Ellipsoid Portion Volume Calculation

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Separation of methods

When calculating the volume of the portion of the ellipsoid, special cases can be calculated very quickly and accurately with a definite integral. The rest of the cases need to be calculated with other methods. In this program, the other methods are a series of Monte Carlo Integrations.

Special cases:

- If the user inputs a sphere ($a\ axis\ =\ b\ axis\ =\ c\ axis$)
- If the user specifies a volume where the values of $\varphi\ and\ \theta$ are both increments of $\frac{\pi}{2}$ (i.e. if they use any and only of the following for φ: $0, \frac{\pi}{2}, \pi$ and the following for θ: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$)

In these special cases, the definite integral can be used to calculate the volume.

The following logic is used to find special cases:

$$Eccentricity - Used\ to\ determine\ how\ spherical\ the\ ellipsoid\ is$$

$$The\ following\ calculations\ are\ based\ off\ the\ equation:$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$IF:\ \frac{c}{a} + \frac{b}{a} = 2, then\ the\ ellipsoid\ is\ a\ sphere$$

$$*\ Note: this\ is\ not\ the\ normal\ definition\ for\ Eccentricity, I've\ augmented\ it\ to\ suite\ my\ needs\ *$$

$$Total\ radian\ measure - Used\ to\ determine\ if\ all\ bounds\ of\ integration\ are\ Increments\ of\ \frac{\pi}{2}$$

$$Total\ radian\ measure = \theta_{start} + \theta_{End} + \varphi_{start} + \varphi_{end}$$

$$IF: \frac{Total\ radian\ measure}{\frac{\pi}{2}} = 0, then\ all\ the\ radian\ measures\ are\ increments\ of\ \frac{\pi}{2}$$

$$\boldsymbol{If\ either\ of\ these\ are\ true, then\ the\ definte\ integral\ is\ used\ (page\ 2)}$$

For explanations about why a Monte Carlo integration is necessary please refer to:

- **Appendix IV: Nature of the Problem with the definite integral (page 17)**
- **Appendix V: Unfinished Definite Integral in Spherical Coordinates (page 19)**
- **Appendix VI: Unfinished Definite Integral in Rectangular Coordinates (page 20)**

Everything else:

- In all other cases, a Monte Carlo Integration is used to estimate the volume.
- At the present functionality, two different Monte Carlo algorithms are used, one for rectangular coordinates and the other using spherical coordinates. The rectangular coordinates are used on more elongated ellipsoids, and the spherical coordinates are used on more spherical ellipsoids

The following logic is used to determine when to use the Monte Carlo algorithm in rectangular coordinates or Spherical coordinates:

$$Using\ the\ same\ formula\ for\ eccentricity\ defined\ above:$$

$$IF: Eccentricity \geq 1, \quad then\ use\ spherical\ Monte\ Carlo\ Algorithm\ in\ Spherical\ Coordinates, otherwise, Use\ Monte\ Carlo\ Algorithm\ in\ in\ Rectangular\ coordinates$$

$$\boldsymbol{Monte\ Carlo\ Algorithm\ for\ Spherical\ Coordinates\ (Page\ 7)}$$

$$\boldsymbol{Monte\ Carlo\ Algorithm\ for\ Rectangular\ Coordinates\ (Page\ 3)}$$

Definite Integral

Definite integral algorithm:

$$Volume = -\left(\frac{1}{3}\right) abc(\cos(\varphi_2) - \cos(\varphi_1))(\theta_2 - \theta_1)$$

Symbolic equation for the volume of an entire ellipsoid using the algorithm

- θ will range from $0\ to\ 2\pi$
- φ will range from $0\ to\ \pi$
- ρ will range from $0\ to\ 1$ (it is eliminated in the triple integral which is why it's not present in the symbolic definite integral, which is the algorithm)

$$Volume = -\left(\frac{1}{3}\right) abc[\cos(\pi) - \cos(0)][2\pi - 0]$$

$$Volume = -\left(\frac{1}{3}\right) abc[(-1) - (1)][2\pi]$$

$$Volume = -\left(\frac{1}{3}\right) abc[-2][2\pi]$$

$$Volume = \frac{4\pi}{3} abc$$

Plugging in bounds for $\varphi\ and\ \theta$ that are increments of $\frac{\pi}{2}$ will also yield a positive accurate answer.

Say for example a user wanted to calculate the upper left quarter of the ellipsoid

- θ will range from $\frac{\pi}{2}\ to\ \frac{3\pi}{2}$
- φ will range from 0 to $\frac{\pi}{2}$
- ρ will range from $0\ to\ 1$ (Again, eliminated from the symbolic definite integral)

$$Volume = -\left(\frac{1}{3}\right) abc\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]\left[\frac{3\pi}{2} - \frac{\pi}{2}\right]$$

