Integration of determinant of jacobian for Kelly and Alan

Determinant :

$$(abc)\rho^2\cos^2(\varphi)\cos^2(\theta)\sin(\varphi) + (abc)\rho^2\cos^2(\varphi)\sin^2(\theta)\sin(\varphi) + (abc)\rho^2\cos^2(\theta)\sin^3(\varphi) + (abc)\rho^2\sin^3(\varphi)\sin^2(\theta)$$

Simplifying using trig identifies:

$$(abc)\rho^2\sin(\varphi)\cos^2(\varphi)[\cos^2(\theta)+\sin^2(\theta)] + (abc)\rho^2\sin^3(\varphi)[\cos^2(\theta)+\sin^2(\theta)]$$

$$(abc)\rho^2\sin(\varphi)\cos^2(\varphi) + (abc)\rho^2\sin^3(\varphi)$$

$$(abc)\rho^2\sin(\varphi)\,[\cos^2(\varphi)+\sin^2(\varphi)]$$

$$(abc)\rho^2\sin(\varphi)$$

Triple Integral:

$$Volume = \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_0^1 [(abc)\rho^2\sin(\varphi)]\, d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} sin(\varphi)\int_0^1 [\rho^2]\, d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\int_0^1 [\rho^2]\, d\rho d\varphi d\theta$$

$$V = abc\int_{\theta_1}^{2\pi}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\, d\varphi d\theta \left[\left(\frac{1}{3}\right)p^3\right]\Bigg|_0^1$$

$\left[\left(\frac{1}{3}\right)(1)^3 - \left(\frac{1}{3}\right)(0)^3\right] = \frac{1}{3}$ This is why ρ goes away

$$V = \left(\frac{1}{3}\right)abc\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2} \sin(\varphi)\, d\varphi d\theta$$

$$V = \left(\frac{1}{3}\right)abc\int_{\theta_1}^{\theta_2} d\theta[-\cos(\varphi)]\Bigg|_{\varphi_1}^{\varphi_2}$$

$$V = \left(\frac{1}{3}\right)abc(-\cos(\varphi)+\cos(\varphi))\int_{\theta_1}^{\theta_2} d\theta$$

Symbolic integration

$$V = -\left(\frac{1}{3}\right)abc(\cos(\varphi)-\cos(\varphi))\int_{\theta_1}^{\theta_2} d\theta$$

Factor out a -1

$$V = -\left(\frac{1}{3}\right)abc[\cos(\varphi)-\cos(\varphi)][\theta_2-\theta_2]$$

Final algorithm