Nature of the problem

Given Jacobian:

$$determinant: -abc\rho^2\sin(\varphi)$$

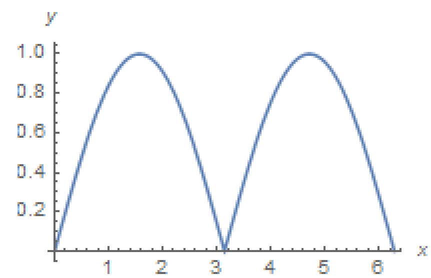
Triple Integral:

- The integrand can be broken up into two parts, the first part will never be negative, the second can be:

$$|-abc\rho^2\sin(\varphi)| = |abc\rho^2(-\sin(\varphi))| = |abc\rho^2| * |-\sin(\varphi)|$$

$|abc|$ Is just a constant and $|\rho^2|$ will always range from 0 to 1 and won't be negative

$|-\sin(\varphi)|$ On the other hand is different from $-\sin(\varphi)$ and looks like this:



- Triple integral:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}[(abc)\rho^2\sin(\varphi)]\,d\rho d\varphi d\theta$$

- From $0\ to\ \pi$, $|-\sin(\varphi)|$ has the same area under the curve as $\sin(\varphi)$. This is why multiplying the jacobian by -1 instead of taking the absolute value did not yield an incorrect volume. It is also why you can just use $\sin(\varphi)$ instead of $|-\sin(\varphi)|$ in the integrand.

$$V = 2abc\int_0^{\pi}\int_0^{\pi} sin(\varphi)\int_0^{1}[\rho^2]\,d\rho d\varphi d\theta$$

$$V = 2abc\int_0^{\pi}\int_0^{\pi}\sin(\varphi)\int_0^{1}[\rho^2]\,d\rho d\varphi d\theta$$

$$V = 2abc\int_0^{\pi}\int_0^{\pi}\sin(\varphi)\,d\varphi d\theta\left[\left(\frac{1}{3}\right)p^3\right]\Bigg|_0^1$$

$$V = \left(\frac{2}{3}\right)abc\int_0^{\pi}\int_0^{\pi}\sin(\varphi)\,d\varphi d\theta$$

$$V = \left(\frac{2}{3}\right)abc\int_0^{\pi}d\theta[-\cos(\varphi)]\Bigg|_0^{\pi}$$

$$V = \left(\frac{2}{3}\right) abc(-(-1) + (1)) \int_0^{\pi} d\theta$$

$$V = \left(\frac{2}{3}\right) abc(2)[\theta]\Big|_0^{\pi}$$

$$V = \left(\frac{4}{3}\right) abc[\pi - 0]$$

$$V = \frac{4\pi}{3} abc$$

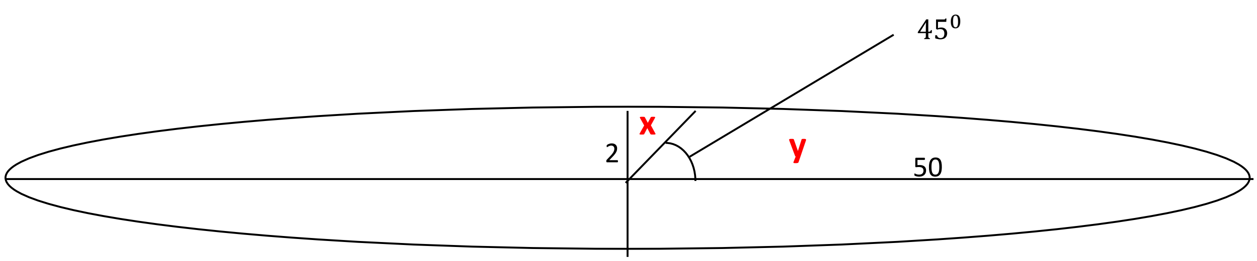
Unfortunately, this still does not solve the problem for values other than increments of $\frac{\pi}{2}$. For example if we integrated the ellipsoid using this jacobian from $0\ to \frac{\pi}{2}$ for φ and $0\ to \frac{\pi}{4}$ for θ, we do not get an accurate volume for that portion of an ellipsoid. This means the way I am approaching this problem is fundamentally wrong.

