Field operator in terms of Fermi Bilinears-6

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1 Fermi-bilinears in terms of $a_{\mathbf{k}}(\mathbf{q})$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,<} = a^{\dagger}_{\mathbf{p}}(\mathbf{q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,>} = a_{\mathbf{p}}(-\mathbf{q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,<} = n_F(\mathbf{p})\ \delta_{\mathbf{q},0} - \sum_{\mathbf{q}_1}\frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>} = \sum_{\mathbf{q}_1}\frac{1}{N_>}\ a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

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$$[a_{\mathbf{p}}(\mathbf{Q}), N_>] = a_{\mathbf{p}}(\mathbf{Q})$$

$$[a^{\dagger}_{\mathbf{p}}(\mathbf{Q}), N_>] = -a^{\dagger}_{\mathbf{p}}(\mathbf{Q})$$

$$f(N_>)\ a_{\mathbf{p}}(\mathbf{Q}) = a_{\mathbf{p}}(\mathbf{Q})\ f(N_> - 1)$$

$$a^{\dagger}_{\mathbf{p}}(\mathbf{Q})\ f(N_>) = f(N_> - 1)\ a^{\dagger}_{\mathbf{p}}(\mathbf{Q})$$

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$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,<} = a^{\dagger}_{\mathbf{p}}(\mathbf{q})$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,>} = a_{\mathbf{p}}(-\mathbf{q})$$

If $\mathbf{q} \neq 0$ then,

$$c_{\mathbf{p}-\mathbf{q}/2,<}c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} = \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ \frac{1}{N_> + 1}\ a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>} = \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\ \frac{1}{N_> + 1}\ a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

Also we define,

$$c_{\mathbf{p},<}c^{\dagger}_{\mathbf{p},<} = lim_{\mathbf{q}\to 0}\ c_{\mathbf{p}-\mathbf{q}/2,<}c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}$$

$$c^{\dagger}_{\mathbf{p},>}c_{\mathbf{p},>} = lim_{\mathbf{q}\to 0}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>}$$

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2 Field operator

Set,

$$e^{iN\theta_{\mathbf{p}}}e^{iN\theta_{\mathbf{p}'}} = 0$$

and

$$e^{iN\theta_{\mathbf{p}}}e^{-iN\theta_{\mathbf{p}'}} = \delta_{\mathbf{p},\mathbf{p}'}$$

This means,

$$c_{\mathbf{p},>} = \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ c_{\mathbf{p}-\mathbf{q}_2,<}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2,<}}}\ \frac{1}{\sqrt{N_> + 1}}\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

and

$$c^{\dagger}_{\mathbf{p},<} = \sqrt{N_> + 1}\ \frac{1}{\sqrt{n_{\mathbf{p},<}}}\ c^{\dagger}_{\mathbf{p},<}\ e^{iN\theta_{\mathbf{p}}} + \sum_{\mathbf{q}_1}\frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1,>}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>}\ \frac{1}{\sqrt{N_> + 1}}\ e^{-iN\theta_{\mathbf{p}+\mathbf{q}_1}}\ a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)$$

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Time evolution of the non-interacting system:
Note that for free fermions,

$$c_{\mathbf{p},s}(t) = c_{\mathbf{p},s}\ e^{-i\epsilon_p\ t}$$

and

$$a_{\mathbf{k}}(\mathbf{q};t) = e^{-i\frac{\mathbf{k}.\mathbf{q}}{m}\ t}\ a_{\mathbf{k}}(\mathbf{q})$$

This means,

$$c_{\mathbf{p},>}(t) = \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}-\mathbf{q}_2}}\ c_{\mathbf{p}-\mathbf{q}_2,<}(t)\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}_2,<}}}\ \frac{1}{\sqrt{N_> + 1}}\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2;t)$$

and

$$c^{\dagger}_{\mathbf{p},<}(t) = \sqrt{N_> + 1}\ \frac{1}{\sqrt{n_{\mathbf{p},<}}}\ c^{\dagger}_{\mathbf{p},<}(t)\ e^{iN\theta_{\mathbf{p}}} + \sum_{\mathbf{q}_1}\frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}_1,>}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>}(t)\ \frac{1}{\sqrt{N_> + 1}}\ e^{-iN\theta_{\mathbf{p}+\mathbf{q}_1}}\ a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1;t)$$

Note that,

$$c_{\mathbf{p}-\mathbf{q}_2,<}(t)\ \cdots\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2;t) \ = \ e^{-i\epsilon_{\mathbf{p}-\mathbf{q}_2}t}e^{-i\frac{(\mathbf{p}-\mathbf{q}_2/2).\mathbf{q}_2}{m}t}\ c_{\mathbf{p}-\mathbf{q}_2,<}\ \cdots\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

$$= \ e^{-i\frac{(\mathbf{p}-\mathbf{q}_2)^2}{2m}t}e^{-i\frac{(\mathbf{p}-\mathbf{q}_2/2).\mathbf{q}_2}{m}t}\ c_{\mathbf{p}-\mathbf{q}_2,<}\ \cdots\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2) \ = \ e^{-i\epsilon_p t}\ c_{\mathbf{p}-\mathbf{q}_2,<}\ \cdots\ a_{\mathbf{p}-\mathbf{q}_2/2}(\mathbf{q}_2)$$

Similarly,

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>}(t)\ \cdots\ a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1;t) \ = \ e^{i\epsilon_p t}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}_1,>}\ \cdots\ a_{\mathbf{p}+\mathbf{q}_1/2}(\mathbf{q}_1)$$

Thus this correspondence gives the correct time evolution. Next we show that this correspondence when back substituted into the equations of the previous section leads to an identity.