$$Volume = -\left(\frac{1}{3}\right) abc[0 - 1][\pi]$$

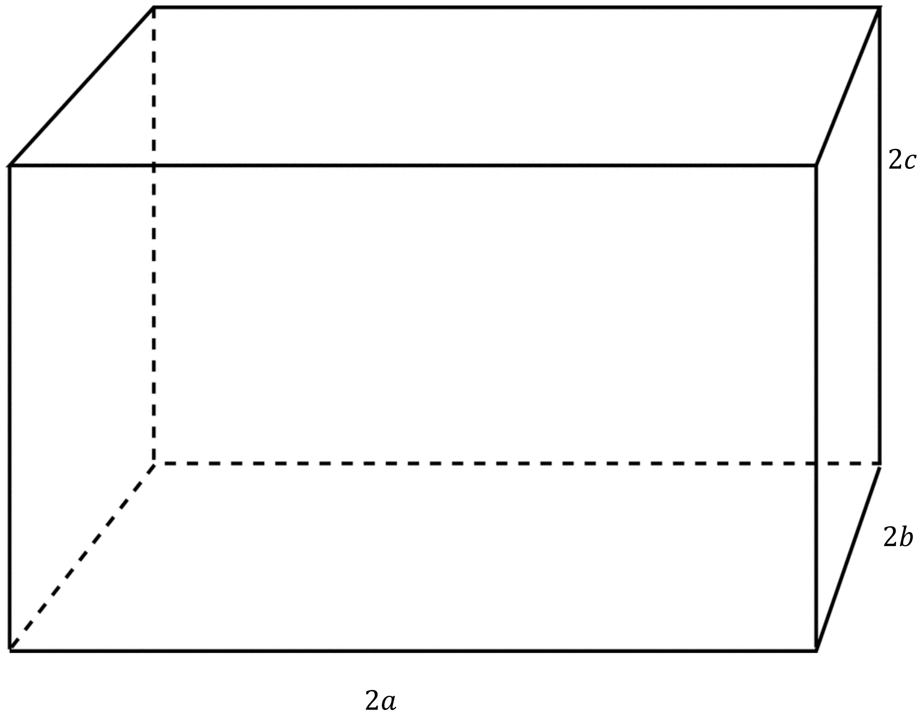
$$Volume = \frac{\pi}{3} abc$$

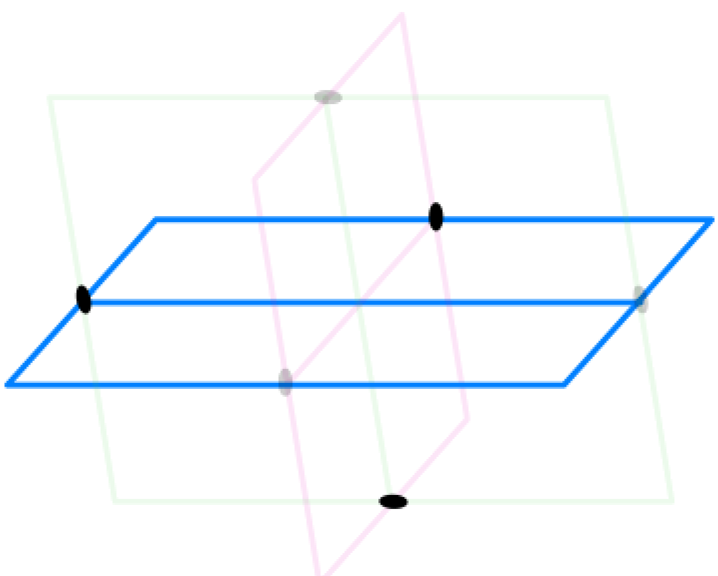
For the Derivation please refer to:

- **Appendix II: Derivation of definite integral (page 14)**
- **Appendix III: Another way to arrive at the same definite integral (page 16)**

Monte Carlo in Rectangular Coordinates

The known volume of the rectangular prism is defined by inscribing the ellipsoid inside of a rectangular prism. The dimensions of the rectangular prism is defined as $V = (2a)(2b)(2c)$ where $a\ b\ and\ c$ are the axes of the ellipsoid.





<u>Random Points Generation in Rectangular Coordinates:</u>

Random points are generated for $x, y\ and\ z$ using the axes of the ellipsoid as follows:

$$x = (a\ axis) * (random\ decimal\ between\ 0\ and\ 1) * (random\ negation)$$

$$y = (b\ axis) * (random\ decimal\ between\ 0\ and\ 1) * (random\ negation)$$

$$z = (c\ axis) * (random\ decimal\ between\ 0\ and\ 1) * (random\ negation)$$

This generates a random sample with uniform distribution

<u>Determine if point is inside of portion of ellipsoid:</u>

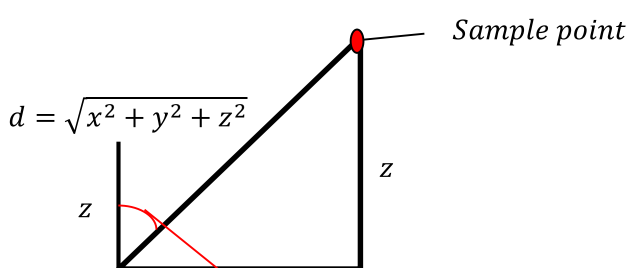
To determine if a point is inside of a portion of the ellipsoid, you must find the value of $phi\ and\ theta$ as well as whether or not the point is inside the equation of the entire ellipsoid.

phi

To find phi, you can draw a right triangle from origin, to the point, then strait down to the xy plane and finally to the z axis. The hypotenuse is defined by using the distance formula in 3 dimensions to the point:

$d = \sqrt{x^2 + y^2 + z^2}$

The vertical leg of the triangle is defined as the value of z so you have:



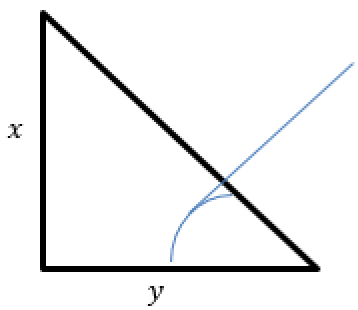
This angle is defined as:

$$\varphi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

This is the phi angle to the random point and will be compared later against the entered bounds of integration for phi

Theta

This angle only exists in the xy plane and is defined as $\theta = \tan^{-1}\left(\frac{y}{x}\right)$



This angle is defined as:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Due to the range and domain restrictions for Arctan, the radian measure need to be adjusted.

