cdots p_N} c_{i_1 j_2 \cdots p_N}|u_{i_1}\rangle|u_{j_2}\rangle \cdots |u_{p_N}\rangle$$

where $c_{i_1 j_2 \cdots p_N} \in \mathbb{C}$ are some constants. then the states in $\mathcal{V}_{N+}$ is given by,

$$|\psi^{\dagger}\rangle \in \mathcal{V}_{N+} \implies |\psi^{\dagger}\rangle = \hat{S}^{\dagger}\sum_{i_1 j_2 \cdots p_N} c_{i_1 j_2 \cdots p_N}|u_{i_1}\rangle|u_{j_2}\rangle \cdots |u_{p_N}\rangle,$$

and the states in V_- is given by,

$$|\psi^-\rangle \in \mathcal{V}_{N-} \implies |\psi^-\rangle = \hat{S}^- \sum_{i_1 j_2 \cdots p_N} c_{i_1 j_2 \cdots p_N} |u_{i_1}\rangle |u_{j_2}\rangle \cdots |u_{p_N}\rangle.$$

Since the particles are identical, applying any permutation to a general state in V_+ gives back the same state. These states called bosonic states. Thus,

$$\hat{P}_\alpha |\psi^\dagger\rangle = |\psi^\dagger\rangle.$$

Applying any permutation to a general state in V_- gives,

$$\hat{P}_\alpha |\psi^-\rangle = \eta_\alpha |\psi^-\rangle.$$

The redundancy in the basis states for systems of identical particles arises from different permutations yielding the same state in $\mathcal{V}_{N+}$ and $\mathcal{V}_{N-}$. This motivates the need for a new representation to minimize repeated states. Hence, the introduction of the occupation number representation aims to eliminate redundancy by focusing on the distinct occupation numbers of single particle states. This representation simplifies the analysis of identical particles.

While associating occupation numbers with basis states, we also ensure that the permutations of the tensor product of states do not alter the physical properties of the system. We will see that this new approach enhances clarity in particle representation.

The occupation number representation uniquely labels states of identical particles, enabling the characterization of any permutation of a basis state using occupation numbers. This method removes redundancy and simplifies analysis in quantum systems.

For bosons, the occupation number represented state reflects how many particles occupy each single particle state. Symbolically given by,

$$|m_1, m_2, ..., m_k, ...\rangle \in \mathcal{V}_{N+}$$

$$= \sqrt{\frac{N!}{m_1!m_2!...m_k!...}}\hat{S}^{\dagger}|u_1\rangle^{m_1}|u_2\rangle^{m_2}...|u_k\rangle^{m_k}...$$

where m_k represents number of particles occupying the kth state. Similarly, we write the state in occupation number representation for fermions, with the caveat that the state must obey the Pauli exclusion principle.

$$|m_1, m_2, ..., m_k, ...\rangle \in \mathcal{V}_{N-}$$

$$= \begin{cases} \sqrt{N!}\hat{S}^{-}|u_1\rangle...|u_i\rangle_{m_i}|u_2\rangle_{m_{i+1}}... & \text{if all } u_i \text{ different} \\ 0 & \text{if two } u_i \text{ equal} \end{cases}$$

As we can see in the case of fermions, the occupation numbers m_i can only be 0 or 1 due to the Pauli exclusion principle.

2.2 Fock space

Fock space (Def. 5.1.1) allows for the description of quantum systems with a variable number of particles, crucial in quantum field theory and statistical mechanics. It combines states of different particle numbers and employs creation and annihilation operators, facilitating calculations while maintaining a fixed particle count. Understanding Fock space is essential for advanced quantum mechanics.

The Fock space, $F(H)$ is defined as the direct sum of state spaces for different particle numbers, forming a basis from individual spaces.

We know from the previous discussion that, the single particle space for identical particles, H consists of

$$H_{=} \begin{cases} \textit{Bosonic subspace}: & \mathcal{V}_{1+} \\ \textit{Fermionic subspace}: & \mathcal{V}_{1-} \end{cases},$$

where we have substituted $N = 1$ for the state spaces discussed above.

Direct sums of vector spaces allow for the combination of dimensions, leading to a new space that encompasses all possible linear combinations as in Eq. (5.2).

Fock space serves as a mathematical tool for fixed particle systems, simplifying calculations while allowing temporary variations in particle number during the process, which we can see in Quantum field theory and Quantum statistical mechanics. Since this notion of state space leads to the notion of particle creation and annihilation, this space can be used to reflect the dynamic nature of quantum systems. Similarly, Quantum statistical mechanics utilizes Fock space to describe systems in equilibrium with particle reservoirs, facilitating the exchange of particles in the grand canonical ensemble.

Thus, we come to an understanding the operators that change the number of particles is essential for navigating the Fock space. These operators are known as creation and annihilation operators.

2.3 Bosonic creation and annihilation operators

Boson creation and annihilation operators are essential in quantum mechanics, allowing for the manipulation of bosonic particle states within Fock space. The creation operator adds a particle to a quantum state, while the annihilation operator removes a particle from the same state.

Suppose, we have single particle state, $|u_i\rangle$ in the single particle space, V_i,

$$|u_i\rangle \in V_i.$$

Since the occupation number representation of the bosonic states is given by,

$$|m_1, m_2, \ldots, m_i, \ldots\rangle \Rightarrow \quad m_i \text{ particles in state } |u_i\rangle,$$

we can define the bosonic creation operator doing the following operation,

$$\hat{a}^\dagger_{u_i}|m_1, m_2, \ldots, m_i, \ldots\rangle = \sqrt{m_i + 1}|m_1, m_2, \ldots, m_i + 1, \ldots\rangle.$$

Figure 2.2: Action of Bosons and Fermions in Fock Space. (Top) Multiple bosons can occupy the same energy state, as shown by the green dots. Creation ($a^\dagger$) and annihilation (a) operators are represented by blue and red arrows, respectively. (Bottom) Fermions obey the Pauli exclusion principle, with only one fermion per state.

The proportionality constant in the creation operator's definition aids in simplifying mathematical expressions, ensuring consistency in calculations. This choice is foundational in quantum mechanics applications ie. in defining the number operator for the state u_i.

There is a shift in the subspace of the Fock space due to the action of the creation operator,

$$H^{\otimes N} \xrightarrow{\hat{a}^\dagger_{u_i}} H^{\otimes N+1}.$$

The action on a single state is given by,

$$\hat{a}^\dagger_{u_i}|m_i\rangle = \sqrt{m_i+1}|m_i+1\rangle.$$

The adjoint operator is given by,

$$(\hat{a}^\dagger_{u_i})^\dagger = \hat{a}_{u_i}$$

Taking inner product $\langle m_i+1|\hat{a}^\dagger_{u_i}|m_i\rangle$, we get

$$\langle m_i+1|\hat{a}^\dagger_{u_i}|m_i\rangle = \sqrt{m_i+1}\,\underbrace{\langle m_i+1|m_i+1\rangle}_{=1} = \sqrt{m_i+1}.$$

Since $\sqrt{m_i+1}$ is real, complex conjugate of $\sqrt{m_i+1}$ is also real, hence taking complex conjugate on both sides gives,

$$\langle m_i+1|\hat{a}^\dagger_{u_i}|m_i\rangle = \langle m_i|\underbrace{(\hat{a}^\dagger_{u_i})^\dagger}_{\hat{a}_{u_i}}|m_i+1\rangle^* = \langle m_i|\hat{a}_{u_i}|m_i+1\rangle^*,$$

$$\implies \langle m_i|\hat{a}_{u_i}|m_i+1\rangle = (\sqrt{m_i+1})^* = \sqrt{m_i+1}$$

and hence, $\hat{a}_{u_i}|m_i+1\rangle \propto |m_i\rangle$.

Thus, we get,

$$\hat{a}_{u_i}|m_i+1\rangle = \sqrt{m_i+1}|m_i\rangle$$

$$\Longrightarrow \hat{a}_{u_i}|m_i\rangle = \sqrt{m_i}|m_i - 1\rangle.$$

2.3.1 Bosonic commutation rules

The commutation relations between any two creation/ annihilation operators for bosons indicate that they commute, meaning their order does not affect the outcome. Here, $\hat{a}_{u_i}$ is written in short as $\hat{a}_i$. The commutation relations are,

$$\hat{a}_i^\dagger \hat{a}_j^\dagger |m_i, m_j\rangle = \sqrt{m_i + 1}\sqrt{m_j + 1}|m_i + 1, m_j + 1\rangle$$
$$\hat{a}_j^\dagger \hat{a}_i^\dagger |m_i, m_j\rangle = \sqrt{m_j + 1}\sqrt{m_i + 1}|m_i + 1, m_j + 1\rangle.$$

Hence,

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0.$$

Similarly,

$$0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]^\dagger = (\hat{a}_i^\dagger \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_i^\dagger)^\dagger = (\hat{a}_i^\dagger \hat{a}_j^\dagger)^\dagger - (\hat{a}_j^\dagger \hat{a}_i^\dagger)^\dagger = \hat{a}_j \hat{a}_i - \hat{a}_i \hat{a}_j = [\hat{a}_j, \hat{a}_i].$$

The commutation relations between $\hat{a}_i$ and $\hat{a}_j^\dagger$ is given by,

(i) $i \neq j$

$$\hat{a}_i \hat{a}_j^\dagger |m_i, m_j\rangle = \sqrt{m_i}\sqrt{m_j + 1}|m_i - 1, m_j + 1\rangle$$

$$\hat{a}_j^\dagger \hat{a}_i |m_i, m_j\rangle = \sqrt{m_j + 1}\sqrt{m_i}|m_i - 1, m_j + 1\rangle.$$

Hence,

$$[\hat{a}_i, \hat{a}_j^\dagger] = 0, \quad i \neq j.$$

(ii) $i = j$

$$\hat{a}_i\hat{a}_i^\dagger|m_i\rangle = \sqrt{m_i+1}\hat{a}_i|m_i+1\rangle = (m_i+1)|m_i\rangle,$$

$$\hat{a}_i^\dagger\hat{a}_i|m_i\rangle = \sqrt{m_i}\hat{a}_i^\dagger|m_i-1\rangle = m_i|m_i\rangle.$$

Hence,

$$[\hat{a}_i, \hat{a}_j^\dagger] = 1, \quad i = j,$$

$$\implies [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{2.2}$$

2.3.2 Bosonic occupation number operator

The occupation number operator is defined as the product of creation and annihilation operators, which reveals the action of creation and annihilation operators on the Fock states and their eigenvalues.

$$\hat{n}_i|m_i\rangle = \hat{a}_i^\dagger\hat{a}_i|m_i\rangle = \sqrt{m_i}\hat{a}_i^\dagger|m_i-1\rangle\sqrt{m_i}|m_i\rangle = m_i|m_i\rangle.$$

The total number of bosons in a state can be given by the sum over all occupation number operators, ie.

$$\hat{N} = \sum_i \hat{n}_i = \sum_i \hat{a}_i^\dagger\hat{a}_i.$$

2.4 Fermionic creation and annihilation operators

The fermionic creation and annihilation operators create and remove particles while adhering to the Pauli exclusion principle.

