Derivatives

$D_x e^x = e^x$
$D_x \sin(x) = \cos(x)$
$D_x \cos(x) = -\sin(x)$
$D_x \tan(x) = \sec^2(x)$
$D_x \cot(x) = -\csc^2(x)$
$D_x \sec(x) = \sec(x)\tan(x)$
$D_x \csc(x) = -\csc(x)\cot(x)$
$D_x \sin^{-1} = \frac{1}{\sqrt{1-x^2}}, x \in [-1,1]$
$D_x \cos^{-1} = \frac{-1}{\sqrt{1-x^2}}, x \in [-1,1]$
$D_x \tan^{-1} = \frac{1}{1+x^2}$
$D_x \sec^{-1} = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
$D_x \sinh(x) = \cosh(x)$
$D_x \cosh(x) = -\sinh(x)$
$D_x \tanh(x) = sech^2(x)$
$D_x \coth(x) = -csch^2(x)$
$D_x sech(x) = -sech(x)\tanh(x)$
$D_x csch(x) = -csch(x)\coth(x)$
$D_x \sinh^{-1} = \frac{1}{\sqrt{x^2+1}}$
$D_x \cosh^{-1} = \frac{-1}{\sqrt{x^2-1}}, x > 1$
$D_x \tanh^{-1} = \frac{1}{1-x^2} - 1 < x < 1$
$D_x sech^{-1} = \frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$
$D_x \ln(x) = \frac{1}{x}$

Integrals

$\int \frac{1}{x} dx = \ln|x| + c$
$\int e^x dx = e^x + c$
$\int a^x dx = \frac{1}{\ln a} a^x + c$
$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$
$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + c$
$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}(x) + c$
$\int \sinh(x) dx = \cosh(x) + c$
$\int \cosh(x) dx = \sinh(x) + c$
$\int \tanh(x) dx = \ln|\cosh(x)| + c$
$\int \tanh(x) sech(x) dx = -sech(x) + c$
$\int sech^2(x) dx = \tanh(x) + c$
$\int csch(x)\coth(x) dx = -csch(x) + c$
$\int \tan(x) dx = -\ln|\cos(x)| + c$
$\int \cot(x) dx = \ln|\sin(x)| + c$
$\int \cos(x) dx = \sin(x) + c$
$\int \sin(x) dx = -\cos(x) + c$
$\int \frac{1}{\sqrt{a^2-u^2}} dx = \sin^{-1}(\frac{u}{a}) + c$
$\int \frac{1}{a^2+u^2} dx = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$

U-Substitution
Let $u = f(x)$ (can be more than one variable).
Determine: $du = \frac{f(x)}{dx} dx$ and solve for dx.
Then, if a definite integral, substitute the bounds for $u = f(x)$ at each bounds
Solve the integral using u.

Integration by Parts
$\int u dv = uv - \int v du$

Fns and Identities

$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$
$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$
$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$
$\tan(\sec^{-1}(x))$
$= (\sqrt{x^2-1} \text{ if } x \geq 1)$
$= (-\sqrt{x^2-1}\, if\, x \leq -1)$
$\sinh^{-1}(x) = \ln x + \sqrt{x^2+1}$
$\sinh^{-1}(x) = \ln x + \sqrt{x^2-1},\ x \geq -1$
$\tanh^{-1}(x) = \frac{1}{2}\ln x + \frac{1+x}{1-x},\ 1 < x < -1$
$sech^{-1}(x) = \ln[\frac{1+\sqrt{1-x^2}}{x}],\ 0 < x \leq -1$
$\sinh(x) = \frac{e^x - e^{-x}}{2}$
$\cosh(x) = \frac{e^x + e^{-x}}{2}$

Trig Identities

$\sin^2(x) + \cos^2(x) = 1$
$1 + \tan^2(x) = \sec^2(x)$
$1 + \cot^2(x) = \csc^2(x)$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$
$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$
$\cosh(n^2 x) - \sinh^2 x = 1$
$1 + \tan^2(x) = \sec^2(x)$
$1 + \cot^2(x) = \csc^2(x)$
$\sin^2(x) = \frac{1-\cos(2x)}{2}$
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$
$\sin(-x) = -\sin(x)$
$\cos(-x) = \cos(x)$
$\tan(-x) = -\tan(x)$

Calculus 3 Concepts

Cartesian coords in 3D

given two points:
(x_1, y_1, z_1) and (x_2, y_2, z_2),
Distance between them:
$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$
Midpoint:
$(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$
Sphere with center (h,k,l) and radius r:
$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Vectors

Vector: $\vec{u}$
Unit Vector: $\hat{u}$
Magnitude: $||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
Unit Vector: $\hat{u} = \frac{\vec{u}}{||\vec{u}||}$

