Nature of the problem

Given Jacobian:Type equation here.

$$determinant: -abc\rho^2\sin(\varphi)$$

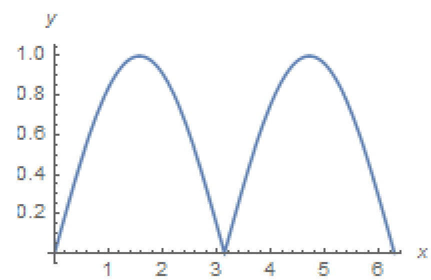
Triple Integral:

- The integrand can be broken up into two parts, the first part will never be negative, the second can be:

$$|-abc\rho^2\sin(\varphi)| = |abc\rho^2(-\sin(\varphi))| = |abc\rho^2| * |-\sin(\varphi)|$$

$|abc|$ Is just a constant and $|\rho^2|$ will always range from 0 to 1 and won't be negative

$|-\sin(\varphi)|$ On the other hand is different from $-\sin(\varphi)$ and looks like this:



- Triple integral:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}[(abc)\rho^2\sin(\varphi)]\,d\rho d\varphi d\theta$$

- From $0\ to\ \pi$, $|-\sin(\varphi)|$ has the same area under the curve as $\sin(\varphi)$. This is why multiplying the jacobian by -1 instead of taking the absolute value did not yield an incorrect volume. It is also why you can just use $\sin(\varphi)$ instead of $|-\sin(\varphi)|$ in the integrand.

$$V = 2abc\int_0^{\pi}\int_0^{\pi} sin(\varphi)\int_0^{1}[\rho^2]\,d\rho d\varphi d\theta$$

$$V = 2abc\int_0^{\pi}\int_0^{\pi} \sin(\varphi)\int_0^{1}[\rho^2]\,d\rho d\varphi d\theta$$

$$V = 2abc\int_0^{\pi}\int_0^{\pi} \sin(\varphi)\,d\varphi d\theta\left[\left(\frac{1}{3}\right)p^3\right]\Bigg|_0^1$$

$$V = \left(\frac{2}{3}\right)abc\int_0^{\pi}\int_0^{\pi} \sin(\varphi)\,d\varphi d\theta$$

$$V = \left(\frac{2}{3}\right)abc\int_0^{\pi} d\theta[-\cos(\varphi)]\Bigg|_0^{\pi}$$

$$V = \left(\frac{2}{3}\right) abc(-(-1) + (1)) \int_0^{\pi} d\theta$$

$$V = \left(\frac{2}{3}\right) abc(2)[\theta]\Big|_0^{\pi}$$

$$V = \left(\frac{4}{3}\right) abc[\pi - 0]$$

$$V = \frac{4\pi}{3} abc$$

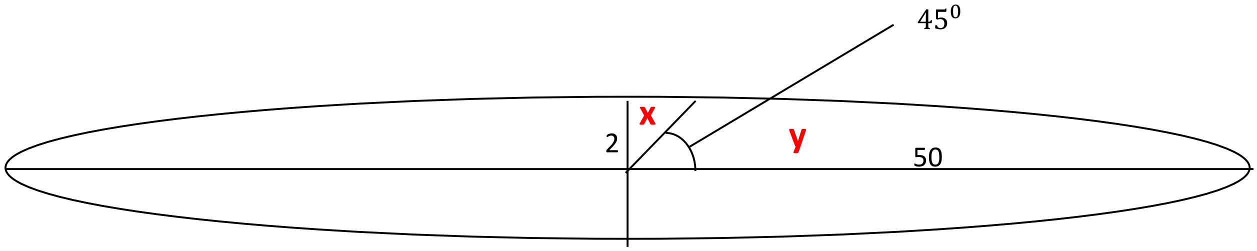
Unfortunately, this still does not solve the problem for values other than increments of $\frac{\pi}{2}$. For example if we integrated the ellipsoid using this jacobian from $0\ to \frac{\pi}{2}$ for φ and $0\ to \frac{\pi}{4}$ for θ, we do not get an accurate volume for that portion of an ellipsoid. This means the way I am approaching this problem is fundamentally wrong.

$$V_{\color{red}{y}} = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 [(abc)\rho^2 \sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12} abc$$

$$V_{\color{red}{x}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 [(abc)\rho^2 \sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12} abc$$

This is only true if $a = b = c$

$$\boxed{But\ what\ if\ a = 50, b = 2\ and\ c = 2?}$$



$$\boxed{\color{red}{x \neq y}}$$

But according to the above integration, it does. This is what I need to fix.

Below is the start of an integration that I cannot finish. It is possible it can be done through iterations but it would be best if a symbolic definite integral can be found

Unfinished triple integral of ellipsoid without using jacobian transformation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{\rho^2\sin^2(\varphi)\sin^2(\theta)}{a^2}+\frac{\rho^2\sin^2(\varphi)\cos^2(\theta)}{b^2}+\frac{\rho^2\cos^2(\varphi)}{c^2}=1$$

$$\rho^2\left[\frac{\sin^2(\varphi)\sin^2(\theta)}{a^2}+\frac{\sin^2(\varphi)\cos^2(\theta)}{b^2}+\frac{\cos^2(\varphi)}{c^2}\right]=1$$

$$\rho=\pm\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_0^{\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}} d\rho d\varphi d\theta$$

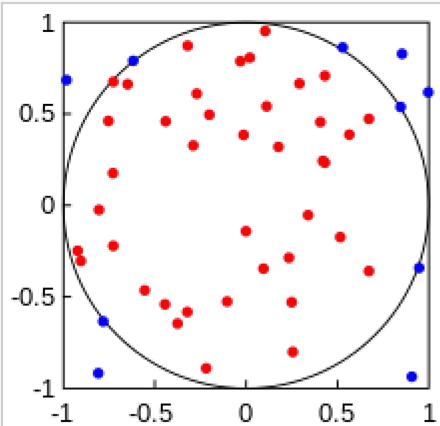
$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}\, d\varphi d\theta$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\sqrt{\csc^2(\varphi)[a^2\csc^2(\theta)+b^2\sec^2(\theta)]+c^2\sec^2(\varphi)}\,\mathrm{d}\varphi\mathrm{d}\theta$$

This is where I get stuck.