Suppose, we have single particle state, $|u_i\rangle$ in the single particle space, V_i,

$$|u_i\rangle \in V_i.$$

States representation for the fermionic states is given by,

$$|u_i, u_j, \ldots, u_k, \ldots\rangle \Rightarrow \quad \text{1 particle in states } |u_i\rangle, |u_j\rangle \ldots |u_k\rangle \ldots \text{ and 0 particles in all other states },$$

we can define the fermionic creation operator doing the following operation,

$$\hat{c}^{\dagger}_{u_i}|u_j, u_k, \ldots, u_p, \ldots\rangle = |u_i, u_j, u_k, \ldots, u_p, \ldots\rangle \; where \; i \; \neq j, k, \ldots, p, \ldots$$

As we can see the proportionality constant is 1 as there can be only one particle in a given state. Also, Pauli's exclusion principle tells us that,

$$\hat{c}^{\dagger}_{u_i}|u_j, u_k, u_i, \ldots, u_p, \ldots\rangle = 0$$

The adjoint operator is given by,

$$(\hat{c}^{\dagger}_{u_i})^{\dagger} = \hat{c}_{u_i}$$

Taking inner product as last time, we get

$$\langle u_i, u_j, u_k, \ldots, u_p, \ldots|\hat{c}^{\dagger}_{u_i}|u_j, u_k, \ldots, u_p, \ldots\rangle = 1.$$

Since 1 is real, complex conjugate of 1 is also real, hence taking complex conjugate on both sides gives,

$$\langle u_i, u_j, u_k, \ldots, u_p, \ldots|\hat{c}^{\dagger}_{u_i}|u_j, u_k, \ldots, u_p, \ldots\rangle = \langle u_i, u_j, u_k, \ldots, u_p, \ldots|(\hat{c}^{\dagger}_{u_i})^{\dagger}|u_j, u_k, \ldots, u_p, \ldots\rangle^*,$$

$$\implies \langle u_j, u_k, \ldots, u_p, \ldots|\hat{c}_{u_i}|u_i, u_j, u_k, \ldots, u_p, \ldots\rangle = 1$$

and hence, $\hat{c}_{u_i}|u_i, u_j, u_k, \ldots, u_p, \ldots\rangle \propto |u_j, u_k, \ldots, u_p, \ldots\rangle$.

Thus, we get,

$$\hat{c}_{u_i}|u_i, u_j, u_k, \ldots, u_p, \ldots\rangle = |u_j, u_k, \ldots, u_p, \ldots\rangle.$$

Note:-

The annihilation operator, denoted as $\hat{c}_{u_i}$, removes a particle from a single particle state in a fermionic system. Its application is sensitive to the order of particles, introducing minus signs during exchanges. For example,

$$\hat{c}_{u_2}|u_1, u_2\rangle = -\hat{c}_{u_2}|u_2, u_1\rangle = |u_1\rangle$$

2.4.1 Fermionic commutation rules

Fermions and bosons exhibit distinct properties in quantum mechanics, particularly in their creation and annihilation operators. Fermions obey anti-commutation relations, while bosons follow commutation relations. This can be shown mathematically, by

$$\text{(i) } i \neq j$$

$$\hat{c}_i^\dagger, \hat{c}_j^\dagger, \ \textit{Assuming} \quad n_i = n_j = 0$$

If $n_i \ \textit{or} \ n_j \neq 0$, we get $\hat{c}_i^\dagger \hat{c}_j^\dagger = \hat{c}_j^\dagger, \hat{c}_i^\dagger = 0$ trivially. Hence we are left with the above case. In this case,

$$\hat{c}_i^\dagger \hat{c}_j^\dagger |u_k, \ldots\rangle = \hat{c}_i^\dagger |u_j, u_k, \ldots\rangle = |u_i, u_j, u_k, \ldots\rangle,$$

$$\hat{c}_j^\dagger \hat{c}_i^\dagger |u_k, \ldots\rangle = \hat{c}_j^\dagger |u_i, u_k, \ldots\rangle = |u_j, u_i, u_k, \ldots\rangle = -|u_i, u_j, u_k, \ldots\rangle.$$

Adding the two equations, we get,

$$(\hat{c}_i^\dagger \hat{c}_j^\dagger + \hat{c}_j^\dagger \hat{c}_i^\dagger)|u_k, \ldots\rangle = 0.$$

We know that the anti commutator is defined as:

$$\hat{A}\hat{B} + \hat{B}\hat{A} = \{\hat{A}, \hat{B}\}.$$

Hence,

$$\implies \quad \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0$$

(ii) $i = j$

The annihilation operator kills the state of fermions since two fermions cannot occupy the same quantum state. Hence,

$$(\hat{c}_i^\dagger)^2 = 0$$

Taking $\{\hat{c}_i^\dagger, \hat{c}_j^\dagger\}^\dagger$, we get

$$\implies \quad \{\hat{c}_i, \hat{c}_j\} = 0.$$

Hence for all i, j we have,

$$\{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0 \tag{2.3}$$

and

$$\{\hat{c}_i, \hat{c}_j\} = 0. \tag{2.4}$$

Since, each particle can occupy a unique state, resulting in occupation numbers of either 0 or 1, we start by taking cases.

For $\hat{c}_i$ and $\hat{c}_j^\dagger$ commutation,

(i) $i \neq j$

$$\hat{c}_i, \hat{c}_j^\dagger, \; Assuming \quad n_j = 0 \; n_i = 1.$$

In all the other values of n_i, n_j, the commutation rules is trivially satisfied. We calculate,

$$\hat{c}_i \hat{c}_j^\dagger |u_i, u_k, \ldots\rangle = \hat{c}_i |u_j, u_i, u_k, \ldots\rangle,$$

$$= -\hat{c}_i |u_i, u_j, u_k, \ldots\rangle = -|u_j, u_k, \ldots\rangle.$$

The operators operated in reverse order is,

$$\hat{c}_j^\dagger \hat{c}_i |u_i, u_k, \ldots\rangle = \hat{c}_j^\dagger |u_k, \ldots\rangle,$$

$$= |u_j, u_k, \ldots\rangle.$$

Thus, adding the two equation derived, we get,

$$(\hat{c}_i \hat{c}_j^\dagger + \hat{c}_j^\dagger \hat{c}_i)|u_i, u_k, \ldots\rangle = 0.$$

Hence, we get,

$$\implies \{\hat{c}_i, \hat{c}_j^\dagger\} = 0.$$

(ii) $i = j$

(a) $n_i = 0$

$$\hat{c}_i \hat{c}_i^\dagger |u_k, \ldots\rangle = \hat{c}_i |u_i, u_k, \ldots\rangle,$$

$$= |u_k, \ldots\rangle.$$

We have,

$$\hat{c}_i^\dagger \hat{c}_i |u_k, \ldots\rangle = 0,$$

as $n_i = 0$.

This implies for $n_i = 0$,

$$\{\hat{c}_i \hat{c}_i^\dagger\} = 1$$

(b) $n_i = 1$

$$\hat{c}_i \hat{c}_i^\dagger |u_k, \dots u_i \dots\rangle = 0,$$

as $n - i = 1$. We have,

$$\hat{c}_i^\dagger \hat{c}_i |u_k, \dots u_i \dots\rangle = \hat{c}_i^\dagger |u_k, \dots\rangle,$$

$$= |u_k, \dots u_i \dots\rangle$$

as $n_i = 1$.

This implies for $n_i = 1$,

$$\{\hat{c}_i, \hat{c}_i^\dagger\} = 1$$

Hence for all i, j we have,

$$\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{i,j} \tag{2.5}$$

2.4.2 Fermionic occupation number operator

The occupation number operator helps identify the number of particles in a state. It operates differently based on whether the occupation number is zero or one, affecting the state accordingly. For a state with zero particles,

$$m_i = 0,$$

the operator $\hat{c}_{u_i}^\dagger \hat{c}_{u_i}$, gives,

$$\hat{n}_{u_i} |u_j, \dots\rangle = \hat{c}_{u_i}^\dagger \hat{c}_{u_i} |u_j, \dots\rangle = 0.$$

And for a state with one particle,

$$m_i = 1,$$

The occupation number operator gives,

$$\hat{n}_{u_i}|u_i, u_j, \ldots\rangle = \hat{c}^\dagger_{u_i}\hat{c}_{u_i}|u_i, u_j, \ldots\rangle,$$

$$= \hat{c}^\dagger_{u_i}|u_j, \ldots\rangle \ = |u_i, u_j, \ldots\rangle,$$

The reason why we don't see any negative sign due to interchange in case of states like $|u_j, \ldots, u_i, \ldots\rangle$ is that,

$$|u_j, \ldots, u_i, \ldots\rangle \xrightarrow{(-1)^p} |u_i, u_j, \ldots\rangle \xrightarrow{\hat{n}_{u_i}} |u_i, u_j, \ldots\rangle \xrightarrow{(-1)^p} |u_j, \ldots, u_i, \ldots\rangle,$$

and hence,

$$(-1)^{2p} = 1.$$

The total number operator is then,

$$\hat{N} = \sum_{u_i} \hat{n}_{u_i} = \sum_{u_i} \hat{c}^\dagger_{u_i}\hat{c}_{u_i}.$$

2.4.3 Quantum field operators

Quantum field operators as the creation and the annihilation operators associated with the position representation.