3 Back substitution

Set,

$$e^{iN\theta_{\mathbf{p}}}e^{iN\theta_{\mathbf{p}'}} \ = \ 0$$

and

$$e^{iN\theta_{\mathbf{p}}}e^{-iN\theta_{\mathbf{p}'}} \ = \ \delta_{\mathbf{p},\mathbf{p}'}$$

This means,

$$c_{\mathbf{p}-\mathbf{q}/2,>} = \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2}}\ c_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}}}\ \frac{1}{\sqrt{N_>+1}}\ a_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)$$

and

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} \ = \ \sqrt{N_>+1}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}}$$
$$+\sum_{\mathbf{q}_1}\frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>}\ \frac{1}{\sqrt{N_>+1}}\ e^{-iN\theta_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1}}\ a_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)$$

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$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>} = \sum_{\mathbf{q}_2} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)\ \frac{1}{\sqrt{N_>+1}}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2}}$$

and

$$c_{\mathbf{p}-\mathbf{q}/2,<} \ = \ e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2}}\ c_{\mathbf{p}-\mathbf{q}/2,<}\frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2,<}}}\ \sqrt{N_>+1}$$
$$+\sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ e^{iN\theta_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1}}\ \frac{1}{\sqrt{N_>+1}}\ c_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>}}}$$

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Off-diagonal:

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,<} \ = \ \sum_{\mathbf{q}_2} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)\ \frac{1}{\sqrt{N_>+1}}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2}}\ c_{\mathbf{p}-\mathbf{q}/2,<}$$

$$= \ \sum_{\mathbf{q}_2} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)\ \frac{1}{\sqrt{N_>+1}}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_2}}\quad e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2}}\ c_{\mathbf{p}-\mathbf{q}/2,<}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2,<}}}\ \sqrt{N_>+1}$$

$$= \ a^{\dagger}_{\mathbf{p}}(\mathbf{q})$$

and

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}c_{\mathbf{p}-\mathbf{q}/2,>} \ = \ \sqrt{N_>+1}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}}\ c_{\mathbf{p}-\mathbf{q}/2,>}$$
$$+\ \sum_{\mathbf{q}_1}\frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1,>}\frac{1}{\sqrt{N_>+1}}\ e^{-iN\theta_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1}}\ a_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ c_{\mathbf{p}-\mathbf{q}/2,>}$$

$$= \ \sqrt{N_>+1}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}}\ \sum_{\mathbf{q}_2} e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2}}\ c_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}}}\ \frac{1}{\sqrt{N_>+1}}\ a_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)$$

$$= \ a_{\mathbf{p}}(-\mathbf{q})$$

Diagonal:

$$c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,>}c_{\mathbf{p}-\mathbf{q}/2,>} \ =$$
$$\sum_{\mathbf{q}_1,\mathbf{q}_2} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\ \frac{1}{\sqrt{N_>+1}}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1}}$$
$$e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2}}\ c_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2,<}}}\ \frac{1}{\sqrt{N_>+1}}\ a_{\mathbf{p}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)$$
$$= \ \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\ \frac{1}{N_>+1}\ a_{\mathbf{p}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

and,

$$c_{\mathbf{p}-\mathbf{q}/2,<}c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<} \ =$$
$$e^{-iN\theta_{\mathbf{p}-\mathbf{q}/2}}\ c_{\mathbf{p}-\mathbf{q}/2,<}\frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2,<}}}\ \sqrt{N_>+1}\ \sqrt{N_>+1}\ \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2,<}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2,<}\ e^{iN\theta_{\mathbf{p}+\mathbf{q}/2}}$$
$$+\sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ e^{iN\theta_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1}}\ \frac{1}{\sqrt{N_>+1}}\ c_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>}\ \frac{1}{\sqrt{n_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1,>}}}$$
$$\times\ \sum_{\mathbf{q}_2}\frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2,>}}}\ c^{\dagger}_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2,>}\ \frac{1}{\sqrt{N_>+1}}\ e^{-iN\theta_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2}}\ a_{\mathbf{p}+\mathbf{q}/2+\mathbf{q}_2/2}(\mathbf{q}_2)$$
$$= \ \sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\ \frac{1}{N_>+1}\ a_{\mathbf{p}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

Thus we have now verified that the correspondence for the Fermi fields in terms of $a_{\mathbf{k}}(\mathbf{q})$ is fully correct.

4 Integral equation for q

Set ($s = >, <$),

$$q_{\mathbf{p},s} \equiv c_{\mathbf{p},s} \, \frac{1}{\sqrt{n_{\mathbf{p},s}}}$$

so that,

$$q_{\mathbf{p},>} \, \sqrt{n_{\mathbf{p},>}} \; = \; \sum_{\mathbf{q_2}} e^{-iN\theta_{\mathbf{p}-\mathbf{q_2}}} \, q_{\mathbf{p}-\mathbf{q_2},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}-\mathbf{q_2}/2}(\mathbf{q_2})$$

and

$$\sqrt{n_{\mathbf{p},<}} \, q^{\dagger}_{\mathbf{p},<} \; = \; \sqrt{N_> + 1} \, q^{\dagger}_{\mathbf{p},<} \, e^{iN\theta_{\mathbf{p}}} \; + \sum_{\mathbf{q_1}} q^{\dagger}_{\mathbf{p}+\mathbf{q_1},>} \, \frac{1}{\sqrt{N_> + 1}} \, e^{-iN\theta_{\mathbf{p}+\mathbf{q_1}}} \, a_{\mathbf{p}+\mathbf{q_1}/2}(\mathbf{q_1})$$

We now have to solve these,

$$q_{\mathbf{p},<} \; = \; \sum_{\mathbf{q_1}} a^{\dagger}_{\mathbf{p}+\mathbf{q_1}/2}(\mathbf{q_1}) \, e^{iN\theta_{\mathbf{p}+\mathbf{q_1}}} \, \frac{1}{\sqrt{N_> + 1}} \, q_{\mathbf{p}+\mathbf{q_1},>} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

and

$$q_{\mathbf{p},>} \, \sqrt{n_{\mathbf{p},>}} \; = \; \sum_{\mathbf{q_2}} e^{-iN\theta_{\mathbf{p}-\mathbf{q_2}}} \, q_{\mathbf{p}-\mathbf{q_2},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}-\mathbf{q_2}/2}(\mathbf{q_2})$$

