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The Fifteen Puzzle—A New Approach

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You probably have seen the 15-puzzle before. This puzzle, also called the Gem Puzzle, Boss Puzzle, Game of Fifteen, Mystic Square, and other names, consists of sliding square tiles numbered 1 to 15 that occupy all but one cell of a 4×4 box (Figure 1). The basic goal is to use the blank cell to slide the square tiles so that the final arrangement is the natural order of the numbers 1 to 15. Do you know how to solve it? Perhaps a trial and error approach might work. Starting from an arbitrary initial arrangement, is it possible to arrive at the natural order? Have you seen an algorithm that you could use for any initial position? This puzzle has a rich history. Sam Loyd claimed that he was the inventor of the puzzle but it may have been invented by Noyes Chapman, a postmaster in Canastota, New York.

Only half of the possible 15! initial arrangements can be restored to the natural order but that was not evident when this puzzle swept the United States and other parts of the world in the 1880's. The major challenge was to achieve the goal starting from a near perfect initial array in which the last row contained 13, 15, 14 in that order—the only difference being the permutation of the last two adjacent tiles. This challenge gripped the people of all ages and walks of life. Sam Loyd even announced a prize of 1000 US dollars for the solver of this challenge.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**14**	**15**	

Figure 1 The 15-puzzle.

It can be proved that of the 15! possible initial arrangements, half of them can be restored to the natural order and the other half to the order in which the first three rows are in natural order and the last row is 13-15-14 (see [**1, 2, 5, 6, 7, 10**]). In particular, the 13-15-14 arrangement cannot be restored to the natural order.

Almost all the proofs use the theory of the alternating group A_{15} but do not exhibit a set of moves to arrive at one of the above two possible arrangements from a given initial arrangement.

There are also several algorithms for solving the puzzle. Even for the general $N \times N$ board, one can find a solution using heuristics, but finding the optimal number of moves in which the general puzzle can be solved is known to be an NP-hard problem (*e.g.*, [**8, 9**]).

In this paper we give a simpler, elementary (nongroup-theoretic) proof that exactly half of the 15! arrangements can be restored to the natural order. We also describe

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an algorithm to restore any arrangement into one of the two arrangements mentioned above. The key strategy we use is “divide and conquer.” We reduce the problem to the 2×4 board and use the results proved for this board to derive the main theorem. Our approach generalizes to the $N \times N$ board as well.

2×4 Puzzle We first consider the 2×4 puzzle in which we arrange the numbers $1, 2, \ldots, 7$ in some order and study which of those arrangements can be restored to the natural order. With any arrangement, we associate a sequence formed by following the arrow in Figure 2. For example, for the arrangement shown in Figure 2, the sequence

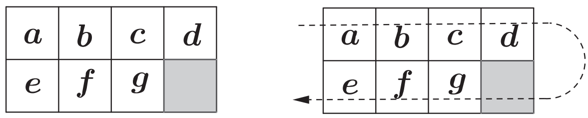

Figure 2 2×4 board and listing order.

associated is (a, b, c, d, g, f, e). Note that we ignore the blank space in forming the sequence.

In any sequence we say that an *inversion* occurs each time a larger number precedes a smaller number. For example, in the sequence $(1, 2, 3, 4, 7, 6, 5)$, the number of inversions is 3 and in $(7, 6, 5, 4, 3, 2, 1)$, the number of inversions is 21.

An arrangement is said to have *even parity* if the corresponding sequence has an even number of inversions and *odd parity* if the sequence has odd number of inversions. Inversions hold the key to determine which arrangements can be restored to the natural order. We prove the following theorem.

Theorem 1. *An arrangement of the numbers in the* 2×4 *board can be restored to the natural order if and only if the arrangement has odd parity.*

We begin with a lemma.

Lemma 1. *If two adjacent numbers in a sequence are interchanged, the resulting sequence has the opposite parity of the original sequence.*

Proof. Suppose that in the sequence $s_1 = (\ldots, a, b, \ldots)$ we interchange a, b to get the sequence $s_2 = (\ldots, b, a, \ldots)$. If $a < b$, then s_2 has one more inversion caused by the larger number b preceding the smaller a, since all other inversions of the sequence s_1 are preserved in s_2. Thus s_2 has opposite parity to the parity of s_1. Also, if $a > b$, the inversion in s_1 caused by the larger a preceding the smaller b is removed in s_2 by the interchange. Thus s_2 has one less inversion than s_1. Thus again, s_2 has opposite parity to the parity of s_1. ■

As a corollary, we make the following observation. The sequences

$$s_1 = (\ldots, a, x_1, x_2, \ldots, x_r, \ldots),$$
$$s_2 = (\ldots, x_1, x_2, \ldots, x_r, a, \ldots)$$

have the same parity if r is even and have opposite parity if r is odd.

