How to Arrive at the Determinant of the Jacobian:

Equation of an ellipsoid: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Transformation into u, v and w space:

- $u=\frac{x}{a}, v=\frac{y}{b}, w=\frac{z}{c}$ $\quad$ *which yields the equation* $u^2+v^2+w^2=1$

This is now the equation of a sphere with a radius of 1 in u, v and w space.

Now for the conversion into spherical coordinates:

Since we are now in rectangular coordinates in u, v and w space, we need to convert u, v and w into spherical coordinates.

- $u=\rho\sin(\varphi)\cos(\theta)\,, v=\rho\sin(\varphi)\sin(\theta)\,, and\ w=\ \rho\cos(\varphi)$

Therefore you have the equations:

- $\rho\sin(\varphi)\cos(\theta)=\frac{x}{a},\ \rho\sin(\varphi)\sin(\theta)=\frac{y}{b},\ and\ \rho\cos(\varphi)=\frac{z}{c}$

Solving for $x, y\ and\ z$ you have:

- $a\rho\sin(\varphi)\cos(\theta)=x,\ \ b\rho\sin(\varphi)\sin(\theta)=y,\ \ and\ \ c\rho\cos(\varphi)=z$

Now for the Jacobean transformation.

The jacobian transformation for a $3x3$ matrix is defined as:

- $$\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \mathrm{a}\sin(\varphi)\cos(\theta) & b\ sin(\varphi)\sin(\theta) & \mathrm{c}\cos(\varphi) \\ -ap\ \sin(\varphi)\sin(\theta) & bp\sin(\varphi)\cos(\theta) & 0 \\ a\rho\cos(\varphi)\cos(\theta) & b\rho\cos(\varphi)\sin(\theta) & -cp\ \sin(\varphi) \end{bmatrix}$$

To find the determinant, use these steps:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

The determinant of the matrix has the form of

$$det(A) = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

$$\begin{aligned} &\text{a}\sin(\varphi)\cos(\theta)\,[bp\sin(\varphi)\cos(\theta)(-cp\sin(\varphi)) - 0)] \\ &\qquad - \ bsin(\varphi)\sin(\theta)\,[-ap\sin(\varphi)\sin(\theta)(-cp\sin(\varphi)) - 0) \\ &\qquad + \text{c}\cos(\varphi)\,[-ap\sin(\varphi)\sin(\theta)\,b\rho\cos(\varphi)\sin(\theta) \\ &\qquad - bp\sin(\varphi)\cos(\theta)\,a\rho\cos(\varphi)\cos(\theta)] \end{aligned}$$

Simplifying:

$$\begin{aligned} determinant = &-abc\rho^2 sin^3(\varphi)\cos^2(\theta)\sin(\varphi) \\ &- abc\rho^2\sin^3(\varphi)\sin^2(\theta) \\ &- abc\rho^2\cos^2(\varphi)\sin^2(\theta)\sin(\varphi) - abc\rho^2\cos^2(\varphi)\cos^2(\theta)\sin(\varphi) \end{aligned}$$

$rearranging\ and\ factoring\ out\ a - 1$:

$$\begin{aligned} -1[&(abc)\rho^2\cos^2(\varphi)\cos^2(\theta)\sin(\varphi) \\ &+ \ (abc)\rho^2\cos^2(\varphi)\sin^2(\theta)\sin(\varphi) \\ &+ (abc)\rho^2\cos^2(\theta)\sin^3(\varphi) + (abc)\rho^2\sin^3(\varphi)\sin^2(\theta)] \end{aligned}$$

$simplfying\ using\ trig\ identities$:

$$-1[(abc)\rho^2\sin(\varphi)\cos^2(\varphi)[\cos^2(\theta) + \sin^2(\theta)] + \ (abc)\rho^2\sin^3(\varphi)[\cos^2(\theta) + \sin^2(\theta)]\ ]$$

$$-1[(abc)\rho^2\sin(\varphi)\cos^2(\varphi) + (abc)\rho^2\sin^3(\varphi)]$$

$$-1[(abc)\rho^2\sin(\varphi)\,[\cos^2(\varphi) + \sin^2(\varphi)]]$$

$$-(abc)\rho^2\sin(\ \varphi)$$

$$Simplified\ determinant = -abc\rho^2\sin(\varphi)$$

Please refer to the document titled “Integration of determinant of jacobian for Kelly and Alan” to see how to integrate using jacobian