Correct Answers are ticked in green
on the answer sheet at the end of this file.

MTH

GS-2012 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 11, 2011**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark; each wrong answer will get a -1 mark, and a question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

3. There are forty (40) questions divided into four parts, Part-A, Part-B, Part-C and Part-D. Each Part consists of 10 True-False questions.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

8. See the back of this page for Notation and Conventions used in this test.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers

$\mathbb{Z}$:= Set of integers

$\mathbb{Q}$:= Set of rational numbers

$\mathbb{R}$:= Set of real numbers

$\mathbb{C}$:= Set of complex numbers

$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$

For a differentiable real valued function $f : \mathbb{R} \to \mathbb{R}$ f' denotes its derivative and $f^{(k)}$ means the k^{th} derivative.

Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and the metric.

INSTRUCTIONS

THERE ARE 4 PARTS AND 40 QUESTIONS IN TOTAL, CONSISTING OF 10 QUESTIONS IN EACH PART.

EVERY CORRECT ANSWER CARRIES $+1$ MARK AND EVERY WRONG ANSWER CARRIES -1 MARK.

PART A

1. If H_1 & H_2 are subgroups of a group G then $H_1.H_2 = \{h_1h_2 \in G | h_1 \in H_1, h_2 \in H_2\}$ is a subgroup of G.

2. There exist polynomials $f(x)$ and $g(x)$, with complex coefficients, such that $\left(\frac{f(x)}{g(x)}\right)^2 = x$.

3. Let f be real valued, differentiable on (a,b) and $f'(x) \neq 0$ for all $x \in (a,b)$. Then f is $1-1$.

4. The inequality $\sum_{n=0}^{\infty} \frac{(\log \log 2)^n}{n!} > \frac{3}{5}$ holds.

5. Every subgroup of order 74 in a group of order 148 is normal.

6. Let u_1, u_2, u_3, u_4 be vectors in $\mathbb{R}^2$ and
$$u = \sum_{j=1}^{4} t_j u_j \quad ; \quad t_j > 0 \text{ and } \sum_{j=1}^{4} t_j = 1.$$
Then three vectors $v_1, v_2, v_3 \in \mathbb{R}^2$ may be chosen from $\{u_1, u_2, u_3, u_4\}$ such that
$$u = \sum_{j=1}^{3} s_j v_j, \quad s_j \geq 0, \quad \sum_{j=1}^{3} s_j = 1.$$

7. The inequality
$$\sqrt{1+x} < 1 + x/2$$
for $x \in (-1, 10)$ is true

8. If n is not a multiple of 23 then the remainder when n^{11} is divided by 23 is $\pm 1 \pmod{23}$.

9. Suppose A is a nilpotent matrix and I is the identity matrix. Then $(I+A)$ is invertible.

10. The equations
$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &= 1 \\ x_1 + \frac{1}{4}x_2 + \frac{1}{9}x_3 &= 1 \\ x_1 + \frac{1}{8}x_2 + \frac{1}{27}x_3 &= 1 \end{aligned}$$
has no solution.

PART B

11. The automorphism group Aut $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ is abelian

12. Let V be the vector space of consisting of polynomials of $\mathbb{R}[t]$ of deg ≤ 2. The map $T : V \to V$ sending $f(t)$ to $f(t) + f^{'}(t)$ is invertible.

13. The polynomials $(t-1)(t-2), (t-2)(t-3), (t-3)(t-4), (t-4)(t-6) \in \mathbb{R}[t]$ are linearly independent.

14. $A \in M_2(\mathbb{C})$ and A is nilpotent then $A^2 = 0$.

15. Let P be an $n \times n$ matrix whose row sums equal 1. Then for any positive integer m the row sums of the matrix P^m equal 1.

16. There is a non trivial group homomorphism from C to R.

17. If the equation

$$xyz = 1$$

holds in a group G, does it follow that

$$yzx = 1.$$

18. Any 3×3 and 5×5 skew-symmetric matrices have always zero determinants.

19. The rank of the matrix

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$

is 2.

20. The number 2 is a prime in $\mathbb{Z}[i]$

PART C

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then the derivative $\frac{\partial^2 f}{\partial x \partial y}$ can exist without $\frac{\partial f}{\partial x}$ existing.

22. If f is continuous on $[0, 1]$ and if $\int_0^1 f(x)x^n \, dx = 0$ for $n = 0, 1, 2, 3, \cdots$. Then $\int_0^1 f^2(x) \, dx = 0$.

23. Suppose that $f \in \mathfrak{L}^2(\mathbb{R})$. Then $f \in \mathfrak{L}^1(\mathbb{R})$.

24. The integral

$$\int_{-\infty}^{+\infty} \frac{e^{-x}}{1 + x^2} \, dx$$

is convergent.

25. If $A \subset \mathbb{R}$ and open then the interior of the closure $\overline{A}^0$ is A.

26. If $f \in C^\infty$ and $f^{(k)}(0) = 0$ for all integer $k \geq 0$, then $f \equiv 0$.

27. Let $f : [0, 1] \to [0, 1]$ be continuous then f assumes the value $\int\limits_0^1 f^2(t)dt$ somewhere in $[0, 1]$.

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{h}$$

exists for all $x \in \mathbb{R}$. Then f is differentiable in $\mathbb{R}$.

29. The functions $f(x) = x|x|$ and $x|\sin x|$ are not differentiable at $x = 0$.

30. The composition of two uniformly continuous functions need not always be uniformly continuous.

PART D

31. $f : [0, \infty] \to [0, \infty]$ is continuous and bounded then f has a fixed point.

32. The polynomial $X^8 + 1$ is irreducible in $\mathbb{R}[X]$.

33.

$$\text{The matrix} \quad \begin{pmatrix} 1 & \pi & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{is diagonalisable}$$

34. If a rectangle $R := \{(x, y) \in \mathbb{R}^2 \mid A \leq x \leq B, C \leq y \leq D\}$ can be covered (allowing overlaps) by 25 discs of radius 1 then it can also be covered by 101 discs of radius $\frac{1}{2}$.

35. Given any integer $n \geq 2$, we can always find an integer m such that each of the $n - 1$ consecutive integers $m + 2$, $m + 3$,..., $m + n$ are composite.

36.

$$\text{The } 10{\times}10 \text{ matrix} \begin{pmatrix} v_1 w_1 & \cdots & v_1 w_{10} \\ v_2 w_1 & \cdots & v_2 w_{10} \\ v_{10} w_1 & \cdots & v_{10} w_{10} \end{pmatrix} \text{ has rank 2, where } v_i, w_i \in \mathbb{C}.$$

37. If every continuos function on $X \subset \mathbb{R}^2$ is bounded, then X is compact.

38. The graph of $xy = 1$ is $\mathbb{C}^2$ is connected.

39. If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ satisfy $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1$, then the least value of $|z_1 - z_2|^2 + |z_1 - z_4|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2$ is 2.

40. Consider the differential equations (with y is a function of x)

$$(1) \quad \begin{aligned} \frac{dy}{dx} &= y \\ y(0) &= 0 \end{aligned} \qquad (2) \quad \begin{aligned} \frac{dy}{dx} &= |y|^{\frac{1}{3}} \\ y(0) &= 0. \end{aligned}$$

Then (1) has infinitely many solutions but (2) has finite number of solutions.

GS-2012 (MATHEMATICS)

ANSWER SHEET

Please see reverse for instructions on filling of answer sheet.

Name	
Ref Code	
Address	
Phone	
Email	

Reference Code :					
1	○	○	○	○	○
2	○	○	○	○	○
3	○	○	○	○	○
4	○	○	○	○	○
5	○	○	○	○	○
6	○	○	○	○	○
7	○	○	○	○	○
8	○	○	○	○	○
9	○	○	○	○	○
0	○	○	○	○	○

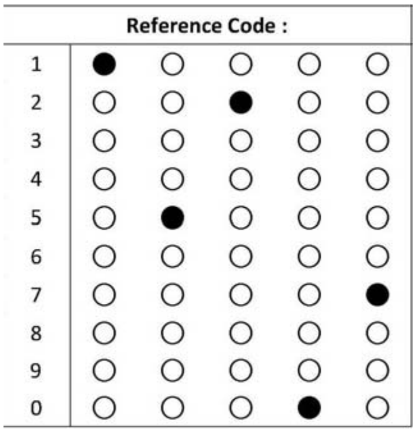
PART-A			**PART-B**			**PART-C**			**PART-D**		
	True	**False**		**True**	**False**		**True**	**False**		**True**	**False**
1	○	✓	1	○	✓	1	✓	○	1	✓	○
2	○	✓	2	✓	○	2	✓	○	2	○	✓
3	✓	○	3	○	✓	3	○	✓	3	✓	○
4	✓	○	4	✓	○	4	○	✓	4	✓	○
5	✓	○	5	✓	○	5	○	✓	5	✓	○
6	✓	○	6	✓	○	6	○	✓	6	○	✓
7	○	✓	7	✓	○	7	○	✓	7	✓	○
8	✓	○	8	✓	○	8	○	✓	8	✓	○
9	✓	○	9	✓	○	9	○	✓	9	✓	○
10	○	✓	10	○	✓	10	○	✓	10	○	✓

Signature of the Student

INSTRUCTIONS

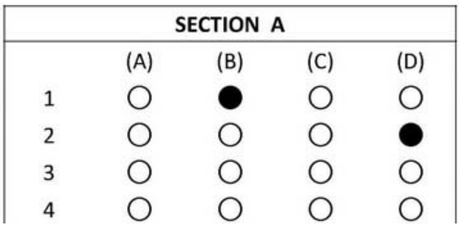
The Answer Sheet is machine-readable. Apart from filling in the details on the answer sheet, please make sure that the Reference Code is filled by blackening the appropriate circles in the box provided on the right-top corner. <u>Only use HB pencils to fill-in the answer sheet.</u>

e.g. if your reference code is 15207 :



Also, the multiple choice questions are to be answered by blackening the appropriate circles as described below

e.g. if your answer to question 1 is (b) and your answer to question 2 is (d) then ....



MTH

GS-2013 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 9, 2012**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark; each wrong answer will get a -1 mark, and a question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

3. There are forty (40) questions divided into four parts, Part-A, Part-B, Part-C and Part-D. Each Part consists of 10 True-False questions.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

8. See the back of this page for Notation and Conventions used in this test.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers

$\mathbb{Z}$:= Set of integers

$\mathbb{Q}$:= Set of rational numbers

$\mathbb{R}$:= Set of real numbers

$\mathbb{C}$:= Set of complex numbers

$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$, the open interval

A sequence is always indexd by natural numbers.

Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology.

INSTRUCTIONS

THERE ARE 4 PARTS AND 40 QUESTIONS IN TOTAL, CONSISTING OF 10 QUESTIONS IN EACH PART.

EVERY CORRECT ANSWER CARRIES +1 MARK AND EVERY WRONG ANSWER CARRIES −1 MARK.

PART A

F 1. Every countable group G has only countably many distinct subgroups.

T 2. Any automorphism of the group $\mathbb{Q}$ under addition is of the form $x \mapsto qx$ for some $q \in \mathbb{Q}$.

T 3. The equation $x^3 + 3x - 4 = 0$ has exactly one real root.

T 4. The equation $x^3 + 10x^2 - 100x + 1729 = 0$ has at least one complex root α such that $|\alpha| > 12$.

F 5. All non-trivial proper subgroups of $(\mathbb{R}, +)$ are cyclic.

F 6. Every infinite abelian group has at least one element of infinite order.

F 7. If A and B are similar matrices then every eigenvector of A is an eigenvector of B.

T 8. If a real square matrix A is similar to a diagonal matrix and satisfies $A^n = 0$ for some n, then A must be the zero matrix.

T 9. There is an element of order 51 in the multiplicative group $(\mathbb{Z}/103\mathbb{Z})^*$.

T 10. Any normal subgroup of order 2 is contained in the center of the group.

PART B

F 11. Consider the sequences

$$x_n = \sum_{j=1}^{n} \frac{1}{j}$$
$$y_n = \sum_{j=1}^{n} \frac{1}{j^2}$$

Then $\{x_n\}$ is Cauchy but $\{y_n\}$ is not.

F 12. $\lim_{x \to 0} \frac{\sin(x^2)}{x^2} \sin\left(\frac{1}{x}\right) = 1.$

F 13. Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ be Riemann integrable functions defined on the closed intervals $[a, b]$ and $[c, d]$ respectively. Then the composite $g \circ f$ is also Riemann integrable.

T 14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x^3$. Then f is continuous but not uniformly continuous.

T 15. Let $x_1 \in (0, 1)$ be a real number between 0 and 1. For $n > 1$, define

$$x_{n+1} = x_n - x_n^{n+1}.$$

Then $\lim_{n \to \infty} x_n$ exists.

T 16. Suppose $\{a_i\}$ is a sequence in $\mathbb{R}$ such that $\sum |a_i||x_i| < \infty$ whenever $\sum |x_i| < \infty$. Then $\{a_i\}$ is a bounded sequence.

T 17. The integral $\int_0^\infty e^{-x^5} dx$ is convergent.

F 18. Let $P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ where n is a large positive integer. Then $\lim_{x \to \infty} \frac{e^x}{P(x)} = 1$.

F 19. Every differentiable function $f : (0, 1) \to [0, 1]$ is uniformly continuous.

T 20. Consider the function $f(x) = ax + b$ with $a, b \in \mathbb{R}$. Then the iteration

$$x_{n+1} = f(x_n); \qquad n \geq 0$$

for a given x_0 converges to $b/(1 - a)$ whenever $0 < a < 1$.

PART C

F 21. Every homeomorphism of the 2-sphere to itself has a fixed point.

F 22. The intervals $[0, 1)$ and $(0, 1)$ are homeomorphic.

F 23. Let X be a complete metric space such that distance between any two points is less than 1. Then X is compact.

F 24. There exists a continuous surjective function from S^1 onto $\mathbb{R}$.

T 25. There exists a complete metric on the open interval $(0, 1)$ inducing the usual topology.

F 26. There exists a continuous surjective map from the complex plane onto the non-zero reals.

T 27. If every differentiable function on a subset $X \subset \mathbb{R}^n$ (i.e., restriction of a differentiable function on a neighbourhood of X) is bounded, then X is compact.

F 28. Let $f : X \to Y$ be a continuous map between metric spaces. If f is a bijection, then its inverse is also continuous.

T 29. Let f be a function on the closed interval $[0, 1]$ defined by

$$f(x) = x \qquad \text{if } x \text{ is rational}$$

$$f(x) = x^2 \qquad \text{if } x \text{ is irrational}$$

Then f is continuous at 0 and 1.

T 30. There exists an infinite subset $S \subset \mathbb{R}^3$ such that any three vectors in S are linearly independent.

PART D

F 31. The inequality

$$\sqrt{n+1}-\sqrt{n}<\frac{1}{\sqrt{n}}$$

is false for all n such that $101 \le n \le 2000$.

F 32. $\lim_{n\to\infty}(n+1)^{1/3}-n^{1/3}=\infty$.

T 33. There exists a bijection between $\mathbb{R}^2$ and the open interval $(0,1)$.

F 34. Let S be the set of all sequences $\{a_1, a_2, ..., a_n, ...\}$ where each entry a_i is either 0 or 1. Then S is countable.

T 35. Let $\{a_n\}$ be any non-constant sequence in $\mathbb{R}$ such that $a_{n+1}=\frac{a_n+a_{n+2}}{2}$ for all $n \geq 1$. Then $\{a_n\}$ is unbounded.

F 36. The function $f:\mathbb{Z}\to\mathbb{R}$ defined by $f(n)=n^3-3n$ is injective.

F 37. The polynomial $x^3+3x-2\pi$ is irreducible over $\mathbb{R}$.

T 38. Let V be the vector space consisting of polynomials with real coefficients in variable t of degree ≤ 9. Let $D: V\to V$ be the linear operator defined by

$$D(f) := \frac{df}{dt}.$$

Then 0 is an eigenvalue of D.