This is clear from the lemma, since we can obtain s_2 by interchanging a with $x_1, x_2, \ldots, x_r$, in that order. Since each interchange reverses the parity, there are r reversals. We first prove that the legal moves do not change the parity.

Lemma 2. *Any legal move does not change the parity of the arrangement.*

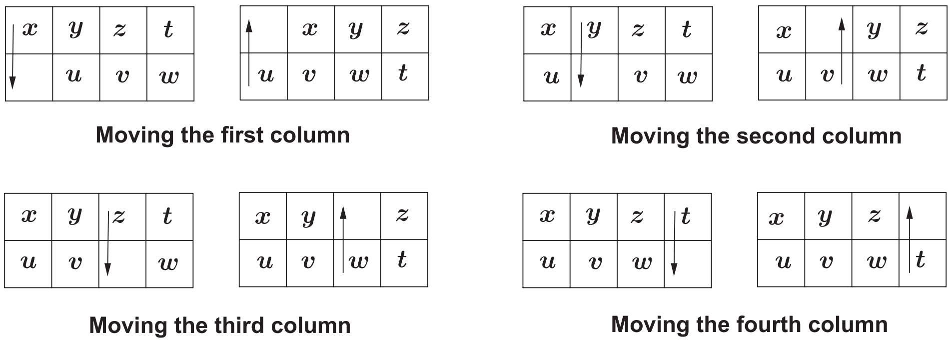

Figure 3 Vertical moves.

Proof. Any horizontal shift does not change the sequence and hence the parity of the sequence is not affected.

Consider the vertical moves (Figure 3).

When we move the x in the first column, the sequence (x, y, z, t, w, v, u) becomes (y, z, t, w, v, u, x) and hence x moves to the final position after 6 interchanges. Hence by the observation above, the resulting sequence has the same parity as the original sequence.

When we move u in the first column, second row to the first column, first row, the sequence changes from (x, y, z, t, w, v, u) to (u, x, y, z, t, w, v) and again u is interchanged six times. Thus in this case also the parity of the sequence does not change.

Similarly, when we move a symbol in the second column, the number of interchanges is 4 and hence the parity is maintained.

When we move a symbol in the third column, the number of interchanges required is 2 and finally, moving a symbol in the last column does not change the sequence at all. Thus in all cases the parity of the sequence is maintained. This completes the proof of the lemma. ■

We now introduce some basic moves that play a key role in the proof of Theorem 1.

$S(\pm n)$: In this move, we slide the symbols either clockwise or anticlockwise so that each symbol moves by n cells. The moves $S(+1)$ and $S(-1)$ are shown in Figure 4.

R_1: This move allows for three adjacent symbols to cycle in a row under certain conditions (as shown in Figure 5). The proof that R_1 can be obtained by a sequence of legal moves is shown in Figure 6.

E_1, E_2: These moves interchange two adjacent columns, as shown in Figure 7. E_1 can be obtained as follows. (See Figure 8.) Shift e, move b down, apply $S(-2)$, push d down, and move a, c anticlockwise. The legal moves constituting E_2 are shown in Figure 9.

T: This move shifts elements in a triangular cycle among two adjacent rows. For example, in Figure 10 we shift a from the top row to the bottom row and shift d from the bottom row to the top row. Figure 11 shows how this can be accomplished using legal moves.

R_2: This moves three elements in the bottom row cyclically, as shown in Figure 12. The legal moves leading to R_2 are given in Figure 13.