We start with a single-particle basis $|u_i\rangle$ that is orthonormal. The associated creation operator is a $\hat{x}^\dagger_{u_i}$ dagger and the associated annihilation the operator is its adjoint. Let us also consider a second single-particle basis. In this case, we write the creation operator as a $\hat{x}^\dagger_{v_i}$ and the annihilation operator as a $\hat{x}_{v_i}$. If this creation and annihilation operator describes identical bosons they obey a set of commutation relations and if they describe fermions they obey a set of anti-commutation relations. Luckily for us, the expressions for the change of bases are the same for both types of particles. The creation operator in the $|v_i\rangle$ basis is:

$$\hat{x}^\dagger_{v_j}|0\rangle = \hat{I}|v_j\rangle = \sum_i |u_i\rangle\langle u_i||v_j\rangle$$

$$= \sum_i \langle u_i|v_j\rangle|u_i\rangle$$

$$= \sum_i \langle u_i|v_j\rangle\hat{x}^\dagger_{u_i}|0\rangle$$

Here, since we can replace the state $|0\rangle$ with any state, we conclude that,

$$\hat{x}^\dagger_{v_j} = \sum_i \langle u_i|v_j\rangle\hat{x}^\dagger_{u_i}. \tag{2.6}$$

Taking † on both sides of Eq.(2.7),

$$\hat{x}_{v_j} = \sum_i \langle u_i|v_j\rangle^*\hat{x}_{u_i} = \sum_i \langle v_j|u_i\rangle\hat{x}_{u_i}, \tag{2.7}$$

We get the equations Eq(2.7) and (2.6) for creation and annihilation operators resp. in a different basis.

Using the above equations and the properties of the position continuous eigenkets, $|r\rangle$,

- Orthonormality:

$$\langle \mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r} - \mathbf{r}')$$

- Completeness- it spans the whole Hilbert space (rigged)

$$\hat{I} = \int_{-\infty}^{\infty} d\mathbf{r}'|\mathbf{r}'\rangle\langle\mathbf{r}'|,$$

gives us the Quantum field operators.

Denoting the wave function as $\langle\mathbf{r}|u_i\rangle = u_i(\mathbf{r})$, the field operator is given by $\hat{\psi}(\mathbf{r})$,

$$\hat{\psi}(\mathbf{r})^\dagger = \sum_i u_i(\mathbf{r})^* \hat{x}_{u_i}^\dagger, \tag{2.8}$$

and,

$$\hat{\psi}(\mathbf{r}) = \sum_i u_i(\mathbf{r}) \hat{x}_{u_i}. \tag{2.9}$$

Conversely, any operator ($\hat{X}_{u_i}$) in the Fock space can hence be written as:

$$\hat{X}_{u_i}^\dagger = \int_{-\infty}^{\infty} d\mathbf{r}\ u_i(\mathbf{r}) \hat{\psi}(\mathbf{r})^\dagger, \tag{2.10}$$

and,

$$\hat{X}_{u_i} = \int_{-\infty}^{\infty} d\mathbf{r}\ u_i(\mathbf{r})^* \hat{\psi}(\mathbf{r}). \tag{2.11}$$

2.5 The One-Body Operator

2.5.1 Notation

Consider a state space of one particle:

$$V_q \rightarrow \text{State space}$$

The total state space for an N-particle system is the tensor product:

$$V = V_1 \otimes \cdots \otimes V_q \otimes V_{q+1} \otimes \cdots \otimes V_N \tag{2.12}$$

An operator $\hat{f}_q$ acts only on the q-th particle's state space. In the total state space V, this is represented as:

$$\hat{F}_q = 1_1 \otimes \cdots \otimes \hat{f}_q \otimes \cdots \otimes 1_N \quad \text{(represented for simplicity as } \hat{f}_q\text{)} \tag{2.13}$$

2.5.2 Recap: Symmetrization Postulate

Since this is a system of N identical particles, exchanging any two particles leads to the exact same physical system. Consequently, the state space is not the full space V, but a subspace of V spanned by totally symmetric states for bosons or totally antisymmetric states for fermions.

Similarly, the operators that act on a system of identical particles must be symmetric with respect to particle exchange. For a one-body operator $\hat{F}$, this means it is a sum of operators acting on each particle individually:

$$\hat{F} = \sum_{q=1}^{N} \hat{f}_q \tag{2.14}$$

2.5.3 Goal: One-Body Operator in Second Quantization

Our goal is to write a one-body operator in the second quantization formalism (occupation number representation). This means we want to write an operator which will act at only one particle "at a time" - individually. To apply this to symmetric/antisymmetric states, we must sum over all particles q, as they are identical.

Let $\{|u_k\rangle\}$ be a complete orthonormal basis for the single-particle state space. The operator $\hat{f}_q$ can be written as:

$$\hat{f}_q = \sum_{k,l} f_{kl} |u_k\rangle_q \langle u_l|_q$$

where the matrix elements are $f_{kl} = \langle u_k|\hat{f}|u_l\rangle$. The total operator is then:

$$\hat{F} = \sum_{q=1}^{N} \hat{f}_q = \sum_{q=1}^{N} \sum_{k,l} f_{kl} |u_k\rangle_q \langle u_l|_q \tag{2.15}$$

Taking the sum over q inside:

$$\hat{F} = \sum_{k,l} f_{kl} \left(\sum_{q=1}^{N} |u_k\rangle_q \langle u_l|_q \right) \tag{2.16}$$

2.5.4 From First to Second Quantization

We start from the occupation number representation. Any state is written as $|n_1, n_2, \dots\rangle$, where n_k is the number of particles in the single-particle state $|u_k\rangle$. For bosons $n_k \in \{0, 1, 2, \dots\}$, for fermions $n_k \in \{0, 1\}$.

In terms of the tensor product of single-particle states, this state is constructed through symmetrization (for bosons) or antisymmetrization (for fermions):

$$|n_1, n_2, \dots\rangle = \sqrt{\frac{N!}{\prod_i n_i!}} \hat{S}_\pm |u_{k_1}\rangle_1 |u_{k_2}\rangle_2 \dots |u_{k_N}\rangle_N \tag{2.17}$$

where $\hat{S}_\pm$ is the (anti)symmetrizer operator. $\hat{S}_+$ is for bosons and $\hat{S}_-$ is for fermions. They are defined using permutation operators $\hat{P}_\alpha$:

$$\hat{S}_+ = \frac{1}{N!} \sum_\alpha \hat{P}_\alpha \quad ; \quad \hat{S}_- = \frac{1}{N!} \sum_\alpha \eta_\alpha \hat{P}_\alpha$$

where $\eta_\alpha = +1$ for even permutations and -1 for odd permutations.

Let us operate with $\sum_q |u_k\rangle_q \langle u_l|_q$ on the state $|n_1, n_2, \dots\rangle$. The operator $\sum_q |u_k\rangle_q \langle u_l|_q$ is symmetric under particle exchange and thus commutes with $\hat{S}_\pm$.

$$\left(\sum_{q=1}^N |u_k\rangle_q \langle u_l|_q \right) |n_1, n_2, \dots\rangle = \sqrt{\frac{N!}{\prod_i n_i!}} \hat{S}_\pm \left(\sum_{q=1}^N |u_k\rangle_q \langle u_l|_q \right) |u_{k_1}\rangle_1 \dots |u_{k_N}\rangle_N$$

The operator $|u_k\rangle_q \langle u_l|_q$ acts on the q-th particle. It changes the state of particle q from $|u_l\rangle$ to $|u_k\rangle$. Summing over q means we do this for every particle that is in state $|u_l\rangle$. There are n_l such particles.

$$\left(\sum_{q=1}^N |u_k\rangle_q \langle u_l|_q \right) |u_{k_1}\rangle_1 \dots |u_{k_N}\rangle_N = n_l \times (\text{a state where one } |u_l\rangle \text{ is replaced by } |u_k\rangle)$$

This resulting state, after symmetrization, corresponds to the occupation number state

$|n_1, \ldots, n_l - 1, \ldots, n_k + 1, \ldots\rangle$. Let's call this new state $|n'\rangle$.

$$|n'\rangle = |n_1, \ldots, n_l' = n_l - 1, \ldots, n_k' = n_k + 1, \ldots\rangle$$

The normalization constant changes accordingly. Since, $|n_1, n_2, \ldots n_l' \ldots n_k' \ldots\rangle = \sqrt{\frac{N!}{\prod_i n_i'!}}\hat{S}_\pm |u_{k_1}\rangle_1 |u_{k_2}\rangle_2 \ldots |u_{k_l}\rangle_{l'} \ldots |u_{k_k}\rangle_{k'} \ldots |u_{k_N}\rangle_N$, writing the state in occupation number representation by taking the normalization to the RHS,

$$\begin{aligned}
\left(\sum_{q=1}^{N} |u_k\rangle_q \langle u_l|_q\right) |n_1, n_2, \ldots\rangle &= n_l \sqrt{\frac{N!}{\prod_i n_i!}} \sqrt{\frac{\prod_i n_i'!}{N!}} |n'\rangle \\
&= n_l \sqrt{\frac{n_l!(n_k)!}{(n_l - 1)!(n_k + 1)!}} |n'\rangle \\
&= n_l \sqrt{\frac{n_l}{n_k + 1}} |n'\rangle \\
&= \sqrt{n_l(n_k + 1)} |n_1, \ldots, n_l - 1, \ldots, n_k + 1, \ldots\rangle
\end{aligned}$$

We know that the annihilation operator $\hat{a}_l$ and creation operator $\hat{a}_k^\dagger$ act as:

$$\hat{a}_k^\dagger \hat{a}_l |n_1, \ldots, n_l, \ldots, n_k, \ldots\rangle = \sqrt{n_l(n_k + 1)} |n_1, \ldots, n_l - 1, \ldots, n_k + 1, \ldots\rangle$$

(This holds for both bosons and fermions, provided $n_k = 0$ for fermions before creation). This means:

$$\sum_{q=1}^{N} |u_k\rangle_q \langle u_l|_q = \hat{a}_k^\dagger \hat{a}_l \tag{2.18}$$

Substituting this back into the expression for the one-body operator:

$$\hat{F} = \sum_{k,l} f_{kl} \hat{a}_k^\dagger \hat{a}_l \tag{2.19}$$

This is the general form of a one-body operator (OBO) in second quantization.

