Resolving identity

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1 Resolve identity

$$\frac{1}{N_>(t_1)} \sum_{\mathbf{p}} c_{\mathbf{p},<}(t_1) c^{\dagger}_{\mathbf{p},<}(t_1) = 1$$

and

$$G_>(\mathbf{k}, t, t^{'}) = < T \ c_{\mathbf{k},>}(t) c^{\dagger}_{\mathbf{k},>}(t^{'}) >$$

or,

$$G_>(\mathbf{k}, t, t^{'}) = \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} c_{\mathbf{k}-\mathbf{q},<}(t) c^{\dagger}_{\mathbf{k}-\mathbf{q},<}(t) \ c_{\mathbf{k},>}(t) c^{\dagger}_{\mathbf{k},>}(t^{'}) >$$

$a_{\mathbf{k}_1}(\mathbf{q}_1) = c^{\dagger}_{\mathbf{k}_1-\mathbf{q}_1/2,<} c_{\mathbf{k}_1+\mathbf{q}_1/2,>}.$ and

$$G_>(\mathbf{k}, t, t^{'}) = \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} c_{\mathbf{k}-\mathbf{q},<}(t) a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}, t) \ c^{\dagger}_{\mathbf{k},>}(t^{'}) >$$

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$$\frac{1}{N_>(t_1)} \sum_{\mathbf{p}} c^{\dagger}_{\mathbf{p},>}(t_1) c_{\mathbf{p},>}(t_1) = 1$$

and

$$G_<(\mathbf{k}, t, t^{'}) = < T \ c_{\mathbf{k},<}(t) c^{\dagger}_{\mathbf{k},<}(t^{'}) >$$

or,

$$G_<(\mathbf{k}, t, t^{'}) = - \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} c^{\dagger}_{\mathbf{k}+\mathbf{q},>}(t) \ c_{\mathbf{k},<}(t) c_{\mathbf{k}+\mathbf{q},>}(t) c^{\dagger}_{\mathbf{k},<}(t^{'}) >$$

$a_{\mathbf{k}_1}(\mathbf{q}_1) = c^{\dagger}_{\mathbf{k}_1-\mathbf{q}_1/2,<} c_{\mathbf{k}_1+\mathbf{q}_1/2,>}.$ and

$$G_<(\mathbf{k}, t, t^{'}) = - \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} c^{\dagger}_{\mathbf{k}+\mathbf{q},>}(t) \ c_{\mathbf{k},<}(t) c_{\mathbf{k}+\mathbf{q},>}(t) c^{\dagger}_{\mathbf{k},<}(t^{'}) >$$

and $a^{\dagger}_{\mathbf{k}_1}(\mathbf{q}_1) = c^{\dagger}_{\mathbf{k}_1+\mathbf{q}_1/2,>} c_{\mathbf{k}_1-\mathbf{q}_1/2,<}.$ and

$$G_<(\mathbf{k}, t, t^{'}) = - \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} \ a^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}, t) \ c_{\mathbf{k}+\mathbf{q},>}(t) c^{\dagger}_{\mathbf{k},<}(t^{'}) >$$

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$$G_>(\mathbf{k}, t, t^{'}) = \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} c_{\mathbf{k}-\mathbf{q},<}(t) a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q}, t) \ c^{\dagger}_{\mathbf{k},>}(t^{'}) >$$

and

$$G_<(\mathbf{k}, t, t^{'}) = - \sum_{\mathbf{q}} \ < T \ \frac{1}{N_>(t)} \ a^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}, t) \ c_{\mathbf{k}+\mathbf{q},>}(t) c^{\dagger}_{\mathbf{k},<}(t^{'}) >$$

2 The 4-Point functions

$$F_>(\mathbf{k},\mathbf{q};t_1,t,t^{'}) = \, <T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>$$

and

$$F_<(\mathbf{k},\mathbf{q};t_1,t,t^{'}) = -\,<T\,e^{-\lambda N>(t)}\,a^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q},t_1)\,c_{\mathbf{k}+\mathbf{q},>}(t)c^{\dagger}_{\mathbf{k},<}(t^{'})>$$

and

$$i\partial_{t_1}F_>(\mathbf{k},\mathbf{q};t_1,t,t^{'}) = i\partial_{t_1}\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>$$

and

$$i\partial_{t_1}F_<(\mathbf{k},\mathbf{q};t_1,t,t^{'}) = -\,i\partial_{t_1}\,<T\,e^{-\lambda N>(t)}\,a^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q},t_1)\,c_{\mathbf{k}+\mathbf{q},>}(t)c^{\dagger}_{\mathbf{k},<}(t^{'})>$$

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$$<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ =$$

$$<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\,\theta(t_1-t)\theta(t_1-t^{'}) + <T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\,\theta(t-t_1)\theta(t_1-t^{'})$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\,\theta(t_1-t)\theta(t^{'}-t_1) + <T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\,\theta(t-t_1)\theta(t^{'}-t_1)$$

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$$<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ =$$

$$<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t_1-t)\theta(t_1-t^{'})$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t-t_1)\theta(t_1-t^{'})$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t_1-t)\theta(t^{'}-t_1)$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t-t_1)\theta(t^{'}-t_1)$$

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$$<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ =$$

$$<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t_1-t)\ \theta(t-t^{'}) + <T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\,\theta(t_1-t^{'})\ \theta(t^{'}-t)$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t-t_1)\theta(t_1-t^{'})$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t^{'}-t_1)\theta(t_1-t)$$

$$+\,<T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t^{'}-t_1)\theta(t-t^{'}) + <T\,e^{-\lambda N>(t)}\,c_{\mathbf{k}-\mathbf{q},<}(t)a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q},t_1)\,c^{\dagger}_{\mathbf{k},>}(t^{'})>\ \theta(t-t_1)\theta(t^{'}-t)$$

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$$< T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ =$$

$$< a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ \theta(t_1-t) \ \theta(t-t^{'}) - < a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t) > \theta(t_1-t^{'}) \ \theta(t^{'}-t)$$

$$+ < e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ \theta(t-t_1)\theta(t_1-t^{'})$$

$$- < c_{\mathbf{k},>}^{\dagger}(t^{'})a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t) > \ \theta(t^{'}-t_1)\theta(t_1-t)$$

$$+ < e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)c_{\mathbf{k},>}^{\dagger}(t^{'})a_{\mathbf{k-q}/2}(\mathbf{q},t_1) > \ \theta(t^{'}-t_1)\theta(t-t^{'}) - < c_{\mathbf{k},>}^{\dagger}(t^{'}) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) > \ \theta(t-t_1)\theta(t^{'}-t)$$

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$$i\partial_{t_1} < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ = \ \frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) >$$

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$$+ < a_{\mathbf{k-q}/2}(\mathbf{q},t) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ i\delta(t_1-t) \ \theta(t-t^{'}) - < e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ \theta(t-t^{'})i\delta(t-t_1) - < c_{\mathbf{k},>}^{\dagger}(t^{'})a_{\mathbf{k-q}/2}(\mathbf{q},t) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)$$

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$$- < a_{\mathbf{k-q}/2}(\mathbf{q},t^{'}) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t) > \ i\delta(t_1-t^{'}) \ \theta(t^{'}-t) + < e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t^{'}) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ \theta(t-t^{'})i\delta(t_1-t^{'}) + < c_{\mathbf{k},>}^{\dagger}(t^{'})a_{\mathbf{k-q}/2}(\mathbf{q},t^{'}) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},}$$

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$$i\partial_{t_1} < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ = \ \frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) >$$

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$$+ < [a_{\mathbf{k-q}/2}(\mathbf{q},t), \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)]c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ i\delta(t_1-t) \ \theta(t-t^{'}) + < c_{\mathbf{k},>}^{\dagger}(t^{'}) \ [e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t), a_{\mathbf{k-q}/2}(\mathbf{q},t)] > \ i\delta(t-t_1)\theta(t^{'}-t)$$