Now we prove Theorem 1. Start with an arbitrary arrangement of $1, 2, \ldots, 7$ in the 2×4 board, as given in Figure 14. By using the moves T and R_1, we see that any symbol in the first row can be interchanged with the first symbol in the second row, with no other symbols switching rows (although their order within the row may change). Again, using E_1 and E_2, we can bring any symbol in the second row to the

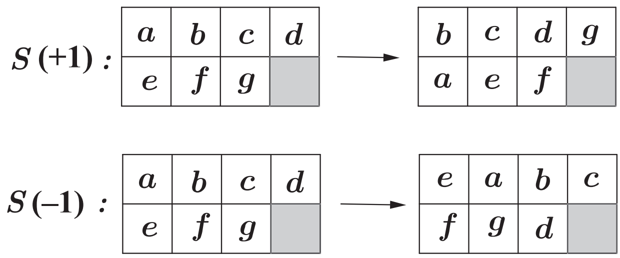

Figure 4 Slide move $S(\pm 1)$.

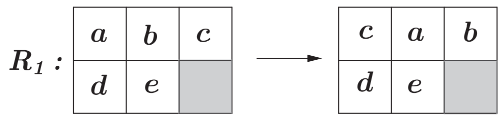

Figure 5 Move R_1.

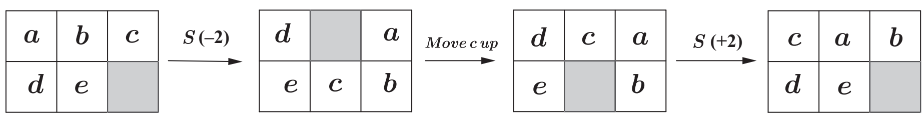

Figure 6 Legal moves for R_1.

first cell in the second row (again without impacting the set of symbols in the individual rows) and hence it follows that we can exchange any pair of symbols between the first and second rows. Thus we can bring the symbols 1, 2, 3, 4 to the first row and 5, 6, 7 to the second row. Now using E_1, E_2, we can assume that 4 is in the first row, fourth column. Using R_2, we can assume that the second row is either 5, 6, 7 in that order or 5, 7, 6. Thus we have reached one of the two arrangements A_1, A_2 in Figure 15 where (x, y, z) is 1, 2, 3 in some order. Suppose that the initial arrangement had odd parity, and we reached the A_1 arrangement. The parity of the sequence $(x, y, z, 4, 7, 6, 5)$ must also be odd and since $(4, 7, 6, 5)$ has three inversions, it follows that (x, y, z) must have an even number of inversions. Hence (x, y, z) must be one of $(1, 2, 3)$, $(3, 1, 2)$, or $(2, 3, 1)$. Now, using R_1, we can move this arrangement to the natural order.

Now suppose that we reach the arrangement A_2. Since $(4, 6, 7, 5)$ has two inversions, the sequence (x, y, z) must have an odd number of inversions. Thus it must be one of $(1, 3, 2)$, $(2, 1, 3)$, or $(3, 2, 1)$. The $(1, 3, 2)$ case is already solved (we use E_1 to interchange the second and third columns and we obtain the natural order). For $(2, 1, 3)$, first use E_1 to interchange second and third columns and then R_1 to obtain the natural order. For $(3, 2, 1)$, we use R_1 to change the first row to $(1, 3, 2)$ and use E_1

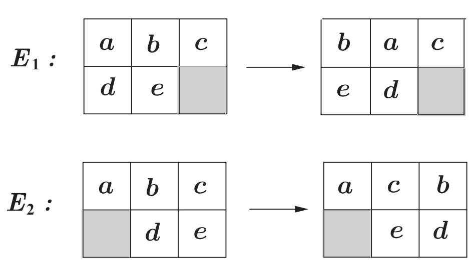

Figure 7 Moves E_1 and E_2.

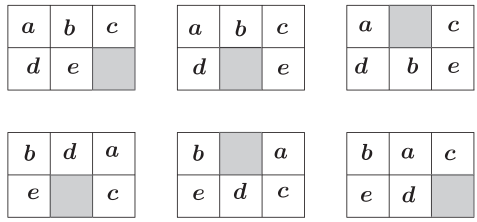

Figure 8 Legal moves for E_1.

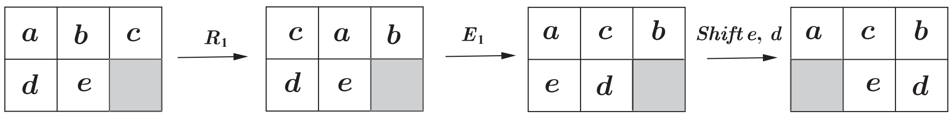

Figure 9 Legal moves for E_2.

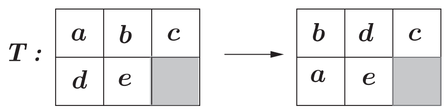

Figure 10 Move T.