2.6 The Two-Body Operator

2.6.1 Notation

Consider a two-particle operator $\hat{g}_{qq'}$ that describes an interaction between particle q and particle q'. It acts on the state space $V_q \otimes V_{q'}$. A symmetric two-body operator for an N-particle system is given by:

$$\hat{G} = \frac{1}{2} \sum_{\substack{q,q'=1 \\ q \neq q'}}^{N} \hat{g}_{qq'} \tag{2.20}$$

The factor of 1/2 avoids double counting, and $q \neq q'$ because a particle does not interact with itself.

2.6.2 Goal: Two-Body Operator in Second Quantization

Our goal is to write the two-body operator $\hat{G}$ in terms of creation and annihilation operators.

2.6.3 Commutation Relations

We will need the (anti)commutation relations for the creation and annihilation operators. Let $\hat{a}_i$ denote either a bosonic operator or a fermionic operator $\hat{c}_i$.

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \quad \text{(bosons)} \qquad \{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij} \quad \text{(fermions)}$$

These can be written in a general form:

$$\hat{a}_i \hat{a}_j^\dagger - \eta \hat{a}_j^\dagger \hat{a}_i = \delta_{ij}$$

where $\eta = +1$ for bosons and $\eta = -1$ for fermions. Similarly, for operators of the same type:

$$\hat{a}_k \hat{a}_l - \eta \hat{a}_l \hat{a}_k = 0 \implies \hat{a}_k \hat{a}_l = \eta \hat{a}_l \hat{a}_k$$

From these, we can derive a useful identity for a product of four operators:

$$\hat{a}_i^\dagger \hat{a}_k \hat{a}_j^\dagger \hat{a}_l = \hat{a}_i^\dagger (\eta \hat{a}_j^\dagger \hat{a}_k + \delta_{jk}) \hat{a}_l = \eta \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l + \delta_{jk} \hat{a}_i^\dagger \hat{a}_l \tag{2.21}$$

2.6.4 Derivation for a General Two-Body Operator

A general two-body operator $\hat{g}_{qq'}$ can be written in the basis $\{|u_k\rangle\}$:

$$\hat{g}_{qq'} = \sum_{i,j,k,l} g_{ijkl} |u_i\rangle_q |u_j\rangle_{q'} \langle u_k|_q \langle u_l|_{q'}$$

where $g_{ijkl} = \langle u_i, u_j | \hat{g} | u_k, u_l \rangle$. The total operator is:

$$\hat{G} = \frac{1}{2} \sum_{\substack{q,q'=1 \\ q \neq q'}}^{N} \sum_{i,j,k,l} g_{ijkl} |u_i\rangle_q |u_j\rangle_{q'} \langle u_k|_q \langle u_l|_{q'}$$

We can rearrange the sums:

$$\hat{G} = \frac{1}{2} \sum_{i,j,k,l} g_{ijkl} \sum_{\substack{q,q'=1 \\ q \neq q'}}^{N} |u_i\rangle_q \langle u_k|_q |u_j\rangle_{q'} \langle u_l|_{q'}$$

The sum over q, q' can be related to creation/annihilation operators.

$$\sum_{\substack{q,q'=1 \\ q \neq q'}}^{N} |u_i\rangle_q \langle u_k|_q |u_j\rangle_{q'} \langle u_l|_{q'} = \sum_{q,q'} |u_i\rangle_q \langle u_k|_q |u_j\rangle_{q'} \langle u_l|_{q'} - \sum_{q} |u_i\rangle_q \langle u_k|_q |u_j\rangle_q \langle u_l|_q$$

Using the one-body operator result $\sum_q |u_i\rangle_q \langle u_k|_q = \hat{a}_i^\dagger \hat{a}_k$:

$$= \left(\sum_q |u_i\rangle_q \langle u_k|_q \right) \left(\sum_{q'} |u_j\rangle_{q'} \langle u_l|_{q'} \right) - \delta_{kl} \sum_q |u_i\rangle_q \langle u_j|_q = \hat{a}_i^\dagger \hat{a}_k \hat{a}_j^\dagger \hat{a}_l - \delta_{jk} \hat{a}_i^\dagger \hat{a}_l$$

Using the four-operator identity from before, Eq. (2.21):

$$\hat{a}_i^\dagger \hat{a}_k \hat{a}_j^\dagger \hat{a}_l - \delta_{jk} \hat{a}_i^\dagger \hat{a}_l = \eta \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$$

For a symmetric interaction, $g_{ijkl} = g_{jilk}$. This allows us to write the final result in a standard form that holds for both bosons and fermions:

$$\hat{G} = \frac{1}{2} \sum_{i,j,k,l} g_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_l \hat{a}_k \tag{2.22}$$

The order of the annihilation operators is important. The state is $|u_k u_l\rangle$, so we first annihilate a particle in state $|u_k\rangle$ and then one in state $|u_l\rangle$. The standard convention is to write the annihilation operators in the reverse order of the kets, hence $\hat{a}_l \hat{a}_k$.

2.6.5 The Hamiltonian

With the results from Sec. 2.5 and 2.6, we can write a general Hamiltonian for an interacting many-body system in second quantization. The Hamiltonian typically consists of a kinetic energy term (one-body), a potential energy term from an external field (one-body), and a particle-particle interaction term (two-body).

$$\hat{H} = \hat{T} + \hat{V} + \hat{W} \tag{2.23}$$

where:

- $\hat{T} = \sum_{q=1}^{N} \hat{t}_q$, with $\hat{t} = \frac{\hat{p}^2}{2m}$ (Kinetic Energy). In second quantization:

$$\hat{T} = \sum_{i,j} t_{ij} \hat{a}_i^\dagger \hat{a}_j \quad \text{where} \quad t_{ij} = \langle u_i | \frac{\hat{p}^2}{2m} | u_j \rangle \tag{2.24}$$

- $\hat{V} = \sum_{q=1}^{N} \hat{v}_q$, with $\hat{v} = v(\hat{\vec{r}})$ (External Potential). In second quantization:

$$\hat{V} = \sum_{i,j} v_{ij} \hat{a}_i^\dagger \hat{a}_j \quad \text{where} \quad v_{ij} = \langle u_i | v(\hat{\vec{r}}) | u_j \rangle \tag{2.25}$$

- $\hat{W} = \frac{1}{2}\sum_{q\neq q'} \hat{w}_{qq'}$, with $\hat{w}_{12} = w(\hat{\vec{r}}_1, \hat{\vec{r}}_2)$ (Two-Body Interaction). In second quantization:

$$\hat{W} = \frac{1}{2}\sum_{i,j,k,l} w_{ijkl}\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_l \hat{a}_k \quad \text{where} \quad w_{ijkl} = \langle u_i u_j | w | u_k u_l \rangle \tag{2.26}$$

Examples of interaction potentials:

- **Coulomb potential:** $w(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}_1 - \vec{r}_2|}$
- **Yukawa potential:** $w(\vec{r}_1, \vec{r}_2) = -g^2\frac{e^{-m|\vec{r}_1-\vec{r}_2|}}{|\vec{r}_1-\vec{r}_2|}$

It is this interaction term $\hat{W}$ which is part of the Hamiltonian that is extremely hard to calculate in many-body problems.

Chapter 3

Non-local particle-hole creation operators

3.1 Outline

3.1.1 Aim

We aim to describe fermions using operators[1] corresponding to creating particle-hole pairs across a Fermi sea. The main goal is to create a formalism using this operator for exact computations of Green functions of many-particle systems[2]. Here, we have used

3.1.2 Notation[8]

1. First, we define our fermionic creation and annihilation operators in the momentum basis, given by $c^{\dagger}_{\mathbf{p}}$ and $c_{\mathbf{p}}$. From this, we infer that the anti-commutation of momentum creation operators is zero,

$$\left\{c^{\dagger}_{\mathbf{p}}, c^{\dagger}_{\mathbf{q}}\right\} = 0 \tag{3.1}$$

[1]It turns out that these operators are non-local.
[2]It is in progress.

and that the anti-commutator of creation and annihilation operators is the Dirac-delta operator of the momenta,

$$\left\{c^\dagger_{\mathbf{p}}, c_{\mathbf{q}}\right\}_{momenta} = \delta_{\mathbf{p},\mathbf{q}} \tag{3.2}$$

. We can now drop our notation for operator usin the ˆ.