Logic for radian measure adjustment:

$$IF: x > 0 \text{ and } y > 0, \text{then radian measure is correct}$$

$$IF: x < 0 \text{ and } y > 0, \text{then add } \pi$$

$$IF: x < 0 \text{ and } y < 0, \text{then add } \pi$$

$$IF: x > 0 \text{ and } y < 0, \text{then add } 2\pi$$

There are two methods for theta, one for start and one for end. This is because if the radian measure comes backs 0, one method should return 0 and the other should return 2π, this will matter later.

There is also other logic present within these methods that accounts for values where arctan is undefined or if the point is on the origin.

$Now\ that\ we\ have\ figured\ out\ the\ value\ of\ \theta\ and\ \varphi, we\ need\ to\ make\ sure\ the\ point\ lays$

$within\ the\ equation\ of\ the\ ellipsoid.$

Given the three dimensional coordinate and the length of the axes, this is fairly simple:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

This is simply a boolean statement.

The algorithm comes down to three boolean statements:

1) $\varphi_{start} \leq \varphi_{sample\ point} \leq \varphi_{end}$
2) $\theta_{start} \leq \theta_{sample\ point} \leq \theta_{end}$
3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

If all of these Boolean statements are true, then the point is inside the portion of the ellipsoid, if not, then it is outside. **Increment** some value each time a point is inside the shape. So after all sample points have been taken you have:

$$ratio = \frac{inside\ shape}{total\ sample}$$

Then to obtain volume you have:

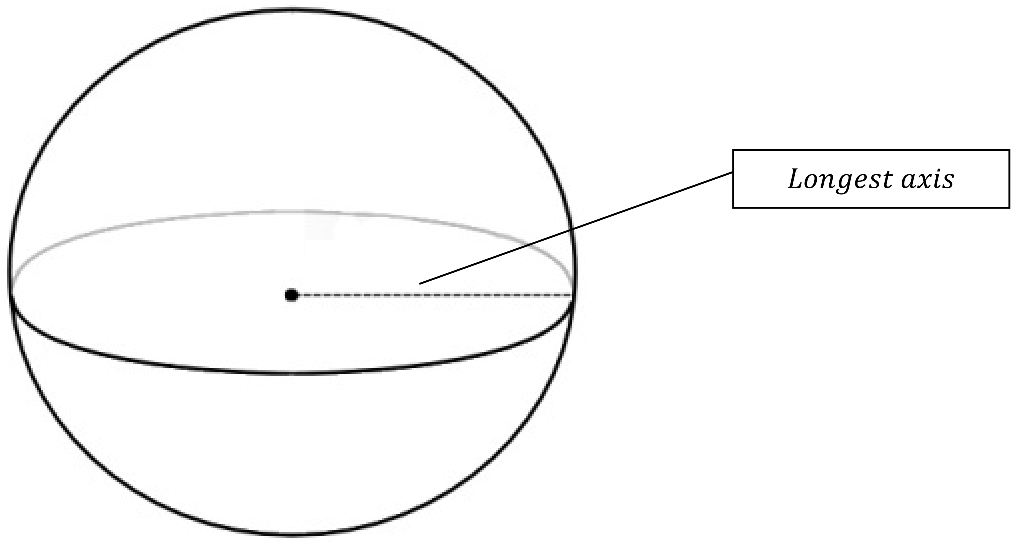
$$V = ratio * (2a)(2b)(2c)$$

With 5 million samples this algorithm is 99% accurate for small numerical volumes and 99.99% accurate for large numerical volumes (this exists on a continuum).

Monte Carlo Algorithm in Spherical Coordinates

This algorithm operates on a very similar set of prinicples that were defined in rectangular coordinates. The main difference is the coordinate system used (spherical) and the shape that the ellipsoid is enscribed inside of (sphere). The volume of the sphere is defined as:

$$V = \frac{4\pi}{3}(longest\ axis)^3$$



First to generate a set of random points with uniform distribution:

- The variables in spherical coordinates are $\varphi, \theta\ and\ \rho$
- Due to bunching at the center of the sphere and at the poles, some special things need to be done to generate a random spherical coordinate with uniform distrubution

$$\varphi = \sin^{-1}(2 * (random\ decimal\ between\ 0\ and\ 1) - 1)$$

$$\theta = 2 * \pi * (random\ decimal\ between\ 0\ and\ 1)$$

$$\rho = (longest\ axis) * \sqrt[3]{(random\ decimal\ between\ 0\ and\ 1)}$$

Determination if point is inside portion of ellipsoid:

Since we already have $\varphi\ and\ \theta$ we simply have two boolean statements that knock out $\frac{2}{3}$ of the tests:

1) $\varphi_{start} \leq \varphi_{sample\ point} \leq \varphi_{end}$
2) $\theta_{start} \leq \theta_{sample\ point} \leq \theta_{end}$

The final test determines whether or not the point lies within the equation of the ellipsoid. This is also a simple Boolean statement:

$$IF\colon \frac{\rho^2\sin^2(\varphi)\cos^2(\theta)}{a^2}+\frac{\rho^2\sin^2(\varphi)\sin^2(\theta)}{b^2}+\frac{\rho^2\cos^2(\varphi)}{c^2}\leq 1, \qquad then\ the\ point\ is\ inside$$