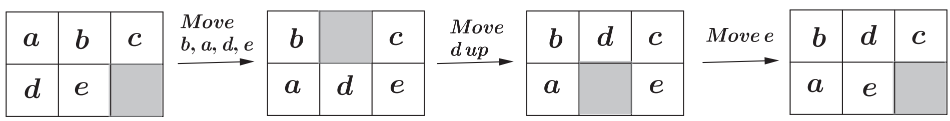

Figure 11 Legal moves for T.

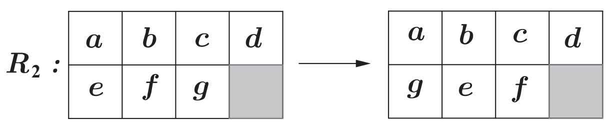

Figure 12 Move R_2.

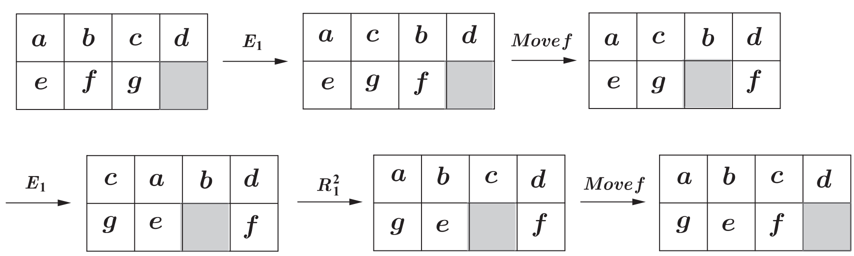

Figure 13 Legal moves for R_2.

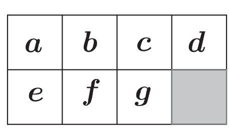

Figure 14 Starting arrangement.

x	y	z	4
5	6	7	

A_1

x	y	z	4
5	7	6	

A_2

Figure 15 Proof of Theorem 1.

to interchange the second and third columns. This results in the natural order. Hence if we start with an arrangement with odd parity, we can reach the natural order.

A similar argument shows that if we start with an arrangement with even parity we reach the arrangement in which the first row is 1, 2, 3, 4 in that order and the second row is 5, 7, 6 in that order. Thus we have proved the following theorem.

Theorem 2. *Starting with any arrangement on the* 2×4 *board, we reach the arrangements* A_{nat} *or* A_{rev} *(Figure 16) when the parity of the original arrangement is odd or even, respectively.*

1	2	3	4
5	6	7	

A_{nat}

1	2	3	4
5	7	6	

A_{rev}

Figure 16 Final positions for 2×4 board.

4 × 4 Puzzle As in the 2×4 case, we first choose a suitable listing order and associate a sequence with any arrangement in such a way that any legal move does not alter the parity of the associated sequence. For the 4×4 board, we choose the listing sequence by following the path shown in Figure 17. With this listing order, the

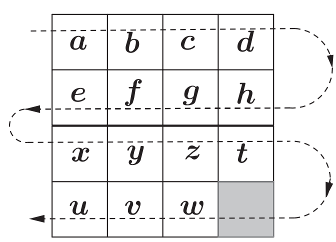

Figure 17 Listing order.

horizontal moves do not change the sequence. For the vertical moves, there are those that do not change the sequence. These are shown in Figure 18. The other vertical moves of an arbitrary element u will move u two, four, or six places in the sequence, as shown in Figures 19, 20, and 21, respectively. In all the cases, the symbols are shifted in the sequence by an even number of places and hence the parity of the sequence is not altered.

Now we are ready to prove the following theorem.

Theorem 3. *An arrangement of the numbers in the* 4×4 *board can be restored to the natural order if and only if the arrangement has odd parity.*

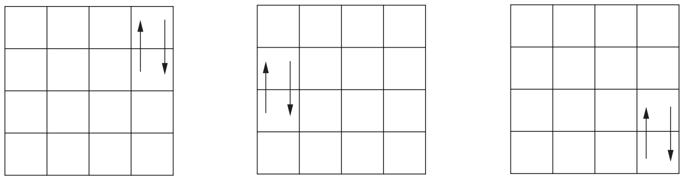

Figure 18 Moves that do not change the sequence.