2. The filled Fermi sea contains N_0, number of fermions and is given by the state $|F.S\rangle$.

3. The momentum distribution[3] of fermions at T=0 for non-interacting fermions is given by:

$$n_F(\mathbf{k}) = \Theta\left(k_F - |\mathbf{k}|\right) \quad . \tag{3.3}$$

This also means that we can write the number of fermions in the filled Fermi Sea, N_0 as a sum over all possible momentum in the N_0 fermion Fermi sea $\mathbf{k}$:

$$N_0 = \sum_{\mathbf{k}} n_F(\mathbf{k}). \tag{3.4}$$

4. We want to create a notation that distinguishes a creation or annihilation operator, with the criteria that it creates or annihilates fermion with $|\mathbf{p}| \leq k_F$ and otherwise is 0 and $|\mathbf{p}| > k_F$ and otherwise is 0. Here, $\mathbf{p}$ is the state's momentum that the creation or annihilation operator creates or annihilates. To facilitate this notation, we form the operators given by:

$$c^\dagger_{\mathbf{p},<} = n_F(\mathbf{p}) c^\dagger_{\mathbf{p}}, \tag{3.5}$$

$$c_{\mathbf{p},<} = n_F(\mathbf{p}) c_{\mathbf{p}} \tag{3.6}$$

and,

$$c^\dagger_{\mathbf{p},>} = (1 - n_F)(\mathbf{p}) c^\dagger_{\mathbf{p}}, \tag{3.7}$$

$$c_{\mathbf{p},>} = (1 - n_F)(\mathbf{p}) c_{\mathbf{p}}. \tag{3.8}$$

5. We want to construct the operator that gives us the number of particle-hole pairs in

[3]probability that a fermion has momentum, $\mathbf{k}$

the state it acts on. Such an operator can be written using the newly defined operators in point 4. The number of particles in a given eigenstate can be found by operating the operator comprising the product of annihilation and creation operators for that eigenstate on the system's state. For counting the number of states with momentum $\mathbf{k}$ below the Fermi momentum (k_F), $c^{\dagger}_{\mathbf{k},<}c_{\mathbf{k},<}$ operated on a state will give either 1 or 0 as we assume these are spinless fermions and no two fermions can occupy the same state. Summing over all momentum states in the N_0 particle Fermi sea gives us the total number of states below the Fermi level. Subtracting this sum from N_0 will give us the number of holes or the particle-hole pairs. Hence,

$$\hat{N}_{>} = N_0 - \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k},<}c_{\mathbf{k},<} = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k},>}c_{\mathbf{k},>} + \hat{N} - N_0, \tag{3.9}$$

where $\hat{N} = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}}c_{\mathbf{k}}$ This can be simplified only in term of $c_{\mathbf{k},<}c^{\dagger}_{\mathbf{k},<}$ using the anti-commutation relations,

$$c_{\mathbf{k}}c^{\dagger}_{\mathbf{k}} + c^{\dagger}_{\mathbf{k}}c_{\mathbf{k}} = \hat{1} \tag{3.10}$$

which gives,

$$\hat{N}_{>} = \sum_{\mathbf{k}} c_{\mathbf{k},<}c^{\dagger}_{\mathbf{k},<} \tag{3.11}$$

Since the $\hat{1}$ or the identity operator summed over all possible momentum states in the N_0 fermion Fermi sea gives just $N_0 \times \hat{1} = \hat{N}$, we get

$$\hat{N}_{>} = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k},>}c_{\mathbf{k},>} \tag{3.12}$$

6. We now move toward defining the Fermionic sea displacement operators. As the name suggests, there is a displacement in momenta; hence, this operator is dependent on two values of momenta. From the above definition of $c^{\dagger}_{\mathbf{k},<}$, $\hat{N}_{>}$ and $c_{\mathbf{k},>}$ we get the Fermionic

sea displacement operator to be:

$$A_{\mathbf{k}}(\mathbf{q}) = c^{\dagger}_{\mathbf{k}-\frac{\mathbf{q}}{2},<} \frac{1}{\sqrt{\hat{N}_{>}}} c_{\mathbf{k}+\frac{\mathbf{q}}{2},>} \tag{3.13}$$

and its adjoint is given by:

$$A^{\dagger}_{\mathbf{k}}(\mathbf{q}) = c^{\dagger}_{\mathbf{k}+\frac{\mathbf{q}}{2},>} \frac{1}{\sqrt{\hat{N}_{>}}} c_{\mathbf{k}-\frac{\mathbf{q}}{2},<} \tag{3.14}$$

7. Since the commutation of the operators,$[A_{\mathbf{k}}(\mathbf{q}), A^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$ is not simple and gives rise to very complicated operators depending on $\mathbf{k} = \mathbf{k}'$, $\mathbf{q} = \mathbf{q}'$ we use a simpler operator defined by:

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>} c_{\mathbf{p}-\mathbf{q}/2,<} \;=\; a^{\dagger}_{\mathbf{p}}(\mathbf{q}) \tag{3.15}$$

and its adjoint is given by:

8.

$$c^{\dagger}_{\mathbf{p}-\mathbf{q}/2,<} c_{\mathbf{p}+\mathbf{q}/2,>} \;=\; a_{\mathbf{p}}(\mathbf{q}), \tag{3.16}$$

3.2 Writing some important operators in terms of the Sea-displacement operators

3.2.1 Writing number conserving operators in terms of Sea-displacement operators

The operator similar to greater than occupation number operator

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>} c_{\mathbf{p}-\mathbf{q}/2,>} \;=\; \sum_{\mathbf{q}_1} \frac{1}{N_{>}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1) = \sum_{\mathbf{q}_1} A^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{q}_1}{2}}(\mathbf{q}_1)\, A_{\mathbf{p}-\mathbf{q}_1}(-\mathbf{q}+\mathbf{q}_1) \tag{3.17}$$

Proof[4] Unwinding the definition of the operators summed over on the right, gives,

$$A^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{q}_1}{2}}\left(\mathbf{q}_1\right)A_{\mathbf{p}-\mathbf{q}_1}\left(-\mathbf{q}+\mathbf{q}_1\right)$$

$$=\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(\frac{1}{\sqrt{N_>}}\right)\underbrace{\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\times\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)}_{1-n_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}}\left(\frac{1}{\sqrt{N_>}}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right).$$

Since, $[N_>, n_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}]=0$, (because the particle or hole number doesn't change) the above expression becomes,

$$A^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{q}_1}{2}}\left(\mathbf{q}_1\right)A_{\mathbf{p}-\mathbf{q}_1}\left(-\mathbf{q}+\mathbf{q}_1\right)$$

$$=\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(\frac{1}{\sqrt{N_>}}\right)\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\times\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(\frac{1}{\sqrt{N_>}}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right)$$

$$=\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(\frac{1}{N_>}\right)\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right)$$

We know that $[N_>, c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}]=0$ from Eq. (3.9) , hence we get,

$$A^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}+\frac{\mathbf{q}_1}{2}}\left(\mathbf{q}_1\right)A_{\mathbf{p}-\mathbf{q}_1}\left(-\mathbf{q}+\mathbf{q}_1\right)$$

$$=\left(\frac{1}{N_>}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q},>}{2}}\right)\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right).$$

$$=\frac{1}{N_>}\;a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$=\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(\frac{1}{N_>}\right)\underbrace{\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)}_{=1-n_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}}\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right)$$

Also, we know from Eqs. (3.5),(3.6) and (3.8) that $[c_{\mathbf{p},>}, n_{\mathbf{q},<}]=0$,

$$=\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right)\left(\frac{1}{N_>}\right)\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)$$

[4] Derived by Prof. Girish Sampath Setlur, verified by the author under his guidance.

Now, taking sum over $\mathbf{q}_1$ gives,

$$\sum_{\mathbf{q}_1} \frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$= \sum_{\mathbf{q}_1} \left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right)\left(\frac{1}{N_>}\right)\left(c_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2}-\mathbf{q}_1,<}\right)$$

$$= \left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},>}\right)$$

The operator similar to the lesser-than-occupation number operator

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} c_{\mathbf{p}-\mathbf{q}/2,<} \ = \ n_F(\mathbf{p})\ \delta_{\mathbf{q},0} - \sum_{\mathbf{q}_1} \frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1) \tag{3.18}$$

Proof[5]

$$\frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$= \left(\frac{1}{N_>}\right)\left(c^{\dagger}_{\mathbf{p}-\frac{\mathbf{q}}{2}+\mathbf{q}_1,>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},<}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2}+\mathbf{q}_1,>}\right).$$

Also, we know from Eqs. (3.5),(3.6) and (3.8) that $[c_{\mathbf{p},>}, c^{1\ or\ \dagger}_{\mathbf{q},<}] = 0$, (since at $\mathbf{p} = \mathbf{q}$, $c_{\mathbf{p},>}c^{\dagger}_{\mathbf{p},<} = 0$), hence we get,

$$\frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$= \left(\frac{1}{N_>}\right)\left(c^{\dagger}_{\mathbf{p}-\frac{\mathbf{q}}{2}+\mathbf{q}_1,>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2}+\mathbf{q}_1,>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},<}\right).$$

Now, taking sum over $\mathbf{q}_1$ gives, since we are in N_0 particle subspace, $\sum_{\mathbf{q}_1}\left(c^{\dagger}_{\mathbf{p}-\frac{\mathbf{q}}{2}+\mathbf{q}_1,>}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2}+\mathbf{q}_1,>}\right) = N_>$. This gives,

$$\sum_{\mathbf{q}_1} \frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$= \left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},<}\right)\left(c^{\dagger}_{\mathbf{p}+\frac{\mathbf{q}}{2},<}\right).$$

[5]Derived by Prof. Girish Sampath Setlur, verified by the author under his guidance.

From the commutation relations, Eq. (2.5),

$$LHS = n_F(\mathbf{p})\ \delta_{\mathbf{q},0} - \left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},<}\right)\left(c^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2},<}\right),$$

$$= n_F(\mathbf{p})\ \delta_{\mathbf{q},0} - \left(n_F(\mathbf{p})\ \delta_{\mathbf{q},0} - \left(c^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2},<}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},<}\right)\right)$$

$$= \left(c^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2},<}\right)\left(c_{\mathbf{p}-\frac{\mathbf{q}}{2},<}\right)$$

3.2.2 Kinetic energy in terms of Sea displacement operators

$$KE = \sum_{\mathbf{p}} \frac{|\mathbf{p}|^2}{2m} c^\dagger_{\mathbf{p}} c_{\mathbf{p}}$$

$$KE = \sum_{\mathbf{p}} \frac{|\mathbf{p}|^2}{2m} n_F(\mathbf{p}) + \sum_{\mathbf{k},\mathbf{q}} \frac{\mathbf{k}\cdot\mathbf{q}}{m} A^\dagger_{\mathbf{k}}(\mathbf{q}) A_{\mathbf{k}}(\mathbf{q}) \tag{3.19}$$

$$KE = \sum_{\mathbf{p}} \frac{|\mathbf{p}|^2}{2m} n_F(\mathbf{p}) + \sum_{\mathbf{k},\mathbf{q}} \frac{\mathbf{k}\cdot\mathbf{q}}{m} \frac{1}{N_>} a^\dagger_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) \tag{3.20}$$