$the\ equation\ of\ the\ ellipsoid.$

Thus the algorithm again comes down to three Boolean statements:

1) $\varphi_{start} \leq \varphi_{sample\ point} \leq \varphi_{end}$
2) $\theta_{start} \leq \theta_{sample\ point} \leq \theta_{end}$
3) $\frac{\rho^2\sin^2(\varphi)\cos^2(\theta)}{a^2}+\frac{\rho^2\sin^2(\varphi)\sin^2(\theta)}{b^2}+\frac{\rho^2\cos^2(\varphi)}{c^2}\leq 1$

If all of these Boolean statements are true, then the point is inside the portion of the ellipsoid, if not, then it is outside. **Increment** some value each time a point is inside the shape. So after all sample points have been taken you have:

$$ratio = \frac{inside\ shape}{total\ sample}$$

Then to obtain volume you have:

$$V = ratio * \frac{4\pi}{3}(longest\ axis)^3$$

It is more accurate than the rectangular algorithm with ellipsoids that are close to spherical.

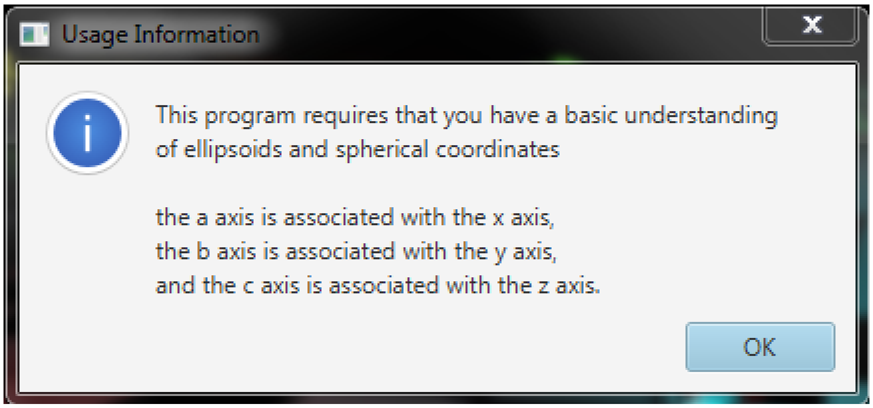
Appendices

Appendix I: Goal of EllipsoidCalc

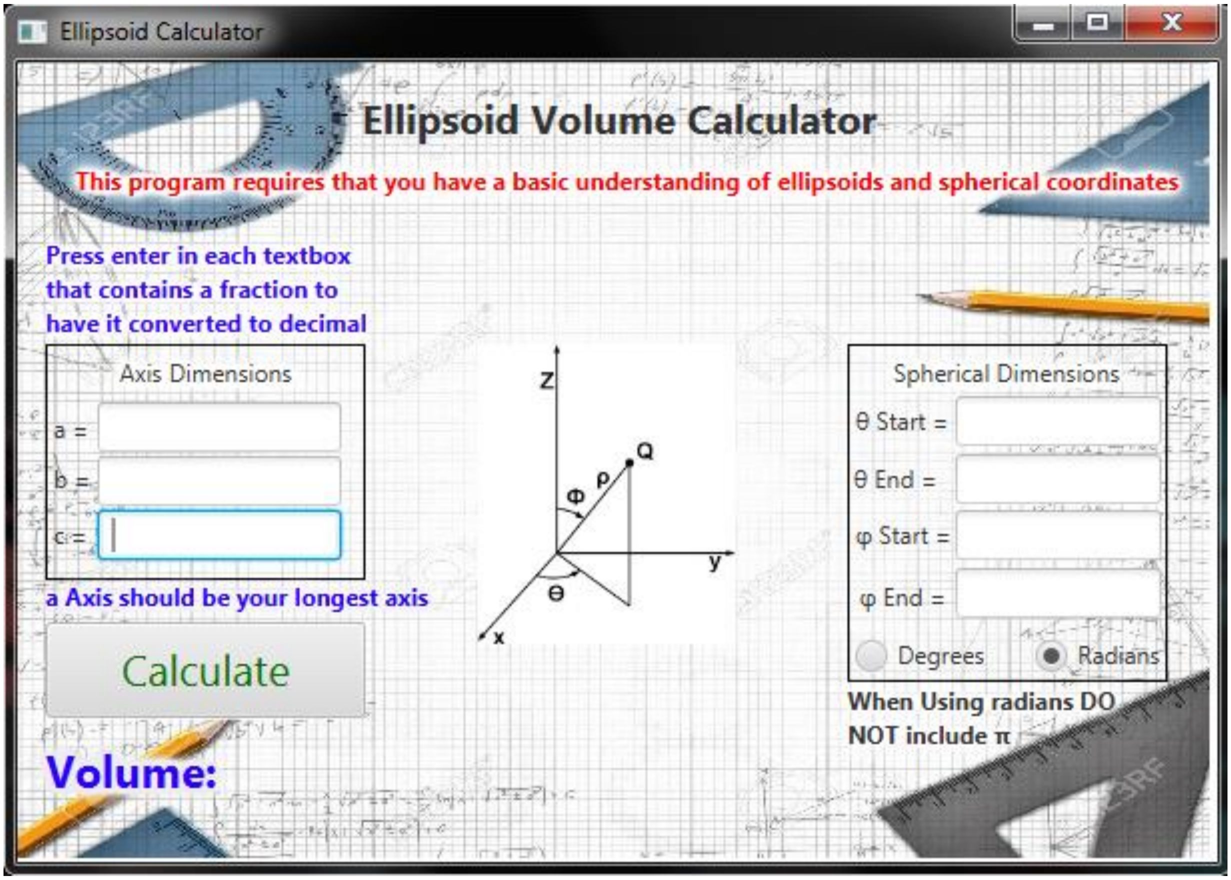
GOAL: The goal of EllipsoidCalc is to calculate the volume of ANY portion of an ellipsoid that is specified in spherical coordinates.