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$$q_{\mathbf{p}+\mathbf{q_1},>} \; = \; \sum_{\mathbf{q_2}} e^{-iN\theta_{\mathbf{p}+\mathbf{q_1}-\mathbf{q_2}}} \, q_{\mathbf{p}+\mathbf{q_1}-\mathbf{q_2},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{q_1}-\mathbf{q_2}/2}(\mathbf{q_2}) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q_1},>}}}$$

Thus we get the following integral equation:

$$q_{\mathbf{p},<} \; = \; \sum_{\mathbf{q_1},\mathbf{q_2}} a^{\dagger}_{\mathbf{p}+\mathbf{q_1}/2}(\mathbf{q_1}) \, \frac{e^{iN\theta_{\mathbf{p}+\mathbf{q_1}}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{q_1}-\mathbf{q_2}}}}{\sqrt{N_> + 1}} \, q_{\mathbf{p}+\mathbf{q_1}-\mathbf{q_2},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{q_1}-\mathbf{q_2}/2}(\mathbf{q_2}) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{q_1},>}}} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

Set, $\mathbf{q_1} - \mathbf{q_2} = \mathbf{Q}$ and $\mathbf{q_1} + \mathbf{q_2} = 2\mathbf{K}$

$$q_{\mathbf{p},<} \; = \; \sum_{\mathbf{K},\mathbf{Q}} a^{\dagger}_{\mathbf{p}+(\mathbf{K}+\mathbf{Q}/2)/2}(\mathbf{K} + \mathbf{Q}/2) \, \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}+\mathbf{Q}/2}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \, q_{\mathbf{p}+\mathbf{Q},<}$$

$$\frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{K}+\mathbf{Q}/2-(\mathbf{K}-\mathbf{Q}/2)/2}(\mathbf{K} - \mathbf{Q}/2) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K}+\mathbf{Q}/2,>}}} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

or,

$$q_{\mathbf{p},<} \; = \; \sum_{\mathbf{K},\mathbf{Q}} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \, q_{\mathbf{p}+\mathbf{Q},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{K}/2+\mathbf{Q}/2}(\mathbf{K} - \mathbf{Q}) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

and

$$q_{\mathbf{p},<} \; = \; \sum_{\mathbf{K}} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} \; e^{-iN\theta_{\mathbf{p}}}}{\sqrt{N_> + 1}} \, q_{\mathbf{p},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

$$+ \sum_{\mathbf{K},\mathbf{Q}\neq 0} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \, q_{\mathbf{p}+\mathbf{Q},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{K}/2+\mathbf{Q}/2}(\mathbf{K} - \mathbf{Q}) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

Solve for $q_{\mathbf{p},<}$ in terms of $q_{\mathbf{p}+\mathbf{Q},<}$ first...

$$a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, q_{\mathbf{p},<} \; = \; c^{\dagger}_{\mathbf{p}+\mathbf{K},>} c_{\mathbf{p},<} \, c_{\mathbf{p},<} \frac{1}{\sqrt{n_{\mathbf{p},<}}} = 0$$

and

$$a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, q_{\mathbf{p}+\mathbf{Q},<} \; = \; q_{\mathbf{p}+\mathbf{Q},<} \, a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K})$$

and

$$q_{\mathbf{p},<} \; = \; \sum_{\mathbf{K},\mathbf{Q}\neq 0} a^{\dagger}_{\mathbf{p}+\mathbf{K}/2}(\mathbf{K}) \, \frac{e^{iN\theta_{\mathbf{p}+\mathbf{K}}} \; e^{-iN\theta_{\mathbf{p}+\mathbf{Q}}}}{\sqrt{N_> + 1}} \, q_{\mathbf{p}+\mathbf{Q},<} \, \frac{1}{\sqrt{N_> + 1}} \, a_{\mathbf{p}+\mathbf{K}/2+\mathbf{Q}/2}(\mathbf{K} - \mathbf{Q}) \, \frac{1}{\sqrt{n_{\mathbf{p}+\mathbf{K},>}}} \, \left(\sqrt{n_{\mathbf{p},<}} \; - \; \sqrt{N_> + 1} \, e^{-iN\theta_{\mathbf{p}}}\right)^{-1}$$

5 Commutators

$$[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})] \ = \ \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} n_F(\mathbf{p}) \ c_{\mathbf{k}+\mathbf{q}/2,>}$$

and

$$[c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \ = \ 0$$

The processes of annihilating a particle below the Fermi surface, annihilating a particle below the surface, and creating a particle above the Fermi surface give the same result when performed together, irrespective of which process is performed first.

and

$$[c_{\mathbf{p},>}, a_{\mathbf{k}}(\mathbf{q})] \ = \ 0$$

and

$$[c_{\mathbf{p},>}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \ = \ \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} \ (1 - n_F(\mathbf{p})) \ c_{\mathbf{k}-\mathbf{q}/2,<}$$

and,

$$[c_{\mathbf{p}}, n_{\mathbf{k}}] \ = \ \delta_{\mathbf{p},\mathbf{k}} \ c_{\mathbf{p}}$$

and,

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')] \ = \ 0 \ ; \quad [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] \ = \ \delta_{\mathbf{k},\mathbf{k}'} \ \delta_{\mathbf{q},\mathbf{q}'} \ (n_{\mathbf{k}-\mathbf{q}/2,<} - n_{\mathbf{k}+\mathbf{q}/2,>}) \ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))$$

and,

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] \ = \ a_{\mathbf{k}}(\mathbf{q}) \ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2})$$

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$$c_{\mathbf{p},<} = \sum_j e^{\sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \ \xi^{<}_{2,j}(\mathbf{p};\mathbf{k},\mathbf{q})} \ e^{\sum_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}}(\mathbf{q}) \ \xi^{<}_{1,j}(\mathbf{p};\mathbf{k},\mathbf{q})} \ \Gamma_{j,<}([n];\mathbf{p})$$

