

How to Arrive at the Determinant of the Jacobian:

Equation of an ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Transformation into u, v and w space:

- $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$ which yields the equation $u^2 + v^2 + w^2 = 1$

This is now the equation of a sphere with a radius of 1 in u, v and w space.

Now for the conversion into spherical coordinates:

Since we are now in rectangular coordinates in u, v and w space, we need to convert u, v and w into spherical coordinates.

- $u = \rho \sin(\varphi) \cos(\theta), v = \rho \sin(\varphi) \sin(\theta), \text{ and } w = \rho \cos(\varphi)$

Therefore you have the equations:

- $\rho \sin(\varphi) \cos(\theta) = \frac{x}{a}, \rho \sin(\varphi) \sin(\theta) = \frac{y}{b}, \text{ and } \rho \cos(\varphi) = \frac{z}{c}$

Solving for x, y and z you have:

- $a\rho \sin(\varphi) \cos(\theta) = x, b\rho \sin(\varphi) \sin(\theta) = y, \text{ and } c\rho \cos(\varphi) = z$

Now for the Jacobian transformation.

The jacobian transformation for a 3x3 matrix is defined as:

$$- \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} a \sin(\varphi) \cos(\theta) & b \sin(\varphi) \sin(\theta) & c \cos(\varphi) \\ -a\rho \sin(\varphi) \sin(\theta) & b\rho \sin(\varphi) \cos(\theta) & 0 \\ a\rho \cos(\varphi) \cos(\theta) & b\rho \cos(\varphi) \sin(\theta) & -c\rho \sin(\varphi) \end{bmatrix}$$

To find the determinant, use these steps:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

The determinant of the matrix has the form of

$$\det(A) = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

$$\begin{aligned} & a \sin(\varphi) \cos(\theta) [bp \sin(\varphi) \cos(\theta)(-cp \sin(\varphi)) - 0] \\ & \quad - b \sin(\varphi) \sin(\theta) [-ap \sin(\varphi) \sin(\theta)(-cp \sin(\varphi)) - 0] \\ & \quad + c \cos(\varphi) [-ap \sin(\varphi) \sin(\theta) b \rho \cos(\varphi) \sin(\theta) \\ & \quad - bp \sin(\varphi) \cos(\theta) ap \cos(\varphi) \cos(\theta)] \end{aligned}$$

Simplifying:

$$\begin{aligned} \text{determinant} &= -abc\rho^2 \sin^3(\varphi) \cos^2(\theta) \sin(\varphi) \\ &\quad - abc\rho^2 \sin^3(\varphi) \sin^2(\theta) \\ &\quad - abc\rho^2 \cos^2(\varphi) \sin^2(\theta) \sin(\varphi) - abc\rho^2 \cos^2(\varphi) \cos^2(\theta) \sin(\varphi) \end{aligned}$$

rearranging and factoring out $a - 1$:

$$\begin{aligned} & -1[(abc)\rho^2 \cos^2(\varphi) \cos^2(\theta) \sin(\varphi) \\ & \quad + (abc)\rho^2 \cos^2(\varphi) \sin^2(\theta) \sin(\varphi) \\ & \quad + (abc)\rho^2 \cos^2(\theta) \sin^3(\varphi) + (abc)\rho^2 \sin^3(\varphi) \sin^2(\theta)] \end{aligned}$$

simplifying using trig identities:

$$\begin{aligned} & -1[(abc)\rho^2 \sin(\varphi) \cos^2(\varphi) [\cos^2(\theta) + \sin^2(\theta)] + (abc)\rho^2 \sin^3(\varphi) [\cos^2(\theta) + \sin^2(\theta)]] \\ & \quad -1[(abc)\rho^2 \sin(\varphi) \cos^2(\varphi) + (abc)\rho^2 \sin^3(\varphi)] \\ & \quad -1[(abc)\rho^2 \sin(\varphi) [\cos^2(\varphi) + \sin^2(\varphi)]] \\ & \quad -(abc)\rho^2 \sin(\varphi) \end{aligned}$$

$$\text{Simplified determinant} = -abc\rho^2 \sin(\varphi)$$

Please refer to the document titled "Integration of determinant of jacobian for Kelly and Alan" to see how to integrate using jacobian