Here is a picture of its User Interface:

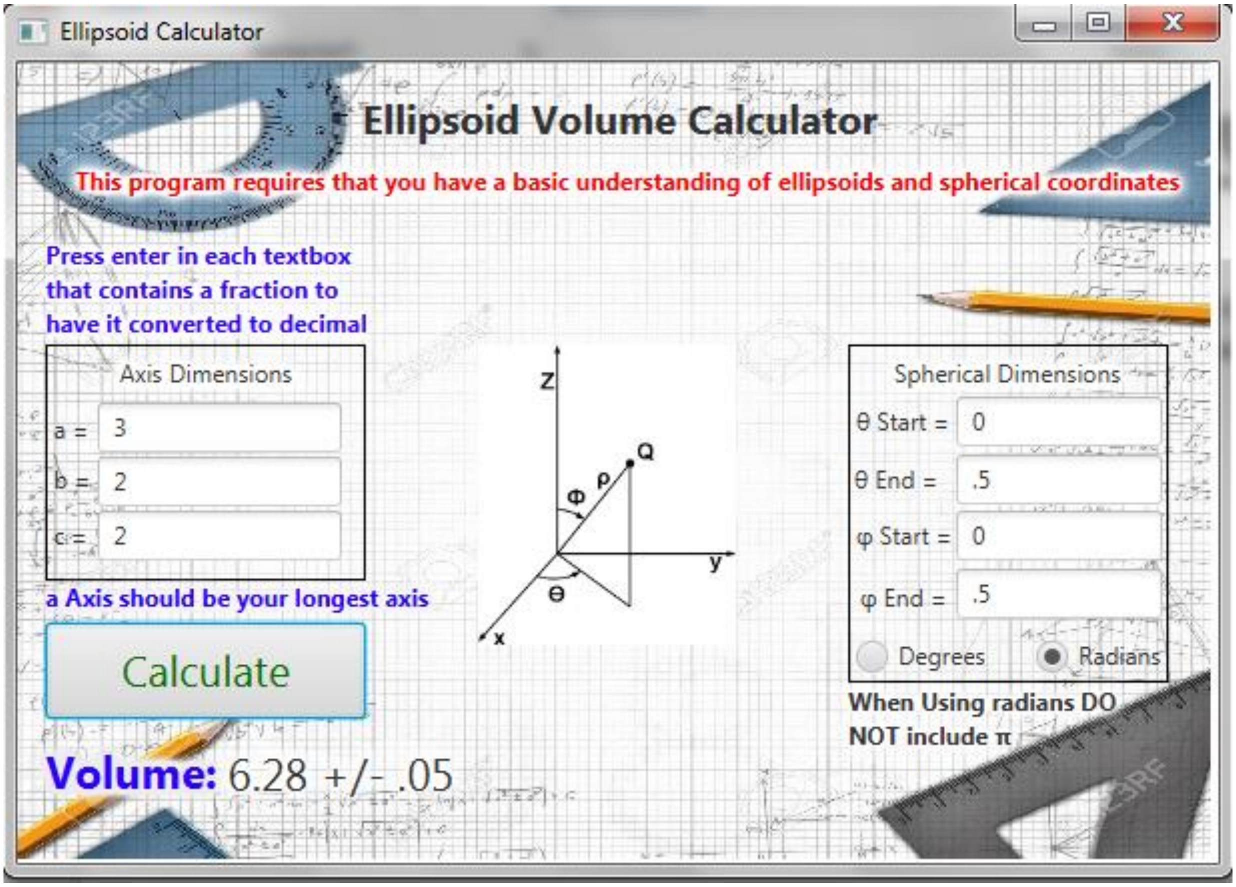
I. This user is first prompted with this message before continuing with the program



II. This UI is where the user specifies the dimensions of the axes and the bounds of integration in spherical coordinates



Example: If a user wanted to specify an ellipsoid with axis dimensions of $(a = 3, b = 2, c = 2)$ and they wanted to calculate the $\underline{upper\ right\ octant}$ of the ellipsoid, their UI should look like this:


Ellipsoid Calculator
Ellipsoid Volume Calculator
This program requires that you have a basic understanding of ellipsoids and spherical coordinates
Press enter in each textbox
that contains a fraction to
have it converted to decimal
Axis Dimensions
a = 3
b = 2
c = 2
a Axis should be your longest axis
Calculate
Volume: 6.28 +/- .05
Z
Q
ρ
Φ
y
Θ
x
Spherical Dimensions
θ Start = 0
θ End = .5
φ Start = 0
φ End = .5
Degrees
Radians
When Using radians DO
NOT include π

Appendix II: Derivation of definite integral

How to Arrive at the Determinant of the Jacobian:

Equation of an ellipsoid: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\boldsymbol{Transformation\ into\ u, v\ and\ w\ space}:$$

- $u=\frac{x}{a}, v=\frac{y}{b}, w=\frac{z}{c}$ $\ \ which\ yields\ the\ equation\ u^2+v^2+w^2=1$

This is now the equation of a sphere with a radius of 1 in u, v and w space.

$$\boldsymbol{Now\ for\ the\ conversion\ into\ spherical\ coordinates}:$$

Since we are now in rectangular coordinates in u, v and w space, we need to convert u, v and w into spherical coordinates.