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$$c_{\mathbf{p},>} = \sum_j \Gamma_{j,>}([n];\mathbf{p}) \ e^{\sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \ \xi^{>}_{2,j}(\mathbf{p};\mathbf{k},\mathbf{q})} \ e^{\sum_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}}(\mathbf{q}) \ \xi^{>}_{1,j}(\mathbf{p};\mathbf{k},\mathbf{q})}$$

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$$c^{\dagger}_{\mathbf{p},<} = \sum_j \Gamma^{*}_{j,<}([n];\mathbf{p}) \ \ e^{\sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \ \xi^{<*}_{1,j}(\mathbf{p};\mathbf{k},\mathbf{q})} \ e^{\sum_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}}(\mathbf{q}) \ \xi^{<*}_{2,j}(\mathbf{p};\mathbf{k},\mathbf{q})}$$

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$$c^{\dagger}_{\mathbf{p},>} = \sum_j e^{\sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \ \xi^{>*}_{1,j}(\mathbf{p};\mathbf{k},\mathbf{q})} \ e^{\sum_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}}(\mathbf{q}) \ \xi^{>*}_{2,j}(\mathbf{p};\mathbf{k},\mathbf{q})} \ \Gamma^{*}_{j,>}([n];\mathbf{p})$$

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6 Exponentiation

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] \ = \ a_{\mathbf{k}}(\mathbf{q}) \ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2})$$

Set,

$$[P_{\mathbf{p}}, n_{\mathbf{p}'}] = i \ \delta_{\mathbf{p},\mathbf{p}'} \ ; \quad [P_{\mathbf{p}}, P_{\mathbf{p}'}] = 0$$

This means,

$$n_{\mathbf{p}} = -i \ \frac{\delta}{\delta P_{\mathbf{p}}}$$

and

$$[\frac{\delta}{\delta P_{\mathbf{p}}}, a_{\mathbf{k}}(\mathbf{q})] \ = \ - i \ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}) \ a_{\mathbf{k}}(\mathbf{q})$$

or,

$$a_{\mathbf{k}}(\mathbf{q}) = e^{-i \ (P_{\mathbf{k}+\mathbf{q}/2} - P_{\mathbf{k}-\mathbf{q}/2})} \ F([n];\mathbf{k},\mathbf{q})$$

and

$$a^{\dagger}_{\mathbf{k}}(\mathbf{q}) = \ F^{*}([n];\mathbf{k},\mathbf{q}) \ e^{i \ (P_{\mathbf{k}+\mathbf{q}/2} - P_{\mathbf{k}-\mathbf{q}/2})}$$

This means,

$$a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) \ = \ F^{*}([n];\mathbf{k},\mathbf{q}) \ \ F([n];\mathbf{k},\mathbf{q})$$

and

$$a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) = n_{\mathbf{k}+\mathbf{q}/2}(1 - n_{\mathbf{k}-\mathbf{q}/2}) \ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))$$

and

$$F([n];\mathbf{k},\mathbf{q}) \ = \ \sqrt{n_{\mathbf{k}+\mathbf{q}/2}(1 - n_{\mathbf{k}-\mathbf{q}/2})} \ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))$$

Examine,

$$a_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}'}(\mathbf{q}') = e^{-i \ (P_{\mathbf{k}+\mathbf{q}/2} - P_{\mathbf{k}-\mathbf{q}/2})} \ F([n];\mathbf{k},\mathbf{q}) e^{-i \ (P_{\mathbf{k}'+\mathbf{q}'/2} - P_{\mathbf{k}'-\mathbf{q}'/2})} \ F([n];\mathbf{k}',\mathbf{q}')$$

or,

$$a_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}'}(\mathbf{q}') = e^{-i \ (P_{\mathbf{k}+\mathbf{q}/2} - P_{\mathbf{k}-\mathbf{q}/2})} e^{-i \ (P_{\mathbf{k}'+\mathbf{q}'/2} - P_{\mathbf{k}'-\mathbf{q}'/2})}$$

$$\sqrt{(n_{\mathbf{k}+\mathbf{q}/2} - (\delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}))(1 - (n_{\mathbf{k}-\mathbf{q}/2} - (\delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2})))}$$

$$\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}'-\mathbf{q}'/2})} \ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2)) \ n_F(\mathbf{k}'-\mathbf{q}'/2)(1 - n_F(\mathbf{k}'+\mathbf{q}'/2))$$

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$$a_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}'}(\mathbf{q}') = e^{-i \ (P_{\mathbf{k}+\mathbf{q}/2} - P_{\mathbf{k}-\mathbf{q}/2})} e^{-i \ (P_{\mathbf{k}'+\mathbf{q}'/2} - P_{\mathbf{k}'-\mathbf{q}'/2})}$$

$$\sqrt{n_{\mathbf{k}+\mathbf{q}/2}(1 - n_{\mathbf{k}-\mathbf{q}/2}) \ n_{\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}'-\mathbf{q}'/2}) - n_{\mathbf{k}+\mathbf{q}/2}\delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2} \ n_{\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}'-\mathbf{q}'/2}) - \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}-\mathbf{q}/2}) \ n_{\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}'-\mathbf{q}'/2}) + \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2}\delta_{\mathbf{k}-}}$$

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$$n_{\mathbf{k}+\mathbf{q}/2}(1 - n_{\mathbf{k}-\mathbf{q}/2}) \ n_{\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}'-\mathbf{q}'/2})$$
$$- n_{\mathbf{k}+\mathbf{q}/2}\delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2} \ n_{\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}-\mathbf{q}/2})$$
$$- \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2}(1 - n_{\mathbf{k}-\mathbf{q}/2}) \ n_{\mathbf{k}+\mathbf{q}/2}(1 - n_{\mathbf{k}'-\mathbf{q}'/2})$$

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7 $a_{\mathbf{k}}(\mathrm{q})$ in terms of bosons