Dot Product
$\vec{u} \cdot \vec{v}$
Produces a Scalar
(Geometrically, the dot product is a vector projection)
$\vec{u} = < u_1, u_2, u_3 >$
$\vec{v} = < v_1, v_2, v_3 >$
$\vec{u} \cdot \vec{v} = \vec{0}$ means the two vectors are Perpendicular θ is the angle between them.
$\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}||\cos(\theta)$
$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$
NOTE:
$\hat{u} \cdot \hat{v} = \cos(\theta)$
$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$
$\vec{u} \cdot \vec{v} = 0$ when $\perp$
Angle Between $\vec{u}$ and $\vec{v}$:
$\theta = \cos^{-1}(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||})$
Projection of $\vec{u}$ onto $\vec{v}$:
$pr_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||^2}$

Cross Product
$\vec{u} \times \vec{v}$
Produces a Vector
(Geometrically, the cross product is the area of a paralellogram with sides $||\vec{u}||$ and $||\vec{v}||$)
$\vec{u} = < u_1, u_2, u_3 >$
$\vec{v} = < v_1, v_2, v_3 >$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\vec{u} \times \vec{v} = \vec{0}$ means the vectors are paralell

Equation of a Plane

(x_0, y_0, z_0) is a point on the plane and $< A, B, C >$ is a normal vector

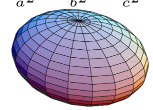
$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$
$< A, B, C > \cdot <$
$x - x_0, y - y_0, z - z_0 >= 0$
$Ax + By + Cz = D$ where
$D = Ax_0 + By_0 + Cz_0$

Coord Sys Conv

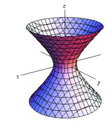
Cylindrical to Rectangular
$x = r\cos(\theta)$
$y = r\sin(\theta)$
$z = z$
Rectangular to Cylindrical
$r = \sqrt{x^2 + y^2}$
$\tan(\theta) = \frac{y}{x}$
$z = z$
Spherical to Rectangular
$x = \rho\sin(\phi)\cos(\theta)$
$y = \rho\sin(\phi)\sin(\theta)$
$z = \rho\cos(\phi)$
Rectangular to Spherical
$\rho = \sqrt{x^2 + y^2 + z^2}$
$\tan(\theta) = \frac{y}{x}$
$\cos(\phi) = \frac{z}{\sqrt{x^2+y^2+z^2}}$
Spherical to Cylindrical
$r = \rho\sin(\phi)$
$\theta = \theta$
$z = \rho\cos(\phi)$
Cylindrical to Spherical
$\rho = \sqrt{r^2 + z^2}$
$\theta = \theta$
$\cos(\phi) = \frac{z}{\sqrt{r^2+z^2}}$

Surfaces

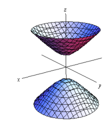
Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



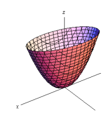
Hyperboloid of One Sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
(Major Axis: z because it follows -)



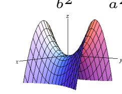
Hyperboloid of Two Sheets
$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
(Major Axis: Z because it is the one not subtracted)



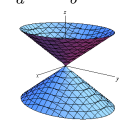
Elliptic Paraboloid
$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
(Major Axis: z because it is the variable NOT squared)



Hyperbolic Paraboloid
(Major Axis: Z axis because it is not squared)
$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$



Elliptic Cone
(Major Axis: Z axis because it's the only one being subtracted)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$



Cylinder
1 of the variables is missing
OR
$(x-a)^2 + (y-b^2) = c$
(Major Axis is missing variable)

Partial Derivatives

Partial Derivatives are simply holding all other variables constant (and act like constants for the derivative) and only taking the derivative with respect to a given variable.
Given z=f(x,y), the partial derivative of z with respect to x is:
$f_x(x,y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$
likewise for partial with respect to y:
$f_y(x,y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$
Notation
For f_{xyy}, work "inside to outside" f_x then f_{xy}, then f_{xyy}
$f_{xyy} = \frac{\partial^3 f}{\partial x \partial^2 y}$,
For $\frac{\partial^3 f}{\partial x \partial^2 y}$, work right to left in the denominator

Gradients

The Gradient of a function in 2 variables is $\nabla f = < f_x, f_y >$
The Gradient of a function in 3 variables is $\nabla f = < f_x, f_y, f_z >$

Chain Rule(s)

Take the Partial derivative with respect to the first-order variables of the function times the partial (or normal) derivative of the first-order variable to the ultimate variable you are looking for summed with the same process for other first-order variables this makes sens for.
Example:

let x = x(s,t), y = y(t) and z = z(x,y).
z then has first partial derivative:
$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
x has the partial derivatives:
$\frac{\partial x}{\partial s}$ and $\frac{\partial x}{\partial t}$
and y has the derivative:
$\frac{dy}{dt}$
In this case (with z containing x and y as well as x and y both containing s and t), the chain rule for $\frac{\partial z}{\partial s}$ is
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$
The chain rule for $\frac{\partial z}{\partial t}$ is
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}$
Note: the use of "d" instead of "∂" with the function of only one independent variable

