To show that $n!$ is not in $O(2^n)$, we want to show that the limit of $\frac{n!}{2^n}$ as n goes to infinity is infinity.

We note that

$$\frac{n!}{2^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{2 \cdot 2 \cdot 2 \cdots 2 \cdots 2} = \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{n-1}{2}\right)\left(\frac{n}{2}\right) > \frac{n}{4}$$

since all the factors between the first and the last are greater than or equal to 1. Thus, since $n/4$ goes to infinity as n goes to infinity, so does $\frac{n!}{2^n}$, because it's bigger.

Exercise 4 [4] (target due date: August 31)

Your job on this Exercise is to consider the functions $f(n) = n^3 2^n$ and $g(n) = n^2 3^n$ and compare their efficiency categories.

Try to prove $f \in O(g)$ and $g \in O(f)$. Decide which of these is true, and whether both are true, and provide fairly formal proofs of whatever you determine.

If you prove both of these (meaning f and g are in the same Θ category), you will be done. Otherwise, use a limit argument to prove that one of them is not in big-O of the other.

You might want to start by experimenting—computing some values of the these functions to arrive at a conjecture as to how their growth rates compare.

$$\frac{\partial}{\partial x} f(n) = \ln(3) \cdot 6^x + \ln(2) \cdot 6^x \quad = \quad \frac{\partial}{\partial x} g(n) = 2x \cdot 3^x + 3^x \ln(3) x^2$$

$$\lim_{n\to\infty}\left(\frac{f(n)'}{g(n)'}\right) = \frac{\ln(3) \cdot 6^x + \ln(2) \cdot 6^x}{2x \cdot 3^x + 3^x \ln(3) x^2} = \frac{\infty}{\infty}$$

$$L'H = \frac{\frac{\partial}{\partial x}\left(\ln(3) \cdot 6^x + \ln(2) \cdot 6^x\right)}{\frac{\partial}{\partial x}\left(2x \cdot 3^x + 3^x \ln(3) x^2\right)} = \frac{\ln(3) \cdot 6^x \ln(2) \cdot 6^x + \ln(2) \cdot 6^x \ln(6)}{2(3^x + \ln(3) \cdot 3^x x) + 3(\ln(3) x^{\ln(3) x + 2}(\ln(x)+1) + 2 \cdot 3^{x \ln(x)} x)}$$

$$= \frac{\infty}{\infty}$$

$$\frac{n^3 2^n}{n^2 3^n} = \frac{n^3}{n^2} \cdot \left(\frac{2}{3}\right)^n$$

$$= n \cdot 0.66\overline{6}^n$$

$$\boxed{\begin{array}{l} \forall n \geq 1, \; n^3 2^n \notin \Theta(n^2 3^n) \\ \forall n \geq 1, \; n^2 3^n \notin \Theta(n^3 2^n) \end{array}}$$