T 39. If A is a complex $n\times n$ matrix with $A^2=A$, then rank $A=$ trace A.

F 40. The series

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$$

is divergent.

Correct answers are ticked in green.

MTH

GS-2014 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 8, 2013**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. There are thirty (30) multiple choice questions divided into two parts. Part I consists of 20 questions and Part II consists of 10 questions. Bachelors students who have applied only for the Integrated Ph.D. program at TIFR CAM, Bangalore will only be evaluated on Part I. All other students (including Bachelors students applying to the Ph.D. programs at both, TIFR, Mumbai and Bangalore) will be evaluated on both Parts I and II.

3. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark. There is no negative marking for wrong answers. A question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

8. Notation and Conventions used in this test are given on page 1 of the question paper.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$:= Set of integers
$\mathbb{Q}$:= Set of rational numbers
$\mathbb{R}$:= Set of real numbers
$\mathbb{C}$:= Set of complex numbers
$\mathbb{R}^*$:= Set of non-zero real numbers
$\mathbb{C}^*$:= Set of non-zero complex numbers
$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$
$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$

A sequence is always indexed by the set of natural numbers.
The cyclic group with n elements is denoted by $\mathbb{Z}/n$.
Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology.
For any set S, the cardinality of the set is denoted by $|S|$.

Part I

1. Let A, B, C be three subsets of $\mathbb{R}$. The negation of the following statement
For every $\epsilon > 1$, there exists $a \in A$ and $b \in B$ such that for all $c \in C$, $|a - c| < \epsilon$ and $|b - c| > \epsilon$

is

A. there exists $\epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ or $|b - c| \leq \epsilon$
B. there exists $\epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ and $|b - c| \leq \epsilon$
C. there exists $\epsilon > 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ and $|b - c| \leq \epsilon$
D. there exists $\epsilon > 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ or $|b - c| \leq \epsilon$. ✓

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function, then:

A. f has to be uniformly continuous
B. there exists an $x \in \mathbb{R}$ such that $f(x) = x$ ✓
C. f cannot be increasing
D. $\lim\limits_{x \to \infty} f(x)$ exists.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim\limits_{x \to +\infty} f'(x) = 1$, then

A. f is bounded
B. f is increasing
C. f is unbounded ✓
D. f' is bounded.

4. Let f be the real valued function on $[0, \infty)$ defined by

$$f(x) = \begin{cases} x^{\frac{2}{3}} \log x \text{ for } x > 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Then

A. f is discontinuous at $x = 0$
B. f is continuous on $[0, \infty)$, but not uniformly continuous on $[0, \infty)$
C. f is uniformly continuous on $[0, \infty)$ ✓
D. f is not uniformly continuous on $[0, \infty)$, but uniformly continuous on $(0, \infty)$.

5. Let $a_n = (n + 1)^{100} e^{-\sqrt{n}}$ for $n \geq 1$. Then the sequence $(a_n)_n$ is

A. unbounded
B. bounded but does not converge
C. bounded and converges to 1
D. bounded and converges to 0. ✓

6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Which of the following statements is always true?

A. $\int_0^1 f^2(x)\, dx = \left(\int_0^1 f(x)\, dx \right)^2$
B. $\int_0^1 f^2(x)\, dx \leq \left(\int_0^1 |f(x)|\, dx \right)^2$
C. $\int_0^1 f^2(x)\, dx \geq \left(\int_0^1 |f(x)|\, dx \right)^2$ ✓
D. $\int_0^1 f^2(x)\, dx \lneq \left(\int_0^1 f(x)\, dx \right)^2$.

7. Let $f_n(x)$, for $n \geq 1$, be a sequence of continuous nonnegative functions on $[0,1]$ such that
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0.$$
Which of the following statements is always correct?

A. $f_n \to 0$ uniformly on $[0,1]$
B. f_n may not converge uniformly but converges to 0 point-wise
C. f_n will converge point-wise and the limit may be non-zero
D. f_n is not guaranteed to have a point-wise limit. ✓

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x)-f(y)| \geq \frac{1}{2}|x-y|$, for all $x, y \in \mathbb{R}$. Then

A. f is both one-to-one and onto ✓
B. f is one-to-one but may not be onto
C. f is onto but may not be one-to-one
D. f is neither one-to-one nor onto.

9. Let $A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\theta \in (0, 2\pi)$. Mark the correct statement below.

A. $A(\theta)$ has eigenvectors in $\mathbb{R}^2$ for all $\theta \in (0, 2\pi)$
B. $A(\theta)$ does not have an eigenvector in $\mathbb{R}^2$, for any $\theta \in (0, 2\pi)$
C. $A(\theta)$ has eigenvectors in $\mathbb{R}^2$, for exactly one value of $\theta \in (0, 2\pi)$ ✓
D. $A(\theta)$ has eigenvectors in $\mathbb{R}^2$, for exactly 2 values of $\theta \in (0, 2\pi)$

10. Let $\mathcal{C} \subset \mathbb{Z} \times \mathbb{Z}$ be the set of integer pairs (a, b) for which the three complex roots r_1, r_2 and r_3 of the polynomial $p(x) = x^3 - 2x^2 + ax - b$ satisfy $r_1^3 + r_2^3 + r_3^3 = 0$. Then the cardinality of $\mathcal{C}$ is

A. $|\mathcal{C}| = \infty$
B. $|\mathcal{C}| = 0$ ✓
C. $|\mathcal{C}| = 1$
D. $1 < |\mathcal{C}| < \infty$.

11. Let A be an $n \times n$ matrix with real entries such that $A^k = 0$ (0-matrix), for some $k \in \mathbb{N}$. Then

A. A has to be the 0 matrix
B. trace(A) could be non-zero
C. A is diagonalizable
D. 0 is the only eigenvalue of A. ✓

12. There exists a map $f : \mathbb{Z} \to \mathbb{Q}$ such that f

A. is bijective and increasing
B. is onto and decreasing
C. is bijective and satisfies $f(n) \geq 0$ if $n \leq 0$ ✓
D. has uncountable image.

13. Let S be the set of all tuples (x, y) with x, y non-negative real numbers satisfying $x + y = 2n$, for a fixed $n \in \mathbb{N}$. Then the supremum value of

$$x^2y^2(x^2 + y^2)$$

on the set S is

A. $3n^6$
B. $2n^6$ ✓
C. $4n^6$
D. n^6.

14. Let G be a group and let H and K be two subgroups of G. If both H and K have 12 elements, which of the following numbers cannot be the cardinality of the set $HK = \{hk : h \in H, k \in K\}$?

A. 72
B. 60
C. 48
D. 36.

15. How many proper subgroups does the group $\mathbb{Z} \oplus \mathbb{Z}$ have?

A. 1
B. 2
C. 3
D. infinitely many.

16. X is a metric space. Y is a closed subset of X such that the distance between any two points in Y is at most 1. Then

A. Y is compact
B. any continuous function from $Y \to \mathbb{R}$ is bounded
C. Y is not an open subset of X
D. none of the above.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let S be a non-empty proper subset of $\mathbb{R}$. Which one of the following statements is always true? (Here $\bar{A}$ denotes the closure of A and A^o denotes the interior of A.)

A. $f(S)^o \subseteq f(S^o)$
B. $f(\bar{S}) \subseteq \overline{f(S)}$
C. $f(\bar{S}) \supseteq \overline{f(S)}$
D. $f(S)^o \supseteq f(S^o)$.

18. What is the last digit of 97^{2013}?

A. 1
B. 3
C. 7 ✓
D. 9.

19. For $n \in \mathbb{N}$, we define

$$s_n = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

Which of the following holds for all $n \in \mathbb{N}$?

A. s_n is an odd integer
B. $s_n = n^2(n+1)^2/4$ ✓
C. $s_n = n(n+1)(2n+1)/6$
D. none of the above.

20. Let C denote the cube $[-1,1]^3 \subset \mathbb{R}^3$. How many rotations are there in $\mathbb{R}^3$ which take C to itself?

A. 6
B. 12
C. 18
D. 24. ✓

Part II

21. Let $f : [0,1] \to [0,\infty)$ be continuous. Suppose

$$\int_0^x f(t)\,\mathrm{d}t \geq f(x), \text{ for all } x \in [0,1].$$

Then

A. no such function exists
B. there are infinitely many such functions
C. there is only one such function
D. there are exactly two such functions.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that $f(x) = 0$ for only finitely many values of x. Which of the following is true?

A. either $f(x) \leq 0$ for all x, or, $f(x) \geq 0$ for all x
B. the map f is onto
C. the map f is one-to-one
D. none of the above.

23. Let S_n be the symmetric group of n letters. There exists an onto group homomorphism

A. from S_5 to S_4
B. from S_4 to S_2
C. from S_5 to $\mathbb{Z}/5$
D. from S_4 to $\mathbb{Z}/4$.

24. Let H_1, H_2 be two distinct subgroups of a finite group G, each of order 2. Let H be the smallest subgroup containing H_1 and H_2. Then the order of H is

A. always 2
B. always 4
C. always 8
D. none of the above. ✓

25. Which of the following groups are isomorphic?

A. $\mathbb{R}$ and $\mathbb{C}$ ✓
B. $\mathbb{R}^*$ and $\mathbb{C}^*$
C. $S_3 \times \mathbb{Z}/4$ and S_4
D. $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$.

26. The number of irreducible polynomials of the form $x^2 + ax + b$, with a, b in the field $\mathbb{F}_7$ of 7 elements is:

A. 7
B. 21 ✓
C. 35
D. 49.

27. X is a topological space of infinite cardinality which is homeomorphic to $X \times X$. Then

A. X is not connected
B. X is not compact
C. X is not homemorphic to a subset of $\mathbb{R}$
D. none of the above. ✓

28. Let X be a non-empty topological space such that every function $f : X \to \mathbb{R}$ is continuous. Then

    A. X has the discrete topology ✓
    B. X has the indiscrete topology
    C. X is compact
    D. X is not connected.

29. Let $f : X \to Y$ be a continuous map between metric spaces. Then $f(X)$ is a complete subset of Y if

    A. the space X is compact ✓
    B. the space Y is compact
    C. the space X is complete
    D. the space Y is complete.

30. How many maps $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ are there, with the property that $\phi(ab) = \phi(a) + \phi(b)$, for all $a, b \in \mathbb{N} \cup \{0\}$?

    A. none
    B. finitely many ✓
    C. countably many
    D. uncountably many.

Correct answers are ticked in green.

MTH

GS-2015 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 14, 2014**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. There are thirty (30) multiple choice questions divided into two parts. Part I consists of 15 questions and Part II consists of 15 questions. Bachelors students who have applied only for the Integrated Ph.D. program at TIFR CAM, Bangalore will only be evaluated on Part I. All other students (including Bachelors students applying to the Ph.D. programs at both TIFR, Mumbai and Bangalore) will be evaluated on both Parts I and II.

3. Indicate your answer ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark. There is no negative marking for wrong answers. A question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

8. Notation and Conventions used in this test are given on page 2 of the question paper.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$:= Set of integers
$\mathbb{Q}$:= Set of rational numbers
$\mathbb{R}$:= Set of real numbers
$\mathbb{C}$:= Set of complex numbers
$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$
$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$
$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$

A sequence is always indexed by the set of natural numbers.
The cyclic group with n elements is denoted by $\mathbb{Z}_n$.
Unless stated otherwise, subsets of $\mathbb{R}^n$ carry the induced topology.
For any set S, the cardinality of S is denoted by $|S|$.

Part I

1. Let A be an invertible 10×10 matrix with real entries such that the sum of each row is 1. Then

   A. The sum of the entries of each row of the inverse of A is 1
   B. The sum of the entries of each column of the inverse of A is 1
   C. The trace of the inverse of A is non-zero
   D. None of the above.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which one of the following sets cannot be the image of $(0, 1]$ under f ?

   A. $\{0\}$
   B. $(0, 1)$
   C. $[0, 1)$
   D. $[0, 1]$.

3. Let A be a 10×10 matrix with complex entries such that all its eigenvalues are non-negative real numbers, and at least one eigenvalue is positive. Which of the following statements is always false ?

   A. There exists a matrix B such that $AB - BA = B$
   B. There exists a matrix B such that $AB - BA = A$
   C. There exists a matrix B such that $AB + BA = A$
   D. There exists a matrix B such that $AB + BA = B$.

4. Let S be the collection of (isomorphism classes of) groups G which have the property that every element of G commutes only with the identity element and itself. Then

A. $|S| = 1$
B. $|S| = 2$
C. $|S| \geq 3$ and is finite
D. $|S| = \infty$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ denote the function defined by $f(x) = (1-x^2)^{\frac{3}{2}}$ if $|x| < 1$, and $f(x) = 0$ if $|x| \geq 1$. Which of the following statements is correct ?

A. f is not continuous
B. f is continuous but not differentiable
C. f is differentiable but f' is not continuous
D. f is differentiable and f' is continuous.

6. Let A be the 2×2 matrix $\begin{pmatrix} \sin\frac{\pi}{18} & -\sin\frac{4\pi}{9} \\ \sin\frac{4\pi}{9} & \sin\frac{\pi}{18} \end{pmatrix}$. Then the smallest number $n \in \mathbb{N}$ such that $A^n = I$ is

A. 3
B. 9
C. 18
D. 27.

7. Let f and g be two functions from $[0, 1]$ to $[0, 1]$ with f strictly increasing. Which of the following statements is always correct ?

A. If g is continuous, then $f \circ g$ is continuous
B. If f is continuous, then $f \circ g$ is continuous
C. If f and $f \circ g$ are continuous, then g is continuous
D. If g and $f \circ g$ are continuous, then f is continuous.

8. Let $f(x) = \frac{e^{-\frac{1}{x}}}{x}$, where $x \in (0,1)$. Then, on $(0,1)$

   A. f is uniformly continuous
   B. f is continuous but not uniformly continuous
   C. f is unbounded
   D. f is not continuous.

9. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \frac{n^2}{2^n}$ for all $n \in \mathbb{N}$. Then

   A. The sequence $\{a_n\}$ may be unbounded
   B. The sequence $\{a_n\}$ is bounded but may not converge
   C. The sequence $\{a_n\}$ has exactly two limit points
   D. The sequence $\{a_n\}$ is convergent.

10. For a group G, let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. Which of the following statements is true ?

    A. $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$
    B. If G is cyclic, then $\mathrm{Aut}(G)$ is cyclic
    C. If $\mathrm{Aut}(G)$ is trivial, then G is trivial
    D. $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

11. Let $\{a_n\}$ be a sequence of real numbers. Which of the following is true ?

    A. If $\sum a_n$ converges, then so does $\sum a_n^4$
    B. If $\sum |a_n|$ converges, then so does $\sum a_n^2$
    C. If $\sum a_n$ diverges, then so does $\sum a_n^3$
    D. If $\sum |a_n|$ diverges, then so does $\sum a_n^2$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function that vanishes at 10 distinct points in $\mathbb{R}$. Suppose $f^{(n)}$ denotes the n-th derivative of f, for $n \geq 1$. Which of the following statements is always true ?

A. $f^{(n)}$ has at least 10 zeros, for $1 \leq n \leq 8$
B. $f^{(n)}$ has at least one zero, for $1 \leq n \leq 9$
C. $f^{(n)}$ has at least 10 zeros, for $n \geq 10$
D. $f^{(n)}$ has at least one zero, for $n \geq 9$.

13. For a real number $t > 0$, let $\sqrt{t}$ denote the positive square root of t. For a real number $x > 0$, let $F(x) = \int_{x^2}^{4x^2} \sin\sqrt{t}\, dt$. If F' is the derivative of F, then

A. $F'(\frac{\pi}{2}) = 0$
B. $F'(\frac{\pi}{2}) = \pi$
C. $F'(\frac{\pi}{2}) = -\pi$
D. $F'(\frac{\pi}{2}) = 2\pi$.

14. Let $n \in \mathbb{N}$ be a six digit number whose base 10 expansion is of the form $abcabc$, where a, b, c are digits between 0 and 9 and a is non-zero. Then

A. n is divisible by 5
B. n is divisible by 8
C. n is divisible by 13
D. n is divisible by 17.

15. The series $\sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$

A. Diverges, for all rational $x \in \mathbb{R}$
B. Diverges, for some irrational $x \in \mathbb{R}$
C. Converges, for some but not all $x \in \mathbb{R}$
D. Converges, for all $x \in \mathbb{R}$.