- $u=\rho\sin(\varphi)\cos(\theta)\ , v=\rho\sin(\varphi)\sin(\theta)\ , and\ w=\ \rho\cos(\varphi)$

Therefore you have the equations:

- $\rho\sin(\varphi)\cos(\theta)=\frac{x}{a},\ \rho\sin(\varphi)\sin(\theta)=\frac{y}{b},\ and\ \rho\cos(\varphi)=\frac{z}{c}$

Solving for $x, y\ and\ z$ you have:

- $a\rho\sin(\varphi)\cos(\theta)=x,\ \ b\rho\sin(\varphi)\sin(\theta)=y,\ \ and\ \ c\rho\cos(\varphi)=z$

$$\boldsymbol{Now\ for\ the\ Jacobian\ transformation}.$$

The jacobian transformation for a $3x3$ matrix is defined as:

- $$\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \mathrm{a}\sin(\varphi)\cos(\theta) & b\ sin(\varphi)\sin(\theta) & \mathrm{c}\cos(\varphi) \\ -ap\sin(\varphi)\sin(\theta) & bp\sin(\varphi)\cos(\theta) & 0 \\ a\rho\cos(\varphi)\cos(\theta) & b\rho\cos(\varphi)\sin(\theta) & -cp\sin(\varphi) \end{bmatrix}$$

$$\boldsymbol{To\ find\ the\ determinant, use\ these\ steps}:$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

The determinant of the matrix has the form of

$$det(A) = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

$$\begin{aligned} \mathrm{a}\sin(\varphi)\cos(\theta)\,&[bp\sin(\varphi)\cos(\theta)(-cp\sin(\varphi)) - 0)] \\ &- \ bsin(\varphi)\sin(\theta)\,[-ap\sin(\varphi)\sin(\theta)(-cp\sin(\varphi)) - 0] \\ &+ \mathrm{c}\cos(\varphi)\,[-ap\sin(\varphi)\sin(\theta)\,b\rho\cos(\varphi)\sin(\theta) \\ &- bp\sin(\varphi)\cos(\theta)\,a\rho\cos(\varphi)\cos(\theta)] \end{aligned}$$

Simplifying:

$$\begin{aligned} determinant = &-abc\rho^2 sin^3(\varphi)\cos^2(\theta) \\ &- abc\rho^2\sin^3(\varphi)\sin^2(\theta) \\ &- abc\rho^2\cos^2(\varphi)\sin^2(\theta)\sin(\varphi) - abc\rho^2\cos^2(\varphi)\cos^2(\theta)\sin(\varphi) \end{aligned}$$

$Factoring\ out\ a - 1$:

$$\begin{aligned} -1[&abc\rho^2 sin^3(\varphi)\cos^2(\theta) \\ &+ abc\rho^2\sin^3(\varphi)\sin^2(\theta) \\ &+ abc\rho^2\cos^2(\varphi)\sin^2(\theta)\sin(\varphi) + abc\rho^2\cos^2(\varphi)\cos^2(\theta)\sin(\varphi)] \end{aligned}$$

$simplfying\ using\ trig\ identities$:

$$-1[(abc)\rho^2\sin(\varphi)\cos^2(\varphi)[\cos^2(\theta) + \sin^2(\theta)] + \ (abc)\rho^2\sin^3(\varphi)[\cos^2(\theta) + \sin^2(\theta)]\ ]$$

$$-1[(abc)\rho^2\sin(\varphi)\cos^2(\varphi)(1) + (abc)\rho^2\sin^3(\varphi)(1)]$$

$$-1[(abc)\rho^2\sin(\varphi)\,[\cos^2(\varphi) + \sin^2(\varphi)]]$$

$$-(abc)\rho^2\sin(\ \varphi)\,(1)$$

$$-(abc)\rho^2\sin(\ \varphi)$$

$$Simplified\ determinant = -abc\rho^2\sin(\varphi)$$

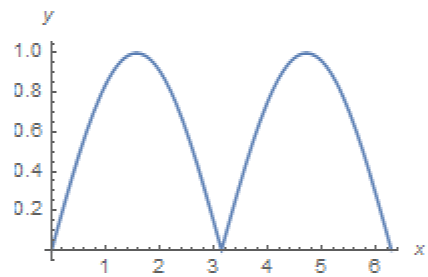
Triple Integral:

- The integrand can be broken up into two parts, the first part will never be negative, the second can be:

$$|-abc\rho^2\sin(\varphi)| = |abc\rho^2(-\sin(\varphi))| = |abc\rho^2| * |-\sin(\varphi)|$$

$|abc|$ Is just a constant and $|\rho^2|$ will always range from 0 to 1 and won't be negative

$|-\sin(\varphi)|$ On the other hand is different from $-\sin(\varphi)$ and looks like this:



- Triple integral:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}[(abc)\rho^2\sin(\varphi)]\,d\rho d\varphi d\theta$$

- From 0 to π, $|-\sin(\varphi)|$ has the same area under the curve as $\sin(\varphi)$. This is why you can just use $\sin(\varphi)$ instead of $|-\sin(\varphi)|$ in the integrand.