Imagine there are exact bosons. In this case the average of the number of these bosons is likely to be,

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q}) b_{\mathbf{k}}(\mathbf{q}) >= \frac{n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))}{e^{\beta \frac{\mathbf{k}.\mathbf{q}}{m}} - 1}$$

This is obtainable from the correspondence,

$$< n_k >= \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

or,

$$\frac{1}{< n_k >} - 1 = e^{\beta(\epsilon_k - \mu)}$$

or,

$$\frac{\frac{1}{< n_{k+q/2} >} - 1}{\frac{1}{< n_{k-q/2} >} - 1} = e^{\beta \frac{\mathbf{k}.\mathbf{q}}{m}}$$

or,

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q}) b_{\mathbf{k}}(\mathbf{q}) >= \frac{n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))}{\frac{\frac{1}{< n_{k+q/2} >}-1}{\frac{1}{< n_{k-q/2} >}-1} - 1}$$

$$= n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) \ \frac{(1- < n_{k-q/2} >) < n_{k+q/2} >}{< n_{k-q/2} > - < n_{k+q/2} >}$$

or,

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q}) b_{\mathbf{k}}(\mathbf{q}) >= n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) \ \frac{< a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) >}{< [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})] >}$$

and

$$< a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) >= n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) < n_{k+q/2} > (1- < n_{k-q/2} >)$$

and

$$< a_{\mathbf{k}}(\mathbf{q}) a^{\dagger}_{\mathbf{k}}(\mathbf{q}) >= n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) < n_{k-q/2} > (1- < n_{k+q/2} >)$$

and

$$< [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})] >= n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))(< n_{k-q/2} > - < n_{k+q/2} >)$$

or,

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q}) b_{\mathbf{k}}(\mathbf{q}) >= n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)) \ \frac{< n_{\mathbf{k}+\mathbf{q}/2} > (1- < n_{\mathbf{k}-\mathbf{q}/2} >)}{< n_{\mathbf{k}-\mathbf{q}/2} > - < n_{\mathbf{k}+\mathbf{q}/2} >}$$

and

$$< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) >= \frac{< n_{\mathbf{k}+\mathbf{q},>} > (n_F(\mathbf{k}) - < n_{\mathbf{k},<} >)}{< n_{\mathbf{k},<} > - < n_{\mathbf{k}+\mathbf{q},>} >}$$

and

$$< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) >= \frac{< n_{\mathbf{k},>} > (n_F(\mathbf{k}-\mathbf{q}) - < n_{\mathbf{k}-\mathbf{q},<} >)}{< n_{\mathbf{k}-\mathbf{q},<} > - < n_{\mathbf{k},>} >}$$

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$$< n_{\mathbf{k},<} >= \frac{< n_{\mathbf{k}+\mathbf{q}} > (< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) > + n_F(\mathbf{k}))}{< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) > + < n_{\mathbf{k}+\mathbf{q}} >} \ (1 - n_F(\mathbf{k}+\mathbf{q}))$$

and

$$< n_{\mathbf{k},>} >= \frac{< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) > \ < n_{\mathbf{k}-\mathbf{q}} >}{< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) > + 1 - < n_{\mathbf{k}-\mathbf{q}} >} \ n_F(\mathbf{k}-\mathbf{q})$$

———

$$< n_{\mathbf{k}} >= \frac{< n_{\mathbf{k}+\mathbf{q}} > (< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) > + n_F(\mathbf{k}))}{< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) > + < n_{\mathbf{k}+\mathbf{q}} >} \ (1 - n_F(\mathbf{k}+\mathbf{q}))$$

$$+ \frac{< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) > \ < n_{\mathbf{k}-\mathbf{q}} >}{< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) > + 1 - < n_{\mathbf{k}-\mathbf{q}} >} \ n_F(\mathbf{k}-\mathbf{q})$$

Hence,

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$$N^0 = \sum_{\mathbf{k}} \frac{< n_{\mathbf{k}} > (< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) > + n_F(\mathbf{k}-\mathbf{q}))}{< b^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}) > + < n_{\mathbf{k}} >} \ (1 - n_F(\mathbf{k})) + \sum_{\mathbf{k}} \frac{< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) > \ < n_{\mathbf{k}} >}{< b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) b_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) > + 1 - < n_{\mathbf{k}} >} \ n_F(\mathbf{k})$$

This is an implicit equation for $< n_{\mathbf{k}} >$ in terms of $< b^{\dagger}_{\mathbf{K}}(\mathbf{Q}) b_{\mathbf{K}}(\mathbf{Q}) >$.
Write $O_{\hat{n}_1,\hat{n}_2}(\mathbf{p}) = c^{\dagger}_{\hat{n}_1|\mathbf{p}|} c_{\hat{n}_2|\mathbf{p}|}$. This all means we may write,

$$\begin{aligned} b_{\mathbf{k}}(\mathbf{q}) &= \sum_{\hat{n}_1,\hat{n}_2} c^{\dagger}_{\hat{n}_1|\mathbf{k}-\mathbf{q}/2|,<} \quad F_{\hat{n}_1,\hat{n}_2}([O];\mathbf{k},\mathbf{q}) \quad c_{\hat{n}_2|\mathbf{k}+\mathbf{q}/2|,>} \\ b^{\dagger}_{\mathbf{k}}(\mathbf{q}) &= \sum_{\hat{n}_1,\hat{n}_2} c^{\dagger}_{\hat{n}_2|\mathbf{k}+\mathbf{q}/2|,>} \quad F^{\dagger}_{\hat{n}_1,\hat{n}_2}([O];\mathbf{k},\mathbf{q}) \quad c_{\hat{n}_1|\mathbf{k}-\mathbf{q}/2|,<} \end{aligned}$$