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$$+ < [c_{\mathbf{k},>}^{\dagger}(t^{'}), a_{\mathbf{k-q}/2}(\mathbf{q},t^{'})] \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t) > \ \theta(t^{'}-t)i\delta(t^{'}-t_1) + < e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)[a_{\mathbf{k-q}/2}(\mathbf{q},t^{'}), c_{\mathbf{k},>}^{\dagger}(t^{'})] > \ \theta(t-t^{'})i\delta(t_1-t^{'})$$

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$$i\partial_{t_1} < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ = \ \frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) >$$

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$$+ < T \ [a_{\mathbf{k-q}/2}(\mathbf{q},t), \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)]c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ i\delta(t_1-t)$$

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$$+ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)[a_{\mathbf{k-q}/2}(\mathbf{q},t^{'}), c_{\mathbf{k},>}^{\dagger}(t^{'})] > \ i\delta(t_1-t^{'})$$

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$$[a_{\mathbf{k-q}/2}(\mathbf{q},t), \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)] \ = \ [a_{\mathbf{k-q}/2}(\mathbf{q},t), \ e^{-\lambda N>(t)}] \ c_{\mathbf{k-q},<}(t) + e^{-\lambda N>(t)} \ [a_{\mathbf{k-q}/2}(\mathbf{q},t), \ c_{\mathbf{k-q},<}(t)]$$

and

$$[a_{\mathbf{k-q}/2}(\mathbf{q},t^{'}), c_{\mathbf{k},>}^{\dagger}(t^{'})] = c_{\mathbf{k-q},<}^{\dagger}(t^{'}) \ (1 - n_F(\mathbf{k}))$$

$$[a_{\mathbf{k-q}/2}(\mathbf{q},t), \ c_{\mathbf{k-q},<}(t)] = -n_F(\mathbf{k-q}) \ c_{\mathbf{k},>}(t)$$

This means,

$$[a_{\mathbf{k-q}/2}(\mathbf{q},t), \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)] \ = \ e^{-\lambda N>(t)} \ (e^{-\lambda} - 1)a_{\mathbf{k-q}/2}(\mathbf{q},t) \ c_{\mathbf{k-q},<}(t) - e^{-\lambda N>(t)} \ n_F(\mathbf{k-q}) \ c_{\mathbf{k},>}(t)$$

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$$i\partial_{t_1} < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) >$$

$$= \ \frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \ c_{\mathbf{k},>}^{\dagger}(t^{'}) >$$

$$+(e^{-\lambda} - 1) \ < T \ e^{-\lambda N>(t)} \ a_{\mathbf{k-q}/2}(\mathbf{q},t) \ c_{\mathbf{k-q},<}(t)c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ i\delta(t_1-t)$$

$$-n_F(\mathbf{k-q}) \ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k},>}(t)c_{\mathbf{k},>}^{\dagger}(t^{'}) > \ i\delta(t_1-t)$$

$$+ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},<}(t) \ c_{\mathbf{k-q},<}^{\dagger}(t^{'}) > \ (1 - n_F(\mathbf{k})) \ i\delta(t_1-t^{'})$$

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$$i\partial_{t_1} < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},>}(t)a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q},t_1) \ c_{\mathbf{k},<}^{\dagger}(t^{'}) > \ = \ \frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ < T \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},>}(t)a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q},t_1) \ c_{\mathbf{k},<}^{\dagger}(t^{'}) >$$

$$+ < a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q},t_1) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},>}(t)c_{\mathbf{k},<}^{\dagger}(t^{'}) > \ i\partial_{t_1} \theta(t_1-t)\theta(t-t^{'})$$

$$- < a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q},t_1)c_{\mathbf{k},<}^{\dagger}(t^{'}) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},>}(t) > \ i\partial_{t_1} \theta(t_1-t^{'})\theta(t^{'}-t)$$

$$- < c_{\mathbf{k},<}^{\dagger}(t^{'}) \ a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q},t_1) \ e^{-\lambda N>(t)} \ c_{\mathbf{k-q},>}(t) > \ i\partial_{t_1} \theta(t_1-t)\theta(t^{'}-t_1)$$

$$+ < e^{-\lambda N>(t)} \ c_{\mathbf{k-q},>}(t)a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q},t_1) \ c_{\mathbf{k},<}^{\dagger}(t^{'}) > \ i\partial_{t_1} \theta(t-t_1)\theta(t_1-t^{'})$$

$$+ <e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) \; c^\dagger_{\mathbf{k},<}(t^{'}) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) > \; i\partial_{t_1} \theta(t^{'}-t_1)\theta(t-t^{'})$$

$$- <c^\dagger_{\mathbf{k},<}(t^{'}) \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) \; a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) > \; i\partial_{t_1} \theta(t-t_1)\theta(t^{'}-t)$$

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$$i\partial_{t_1} < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},<}(t^{'}) > \; = \; \frac{(\mathbf{k}-\mathbf{q}/2)\cdot\mathbf{q}}{m} \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},<}(t^{'}) >$$

$$+ <[a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t), \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t)]c^\dagger_{\mathbf{k},<}(t^{'}) > \; i\delta(t_1-t)\theta(t-t^{'})$$

$$+ <c^\dagger_{\mathbf{k},<}(t^{'}) \; [e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) \;, a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t)] > \; i\delta(t-t_1)\theta(t^{'}-t)$$

$$+ <e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t)[a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t^{'}), c^\dagger_{\mathbf{k},<}(t^{'})] > \; i\theta(t-t^{'})\delta(t_1-t^{'})$$

$$+ <[c^\dagger_{\mathbf{k},<}(t^{'}), a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t^{'})] \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) > \; i\theta(t^{'}-t)\delta(t^{'}-t_1)$$

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$$i\partial_{t_1} < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},<}(t^{'}) > \; = \; \frac{(\mathbf{k}-\mathbf{q}/2)\cdot\mathbf{q}}{m} \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},<}(t^{'}) >$$

$$+ < T \; [a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t), \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t)]c^\dagger_{\mathbf{k},<}(t^{'}) > \; i \; \delta(t_1-t)$$

$$+ < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t)[a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t^{'}), c^\dagger_{\mathbf{k},<}(t^{'})] > \; i \; \delta(t_1-t^{'})$$

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$$i\partial_{t_1} < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},<}(t^{'}) > \; = \; \frac{(\mathbf{k}-\mathbf{q}/2)\cdot\mathbf{q}}{m} \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},<}(t^{'}) >$$

$$+(e^{\lambda} - 1) \; < T \; c^\dagger_{\mathbf{k-q},>}(t) \; e^{-\lambda N>(t)} \; c_{\mathbf{k},<}(t) \; c_{\mathbf{k-q},>}(t)c^\dagger_{\mathbf{k},<}(t^{'}) > \; i \; \delta(t_1-t)$$

$$-(1 - n_F(\mathbf{k}-\mathbf{q})) \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k},<}(t)c^\dagger_{\mathbf{k},<}(t^{'}) > \; i \; \delta(t_1-t)$$

$$+n_F(\mathbf{k}) \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},>}(t)c^\dagger_{\mathbf{k-q},>}(t^{'}) > \; i \; \delta(t_1-t^{'})$$

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$$i\partial_{t_1} < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},<}(t) a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},>}(t^{'}) > \; = \; \frac{(\mathbf{k}-\mathbf{q}/2)\cdot\mathbf{q}}{m} \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},<}(t) a_{\mathbf{k-q}/2}(\mathbf{q},t_1) \; c^\dagger_{\mathbf{k},>}(t^{'}) >$$

$$+(e^{-\lambda} - 1) \; < T \; e^{-\lambda N>(t)} \; a_{\mathbf{k-q}/2}(\mathbf{q},t) \; c_{\mathbf{k-q},<}(t)c^\dagger_{\mathbf{k},>}(t^{'}) > \; i\delta(t_1-t)$$

$$-n_F(\mathbf{k}-\mathbf{q}) \; < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k},>}(t)c^\dagger_{\mathbf{k},>}(t^{'}) > \; i\delta(t_1-t)$$

$$+ < T \; e^{-\lambda N>(t)} \; c_{\mathbf{k-q},<}(t) \; c^\dagger_{\mathbf{k-q},<}(t^{'}) > \; (1 - n_F(\mathbf{k})) \; i\delta(t_1-t^{'})$$