Part II

16. Let X be a proper closed subset of $[0, 1]$. Which of the following statements is always true ?

    A. The set X is countable
    B. There exists $x \in X$ such that $X \setminus \{x\}$ is closed
    C. The set X contains an open interval
    D. None of the above. ✓

17. In how many ways can the group $\mathbb{Z}_5$ act on the set $\{1, 2, 3, 4, 5\}$?

    A. 5
    B. 24
    C. 25 ✓
    D. 120.

18. Let f be a function from $\{1, 2, \ldots, 10\}$ to $\mathbb{R}$ such that

$$\left(\sum_{i=1}^{10} \frac{|f(i)|}{2^i}\right)^2 = \left(\sum_{i=1}^{10} |f(i)|^2\right)\left(\sum_{i=1}^{10} \frac{1}{4^i}\right).$$

    Mark the correct statement.

    A. There are uncountably many f with this property ✓
    B. There are only countably infinitely many f with this property
    C. There is exactly one such f
    D. There is no such f.

19. Let $U_1 \supset U_2 \supset \cdots$ be a decreasing sequence of open sets in Euclidean 3-space $\mathbb{R}^3$. What can we say about the set $\cap U_i$?

A. It is infinite
B. It is open
C. It is non-empty
D. None of the above.

20. Let $n \geq 1$ and let A be an $n \times n$ matrix with real entries such that $A^k = 0$, for some $k \geq 1$. Let I be the identity $n \times n$ matrix. Then

A. $I + A$ need not be invertible
B. $\mathrm{Det}(I + A)$ can be any non-zero real number
C. $\mathrm{Det}(I + A) = 1$
D. A^n is a non-zero matrix.

21. Let $f : [0, 1] \to \mathbb{R}$ be a fixed continuous function such that f is differentiable on $(0, 1)$ and $f(0) = f(1) = 0$. Then the equation $f(x) = f'(x)$ admits

A. No solution $x \in (0, 1)$
B. More than one solution $x \in (0, 1)$
C. Exactly one solution $x \in (0, 1)$
D. At least one solution $x \in (0, 1)$.

22. A complex number $\alpha \in \mathbb{C}$ is called *algebraic* if there is a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ with rational coefficients such that $P(\alpha) = 0$. Which of the following statements is true ?

A. There are only finitely many algebraic numbers
B. All complex numbers are algebraic
C. $\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{4})$ is algebraic
D. None of the above.

23. For a group G, let $F(G)$ denote the collection of all subgroups of G. Which one of the following situations can occur ?

A. G is finite but $F(G)$ is infinite
B. G is infinite but $F(G)$ is finite
C. G is countable but $F(G)$ is uncountable
D. G is uncountable but $F(G)$ is countable.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $A \subset \mathbb{R}$ be defined by

$$A = \{y \in \mathbb{R} : y = \lim_{n\to\infty} f(x_n), \text{ for some sequence } x_n \to +\infty\}.$$

Then the set A is necessarily

A. A connected set
B. A compact set
C. A singleton set
D. None of the above.

25. How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that each $x_i = 1$ or 2, and $\sum_{i=1}^{m} x_i = 10$?

A. 89
B. 91
C. 92
D. 120.

26. Let (X, d) be a path connected metric space with at least two elements, and let $S = \{d(x, y) : x, y \in X\}$. Which of the following statements is not necessarily true ?

A. S is infinite
B. S contains a non-zero rational number
C. S is connected
D. S is a closed subset of $\mathbb{R}$.

27. Let $X \subset \mathbb{R}$ and let $f, g : X \to X$ be continuous functions such that $f(X) \cap g(X) = \emptyset$ and $f(X) \cup g(X) = X$. Which one of the following sets cannot be equal to X ?

A. $[0, 1]$
B. $(0, 1)$
C. $[0, 1)$
D. $\mathbb{R}$.

28. Let $X = \{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 = 1\}$. Endow $\mathbb{R}^2$ with the discrete topology, and X with the subspace topology. Then

A. X is a compact subset of $\mathbb{R}^2$ in this topology
B. X is a connected subset of $\mathbb{R}^2$ in this topology
C. X is an open subset of $\mathbb{R}^2$ in this topology
D. None of the above.

29. Let G be a group. Suppose $|G| = p^2 q$, where p and q are distinct prime numbers satisfying $q \not\equiv 1 \mod p$. Which of the following is always true ?

A. G has more than one p-Sylow subgroup
B. G has a normal p-Sylow subgroup
C. The number of q-Sylow subgroups of G is divisible by p
D. G has a unique q-Sylow subgroup.

30. Let $d(x, y)$ be the usual Euclidean metric on $\mathbb{R}^2$. Which of the following metric spaces is complete ?

A. $\mathbb{Q}^2 \subset \mathbb{R}^2$ with the metric $d(x, y)$
B. $[0, 1] \times [0, \infty) \subset \mathbb{R}^2$ with the metric $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$
C. $(0, \infty) \times [0, \infty) \subset \mathbb{R}^2$ with the metric $d(x, y)$
D. $[0, 1] \times [0, 1) \subset \mathbb{R}^2$ with the metric $d''(x, y) = \min\{1, d(x, y)\}$.

MTH

GS-2016 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 13, 2015**

For the Ph.D. Programs at TIFR (Mumbai, and CAM and ICTS, Bangalore)
and for the Int. Ph.D. Programs at TIFR (CAM, Bangalore and Mumbai)

Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. There are thirty (30) multiple choice questions divided into two parts. Part I consists of 20 questions and Part II consists of 10 questions.

3. Bachelors students who have applied only for the Integrated Ph.D. program at TIFR CAM, Bangalore will only be evaluated on Part I. All other students (including Bachelors students applying for the Integrated Ph.D. programs at TIFR, Mumbai) will be evaluated on both Parts I and II.

4. Indicate your answer ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark. There is no negative marking for wrong answers. A question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

5. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

6. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator.

7. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

8. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

9. Notation and Conventions used in this test are given on page 2 of the question paper.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$:= Set of integers
$\mathbb{Q}$:= Set of rational numbers
$\mathbb{R}$:= Set of real numbers
$\mathbb{C}$:= Set of complex numbers
$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$
$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$
$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$

A sequence is always indexed by the set of natural numbers.
The cyclic group with n elements is denoted by $\mathbb{Z}/n\mathbb{Z}$.
Unless stated otherwise, subsets of $\mathbb{R}^n$ carry the induced topology.
For any set S, the cardinality of S is denoted by $|S|$.

Part I

1. The value of the product $\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots\right)\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\right)$ is

   A. 1
   B. e^2
   C. 0
   D. $\log_e 2$.

2. Which of the following is false ?

   A. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges
   B. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$ converges
   C. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$ diverges
   D. $\sum_{n=1}^{\infty} \cos\frac{1}{n^2}$ converges.

3. The value of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is

   A. 1
   B. 2
   C. 3
   D. 4.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \dfrac{\sin x}{|x| + \cos x}$. Then

   A. f is differentiable at all $x \in \mathbb{R}$
   B. f is not differentiable at $x = 0$
   C. f is differentiable at $x = 0$ but f' is not continuous at $x = 0$
   D. f is not differentiable at $x = \frac{\pi}{2}$.

5. Which of the following continuous functions $f : (0, \infty) \to \mathbb{R}$ can be extended to a continuous function on $[0, \infty)$?

A. $f(x) = \sin \frac{1}{x}$
B. $f(x) = \frac{1 - \cos x}{x^2}$ ✓
C. $f(x) = \cos \frac{1}{x}$
D. $f(x) = \frac{1}{x}$.

6. Let V be the vector space over $\mathbb{R}$ consisting of polynomials $p(t)$ over $\mathbb{R}$ of degree less than or equal to 4. Let $D : V \to V$ be the linear operator that takes any polynomial $p(t)$ to its derivative $p'(t)$. Then the characteristic polynomial $f(x)$ of D is

A. x^4
B. x^5 ✓
C. $x^3(x-1)$
D. $x^4(x-1)$.

7. Let $A = \{\sum_{i=1}^{\infty} \frac{a_i}{5^i} : a_i = 0, 1, 2, 3 \text{ or } 4\} \subset \mathbb{R}$. Then

A. A is a finite set
B. A is countably infinite
C. A is uncountable but does not contain an open interval
D. A contains an open interval. ✓

8. The number of group homomorphisms from $\mathbb{Z}/20\mathbb{Z}$ to $\mathbb{Z}/29\mathbb{Z}$ is

A. 1 ✓
B. 20
C. 29
D. 580.

9. Let $p(x)$ be a polynomial of degree 3 with real coefficients. Which of the following is possible ?

A. $p(x)$ has no real roots
B. $p(x)$ has exactly 2 real roots
C. $p(1) = -1$, $p(2) = 1$, $p(3) = 11$ and $p(4) = 35$ ✓
D. $i - 1$ and $i + 1$ are roots of $p(x)$, where i is the square root of -1

10. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that the series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Then the series $\sum_{n=1}^{\infty} a_n b_n$

A. is absolutely convergent ✓
B. may not converge
C. is always convergent, but may not converge absolutely
D. converges to 0.

11. Let $v_i = (v_i^{(1)}, v_i^{(2)}, v_i^{(3)}, v_i^{(4)})$, for $i = 1, 2, 3, 4$, be four vectors in $\mathbb{R}^4$ such that $\sum_{i=1}^{4} v_i^{(j)} = 0$, for each $j = 1, 2, 3, 4$. Let W be the subspace of $\mathbb{R}^4$ spanned by $\{v_1, v_2, v_3, v_4\}$. Then the dimension of W over $\mathbb{R}$ is always

A. either equal to 1 or equal to 4
B. less than or equal to 3 ✓
C. greater than or equal to 2
D. either equal to 0 or equal to 4.

12. Let A be a subset of $[0, 1]$ with non-empty interior, and let $\mathbb{Q} + A = \{q + a : q \in \mathbb{Q},\ a \in A\}$. Which of the following is true ?

A. $\mathbb{Q} + A = \mathbb{R}$ ✓
B. $\mathbb{Q} + A$ can be a proper subset of $\mathbb{R}$
C. $\mathbb{Q} + A$ need not be closed in $\mathbb{R}$
D. $\mathbb{Q} + A$ need not be open in $\mathbb{R}$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $|f(x) - f(y)| \geq |x - y|$, for all $x, y \in \mathbb{R}$. Then the equation $f'(x) = \frac{1}{2}$

A. has exactly one solution
B. has no solution
C. has a countably infinite number of solutions
D. has uncountably many solutions.

14. Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function such that $g(x) = (f(x))^2$ is uniformly continuous. Which of the following statements is always true ?

A. f is bounded
B. f may not be uniformly continuous
C. f is uniformly continuous
D. f is unbounded.

15. Which of the following sequences of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly ?

A. $f_n(x) = x^n$ on $[0, 1]$
B. $f_n(x) = 1 - x^n$ on $[\frac{1}{2}, 1]$
C. $f_n(x) = \dfrac{1}{1 + nx^2}$ on $[0, \frac{1}{2}]$
D. $f_n(x) = \dfrac{1}{1 + nx^2}$ on $[\frac{1}{2}, 1]$.

16. Let S be a collection of subsets of $\{1, 2, \ldots, 100\}$ such that the intersection of any two sets in S is non-empty. What is the maximum possible cardinality $|S|$ of S ?

A. 100
B. 2^{100}
C. 2^{99}
D. 2^{98}.

17. Let S be the set of all 3×3 matrices A with integer entries such that the product AA^t is the identity matrix. Here A^t denotes the transpose of A. Then $|S| =$

A. 12
B. 24
C. 48 ✓
D. 60.

18. Let A be a 3×3 matrix with integer entries such that $\det(A) = 1$. What is the maximum possible number of entries of A that are even ?

A. 2
B. 3
C. 6 ✓
D. 8.

19. The limit

$$\lim_{n \to \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \right) =$$

A. e
B. 2
C. $\log_e 2$ ✓
D. e^2.

20. Let $G = \mathbb{Z}/100\mathbb{Z}$ and let $S = \{h \in G : \text{Order}(h) = 50\}$. Then $|S|$ equals

A. 20 ✓
B. 25
C. 30
D. 50.

Part II

21. Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$ be an infinite sequence of non-empty subsets of $\mathbb{R}^3$. Which of the following conditions ensures that their intersection is non-empty ?

A. Each A_i is uncountable
B. Each A_i is open
C. Each A_i is connected
D. Each A_i is compact. ✓

22. Let (X, d) be a metric space. Which of the following is possible ?

A. X has exactly 3 dense subsets
B. X has exactly 4 dense subsets ✓
C. X has exactly 5 dense subsets
D. X has exactly 6 dense subsets.

23. Let $\{f_n\}_{n=1}^{\infty}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = n^2 x^n$. Let A be the set of all points a in $\mathbb{R}$ such that the sequence $\{f_n(a)\}_{n=1}^{\infty}$ converges. Then

A. $A = \{0\}$
B. $A = [0, 1)$
C. $A = \mathbb{R} \setminus \{-1, 1\}$
D. $A = (-1, 1)$. ✓

24. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(i) = 0$, for all $i \in \mathbb{Z}$. Which of the following statements is always true ?

A. Image(f) is closed in $\mathbb{R}$
B. Image(f) is open in $\mathbb{R}$
C. f is uniformly continuous
D. None of the above.

25. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Which of the following is false ? Any continuous function from S^1 to $\mathbb{R}$

A. is bounded
B. is uniformly continuous
C. has image containing a non-empty open subset of $\mathbb{R}$
D. has a point $z \in S^1$ such that $f(z) = f(-z)$.

26. Which of the following is false ?

A. Any continuous function from $[0, 1]$ to $[0, 1]$ has a fixed point
B. Any homeomorphism from $[0, 1)$ to $[0, 1)$ has a fixed point
C. Any bounded continuous function from $[0, \infty)$ to $[0, \infty)$ has a fixed point
D. Any continuous function from $(0, 1)$ to $(0, 1)$ has a fixed point.

27. For $n \geq 1$, let S_n denote the group of all permutations on n symbols. Which of the following statements is true ?

A. S_3 has an element of order 4
B. S_4 has an element of order 6
C. S_4 has an element of order 5
D. S_5 has an element of order 6.

28. Which of the following rings is an integral domain ?

A. $\mathbb{R}[x]/(x^2 + x + 1)$
B. $\mathbb{R}[x]/(x^2 + 5x + 6)$
C. $\mathbb{R}[x]/(x^3 - 2)$
D. $\mathbb{R}[x]/(x^7 + 1)$.

29. Let $f : \mathbb{R} \to (0, \infty)$ be a twice differentiable function such that $f(0) = 1$ and $\int_a^b f(x)\, dx = \int_a^b f'(x)\, dx$, for all a, $b \in \mathbb{R}$, with $a \leq b$. Which of the following statements is false ?

A. f is one to one
B. The image of f is compact
C. f is unbounded
D. There is only one such function.

30. For $X \subset \mathbb{R}^n$, consider X as a metric space with metric induced by the usual Euclidean metric on $\mathbb{R}^n$. Which of the following metric spaces X is complete?

A. $X = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$
B. $X = \mathbb{Q} \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$
C. $X = (-\pi, \pi) \cap \mathbb{Q} \subset \mathbb{R}$
D. $X = [-\pi, \pi] \cap (\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{R}$.

MTH

GS-2017 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 11, 2016**

For the Ph.D. Programs at TIFR, Mumbai and CAM & ICTS, Bangalore and for the Int. Ph.D. Programs at TIFR, Mumbai and CAM, Bangalore.