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$$b_{\mathbf{k}}(\mathbf{q}) = \sum_{\hat{n}_1,\hat{n}_2} c^{\dagger}_{\hat{n}_1|\mathbf{k}-\mathbf{q}/2|,<} \quad F_{\hat{n}_1,\hat{n}_2}([O];\mathbf{k},\mathbf{q}) \quad c_{\hat{n}_2|\mathbf{k}+\mathbf{q}/2|,>}$$

$$b^{\dagger}_{\mathbf{k}'}(\mathbf{q}') = \sum_{\hat{n}'_1,\hat{n}'_2} c^{\dagger}_{\hat{n}'_2|\mathbf{k}'+\mathbf{q}'/2|,>} \quad F^{\dagger}_{\hat{n}'_1,\hat{n}'_2}([O];\mathbf{k}',\mathbf{q}') \quad c_{\hat{n}'_1|\mathbf{k}'-\mathbf{q}'/2|,<}$$

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$$< b^{\dagger}_{\mathbf{k}'}(\mathbf{q}')b_{\mathbf{k}}(\mathbf{q}) > = \sum_{\hat{n}_1,\hat{n}_2}\sum_{\hat{n}'_1,\hat{n}'_2} < c^{\dagger}_{\hat{n}'_2|\mathbf{k}'+\mathbf{q}'/2|,>} \quad F^{\dagger}_{\hat{n}'_1,\hat{n}'_2}([O];\mathbf{k}',\mathbf{q}') \quad c_{\hat{n}'_1|\mathbf{k}'-\mathbf{q}'/2|,<} c^{\dagger}_{\hat{n}_1|\mathbf{k}-\mathbf{q}/2|,<} \quad F_{\hat{n}_1,\hat{n}_2}([O];\mathbf{k},\mathbf{q}) \quad c_{\hat{n}_2|\mathbf{k}+\mathbf{q}/2|,>} >$$

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$$< b_{\mathbf{k}}(\mathbf{q})b^{\dagger}_{\mathbf{k}'}(\mathbf{q}') > = \sum_{\hat{n}_1,\hat{n}_2}\sum_{\hat{n}'_1,\hat{n}'_2} < c^{\dagger}_{\hat{n}_1|\mathbf{k}-\mathbf{q}/2|,<} \quad F_{\hat{n}_1,\hat{n}_2}([O];\mathbf{k},\mathbf{q}) \quad c_{\hat{n}_2|\mathbf{k}+\mathbf{q}/2|,>} c^{\dagger}_{\hat{n}'_2|\mathbf{k}'+\mathbf{q}'/2|,>} \quad F^{\dagger}_{\hat{n}'_1,\hat{n}'_2}([O];\mathbf{k}',\mathbf{q}') \quad c_{\hat{n}'_1|\mathbf{k}'-\mathbf{q}'/2|,<} >$$

and

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q})b_{\mathbf{k}}(\mathbf{q}) > = n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\frac{< n_{\mathbf{k}+\mathbf{q}/2} > (1- < n_{\mathbf{k}-\mathbf{q}/2} >)}{< n_{\mathbf{k}-\mathbf{q}/2} > - < n_{\mathbf{k}+\mathbf{q}/2} >}$$

Let,

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q})b_{\mathbf{k}}(\mathbf{q}) > = < f(n_{\mathbf{k}+\mathbf{q}/2}, n_{\mathbf{k}-\mathbf{q}/2};\mathbf{k},\mathbf{q}) >$$

or,

$$< b^{\dagger}_{\mathbf{k}}(\mathbf{q})b_{\mathbf{k}}(\mathbf{q}) > = \frac{\sum_{n_{\mathbf{k}\pm\mathbf{q}/2}=0,1} e^{-\beta(\epsilon_{\mathbf{k}+\mathbf{q}/2}-\mu)n_{\mathbf{k}+\mathbf{q}/2}} e^{-\beta(\epsilon_{\mathbf{k}-\mathbf{q}/2}-\mu)n_{\mathbf{k}-\mathbf{q}/2}} f(n_{\mathbf{k}+\mathbf{q}/2}, n_{\mathbf{k}-\mathbf{q}/2};\mathbf{k},\mathbf{q})}{\sum_{n_{\mathbf{k}\pm\mathbf{q}/2}=0,1} e^{-\beta(\epsilon_{\mathbf{k}+\mathbf{q}/2}-\mu)n_{\mathbf{k}+\mathbf{q}/2}} e^{-\beta(\epsilon_{\mathbf{k}-\mathbf{q}/2}-\mu)n_{\mathbf{k}-\mathbf{q}/2}}}$$

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$$1 = \frac{1}{N_>(t)}\sum_{\mathbf{p}'} c_{\mathbf{p}',<}(t)c^{\dagger}_{\mathbf{p}',<}(t)$$

$$< T\ c_{\mathbf{p},>}(t)c^{\dagger}_{\mathbf{p}',>}(t') > = \sum_{\mathbf{p}'} < T\ \frac{1}{N_>(t)}\ c_{\mathbf{p}',<}(t)c^{\dagger}_{\mathbf{p}',<}(t)c_{\mathbf{p},>}(t)c^{\dagger}_{\mathbf{p}',>}(t') >$$

8 Approximate Exponentiation

$$[c_{\mathbf{p},<}, a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})] = n_F(\mathbf{p})\ c_{\mathbf{p}+\mathbf{q},>}$$

and

$$[c_{\mathbf{p},>}, a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q})] = (1-n_F(\mathbf{p}))\ c_{\mathbf{p}-\mathbf{q},<}$$