3 Nonlocal Green functions

$$i\partial_{t_1} < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}(t)a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t_1)\ c^{\dagger}_{\mathbf{k},<}(t^{'}) > \quad = \quad \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}(t)a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t_1)\ c^{\dagger}_{\mathbf{k},<}(t^{'}) >$$

$$+(e^{\lambda}-1)\ < T\ e^{-\lambda N} >^{(t)}\ a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t)\ c_{\mathbf{k-q},>}(t)c^{\dagger}_{\mathbf{k},<}(t^{'}) >\ i\ \delta(t_1-t)$$

$$-(1-n_F(\mathbf{k-q}))\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k},<}(t)c^{\dagger}_{\mathbf{k},<}(t^{'}) >\ i\ \delta(t_1-t)$$

$$+n_F(\mathbf{k})\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}(t)c^{\dagger}_{\mathbf{k-q},>}(t^{'}) >\ i\ \delta(t_1-t^{'})$$

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$$i\partial_{t_1} < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1)\ c^{\dagger}_{\mathbf{k},>}(t^{'}) > \quad = \quad \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}(t)a_{\mathbf{k-q}/2}(\mathbf{q},t_1)\ c^{\dagger}_{\mathbf{k},>}(t^{'}) >$$

$$+(e^{-\lambda}-1)\ < T\ e^{-\lambda N} >^{(t)}\ a_{\mathbf{k-q}/2}(\mathbf{q},t)\ c_{\mathbf{k-q},<}(t)c^{\dagger}_{\mathbf{k},>}(t^{'}) >\ i\delta(t_1-t)$$

$$-n_F(\mathbf{k-q})\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k},>}(t)c^{\dagger}_{\mathbf{k},>}(t^{'}) >\ i\delta(t_1-t)$$

$$+ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}(t)\ c^{\dagger}_{\mathbf{k-q},<}(t^{'}) >\ (1-n_F(\mathbf{k}))\ i\delta(t_1-t^{'})$$

——

These Green functions are periodic in t_1 with BOSONIC Matsubara frequencies even though they describe fermions.

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3.1 Non-local Green function: Simplified

$$< T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k-q},>}(t) a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q}, t_1) \ c^\dagger_{\mathbf{k},<}(t^{'}) > \ =$$

$$= \ \frac{ie^{-i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t_1}}{1 - e^{\beta\frac{(\mathbf{k-q}/2).\mathbf{q}}{m}}}$$

$$(-(1 - e^{\beta\frac{(\mathbf{k-q}/2).\mathbf{q}}{m}})(((e^\lambda - 1) \ < T \ e^{-\lambda N_>(t)} \ a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q}, t) \ c_{\mathbf{k-q},>}(t) c^\dagger_{\mathbf{k},<}(t^{'}) > \ i - (1 - n_F(\mathbf{k - q})) \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},<}(t) c^\dagger_{\mathbf{k},<}(t^{'}) > \ i) \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t} \theta(t_1 - t)$$

$$+ n_F(\mathbf{k}) \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k-q},>}(t) c^\dagger_{\mathbf{k-q},>}(t^{'}) > \ i \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t^{'}} \theta(t_1 - t^{'}))$$

$$+ ((e^\lambda - 1) \ < T \ e^{-\lambda N_>(t)} \ a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q}, t) \ c_{\mathbf{k-q},>}(t) c^\dagger_{\mathbf{k},<}(t^{'}) > \ i - (1 - n_F(\mathbf{k - q})) \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},<}(t) c^\dagger_{\mathbf{k},<}(t^{'}) > \ i) \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t}$$

$$+ n_F(\mathbf{k}) \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k-q},>}(t) c^\dagger_{\mathbf{k-q},>}(t^{'}) > \ i \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t^{'}})$$

_____

$$< T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k-q},<}(t) a_{\mathbf{k-q}/2}(\mathbf{q}, t_1) \ c^\dagger_{\mathbf{k},>}(t^{'}) > \ = \ \frac{ie^{-i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t_1}}{1 - e^{\beta\frac{(\mathbf{k-q}/2).\mathbf{q}}{m}}}$$

$$(-(1 - e^{\beta\frac{(\mathbf{k-q}/2).\mathbf{q}}{m}})(((e^{-\lambda} - 1) \ < T \ e^{-\lambda N_>(t)} \ a_{\mathbf{k-q}/2}(\mathbf{q}, t) \ c_{\mathbf{k-q},<}(t) c^\dagger_{\mathbf{k},>}(t^{'}) > \ i - n_F(\mathbf{k - q}) \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},>}(t) c^\dagger_{\mathbf{k},>}(t^{'}) > \ i) \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t} \theta(t_1 - t)$$

$$+ \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k-q},<}(t) \ c^\dagger_{\mathbf{k-q},<}(t^{'}) > \ (1 - n_F(\mathbf{k})) \ i \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t^{'}} \theta(t_1 - t^{'}))$$

$$+ ((e^{-\lambda} - 1) \ < T \ e^{-\lambda N_>(t)} \ a_{\mathbf{k-q}/2}(\mathbf{q}, t) \ c_{\mathbf{k-q},<}(t) c^\dagger_{\mathbf{k},>}(t^{'}) > \ i - n_F(\mathbf{k - q}) \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},>}(t) c^\dagger_{\mathbf{k},>}(t^{'}) > \ i) e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t}$$

$$+ \ < T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k-q},<}(t) \ c^\dagger_{\mathbf{k-q},<}(t^{'}) > \ (1 - n_F(\mathbf{k})) \ i \ e^{i\frac{(\mathbf{k-q}/2).\mathbf{q}}{m} \ t^{'}})$$

$$< e^{-\lambda N} >(t) \ c_{\mathbf{k-q},>}(t) a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t) \ c^{\dagger}_{\mathbf{k},<}(t') > \quad =$$

$$= \frac{i e^{-i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t}}{1 - e^{\beta \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$(-(1 - e^{\beta \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}})(n_F(\mathbf{k}) \ < T \ e^{-\lambda N} >(t) \ c_{\mathbf{k-q},>}(t) c^{\dagger}_{\mathbf{k-q},>}(t') > \ i \ e^{i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t'} \theta(t - t'))$$

$$+((e^{\lambda} - 1) \ < T \ e^{-\lambda N} >(t) \ a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t) \ c_{\mathbf{k-q},>}(t) c^{\dagger}_{\mathbf{k},<}(t') > \ i - (1 - n_F(\mathbf{k-q})) \ < T \ e^{-\lambda N} >(t) \ c_{\mathbf{k},<}(t) c^{\dagger}_{\mathbf{k},<}(t') > \ i) \ e^{i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t}$$

$$+ n_F(\mathbf{k}) \ < T \ e^{-\lambda N} >(t) \ c_{\mathbf{k-q},>}(t) c^{\dagger}_{\mathbf{k-q},>}(t') > \ i \ e^{i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t'})$$

