A New Look at the Fifteen Puzzle
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A NEW LOOK AT THE FIFTEEN PUZZLE

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1. Introduction. Almost ninety years ago, Sam Loyd invented a very simple puzzle consisting of fifteen blocks to be slid around in a square tray big enough to hold sixteen blocks. For a short time, the puzzle became immensely popular, as people tried to slide the blocks into certain configurations which seemed especially challenging. Then two articles [2, 3] appeared in the American Journal of Mathematics. The first [2] showed that some of those challenging configurations were actually impossible to achieve. The second [3] went on to determine exactly what configurations could be obtained.

As the editors of the Journal pointed out, the puzzle furnished an extremely interesting illustration of the difference between even and odd permutations. Those configurations which could be obtained were closely associated with even permutations of fifteen objects. Those which could not be obtained were closely associated with odd permutations of fifteen objects.

Since that time, interest in the puzzle has diminished, though one can obtain it in several different forms in novelty shops. Also, our view of permutations has changed. We use cyclic notation to write down permutations, and we view the set of all even permutations of n objects as the alternating group A_n, a group with no proper normal subgroups if $n \geqq 5$. It turns out that these new ways of regarding permutations are very closely related to the behavior of the fifteen puzzle, and so in turn the puzzle furnishes a good illustration of these ideas. In this article we examine the puzzle from these new points of view.

2. The puzzle. In its most common form, the puzzle is a shallow square tray about two inches along a side. Within the tray are fifteen blocks each about one half inch square, with two sides grooved and two sides ridged. The blocks are fitted into the tray so that they may be slid freely, but the grooves and ridges prevent their being removed. The blocks are numbered 1, 2, $\cdots$, 15, and it is easy to slide them into numerical order, as in Figure 1. The question then arises: Can one slide the blocks around so as to achieve an arrangement such as in Figure 2, where just the blocks 14 and 15 have been interchanged?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

FIG. 1.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

FIG. 2.

3. Configurations unattainable. The very simple argument of this section was given by Ball [1]. Consider the puzzle in any configuration in which the empty space is in the lower right hand corner. Now consider any series of moves rearranging the blocks at the end of which the empty space is returned to the lower right hand corner. The effect of such a sequence of moves is a permutation of the fifteen blocks. We show that the only such permutations which can be obtained by sliding the blocks within the tray are even permutations.

Notice that the permutation of the blocks is completely determined by the path of the empty space. But any path of the empty space can be considered as being composed of several paths, in succession, of an especially simple kind: a closed loop possibly on the end of a path leading from and back to the lower right hand corner. An example appears in Figures 3 to 5, in which the path in Figure 3 can be regarded as the result of taking the path of Figure 4 and then taking the path of Figure 5. The permutation resulting from following one of these simple closed loops is a single cycle, permuting every block along the loop except for the one at the space where the path joining the loop to the lower right hand corner enters. Thus any permutation given by a closed-loop path is cyclic of odd length, so is a product of an even number of transpositions, and so is an even permutation. Since any path can be given as the result of a series of these simple closed loop paths, the corresponding permutation can be written as the product of even permutations, and so must be an even permutation. Thus it is impossible to obtain the configuration in Figure 2 from that of Figure 1, since the permutation, a single transposition, is odd.

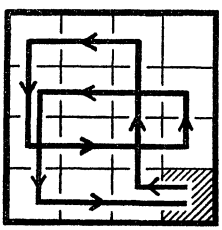

FIG. 3.

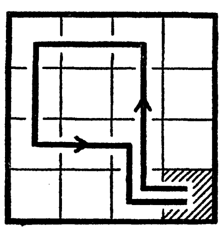

FIG. 4.

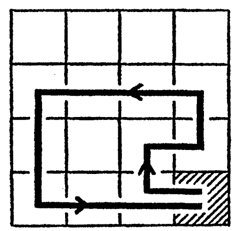

FIG. 5.

4. Configurations attainable. The converse to the preceding result is true, namely if a rearrangement of the blocks which returns the empty space to the lower right hand corner gives an even permutation, then the rearrangement can be achieved by sliding the blocks within the tray. The proof we give here is remarkably closely tied to the fact that the alternating group A_{15} has no proper normal subgroups.

First we show that if three blocks are lined up next to each other in a row or column, then it is possible to permute them cyclically while returning the rest of the puzzle to its initial position. Suppose first that the blocks are lined up in a row. Move the empty space so that it lies either directly above or directly below the rightmost of the three blocks. If it lies below, one cyclic permutation is obtained by the process illustrated in Figures 6 to 9, and the other can be gotten by repeating this process. When this is done, take the empty space back along the route it came, and the rest of the puzzle will return to the initial position. If the empty space lies above, the process to be used is that of Figures 6 to 9 turned upside down. If the blocks are lined up in a column, rotate the puzzle through 90° so that they lie along a row and proceed as above.