==

$$[c_{\mathbf{p},<}, a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})] = n_F(\mathbf{p})\ c_{\mathbf{p}+\mathbf{q},>}$$

and

$$[c_{\mathbf{p}+\mathbf{q},>}, a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})] = (1-n_F(\mathbf{p}+\mathbf{q}))\ c_{\mathbf{p},<}$$

and

$$c_{\mathbf{p},<} = \int D[C_1]\int D[C_2]\ W^<[\mathbf{p};C_1,C_2;[n]]\ e^{\sum_{\mathbf{q}_1} a_{\mathbf{k}_1}(\mathbf{q}_1)\ C_1(\mathbf{k}_1,\mathbf{q}_1)} e^{\sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{k}_1}(\mathbf{q}_1)\ C_2(\mathbf{k}_1,\mathbf{q}_1)}$$

and

$$c_{\mathbf{p},>} = \int D[C_1]\int D[C_2]\ W^>[\mathbf{p};C_1,C_2;[n]]\ e^{\sum_{\mathbf{q}_1} a_{\mathbf{k}_1}(\mathbf{q}_1)\ C_1(\mathbf{k}_1,\mathbf{q}_1)} e^{\sum_{\mathbf{q}_1} a^{\dagger}_{\mathbf{k}_1}(\mathbf{q}_1)\ C_2(\mathbf{k}_1,\mathbf{q}_1)}$$

9 Correlation functions

$$G_d(\mathbf{k}',\mathbf{q}';\mathbf{k},\mathbf{q};\lambda) = < e^{-\lambda N_>}\ a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q}) >$$

and

$$G_o(\mathbf{k}',\mathbf{q}';\mathbf{k},\mathbf{q};\lambda) = < e^{-\lambda N_>}\ a_{\mathbf{k}'}(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q}) >$$

$$H = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} c^{\dagger}_{\mathbf{p}} c_{\mathbf{p}} + \sum_{\mathbf{q}} \frac{v_q}{2V}(a_{.}(\mathbf{q}) + a^{\dagger}_{.}(-\mathbf{q}))(a_{.}(-\mathbf{q}) + a^{\dagger}_{.}(\mathbf{q}))$$

$$G_d(\mathbf{k}',\mathbf{q}';\mathbf{k},\mathbf{q};\lambda) = \frac{Tr(e^{-\beta(H-\mu N)}\ e^{-\lambda N_>}\ a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q}))}{Tr(e^{-\beta(H-\mu N)})}$$

and

$$G_o(\mathbf{k}',\mathbf{q}';\mathbf{k},\mathbf{q};\lambda) = \frac{Tr(e^{-\beta(H-\mu N)}\ e^{-\lambda N_>}\ a_{\mathbf{k}'}(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q})}{Tr(e^{-\beta(H-\mu N)})}$$

$$i\partial_t a_{\mathbf{k}}(\mathbf{q}) = \frac{\mathbf{k}.\mathbf{q}}{m}\ a_{\mathbf{k}}(\mathbf{q}) + \frac{v_q}{V}[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{.}(\mathbf{q})]\ (a_{.}(\mathbf{q}) + a^{\dagger}_{.}(-\mathbf{q}))$$

Regular RPA:

$$a_{\mathbf{k}}(\mathbf{q},t) - a_{\mathbf{k}}(\mathbf{q},0)\ e^{-i\frac{\mathbf{k}.\mathbf{q}}{m}t} = -i\ \frac{v_q}{V} n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\int_0^t dt'\ e^{i\frac{\mathbf{k}.\mathbf{q}}{m}(t'-t)}\ (a_{.}(\mathbf{q},t') + a^{\dagger}_{.}(-\mathbf{q},t'))$$

$$a_{.}(\mathbf{q},t) - \sum_{\mathbf{k}} a_{\mathbf{k}}(\mathbf{q},0)\ e^{-i\frac{\mathbf{k}.\mathbf{q}}{m}t} = -i\ v_q\int_0^t dt'\left(\frac{1}{V}\sum_{\mathbf{k}} n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\ e^{i\frac{\mathbf{k}.\mathbf{q}}{m}(t'-t)}\right)\ (a_{.}(\mathbf{q},t') + a^{\dagger}_{.}(-\mathbf{q},t'))$$

9.1 In 3D

$$a_{.}(\mathbf{q},t)-\sum_{\mathbf{k}}\ a_{\mathbf{k}}(\mathbf{q},0)\ e^{-i\frac{\mathbf{k.q}}{m}t}=-i\ v_q\int_0^t dt'\ \left(\frac{k_F^2((v_Fq(t-t')-i)\sin(v_Fq(t-t'))+(1+iv_Fq(t-t'))\cos(v_Fq(t-t'))-1)}{4\pi^2qv_F^2(t-t')^2}\right)\ (a_{.}(\mathbf{q},t')+a_{.}^{\dagger}(-\mathbf{q},t'))$$

$$(a_{.}(\mathbf{q},t)+a_{.}^{\dagger}(-\mathbf{q},t))-\sum_{\mathbf{k}}\ (a_{\mathbf{k}}(\mathbf{q},0)+a_{\mathbf{k}}^{\dagger}(-\mathbf{q},0))\ e^{-i\frac{\mathbf{k.q}}{m}t}=\int_0^t dt'\ \left(\frac{k_F^2\ v_q\ (v_Fq(t-t')\cos(v_Fq(t-t'))-\sin(v_Fq(t-t')))}{2\pi^2qv_F^2(t-t')^2}\right)\ (a_{.}(\mathbf{q},t')+a_{.}^{\dagger}(-\mathbf{q},t'))$$

$$\left(\frac{k_F^2\ v_q\ (v_Fq(t-t')\cos(v_Fq(t-t'))-\sin(v_Fq(t-t')))}{2\pi^2qv_F^2(t-t')^2}\right)\ =\ \int\frac{d\omega}{\sqrt{2\pi}}e^{i\omega(t-t')}\ W(\omega)$$

$$W(\omega)\ =\ -\frac{ik_F^2v_q\omega(\mathrm{sgn}(v_Fq-\omega)+\mathrm{sgn}(v_Fq+\omega))}{4\sqrt{2}\pi^{3/2}qv_F^2}$$

$$(a_{.}(\mathbf{q},t)+a_{.}^{\dagger}(-\mathbf{q},t))-\sum_{\mathbf{k}}\ (a_{\mathbf{k}}(\mathbf{q},0)+a_{\mathbf{k}}^{\dagger}(-\mathbf{q},0))\ e^{-i\frac{\mathbf{k.q}}{m}t}\ =\ \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\ e^{i\omega t}\ W(\omega)\ \int_0^t dt'\ e^{-i\omega t'}\ (a_{.}(\mathbf{q},t')+a_{.}^{\dagger}(-\mathbf{q},t'))$$

