

Ellipsoid Portion Volume Calculation

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Separation of methods

When calculating the volume of the portion of the ellipsoid, special cases can be calculated very quickly and accurately with a definite integral. The rest of the cases need to be calculated with other methods. In this program, the other methods are a series of Monte Carlo Integrations.

Special cases:

- If the user inputs a sphere ($a \text{ axis} = b \text{ axis} = c \text{ axis}$)
- If the user specifies a volume where the values of φ and θ are both increments of $\frac{\pi}{2}$ (i.e. if they use any and only of the following for φ : $0, \frac{\pi}{2}, \pi$ and the following for θ : $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$)

In these special cases, the definite integral can be used to calculate the volume.

The following logic is used to find special cases:

Eccentricity – Used to determine how spherical the ellipsoid is

The following calculations are based off the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

IF: $\frac{c}{a} + \frac{b}{a} = 2$, then the ellipsoid is a sphere

** Note: this is not the normal definition for Eccentricity, I've augmented it to suite my needs **

Total radian measure – Used to determine if all bounds of integration are Increments of $\frac{\pi}{2}$

$$\text{Total radian measure} = \theta_{\text{start}} + \theta_{\text{End}} + \varphi_{\text{start}} + \varphi_{\text{end}}$$

IF: $\frac{\text{Total radian measure}}{\frac{\pi}{2}} = 0$, then all the radian measures are increments of $\frac{\pi}{2}$

If either of these are true, then the definite integral is used (page 2)

Everything else:

- In all other cases, a Monte Carlo Integration is used to estimate the volume.
- At the present functionality, two different Monte Carlo algorithms are used, one for rectangular coordinates and the other using spherical coordinates. The rectangular coordinates are used on more elongated ellipsoids, and the spherical coordinates are used on more spherical ellipsoids

The following logic is used to determine when to use the Monte Carlo algorithm in rectangular coordinates or Spherical coordinates:

Using the same formula for eccentricity defined above:

IF: Eccentricity ≥ 1 , then use spherical Monte Carlo Algorithm in Spherical Coordinates, otherwise, Use Monte Carlo Algorithm in in Rectangular coordinates

Monte Carlo Algorithm for Spherical Coordinates (Page 7)

Monte Carlo Algorithm for Rectangular Coordinates (Page 3)

Definite Integral

Definite integral algorithm:

$$Volume = -\left(\frac{1}{3}\right)abc(\cos(\varphi_2) - \cos(\varphi_1))(\theta_2 - \theta_1)$$

Symbolic equation for the volume of an entire ellipsoid using the algorithm

- θ will range from 0 to 2π
- φ will range from 0 to π
- ρ will range from 0 to 1 (it is eliminated in the triple integral which is why it's not present in the symbolic definite integral, which is the algorithm)

$$Volume = -\left(\frac{1}{3}\right)abc[\cos(\pi) - \cos(0)][2\pi - 0]$$

$$Volume = -\left(\frac{1}{3}\right)abc[(-1) - (1)][2\pi]$$

$$Volume = -\left(\frac{1}{3}\right)abc[-2][2\pi]$$

$$Volume = \frac{4\pi}{3} abc$$

Plugging in bounds for φ and θ that are increments of $\frac{\pi}{2}$ will also yield a positive accurate answer.

Say for example a user wanted to calculate the upper left quarter of the ellipsoid

- θ will range from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$
- φ will range from 0 to $\frac{\pi}{2}$
- ρ will range from 0 to 1 (Again, eliminated from the symbolic definite integral)