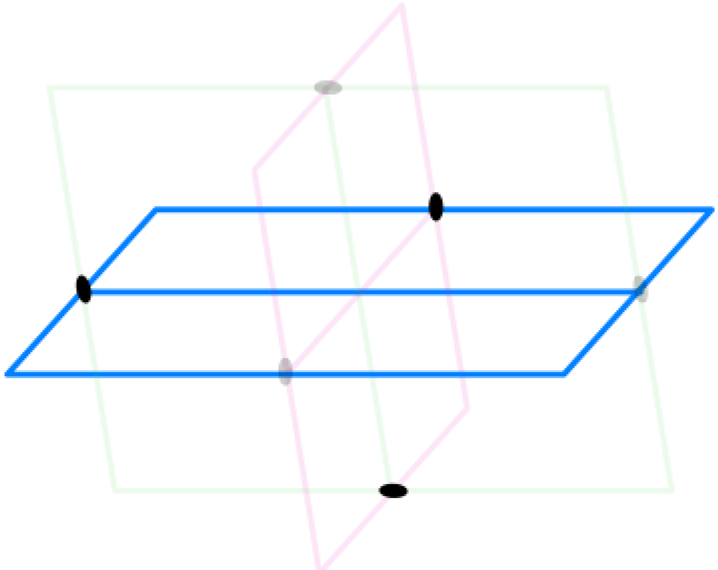
I have since abandoned this idea in light of another alternative: Monte Carlo Integration.

A Monte Carlo integration take a series of sample points in a known area (or volume) and determines if the point is inside or outside the shape. Using statistics, you are able to approximate the area (or volume) of a shape:



An illustration of Monte Carlo integration. In this example, the domain *D* is the inner circle and the domain E is the square. Because the square's area (4) can be easily calculated, the area of the circle ($\pi * 1^2$) can be estimated by the ratio (0.8) of the points inside the circle (40) to the total number of points (50), yielding an approximation for the circle's area of $4*0.8 = 3.2 \approx \pi * 1^2$.

Using this same principle, we can extend this into three dimensions. The coordinate system is comprised of x y and z. The known volume of the rectangular prism is defined by inscribing the ellipsoid inside of a rectangular prism. The dimensions of the rectangular prism is defined as $V = (2a)(2b)(2c)$ where a b c are the axes of the ellipsoid



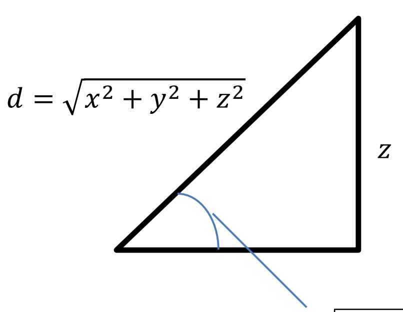
Random points are generated using x, y and z. To determine if a point is inside of a portion of the ellipsoid, you must find the value of $phi, theta\ and$ ρ.

phi

To find phi, you can draw a right triangle from origin, to the point, then strait down to the xy plane and finally to the z axis. The hypotenuse is defined by using the distance formula in 3 dimentions to the point:

$$d = \sqrt{x^2 + y^2 + z^2}$$

The verical leg of the triangle is defined as the value of z so you have:

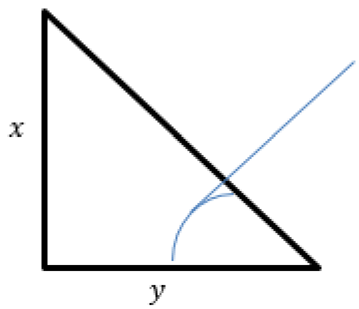


$This\ angle\ is\ defined\ as:$

$$\varphi = \sin^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Theta

This angle only exists in the xy plane and is defined as $\theta = \tan^{-1}\left(\frac{y}{x}\right)$



This angle is defined as:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now that we have figured out the value of θ *and* φ, we need to make sure the point lays within the equation of the ellipsoid.

Given the three dimensional coordinate and the length of the axes, this is fairly simple:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

This is simply a boolean statement.

The algorithm comes down to three Boolean statement:

1) $\varphi_{start} \leq \varphi \leq \varphi_{end}$
2) $\theta_{start} \leq \theta \leq \theta_{end}$
3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

If all of these Boolean statements are true, then the point is inside the portion of the ellipsoid, if not, then it is outside. **Increment** some value each time a point is inside the shape. So after all sample points have been taken you have:

$$ratio = \frac{inside\ shape}{total\ sample}$$

Then to obtain volume you have:

$$V = ratio * (2a)(2b)(2c)$$

With 5 million samples this algorithm is accurate to $\pm.1$

I am developing a similar algorithm with an ellipsoid inscribed inside of a sphere using spherical coordinates, but it is not working yet.