———

$$< T \ e^{-\lambda N} >(t) \ c_{\mathbf{k-q},<}(t) a_{\mathbf{k-q}/2}(\mathbf{q},t) \ c^{\dagger}_{\mathbf{k},>}(t') > \quad = \quad \frac{i e^{-i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t}}{1 - e^{\beta \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$(-(1 - e^{\beta \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}})(< T \ e^{-\lambda N} >(t) \ c_{\mathbf{k-q},<}(t) \ c^{\dagger}_{\mathbf{k-q},<}(t') > \ (1 - n_F(\mathbf{k})) \ i \ e^{i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t'} \theta(t - t'))$$

$$+((e^{-\lambda} - 1) \ < T \ e^{-\lambda N} >(t) \ a_{\mathbf{k-q}/2}(\mathbf{q},t) \ c_{\mathbf{k-q},<}(t) c^{\dagger}_{\mathbf{k},>}(t') > \ i - n_F(\mathbf{k-q}) \ < T \ e^{-\lambda N} >(t) \ c_{\mathbf{k},>}(t) c^{\dagger}_{\mathbf{k},>}(t') > \ i) e^{i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t}$$

$$+ < T \ e^{-\lambda N} >(t) \ c_{\mathbf{k-q},<}(t) \ c^{\dagger}_{\mathbf{k-q},<}(t') > \ (1 - n_F(\mathbf{k})) \ i \ e^{i \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} t'})$$

$$a_{\mathbf{k-q}/2}(\mathbf{q},t) \ c_{\mathbf{k-q},<}(t) = c_{\mathbf{k-q},<}(t) \ a_{\mathbf{k-q}/2}(\mathbf{q},t) - c_{\mathbf{k},>}(t)$$

$$a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t) \ c_{\mathbf{k-q},>}(t) = c_{\mathbf{k-q},>}(t) \ a^{\dagger}_{\mathbf{k-q}/2}(-\mathbf{q},t) - c_{\mathbf{k},<}$$

$$< e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}^{(t)} a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q}, t)\ c_{\mathbf{k},<}^{\dagger}(t') >$$

$$= \frac{ie^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t}}{1 - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$(-(1 - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}})n_F(\mathbf{k})\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}^{(t)} c_{\mathbf{k-q},>}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t'}\theta(t - t')$$

$$+(e^{\lambda} - 1)\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}^{(t)}\ a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q}, t)\ c_{\mathbf{k},<}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t} - (e^{\lambda} - 1)\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k},<}^{(t)}\ c_{\mathbf{k},<}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t}$$

$$-(1 - n_F(\mathbf{k - q}))\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k},<}^{(t)} c_{\mathbf{k},<}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t}$$

$$+n_F(\mathbf{k})\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},>}^{(t)} c_{\mathbf{k-q},>}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t'})$$

$$< T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}^{(t)} a_{\mathbf{k-q}/2}(\mathbf{q}, t)\ c_{\mathbf{k},>}^{\dagger}(t') >\ =\ \frac{ie^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t}}{1 - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$(-(1 - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}})\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}^{(t)}\ c_{\mathbf{k-q},<}^{\dagger}(t') >\ (1 - n_F(\mathbf{k}))\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t'}\theta(t - t')$$

$$+(e^{-\lambda} - 1)\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}^{(t)}\ a_{\mathbf{k-q}/2}(\mathbf{q}, t) c_{\mathbf{k},>}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t} - (e^{-\lambda} - 1)\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k},>}^{(t)} c_{\mathbf{k},>}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t}$$

$$-n_F(\mathbf{k - q})\ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k},>}^{(t)} c_{\mathbf{k},>}^{\dagger}(t') >\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t}$$

$$+ < T\ e^{-\lambda N} >^{(t)}\ c_{\mathbf{k-q},<}^{(t)}\ c_{\mathbf{k-q},<}^{\dagger}(t') >\ (1 - n_F(\mathbf{k}))\ i\ e^{i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ t'})$$

$$a_{\mathbf{k-q}/2}(\mathbf{q}, t)\ c_{\mathbf{k-q},<}(t) = c_{\mathbf{k-q},<}(t)\ a_{\mathbf{k-q}/2}(\mathbf{q}, t) - c_{\mathbf{k},>}(t)$$
$$a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q}, t)\ c_{\mathbf{k-q},>}(t) = c_{\mathbf{k-q},>}(t)\ a_{\mathbf{k-q}/2}^{\dagger}(-\mathbf{q}, t) - c_{\mathbf{k},<}$$

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},>}(t)\ a^\dagger_{\mathbf{k-q}/2}(-\mathbf{q},t)\ c^\dagger_{\mathbf{k},<}(t^{'}) >$$

$$= \ -n_F(\mathbf{k})\ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},>}(t)c^\dagger_{\mathbf{k-q},>}(t^{'}) >\ e^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ (t-t^{'})}\ \frac{\theta(t-t^{'})\ e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t^{'}-t)}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$+ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t^{'}) >\ (1 - n_F(\mathbf{k-q}))\ \frac{e^\lambda}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},<}(t)\ a_{\mathbf{k-q}/2}(\mathbf{q},t)c^\dagger_{\mathbf{k},>}(t^{'}) > \quad =$$

$$- \ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},<}(t)\ c^\dagger_{\mathbf{k-q},<}(t^{'}) >\ (1 - n_F(\mathbf{k}))\ e^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ (t-t^{'})}\ \frac{\theta(t-t^{'})e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t^{'}-t)}{e^{-\lambda} - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$+ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},>}(t)c^\dagger_{\mathbf{k},>}(t^{'}) >\ n_F(\mathbf{k-q})\ \frac{e^{-\lambda}}{e^{-\lambda} - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

This means,

$$- < T\ e^{-\lambda N_>(t)}\ N_>(t)c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t^{'}) >$$

$$= \ -n_F(\mathbf{k})\ \sum_{\mathbf{q}}\ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},>}(t)c^\dagger_{\mathbf{k-q},>}(t^{'}) >\ e^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ (t-t^{'})}\ \frac{\theta(t-t^{'})\ e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t^{'}-t)}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$+ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t^{'}) >\ \sum_{\mathbf{q}}(1 - n_F(\mathbf{k-q}))\ \frac{e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$< T\ e^{-\lambda N_>(t)}\ N_>(t)\ c_{\mathbf{k},>}(t)c^\dagger_{\mathbf{k},>}(t^{'}) > \quad =$$

$$- \sum_{\mathbf{q}}\ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},<}(t)\ c^\dagger_{\mathbf{k-q},<}(t^{'}) >\ (1 - n_F(\mathbf{k}))\ e^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}\ (t-t^{'})}\ \frac{\theta(t-t^{'})e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t^{'}-t)}{e^{-\lambda} - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$+ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},>}(t)c^\dagger_{\mathbf{k},>}(t^{'}) >\ \sum_{\mathbf{q}} n_F(\mathbf{k-q})\ \frac{e^{-\lambda}}{e^{-\lambda} - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

The exact solution is,

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t^{'}) >$$

$$= \ e^{-i\epsilon_{\mathbf{k}}(t-t^{'})}\ n_F(\mathbf{k})\ \frac{(\theta(t-t^{'}) - \theta(t^{'}-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}$$

$$\prod_{\mathbf{p}}\frac{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)})}$$

===

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},>}(t)\ c^\dagger_{\mathbf{k},>}(t^{'}) >$$

$$= \ e^{-i\epsilon_{\mathbf{k}}(t-t^{'})}\ (1 - n_F(\mathbf{k}))\ \frac{(\theta(t-t^{'}) - \theta(t^{'}-t)\ e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}$$

$$\prod_{\mathbf{p}}\frac{(1 + e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_p-\mu)})}$$

===
Changing $\mathbf{k} \rightarrow \mathbf{k-q}$,

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},<}(t)\ c^\dagger_{\mathbf{k-q},<}(t^{'}) >$$

$$= \ e^{-i\epsilon_{\mathbf{k-q}}(t-t^{'})}\ n_F(\mathbf{k-q})\ \frac{(\theta(t-t^{'}) - \theta(t^{'}-t)e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)})}$$

$$\prod_{\mathbf{p}}\frac{(1 + e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_p-\mu)})}$$

===
and,

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},>}(t)\ c^\dagger_{\mathbf{k-q},>}(t^{'}) >$$

$$= \ e^{-i\epsilon_{\mathbf{k-q}}(t-t^{'})}\ (1 - n_F(\mathbf{k-q}))\ \frac{(\theta(t-t^{'}) - \theta(t^{'}-t)\ e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}$$

$$\prod_{\mathbf{p}}\frac{(1 + e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_p-\mu)})}$$