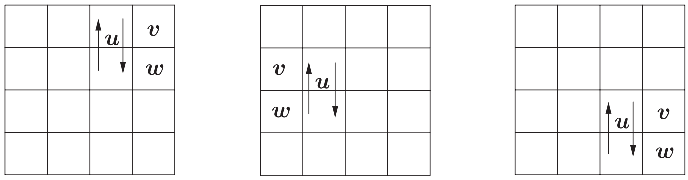

Figure 19 Moves that shift a symbol by 2 places.

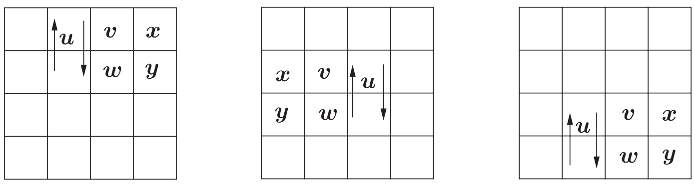

Figure 20 Moves that shift a symbol by 4 places.

The "only if" part is obvious—if the arrangement can be restored to the natural order using legal moves, then it necessarily has odd parity since the sequence associated with the natural order

$$(1, 2, 3, 4, 8, 7, 6, 5, 9, 10, 11, 12, 15, 14, 13)$$

has 9 inversions and parity is not changed by legal moves.

Now suppose that we start with an arrangement with odd parity. Let us call the 2×4 board consisting of the first and second rows of the board B_{top} and the 2×4 board consisting of the third and fourth rows B_{bot}. Using the move T repeatedly, we can move all numbers $1, 2, 3, \dots, 8$ to B_{top} and the numbers $9, 10, \dots, 15$ to the B_{bot}. Again through a combination of the moves E_1, E_2, R_1, R_2, we can assume that 4 is at the first row, fourth column, 8 is at the second row, fourth column, and 12 is at the third row, fourth column with blank occupying the last row, last column. Now since the parity of the board is odd, we have two cases to consider:

1. B_{top} has odd parity, B_{bot} has even parity, and
2. B_{top} has even parity, B_{bot} has odd parity.

1. B_{top} has odd parity, B_{bot} has even parity: Bring the blank cell to the second row,

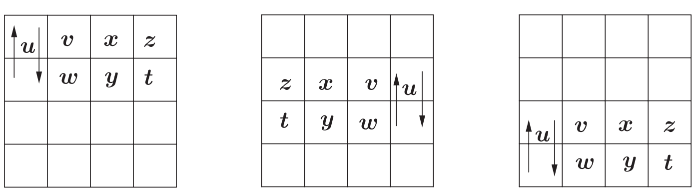

Figure 21 Moves that shift a symbol by 6 places.

fourth column by pushing 12 to the fourth row, fourth column and 8 to third row, fourth column. Since B_{top} has odd parity, the sequence $(a, b, c, 4, 8, x, y, z)$ has odd parity. Since 8 is involved in three inversions (all x, y, z are less than 8), the sequence $(a, b, c, 4, x, y, z)$ has even parity. Thus by Theorem 2, we can bring the board B_{top} to the arrangement A_2 (Figure 22). Now move 8 back to second row, fourth column, 12 to third row, fourth column, and consider the bottom board B_{bot}. Since this has even parity, again by Theorem 2, we can bring the board B_{bot} to the arrangement A'_2 (Figure 22). Now, applying the move E_1 in A'_2, we can change it to A''_2 (Figure 23). Stitching together A_2 and A''_2, and moving 12, we obtain the arrangement in A_3. Now, applying the move E_1 to the second and third rows, we obtain the natural order. This completes proof for this case.

1	**2**	**3**	**4**
5	**7**	**6**	

A_2

9	**10**	**11**	**12**
13	**15**	**14**	

A'_2

Figure 22 B_{top} and B_{bot} arrangement.

9	**11**	**10**	**12**
13	**14**	**15**	

A''_2

1	**2**	**3**	**4**
5	**7**	**6**	**8**
9	**11**	**10**	
13	**14**	**15**	**12**

A_3

Figure 23 B_{bot} arrangement and full board.

2. B_{top} has even parity, B_{bot} has odd parity

   Since B_{top} has even parity, the sequence $(a, b, c, 4, 8, x, y, z)$ has even parity. Since 8 is involved in three inversions (all x, y, z are less than 8), the sequence $(a, b, c, 4, x, y, z)$ has odd parity. Sliding 8 and 12 down, applying Theorem 2, and sliding 8 and 12 up show that we can bring the board B_{top} to the natural order. Also, since B_{bot} has odd parity, it can be brought to the natural order by Theorem 2. Now, stitching these together, we obtain the natural order on $1, 2, \ldots, 15$.