Duration : Three hours (3 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet for Part I as well on the answer booklet of Part II . The Answer Sheet for Part I is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. **PART I** - There are thirty (30) True/False type questions in Part I of the question paper. Allotted time for Part I is 90 minutes. The answer sheet for Part I will be collected at the end of 90 minutes. Part I questions carry +2 for a correct answer, -1 (negative marks) for a wrong answer and 0 for not answering.

   Indicate your answer ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.

   We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

3. **PART II** – 10 problems to be solved. The solutions should be written in the Answer Booklet for Part II that is provided. Extra blank sheets will be provided if needed. All Part II questions carry equal marks, and there are no negative marks. Partial credit will be given for partial solutions.

   Candidate can begin answering questions on Part II anytime. The answer booklet for Part II will be collected at the end of the exam.

4. **Selection Procedure** : The answers for Part I will be machine-graded. Part I will score will be used to decide a cut-off. Answer papers for Part II will be graded only for those candidates whose score is above the cut-off. List of candidates to be called for interview for the final selection for admission in the various programs will be decided based on the combined performance in Part I and II, weighted appropriately for each program.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator.

6. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

8. Notation and Conventions used in this test are given on page 2 of the question paper.

Mathematics Question Paper, GS2017
Parts I and II

Notation and Conventions:

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \cdots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rationals, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and metric. For a vector $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$, the norm $||v||$ is defined by $||v||^2 = v_1^2 + \cdots + v_n^2$.
- $M_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices with the Euclidean metric.
- All logarithms are natural logarithms.

Part I

Answer whether the following statements are True or False. Mark your answer on the machince checkable answer sheet that is provided.

Note: $+2$ marks for a correct answer, -1 mark (negative marks) for a wrong answer, 0 marks for not answering.

1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) \geq x^3$ for all $x \in [0, 1]$ with $\int_0^1 f(x)dx = \frac{1}{4}$. Then $f(x) = x^3$ for all $x \in \mathbb{R}$. True

2. Suppose $a,\ b,\ c$ are positive real numbers such that True

$$(1 + a + b + c)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 16.$$

Then $a + b + c = 3$.

3. There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying, False

$$f(-1) = -1, \quad f(1) = 1 \quad \text{and} \quad |f(x)-f(y)| \leq |x-y|^{\frac{3}{2}}, \quad \text{for all} \quad x, y \in \mathbb{R}.$$

4. Over the real line,

$$\lim_{x\to\infty} \log \left(1+\sqrt{4+x}-\sqrt{1+x}\right) = \log(2).$$

False

5. Suppose f is a continuously differentiable function on $\mathbb{R}$ such that $f(x) \to 1$ and $f'(x) \to b$ as $x \to \infty$. Then $b = 1$.

False

6. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and bijective, then f^{-1} is also differentiable.

False

7. Let $H_1,\ H_2,\ H_3,\ H_4$ be four hyperplanes in $\mathbb{R}^3$. The maximum possible number of connected components of $\mathbb{R}^3 - (H_1 \cup H_2 \cup H_3 \cup H_4)$ is 14.

False

8. Let $n \geq 2$ be a natural number. Let S be the set of all $n \times n$ real matrices whose entries are only 0, 1 or 2. Then the average determinant of a matrix in S is greater than or equal to 1.

False

9. For any metric space (X, d) with X finite, there exists an isometric embedding $f : X \to \mathbb{R}^4$.

False

10. There exists a non-negative continuous function $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f^n dx \to 2$ as $n \to \infty$.

False

11. There exists a subset A of $\mathbb{N}$ with exactly five elements such that the sum of any three elements of A is a prime number.

False

12. There exists a finite abelian group G containing exactly 60 elements of order 2.

False

13. Let $\alpha,\ \beta$ be complex numbers with non-positive real parts. Then

$$|e^\alpha - e^\beta| \leq |\alpha - \beta|.$$

True

14. Every 2×2-matrix over $\mathbb{C}$ is a square of some matrix.

False

15. Under the projection map $\mathbb{R}^2 \to \mathbb{R}$ sending (x, y) to x, the image of any closed set is closed. False

16. The number of ways a 2×8 rectangle can be tiled with rectangular tiles of size 2×1 is 34. True

17. Over the real line, True
$$\lim_{x\to\infty} \left(\frac{x + \log 9}{x - \log 9}\right)^x = 81.$$

18. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with $\lim_{x\to\infty} f(x) = 0$. Then f has a maximum value in $[0, \infty)$. False

19. Given a continuous function $f : \mathbb{Q} \to \mathbb{Q}$, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that the restriction of g to $\mathbb{Q}$ is f. False

20. For all positive integers m and n, if A is an $m \times n$ real matrix, and B is an $n \times m$ real matrix such that $AB = I$, then $BA = I$. False

21. There is a continuous onto function $f : S^2 \to S^1$ from the unit sphere in $\mathbb{R}^3$ to the unit sphere in $\mathbb{R}^2$, where $S^n = \{v \in \mathbb{R}^{n+1} \mid ||v|| = 1\}$ denotes the unit sphere in $\mathbb{R}^{n+1}$. True

22. Let P be a monic, non-zero, polynomial of even degree, and $K > 0$. Then the function $P(x) - Ke^x$ has a real zero. True

23. A p-Sylow subgroup of the underlying additive group of a finite commutative ring R is an ideal in R. True

24. Suppose A is an $n \times n$-real matrix, all whose eigenvalues have absolute value less than 1. Then for any $v \in \mathbb{R}^n$, $||Av|| \leq ||v||$. False

25. For any $x \in \mathbb{R}$, the sequence $\{a_n\}$, where $a_1 = x$ and $a_{n+1} = \cos(a_n)$ for all n, is convergent. True

26. Suppose $A_1, \cdots, A_m$ are distinct $n \times n$ real matrices such that $A_iA_j = 0$ for all $i \neq j$. Then $m \leq n$. False

27. In the symmetric group S_n any two elements of the same order are conjugate. False

28. If a particle moving on the Euclidean line traverses distance 1 in time 1 starting and ending at rest, then at some time $t \in [0, 1]$, the absolute value of its acceleration should be at least 4. True

29. Let $y(t)$ be a real valued function defined on the real line such that $y' = y(1-y)$, with $y(0) \in [0, 1]$. Then $\lim_{t\to\infty} y(t) = 1$. False

30. The matrices

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}, \quad x \neq y,$$

True

for any $x,\ y \in \mathbb{R}$ are conjugate in $M_2(\mathbb{R})$.

Part II

Write your solutions in the answer booklet provided. All questions carry equal marks. There are no negative marks, and partial credit will be given for partial solutions.

1. Show that the subset $GL_n(\mathbb{R})$ of $M_n(\mathbb{R})$ consisting of all invertible matrices is dense in $M_n(\mathbb{R})$.

2. Let f be a continuous function on $\mathbb{R}$ satisfying the relation

$$f(f(f(x))) = x \quad \text{for all } x \in \mathbb{R}.$$

Prove or disprove that f is the identity function.

3. Prove or disprove: the group of positive rationals under multiplication is isomorphic to its subgroup consisting of rationals which can be expressed as p/q, where both p and q are odd positive integers.

4. Show that the only elements in $M_n(\mathbb{R})$ commuting with every idempotent matrix are the scalar matrices. (A matrix P in $M_n(\mathbb{R})$ is said to be idempotent if $P^2 = P$.)

5. Prove or disprove the following: let $f : X \to X$ be a continuous function from a complete metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Then f has a fixed point.

6. How many isomorphism classes of associative rings (with identity) are there with 35 elements? Prove your answer.

7. Prove or disprove: If G is a finite group and $g, h \in G$, then g, h have the same order if and only if there exists a group H containing G such that g and h are conjugate in H.

8. Prove or disprove: there exists $A \subset \mathbb{N}$ with exactly five elements, such that sum of any three elements of A is a prime number.

9. Show that there does not exist any continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every value exactly twice.

10. For which positive integers n does there exist a $\mathbb{R}$-linear ring homomorphism $f : \mathbb{C} \to M_n(\mathbb{R})$? Justify your answer.

MTH

GS-2018 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 10, 2017**

For the Ph.D. Programs at TIFR, Mumbai and CAM & ICTS, Bangalore and for the Int. Ph.D. Programs at TIFR, Mumbai and CAM, Bangalore.

Duration: Two hours (2 hours)

Name: ____ Ref. Code: ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet for. The Answer Sheet is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question: this will be treated as a wrong answer.

3. There are twenty-five (25) True/False type questions in **PART A** of the question paper. **PART B** contains 15 multiple choice questions. Questions in both Parts carry +1 for a correct answer, -1 (negative marks) for a wrong answer and 0 for not answering.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator.

6. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

8. Notation and Conventions used in this test are given on page 2 of the question paper.

Test structure

The duration of this test is two hours. It has two parts, Part A and Part B. Part A has 25 'True or False' questions. Part B has 15 multiple choice questions. Each multiple choice question comes with four options, of which exactly one is correct.

Marking scheme

In both Part A and Part B, a correct answer will get 1 point, a wrong answer or an invalid answer (such as ticking multiple boxes) will get -1 point, and not attempting a particular question will get 0 points.

Notation and conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \cdots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rationals, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and metric.
- $M_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices with the Euclidean metric, and I denotes the identity matrix.
- For any prime number p, $\mathbb{F}_p$ denotes the finite field with p elements.
- All rings are associative, with a multiplicative identity.
- All logarithms are natural logarithms.

Part A

Answer whether the following statements are True or False. Mark your answer on the machine checkable answer sheet that is provided.

1. Let A be a countable subset of $\mathbb{R}$ which is well-ordered with respect to the usual ordering on $\mathbb{R}$ (where 'well-ordered' means that every nonempty subset has a minimum element in it). Then A has an order preserving bijection with a subset of $\mathbb{N}$. F

2. $\lim_{x\to 0} \dfrac{\sin x}{\log(1+\tan x)} = 1.$ T

3. For any closed subset $A \subset \mathbb{R}$, there exists a continuous function f on $\mathbb{R}$ which vanishes exactly on A. T

4. Let f be a nonnegative continuous function on $\mathbb{R}$ such that $\int_0^\infty f(t)dt$ is finite. Then $\lim_{x\to\infty} f(x) = 0$. F

5. The function $f(x) = \cos(e^x)$ is not uniformly continuous on $\mathbb{R}$. T

6. Let A be a 3×3 real symmetric matrix such that $A^6 = I$. Then, $A^2 = I$. T

7. In the vector space $\{f \mid f : [0,1] \to \mathbb{R}\}$ of real-valued functions on the closed interval $[0,1]$, the set $S = \{\sin(x), \cos(x), \tan(x)\}$ is linearly independent. T

8. Let f be a twice differentiable function on $\mathbb{R}$ such that both f and f'' are strictly positive on $\mathbb{R}$. Then $\lim_{x\to\infty} f(x) = \infty$. F

9. Let G, H be finite groups. Then any subgroup of $G \times H$ is equal to $A \times B$ for some subgroups $A < G$ and $B < H$. F

10. Let g be a continuous function on $[0,1]$ such that $g(1) = 0$. Then the sequence of functions $f_n(x) = x^n g(x)$ converges uniformly on $[0,1]$. T

11. Let $A, B, C \in M_3(\mathbb{R})$ be such that A commutes with B, B commutes with C and B is not a scalar matrix. Then A commutes with C. F

12. If $A \in M_n(\mathbb{R})$ (with $n \geq 2$) has rank 1, then the minimal polynomial of A has degree 2. T

13. Let V be the vector space over $\mathbb{R}$ consisting of polynomials of degree less than or equal to 3. Let $T : V \to V$ be the operator sending $f(t)$ to $f(t+1)$, and $D : V \to V$ the operator sending $f(t)$ to $df(t)/dt$. Then T is a polynomial in D. T

14. Let V be the subspace of the real vector space of real valued functions on $\mathbb{R}$, spanned by $\cos t$ and $\sin t$. Let $D : V \to V$ be the linear map sending $f(t) \in V$ to $df(t)/dt$. Then D has a real eigenvalue. F

15. The set of nilpotent matrices in $M_3(\mathbb{R})$ spans $M_3(\mathbb{R})$ considered as an $\mathbb{R}$-vector space (a matrix A is said to be nilpotent if there exists $n \in \mathbb{N}$ such that $A^n = 0$). F

16. Let G be a finite group with a normal subgroup H such that G/H has order 7. Then $G \cong H \times G/H$. F

17. The multiplicative group $\mathbb{F}_7^\times$ is isomorphic to a subgroup of the multiplicative group $\mathbb{F}_{31}^\times$. T

18. Any linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^4$ has a proper non-zero invariant subspace. T

19. Let $A, B \in M_n(\mathbb{R})$ be such that $A + B = AB$. Then $AB = BA$. T

20. Let $A \in M_n(\mathbb{R})$ be upper triangular with all diagonal entries 1 such that $A \neq I$. Then A is not diagonalizable. T

21. A countable group can have only countably many distinct subgroups. F

22. There exists a continuous surjection from $\mathbb{R}^3 - S^2$ to $\mathbb{R}^2 - \{(0,0)\}$ (here $S^2 \subset \mathbb{R}^3$ denotes the unit sphere defined by the equation $x^2 + y^2 + z^2 = 1$). T

23. The permutation group S_{10} has an element of order 30. T

24. Let G be a finite group and $g \in G$ an element of even order. Then we can colour the elements of G with two colours in such a way that x and gx have different colours for each $x \in G$. T

25. Let $f(x)$ and $g(x)$ be uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then their pointwise product $f(x)g(x)$ is uniformly continuous. F

Part B

Answer the following multiple choice questions, by appropriately marking your answer on the machine checkable answer sheet that is provided.

1. The set of real numbers in the open interval $(0, 1)$ which have more than one decimal expansion is

   (a) empty.

   (b) non-empty but finite.

   (c) countably infinite.

   (d) uncountable.

2. How many zeroes does the function $f(x) = e^x - 3x^2$ have in $\mathbb{R}$?

   (a) 0

   (b) 1

   (c) 2

   (d) 3.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \textit{and} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ with } m, n \in \mathbb{Z}, n > 0, \text{ and } \gcd(m, n) = 1. \end{cases}$$

   Which of the following statements is true?

   (a) f is continuous everywhere except at 0.

   (b) f is continuous only at the irrationals.

   (c) f is continuous only at the non-zero rationals.

   (d) f is not continuous anywhere.

4. Suppose p is a degree 3 polynomial such that $p(0) = 1$, $p(1) = 2$, and $p(2) = 5$. Which of the following numbers cannot equal $p(3)$?

   (a) 0

   (b) 2

   (c) 6

   (d) 10.

5. Let A be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following two properties:

   - f has derivatives of all orders, and
   - for all $x, y \in \mathbb{R}$,

   $$f(x+y) - f(y-x) = 2xf'(y).$$

   Which of the following sentences is true?

   (a) Any $f \in A$ is a polynomial of degree less than or equal to 1.

   (b) Any $f \in A$ is a polynomial of degree less than or equal to 2. ✓

   (c) There exists $f \in A$ which is not a polynomial.

   (d) There exists $f \in A$ which is a polynomial of degree 4.

6. Denote by $\mathfrak{A}$ the set of all $n \times n$ complex matrices A ($n \geq 2$ a natural number) having the property that 4 is the only eigenvalue of A. Consider the following four statements.

   - $(A - 4I)^n = 0$,
   - $A^n = 4^n I$,
   - $(A^2 - 5A + 4I)^n = 0$,
   - $A^n = 4nI$.

   How many of the above statements are true for all $A \in \mathfrak{A}$?

   (a) 0

   (b) 1

   (c) 2 ✓

   (d) 3.

7. Let A be the set of all continuous functions $f : [0,1] \to [0, \infty)$ satisfying the following condition:

   $$\int_0^x f(t)\,dt \geq f(x), \text{ for all } x \in [0,1].$$

   Then which of the following statements is true?