===

To get the LHS of the first equation $- < T \ e^{-\lambda N_>(t)} \ N_>(t) c_{\mathbf{k},<}(t) \ c^\dagger_{\mathbf{k},<}(t') >$, we need to differentiate $< T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},<}(t) \ c^\dagger_{\mathbf{k},<}(t') >$ w.r.t λ, ie.,

$$- < T \ e^{-\lambda N_>(t)} \ N_>(t) c_{\mathbf{k},<}(t) \ c^\dagger_{\mathbf{k},<}(t')$$

$$= -e^{-i\epsilon_{\mathbf{k}}(t-t')} \ n_F(\mathbf{k}) \ \frac{(\theta(t-t') - \theta(t'-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}$$

$$\frac{d}{d\lambda} \left(\prod_{\mathbf{p}} \frac{(1+ \ e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)})}{(1+e^{-\beta(\epsilon_p-\mu)})} \right)$$

where we can denote the expression as $Z(\lambda)$:

$$Z(\lambda) = \prod_{\mathbf{p}} \frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}}$$

Now, we want to compute $\frac{dZ(\lambda)}{d\lambda}$.
Taking the logarithm of $Z(\lambda)$ simplifies the differentiation:

$$\ln Z(\lambda) = \sum_{\mathbf{p}} \ln \left(\frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}} \right)$$

The derivative of $\ln Z(\lambda)$ with respect to λ is:

$$\frac{d\ln Z(\lambda)}{d\lambda} = \sum_{\mathbf{p}} \frac{d}{d\lambda} \ln \left(\frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}} \right)$$

The derivative of the logarithm is given by:

$$\frac{d}{d\lambda} \ln \left(\frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}} \right)$$

$$= \frac{1}{\frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}}} \cdot \frac{d}{d\lambda} \left(\frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}} \right)$$

This simplifies to:

$$= \frac{1+e^{-\beta(\epsilon_p-\mu)}}{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}} \cdot \frac{-\theta(p-k_F)e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}}$$

This further simplifies to:

$$= -\frac{\theta(p-k_F)e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}$$

Thus:

$$\frac{d\ln Z(\lambda)}{d\lambda} = -\sum_{\mathbf{p}} \frac{\theta(p-k_F)e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}$$

Finally, we have:

$$\frac{dZ(\lambda)}{d\lambda} = Z(\lambda) \cdot \frac{d\ln Z(\lambda)}{d\lambda}$$

$$= -Z(\lambda) \sum_{\mathbf{p}} \frac{\theta(p-k_F)e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}$$

$$= -\prod_{\mathbf{p}} \frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}} \sum_{\mathbf{q}} \frac{\theta(q-k_F)e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}}{1+e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}}$$

===
Therefore,

$$- < T \ e^{-\lambda N_>(t)} \ N_>(t) c_{\mathbf{k},<}(t) \ c^\dagger_{\mathbf{k},<}(t')$$

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')} \ n_F(\mathbf{k}) \ \frac{(\theta(t-t') - \theta(t'-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}$$

$$\times \sum_{\mathbf{q}} \frac{\theta(q-k_F)e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}}{1+e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}} \prod_{\mathbf{p}} \frac{1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1+e^{-\beta(\epsilon_p-\mu)}}$$

===
Similarly for, $< T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},>}(t) \ c^\dagger_{\mathbf{k},>}(t') >$ since we have two terms depending on λ, we can use chain rule to get,

$$< T \ e^{-\lambda N_>(t)} \ c_{\mathbf{k},>}(t) \ c^\dagger_{\mathbf{k},>}(t') >$$

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')} \ (1-n_F(\mathbf{k})) \ \frac{d}{d\lambda} \left(\frac{(\theta(t-t') - \theta(t'-t) \ e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})} \right)$$

$$\times \prod_{\mathbf{p}} \frac{(1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)})}{(1+e^{-\beta(\epsilon_p-\mu)})}$$

$$+ e^{-i\epsilon_{\mathbf{k}}(t-t')} \ (1-n_F(\mathbf{k})) \left(\frac{(\theta(t-t') - \theta(t'-t) \ e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})} \right)$$

$$\times \frac{d}{d\lambda} \prod_{\mathbf{p}} \frac{(1 + e^{-\beta(\epsilon_p - \mu) - \lambda\theta(p - k_F)})}{(1 + e^{-\beta(\epsilon_p - \mu)})}$$

===

This gives,

$$< T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},>}(t)\ c^\dagger_{\mathbf{k},>}(t') >$$

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')}\ (1 - n_F(\mathbf{k}))$$

$$\left(\frac{e^{-\lambda-\beta(-\mu+\epsilon_{\mathbf{k}})} \cdot \theta(-t+t')}{1 + e^{-\lambda-\beta(-\mu+\epsilon_{\mathbf{k}})}} + \frac{e^{-\lambda-\beta(-\mu+\epsilon_{\mathbf{k}})} \cdot \left(\theta(t-t') - e^{-\lambda-\beta(-\mu+\epsilon_{\mathbf{k}})} \cdot \theta(-t+t')\right)}{\left(1 + e^{-\lambda-\beta(-\mu+\epsilon_{\mathbf{k}})}\right)^2} \right.$$

$$\left. - \frac{(\theta(t-t') - \theta(t'-t)\ e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})} \sum_{\mathbf{q}} \frac{\theta(q - k_F)e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}} \right)$$

$$\times \prod_{\mathbf{p}} \frac{1 + e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1 + e^{-\beta(\epsilon_p-\mu)}}$$

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3.2 Verification of equation 1

$$- < T\ e^{-\lambda N_>(t)}\ N_>(t)c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t') >$$

$$= -n_F(\mathbf{k}) \sum_{\mathbf{q}}\ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k-q},>}(t)c^\dagger_{\mathbf{k-q},>}(t') >\ e^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}(t-t')}\ \frac{\theta(t-t')\ e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t'-t)}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

$$+ < T\ e^{-\lambda N_>(t)}\ c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t') >\ \sum_{\mathbf{q}}(1 - n_F(\mathbf{k-q}))\ \frac{e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}$$

3.2.1 LHS of eqn 1

$$- < T\ e^{-\lambda N_>(t)}\ N_>(t)c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t')$$

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')}\ n_F(\mathbf{k})\ \left(\frac{(\theta(t-t') - \theta(t'-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})} \right.$$

$$\left. \times \sum_{\mathbf{q}} \frac{\theta(q - k_F)e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda\theta(q-k_F)}} \right) \prod_{\mathbf{p}} \frac{1 + e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)}}{1 + e^{-\beta(\epsilon_p-\mu)}}$$

3.2.2 RHS of eqn 1

$$- < T\ e^{-\lambda N_>(t)}\ N_>(t)c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t') >$$

$$= n_F(\mathbf{k})\ \left(- \sum_{\mathbf{q}}\ e^{-i\epsilon_{\mathbf{k-q}}(t-t')}\ (1 - n_F(\mathbf{k-q}))\ \frac{(\theta(t-t') - \theta(t'-t)\ e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})} \right.$$

$$\times e^{-i\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}(t-t')}\ \frac{\theta(t-t')\ e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t'-t)}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}} +\ e^{-i\epsilon_{\mathbf{k}}(t-t')}\ n_F(\mathbf{k})\ \frac{(\theta(t-t') - \theta(t'-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}$$

$$\times \sum_{\mathbf{q}}(1 - n_F(\mathbf{k-q}))\ \frac{e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}} \right) \times \prod_{\mathbf{p}} \frac{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)})}$$

Using,

$$\epsilon_{\mathbf{k-q}} + \frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m} = \frac{k^2}{2m} = \epsilon_{\mathbf{k}}$$

RHS simplifies to,

$$- < T\ e^{-\lambda N_>(t)}\ N_>(t)c_{\mathbf{k},<}(t)\ c^\dagger_{\mathbf{k},<}(t') >$$

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')}\ n_F(\mathbf{k})\ \left(- \sum_{\mathbf{q}}\ (1 - n_F(\mathbf{k-q}))\ \frac{(\theta(t-t') - \theta(t'-t)\ e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})} \right.$$

$$\times \frac{\theta(t-t')\ e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}} + \theta(t'-t)}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}} +\ \frac{(\theta(t-t') - \theta(t'-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}$$

$$\left. \times \sum_{\mathbf{q}}(1 - n_F(\mathbf{k-q}))\ \frac{e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}}{e^\lambda - e^{\beta\frac{(\mathbf{k-q}/2)\cdot\mathbf{q}}{m}}} \right) \prod_{\mathbf{p}} \frac{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)})}$$