Triple Integral:

$$Volume = \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_0^{1}[(abc)\rho^2\sin(\varphi)]\,d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} sin(\varphi)\int_0^{1}[\rho^2]\,d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\int_0^{1}[\rho^2]\,d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{2\pi}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\,d\varphi d\theta\left[\left(\frac{1}{3}\right)p^3\right]\Bigg|_0^1$$

$\left[\left(\frac{1}{3}\right)(1)^3 - \left(\frac{1}{3}\right)(0)^3\right] = \frac{1}{3}$ This is why ρ goes away

$$V = \left(\frac{1}{3}\right)abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\,d\varphi d\theta$$

Symbolic integration

$$V = \left(\frac{1}{3}\right) abc \int_{\theta_1}^{\theta_2} d\theta [-\cos(\varphi)] \Bigg|_{\varphi_1}^{\varphi_2}$$

$$V = \left(\frac{1}{3}\right) abc(-\cos(\varphi_2) + \cos(\varphi_1)) \int_{\theta_1}^{\theta_2} d\theta$$

$$V = -\left(\frac{1}{3}\right) abc(\cos(\varphi_2) - \cos(\varphi_1)) \int_{\theta_1}^{\theta_2} d\theta$$

$$V = -\left(\tfrac{1}{3}\right) abc[\cos(\varphi_2) - \cos(\varphi_1)][\theta_2 - \theta_1]$$ Final algorithm

Appendix III: Another way to arrive at the same definite integral

Equation of an ellipsoid: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\boldsymbol{Transformation\ into\ u, v\ and\ w\ space}:$$

- $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$ *which yields the equation* $u^2 + v^2 + w^2 = 1$

This is now the equation of a sphere with a radius of 1 in u, v and w space.

Solve the equations for $x, y\ and\ z$ you have:

$$ua = x, \quad vb = y \quad and \quad wc = z$$

$$\boldsymbol{Now\ for\ the\ jacobian\ transformation}:$$

- $$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} \mathrm{a} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \ determinant = abc$$

Conversion of $u^2 + v^2 + w^2 = 1$ in to spherical coordinates:

$$u = \rho \sin(\varphi)\cos(\theta), v = \rho \sin(\varphi)\sin(\theta), and\ w = \rho\cos(\varphi)$$

Convert equation:

$$\rho^2 \sin^2(\varphi)\sin^2(\theta) + \rho^2 \sin^2(\varphi)\cos^2(\theta) + \rho^2 \cos^2(\varphi) = 1$$

Simplify equation:

$$\rho^2 \sin^2(\varphi)[\sin^2(\theta) + \cos^2(\theta)] + \rho^2 \cos^2(\varphi) = 1$$

$$\rho^2 \sin^2(\varphi) + \rho^2 \cos^2(\varphi) = 1$$

$$\rho^2(\sin^2(\varphi) + \cos^2(\varphi)) = 1$$

$$\rho^2 = 1$$

$$\rho = 1$$

$\boldsymbol{Triple\ integral\ after\ conversion\ to\ spherical\ coordinates}$:

- The integrand $\rho^2 \sin(\varphi)$ is derived from a second jacobian but generally, anytime one converts to spherical coordinates, it can be placed in the integrand

$$Volume = \iiint_Q \rho^2 \sin(\varphi)|jacobian|dV$$

$$Volume = \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_0^1 \rho^2 \sin(\varphi)|abc|d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} sin(\varphi)\int_0^1 [\rho^2]\, d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\int_0^1 [\rho^2]\, d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{2\pi}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\, d\varphi d\theta \left[\left(\frac{1}{3}\right)p^3\right]$$

$$V = \left(\frac{1}{3}\right)abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\, d\varphi d\theta$$

$$V = \left(\frac{1}{3}\right)abc\int_{\theta_1}^{\theta_2} d\theta[-\cos(\varphi)]$$

$$V = \left(\frac{1}{3}\right)abc(-\cos(\varphi_2) + \cos(\varphi_1))\int_{\theta_1}^{\theta_2} d\theta$$

$$V = -\left(\frac{1}{3}\right)abc(\cos(\varphi_2) - \cos(\varphi_1))\int_{\theta_1}^{\theta_2} d\theta$$

$$\boxed{V = -\left(\frac{1}{3}\right)abc[\cos(\varphi_2) - \cos(\varphi_1)][\theta_2 - \theta_1]}$$

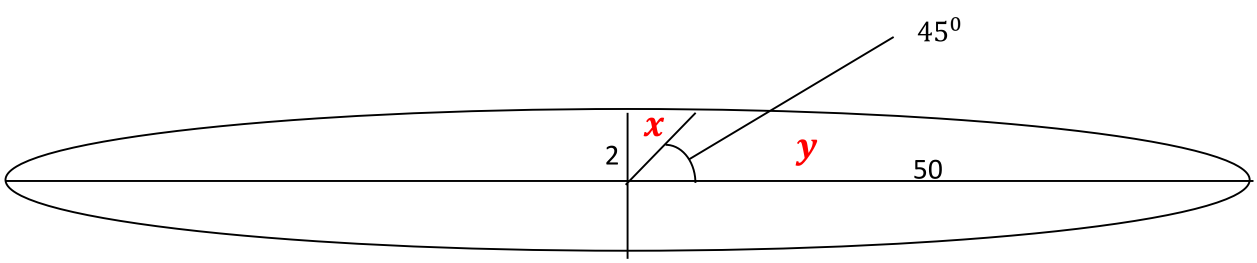
Appendix IV: Nature of the Problem with the Definite Integral

If we integrated the ellipsoid using the defintie integral from $0\ to \frac{\pi}{2}$ for φ and $0\ to \frac{\pi}{4}$ for θ, we do not get an accurate volume for that portion of an ellipsoid. This means this way of approaching this problem is fundamentally wrong.

$$V_y = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^1 [(abc)\rho^2\sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12}abc$$

$$V_x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 [(abc)\rho^2\sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12}abc$$

$$\boxed{But\ what\ if\ a = 50, b = 2\ and\ c = 2?}$$



$$\boxed{x \neq y}$$

But according to the above integration, it does. This is why a Monte Carlo Integration is necessary.

