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Sliding puzzles and rotating puzzles on graphs[☆]

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ABSTRACT

Sliding puzzles on graphs are generalizations of the Fifteen Puzzle. Wilson has shown that the sliding puzzle on a 2-connected graph always generates all even permutations of the tiles on the vertices of the graph, unless the graph is isomorphic to a cycle or the graph θ_0 [R.M. Wilson, Graph puzzles, homotopy, and the alternating group, J. Combin. Theory Ser. B 16 (1974) 86–96]. In a rotating puzzle on a graph, tiles are allowed to be rotated on some of the cycles of the graph. It was shown by Scherphuis that all even permutations of the tiles are also obtainable for the rotating puzzle on a 2-edge-connected graph, except for a few cases. In this paper, Scherphuis' Theorem is generalized to every connected graph, and Wilson's Theorem is derived from the generalized Scherphuis' Theorem, which will give a uniform treatise for these two families of puzzles and reveal the structural relation of the graphs of the two puzzles.

1. Introduction

Sliding puzzles on graphs are generalizations of the famous Fifteen Puzzle. Let G be a simple undirected graph with vertex set $V = \{0, 1, 2, \dots, n\}$. Initially, for each i ($0 \leqslant i \leqslant n$), a tile indexed i is placed on the vertex i of the graph. The tile indexed 0 is regarded as an empty tile. A legal move of the puzzle is the transposition of the empty tile and an adjacent tile. When the empty tile is back to vertex 0 again, the positions of all the other tiles can be viewed as a permutation f on $[n] = \{1, 2, \dots, n\}$, where $f(i) = j$ means that the tile indexed j is placed on vertex i. Let $S(G)$ be the set of all permutations available in the sliding puzzle on the graph G when the empty tile returns to vertex 0, which forms a group.

Let A_n and S_n denote the alternating group and the symmetric group on $[n]$, respectively. Wilson [6] proved the following theorem.

Theorem 1 (Wilson). *Let G be a 2-connected graph with vertex set $\{0, 1, 2, \dots, n\}$, then*

$$S(G) \supseteq A_n,$$

unless G is a cycle of length at least 5, or is isomorphic to the graph θ_0 illustrated in Fig. 1. Moreover, $S(G) = S_n$ if and only if G is non-bipartite.

Before introducing the rotating puzzles, let us review the basic concept of the cycle space of a graph. Let $\mathbb{F}_2 = \{0, 1\}$ be a finite field, and let G be a graph with edge set $E(G)$. The set of all functions $f : E(G) \to \mathbb{F}_2$ forms a vector space over $\mathbb{F}_2$. This is called the *edge space* of the graph G. A vector f in the edge space can be also viewed as an edge subset $F = \{e \in E(G) | f(e) = 1\}$ of G. The sum of two vectors is the symmetric difference of the two edge sets. The *cycle space* of G is a subspace of the edge space generated by the set $\{E(C) | C \text{ is a cycle of } G\}$. The dimension of the cycle space of a connected graph G is $m(G) - n(G) + 1$, where $n(G)$ and $m(G)$ denote the number of vertices and edges of the graph G, respectively.

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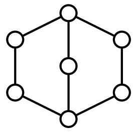

Fig. 1. The graph θ_0.

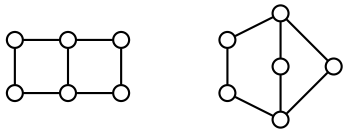

Fig. 2. The graphs θ_1 and θ_2.