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')}\ n_F(\mathbf{k})\ \left(- \sum_{\mathbf{q}}\ (1 - n_F(\mathbf{k-q}))\ \frac{(\theta(t-t') - \theta(t'-t)\ e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})} \right.$$

$$\times \frac{\theta(t-t')\ e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})} + \theta(t'-t)}{e^\lambda - e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}} +\ \frac{(\theta(t-t') - \theta(t'-t)e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}$$

$$\left. \times \sum_{\mathbf{q}}(1 - n_F(\mathbf{k-q}))\ \frac{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}}{e^\lambda - e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}} \right) \prod_{\mathbf{p}} \frac{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)-\lambda\theta(p-k_F)})}{(1 + e^{-\beta(\epsilon_{\mathbf{p}}-\mu)})}$$

$$= - e^{-i\epsilon_{\mathbf{k}}(t-t')} \, n_F(\mathbf{k}) \, \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q}))$$

$$\left(\frac{(\theta(t - t') - \theta(t' - t) \, e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \, \frac{\theta(t - t') \, e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})} + \theta(t' - t)}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right.$$

$$\left. - \, \frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right) \prod_{\mathbf{p}} \frac{(1 + e^{-\beta(\epsilon_{\mathbf{p}} - \mu) - \lambda \theta(p - k_F)})}{(1 + e^{-\beta(\epsilon_{\mathbf{p}} - \mu)})}$$

Comparing with the LHS,

$$= e^{-i\epsilon_{\mathbf{k}}(t-t')} \, n_F(\mathbf{k}) \, \left(\frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \right.$$

$$\left. \times \sum_{\mathbf{q}} \frac{\theta(q - k_F) e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}} \right) \prod_{\mathbf{p}} \frac{1 + e^{-\beta(\epsilon_p - \mu) - \lambda \theta(p - k_F)}}{1 + e^{-\beta(\epsilon_p - \mu)}},$$

we need to show,

$$\frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \sum_{\mathbf{q}} \frac{\theta(q - k_F) e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}$$

$$= - \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \left(\frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right.$$

$$\left. - \frac{(\theta(t - t') - \theta(t' - t) \, e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \, \frac{\theta(t - t') \, e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})} + \theta(t' - t)}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right).$$

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Taking $t > t'$, LHS is,

$$\frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \sum_{\mathbf{q}} \frac{\theta(q - k_F) e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}$$

The RHS taking $t > t'$ becomes,

$$= - \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \left(- \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right.$$

$$\left. + \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right)$$

$$= - \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \left(\frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} - \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \right)$$

$$= - \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \left(\frac{e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda} - e^{-\beta(\epsilon_{\mathbf{k}} - \mu)}}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \right)$$

$$= - \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \, e^{-\beta(\epsilon_{\mathbf{k}} - \mu) - \lambda} \left(\frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})} - e^{\lambda}}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \right)$$

$$= - \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \, e^{-\beta(\epsilon_{\mathbf{k}} - \mu) - \lambda} \left(\frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})} - e^{\lambda}}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \right)$$

$$= \frac{1}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \left(\frac{e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda}}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \right).$$

This matches with the LHS, as we can just shift the sum from $\mathbf{q} \to \mathbf{k} - \mathbf{q}$

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We need to show,

$$\frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \sum_{\mathbf{q}} \frac{\theta(q - k_F) e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}$$

$$= - \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \left(\frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right.$$

$$\left. - \frac{(\theta(t - t') - \theta(t' - t) \, e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \, \frac{\theta(t - t') \, e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})} + \theta(t' - t)}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right).$$

Taking $t' > t$, LHS is,

$$- \frac{(e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \sum_{\mathbf{q}} \frac{\theta(q - k_F) e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}{1 + e^{-\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda \theta(q - k_F)}}$$

$$= - \frac{1}{(1 + e^{\beta(\epsilon_{\mathbf{k}} - \mu)})} \sum_{\mathbf{q}} \frac{\theta(q - k_F)}{1 + e^{\beta(\epsilon_{\mathbf{q}} - \mu) - \lambda}}$$

The RHS is,

$$= - \sum_{\mathbf{q}} \, (1 - n_F(\mathbf{k} - \mathbf{q})) \left(\frac{(-e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})} \, \frac{e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right.$$

$$\left. - \frac{(- e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})}{(1 + e^{-\beta(\epsilon_{\mathbf{k-q}} - \mu) - \lambda})} \, \frac{1}{e^{\lambda} - e^{\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k-q}})}} \right).$$

For this we need to simplify the expression,

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$$\frac{(-e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})} \ \frac{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}}{e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}} - \frac{(-e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}{(1+e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})} \ \frac{1}{e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}}$$

Simplifying the first term,

$$\frac{-e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}}{1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}} \cdot \frac{e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}}{e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}} = \frac{-e^{-\beta(\epsilon_{\mathbf{k}}-\mu)} \cdot e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})}}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}) \cdot (e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})})}$$

$$= \frac{-e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)}}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}) \cdot (e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})})}$$

The simplified expression becomes:

$$\frac{-e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)}}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}) \cdot (e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})})} - \frac{-e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda}}{(1+e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda}) \cdot (e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})})}$$

$$= -\frac{e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)}}{(e^{\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k-q}})})} \Big(\frac{(1+e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})-(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})e^{-\lambda}}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})(1+e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}\Big)$$

$$= -\frac{1}{(e^{\beta(\epsilon_{\mathbf{k-q}}-\mu)+\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\mu)})} \Big(\frac{(1+e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})-(e^{-\lambda}+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})(1+e^{-\beta(\epsilon_{\mathbf{k-q}}-\mu)-\lambda})}\Big)$$

Taking $\mathbf{k} - \mathbf{q} = \mathbf{Q}$,

$$= -\frac{1}{(e^{\beta(\epsilon_{\mathbf{Q}}-\mu)+\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\mu)})} \Big(\frac{(1+e^{-\beta(\epsilon_{\mathbf{Q}}-\mu)-\lambda})-(e^{-\lambda}+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})(1+e^{-\beta(\epsilon_{\mathbf{Q}}-\mu)-\lambda})}\Big)$$

$$= -\frac{e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})} \Big(\frac{e^{\beta(\epsilon_{\mathbf{k}}-\mu)}(1-e^{-\lambda})+e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{Q}})} \cdot e^{-\lambda}-e^{-\lambda}}{(e^{\beta(\epsilon_{\mathbf{Q}}-\mu)+\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\mu)})(1+e^{-\beta(\epsilon_{\mathbf{Q}}-\mu)-\lambda})}\Big)$$

$$= -\frac{e^{-\beta(\epsilon_{\mathbf{k}}-\mu)}}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)})} \Big(\frac{(1-e^{-\lambda})e^{\beta(\epsilon_{\mathbf{k}}-\mu)}+(e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{Q}})}-1) \cdot e^{-\lambda}}{(e^{\beta(\epsilon_{\mathbf{Q}}-\mu)+\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\mu)})+(1-e^{\beta(\epsilon_{\mathbf{k}}-\mathbf{Q})-\lambda})}\Big)$$

This must be equal to,

$$= -\frac{1}{(1+e^{\beta(\epsilon_{\mathbf{k}}-\mu)})} \ \frac{1}{1+e^{\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda}}.$$

For this,

$$\Big(\frac{(1-e^{-\lambda})e^{\beta(\epsilon_{\mathbf{k}}-\mu)}+(e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{Q}})}-1) \cdot e^{-\lambda}}{(e^{\beta(\epsilon_{\mathbf{Q}}-\mu)+\lambda}-e^{\beta(\epsilon_{\mathbf{k}}-\mu)})+(1-e^{\beta(\epsilon_{\mathbf{k}}-\mathbf{Q})-\lambda})}\Big) = \frac{1}{1+e^{\beta(\epsilon_{\mathbf{q}}-\mu)-\lambda}}$$