$$V_y = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 [(abc)\rho^2 \sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12} abc$$

This is only true if $a = b = c$

$$V_x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 [(abc)\rho^2 \sin(\varphi)]\, d\rho d\varphi d\theta = \frac{\pi}{12} abc$$

$$\boxed{But\ what\ if\ a = 50, b = 2\ and\ c = 2?}$$



$$\boxed{x \neq y}$$

But according to the above integration, it does. This is what I need to fix.

Below is the start of an integration that I cannot finish. It is possible it can be done through iterations but it would be best if a symbolic definite integral can be found

Unfinished triple integral of ellipsoid without using jacobian transformation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{\rho^2\sin^2(\varphi)\sin^2(\theta)}{a^2}+\frac{\rho^2\sin^2(\varphi)\cos^2(\theta)}{b^2}+\frac{\rho^2\cos^2(\varphi)}{c^2}=1$$

$$\rho^2\left[\frac{\sin^2(\varphi)\sin^2(\theta)}{a^2}+\frac{\sin^2(\varphi)\cos^2(\theta)}{b^2}+\frac{\cos^2(\varphi)}{c^2}\right]=1$$

$$\rho=\pm\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_0^{\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}}d\rho d\varphi d\theta$$

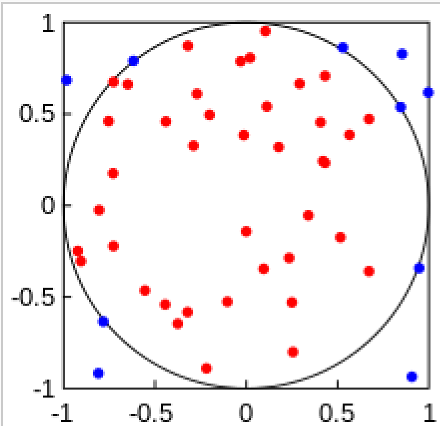
$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\sqrt{a^2\csc^2(\varphi)\csc^2(\theta)+b^2\csc^2(\varphi)\sec^2(\theta)+c^2\sec^2(\varphi)}\,d\varphi d\theta$$

$$V=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\sqrt{\csc^2(\varphi)[a^2\csc^2(\theta)+b^2\sec^2(\theta)]+c^2\sec^2(\varphi)}\,\mathrm{d}\varphi\mathrm{d}\theta$$

This is where I get stuck.

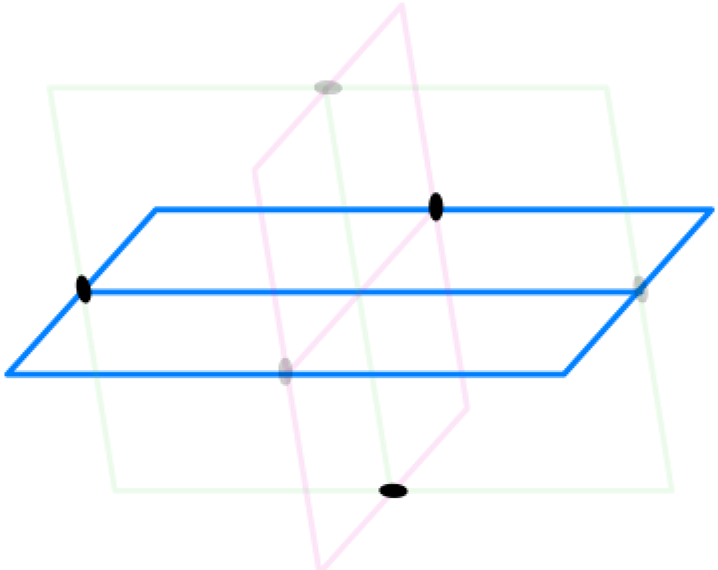
I have since abandoned this idea in light of another alternative: Monte Carlo Integration.