Proof[6]

$$A^\dagger_{\mathbf{k}}(\mathbf{q}) A_{\mathbf{k}}(\mathbf{q}) = c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>} \frac{1}{\sqrt{N_>}} \underbrace{c_{\mathbf{k}-\frac{\mathbf{q}}{2},<} c^\dagger_{\mathbf{k}-\frac{\mathbf{q}}{2},<}}_{1-n_{\mathbf{k}-\frac{\mathbf{q}}{2},<}} \frac{1}{\sqrt{N_>}} c_{\mathbf{k}+\frac{\mathbf{q}}{2},>}$$

Since, $[N_>, n_{\mathbf{k}-\frac{\mathbf{q}}{2},<}] = 0$, and

$$A^\dagger_{\mathbf{k}}(\mathbf{q}) A_{\mathbf{k}}(\mathbf{q})$$

$$= c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>} \frac{1}{N_>} c_{\mathbf{k}-\frac{\mathbf{q}}{2},<} c^\dagger_{\mathbf{k}-\frac{\mathbf{q}}{2},<} c_{\mathbf{k}+\frac{\mathbf{q}}{2},>}.$$

Since, $[c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>}, N_>] = 0$ from Eq. (3.9),

$$A^\dagger_{\mathbf{k}}(\mathbf{q}) A_{\mathbf{k}}(\mathbf{q}) = \frac{1}{N_>} c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>} c_{\mathbf{k}-\frac{\mathbf{q}}{2},<} c^\dagger_{\mathbf{k}-\frac{\mathbf{q}}{2},<} c_{\mathbf{k}+\frac{\mathbf{q}}{2},>},$$

$$A^\dagger_{\mathbf{k}}(\mathbf{q}) A_{\mathbf{k}}(\mathbf{q}) = \frac{1}{N_>} a^\dagger_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}).$$

[6] Derived by Prof. Girish Sampath Setlur, verified by the author under his guidance.

Also, since $[c_{\mathbf{k}+\frac{\mathbf{q}}{2},>}, n_{\mathbf{k}-\frac{\mathbf{q}}{2},<}] = 0$ and $[c_{\mathbf{k}+\frac{\mathbf{q}}{2},>}, N_>] = 0$,

$$A^\dagger_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})$$

$$= c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>}\frac{1}{N_>}\underbrace{c_{\mathbf{k}-\frac{\mathbf{q}}{2},<}c^\dagger_{\mathbf{k}-\frac{\mathbf{q}}{2},<}}_{1-n_{\mathbf{k}-\frac{\mathbf{q}}{2},<}}c_{\mathbf{k}+\frac{\mathbf{q}}{2},>},$$

$$= c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>}c_{\mathbf{k}+\frac{\mathbf{q}}{2},>}\frac{1}{N_>}c_{\mathbf{k}-\frac{\mathbf{q}}{2},<}c^\dagger_{\mathbf{k}-\frac{\mathbf{q}}{2},<}.$$

Substituting the above in the LHS of the equation,

$$LHS = \sum_{\mathbf{p}}\frac{|\mathbf{p}|^2}{2m}n_F(\mathbf{p}) + \sum_{\mathbf{q}}\sum_{\mathbf{k}}\frac{\mathbf{k}\cdot\mathbf{q}}{m}c^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},>}c_{\mathbf{k}+\frac{\mathbf{q}}{2},>}\frac{1}{N_>}c^\dagger_{\mathbf{k}-\frac{\mathbf{q}}{2},<}c_{\mathbf{k}-\frac{\mathbf{q}}{2},<}$$

Uisng the formula,

$$\mathbf{k}\cdot\mathbf{q} = \frac{1}{2}\left(\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)^2 - \left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)^2\right),$$

we get,

$$LHS = \sum_{\mathbf{p}}\frac{|\mathbf{p}|^2}{2m}n_F(\mathbf{p}) + \sum_{\mathbf{q}}\sum_{\mathbf{k}}\frac{\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)^2}{2m}c^\dagger_{\mathbf{k}+\frac{q}{2},>}c_{\mathbf{k}+\frac{q}{2},>}\frac{1}{N_>}c^\dagger_{\mathbf{k}-\frac{q}{2},<}c_{\mathbf{k}-\frac{q}{2},<}$$

$$-\sum_{q}\sum_{\mathbf{k}}\frac{\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)^2}{2m}c^\dagger_{\mathbf{k}+\frac{q}{2},>}c_{\mathbf{k}+\frac{q}{2},>}\frac{1}{N_>}c^\dagger_{\mathbf{k}-\frac{q}{2},<}c_{\mathbf{k}-\frac{q}{2},<}$$

We can separate the two double sums, taking,

$$\mathbf{k}' = \mathbf{k}+\frac{\mathbf{q}}{2}$$

Calling the first double sum S_1, we get,

$$S_1 = \sum_{\mathbf{k}'}\frac{\mathbf{k}'^2}{2m}\hat{c}^\dagger_{\mathbf{k}',>}\hat{c}_{\mathbf{k}',>}\sum_{\mathbf{q}}\frac{1}{N_>}\hat{c}_{\mathbf{k}'-\mathbf{q},<}\hat{c}^\dagger_{\mathbf{k}'-\mathbf{q},<}$$

and summing over $\mathbf{q}$,

$$S_1 = \sum_{\mathbf{k}'} \frac{(\mathbf{k}')^2}{2m} \hat{c}^\dagger_{\mathbf{k}',>} \hat{c}_{\mathbf{k}',>} \frac{1}{N_>} N_> = \sum_{\mathbf{k}'} \frac{(\mathbf{k}')^2}{2m} \hat{c}^\dagger_{\mathbf{k}',>} \hat{c}_{\mathbf{k}',>}.$$

Calling the second double sum S_2, we get,

$$\mathbf{k}' = \mathbf{k} - \frac{\mathbf{k}}{2},$$

$$\implies S_2 = \sum_{\mathbf{k}'} \sum_{\mathbf{q}} \frac{(\mathbf{k}')^2}{2m} \hat{c}^\dagger_{\mathbf{k}'+\mathbf{q},>} \hat{c}_{\mathbf{k}'+\mathbf{q},>} \frac{1}{N_>} \hat{c}^\dagger_{\mathbf{k}',<} \hat{c}_{\mathbf{k}',<},$$

and summing over $\mathbf{q}$,

$$S_2 = \sum_{\mathbf{k}'} \frac{(\mathbf{k}')^2}{2m} \left(\frac{N_> - N_0 + \hat{N}}{N_>} \right) \hat{c}_{\mathbf{k}',<} \hat{c}^\dagger_{\mathbf{k}',<} = \sum_{\mathbf{k}'} \frac{(\mathbf{k}')^2}{2m} \hat{c}_{\mathbf{k}',<} \hat{c}^\dagger_{\mathbf{k}',<}.$$

Therefore, working in the N_0 particle subspace, the LHS becomes,

$$LHS = \sum_{\mathbf{p}} \frac{|\mathbf{p}|^2}{2m} n_F(\mathbf{p}) + \sum_{\mathbf{k}'} \frac{(\mathbf{k}')^2}{2m} \hat{c}^\dagger_{\mathbf{k}',>} \hat{c}_{\mathbf{k}',>} - \sum_{\mathbf{k}'} \frac{(\mathbf{k}')^2}{2m} \hat{c}_{\mathbf{k}',<} \hat{c}^\dagger_{\mathbf{k}',<}$$

Writing $n_F(\mathbf{k}) \times I = \hat{c}^\dagger_{\mathbf{k}',<} \hat{c}_{\mathbf{k}',<} + \hat{c}_{\mathbf{k}',<} \hat{c}^\dagger_{\mathbf{k}',<}$, LHS equals,

$$LHS = \sum_{\mathbf{k}'} \frac{|\mathbf{k}'|^2}{2m} \hat{c}^\dagger_{\mathbf{k}',<} \hat{c}_{\mathbf{k}',<} + \sum_{\mathbf{k}'} \frac{|\mathbf{k}'|^2}{2m} \hat{c}^\dagger_{\mathbf{k}',>} \hat{c}_{\mathbf{k}',>} = \sum_{\mathbf{k}'} \frac{|\mathbf{k}'|^2}{2m} \hat{c}^\dagger_{\mathbf{k}'} \hat{c}_{\mathbf{k}'}.$$

3.3 Important commutation properties of Sea-displacement operators

We find the commutators of $a_{\mathbf{p}}(\mathbf{q})$ and $a^\dagger_{\mathbf{p}}(\mathbf{q})$ so that we can use this algebra to substitute and find correlation function $< e^{-\lambda N_>} a^\dagger_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) >$ and further derive the Fermi-Dirac distribution in Section. 3.5.

- We find what is the commutator, $[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')]$

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')]$$

$$= [c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<} c_{\mathbf{k}+\mathbf{q}/2,>} \ , c^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2,<} c_{\mathbf{k}'+\mathbf{q}'/2,>} \ ].$$

Using the formula,

$$[AB, CD] = A\{B, C\}D - AC\{B, D\} + \{A, C\}DB - C\{A, D\}B, \tag{3.21}$$

we see that the second and third term cancels from Eq. (2.3) and Eq. (2.4) respectively. Hence,

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')]$$

$$= c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<} \{c_{\mathbf{k}+\mathbf{q}/2,>} \ , c^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2,<}\} c_{\mathbf{k}'+\mathbf{q}'/2,>} \ - c^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2,<} \{c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<}, c_{\mathbf{k}'+\mathbf{q}'/2,>} \ \} c_{\mathbf{k}+\mathbf{q}/2,>} \ .$$

From Eq. (3.5) and Eq.(3.8), we get,

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')] = 0 \tag{3.22}$$

We find what is the commutator, $[a^{\dagger}_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$,

$$[a^{\dagger}_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$$

$$= [c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,>} c_{\mathbf{k}-\mathbf{q}/2,<} \ , c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>} c_{\mathbf{k}'-\mathbf{q}'/2,<} \ ].$$

Similarly from Eq. (3.21), and Eq. (2.3) and Eq.(2.4), we get,

$$[a^{\dagger}_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$$

$$= c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,>} \{c_{\mathbf{k}-\mathbf{q}/2,<} \ , c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>}\} c_{\mathbf{k}'-\mathbf{q}'/2,<} \ - c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>} \{c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,>}, c_{\mathbf{k}'-\mathbf{q}'/2,<} \ \} c_{\mathbf{k}-\mathbf{q}/2,<} \ .$$