The rotating puzzle is another type of puzzle on graphs. Let G be a 2-edge-connected graph with vertex set $V = \{1, 2, \ldots, n\}$, with a tile indexed i on each vertex i initially. Let $\mathcal{C}$ be a set of cycles which forms a base of the cycle space of G. Let $c = (x_0x_1 \cdots x_{m-1})$ be a cycle in $\mathcal{C}$. For convenience, we also view c as the cyclic permutation which maps x_i to $x_{(i+1)(\mathrm{mod}\ m)}$. A move of the rotating puzzle is to rotate the tiles of one of the cycles in $\mathcal{C}$. In exact words, by rotating the cycle c, the tile on vertex x_i is moved to the vertex $x_{(i+1)(\mathrm{mod}\ m)}$. Similar to the sliding puzzles, the positions of the tiles in a rotating puzzle can be represented by a permutation f, where $f(i) = j$ means that tile j is on vertex i. Thus, if the positions of the tiles before and after the rotation of the cycle c are represented by permutations f and g, respectively, we have $g = f \circ c^{-1}$. Let $R(G, \mathcal{C})$ be the set of permutations attainable by rotating the cycles in $\mathcal{C}$ from the initial position. Then the set $\mathcal{C}$ can be also viewed as a generating set for $R(G, \mathcal{C})$.

Scherphuis proved in his web articles [3,4] a result analogous to Wilson's Theorem for rotating puzzles, though he did not state it explicitly.

Theorem 2 (*Scherphuis*)**.** *Let G be a 2-edge-connected graph with vertex set $\{1, 2, \ldots, n\}$, and let $\mathcal{C}$ be a set of cycles defined as above, then*

$$R(G, \mathcal{C}) \supseteq A_n,$$

unless G is a cycle with length at least 4, or is isomorphic to one of the two graphs θ_1 and θ_2 in Fig. 2, and $\mathcal{C}$ is the set of two cycles corresponding to the two inner faces. Moreover, $R(G, \mathcal{C}) = S_n$ if and only if $\mathcal{C}$ contains an even cycle.

Note that $R(G, \mathcal{C}_1)$ and $R(G, \mathcal{C}_2)$ may be different for the same graph G with distinct sets of cycles $\mathcal{C}_1$ and $\mathcal{C}_2$. A 2-connected graph whose cycle space has dimension 2 is called a *θ-graph*. The graph θ_0, θ_1 and θ_2 are all θ-graphs.

The proof of Scherphuis' Theorem for rotating puzzles is essentially the same as that of the sliding puzzle, by induction on the dimension of the cycle space of the graph. In this paper, the author studies the relation of these two theorems. First, a generalization of Scherphuis' Theorem to arbitrary connected graphs is given in Section 2. Then, in Section 3, Wilson's Theorem is derived from the generalized Scherphuis' Theorem.

Wilson's Theorem is frequently cited in the literature, especially in the fields of recreational mathematics and combinatorial games theory, for example [2]. Our treatment of the theorem has split the group-theoretical aspect and the graph-theoretical aspect apart, which makes the proof a bit more comprehensible and provides some insight into the structural properties of the graphs.

2. Generating A_n

It is a simple exercise in permutation group theory to show that the set of 3-cycles $\{(123), (124), \ldots, (12n)\}$ generates the alternating group A_n. This fact can be viewed as a special case of Theorem 2, since the set of generators correspond to a base of the cycle space of the graph $K_2 + \overline{K}_{n-2}$ obtained by joining $n - 2$ independent vertices to an edge.

Theorem 2 is interesting in a way that it shows a connection between graph theory and permutation group theory. It is actually a sufficient condition, in graph-theoretical terms, for a set of permutations to generate the alternating group. In this section, a set of permutations is defined on every connected graph, rather than on 2-edge-connected one, and is shown to be the generating set of the alternating group or the symmetric group.

Let G be a connected graph. A *block* is a maximal subgraph without any cut-vertex. Suppose B is a block of G, then B is either a 2-connected (hence 2-edge-connected) subgraph, or an edge. If B is 2-connected, let $\mathcal{C}_B$ be a set of cycles which is a base of the cycle space of B. If B is a single edge ab, let $\mathcal{C}_B = \{(ab)\}$, which contains a single transposition. Let

$$\mathcal{C}_G = \bigcup_B \mathcal{C}_B,$$

where the union is taken over all blocks of G. In this way, $\mathcal{C}_G$ defines a generalized rotating puzzle on the graph G. In a generalized rotating puzzle, transposition of two tiles is allowed if they are on the endpoints of a bridge. The permutation group generated by $\mathcal{C}_G$ is exactly the set of all permutations of the tiles obtainable on the generalized rotating puzzles on G. The set of all permutations for the generalized rotating puzzles on G with $\mathcal{C}_G$ is also denoted by $R(G, \mathcal{C}_G)$. Without introducing confusion, we will refer to the generalized rotating puzzles as rotating puzzles for short.