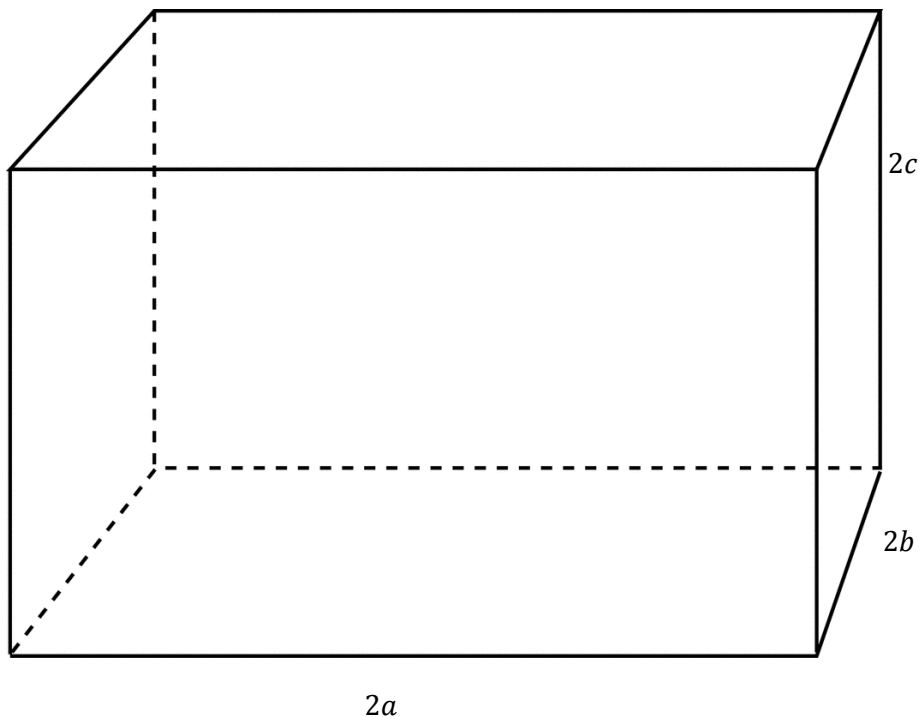
$$Volume = -\left(\frac{1}{3}\right) abc \left[\cos\left(\frac{\pi}{2}\right) - \cos(0) \right] \left[\frac{3\pi}{2} - \frac{\pi}{2} \right]$$

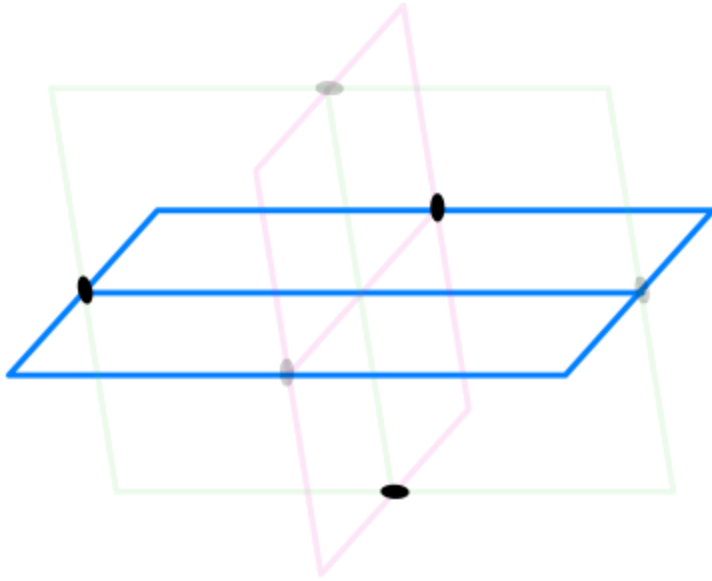
$$Volume = -\left(\frac{1}{3}\right) abc [0 - 1] [\pi]$$

$$Volume = \frac{\pi}{3} abc$$

Monte Carlo in Rectangular Coordinates

The known volume of the rectangular prism is defined by inscribing the ellipsoid inside of a rectangular prism. The dimensions of the rectangular prism is defined as $V = (2a)(2b)(2c)$ where a , b and c are the axes of the ellipsoid.