A similar argument proves the following result.

Theorem 4. *An arrangement of the numbers in the* 4×4 *board can be restored to the order* $13 - 15 - 14$ *if and only if the arrangement has even parity.*

The above proof also gives an algorithm for restoring the board to one of the two arrangements. The steps are as follows:

1. Move $1, 2, \ldots, 8$ to the top two rows, bring 4, 8, and 12 into their proper positions, slide 8 and 12 down, bring $1, 2, \ldots, 7$ into the natural order or $5 - 7 - 6$ order, slide 8 and 12 back up.
2. (a) top board is in natural order, bottom board is in natural order: the full board is in natural order

(b) top board is in natural order, bottom board is in $13 - 15 - 14$ order: the full board is in $13 - 15 - 14$ order

(c) top board is in $5 - 7 - 6$ order, bottom board is in natural order: Use E_1 in second and third rows and again in third and fourth rows. The board will be in $13 - 15 - 14$ order

(d) top board is in $5 - 7 - 6$ order, bottom board is in $13 - 15 - 14$ order: Use E_1 in second and third rows and again in third and fourth rows. The board will be in natural order.

The above algorithm may not yield the optimal number of moves required to restore the board. It is known that the 15-puzzle can be solved in maximum of 80 single tile moves [**11**].

In general, permutation puzzles (*e.g.,* [**3, 4, 6**]) are interesting and any discussion of them involves some amount of group theory. Another famous permutation puzzle, Rubik's cube, has been studied extensively and there are algorithms for solving the cube. The usual proofs in the literature for the 15-puzzle are all existence proofs—that is, using the theory of permutation groups, one shows that there exists a set of moves—without explicitly describing the moves—to restore any starting arrangement to one of the two arrangements mentioned in the beginning of this paper. An arrangement is first mapped to a permutation in the symmetric group S_{15} on 15 symbols (the blank cell is assumed to be at the 16th place). One shows that an arrangement can be restored to the natural order of the numbers 1 to 15 if and only if the mapped permutation is even. The proof uses the fact that the subgroup A_{15} of even permutations is generated by the 3-cycles [**1**]. For the 15-puzzle, there are algorithms [**6, 10**] but none of the published algorithms use a divide and conquer approach. They also appear to be specific to 4×4 board. The proof given in this paper provides a better intuition for solving the puzzle and also extends to the more general $N \times N$ board.

Generalization It is easy to see that the above proof can be modified to prove that for any board of size $m \times n$, two arrangements can be reached from one another if and only if they have the same parity.

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Summary. We give an elementary, nongroup theoretic proof that exactly half of the 15! arrangements of the fifteen puzzle can be restored to the natural order.

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1 A	2 D	3 O	4 S			5 S	6 N	7 A	8 P			9 S	10 P	11 I	12 N
13 W	I	S	P		14 B	O	O	B	O	15 O		16 I	O	T	A
17 O	V	A	L		18 O	N	T	A	P	E		19 D	I	S	H
20 L	O	G	I	21 S	T	I	C	C	U	R	22 V	E			
23 S	T	E	N	O		24 C	H	I	P		25 O	V	26 I	27 N	28 E
			29 E	D	30 U				31 M	32 A	X	I	M	A	L
33 P	34 T	35 A		36 O	N	37 E	38 O	39 V	E	R		40 E	M	I	L
41 L	O	N	42 G	I	T	U	D	I	N	A	43 L	43 W	A	V	E
44 O	L	G	A		45 I	C	E	C	U	B	E		46 D	E	N
47 T	E	L	L	48 A	L	L				49 S	I	50 N			
51 S	T	E	I	N		52 I	53 D	54 E	55 M		56 C	O	57 S	58 E	59 C
			60 L	O	61 G	D	E	R	I	62 V	A	T	I	V	E
63 M	64 A	65 Z	E		66 N	E	B	U	L	A		67 N	E	E	D
68 A	R	E	A		69 C	A	R	P	E	T		70 O	G	R	E
71 V	E	N	N			72 N	A	T	S			73 W	E	T	S

SOLUTION TO PINEMI PUZZLE

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