4 Deriving the exact expectation value

$$\hat{N}_> = \sum_{\mathbf{k}} c_{\mathbf{k},>} c_{\mathbf{k},>}^{\dagger}$$

$$c_{\mathbf{p},s}(t) = c_{\mathbf{p},s}\ e^{-i\epsilon_p\ t}$$

4.1 First equation

$$< T\ e^{-\lambda N_>}(t)\ c_{\mathbf{k},<}(t)\ c_{\mathbf{k},<}^{\dagger}(t^{'})\ >$$

$$= \theta(t-t^{'})\ < e^{-\lambda N_>}(t)\ \underbrace{c_{\mathbf{k},<}(t)\ c_{\mathbf{k},<}^{\dagger}(t^{'})}_{1-c_{\mathbf{k},<}^{\dagger}(t^{'})c_{\mathbf{k},<}(t)}\ > +\theta(t^{'}-t)\ < \underbrace{c_{\mathbf{k},<}^{\dagger}(t^{'})e^{-\lambda N_>}(t)}_{[c_{\mathbf{k},<}^{\dagger}(t^{'}),e^{-\lambda N_>}(t)]=0}\ c_{\mathbf{k},<}(t)\ >$$

$$= \theta(t-t^{'})\ < e^{-\lambda N_>}(t)\ (1-c_{\mathbf{k},<}^{\dagger}(t^{'})c_{\mathbf{k},<}(t)) > +\theta(t^{'}-t)\ < e^{-\lambda N_>}(t)\ c_{\mathbf{k},<}^{\dagger}(t^{'})c_{\mathbf{k},<}(t)\ >$$

Since, $c_{\mathbf{k},s}(t) = c_{\mathbf{k},s}\ e^{-i\epsilon_k\ t}$ and $N_> = \sum_{\mathbf{p}} \theta(k_F-|p|)n_p$,

$$< T\ e^{-\lambda N_>}(t)\ c_{\mathbf{k},<}(t)\ c_{\mathbf{k},<}^{\dagger}(t^{'})\ >$$

$$= e^{i\epsilon_k\ (t^{'}-t)}\left(\theta(t-t^{'})\ < e^{-\lambda \sum_{\mathbf{p}} \theta(k_F-|p|)n_p}\ (1-n_{\mathbf{k},<}\ ) > +\theta(t^{'}-t)\ < e^{-\lambda \sum_{\mathbf{p}} \theta(k_F-|p|)n_p}\ n_{\mathbf{k},<}\ >\right)$$

The term $< e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p}\ (1-n_{\mathbf{k},<}\ ) >$ can be written as $< e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p}\ > - < e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p}\ n_{\mathbf{k},<}\ >$, hence,

$$< T\ e^{-\lambda N_>}(t)\ c_{\mathbf{k},<}(t)\ c_{\mathbf{k},<}^{\dagger}(t^{'})\ >$$

$$= e^{i\epsilon_k\ (t^{'}-t)}\left(\theta(t-t^{'})\ \left(< e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p}\ > - < e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p}\ n_{\mathbf{k},<}\ >\right)\right.$$

$$\left. +\theta(t^{'}-t)\ < e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p}\ n_{\mathbf{k},<}(t)\ >\right)$$

===

Taking,

$$< e^{-\lambda N}>\ n_{k,<}\ > = \theta(k_F-|k|)\ \frac{Tr(e^{-\sum_p n_p[\beta(\epsilon_p-\mu)+\lambda\ \theta(|p|-k_F)]}\ n_k)}{Tr(e^{-\beta \sum_p n_p(\epsilon_p-\mu)})}$$

$$< e^{-\lambda N}>\ n_{k,<}\ > = \theta(k_F-|k|)\ \frac{Tr(e^{-n_k[\beta(\epsilon_k-\mu)+\lambda\ \theta(|k|-k_F)]}\ n_k)}{Tr(e^{-\beta n_k(\epsilon_k-\mu)})}\ \frac{Tr(e^{-\sum_{p\neq k} n_p[\beta(\epsilon_p-\mu)+\lambda\ \theta(|p|-k_F)]})}{Tr(e^{-\beta \sum_{p\neq k} n_p(\epsilon_p-\mu)})}$$

$$< e^{-\lambda N}>\ n_{k,<}\ > = \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ \theta(|k|-p_F)]}}{(1+e^{-\beta(\epsilon_k-\mu)})}\ \frac{Tr(e^{-\sum_{p\neq k} n_p[\beta(\epsilon_p-\mu)+\lambda\ \theta(|p|-k_F)]})}{Tr(e^{-\beta \sum_{p\neq k} n_p(\epsilon_p-\mu)})}$$

$$< e^{-\lambda N}>\ n_{k,<}\ > = \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ \theta(|k|-k_F)]}}{(1+e^{-\beta(\epsilon_k-\mu)})}\ \frac{\Pi_{p\neq k}(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ \theta(|p|-k_F)]})}{\Pi_{p\neq k}(1+e^{-\beta(\epsilon_p-\mu)})}$$

$$< e^{-\lambda N}>\ n_{k,<}\ > = \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ \theta(|k|-k_F)]}}{(1+e^{-\beta(\epsilon_k-\mu)})}\ e^{\sum_{p\neq k}\ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ \theta(|p|-k_F)]})}{(1+e^{-\beta(\epsilon_p-\mu)})}\right)}$$

$$= \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]}}{(1+e^{-\beta(\epsilon_k-\mu)})}\ e^{\sum_p\ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ (1-\theta(k_F-|p|))]})}{(1+e^{-\beta(\epsilon_p-\mu)})}\right)-\log\left(\frac{(1+e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]})}{(1+e^{-\beta(\epsilon_k-\mu)})}\right)}$$

Inverting the argument of the log and making the $-$ sign a $+$ sign,

$$< e^{-\lambda N}>\ n_{k,<}\ >$$

$$= \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]}}{(1+e^{-\beta(\epsilon_k-\mu)})}\ e^{\sum_p\ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ (1-\theta(k_F-|p|))]})}{(1+e^{-\beta(\epsilon_p-\mu)})}\right)+\log\left(\frac{(1+e^{-\beta(\epsilon_k-\mu)})}{(1+e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]})}\right)}$$

Taking the last $e^{\log}$ down and canceling with the denominator of the first term,

$$< e^{-\lambda N}>\ n_{k,<}\ >$$

$$= \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]}}{(1+e^{-\beta(\epsilon_k-\mu)})}\ e^{\sum_p\ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ (1-\theta(k_F-|p|))]})}{(1+e^{-\beta(\epsilon_p-\mu)})}\right)}\ \frac{(1+e^{-\beta(\epsilon_k-\mu)})}{(1+e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]})}$$

we get,

$$< e^{-\lambda N}>\ n_{k,<}\ >$$

$$= \theta(k_F-|k|)\ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]}}{(1+e^{-[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]})}\ e^{\sum_p\ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ (1-\theta(k_F-|p|))]})}{(1+e^{-\beta(\epsilon_p-\mu)})}\right)}$$

or,

$$< e^{-\lambda N}>\ n_{k,<}\ >$$

$$= \theta(k_F-|k|)\ \frac{1}{(1+e^{[\beta(\epsilon_k-\mu)+\lambda\ (1-\theta(k_F-|k|))]})}\ e^{\sum_p\ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda\ (1-\theta(k_F-|p|))]})}{(1+e^{-\beta(\epsilon_p-\mu)})}\right)}.$$