Theorem 3. *Let G be a connected graph on vertex set $[n]$ with at least 2 blocks, and let $\mathcal{C}_G$ be a set of permutations based on G defined as above. Then*

$$R(G, \mathcal{C}_G) \supseteq A_n.$$

Moreover, $R(G, \mathcal{C}_G) = S_n$ if and only if $\mathcal{C}_G$ contains an even cycle.

Let us introduce some useful lemmas from Wilson and Scherphuis.

Lemma 1 ([6]). *Let G be a 2-transitive group on $\{1, 2, \ldots, n\}$ and contains a 3-cycle, then $A_n \subseteq G$.*

Lemma 2. *The transposition $(1, 2)$ and the cycle $(1\ 2\ 3\ \cdots\ n)$ generate S_n. The transposition $(1, 2)$ and the cycle $(2\ 3\ \cdots\ 4n)$ also generate S_n.*

Lemma 3. *Let A_n be an alternating group with $n \geqslant 3$. Let c be the cycle $(kk + 1k + 2 \cdots m)$, where $k \leqslant n < m$. Let $\langle A_n, c\rangle$ denote the group generated by A_n and c. Then $A_m \subseteq < A_n, c >$.*

Proof. It is easy to show that the group $\langle A_n, c\rangle$ is 2-transitive and contains a 3-cycle, thus $\langle A_n, c\rangle \supseteq A_m$ by Lemma 1. □

Proof of Theorem 3. We shall induct on the number of blocks.

First, suppose G has two blocks B_1 and B_2. If neither B_1 nor B_2 is an edge, then G is 2-edge-connected. So G is not isomorphic to a cycle or the graph θ_1 or θ_2, since both the cycles and the θ-graphs have only one block. Hence $R(G, \mathcal{C}_G) \supseteq A_n$ by Theorem 2. If at least one of the two blocks is an edge, the theorem holds by applying Lemmas 2 and 3 starting at the block which is an edge.

If G has three or more blocks, then use the induction hypothesis and apply Lemma 3 to obtain the theorem. □

Recall another fact from permutation group theory that the transpositions $(12), (13), \cdots, (1n)$ generate S_n. This is also a special case of Theorem 3, since these transpositions correspond to the star graph $K_{1,n-1}$. In fact, let T be any tree on $\{1, 2, \ldots, n\}$, then $\mathcal{C}_T$ generates S_n. A set of permutations $\mathcal{D}$ is said to be *realized* by the graph G if $\mathcal{D} = \mathcal{C}_G$. Thus, the set of 2-cycles $\{(12), (13), \ldots, (1n)\}$ is realized by the star graph $K_{1,n-1}$.

3. Wilson's Theorem

In order to prove Wilson's Theorem using Theorems 2 and 3, we should find a generating set of $S(G)$ (or a subset of the generating set) such that it is realized by some graph. In [1], a Hamiltonian path is used to find a generating set for the Fifteen Puzzle. We generalize this method by using a spanning tree with a special property.

All through this section, let G be a graph with vertex set $\{0, 1, 2, \ldots, n\}$. Let T be a spanning tree of G. Note that in a sliding puzzle on G, if the transposition of the empty tile and its neighbor tile is restricted to the edges of T, the positions of the tiles will not change when the empty tile returns to vertex 0. Changes occur when the transposition is taking place on an edge of $G - E(T)$.

Following Wilson [6], we say that the two walks are *homotopic* if one is obtained from the other by replacing a vertex y with a walk yxy or by replacing the subwalk yxy with the vertex y. Two walks are also homotopic if they are related by any sequence of such operations.