9.2 In 1D

$$a_{.}(q,t)-\sum_{\mathbf{k}}\ a_k(q,0)\ e^{-iv_F|q|t}=-i\ v_q\int_0^t dt'\ \left(\frac{1}{L}\sum_{\mathbf{k}}n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\right)\ e^{iv_F|q|(t'-t)}\ (a_{.}(q,t')+a_{.}^{\dagger}(-q,t'))$$

or,

$$\frac{1}{L}\sum_{\mathbf{k}}n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\ =\ \frac{|q|}{2\pi}$$

or,

$$a_{.}(q,t)-\sum_{\mathbf{k}}\ a_k(q,0)\ e^{-iv_F|q|t}=-i\ v_q\int_0^t dt'\ \frac{|q|}{2\pi}\ e^{iv_F|q|(t'-t)}\ (a_{.}(q,t')+a_{.}^{\dagger}(-q,t'))$$

or,

$$a_{.}(q,t)-\sum_{k}\ a_k(q,0)\ e^{-iv_F|q|t}=-i\ v_q\int_0^t dt'\ \frac{|q|}{2\pi}\ e^{iv_F|q|(t'-t)}\ (a_{.}(q,t')+a_{.}^{\dagger}(-q,t'))$$

$$a_{.}^{\dagger}(-q,t)-\sum_{k}\ a_k^{\dagger}(-q,0)\ e^{iv_F|q|t}=i\ v_q\int_0^t dt'\ \frac{|q|}{2\pi}\ e^{-iv_F|q|(t'-t)}\ (a_{.}(q,t')+a_{.}^{\dagger}(-q,t'))$$

and

$$X(q,t')=(a_{.}(q,t')+a_{.}^{\dagger}(-q,t'))$$

and

$$X_{in}(q,t)=a_{.}(q,0)\ e^{-iv_F|q|t}+a_{.}^{\dagger}(-q,0)\ e^{iv_F|q|t}$$

or,

$$X(q,t)-X_{in}(q,t)=-i\ v_q\int_0^t dt'\ \frac{|q|}{2\pi}\ e^{iv_F|q|(t'-t)}\ X(q,t')+i\ v_q\int_0^t dt'\ \frac{|q|}{2\pi}\ e^{-iv_F|q|(t'-t)}\ X(q,t')$$

or,

$$X(q,t)-X_{in}(q,t)=i\ v_q\frac{|q|}{2\pi}\ \int_0^t dt'\,(-e^{iv_F|q|(t'-t)}\ +\ e^{-iv_F|q|(t'-t)})\ X(q,t')$$

or,

$$X(q,t)-X_{in}(q,t)=i\ v_q\frac{|q|}{2\pi}\ \int_0^t dt'\,(-2\sin(iv_F|q|(t'-t))\ X(q,t')$$

Using the formula to differentiate under an integral,

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)}f(t,t')\,dt'\right)=f(t,b(t))\cdot\frac{d}{dt}b(t)-f(t,a(t))\cdot\frac{d}{dt}a(t)+\int_{a(t)}^{b(t)}\frac{\partial}{\partial t}f(t,t')\,dt' \tag{1}$$

This means differentiating; under the integral, we would get the first two terms to be 0, ie.

$$\frac{d}{dt}\,(X(q,t)-X_{in}(q,t))$$

$$=i\ v_q\frac{|q|}{2\pi}\left((-2\sin(iv_F|q|(t-t))\ X(q,t)\cdot\frac{d}{dt}t-(-2\sin(iv_F|q|t))\ X(q,t)\ \cdot\frac{d}{dt}0+\int_0^t\frac{\partial}{\partial t}(-2\sin(iv_F|q|(t'-t))\ X(q,t')\ \ dt'\right)$$

$$=i\ v_q\frac{|q|}{2\pi}\left(\int_0^t 2\cos(iv_F|q|(t'-t))\ X(q,t')\ \ dt'\right)$$

Taking a double derivative gives us,

$$\frac{d^2}{dt^2}\,(X(q,t)-X_{in}(q,t))$$

$$=i\ v_q\frac{|q|}{2\pi}\left(2X(q,t)+\int_0^t\frac{\partial}{\partial t}(2\cos(iv_F|q|(t'-t))\ X(q,t'))\,dt'\right)$$

$$=i\ v_q\frac{|q|}{2\pi}\left(2X(q,t)+\int_0^t 2\sin(iv_F|q|(t'-t))\ X(q,t')\,dt'\right)$$

Since,

$$X(q,t)-X_{in}(q,t)=i\ v_q\frac{|q|}{2\pi}\ \int_0^t dt'\,(-2\sin(iv_F|q|(t'-t))\ X(q,t')$$

We get the differential equation,

$$\frac{d^2}{dt^2}\,(X(q,t)-X_{in}(q,t))=i\ v_q\frac{|q|}{2\pi}\,(X(q,t)+X_{in}(q,t))$$

From the definition of X_{in},

$$\frac{d^2}{dt^2}X_{in}(q,t)=-(v_F|q|)^2X_{in}(q,t)$$

This gives,

$$\frac{d^2}{dt^2}X(q,t)-i\ v_q\frac{|q|}{2\pi}X(q,t)=\left(-(v_F|q|)^2+i\ v_q\frac{|q|}{2\pi}\right)X_{in}(q,t).$$

Where,

$$X_{in}(q,t)=a_{.}(q,0)\ e^{-iv_F|q|t}+a_{.}^{\dagger}(-q,0)\ e^{iv_F|q|t}.$$

On putting this in Mathematica, we get,

$$X(q,t)=C_1e^{\sqrt{-iv_F|q|}t}+C_2e^{\sqrt{-iv_F|q|}t}$$