   (a) A has cardinality 1. ✓

   (b) A has cardinality 2.

   (c) A is infinite.

   (d) A is empty.

8. Consider the following four sets of maps $f : \mathbb{Z} \to \mathbb{Q}$:

   (i) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is bijective and increasing}\}$,

   (ii) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is onto and increasing}\}$,

   (iii) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is bijective, and satisfies that } \forall\, n \leq 0, f(n) \geq 0\}$, and

   (iv) $\{f : \mathbb{Z} \to \mathbb{Q} \mid f \text{ is onto and decreasing}\}$.

   How many of these sets are empty?

   (a) 0

   (b) 1

   (c) 2

   (d) 3. ✓

9. What are the last 3 digits of 2^{2017}?

   (a) 072 ✓

   (b) 472

   (c) 512

   (d) 912.

10. The minimal polynomial of $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}$ is

    (a) $(x-2)(x-5)$.

    (b) $(x-2)^2(x-5)$. ✓

    (c) $(x-2)^3(x-5)$.

    (d) none of the above.

11. Consider a cube C centered at the origin in $\mathbb{R}^3$. The number of invertible linear transformations of $\mathbb{R}^3$ which map C onto itself is

    (a) 72

    (b) 48 ✓

    (c) 24

    (d) 12.

12. The number of rings of order 4, up to isomorphism, is:

   (a) 1
   (b) 2
   (c) 3
   (d) 4.

13. For a sequence $\{a_n\}$ of real numbers, which of the following is a negation of the statement '$\lim_{n\to\infty} a_n = 0$'?

   (a) There exists $\varepsilon > 0$ such that the set $\{n \in \mathbb{N} \mid |a_n| > \varepsilon\}$ is infinite.
   (b) For any $M > 0$, there exists $N \in \mathbb{N}$ such that $|a_n| > M$ for all $n \geq N$.
   (c) There exists a nonzero real number a such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $|a_n - a| < \varepsilon$ for all $n \geq N$.
   (d) For any $a \in \mathbb{R}$, and every $\varepsilon > 0$, there exist infinitely many n such that $|a_n - a| > \varepsilon$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then which of the following statements implies that $f(0) = 0$?

   (a) $\displaystyle\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0.$
   (b) $\displaystyle\lim_{n\to\infty} \int_0^1 f(x/n) \, dx = 0.$
   (c) $\displaystyle\lim_{n\to\infty} \int_0^1 f(nx) \, dx = 0.$
   (d) None of the above.

15. Consider the following maps from $\mathbb{R}^2$ to $\mathbb{R}^2$:

   (i) the map $(x, y) \mapsto (2x + 5y + 1, x + 3y)$,
   (ii) the map $(x, y) \mapsto (x + y^2, y + x^2)$, and
   (iii) the map given in polar coordinates as $(r, \theta) \mapsto (r, \theta + r^3)$ for $r \neq 0$, with the origin mapping to the origin.

   The number of maps in the above list that preserve areas is:

   (a) 0
   (b) 1
   (c) 2
   (d) 3.

GS2019 - Mathematics Question Paper

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $M_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices with the Euclidean metric, and I denotes the identity matrix in $M_n(\mathbb{R})$.
- All rings are associative, with a multiplicative identity.
- For any prime number p, $\mathbb{F}_p$ denotes the finite field with p elements.
- If A and B are sets, then $A - B$ refers to $\{x \in A \mid x \notin B\}$.
- For a ring R, $R[x]$ denotes the polynomial ring in one variable over R, and $R[x, y]$ denotes the polynomial ring in two variables over R.

PART A

Answer the following multiple choice questions.

1. The following sum of numbers (expressed in decimal notation)

$$1 + 11 + 111 + \cdots + \underbrace{11\ldots1}_{n}$$

is equal to

(a) $(10^{n+1} - 10 - 9n)/81$
(b) $(10^{n+1} - 10 + 9n)/81$
(c) $(10^{n+1} - 10 - n)/81$
(d) $(10^{n+1} - 10 + n)/81$

2. For $n \geq 1$, the sequence $\{x_n\}_{n=1}^{\infty}$, where:

$$x_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

is

(a) decreasing
(b) increasing
(c) constant
(d) oscillating

3. Define a function:

$$f(x) = \begin{cases} x + x^2\cos(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Consider the following statements:

(i) $f'(0)$ exists and is equal to 1
(ii) f is not increasing in any neighborhood of 0
(iii) $f'(0)$ does not exist
(iv) f is increasing on $\mathbb{R}$.

How many of the above statements is/are true?

(a) 0
(b) 1
(c) 2
(d) 3

4. Consider differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the property that for all $a, b \in \mathbb{R}$ we have:
$$f(b) - f(a) = (b-a)f'\left(\frac{a+b}{2}\right).$$
Then which one of the following sentences is true?

   ✓ (a) Every such f is a polynomial of degree less than or equal to 2
   (b) There exists such a function f which is a polynomial of degree bigger than 2
   (c) There exists such a function f which is not a polynomial
   (d) Every such f satisfies the condition $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$ for all $a, b \in \mathbb{R}$

5. Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation such that
$$\text{Rank } T \leq \text{Rank } T^3.$$
Then which one of the following statements is necessarily true?

   (a) Null space(T) = Range(T)
   ✓ (b) Null space$(T) \cap$ Range$(T) = \{0\}$
   (c) There exists a nonzero subspace W of V such that Null space$(T) \cap$ Range$(T) = W$
   (d) Null space$(T) \subseteq$ Range(T)

6. The limit
$$\lim_{n\to\infty} n^2 \int_0^1 \frac{1}{(1+x^2)^n} dx$$
is equal to

   (a) 1
   (b) 0
   ✓ (c) $+\infty$
   (d) 1/2

7. Let A be an $n \times n$ matrix with rank k. Consider the following statements:

   (i) If A has real entries, then AA^t necessarily has rank k
   (ii) If A has complex entries, then AA^t necessarily has rank k.

   Then

   (a) (i) and (ii) are true
   (b) (i) and (ii) are false
   ✓ (c) (i) is true and (ii) is false
   (d) (i) is false and (ii) is true

8. Consider the following two statements:

(E) Continuous functions on $[1, 2]$ can be approximated uniformly by a sequence of even polynomials (i.e., polynomials $p(x) \in \mathbb{R}[x]$ such that $p(-x) = p(x)$).

(O) Continuous functions on $[1, 2]$ can be approximated uniformly by a sequence of odd polynomials (i.e., polynomials $p(x) \in \mathbb{R}[x]$ such that $p(-x) = -p(x)$).

Choose the correct option below.

(a) (E) and (O) are both false

✓ (b) (E) and (O) are both true

(c) (E) is true but (O) is false

(d) (E) is false but (O) is true

9. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{\sin(x^3)}{x}$. Then f is

✓ (a) bounded and uniformly continuous

(b) bounded but not uniformly continuous

(c) not bounded but uniformly continuous

(d) not bounded and not uniformly continuous

10. Let

$$S = \{x \in \mathbb{R} \mid x = \text{Trace}(A) \text{ for some } A \in M_4(\mathbb{R}) \text{ such that } A^2 = A\}.$$

Then which of the following describes S?

(a) $S = \{0, 2, 4\}$

(b) $S = \{0, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4\}$

✓ (c) $S = \{0, 1, 2, 3, 4\}$

(d) $S = [0, 4]$

11. Let f be a continuous function on $[0, 1]$. Then the limit $\lim_{n\to\infty} \int_0^1 nx^n f(x)\, dx$ is equal to

(a) $f(0)$

✓ (b) $f(1)$

(c) $\sup_{x\in[0,1]} f(x)$

(d) The limit need not exist

12. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$, defined by

$$f_n(x) = \frac{1}{n} \exp(-n^2 x^2).$$

Then which one of the following statements is true?

(a) Both the sequences $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $\mathbb{R}$

(b) Neither $\{f_n\}$ nor $\{f_n'\}$ converges uniformly on $\mathbb{R}$

(c) $\{f_n\}$ converges pointwise but not uniformly on any interval containing the origin

✔ (d) $\{f_n'\}$ converges pointwise but not uniformly on any interval containing the origin

13. Let the sequence $\{x_n\}_{n=1}^{\infty}$ be defined by $x_1 = \sqrt{2}$ and $x_{n+1} = (\sqrt{2})^{x_n}$ for $n \geq 1$. Then which one of the following statements is true?

✔ (a) The sequence $\{x_n\}$ is monotonically increasing and $\lim_{n\to\infty} x_n = 2$

(b) The sequence $\{x_n\}$ is neither monotonically increasing nor monotonically decreasing

(c) $\lim_{n\to\infty} x_n$ does not exist

(d) $\lim_{n\to\infty} x_n = \infty$

14. Consider functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $|f(x) - f(y)| \leq 4321|x - y|$ for all real numbers x, y. Then which one of the following statements is true?

(a) f is always differentiable

(b) There exists at least one such f that is continuous and such that $\lim_{x\to\pm\infty} \frac{f(x)}{|x|} = \infty$

✔ (c) There exists at least one such f that is continuous, but is non-differentiable at exactly 2018 points, and satisfies $\lim_{x\to\pm\infty} \frac{f(x)}{|x|} = 2018$

(d) It is not possible to find a sequence $\{x_n\}$ of real numbers such that $\lim_{n\to\infty} x_n = \infty$ and further satisfying $\lim_{n\to\infty} \left|\frac{f(x_n)}{x_n}\right| \leq 10000$

15. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$, defined by

$$f_n(x) = \frac{\sqrt{1 + (nx)^2}}{n}.$$

Then which one of the following statements is true?

(a) $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $\mathbb{R}$

(b) $\{f_n'\}$ converges uniformly on $\mathbb{R}$ but $\{f_n\}$ does not

✔ (c) $\{f_n\}$ converges uniformly on $\mathbb{R}$ but $\{f_n'\}$ does not

(d) $\{f_n\}$ converges uniformly to a differentiable function on $\mathbb{R}$

16. The number of ring homomorphisms from $\mathbb{Z}[x, y]$ to $\mathbb{F}_2[x]/(x^3 + x^2 + x + 1)$ equals

✔ (a) 2^6

(b) 2^{18}

(c) 1

(d) 2^9

17. Let $X \subset \mathbb{R}^2$ be the subset

$$X = \{(x, y) \mid x = 0, |y| \leq 1\} \cup \left\{(x, y) \mid 0 < x \leq 1, y = \sin\frac{1}{x}\right\}.$$

Consider the following statements:

(i) X is compact

(ii) X is connected

(iii) X is path connected.

How many of the statements (i)-(iii) is/are true?

(a) 0

(b) 1

(c) 2

(d) 3

18. Consider the different ways to colour the faces of a cube with six given colours, such that each face is given exactly one colour and all the six colours are used. Define two such colouring schemes to be equivalent if the resulting configurations can be obtained from one another by a rotation of the cube. Then the number of inequivalent colouring schemes is

(a) 15

(b) 24

(c) 30

(d) 48

19. Let $C^\infty(0,1)$ stand for the set of all real-valued functions on $(0,1)$ that have derivatives of all orders. Then the map $C^\infty(0,1) \to C^\infty(0,1)$ given by

$$f \mapsto f + \frac{df}{dx}$$

is

(a) injective but not surjective

(b) surjective but not injective

(c) neither injective nor surjective

(d) both injective and surjective

20. A stick of length 1 is broken into two pieces by cutting at a randomly chosen point. What is the expected length of the smaller piece?

(a) 1/8

(b) 1/4

(c) $1/e$

(d) $1/\pi$

PART B

Answer whether the following statements are True or False.

F 1. There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{R} - \mathbb{Q}$ and $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$.

T 2. If $A \in M_{10}(\mathbb{R})$ satisfies $A^2 + A + I = 0$, then A is invertible.

F 3. Let $X \subseteq \mathbb{Q}^2$. Suppose each continuous function $f : X \to \mathbb{R}^2$ is bounded. Then X is necessarily finite.

F 4. If A is a 2×2 complex matrix that is invertible and diagonalizable, and such that A and A^2 have the same characteristic polynomial, then A is the 2×2 identity matrix.

F 5. Suppose A, B, C are 3×3 real matrices with Rank $A = 2$, Rank $B = 1$, Rank $C = 2$. Then Rank $(ABC) = 1$.

F 6. For any $n \geq 2$, there exists an $n \times n$ real matrix A such that the set $\{A^p \mid p \geq 1\}$ spans the $\mathbb{R}$-vector space $M_n(\mathbb{R})$.

T 7. The matrices

$$\begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

are similar.

F 8. Consider the set $A \subset M_3(\mathbb{R})$ of 3×3 real matrices with characteristic polynomial $x^3 - 3x^2 + 2x - 1$. Then A is a compact subset of $M_3(\mathbb{R}) \cong \mathbb{R}^9$.

F 9. There exists an injective ring homomorphism from the product ring $\mathbb{R} \times \mathbb{R}$ into $C(\mathbb{R})$, where $C(\mathbb{R})$ denotes the ring of all continuous functions $\mathbb{R} \to \mathbb{R}$ under pointwise addition and multiplication.

T 10. $\mathbb{R}$ and $\mathbb{R} \oplus \mathbb{R}$ are isomorphic as vector spaces over $\mathbb{Q}$.

T 11. If 0 is a limit point of a set $A \subseteq (0, \infty)$, then the set of all $x \in (0, \infty)$ that can be expressed as a sum of (not necessarily distinct) elements of A is dense in $(0, \infty)$.

F 12. The only idempotents in the ring $\mathbb{Z}_{51}$ (i.e., $\mathbb{Z}/51\mathbb{Z}$) are 0 and 1. (An idempotent is an element x such that $x^2 = x$).

T 13. Let A be a commutative ring with 1, and let $a, b, c \in A$. Suppose there exist $x, y, z \in A$ such that $ax + by + cz = 1$. Then there exist $x', y', z' \in A$ such that $a^{50}x' + b^{20}y' + c^{15}z' = 1$.

F 14. The ring $\mathbb{R}[x]/(x^5 + x - 3)$ is an integral domain.

F 15. Given any group G of order 12, and any n that divides 12, there exists a subgroup H of G of order n.

T 16. Let H, N be subgroups of a finite group G, with N a normal subgroup of G. If the orders of G/N and H are relatively prime, then H is necessarily contained in N.

F 17. If every proper subgroup of an infinite group G is cyclic, then G is cyclic.

T 18. Each solution of the differential equation

$$y'' + e^x y = 0$$

remains bounded as $x \to \infty$.

F 19. There exists a uniformly continuous function $f : (0, \infty) \to (0, \infty)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converges.

F 20. Let $v : \mathbb{R} \to \mathbb{R}^2$ be C^∞ (i.e., has derivatives of all orders). Then there exists $t_0 \in (0, 1)$ such that $v(1) - v(0)$ is a scalar multiple of $\frac{dv}{dt}\big|_{t=t_0}$.

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $\mathrm{M}_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices, and $\mathrm{M}_n(\mathbb{C})$ the complex vector space of $n \times n$ complex matrices. I denotes the identity matrix in $\mathrm{M}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{C})$.
- For any $A \in \mathrm{M}_n(\mathbb{C})$, we denote by $\mathrm{tr}(A)$ the trace of A and by $\det(A)$ the determinant of A.
- All rings are associative, with a multiplicative identity.
- For a ring R, $R[x]$ denotes the polynomial ring in one variable over R, and $R^\times$ denotes the multiplicative group of units of R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.

PART A

Answer the following multiple choice questions.

1. Consider the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ defined by

$$a_n = (2^n + 3^n)^{1/n} \text{ and } b_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}.$$

What is the limit of $\{b_n\}_{n=1}^{\infty}$?

(a) 2.

✓ (b) 3.

(c) 5.

(d) The limit does not exist.

2. Consider the set of continuous functions $f : [0,1] \to \mathbb{R}$ that satisfy:

$$\int_0^1 f(x)(1 - f(x))\, dx = \frac{1}{4}.$$

Then the cardinality of this set is:

(a) 0.