Let $g \in S(G)$ be a permutation, then g is obtained by sequential movement of the empty tile along a closed walk $W = x_1x_2 \cdots x_m$ of the graph G, where $x_1 = x_m = 0$. Let $x_{i_1}x_{i_1+1}, x_{i_2}x_{i_2+1}, \ldots, x_{i_k}x_{i_k+1}$ be all the edges of W which do not belong to T, in the order of their appearance in W. Let $T(x, y)$ denote the unique path from x to y in T, and let $\oplus$ denote the operation of walk catenation. Define k closed walks as below,

$$W_1 = x_1x_2 \cdots x_{i_1}x_{i_1+1} \oplus T(x_{i_1+1}, 0);$$
$$W_2 = T(0, x_{i_1+1}) \oplus x_{i_1+1}x_{i_1+2} \cdots x_{i_2}x_{i_2+1} \oplus T(x_{i_2+1}, 0);$$
$$\cdots$$
$$W_{k-1} = T(0, x_{i_{k-2}+1}) \oplus x_{i_{k-2}+1}x_{i_{k-2}+2} \cdots x_{i_{k-1}}x_{i_{k-1}+1} \oplus T(x_{i_{k-1}+1}, 0);$$
$$W_k = T(0, x_{i_{k-1}+1}) \oplus x_{i_{k-1}+1}x_{i_{k-1}+2} \cdots x_{i_k}x_{i_k+1}x_{i_k+2} \cdots x_m.$$

Then W is homotopic to $W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Note that for each j, W_j has exactly one edge outside T, and W_j is homotopic to $T(0, x_{i_j}) \oplus x_{i_j}x_{i_j+1} \oplus T(x_{i_j+1}, 0)$. Thus we have decomposed the walk W into a series of walks each having exactly one edge outside T. The effect of W is the composition of the effect of W_j $(1 \leqslant j \leqslant k)$.

Thus, it suffices to study the effects of the empty tile moving along the close walk $W' = T(0, x) \oplus xy \oplus T(y, 0)$, where xy is an edge outside T. Let $T(x, y) = x_0x_1 \cdots x_s$ be the path from x to y in T, then $T(x, y) \oplus yx$ forms a cycle. Let x_j be the

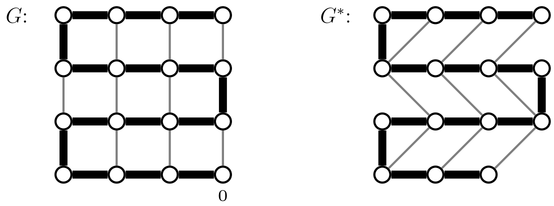

Fig. 3. The graphs G and G^* for the Fifteen Puzzle.

nearest vertex to 0 in T among the vertices of $T(x, y) \oplus yx$, which is said to be the *gate* of the cycle $T(x, y) \oplus yx$ with respect to the vertex 0. Then, $W' = T(0, x_j) \oplus x_j x_{j-1} \cdots x_0 x_s x_{s-1} \cdots x_j \oplus T(x_j, 0)$. It is easy to see that the effect of the empty tile moving along W' and back to the vertex 0 is that, the tiles on all but one vertex (the gate vertex x_j) of the cycle $T(x, y) \oplus yx$ move cyclically in the manner $x_{j-1} \to x_{j+1} \to x_{j+2} \to x_{j+3} \to \cdots \to x_s \to x_0 \to x_1 \cdots \to x_{j-1}$, where $x' \to x''$ means the tile on vertex x' is moved to x''.

In this way, we find a generating set of $m(G) - n(G) + 1$ cycles (these may include cycles of length two) for the group $S(G)$, one cycle for each $e \in E(G) - E(T)$. Denote the set of generators by $\mathcal{D}_T$, we will show that a subset of $\mathcal{D}_T$ is realized by some graph G^* in the next few paragraphs, for some special spanning tree T.