Limits and Continuity

Limits in 2 or more variables
Limits taken over a vectorized limit just evaluate separately for each component of the limit.
Strategies to show limit exists
1. Plug in Numbers, Everything is Fine
2. Algebraic Manipulation
-factoring/dividing out
-use trig identites
3. Change to polar coords

$if(x,y) \to (0,0) \Leftrightarrow r \to 0$
Strategies to show limit DNE
1. Show limit is different if approached from different paths
(x=y, $x = y^2$, etc.)
2. Switch to Polar coords and show the limit DNE.

Continuity
A fn, $z = f(x,y)$, is continuous at (a,b) if
$f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$
Which means:
1. The limit exists
2. The fn value is defined
3. They are the same value

Directional Derivatives
Let z=f(x,y) be a fuction, (a,b) ap point in the domain (a valid input point) and $\hat{u}$ a unit vector (2D).
The Directional Derivative is then the derivative at the point (a,b) in the direction of $\hat{u}$ or:
$D_{\vec{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$
This will return a *scalar.* 4-D version:
$D_{\vec{u}} f(a,b,c) = \hat{u} \cdot \nabla f(a,b,c)$

Tangent Planes
let F(x,y,z) = k be a surface and P = (x_0, y_0, z_0) be a point on that surface.
Equation of a Tangent Plane:
$\nabla F(x_0, y_0, z_0) \cdot < x - x_0, y - y_0, z - z_0 >$

Approximations
let $z = f(x,y)$ be a differentiable function total differential of f = dz
$dz = \nabla f \cdot < dx, dy >$
This is the *approximate* change in z
The actual change in z is the difference in z values:
$\Delta z = z - z_1$

Maxima and Minima
Internal Points
1. Take the Partial Derivatives with respect to X and Y (f_x and f_y) (Can use gradient)
2. Set derivatives equal to 0 and use to solve system of equations for x and y
3. Plug back into original equation for z.

Use Second Derivative Test for whether points are local max, min, or saddle

Second Partial Derivative Test
1. Find all (x,y) points such that $\nabla f(x,y) = \vec{0}$
2. Let $D = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y)$
IF (a) D > 0 AND $f_{xx} < 0$, f(x,y) is local max value
(b) D > 0 AND $f_{xx}(x,y) > 0$ f(x,y) is local min value
(c) D < 0, (x,y,f(x,y)) is a saddle point
(d) D = 0, test is inconclusive
3. Determine if any boundary point gives min or max. Typically, we have to parametrize boundary and then reduce to a Calc 1 type of min/max problem to solve.

The following only apply only if a boundary is given
1. check the corner points
2. Check each line ($0 \leq$ x ≤ 5 would give x=0 and x=5)
On Bounded Equations, this is the global min and max...second derivative test is not needed.

Lagrange Multipliers
Given a function f(x,y) with a constraint g(x,y), solve the following system of equations to find the max and min points on the constraint (NOTE: may need to also find internal points.):
$\nabla f = \lambda \nabla g$
$g(x,y) = 0 (orkifgiven)$

Double Integrals
With Respect to the xy-axis, if taking an integral, $\int\int dydx$ is cutting in vertical rectangles, $\int\int dxdy$ is cutting in horizontal rectangles

Polar Coordinates When using polar coordinates, $dA = rdrd\theta$

Surface Area of a Curve
let z = f(x,y) be continuous over S (a closed Region in 2D domain)
Then the surface area of z = f(x,y) over S is:
$SA = \int\int_S \sqrt{f_x^2 + f_y^2 + 1} dA$

Triple Integrals
$\int\int\int_s f(x,y,z) dv =$
$\int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z) dzdydx$
Note: dv can be exchanged for $dxdydz$ in any order, but you must then choose your limits of integration according to that order

Jacobian Method
$\int\int_G f(g(u,v), h(u,v)) |J(u,v)| dudv =$
$\int\int_R f(x,y) dxdy$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Common Jacobians:
Rect. to Cylindrical: r
Rect. to Spherical: $\rho^2 \sin(\phi)$

Vector Fields
let $f(x,y,z)$ be a scalar field and
$\vec{F}(x,y,z) =$
$M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$
be a vector field,
Grandient of f = $\nabla f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$
Divergence of $\vec{F}$:
$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
Curl of $\vec{F}$:
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Other Information
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
Where a Cone is defined as
$z = \sqrt{a(x^2 + y^2)}$,
In Spherical Coordinates,
$\phi = \cos^{-1}(\frac{a}{1+a})$

Unit Circle
(cos, sin)