Nature of the problem

Given Jacobian:
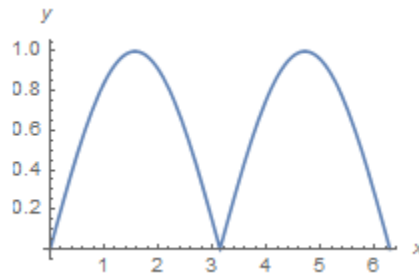
$$determinant: -abc\rho^2\sin(\varphi)$$

Triple Integral:

- The triple integral will need to be broken up into 2 integrals

$$|abc\rho^2(-\sin(\varphi))| = |abc\rho^2| * |-\sin(\varphi)|$$

$|abc|$ Is just a constant and $|\rho^2|$ will always range from 0 to 1 and won't be negative

$|-\sin(\varphi)|$ On the other hand is different from $-\sin(\varphi)$ and looks like this:

- Triple integral:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 [(abc)\rho^2 \sin(\varphi)] \, d\rho d\varphi d\theta$$

- From 0 to π, $|-\sin(\varphi)|$ has the same area under the curve as $\sin(\varphi)$. This is why multiplying the jacobian by -1 instead of taking the absolute value did not yield an incorrect volume. It is also why you can just use $\sin(\varphi)$ instead of $|-\sin(\varphi)|$ in the integrand.

$$V = 2abc \int_0^{\pi} \int_0^{\pi} \sin(\varphi) \int_0^1 [\rho^2] \, d\rho d\varphi d\theta$$

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$$V = 2abc \int_0^{\pi} \int_0^{\pi} \sin(\varphi) \, d\varphi d\theta \left[\left(\frac{1}{3}\right) \rho^3 \right]\Big|_0^1$$

$$V = \left(\frac{2}{3}\right) abc \int_0^{\pi} \int_0^{\pi} \sin(\varphi) \, d\varphi d\theta$$

$$V = \left(\frac{2}{3}\right) abc \int_0^{\pi} d\theta [-\cos(\varphi)] \Big|_0^{\pi}$$

$$V = \left(\frac{2}{3}\right) abc(-(-1) + (1)) \int_0^\pi d\theta$$

$$V = \left(\frac{2}{3}\right) abc(2)[\theta]\Big|_0^\pi$$

$$V = \left(\frac{4}{3}\right) abc[\pi - 0]$$

$$V = \frac{4\pi}{3} abc$$

Unfortunately, this still does not solve the problem for values other than increments of $\frac{\pi}{2}$. For example if we integrated the ellipsoid using this jacobian from $0\ to\ \frac{\pi}{2}$ for φ and $0\ to\ \frac{\pi}{4}$ for θ, we do not get an accurate volume for that portion of an ellipsoid. This means the way I am approaching this problem is fundamentally wrong.

$$V_y = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 [(abc)\rho^2 \sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12} abc$$

This is only true if $a = b = c$

$$V_x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 [(abc)\rho^2 \sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12} abc$$

But what if $a = 50, b = 2$ and $c = 2$?

45^0

X

2

y

50
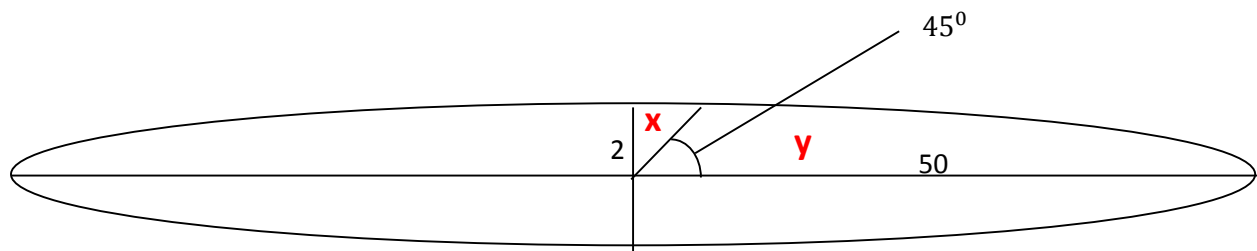
$$x \neq y$$

But according to the above integration, it does. This is what I need to fix.

Below is the start of an integration that I cannot finish. It is possible it can be done through iterations but it would be best if a symbolic definite integral can be found

Unfinished triple integral of ellipsoid without using jacobian transformation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{\rho^2 \sin^2(\varphi) \sin^2(\theta)}{a^2} + \frac{\rho^2 \sin^2(\varphi) \cos^2(\theta)}{b^2} + \frac{\rho^2 \cos^2(\varphi)}{c^2} = 1$$

$$\rho^2 \left[\frac{\sin^2(\varphi) \sin^2(\theta)}{a^2} + \frac{\sin^2(\varphi) \cos^2(\theta)}{b^2} + \frac{\cos^2(\varphi)}{c^2} \right] = 1$$

$$\rho = \pm\sqrt{a^2 \csc^2(\varphi) \csc^2(\theta) + b^2 \csc^2(\varphi) \sec^2(\theta) + c^2 \sec^2(\varphi)}$$

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{a^2 \csc^2(\varphi) \csc^2(\theta) + b^2 \csc^2(\varphi) \sec^2(\theta) + c^2 \sec^2(\varphi)}} d\rho \, d\varphi \, d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi} \sqrt{a^2 \csc^2(\varphi) \csc^2(\theta) + b^2 \csc^2(\varphi) \sec^2(\theta) + c^2 \sec^2(\varphi)} \, d\varphi \, d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi} \sqrt{\csc^2(\varphi)[a^2 \csc^2(\theta) + b^2 \sec^2(\theta)] + c^2\sec^2(\varphi)} \, d\varphi \, d\theta$$

This is where I get stuck.