For an edge $e \in E(G) - T$, the unique cycle C contained in $T + e$ is said to be a *basic cycle*. A basic cycle is said to be *facing the vertex 0*, or a *good* cycle for short, if the gate of C is incident with e.

Lemma 4. *There exists a spanning tree T of G such that, each basic cycle with respect to T is facing the vertex* 0.

Proof. Let T be the tree obtained by depth-first search starting from vertex 0. (cf. Lemma 4.1.22 in [5]) $\square$

A spanning tree with all basic cycles good is called a *good* spanning tree with respect to the vertex 0. From now on, let T always be a good spanning tree. We will construct a graph G^* from G such that G^* will realize a subset of $\mathcal{D}_T$. The vertex set of G^* is $V(G) - 0 = \{1, 2, \ldots, n\}$. For each basic cycle $C = x_1 x_2 \cdots x_s$, let x_j be the gate of C, and let $x_j x_{j+1}$ be the unique edge of C lying outside T. We have shown earlier in this section that the effect of the close walk W containing the edge $x_j x_{j+1}$ is the rotation of tiles of vertices of $C - \{x_j\}$. Hence, in order to realized this rotation, we add $s - 1$ edges $x_1x_2, x_2x_3, \ldots, x_{j-1}x_{j+1}, \ldots, x_{s-1}x_s, x_s x_1$ to G^*. We apply this for every basic cycle of G to get the edge set of G^*, and delete any duplicate edges. Fig. 3 illustrates the graph G^* constructed from G in the case of the Fifteen Puzzle, where the spanning tree is a Hamiltonian path represented by bold lines.

The next few lemmas will reveal some features of G^* constructed in this manner from a 2-connected graph G.

Lemma 5. *Let G be a* 2*-connected graph and let T be a good spanning tree, then the vertex 0 is a leaf of T.*

Proof. Suppose to the contrary that 0 is not a leaf, then it is a cut-vertex of T. Since G is 2-connected, there is an edge $e \in E(G) - T$ linking two components of $T - 0$. Hence the basic cycle in $T + e$ contains the vertex 0, and is not a good cycle, a contradiction. $\square$

As a consequence, the edges incident with 0 in G will not show up in G^*, this is what we expected.

Lemma 6. *Let G be a* 2*-connected graph and let T be a good spanning tree, then all edges in T except the one incident with the vertex 0 are retained as edges of G^*.*

Proof. Let $e \in E(T - 0)$. It suffices to show that e lies in a basic cycle, and e is not incident with the gate of the cycle. Let $d_T(x, y)$ denote the distance between the vertices x and y in T. Suppose $e = xy$ with $d_T(0, y) = d_T(0, x) + 1$. Then x is a cut-vertex of G, and let e' be the edge of $E(G) - T$ linking two components of $T - x$, one component contains 0 and the other contains y. Then the basic cycle in $T + e'$ is the one we desired. $\square$

By the previous lemma, $T - 0$ is a spanning tree of G^*, and G^* is obtained by adding edges to $T - 0$. Let $e = xy \in E(G) - T$, and let $C_e = x_1 x_2 \cdots x_s x_1$ be the basic cycle in $T + e$, where $x_1 = x$ and $x_2 = y$. Then the effect of the closed walk $W' = T(0, x) \oplus xy \oplus T(y, 0)$ is a rotation of tiles on the vertex $x_s, x_{s-1}, \ldots, x_3, x_2, x_s$. By the definition of G^*, $x_s x_{s-1}, x_{s-1}x_{s-2}, \ldots, x_2 x_s$ are $s - 1$ edges of G^*. All but $x_2 x_s$ are already in $T - 0$. Hence, for each basic cycle C_e, an edge $x_2 x_s$ is added to G^* if the length of the cycle C_e is four or more. The edge $x_2 x_s$ is already in $T - 0$ if the cycle C_e is of length three. In this case, the edge $x_2 x_s$ in $T - 0$ is called a *thick edge*.