Now we show that any three blocks can be permuted cyclically while return-

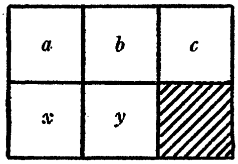

FIG. 6.

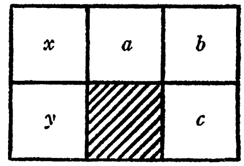

FIG. 7.

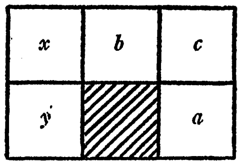

FIG. 8.

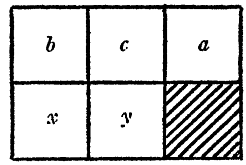

FIG. 9.

ing the rest of the puzzle to its initial position. Let the three blocks to be moved be numbered a, b, c. By 3-cycle permutations in rows and columns, we can move block a to the first row, block b to the second row, and block c to the third row. Then we can move blocks a, b, c over to the column on the extreme right. There they can be permuted cyclically, so that block b replaces a, block c replaces b, and block a replaces c. Then we can reverse the process which brought the blocks to the right hand column, so that block b moves back until it fits into a's initial position, c fits into b's initial position, and a fits into c's initial position. The rest of the blocks have, of course, returned to their initial positions.

Since we can achieve these 3-cycle permutations starting from any initial position, it follows that the set of all configurations which can be obtained from Figure 1 by sliding the blocks and returning the empty space to the lower right hand corner must contain the set of all products of 3-cycles, which is a subgroup of A_{15}. Now every conjugate of the 3-cycle $(1, 2, 3)$ is a 3-cycle, and so this latter subgroup contains the subgroup consisting of all products of conjugates of $(1, 2, 3)$. But this subgroup is normal in A_{15} and so must be all of A_{15}.

Everything which we have done here, with the possible exception of the last two sentences, depends only on results commonly taught in modern algebra courses at the undergraduate level. These two sentences can be replaced as follows. Instead of using the fact that A_{15} has no proper normal subgroups, we prove and use a key lemma from one of the proofs of this fact. We know that the set of configurations attainable from Figure 1 contains the set of all products of 3-cycles. Thus all we have to do is show that every even permutation is a product of 3-cycles. Write out the permutation as a product of an even number of transpositions. It then suffices to show that every product of two transpositions can be written as a product of 3-cycles. If the product consists of two transpositions with a letter in common, we have $(a, b)\ (b, c) = (a, c, b)$. If the product consists of two disjoint transpositions, we have $(a, b)(c, d) = (a, c, b)(b, d, c)$.

Now that we know what rearrangements of Figure 1 we can obtain with the blank space in the lower right hand corner, it is easy to determine whether a given configuration can be achieved, by mentally sliding the blocks so as to put the blank space in the lower right hand corner.

5. Two questions. While the preceding section gives a practical method of proof, that method is not very practical to use if we wish to slide the puzzle into a particular arrangement in a short amount of time. If we define a *move* to mean sliding a block into the blank space next to it, the fewer moves one takes, the more efficient he is at working the puzzle. We can therefore ask: What is the greatest number of moves that will ever be necessary to obtain an attainable configuration from Figure 1? A fairly crude calculation shows that more than 241 moves will never be necessary but the exact number is probably considerably below this.

A second question is: Is it possible to go through the entire list of $\frac{1}{2}\cdot 16!$ configurations attainable without ever repeating one? This would be the most efficient way of obtaining all attainable configurations, since precisely $\frac{1}{2}\cdot 16!$ moves would be used. A check on the time involved shows, however, that if such a thing is possible, it is not very practical. A variant of this second question which should be more vulnerable to analysis is: If we consider only those configurations with the blank space in the lower right hand corner, is it possible to go through the entire list of these without repeating one?

References

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SPACELAND; AS VIEWED INFORMALLY FROM THE FOURTH DIMENSION

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Purpose. This note presents a setting for an informal talk at the undergraduate level, hopefully leading to a lively discussion of n-dimensional geometry. For those students who wish to pursue the topic at a more sophisticated level, we suggest treatises by either Coxeter [2], including historical notes and a group theory approach, or Sommerville [3].

Introduction. In a delightful little book [1] by Edwin A. Abbott, we are given an insight into the two-dimensional world of the Flatlander. Introducing a time reference, thus putting Flatland in a third dimension, we make some simple observations that relate certain seemingly different objects. With this in mind we take a retrospective look at our own universe, that of three-dimensional space and time.

Employing the same arguments that we feel should be effective in Flatland, we suggest some intriguing ideas and hope to tempt the reader into further investigation. We conclude with the thought that, fascinating and strange as these observations might be, we suspect that a four-dimensional creature might classify them as "obvious."