Specialised random phase approximation (SRPA) vs. Generalised random phase approximation (GRPA)

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1 SRPA versus GRPA - I

Defn: $a_{\mathbf{k}}(\mathbf{q}) = c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<}c_{\mathbf{k}+\mathbf{q}/2,>}$, $n_{\mathbf{p}} = c^{\dagger}_{\mathbf{p}}c_{\mathbf{p}}$.

This means (these are exact),

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}^{'})] = 0$$

and

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q})\ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2})$$

and

$$[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})] \ = \ n_F(\mathbf{p})\ c_{\mathbf{p}+\mathbf{q},>}\ \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}$$

$$[c_{\mathbf{p},>}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \ = \ (1 - n_F(\mathbf{p}))\ c_{\mathbf{p}-\mathbf{q},<}\ \delta_{\mathbf{k},\mathbf{p}-\mathbf{q}/2}$$

and

$$[c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \ = \ [c_{\mathbf{p},>}, a_{\mathbf{k}}(\mathbf{q})] \ = \ 0$$

Now the big question is this, should we say (these are approximate),

SRPA: $[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))$

OR

GRPA: $[a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))(n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2})$.

Suppose we select the simple **SRPA**.

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \ = \ [c_{\mathbf{p},<}, \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))] \ = \ 0$$

On the other hand,

$$[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \ = \ [[c_{\mathbf{p},<}, a_{\mathbf{k}}(\mathbf{q})], a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] + [a_{\mathbf{k}}(\mathbf{q}), [c_{\mathbf{p},<}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]]$$

this means,

$$\begin{aligned}[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \ &= \ [n_F(\mathbf{p})\ c_{\mathbf{p}+\mathbf{q},>}\ \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}, a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] + 0 \\ &= \ n_F(\mathbf{p})\ \ \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}/2}\ (1 - n_F(\mathbf{p}+\mathbf{q}))\ c_{\mathbf{p}+\mathbf{q}-\mathbf{q}',<}\ \delta_{\mathbf{k}',\mathbf{p}+\mathbf{q}-\mathbf{q}'/2}\end{aligned}$$

Thus we have reached a contradiction.
Suppose we select the **GRPA**.

$$\begin{aligned}[c_{\mathbf{p},<}, [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \ &= \ \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))\ [c_{\mathbf{p},<}, n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}] \ = \\ &= \ \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\ n_F(\mathbf{k}-\mathbf{q}/2)(1 - n_F(\mathbf{k}+\mathbf{q}/2))\ n_F(\mathbf{p})\ (\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\ c_{\mathbf{p}}\end{aligned}$$

On the other hand,

$$[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \;=\; [[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] + [a_{\mathbf{k}}(\mathbf{q}),[c_{\mathbf{p},<},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]]$$

this means,

$$\begin{aligned}[c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] \;&=\; [n_F(\mathbf{p})\; c_{\mathbf{p+q},>}\;\delta_{\mathbf{k},\mathbf{p+q}/2},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] + 0\\ &=\; n_F(\mathbf{p})\;\;\delta_{\mathbf{k},\mathbf{p+q}/2}\;(1-n_F(\mathbf{p+q}))\;c_{\mathbf{p+q-q}',<}\;\delta_{\mathbf{k}',\mathbf{p+q-q}'/2}\end{aligned}$$

This is not contradiction at least for $\mathbf{k}=\mathbf{k}^{'}$ and $\mathbf{q}=\mathbf{q}^{'}$.

2 SRPA versus GRPA - II

$$[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})] \;=\; n_F(\mathbf{p})\;c_{\mathbf{p+q},>}\;\delta_{\mathbf{k},\mathbf{p+q}/2}$$

$$[c_{\mathbf{p},>},a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \;=\; (1-n_F(\mathbf{p}))\;c_{\mathbf{p-q},<}\;\delta_{\mathbf{k},\mathbf{p-q}/2}$$

Take a further commutator with $a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})$ and $a_{\mathbf{k}'}(\mathbf{q}^{'})$,

$$[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] \;=\; n_F(\mathbf{p})\;[c_{\mathbf{p+q},>},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]\;\delta_{\mathbf{k},\mathbf{p+q}/2}$$

$$[[c_{\mathbf{p},>},a^{\dagger}_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}(\mathbf{q}^{'})] \;=\; (1-n_F(\mathbf{p}))\;[c_{\mathbf{p-q},<},a_{\mathbf{k}'}(\mathbf{q}^{'})]\;\delta_{\mathbf{k},\mathbf{p-q}/2}$$