Appendix V: Unfinished Definite Integral in Spherical Coordinates

Unfinished triple integral of ellipsoid without using jacobian transformation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{\rho^2\sin^2(\varphi)\sin^2(\theta)}{a^2}+\frac{\rho^2\sin^2(\varphi)\cos^2(\theta)}{b^2}+\frac{\rho^2\cos^2(\varphi)}{c^2}=1$$

$$\rho^2\left[\frac{\sin^2(\varphi)\sin^2(\theta)}{a^2}+\frac{\sin^2(\varphi)\cos^2(\theta)}{b^2}+\frac{\cos^2(\varphi)}{c^2}\right]=1$$

$$\rho=\pm\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_0^{\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}} d\rho d\varphi d\theta$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}\,d\varphi d\theta$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\sqrt{\csc^2(\varphi)[a^2\csc^2(\theta)+b^2\sec^2(\theta)]+c^2\sec^2(\varphi)}\,\mathrm{d}\varphi\mathrm{d}\theta$$

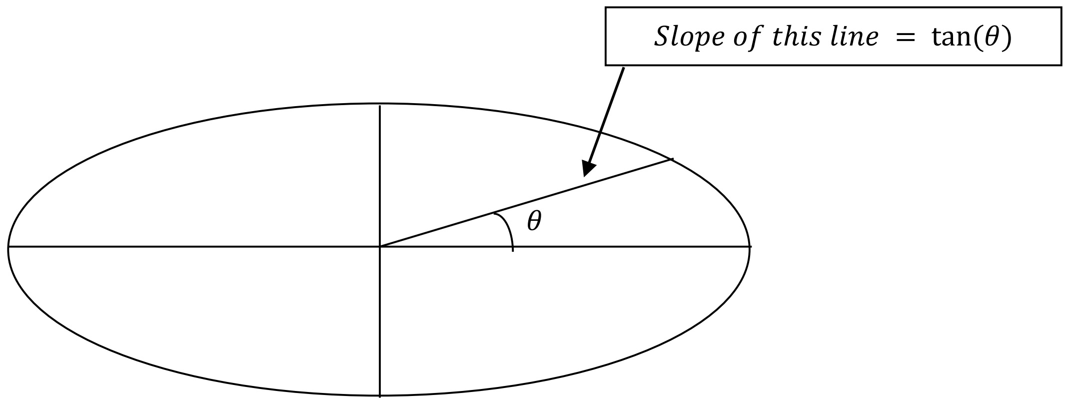
This is where I get stuck.

Appendix VI: Unfinished Definite Integral in Rectangular Coordinates

Partial Solution to ellipsoid Problem

Given:

- user HAS to specify a φ value of $\frac{\pi}{2}$ **or** π
- User specifies a value for θ with the idea of spherical coordinates in mind



Equation of the line from the origin

- $y = \tan(\theta)\, x$
- $\frac{1}{x} = \frac{\tan(\theta)}{y}$
- $x = \frac{y}{\tan(\theta)}$ Lower bound for x

Equation of ellipse in $xy\ plane$:

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solving the equation for x in the $xy\ plane$:

- $\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$
- $x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$ Upper bound for x

Solving for x bounds:

- $x = m(y^2) + a$
- $0 = m(b^2) + a$ When x is $0, y$ is b^2
- $-\frac{a}{b^2} = m$ Solve for m
- $x = -\frac{a}{b^2}(y^2) + a$

Setting the equations equal to each other to find the $y\ value$ where the two equations intersect

- $\frac{y}{\tan(\theta)} = a\sqrt{1 - \frac{y^2}{b^2}}$
- $\frac{y^2}{\tan^2(\theta)} = a^2\left(1 - \frac{y^2}{b^2}\right)$
- $\frac{y^2}{\mathrm{a}^2\tan^2(\theta)} = \left(1 - \frac{y^2}{b^2}\right)$
- $\frac{y^2}{\mathrm{a}^2\tan^2(\theta)} + \frac{y^2}{b^2} = 1$
- $y^2\left(\frac{1}{a^2\tan^2(\theta)} + \frac{1}{b^2}\right) = 1$
- $y^2\left(\frac{b^2}{a^2b^2\tan^2(\theta)} + \frac{a^2\tan^2(\theta)}{a^2b^2\tan^2(\theta)}\right) = 1$
- $y^2 = \frac{a^2b^2\tan^2(\theta)}{b^2+a^2\tan^2(\theta)}$
- $y = \frac{ab(\tan(\theta))}{\sqrt{b^2+a^2\tan^2(\theta)}}$ Upper bound for y

Triple integral:

$$\int_0^{\frac{ab(\tan(\theta))}{\sqrt{b^2+a^2\tan^2(\theta)}}} \int_{\frac{y}{\tan(\theta)}}^{a\sqrt{1-\frac{y^2}{b^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdxdy$$

This definite integral will be the algorithm