A Monte Carlo integration take a series of sample points in a known area (or volume) and determines if the point is inside or outside the shape. Using statistics, you are able to approximate the area (or volume) of a shape:



An illustration of Monte Carlo integration. In this example, the domain *D* is the inner circle and the domain E is the square. Because the square's area (4) can be easily calculated, the area of the circle ($\pi*1^2$) can be estimated by the ratio (0.8) of the points inside the circle (40) to the total number of points (50), yielding an approximation for the circle's area of $4*0.8 = 3.2 \approx \pi*1^2$.

Using this same principle, we can extend this into three dimensions. The coordinate system is comprised of x y and z. The known volume of the rectangular prism is defined by inscribing the ellipsoid inside of a rectangular prism. The dimensions of the rectangular prism is defined as $V = (2a)(2b)(2c)$ where a b c are the axes of the ellipsoid



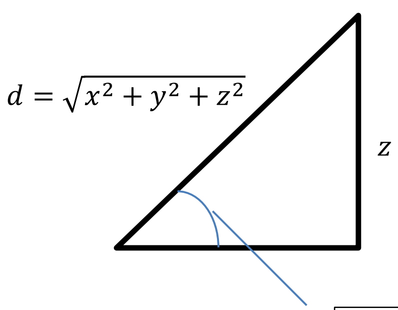
Random points are generated using x, y and z. To determine if a point is inside of a portion of the ellipsoid, you must find the value of $phi, theta\ and\ \rho$.

phi

To find phi, you can draw a right triangle from origin, to the point, then strait down to the xy plane and finally to the z axis. The hypotenuse is defined by using the distance formula in 3 dimentions to the point:

$$d = \sqrt{x^2 + y^2 + z^2}$$

The verical leg of the triangle is defined as the value of z so you have:

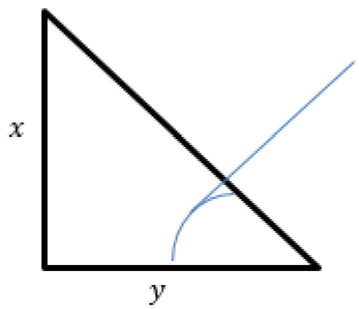


$This\ angle\ is\ defined\ as:$

$$\varphi = \sin^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Theta

This angle only exists in the xy plane and is defined as $\theta = \tan^{-1}\left(\frac{y}{x}\right)$



This angle is defined as:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now that we have figured out the value of θ *and* φ, we need to make sure the point lays within the equation of the ellipsoid.

Given the three dimensional coordinate and the length of the axes, this is fairly simple:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

This is simply a boolean statement.

The algorithm comes down to three boolean statements:

1) $\varphi_{start} \leq \varphi \leq \varphi_{end}$
2) $\theta_{start} \leq \theta \leq \theta_{end}$
3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

If all of these Boolean statements are true, then the point is inside the portion of the ellipsoid, if not, then it is outside. **Increment** some value each time a point is inside the shape. So after all sample points have been taken you have:

$$ratio = \frac{inside\ shape}{total\ sample}$$

Then to obtain volume you have:

$$V = ratio * (2a)(2b)(2c)$$

With 5 million samples this algorithm is 99% accurate for small numerical volumes and 99.99% accurate for large numerical volumes (this exists on a continuum).

I have developed similar algorithms for an ellipsoid inscribed inside of a sphere and a cylinder which use spherical coordinates and cylindrical coordinates respectively.

If the ellipsoid is relatively spherical, the spherical coordinates are used. If it is more elongated, rectangular coordinates are used with their respective Monte Carlo Algorithm. If 8^{ths} of an ellipsoid is specified or a sphere is specified, the jacobian integration algorithm is used.