✓ (b) 1.

(c) 2.

(d) more than 2.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0, \text{ and} \\ 0, & \text{if } x = 0. \end{cases}$$

Which of the following statements is correct?

✓ (a) f is a surjective function.

(b) f is bounded.

(c) The derivative of f exists and is continuous on $\mathbb{R}$.

(d) $\{x \in \mathbb{R} \mid f(x) = 0\}$ is a finite set.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing bounded sequence of real numbers such that $\lim_{n\to\infty} a_n = A$. Let $f : [a_1, A] \to \mathbb{R}$ be a continuous function such that for each positive integer i, $f|_{[a_i, a_{i+1}]} : [a_i, a_{i+1}] \to \mathbb{R}$ is either strictly increasing or strictly decreasing. Consider the set

$$B = \{M \in \mathbb{R} \mid \text{ there exist infinitely many } x \in [a_1, A] \text{ such that } f(x) = M\}.$$

Then the cardinality of B is:

(a) necessarily 0.

✓ (b) at most 1.

(c) possibly greater than 1, but finite.

(d) possibly infinite.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies:

$$|f(x) - f(y)| \leq |x - y||\sin(x - y)|, \text{ for all } x, y \in \mathbb{R}.$$

Which of the following statements is correct?

(a) f is continuous but need not be uniformly continuous.

(b) f is uniformly continuous but not necessarily differentiable.

(c) f is differentiable, but its derivative may not be continuous.

✓ (d) f is constant.

6. Let

$$\mathcal{C} = \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable, and } \lim_{x \to \infty} (2f(x) + f'(x)) = 0 \right\}.$$

Which of the following statements is correct?

(a) For each L with $0 \neq L < \infty$, there exists $f \in \mathcal{C}$ such that $\lim_{x \to \infty} f(x) = L$.

✓ (b) For all $f \in \mathcal{C}$, $\lim_{x \to \infty} f(x) = 0$.

(c) There exists $f \in \mathcal{C}$ such that $\lim_{x \to \infty} f(x)$ does not exist.

(d) There exists $f \in \mathcal{C}$ such that $\lim_{x \to \infty} f(x) = \frac{1}{2}$.

7. Let $f(x) = \dfrac{\log(2 + x)}{\sqrt{1 + x}}$ for $x \geq 0$, and $a_m = \dfrac{1}{m} \int_0^m f(t)\, dt$ for every positive integer m. Then the sequence $\{a_m\}_{m=1}^{\infty}$

(a) diverges to $+\infty$.

(b) has more than one limit point.

(c) converges and satisfies $\lim_{m \to \infty} a_m = \frac{1}{2} \log 2$.

✓ (d) converges and satisfies $\lim_{m \to \infty} a_m = 0$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that:

$$|f(x) - f(y)| \geq \log(1 + |x - y|), \text{ for all } x, y \in \mathbb{R}.$$

Then:

(a) f is injective but not surjective.

(b) f is surjective but not injective.

(c) f is neither injective nor surjective.

✓ (d) f is bijective.

9. What is the greatest integer less than or equal to

$$\sum_{n=1}^{9999} \frac{1}{\sqrt[4]{n}}?$$

(a) 1332

(b) 1352

(c) 1372

(d) 1392

10. Consider the following sentences:

(I) For every connected subset Y of a metric space X, its interior Y° is connected.

(II) For every connected subset Y of a metric space X, its boundary ∂Y is connected.

Which of the following options is correct?

(a) (I) is true, but (II) is false.

(b) (II) is true, but (I) is false.

(c) (I) and (II) are both true.

(d) (I) and (II) are both false.

11. Consider a set $\{A_1, \ldots, A_n\}$ of events, $n > 1$. Suppose that one of the events in $\{A_1, \ldots, A_n\}$ is certain to occur, but that no more than two of them can occur. Suppose that for each $1 \leq r, s \leq n$ such that $r \neq s$, the probability of A_r occurring is p, while the probability of both A_r and A_s occurring is q. Then:

(a) $p \leq 1/n$ and $q \leq 2/n$.

(b) $p \leq 1/n$ and $q \geq 2/n$.

(c) $p \geq 1/n$ and $q \leq 2/n$.

(d) $p \geq 1/n$ and $q \geq 2/n$.

12. Let $\{z_1, z_2, \ldots, z_7\}$ be a set of seven complex numbers with unit modulus. Assume that they form the vertices of a regular heptagon in the complex plane. Define

$$w = \sum_{i<j} z_i z_j.$$

Then:

(a) $w = 0$.

(b) $|w| = \sqrt{7}$.

(c) $|w| = 7$.

(d) $|w| = 1$.

13. Consider $\mathbb{R}^3$ as the space of 3×1 real matrices. The multiplicative group $\mathrm{GL}_3(\mathbb{R})$ of invertible 3×3 real matrices acts on this space by left multiplication. What is the number of orbits for this action?

    (a) 1.

    ✓ (b) 2.

    (c) 4.

    (d) ∞.

14. Let V be a finite dimensional vector space over $\mathbb{R}$, and $W \subset V$ a subspace. Then $W \cap T(W) \neq \{0\}$ for every linear automorphism $T : V \to V$ if and only if:

    (a) $W = V$.

    (b) $\dim W < \dfrac{1}{2} \dim V$.

    (c) $\dim W = \dfrac{1}{2} \dim V$.

    ✓ (d) $\dim W > \dfrac{1}{2} \dim V$.

15. Let $A \in \mathrm{M}_n(\mathbb{C})$. Then $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is diagonalizable if and only if:

    ✓ (a) $A = 0$.

    (b) $A = I$.

    (c) $n = 2$.

    (d) None of the other three options.

16. Let $T : \mathbb{C} \to \mathbb{R}$ be the map defined by $T(z) = z + \bar{z}$. For a $\mathbb{C}$-vector space V, consider the map

$$\varphi : \{f : V \to \mathbb{C} \mid f \text{ is } \mathbb{C}\text{-linear}\} \to \{g : V \to \mathbb{R} \mid g \text{ is } \mathbb{R}\text{-linear}\},$$

    defined by $\varphi(f) = T \circ f$. Then this map is

    (a) injective, but not surjective.

    (b) surjective, but not injective.

    ✓ (c) bijective.

    (d) neither injective nor surjective.

17. Which of the following statements is correct for every linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T^3 - T^2 - T + I = 0$?

    (a) T is invertible as well as diagonalizable.

    ✓ (b) T is invertible, but not necessarily diagonalizable.

    (c) T is diagonalizable, but not necessarily invertible.

    (d) None of the other three statements.

18. Let $n \geq 2$. Which of the following statements is true for every $n \times n$ real matrix A of rank one?

    (a) There exist matrices $P, Q \in \mathrm{M}_n(\mathbb{R})$ such that all the entries of the matrix PAQ are equal to 1.

    (b) There exists an invertible matrix $P \in \mathrm{M}_n(\mathbb{R})$ such that PAP^{-1} is a diagonal matrix.

    (c) A has a nonzero eigenvalue.

    (d) The vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$ is an eigenvector for A.

19. Let m, n be positive integers. Then the greatest common divisor (gcd) of the polynomials $x^m - 1$ and $x^n - 1$ in the ring $\mathbb{C}[x]$ equals

    (a) $x^{\min(m,n)} - 1$.

    (b) $x - 1$.

    (c) $x^{\gcd(m,n)} - 1$.

    (d) None of the other three options.

20. Let A_4 denote the group of even permutations of $\{1, 2, 3, 4\}$. Consider the following statements:

    (I) There exists a surjective group homomorphism $A_4 \to \mathbb{Z}/4\mathbb{Z}$.

    (II) There exists a surjective group homomorphism $A_4 \to \mathbb{Z}/3\mathbb{Z}$.

    Which of the following statements is correct?

    (a) (I) is true and (II) is false.

    (b) (II) is true and (I) is false.

    (c) (I) and (II) are both true.

    (d) (I) and (II) are both false.

PART B

True/False Questions.

F 1. There exists no monotone function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every rational number.

T 2. Let $C([0, 1])$ denote the set of continuous real valued functions on $[0, 1]$, and $\mathbb{R}^{\mathbb{N}}$ the set of all sequences of real numbers. Then there exists an injective map from $C([0, 1])$ to $\mathbb{R}^{\mathbb{N}}$.

T 3. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of positive real numbers. Then:

$$\limsup_{n\to\infty} \frac{1}{a_n} = \frac{1}{\liminf_{n\to\infty} a_n}.$$

T 4. Let $C([0,1])$ denote the metric space of continuous real valued functions on $[0,1]$ under the supremum metric - i.e., the distance between f and g in $C([0,1])$ equals

$$\sup\{|f(x) - g(x)| \mid x \in [0,1]\}.$$

Let $Q \subset C([0,1])$ be the set of all polynomials in $\mathbb{R}[x]$ in which the coefficient of x^2 is 0. Then Q is dense in $C([0,1])$.

F 5. If X is a metric space such that every continuous function $f : X \to \mathbb{R}$ is uniformly continuous, then X is compact.

T 6. Let X be a metric space, and let $C(X)$ denote the $\mathbb{R}$-vector space of continuous real valued functions on X. Then X is infinite if and only if $\dim_{\mathbb{R}} C(X) = \infty$.

T 7. Let A be a countable union of lines in $\mathbb{R}^3$. Then $\mathbb{R}^3 \setminus A$ is connected.

T 8. An invertible linear map from $\mathbb{R}^2$ to itself takes parallel lines to parallel lines.

F 9. For any matrix C with entries in $\mathbb{C}$, let $m(C)$ denote the minimal polynomial of C, and $p(C)$ its characteristic polynomial. Then for any $n \in \mathbb{N}$, two matrices $A, B \in \mathrm{M}_n(\mathbb{C})$ are similar if and only if $m(A) = m(B)$ and $p(A) = p(B)$.

T 10. Let $A, B \in \mathrm{M}_3(\mathbb{R})$. Then

$$\det(AB - BA) = \frac{\mathrm{tr}[(AB - BA)^3]}{3}.$$

F 11. There exist an integer $r \geq 1$ and a symmetric matrix $A \in \mathrm{M}_r(\mathbb{R})$ such that for all $n \in \mathbb{N}$, we have:

$$2^{\sqrt{n}} \leq |\mathrm{tr}(A^n)| \leq 2020 \cdot 2^{\sqrt{n}}.$$

T 12. The polynomial $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{101}}{101!}$ is irreducible in $\mathbb{Q}[x]$.

F 13. There exists an integer $n > 3$ such that the group of units of the ring $\mathbb{Z}/2^n\mathbb{Z}$ is cyclic.

F 14. For every surjective ring homomorphism $\varphi : R \to S$, we have $\varphi(R^\times) = S^\times$.

F 15. Let G be a finite group and P a p-Sylow subgroup of G, where p is a prime number. Then for every subgroup H of G, $H \cap P$ is a p-Sylow subgroup of H.

T 16. Let G be an abelian group, with identity element e. If

$$\{g \in G \mid g = e \text{ or } g \text{ has infinite order}\}$$

is a subgroup of G, then either all elements of $G \setminus \{e\}$ have infinite order, or all elements of G have finite order.

F 17. There exists a natural number n, with $1 < n \leq 10$, such that x^n and x are conjugate for every element x of S_7, the group of permutations of $\{1, \ldots, 7\}$.

F 18. Every noncommutative ring has at least 10 elements.

T 19. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in $\{0,1\}$ such that for all positive integers n, $\sum_{i=n}^{n+9} a_i$ is divisible by 3. Then there exists a positive integer k such that $a_{n+k} = a_n$ for all positive integers n.

T 20. The interior of any strip bounded by two parallel lines in $\mathbb{R}^2$, of width strictly greater than 1, contains a point with integer coordinates.

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $\mathrm{M}_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices, and $\mathrm{M}_n(\mathbb{C})$ the complex vector space of $n \times n$ complex matrices. $\mathrm{M}_n(\mathbb{R})$ gets the topology transferred from any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, and its subsets get the subspace topology.
- Id denotes the identity matrix in $\mathrm{M}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{C})$.
- A matrix $A \in \mathrm{M}_n(\mathbb{R})$ is called idempotent if $A^2 = A$, and nilpotent if $A^m = 0$ for some positive integer m.
- All rings are associative, with a multiplicative identity.
- For a ring R, $R[x]$ denotes the polynomial ring in one variable over R, and $R^\times$ denotes the multiplicative group of units of R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- An isometry from a metric space (X, d) to a metric space (X', d') is a (not necessarily surjective) map $f : X \to X'$ such that for all $a, b \in X$, we have $d'(f(a), f(b)) = d(a, b)$.

PART A

Answer the following multiple choice questions.

1. For each positive integer n, let

$$s_n = \frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}}.$$

   Then $\lim_{n\to\infty} s_n$ equals

   (a) $\pi/2$
   (b) $\pi/6$
   (c) $1/2$
   (d) ∞

2. The number of bijective maps $g : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^2} < \infty$$

   is

   (a) 0
   (b) 1
   (c) 2
   (d) ∞

3. The value of

$$\lim_{n\to\infty} \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)$$

   is

   (a) $1/2$
   (b) 1
   (c) $1/4$
   (d) 0

4. The set

$$S = \{x \in \mathbb{R} \mid x > 0 \text{ and } (1 + x^2)\tan(2x) = x\}$$

   is

   (a) empty
   (b) nonempty but finite
   (c) countably infinite

(d) uncountable

5. The dimension of the real vector space

$V = \{f : (-1,1) \to \mathbb{R} \mid f \text{ is infinitely differentiable on } (-1,1) \text{ and } f^{(n)}(0) = 0 \text{ for all } n \geq 0\}$

is

(a) 0
(b) 1
(c) greater than one, but finite
✔ (d) infinite

6. For a positive integer n, let a_n denote the unique positive real root of $x^n + x^{n-1} + \cdots + x - 1 = 0$. Then

(a) the sequence $\{a_n\}_{n=1}^{\infty}$ is unbounded
(b) $\lim\limits_{n\to\infty} a_n = 0$
✔ (c) $\lim\limits_{n\to\infty} a_n = 1/2$
(d) $\lim\limits_{n\to\infty} a_n$ does not exist

7. Let A be the set of all real numbers $\lambda \in [0,1]$ such that

$$\lim_{p\to 0} \frac{\log(\lambda 2^p + (1-\lambda)3^p)}{p} = \lambda \log 2 + (1-\lambda)\log 3.$$

Then

(a) $A = \{0,1\}$
(b) $A = \left\{0, \frac{1}{2}, 1\right\}$
(c) $A = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$
✔ (d) $A = [0,1]$

8. Let $X \subseteq \mathbb{R}$ be a subset. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : X \to \mathbb{R}$, that converges uniformly to a function $f : X \to \mathbb{R}$. For each positive integer n, let $D_n \subseteq X$ denote the set of points at which f_n is not continuous. Let $D \subseteq X$ denote the set of points at which f is not continuous. Which one of the following statements is correct?

(a) If each D_n is finite, then D is finite.
✔ (b) If each D_n has at most 7 elements, then D has at most 7 elements.
(c) If each D_n is uncountable, then D is uncountable.
(d) None of the other three statements is correct.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Consider the following assertions:

(I) f is continuous.

(II) The set

$$\mathrm{Graph}(f) = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

is a connected subset of $\mathbb{R}^2$.

Which one of the following statements is correct?

(a) (I) implies (II) but (II) does not imply (I).

(b) (II) implies (I) but (I) does not imply (II).

(c) (I) implies (II) and (II) implies (I).

(d) (I) does not imply (II), and (II) does not imply (I).