Since $\theta(k_F - |k|) = 1$, hence $(1 - \theta(k_F - |k|)) = 0$,

$$< e^{-\lambda N} > \; n_{k,<} \; > \; = \; \frac{\theta(k_F - |k|)}{(1 + e^{\beta(\epsilon_k - \mu)})} \; e^{\sum_{|p| > k_F} \; \log\left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)}.$$

$$< e^{-\lambda N} > \; n_{k,<} \; > \; = \; \frac{n_F(\mathbf{k})}{(1 + e^{\beta(\epsilon_k - \mu)})} \; e^{\sum_{|p| > k_F} \; \log\left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)}$$

$$< e^{-\lambda N} > \; n_{k,<} \; > \; = \; \frac{n_F(\mathbf{k})}{(1 + e^{\beta(\epsilon_k - \mu)})} \; \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)$$

===

$$< e^{-\lambda N} > \; n_{k,<} \; > \; = n_F(\mathbf{k}) \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \; \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)$$

===
Also, $< e^{-\lambda N} > \; >$ can be written as,

$$< e^{-\lambda N} > \; > \; = \; \frac{Tr(e^{-\sum_p n_p [\beta(\epsilon_p - \mu) + \lambda \; \theta(|p| - k_F)]})}{Tr(e^{-\beta \sum_p n_p (\epsilon_p - \mu)})}$$

$$= \; \frac{\prod_p (1 + e^{-[\beta(\epsilon_p - \mu) + \lambda \; \theta(|p| - k_F)]})}{\prod_p (1 + e^{-\beta(\epsilon_p - \mu)})}$$

===

$$< e^{-\lambda N} > \; > \; = \; \prod_{|\mathbf{p}| > k_F} \frac{(1 + e^{-[\beta(\epsilon_p - \mu) + \lambda \;]})}{(1 + e^{-\beta(\epsilon_p - \mu)})}$$

===
Finally,

$$< T \; e^{-\lambda N} >^{(t)} \; c_{\mathbf{k},<}(t) \; c^{\dagger}_{\mathbf{k},<}(t') >$$

$$= e^{i\epsilon_k \; (t' - t)} \Big(\theta(t - t') \; \Big(< e^{-\lambda \sum_{\mathbf{p}} \theta(|p| - k_F) n_p} \; > \; - \; < e^{-\lambda \sum_{\mathbf{p}} \theta(|p| - k_F) n_p} n_{k,<} \; > \Big)$$

$$+ \theta(t' - t) n_F(\mathbf{k}) \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)\Big)$$

$$= e^{i\epsilon_k \; (t' - t)} \Big(\theta(t - t') \; \Big(\prod_{|\mathbf{p}| > k_F} \frac{(1 + e^{-[\beta(\epsilon_p - \mu) + \lambda \;]})}{(1 + e^{-\beta(\epsilon_p - \mu)})} - n_F(\mathbf{k}) \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)\Big)$$

$$+ \theta(t' - t) n_F(\mathbf{k}) \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)\Big)$$

$$= e^{i\epsilon_k \; (t' - t)} n_F(\mathbf{k}) \Big(\theta(t - t') \; \Big(1 - \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \Big)$$

$$+ \theta(t' - t) \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \Big) \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)$$

===

$$< T \; e^{-\lambda N} >^{(t)} \; c_{\mathbf{k},<}(t) \; c^{\dagger}_{\mathbf{k},<}(t') > = e^{i\epsilon_k \; (t' - t)} n_F(\mathbf{k}) \Big(\theta(t - t') \; \Big(\frac{1}{(1 + e^{-\beta(\epsilon_k - \mu)})} \Big)$$

$$+ \theta(t' - t) \frac{e^{-\beta(\epsilon_k - \mu)}}{(1 + e^{-\beta(\epsilon_k - \mu)})} \Big) \prod_{|\mathbf{p}| > k_F} \left(\frac{1 + e^{-\beta(\epsilon_p - \mu)} e^{-\lambda}}{1 + e^{-\beta(\epsilon_p - \mu)}}\right)$$

===
Thus, we have verified the answer,

$$< T \; e^{-\lambda N} >^{(t)} \; c_{\mathbf{k},<}(t) \; c^{\dagger}_{\mathbf{k},<}(t') >$$

$$= \; e^{-i\epsilon_{\mathbf{k}}(t - t')} \; n_F(\mathbf{k}) \; \frac{(\theta(t - t') - \theta(t' - t) e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}{(1 + e^{-\beta(\epsilon_{\mathbf{k}} - \mu)})}$$

$$\prod_{\mathbf{p}} \frac{(1 + e^{-\beta(\epsilon_{\mathbf{p}} - \mu) - \lambda \theta(p - k_F)})}{(1 + e^{-\beta(\epsilon_{\mathbf{p}} - \mu)})}$$

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4.2 Second equation

Using similar steps as above, we get,

$$< T \ e^{-\lambda N >} (t) \ c_{\mathbf{k},>}(t) \ c_{\mathbf{k},>}^{\dagger}(t') >$$

$$= e^{i\epsilon_k \ (t'-t)} \left(\theta(t-t') \ \left(< e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p} > \ - \ < e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p} \ n_{\mathbf{k},>} > \right) \right.$$

$$\left. + \theta(t'-t) \ < e^{-\lambda \sum_{\mathbf{p}} \theta(|p|-k_F)n_p} \ n_{\mathbf{k},>}(t) > \right)$$

==

Similar steps as above gives,

$$< e^{-\lambda N} \ n_{k,>} > \ = \ \theta(|k|-k_F) \ \frac{e^{-[\beta(\epsilon_k-\mu)+\lambda \ \theta(|k|-k_F)]}}{(1+e^{-\beta(\epsilon_k-\mu)})} \ e^{\sum_{p\neq k} \ \log\left(\frac{(1+e^{-[\beta(\epsilon_p-\mu)+\lambda \ \theta(|p|-k_F)]})}{(1+e^{-\beta(\epsilon_p-\mu)})} \right)}$$

$$< e^{-\lambda N} \ n_{k,>} > \ = \ \frac{\theta(|k|-k_F)}{(1+e^{\beta(\epsilon_k-\mu)}e^{\lambda})} \ e^{\sum_{|p|>k_F} \ \log\left(\frac{(1+e^{-\beta(\epsilon_p-\mu)}e^{-\lambda})}{(1+e^{-\beta(\epsilon_p-\mu)})} \right)}$$

==

$$< e^{-\lambda N} \ n_{k,>} > \ = \ \frac{(1-n_F(\mathbf{k}))}{(1+e^{\beta(\epsilon_k-\mu)}e^{\lambda})} \ \prod_{|p|>k_F} \left(\frac{(1+e^{-\beta(\epsilon_p-\mu)}e^{-\lambda})}{(1+e^{-\beta(\epsilon_p-\mu)})} \right)$$

==

Hence,

$$< T \ e^{-\lambda N >} (t) \ c_{\mathbf{k},>}(t) \ c_{\mathbf{k},>}^{\dagger}(t') >$$

$$= e^{i\epsilon_k \ (t'-t)} (1-n_F(\mathbf{k})) \left(\theta(t-t') \ \left(\frac{1}{(1+e^{-\beta(\epsilon_k-\mu)-\lambda})} \right) \right.$$

$$\left. + \theta(t'-t) \frac{e^{-\beta(\epsilon_k-\mu)-\lambda}}{(1+e^{-\beta(\epsilon_k-\mu)-\lambda})} \right) \prod_{|p|>k_F} \left(\frac{(1+e^{-\beta(\epsilon_p-\mu)}e^{-\lambda})}{(1+e^{-\beta(\epsilon_p-\mu)})} \right)$$

This gives us,

$$< T \ e^{-\lambda N >} (t) \ c_{\mathbf{k},>}(t) \ c_{\mathbf{k},>}^{\dagger}(t') > = \ e^{-i\epsilon_{\mathbf{k}}(t-t')} \ (1-n_F(\mathbf{k})) \ \frac{(\theta(t-t')-\theta(t'-t) \ e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}{(1+e^{-\beta(\epsilon_{\mathbf{k}}-\mu)-\lambda})}$$

$$\prod_{\mathbf{p}} \frac{(1+e^{-\beta(\epsilon_p-\mu)-\lambda\theta(p-k_F)})}{(1+e^{-\beta(\epsilon_p-\mu)})}$$

==

Therefore, the solutions are correct.