From Eq. (3.6) and Eq.(3.7), we get,

$$[a^{\dagger}_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = 0 \tag{3.23}$$

- We find what is the commutator, $[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$$

$$= [c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<} c_{\mathbf{k}+\mathbf{q}/2,>} \ , c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>} c_{\mathbf{k}'-\mathbf{q}'/2,<} \ ].$$

Similarly from Eq. (3.21), and Eq. (2.3) and Eq.(2.4), we get,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$$

$$= c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<} \{c_{\mathbf{k}+\mathbf{q}/2,>} \ , c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>}\} c_{\mathbf{k}'-\mathbf{q}'/2,<} \ - c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>} \{c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<}, c_{\mathbf{k}'-\mathbf{q}'/2,<} \ \} c_{\mathbf{k}+\mathbf{q}/2,>} \ .$$

From Eq. (2.5), we get,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$$

$$= c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<} c_{\mathbf{k}'-\mathbf{q}'/2,<} \ \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2,>} c_{\mathbf{k}+\mathbf{q}/2,>} \ \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}$$

$$= n_{\mathbf{k}'-\mathbf{q}'/2,<} \ \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - n_{\mathbf{k}+\mathbf{q}/2,>} \ \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}.$$

To summarize, we have,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = n_{\mathbf{k}'-\mathbf{q}'/2,<} \ \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - n_{\mathbf{k}+\mathbf{q}/2,>} \ \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}. \tag{3.24}$$

3.4 Approximations for commutation of Sea- displacement operator: a and its adjoint

We have derived the exact result,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = n_{\mathbf{k}'-\mathbf{q}'/2,<}\ \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - n_{\mathbf{k}+\mathbf{q}/2,>}\ \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}.$$

On closer inspection, we notice that the first term depends on the parameters of $a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')$, even when $\mathbf{k} \neq \mathbf{k}'$ and $\mathbf{q} \neq \mathbf{q}'$. Similarly, the second term depends on the parameters of $a_{\mathbf{k}}(\mathbf{q})$ even when $\mathbf{k} \neq \mathbf{k}'$ and $\mathbf{q} \neq \mathbf{q}'$.

This becomes a problem as for $\mathbf{k} \neq \mathbf{k}'$ and $\mathbf{q} \neq \mathbf{q}'$, the commutator, $[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]$ doesn't vanish.

Hence, there are two possibilities, first being that we work with this exact result as the commutator. However, the mathematics becomes very complicated.

The second possibility is to make an appropriate approximation from the physical aspect of systems, which gives rise to the Random Phase Approximation (RPA) and General Random Phase Approximation (GRPA).

3.4.1 SRPA versus GRPA - I

[7] We know that,

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}^{'})] = 0.$$

Using the commutation and anti-commutation algebra (Eq. (3.21)), we can calculate the commutator of $a_{\mathbf{k}}(\mathbf{q})$ with $n_{\mathbf{p}}$ as ,

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q})\ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}).$$

[7] Derived by Prof. Girish Sampath Setlur, verified by the author under his guidance.

and

$$[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})] \;=\; n_F(\mathbf{p})\; c_{\mathbf{p}+\mathbf{q},>}\; \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2},$$

$$[c_{\mathbf{p},>}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \;=\; (1 - n_F(\mathbf{p}))\; c_{\mathbf{p}-\mathbf{q},<}\; \delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

and

$$[c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \;=\; [c_{\mathbf{p},>}, a_{\mathbf{k}}(\mathbf{q})] \;=\; 0$$

SRPA is given by,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\; n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)). \tag{3.25}$$

and GRPA is given by,

$$[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\; n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))(n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}) \tag{3.26}$$

Suppose we select the simple **SRPA**.

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \;=\; [c_{\mathbf{p},<}, \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\; n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))] \;=\; 0$$

On the other hand,

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \;=\; [[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})], a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] + [a_{\mathbf{k}}(\mathbf{q}), [c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]]$$

this means,

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \;=\; [n_F(\mathbf{p})\; c_{\mathbf{p}+\mathbf{q},>}\; \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] + 0$$

$$=\; n_F(\mathbf{p})\;\; \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}\; (1 - n_F(\mathbf{p}+\mathbf{q}))\; c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<}\; \delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}$$

Thus we have reached a contradiction.

Suppose we select the **GRPA**.

$$[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\; n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\;[c_{\mathbf{p},<},n_{\mathbf{k}-\mathbf{q}/2}-n_{\mathbf{k}+\mathbf{q}/2}] \;=$$

$$=\;\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\; n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\;n_F(\mathbf{p})\;(\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}-\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\;c_{\mathbf{p}}$$

On the other hand,

$$[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; [[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]+[a_{\mathbf{k}}(\mathbf{q}),[c_{\mathbf{p},<},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]]$$

this means,

$$[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]] \;=\; [n_F(\mathbf{p})\;c_{\mathbf{p}+\mathbf{q},>}\;\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]+0$$

$$=\;n_F(\mathbf{p})\;\;\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}\;(1-n_F(\mathbf{p}+\mathbf{q}))\;c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<}\;\delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}$$

This is not contradiction at least for $\mathbf{k}=\mathbf{k}'$ and $\mathbf{q}=\mathbf{q}'$.

3.4.2 SRPA versus GRPA - II

[8] Using Eq. (3.21), we have,

$$[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})] \;=\; n_F(\mathbf{p})\;c_{\mathbf{p}+\mathbf{q},>}\;\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2},$$

and,

$$[c_{\mathbf{p},>},a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \;=\; (1-n_F(\mathbf{p}))\;c_{\mathbf{p}-\mathbf{q},<}\;\delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}.$$

Take a further commutator with $a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')$ and $a_{\mathbf{k}'}(\mathbf{q}')$,

$$[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] \;=\; n_F(\mathbf{p})\;[c_{\mathbf{p}+\mathbf{q},>},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]\;\delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

$$[[c_{\mathbf{p},>},a^{\dagger}_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}(\mathbf{q}')] \;=\; (1-n_F(\mathbf{p}))\;[c_{\mathbf{p}-\mathbf{q},<},a_{\mathbf{k}'}(\mathbf{q}')]\;\delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

[8] Derived by Prof. Girish Sampath Setlur, verified by the author under his guidance.

In **SRPA**,

$$[[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})], a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] \;=\; [c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] + [[c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})], a_{\mathbf{k}}(\mathbf{q})]$$

$$= [c_{\mathbf{p},<}, \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'} n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))] + 0 = 0$$

whereas,

$$n_F(\mathbf{p})\ [c_{\mathbf{p}+\mathbf{q},>}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]\ \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2} = n_F(\mathbf{p})\ (1-n_F(\mathbf{p}+\mathbf{q}))\ c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<}\ \delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}\ \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

which is a contradiction.

Whereas in **GRPA**,

$$[[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})], a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] \;=\; [c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] + [[c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})], a_{\mathbf{k}}(\mathbf{q})]$$

$$= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'} n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\ (\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\ c_{\mathbf{p},<}$$

These two are equal when $\mathbf{q}^{'} = \mathbf{q}$ (**GRPA**). Also,

$$[[c_{\mathbf{p},>}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})], a_{\mathbf{k}'}(\mathbf{q}^{'})] \;=\; -\,n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ [c_{\mathbf{p},>}, n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}]$$

$$=\; -\,n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ (\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\ c_{\mathbf{p},>}$$

and

$$(1-n_F(\mathbf{p}))\ [c_{\mathbf{p}-\mathbf{q},<}, a_{\mathbf{k}'}(\mathbf{q}^{'})]\ \delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2} \;=\; (1-n_F(\mathbf{p}))\ n_F(\mathbf{p}-\mathbf{q})\ c_{\mathbf{p}-\mathbf{q}+\mathbf{q}',>}\ \delta_{\mathbf{k}',\mathbf{p}-\mathbf{q}+\mathbf{q}'/2}\ \delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

These two are equal when $\mathbf{q} = \mathbf{q}^{'}$ (**GRPA**).

Thus, we conclude that GRPA is a more realistic option to approximate the commutation Eq.(3.24). We will now use this to derive the fermi-Dirac equation.

3.5 Deriving the Fermi-Dirac distribution[8]

To compute the momentum distribution at finite temperature, it is better to calculate the following finite temperature correlation function:

$$G(\mathbf{k},\mathbf{q};\lambda) =< e^{-\lambda N_>} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) >\equiv \frac{\mathrm{Tr}\left(e^{-\beta(H-\mu N)} e^{-\lambda N_>} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q})\right)}{\mathrm{Tr}\left(e^{-\beta(H-\mu N)}\right)} \tag{3.27}$$

Therefore,

$$\int_{\infty}^{\lambda} d\lambda' G\left(\mathbf{k},\mathbf{q};\lambda'\right) = - < e^{-\lambda N_>} \frac{1}{N_>} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) > \tag{3.28}$$

and,

$$< \hat{n}_{\mathbf{k},\lambda} >=< e^{-\lambda N_>} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} >= n_F(\mathbf{k}) < e^{-\lambda N_>} >$$
$$-\sum_{\mathbf{q}} \int_{\infty}^{\lambda} d\lambda' G\left(\mathbf{k}-\mathbf{q}/2,\mathbf{q};\lambda'\right) + \sum_{\mathbf{q}} \int_{\infty}^{\lambda} d\lambda' G\left(\mathbf{k}+\mathbf{q}/2,\mathbf{q};\lambda'\right)$$

Using the cyclic permutation property of the trace and the RPA algebra, we obtain the following expression for G.

$$G(\mathbf{k},\mathbf{q};\lambda) = \frac{e^{-\lambda} e^{-\beta\frac{\mathbf{k}\cdot\mathbf{q}}{m}}}{\left(1 - e^{-\lambda} e^{-\beta\frac{\mathbf{k}\cdot\mathbf{q}}{m}}\right)} \left(< \hat{n}_{\mathbf{k}-\mathbf{q}/2,\lambda} > - < \hat{n}_{\mathbf{k}+\mathbf{q}/2,\lambda} >\right)$$
$$n_F(\mathbf{k}-\mathbf{q}/2)\left(1 - n_F(\mathbf{k}+\mathbf{q}/2)\right)$$