Random Points Generation in Rectangular Coordinates:

Random points are generated for x , y and z using the axes of the ellipsoid as follows:

$$x = (a \text{ axis}) * (\text{random decimal between } 0 \text{ and } 1) * (\text{random negation})$$

$$y = (b \text{ axis}) * (\text{random decimal between } 0 \text{ and } 1) * (\text{random negation})$$

$$z = (c \text{ axis}) * (\text{random decimal between } 0 \text{ and } 1) * (\text{random negation})$$

This generates a random sample with uniform distribution

Determine if point is inside of portion of ellipsoid:

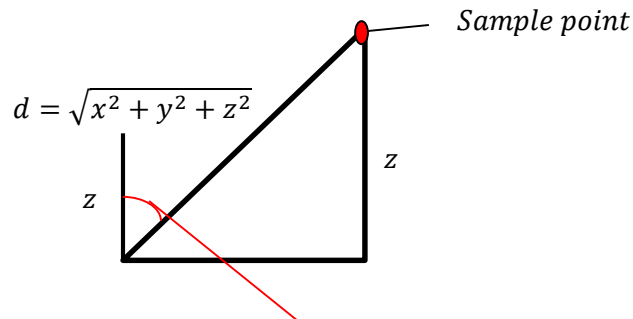
To determine if a point is inside of a portion of the ellipsoid, you must find the value of ϕ and θ as well as whether or not the point is inside the equation of the entire ellipsoid.

phi

To find *phi*, you can draw a right triangle from origin, to the point, then straight down to the *xy* plane and finally to the *z* axis. The hypotenuse is defined by using the distance formula in 3 dimensions to the point:

$$d = \sqrt{x^2 + y^2 + z^2}$$

The vertical leg of the triangle is defined as the value of *z* so you have:



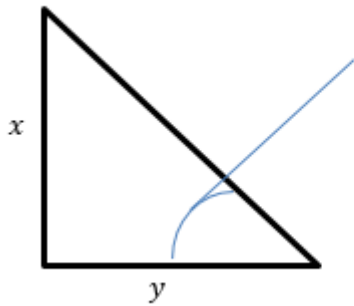
This angle is defined as:

$$\phi = \cos^{-1} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

This is the *phi* angle to the random point and will be compared later against the entered bounds of integration for *phi*

Theta

This angle only exists in the xy plane and is defined as $\theta = \tan^{-1}\left(\frac{y}{x}\right)$



This angle is defined as:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Due to the range and domain restrictions for Arctan, the radian measure need to be adjusted.

Logic for radian measure adjustment:

IF: $x > 0$ and $y > 0$, then radian measure is correct

IF: $x < 0$ and $y > 0$, then add π

IF: $x < 0$ and $y < 0$, then add π

IF: $x > 0$ and $y < 0$, then add 2π

There are two methods for theta, one for start and one for end. This is because if the radian measure comes back 0, one method should return 0 and the other should return 2π , this will matter later.

There is also other logic present within these methods that accounts for values where arctan is undefined or if the point is on the origin.

Now that we have figured out the value of θ and φ , we need to make sure the point lays within the equation of the ellipsoid.

Given the three dimensional coordinate and the length of the axes, this is fairly simple:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

This is simply a boolean statement.

The algorithm comes down to three boolean statements:

- 1) $\varphi_{start} \leq \varphi_{sample\ point} \leq \varphi_{end}$
- 2) $\theta_{start} \leq \theta_{sample\ point} \leq \theta_{end}$
- 3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

If all of these Boolean statements are true, then the point is inside the portion of the ellipsoid, if not, then it is outside. **Increment** some value each time a point is inside the shape. So after all sample points have been taken you have:

$$ratio = \frac{inside\ shape}{total\ sample}$$

Then to obtain volume you have:

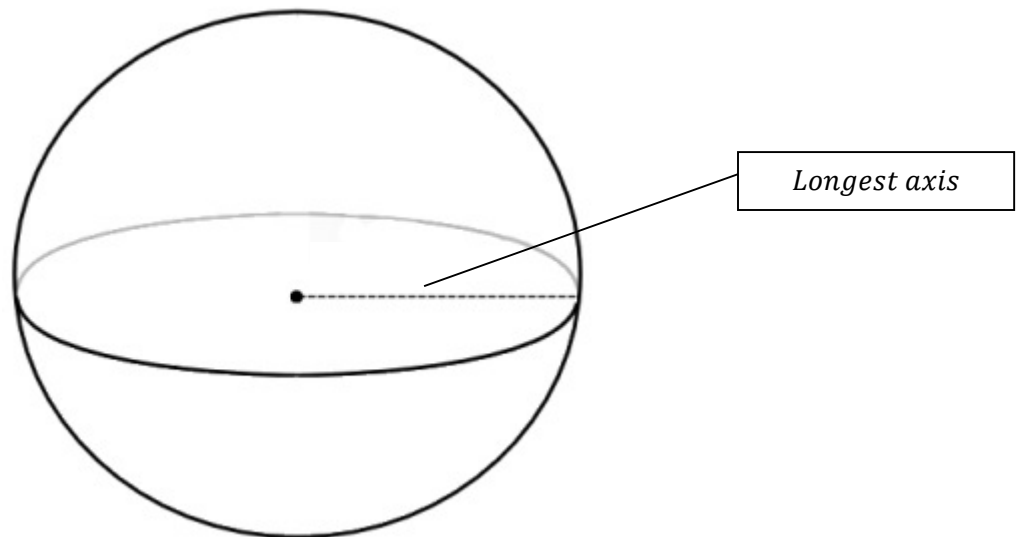
$$V = ratio * (2a)(2b)(2c)$$

With 5 million samples this algorithm is 99% accurate for small numerical volumes and 99.99% accurate for large numerical volumes (this exists on a continuum).

Monte Carlo Algorithm in Spherical Coordinates

This algorithm operates on a very similar set of principles that were defined in rectangular coordinates. The main difference is the coordinate system used (spherical) and the shape that the ellipsoid is inscribed inside of (sphere). The volume of the sphere is defined as:

$$V = \frac{4\pi}{3}(\text{longest axis})^3$$



First to generate a set of random points with uniform distribution:

- The variables in spherical coordinates are φ , θ and ρ
- Due to bunching at the center of the sphere and at the poles, some special things need to be done to generate a random spherical coordinate with uniform distribution

$$\varphi = \sin^{-1}(2 * (\text{random decimal between 0 and 1}) - 1)$$

$$\theta = 2 * \pi * (\text{random decimal between 0 and 1})$$

$$\rho = (\text{longest axis}) * \sqrt[3]{(\text{random decimal between 0 and 1})}$$

Determination if point is inside portion of ellipsoid:

Since we already have φ and θ we simply have two boolean statements that knock out $\frac{2}{3}$ of the tests:

- 1) $\varphi_{start} \leq \varphi_{sample\ point} \leq \varphi_{end}$
- 2) $\theta_{start} \leq \theta_{sample\ point} \leq \theta_{end}$

The final test determines whether or not the point lies within the equation of the ellipsoid. This is also a simple Boolean statement:

$$IF: \frac{\rho^2 \sin^2(\varphi) \cos^2(\theta)}{a^2} + \frac{\rho^2 \sin^2(\varphi) \sin^2(\theta)}{b^2} + \frac{\rho^2 \cos^2(\varphi)}{c^2} \leq 1, \quad \text{then the point is inside}$$

the equation of the ellipsoid.

Thus the algorithm again comes down to three Boolean statements:

- 1) $\varphi_{start} \leq \varphi_{sample\ point} \leq \varphi_{end}$
- 2) $\theta_{start} \leq \theta_{sample\ point} \leq \theta_{end}$
- 3) $\frac{\rho^2 \sin^2(\varphi) \cos^2(\theta)}{a^2} + \frac{\rho^2 \sin^2(\varphi) \sin^2(\theta)}{b^2} + \frac{\rho^2 \cos^2(\varphi)}{c^2} \leq 1$

If all of these Boolean statements are true, then the point is inside the portion of the ellipsoid, if not, then it is outside. **Increment** some value each time a point is inside the shape. So after all sample points have been taken you have:

$$ratio = \frac{inside\ shape}{total\ sample}$$

Then to obtain volume you have:

$$V = ratio * \frac{4\pi}{3} (longest\ axis)^3$$

It is more accurate than the rectangular algorithm with ellipsoids that are close to spherical.