10. Let $\mathcal{C}$ denote the set of colorings of an 8×8 chessboard, where each square is colored either black or white. Let $\sim$ denote the equivalence relation on $\mathcal{C}$ defined as follows: two colorings are equivalent if and only if one of them can be obtained from the other by a rotation of the chessboard. The cardinality of the set $\mathcal{C}/\sim$ of equivalence classes of elements of $\mathcal{C}$ under $\sim$ is

(a) 2^{62}

(b) $2^{62} + 2^{30} + 2^{15}$

(c) $2^{64} - 2^{32} + 2^{16}$

(d) $2^{63} - 2^{31} + 2^{15}$

11. What is the number of surjective maps from the set $\{1, \ldots, 10\}$ to the set $\{1, 2\}$?

(a) 90

(b) 1022

(c) 98

(d) 1024

12. Let V be a vector space over a field F. Consider the following assertions:

(I) V is finite dimensional.

(II) For every linear transformation $T : V \to V$, there exists a nonzero polynomial $p(x) \in F[x]$ such that $p(T) : V \to V$ is the zero map.

Which one of the following statements is correct?

(a) (I) implies (II) but (II) does not imply (I).

(b) (II) implies (I) but (I) does not imply (II).

(c) (I) implies (II) and (II) implies (I).

(d) (I) does not imply (II), and (II) does not imply (I).

13. $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be the $\mathbb{C}$-linear transformation defined on the complex vector space $\mathbb{C}[x]$ of one variable complex polynomials by $Tf(x) = f(x + 1)$. How many eigenvalues does T have?

(a) 1
(b) finite but more than 1
(c) countably infinite
(d) uncountable

14. Let $\mathbb{R}^{\mathbb{N}}$ denote the real vector space of sequences $(x_0, x_1, x_2, \dots)$ of real numbers. Define a linear transformation $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$(x_0, x_1, \dots,) \mapsto (x_0 + x_1, x_1 + x_2, \dots).$$

Which one of the following statements is correct?

(a) The kernel of T is infinite dimensional.
(b) The image of T is infinite dimensional.
(c) The quotient vector space $\mathbb{R}^{\mathbb{N}}/T(\mathbb{R}^{\mathbb{N}})$ is infinite dimensional.
(d) None of the other three statements is correct.

15. Which one of the following statements is correct?

(a) There exists a $\mathbb{C}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$.
(b) There exists no $\mathbb{C}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$, but there exists an $\mathbb{R}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$.
(c) There exists no $\mathbb{R}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$, but there exists a $\mathbb{Q}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$.
(d) None of the other three statements is correct.

16. The matrix

$$\begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

is

(a) diagonalizable
(b) nilpotent
(c) idempotent
(d) none of the other three options

17. Which of the following is a necessary and sufficient condition for two real 3×3 matrices A and B to be similar (i.e., $PAP^{-1} = B$ for an invertible real 3×3 matrix P)?

(a) They have the same characteristic polynomial.
(b) They have the same minimal polynomial.
(c) They have the same minimal and characteristic polynomials.
(d) None of the other three conditions.

18. Consider the following two subgroups A, B of the group $\mathbb{Q}[x]$ of one variable rational polynomials under addition:

$$A = \{p(x) \in \mathbb{Z}[x] \mid p \text{ has degree at most } 2\}, \text{ and}$$

$$B = \{p(x) \in \mathbb{Q}[x] \mid p \text{ has degree at most 2, and } p(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

Then the index $[B : A]$ of A in B equals

(a) 1
✓ (b) 2
(c) 4
(d) none of the other three options

19. Let G be any finite group of order 2021. For which of the following positive integers m is the map $G \to G$, given by $g \mapsto g^m$, a bijection?

(a) 43
✓ (b) 45
(c) 47
(d) none of the other three options

20. How many subgroups does $(\mathbb{Z}/13\mathbb{Z}) \times (\mathbb{Z}/13\mathbb{Z})$ have?

(a) 13
✓ (b) 16
(c) 4
(d) 25

PART B

Answer whether the following statements are True or False.

F 1. Let $f_n : [0, 1] \to \mathbb{R}$ be a continuous function for each positive integer n. If

$$\lim_{n\to\infty} \int_0^1 f_n(x)^2 \, dx = 0,$$

then

$$\lim_{n\to\infty} f_n\left(\frac{1}{2}\right) = 0.$$

T 2. Let (X, d) be an infinite compact metric space. Then there exists no function $f : X \to X$, continuous or otherwise, with the property that $d(f(x), f(y)) > d(x, y)$ for all $x \neq y$.

T 3. Every infinite closed subset of $\mathbb{R}^n$ is the closure of a countable set.

T 4. If X is a compact metric space, there exists a surjective (not necessarily continuous) function $\mathbb{R} \to X$.

T 5. If X is a compact metric space, then every isometry $f : X \to X$ is surjective.

F 6. Define a metric on the set of finite subsets of $\mathbb{Z}$ as follows:

$$d(A, B) = \text{ the cardinality of } (A \cup B \setminus (A \cap B)).$$

The resulting metric space admits an isometry into $\mathbb{R}^n$, for some positive integer n.

F 7. There exists a continuous function

$$f : [0, 1] \to \{A \in \mathrm{M}_2(\mathbb{R}) \mid A^2 = A\}$$

such that $f(0) = 0$ and $f(1) = \mathrm{Id}$.

T 8. Let $f : [0, 1] \to \mathbb{R}$ be a monotone increasing (not necessarily continuous) function such that $f(0) > 0$ and $f(1) < 1$. Then there exists $x \in [0, 1]$ such that $f(x) = x$.

F 9. The set

$$\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x^y \text{ divides } y^x, x \neq y, xy \neq 0, x \neq 1\}$$

is finite.

T 10. Suppose a line segment of a fixed length L is given. It is possible to construct a triangle of perimeter L, whose angles are $105°, 45°$ and $30°$, using only a straight edge and a compass.

T 11. The real vector space $\mathrm{M}_n(\mathbb{R})$ cannot be spanned by nilpotent matrices, for any positive integer n.

T 12. Let $S \subseteq \mathrm{M}_n(\mathbb{R})$ be a nonempty finite set closed under matrix multiplication. Then there exists $A \in S$ such that the trace of A is an integer.

F 13. Given a linear transformation $T : \mathbb{Q}^4 \to \mathbb{Q}^4$, there exists a nonzero proper subspace V of $\mathbb{Q}^4$ such that $T(V) \subseteq V$.

T 14. If G is a finite group such that the group $\mathrm{Aut}(G)$ of automorphisms of G is cyclic, then G is abelian.

T 15. There exists a countable group having uncountably many subgroups.

F 16. There exists a nonzero ideal $I \subseteq \mathbb{Z}[i]$ such that the quotient ring $\mathbb{Z}[i]/I$ is infinite (here i is a square root of -1 in $\mathbb{C}$).

F 17. There exists an injective ring homomorphism from the ring $\mathbb{Q}[x, y]/(x^2 - y^2)$ into the ring $\mathbb{Q}[x, y]/(x - y^2)$.

F 18. The set

$$\{n \in \mathbb{N} \mid n \text{ divides } a^3 - 1, \text{ for all integers } a \text{ such that } \gcd(a, n) = 1\}$$

is infinite.

T 19. The set of polynomials in the ring $\mathbb{Z}[x]$, the sum of whose coefficients is zero, forms an ideal of the ring $\mathbb{Z}[x]$.

T 20. Let $c_1, c_2 > 0$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be functions (not assumed to be continuous) such that for all $x \in \mathbb{R}$

$$f(x + c_1) = f(x) \qquad \text{and} \qquad g(x + c_2) = g(x).$$

Further, assume that

$$\lim_{x\to\infty} (f(x) - g(x)) = 0.$$

Then $f = g$.

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $\mathrm{M}_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices, and $\mathrm{M}_n(\mathbb{C})$ the complex vector space of $n \times n$ complex matrices. $\mathrm{M}_n(\mathbb{R})$ gets the topology transferred from any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, and its subsets get the subspace topology.
- Id denotes the identity matrix in $\mathrm{M}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{C})$.
- For a metric space X, $C(X, \mathbb{R})$ denotes the set of continuous functions from X to $\mathbb{R}$, viewed as a ring under pointwise addition and multiplication. Similarly, $C(X, \mathbb{C})$ will denote the ring of continuous functions from X to $\mathbb{C}$, again with pointwise addition and multiplication.
- A matrix $T \in \mathrm{M}_n(\mathbb{C})$, or a linear transformation $T : V \to V$ from a vector space V to itself, is called idempotent if $T^2 = T$, and nilpotent if $T^m = 0$ for some positive integer m.
- All rings are associative, with a multiplicative identity. A subring of a ring R is assumed by definition to contain the multiplicative identity of R.
- If q is a power of a prime number, $\mathbb{F}_q$ will stand for the finite field with q elements.
- For a ring R, $R[x_1, \dots, x_n]$ denotes the polynomial ring in n variables $x_1, \dots, x_n$ over R, and $R^\times$ denotes the multiplicative group of units of R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- By the group of isometries of a metric space (X, d), we mean the group whose elements are bijective maps $f : X \to X$ satisfying $d(x, y) = d(f(x), f(y))$ for all $x, y \in X$, and whose multiplication is defined by composition.
- If f is a real or complex valued function defined on an open interval (a, b) of $\mathbb{R}$, then f is said to be continuously differentiable on (a, b) if its derivative exists and is continuous on (a, b).
- If f is a real or complex valued function defined on an open interval of $\mathbb{R}$, then f' will stand for the first derivative of f (wherever it exists), and f'' for the second derivative of f (wherever it exists).

PART A

Answer the following multiple choice questions.

1. Consider the following properties of a metric space (X, d):

   (I) (X, d) is complete as a metric space.

   (II) For any sequence $\{Z_n\}_{n \in \mathbb{N}}$ of closed nonempty subsets of X, such that $Z_1 \supseteq Z_2 \supseteq \dots$ and

   $$\lim_{n \to \infty} \left(\sup_{x,y \in Z_n} d(x, y) \right) = 0,$$

   $\bigcap_{n=1}^{\infty} Z_n$ is a singleton set.

   Which of the following sentences is true?

   (a) (I) implies (II) and (II) implies (I).

   (b) (I) implies (II) but (II) does not imply (I).

   (c) (I) does not imply (II) but (II) implies (I).

   (d) (I) does not imply (II) and (II) does not imply (I).

2. Consider the following assertions:

   (I) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ is connected.

   (II) $\{(x, y) \in \mathbb{C}^2 \mid xy = 1\}$ is connected.

   Which of the following sentences is true?

   (a) Both (I) and (II) are true.

   (b) (I) is true but (II) is false.

   (c) (I) is false but (II) is true.

   (d) Both (I) and (II) are false.

3. What is the number of solutions of:

   $$x = \frac{x^2}{50} - \cos\frac{x}{2} + 2$$

   in $[0, 10]$?

   (a) 0

   (b) 1

   (c) 2

   (d) ∞

4. Let A be an element of $\mathrm{M}_4(\mathbb{R})$ with characteristic polynomial $t^4 - t$. What is the characteristic polynomial of A^2?

   (a) $t^4 - t$

(b) $t^4 - 2t^3 + t^2$

(c) $t^4 - t^2$

(d) None of the other three options

5. Let n be a positive integer, and let $V = \{f \in \mathbb{R}[x] \mid \deg f \leq n\}$ be the real vector space of real polynomials of degree at most n. Let $\mathrm{End}_{\mathbb{R}}(V)$ denote the real vector space of linear transformations from V to itself. For $m \in \mathbb{Z}$, let $T_m \in \mathrm{End}_{\mathbb{R}}(V)$ be such that $(T_m f)(x) = f(x+m)$ for all $f \in V$. Then the dimension of the vector subspace of $\mathrm{End}_{\mathbb{R}}(V)$ given by
$$\mathrm{Span}\left(\{T_m \mid m \in \mathbb{Z}\}\right)$$
is

(a) 1

(b) n

(c) $n+1$

(d) n^2

6. Let T be the linear transformation from the real vector space $\mathbb{R}[x]$ to itself, given by $T(f) = f'$, where f' is the derivative of f. Consider the following statements about T:

(I) T is nilpotent.

(II) The only eigenvalue of T is 0.

Which of the following sentences is true?

(a) Both (I) and (II) are true.

(b) (I) is true but (II) is false.

(c) (I) is false but (II) is true.

(d) Both (I) and (II) are false.

7. What is the cardinality of the set of $\theta \in [0, 2\pi)$ such that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix:
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
has an eigenvector in $\mathbb{R}^2$?

(a) 1

(b) 2

(c) 4

(d) ∞

8. Let p be a prime number, and let A equal $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, viewed as a 2×2 matrix with integer entries. What is the smallest positive integer n such that the matrix A^n is congruent to the 2×2 identity matrix modulo p?

(a) $p^2 - 1$

(b) $p - 1$

(c) p

(d) $p + 1$

9. What is the largest value of n for which there exists a set $\{A_1, \ldots, A_n\}$ of (distinct) nonzero matrices in $\mathrm{M}_2(\mathbb{C})$ such that $A_i^* A_j$ has trace zero for all $1 \leq i < j \leq n$?

(a) 1

(b) Greater than 1 but at most 4

(c) Greater than 4 but finite

(d) ∞

10. Let p be a prime number. What is the number of elements in the group $\mathbb{Q}/\mathbb{Z}$ that have order exactly p?

(a) 0

(b) $p - 1$

(c) p

(d) ∞

11. Consider the real polynomial

$$f(x) = x^{11} - x^7 + x^2 - 1.$$

Which of the following sentences is correct?

(a) $f(x)$ has exactly one positive root.

(b) $f(x)$ has exactly two positive roots.

(c) $f(x)$ has at least three positive roots.

(d) None of the other three options.

12. Consider polynomials

$$f_1(x, y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j \qquad \text{and} \qquad f_2(x, y) = \sum_{i,j=0}^{\infty} b_{ij} x^i y^j \in \mathbb{R}[x, y]$$

(where $a_{ij} = b_{ij} = 0$ for all but finitely many $(i, j) \in \mathbb{N}^2$), such that $f_1(p, q) = f_2(p, q)$ for all $(p, q) \in \mathbb{R}^2$ satisfying $p^2 = q^2$. Which of the following sentences is true for all such f_1 and f_2?

(a) $a_{00} = b_{00}$, but we may not have $a_{ij} = b_{ij}$ for all (i, j) with $i + j = 1$.

(b) $a_{ij} = b_{ij}$ if $i + j \leq 1$, but we may not have $a_{ij} = b_{ij}$ for all (i, j) with $i + j = 2$.

(c) $a_{ij} = b_{ij}$ if $i + j \leq 2$, but we may not have $a_{ij} = b_{ij}$ for all (i, j) with $i + j = 3$.

(d) $a_{ij} = b_{ij}$ if $i + j \leq 3$, but we may not have $f_1 = f_2$.

13. What is the number of bijections $f : \{1, 2, \dots, 9\} \to \{1, 2, \dots, 9\}$ such that, for all distinct $i, j \in \{1, \dots, 9\}$, whenever the squares labelled i and j in the diagram below share an edge, the squares labelled $f(i)$ and $f(j)$ share an edge too?

1	2	3
4	5	6
7	8	9

(a) 8

(b) 9

(c) 4

(d) 24

14. Let m be the number of positive integers n such that $1 \le n \le 2022$ and such that n has an odd number of (positive integer) divisors. Then m is

(a) 22

(b) 33

(c) 44

(d) 55

15. Let S be the set of nonnegative continuous functions f on $[0, 1]$ satisfying

$$\int_0^1 \sin^2(x) f(x)\, dx = \int_0^1 \sin(x)\cos(x) f(x)\, dx = \int_0^1 \cos^2(x) f(x)\, dx = 1.$$

Then S is:

(a) an uncountable set

(b) a countably infinite set

(c) a finite and nonempty set

(d) the empty set

16. Let $f(x) = 1 - \sin x$ for $x \in \mathbb{R}$. Define

$$a_n = \sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \dots f\left(\frac{n}{n}\right)}.$$

Then

(a) $\{a_n\}_n$ converges to 0

(b) $\{a_n\}_n$ diverges to ∞

(c) $\{a_n\}_n$ converges and $\lim_{n\to\infty} a_n > 0$

(d) none of the other three options is correct

17. Which of the following is true for every function $u : \mathbb{R} \to \mathbb{R}$ which is continuously differentiable on $\mathbb{R}$ (i.e., u is diffferentiable on $\mathbb{R}$ and its derivative u' is continuous on $\mathbb{R}$), and satisfies

$$u(y) \geq u(x) + u'(x)(y - x)$$

for all $x, y \in \mathbb{R}$?