Let $D(\varepsilon)$ be the density of states of the free theory. Thus, $D(\varepsilon)d\varepsilon = \frac{V}{(2\pi)^d}\Omega_d k^{(d-1)}dk$. Note that $D(\varepsilon)$ is an extensive quantity as is the summation $\sum_{\mathbf{q}}$. Thus we have to ensure that the dependence of $n(\lambda,\varepsilon)$ on λ is such that when integrated over λ, leads to an extensive quantity in the denominator. This matter may be made more explicit by differentiating with respect to λ.

$$\frac{d}{d\lambda}n_<(\lambda,\varepsilon) = -\theta\left(\varepsilon_F-\varepsilon\right)u(\lambda)$$
$$+\int_{\varepsilon_F}^{\infty} d\varepsilon' D\left(\varepsilon'\right)\frac{1}{\left(e^{\lambda}e^{\beta(\varepsilon'-\varepsilon)}-1\right)}\left(n_<(\lambda,\varepsilon)-n_>\left(\lambda,\varepsilon'\right)\right)\theta\left(\varepsilon_F-\varepsilon\right)$$
$$\frac{d}{d\lambda}n_>(\lambda,\varepsilon) =$$
$$-\int_{0}^{\varepsilon_F} d\varepsilon' D\left(\varepsilon'\right)\frac{1}{\left(e^{\lambda}e^{\beta(\varepsilon-\varepsilon')}-1\right)}\left(n_<\left(\lambda,\varepsilon'\right)-n_>(\lambda,\varepsilon)\right)\theta\left(\varepsilon-\varepsilon_F\right)$$
$$u(\lambda)=\left\langle e^{-\lambda N_>}N_>\right\rangle = N^0\left\langle e^{-\lambda N_>}\right\rangle - \int_0^{\varepsilon_F} d\varepsilon D(\varepsilon)n_<(\lambda,\varepsilon)$$

$$= \int_{\varepsilon_F}^{\infty} d\varepsilon D(\varepsilon)n_>(\lambda,\varepsilon) \tag{3.29}$$

We may suspect, that these equations can be solved by the following ansatz.

$$n_{>,<}(\lambda,\varepsilon) = \tilde{n}_{>,<}(\lambda,\varepsilon)e^{I(\lambda)} \tag{3.30}$$

where the function $I(\lambda)$ is extensive and is independent of the energy variable ε, whereas $\tilde{n}$ is intensive and depends on both the variables in general. Substituting this ansatz into the equations we find,

$$I'(\lambda)\tilde{n}_<(\lambda,\varepsilon)+\frac{d}{d\lambda}\tilde{n}_<(\lambda,\varepsilon) = -\theta\left(\varepsilon_F-\varepsilon\right)\tilde{u}(\lambda)$$
$$+\int_{\varepsilon_F}^{\infty} d\varepsilon' D\left(\varepsilon'\right)\frac{1}{\left(e^{\lambda}e^{\beta(\varepsilon'-\varepsilon)}-1\right)}\left(\tilde{n}_<(\lambda,\varepsilon)-\tilde{n}_>\left(\lambda,\varepsilon'\right)\right)\theta\left(\varepsilon_F-\varepsilon\right)$$
$$I'(\lambda)\tilde{n}_>(\lambda,\varepsilon)+\frac{d}{d\lambda}\tilde{n}_>(\lambda,\varepsilon) =$$
$$-\int_{0}^{\varepsilon_F} d\varepsilon' D\left(\varepsilon'\right)\frac{1}{\left(e^{\lambda}e^{\beta(\varepsilon-\varepsilon')}-1\right)}\left(\tilde{n}_<\left(\lambda,\varepsilon'\right)-\tilde{n}_>(\lambda,\varepsilon)\right)\theta\left(\varepsilon-\varepsilon_F\right)$$
$$\tilde{u}(\lambda) = \int_{\varepsilon_F}^{\infty} D(\varepsilon)d\varepsilon\tilde{n}_>(\lambda,\varepsilon).$$

Since $I'(\lambda),\tilde{u}$ and $D(\varepsilon)$ are extensive and $\tilde{n}$ is intensive, we may write after setting $\lambda=0$,

$$I'(0)\tilde{n}_{<}(0,\varepsilon) = -\theta\left(\varepsilon_F - \varepsilon\right)\tilde{u}(0)$$
$$+\int_{\varepsilon_F}^{\infty} d\varepsilon' D\left(\varepsilon'\right)\frac{1}{\left(e^{\beta(\varepsilon'-\varepsilon)}-1\right)}\left(\tilde{n}_{<}(0,\varepsilon)-\tilde{n}_{>}\left(0,\varepsilon'\right)\right)\theta\left(\varepsilon_F-\varepsilon\right)$$
$$I'(0)\tilde{n}_{>}(0,\varepsilon) =$$
$$-\int_{0}^{\varepsilon_F} d\varepsilon' D\left(\varepsilon'\right)\frac{1}{\left(e^{\beta(\varepsilon-\varepsilon')}-1\right)}\left(\tilde{n}_{<}\left(0,\varepsilon'\right)-\tilde{n}_{>}(0,\varepsilon)\right)\theta\left(\varepsilon-\varepsilon_F\right).$$

Dividing both sides of Eq. (3.5) by $\tilde{n}_{>}(0,\varepsilon)$ allows us to suspect that it should be possible to write

$$\left(\frac{\tilde{n}_{<}\left(0,\varepsilon'\right)}{\tilde{n}_{>}(0,\varepsilon)}-1\right)=\left(e^{\beta(\varepsilon-\varepsilon')}-1\right)h\left(\varepsilon'\right) \tag{3.31}$$

so that for some h we have,

$$I'(0) = -\int_{0}^{\varepsilon_F} d\varepsilon' D\left(\varepsilon'\right)h\left(\varepsilon'\right). \tag{3.32}$$

Interchanging ε and ε' in Eq. (3.31) and substituting into Eq. (3.5) we obtain

$$I'(0)\tilde{n}_{<}(0,\varepsilon) = -\theta\left(\varepsilon_F-\varepsilon\right)\tilde{u}(0)+h(\varepsilon)\tilde{u}(0)\theta\left(\varepsilon_F-\varepsilon\right). \tag{3.33}$$

We now multiply by the density of states and integrate to obtain, $I'(0)N^{0,<} = -N^0N^{0,>} - I'(0)N^{0,>}$, where the notation is self-explanatory. Thus $I'(0) = -N^{0,>} = -\tilde{u}(0)$. In other words, $\tilde{n}_{<}(0,\varepsilon) = 1 - h(\varepsilon)$. Hence we find,

$$\tilde{n}_{>}(0,\varepsilon) = \frac{1}{1+\frac{h(\varepsilon')}{1-h(\varepsilon')}e^{\beta(\varepsilon-\varepsilon')}} \tag{3.34}$$

Therefore, we may conclude that there exists a constant μ such that, $\frac{h(\varepsilon')}{1-h(\varepsilon')} = e^{\beta(\varepsilon'-\mu)}$, or $h\left(\varepsilon'\right) = \frac{1}{e^{-\beta(\varepsilon'-\mu)}+1}$. Thus $\tilde{n}_{>}(0,\varepsilon) = \tilde{n}_{<}(0,\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$. It is remarkable indeed that the Fermi-Dirac distribution emerges from a theory that is bosonic in character. However, it is important to impress upon the reader that it is the generalized RPA that takes into account fluctuations in the number of particle-hole pairs in a self-consistent manner that leads to the Fermi-Dirac distribution, whereas the simple-minded RPA fails to do so. This latter fact is easily seen by replacing the commutator $\left[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})\right] = n_F(\mathbf{k}-\mathbf{q}/2)\,(1-n_F(\mathbf{k}+\mathbf{q}/2))$, which is nothing but the simple-minded RPA.

Chapter 4

Conclusions and future work

There is a lot of ongoing research on this topic and in various directions. Using the operators $a_{\mathbf{k}}(\mathbf{q})$, we wish to create a formalism to re-derive the zero temperature, Green's function in the interaction picture and the Fermi-Dirac distribution. This might give us a further idea of how to take up research topics like the finite-temperature Green's function for the interaction picture.

As an end goal, we want to generalize the results obtained in the Luttinger liquid model to many dimensions while resolving seemingly conflicting principles, that is, that the Sea-Boson formalism leads to a violation of the Area law of entanglement entropy. We hence need to work around this by using non-linear operators, which are non-local, justifying the name of this report.

We conclude by saying that the path ahead of us is difficult, but we hope that as time progresses, we are able to explain in the least some news ways about thinking about many-body physics.

Chapter 5

Mathematical preliminaries

5.1 Second quantization

Definition 5.1.1 (Fock space)**.** The Fock space is the (Hilbert) direct sum of tensor products of copies of a single-particle Hilbert space [9].

$$F(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n} = \mathbb{C} \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus \cdots \tag{5.1}$$

Here $\mathbb{C}$, the complex scalars consist of the states corresponding to no particles, H the states of one particle, $(H \otimes H)$ the states of two identical particles etc. A general state in $F(H)$ is given by

$$|\Psi\rangle = |\Psi_0\rangle \oplus |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots = a|0\rangle \oplus \sum_i a_i|\psi_i\rangle \oplus \sum_{ij} a_{ij}|\psi_i, \psi_j\rangle \oplus \cdots \tag{5.2}$$

where

- capital $|\Psi_i\rangle$ (tensor product states) are expanded in terms of small $|\psi\rangle$ the single-particle Hilbert space states,
- $|0\rangle$ is a vector of length 1 called the vacuum state and $a \in \mathbb{C}$ is a complex coefficient,
- $|\psi_i\rangle \in H$ is a state in the single particle Hilbert space and $a_i \in \mathbb{C}$ is a complex coefficient,

- $|\psi_i, \psi_j\rangle = a_{ij}|\psi_i\rangle \otimes |\psi_j\rangle + a_{ji}|\psi_j\rangle \otimes |\psi_i\rangle \in S(H \otimes H)$, and $a_{ij} = \nu a_{ji} \in \mathbb{C}$ is a complex coefficient, etc.

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