Lemma 7. *Let G be a* 2*-connected graph, G^* is constructed from G with respect to a good spanning tree T. Then every bridge of G^* is a thick edge.*

Proof. Suppose $e = xy$ with $d_T(0, y) = d_T(0, x) + 1$. Let T_1 and T_2 be the components of $T - x$ such that $0 \in T_1$ and $y \in T_2$, and let $z \in T_1$ be the vertex adjacent to x in T. By the hypothesis that G is 2-connected and T is good, there must be an edge e' of G linking T_1 and T_2. If $e' = zy$, then e is a thick edge of G^*. If $e' \neq zy$, e will lie on a cycle, obtained from e', of G^*, a contradiction. $\square$

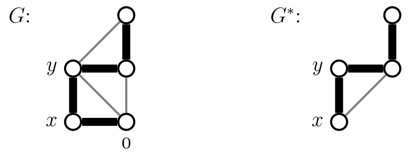

Fig. 4. Example to show that G^* need not be 2-connected.

Note that the converse of Lemma 7 is not true, there may be a thick edge that is not a bridge of G^*. The edge xy illustrated in Fig. 4 is such an example. It is also an example that G^* may not be 2-connected.

Let $\mathcal{C}_{G^*}$ be a set of cycles for G^* as defined in Section 2, where the base of cycle space for each non-bridge block is the set of basic cycles with respect to the spanning tree $T - 0$.

Lemma 8. *Let G be a 2-connected graph, G^* is constructed from G with respect to a good spanning tree T. Then*

$$\mathcal{D}_T \supseteq \mathcal{C}_{G^*}.$$

Proof. Let $C \in \mathcal{C}_{G^*}$, if C is of length 3 or more, then $C \in \mathcal{D}_T$ by the construction of the graph G^*. If C is a transposition, then $C \in \mathcal{D}_T$ by Lemma 7. □

Note that $\mathcal{C}_{G^*}$ may be a proper subset of $\mathcal{D}_T$ in the last lemma, since a basic cycle of length 3 in G may become a thick edge which is not a bridge in G^*. Fig. 4 is an example for this.

Now we are ready to prove Wilson's Theorem.

Proof of Theorem 1. If G^* has two or more blocks, then $\mathcal{C}_{G^*}$ suffices to generate A_n by Theorem 3. If G^* has only one block, but G^* is neither a cycle of length at least four nor either of the graph θ_1 or θ_2, then $\mathcal{C}_{G^*}$ also generates A_n by Theorem 2. If G^* is isomorphic to a cycle of length at least four or one of the graphs θ_1 or θ_2, and G^* has at least one thick edge, $\mathcal{C}_{G^*}$ does not generate A_n, but by Lemmas 2 and 3, $\mathcal{D}_T$ does.

The only case that $\mathcal{D}_T$ does not generate A_n is that G^* is isomorphic to a cycle of length at least four or one of the graphs θ_1 or θ_2, and G^* has no thick edges. And it is easy to see in this case, G is isomorphic to a cycle of length at least five or the graph θ_0. In fact, if G^* is isomorphic to C_n, then G is isomorphic to C_{n+1}. If G^* is isomorphic to θ_1, then G must be a θ-graph of order 7, with two basic cycles of length both 5. The only graph with this property is θ_0. A similar argument applies when G^* is isomorphic to θ_2.

For the second part of the theorem, it suffices to show that G is non-bipartite if and only if $\mathcal{D}_T$ contains an even cycle. If G is bipartite, then every basic cycle is an even cycle. Thus $\mathcal{D}_T$ contains no even cycle, since each cycle in $\mathcal{D}_T$ is one less in length than the corresponding basic cycle. If G is non-bipartite, then G must have a basic cycle C_0 with odd length. Suppose to the contrary that all basic cycles are even. Since G is non-bipartite, there is an odd cycle C which is the symmetric difference of several basic cycles, namely $C = C_1 \triangle C_2 \triangle \cdots \triangle C_k$, where C_i is a basic cycle for each i. But by induction on k, C has even number of edges, which is a contradiction. Thus the odd basic cycle C_0 becomes an even cycle of $\mathcal{D}_T$. □

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