In **SRPA**,

$$\begin{aligned}[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] \;&=\; [c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] + [[c_{\mathbf{p},<},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})],a_{\mathbf{k}}(\mathbf{q})]\\ &= [c_{\mathbf{p},<},\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))] + 0 = 0\end{aligned}$$

whereas,

$$n_F(\mathbf{p})\;[c_{\mathbf{p+q},>},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]\;\delta_{\mathbf{k},\mathbf{p+q}/2} = n_F(\mathbf{p})\;(1-n_F(\mathbf{p+q}))\;c_{\mathbf{p+q-q}',<}\;\delta_{\mathbf{k}',\mathbf{p+q-q}'/2}\;\delta_{\mathbf{k},\mathbf{p+q}/2}$$

which is a contradiction. Whereas in **GRPA**,

$$\begin{aligned}[[c_{\mathbf{p},<},a_{\mathbf{k}}(\mathbf{q})],a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] \;&=\; [c_{\mathbf{p},<},[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]] + [[c_{\mathbf{p},<},a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})],a_{\mathbf{k}}(\mathbf{q})]\\ &= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\;(\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}-\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\;c_{\mathbf{p},<}\end{aligned}$$

These two are equal when $\mathbf{q}^{'}=\mathbf{q}$ (**GRPA**). Also,

$$\begin{aligned}[[c_{\mathbf{p},>},a^{\dagger}_{\mathbf{k}}(\mathbf{q})],a_{\mathbf{k}'}(\mathbf{q}^{'})] \;&=\; -\,n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\;[c_{\mathbf{p},>},n_{\mathbf{k}-\mathbf{q}/2}-n_{\mathbf{k}+\mathbf{q}/2}]\\ &=\; -\,n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\;(\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}-\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2})\;c_{\mathbf{p},>}\end{aligned}$$

and

$$(1-n_F(\mathbf{p}))\;[c_{\mathbf{p-q},<},a_{\mathbf{k}'}(\mathbf{q}^{'})]\;\delta_{\mathbf{k},\mathbf{p-q}/2} \;=\; (1-n_F(\mathbf{p}))\;n_F(\mathbf{p-q})\;c_{\mathbf{p-q+q}',>}\;\delta_{\mathbf{k}',\mathbf{p-q+q}'/2}\;\delta_{\mathbf{k},\mathbf{p-q}/2}$$

These two are equal when $\mathbf{q}=\mathbf{q}^{'}$ (**GRPA**).

3 Exact bosons

The goal now is to find exact bosons $b_{\mathbf{k}}(\mathbf{q})$ such that,

$$[b_{\mathbf{k}}(\mathbf{q}),b_{\mathbf{k}'}(\mathbf{q}^{'})]=0;\quad [b_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]=n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\;\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}$$

whereas the starting point is **GRPA**,

$$[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}(\mathbf{q}^{'})]=0;\quad [a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})]=n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\;(n_{\mathbf{k}-\mathbf{q}/2}-n_{\mathbf{k}+\mathbf{q}/2})\;\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}$$

and

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q})\ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2})$$

Note that,

$$n_{\mathbf{p}} \neq n_F(\mathbf{p}) + \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) - \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})$$

whereas,

$$n_{\mathbf{p}} = n_F(\mathbf{p}) + \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) - \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})$$

and

$$(N_>)^2 \ = \ \sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q})$$

Set,

$$a_{\mathbf{k}}(\mathbf{q}) \ = \ \Gamma(b^{\dagger}b; \mathbf{k}, \mathbf{q}) b_{\mathbf{k}}(\mathbf{q})$$

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}^{'})] = 0; \quad [a_{\mathbf{k}}(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q}^{'})] = n_F(\mathbf{k}-\mathbf{q}/2)(1-n_F(\mathbf{k}+\mathbf{q}/2))\ (n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2})\ \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}$$

and

$$[a_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = a_{\mathbf{k}}(\mathbf{q})\ (\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2})$$

$$n_{\mathbf{p}} = n_F(\mathbf{p}) + \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q}) - \frac{1}{N_>} \sum_{\mathbf{q}} a^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q}) a_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})$$

and

$$(N_>)^2 \ = \ \sum_{\mathbf{k},\mathbf{q}} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q})$$