(a) u' is nonnegative.

(b) u attains a minimum at some $x \in \mathbb{R}$.

(c) u' is nondecreasing.

(d) u' is nonincreasing.

18. Let $\{x_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} x_n = x$. Define

$$z_n = \frac{1}{n}\left[x_1\left(1 + \frac{x}{n}\right)^n + x_2\left(1 + \frac{x}{n-1}\right)^{n-1} + \cdots + x_n(1 + x)\right].$$

Then

(a) $\{z_n\}$ converges to xe

(b) $\{z_n\}$ converges to e^x

(c) $\{z_n\}$ does not have a limit

(d) $\{z_n\}$ converges to xe^x

19. The value of

$$\lim_{n\to\infty} \int_0^1 \frac{ne^x}{1 + n^2x^2}\, dx$$

is

(a) πe

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{2}e$

(d) π

20. What is the number of real solutions of the equation

$$e^{\sin x} = \pi?$$

(a) 0

(b) 1

(c) Countably infinite

(d) Uncountable

PART B

Answer whether the following statements are True or False.

1. $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected but not path-connected. False
2. If X is a connected metric space, and F is a subring of $C(X, \mathbb{R})$ that is a field, then every element of $C(X, \mathbb{R})$ that belongs to F is a constant function. True
3. Let $K \subseteq [0, 1]$ be the Cantor set. Then there exists no injective ring homomorphism $C([0, 1], \mathbb{R}) \to C(K, \mathbb{R})$. False
4. There exists a metric space (X, d) such that the group of isometries of X is isomorphic to $\mathbb{Z}$. True
5. Let $A \subset \mathbb{R}^2$ be a nonempty subset such that any continuous function $f : A \to \mathbb{R}$ is constant. Then A is a singleton set. True
6. For a nilpotent matrix $A \in \mathrm{M}_n(\mathbb{R})$, let

   $$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!} = \mathrm{Id} + \frac{A}{1!} + \frac{A^2}{2!} + \cdots \in \mathrm{M}_n(\mathbb{R}).$$

   If A is a nilpotent matrix such that $\exp(A) = \mathrm{Id}$, then A is the zero matrix. True
7. There exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$, with $A^2 = A \neq 0$, such that

   $$|a| + |b| < 1 \qquad \text{and} \qquad |c| + |d| < 1.$$ False

8. If $A \in \mathrm{M}_3(\mathbb{C})$ is such that A^i has trace zero for all positive integers i, then A is nilpotent. True
9. For any finite cyclic group G, there exists a prime power q such that G is a subgroup of $\mathbb{F}_q^{\times}$. True
10. There are only finitely many isomorphism classes of finite nonabelian groups, all of whose proper subgroups are abelian. False
11. Every subring of a unique factorization domain is a unique factorization domain. False
12. Let $f_1, f_2, f_3, f_4 \in \mathbb{R}[x]$ be monic polynomials each of degree exactly two. Then there exist a real polynomial $p \in \mathbb{R}[x]$ and a subset $\{i, j\} \subset \{1, 2, 3, 4\}$ with $i \neq j$, such that $f_i \circ p = c f_j$ for some $c \in \mathbb{R}$. True
13. There exists a finite abelian group G such that the group $\mathrm{Aut}(G)$ of automorphisms of G is isomorphic to $\mathbb{Z}/7\mathbb{Z}$. False
14. There exists an integral domain R and a surjective homomorphism $R \to R$ of rings that is not injective. True
15. There exists $f \in C([0, 1], \mathbb{R})$ satisfying the following two conditions:

(i) $\int_0^1 f(x)\,dx = 1$; and

(ii) $\lim_{n\to\infty} \int_0^1 f(x)^n\,dx = 0$. False

16. Let $a_n \geq 0$ for each positive integer n. If the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, then so does the series $\sum_{n=1}^{\infty} \frac{a_n}{n^{1/4}}$. True

17. There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = 2 \quad \text{and} \quad \lim_{x\to\infty} f'(x) = 1.$$

False

18. Let $f : [0,1] \to [0,\infty)$ be continuous on $[0,1]$ and twice differentiable in $(0,1)$. If $f''(x) = 7f(x)$ for all $x \in (0,1)$, then $f(x) \leq \max\{f(0), f(1)\}$ for all $x \in [0,1]$. True

19. There are N balls in a box, out of which n are blue $(1 < n < N)$ and the rest are red. Balls are drawn from the box one by one at random, and discarded. Then the probability of picking all the blue balls in the first n draws is the same as the probability of picking all the red balls in the first $(N-n)$ draws. True

20. The set $\{f(x) \in \mathbb{R}[x] \mid f(n) \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}\}$ is uncountable. False

GS2023 - Mathematics Question Paper

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the standard Euclidean norm of x, i.e., the distance from x to 0.
- All rings are associative, with a multiplicative identity.
- For any ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. The identity matrix in $\mathrm{M}_n(R)$ will be denoted by Id.
- $\mathrm{M}_n(\mathbb{R})$ will also be viewed as a real vector space, and $\mathrm{M}_n(\mathbb{C})$ as a complex vector space. $\mathrm{M}_n(\mathbb{R})$ is given the topology such that any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ is a homeomorphism. Subsets of $\mathrm{M}_n(\mathbb{R})$ are given the subspace topology.
- For a ring R, $R[x_1, \dots, x_n]$ denotes the polynomial ring in n variables $x_1, \dots, x_n$ over R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- Let G be a finite group, and let $S \subset G$. We say that S generates G if no proper subgroup of G contains S.
- If $f : X \to Y$ is a map of sets, and $X_1 \subset X$, then $f|_{X_1}$ denotes the restriction of f to X_1.

PART A

Answer the following multiple choice questions.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = (3x^2 + 1)/(x^2 + 3)$. Let $f^{\circ 1} = f$, and let $f^{\circ n} = f^{\circ(n-1)} \circ f$ for all integers $n \geq 2$. Which of the following statements is correct?

   (a) $\lim_{n\to\infty} f^{\circ n}(1/2) = 1$, and $\lim_{n\to\infty} f^{\circ n}(2) = 1$.
   (b) $\lim_{n\to\infty} f^{\circ n}(1/2) = 1$, but $\lim_{n\to\infty} f^{\circ n}(2)$ does not exist.
   (c) $\lim_{n\to\infty} f^{\circ n}(1/2)$ does not exist, but $\lim_{n\to\infty} f^{\circ n}(2) = 1$.
   (d) Neither $\lim_{n\to\infty} f^{\circ n}(1/2)$ nor $\lim_{n\to\infty} f^{\circ n}(2)$ exists.

2. Consider the following properties of a sequence $\{a_n\}_n$ of real numbers.

   (I) $\lim_{n\to\infty} a_n = 0$.
   (II) There exists a sequence $\{i_n\}_n$ of positive integers such that $\sum_{n=1}^{\infty} a_{i_n}$ converges.

   Which of the following statements is correct?

   (a) (I) implies (II), and (II) implies (I).
   (b) (I) implies (II), but (II) does not imply (I).
   (c) (I) does not imply (II), but (II) implies (I).
   (d) (I) does not imply (II), and (II) does not imply (I).

3. Consider sequences $\{x_n\}_n$ of real numbers such that

   $$\lim_{n\to\infty}(x_{2n-1} + x_{2n}) = 2 \qquad \text{and} \qquad \lim_{n\to\infty}(x_{2n} + x_{2n+1}) = 3.$$

   Which of the following statements is correct?

   (a) For every such sequence $\{x_n\}_n$, $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = 1$.
   (b) For every such sequence $\{x_n\}_n$, $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = -1$.
   (c) For every such sequence $\{x_n\}_n$, $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = 3/2$.
   (d) There exists such a sequence $\{x_n\}_n$, for which $\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}}$ does not exist.

4. Consider the function $f : (0, \infty) \to (0, \infty)$ given by $f(x) = xe^x$. Let $L : (0, \infty) \to (0, \infty)$ be its inverse function. Which of the following statements is correct?

   (a) $\lim_{x\to\infty} \frac{L(x)}{\log x} = 1$.
   (b) $\lim_{x\to\infty} \frac{L(x)}{(\log x)^2} = 1$.
   (c) $\lim_{x\to\infty} \frac{L(x)}{\sqrt{\log x}} = 1$.
   (d) None of the remaining three options is correct.

5. Let $\{b_n\}_n$ be a monotonically increasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$. Which of the following statements is true about

   $$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \frac{b_k}{k^2}?$$

(a) The limit exists for all such sequences, and its value is always $+\infty$.

(b) The limit exists for all such sequences, and its value is always 0.

(c) The limit exists for all such sequences, and its value is always 1.

(d) None of the remaining three options is correct.

6. For every positive integer n, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \frac{\sin(n^2 x)+\cos(e^n x)}{1+n^2x^2}$. Then

$$\lim_{n\to\infty} \int_0^{1-\sin(1/n)} f_n(x)\,dx$$

equals

(a) 1.

(b) 0.

(c) ∞.

(d) 1/2.

7. Consider the functions $f_1, f_2 : (0,\infty) \to \mathbb{R}$ defined by

$$f_1(x) = \sqrt{x}, \qquad \text{and} \qquad f_2(x) = \sqrt{x}\sin x.$$

Which of the following statements is correct?

(a) f_1 and f_2 are uniformly continuous.

(b) f_1 is uniformly continuous, but f_2 is not.

(c) f_2 is uniformly continuous, but f_1 is not.

(d) Neither f_1 nor f_2 is uniformly continuous.

8. Let $x_1 \in \mathbb{R}^2 \setminus \{0\}$ be fixed, and inductively define $x_{n+1} = Ax_n$ for $n \geq 1$, where A is the 2×2 real matrix given by

$$A := \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Which of the following statements is correct?

(a) $\{x_n\}_n$ is a convergent sequence.

(b) $\{x_n\}_n$ is not a convergent sequence, but it has a convergent subsequence.

(c) $\lim_{n\to\infty} \|x_n\| = 0$.

(d) None of the remaining three options is correct.

9. Let $T : \mathrm{M}_3(\mathbb{R}) \to \mathbb{R}^3$ be the linear map defined by $T(A) = A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Then the dimension of the kernel of T equals

(a) 2.

(b) 8.

(c) 1.

(d) None of the remaining three options.

10. Let $V = \{f(x) \in \mathbb{R}[x] \mid f(0) = 0\}$, viewed as a real vector space. Consider the following assertions:

(I) V contains three linearly independent polynomials of degree 2.

(II) V contains two linearly independent polynomials of degree 3.

Which of the following statements is correct?

(a) Both (I) and (II) are true.
(b) (I) is true, but (II) is false.
(c) (I) is false, but (II) is true.
(d) Neither (I) nor (II) is true.

11. Let $C([-1,1],\mathbb{R})$ denote the real vector space of continuous functions from $[-1,1]$ to $\mathbb{R}$, and consider the subspace

$$V = \{f \in C([-1,1],\mathbb{R}) \mid f(-x) = f(x) \text{ for all } x \in [-1,1]\}.$$

Define an inner product on $C([-1,1],\mathbb{R})$ by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

What is the orthogonal complement of V in $C([-1,1],\mathbb{R})$?

(a) $\{f \in C([-1,1],\mathbb{R}) \mid f(-x) = -f(x) \text{ for all } x \in [-1,1]\}$.
(b) $\{f \in C([-1,1],\mathbb{R}) \mid f(0) = 0\}$.
(c) V does not have an orthogonal complement in $C([-1,1],\mathbb{R})$.
(d) None of the remaining three options.

12. Consider pairs (X, d), where X is a set with 100 elements, and $d : X \times X \to \mathbb{R}$ is a function such that $d(x,y) = d(y,x) > 0$ if $x, y \in X$ are distinct, and $d(x,x) = 0$ for all $x \in X$. For $n < 100$, let A_n be the statement:

For every such pair (X, d), there exists a subset X_1 of X, with n elements, such that $(X_1, d|_{X_1\times X_1})$ is a metric space.

Which of the following statements is correct?

(a) A_2 is true, but A_3 is not true.
(b) A_3 is true, but A_4 is not true.
(c) A_n is true for all $n \leq 10$, but not for all $n \leq 25$.
(d) A_n is true for all $n \leq 25$.

13. Let $\{x_n\}_n$ be a sequence in a metric space (X, d). Let $f : X \to \mathbb{R}$ be defined by

$$f(x) = \inf\{d(x, x_n) \mid n \in \mathbb{N}\}.$$

Which of the following statements is correct?

(a) f is uniformly continuous on X.
(b) f is continuous on X, but not necessarily uniformly continuous.
(c) f is continuous on X if and only if X is compact.
(d) None of the remaining three options is correct.

14. The number of finite groups, up to isomorphism, with exactly two conjugacy classes, equals

(a) 1.

(b) 2.

(c) Greater than 2, but finite.

(d) Infinite.

15. Consider the following assertions about a commutative ring R with identity and elements $a, b \in R$:

    (I) There exist $p, q \in R$ such that $ap + bq = 1$.

    (II) There exist $p, q \in R$ such that $a^2p + b^2q = 1$.

    Then:

    (a) (I) implies (II), and (II) implies (I).

    (b) (I) implies (II), but (II) does not imply (I).

    (c) (I) does not imply (II), but (II) implies (I).

    (d) (I) does not imply (II), and (II) does not imply (I).

16. The number of elements of finite order in the group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

    is

    (a) 1.

    (b) Finite, but not 1.

    (c) Countably infinite.

    (d) Uncountably infinite.

17. The value of

$$\max \left(\bigcup_{\substack{k \in \mathbb{N} \\ k \geq 1}} \{x_1 x_2 \ldots x_k \mid x_1, \ldots, x_k \in \mathbb{N}, \text{ and } x_1 + \cdots + x_k = 100\} \right)$$

    equals

    (a) 4×3^{32}.

    (b) 2^{50}.

    (c) $2^{26} \times 3^{16}$.

    (d) None of the remaining three options.

18. Choose the option that completes the sentence correctly: There exists a 10×10 real symmetric matrix A, all of whose entries are nonnegative and all of whose diagonal entries are positive, such that A^{10} has

    (a) exactly 67 positive entries.

    (b) exactly 68 positive entries.

    (c) exactly 69 positive entries.

    (d) exactly 70 positive entries.

19. The number of (nondegenerate Euclidean) triangles with sides of integer length and perimeter 8, up to congruence, is

(a) 1.
(b) 2.
(c) 3.
(d) 4.

20. Let

$$A = \{(\alpha, \beta) \in \mathbb{Z}^2 \mid \text{ the roots } r_1, r_2, r_3 \text{ of the polynomial } p(x) = x^3 - 2x^2 + \alpha x - \beta \text{ satisfy } r_1^3 + r_2^3 + r_3^3 = 0\}.$$

Which of the following statements is correct?

(a) A is infinite.
(b) A is empty.
(c) A is singleton.
(d) A is finite, but neither empty nor singleton.

PART B

Answer whether the following statements are True or False.

1. Let α be a positive real number, and let $f : (0,1) \to \mathbb{R}$ be a function such that $|f(x)-f(y)| \leq |x-y|^\alpha$ for all $x, y \in (0,1)$. Then f can be extended to a continuous function $[0,1] \to \mathbb{R}$. *True*

2. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f^2 + g^2$ is uniformly continuous. Then at least one of the two functions f and g is uniformly continuous. *False*

3. Let $\{f_n\}_n$ be a sequence of (not necessarily continuous) functions from $[0,1]$ to $\mathbb{R}$. Let $f : [0,1] \to \mathbb{R}$ be such that for any $x \in [0,1]$ and any sequence $\{x_n\}_n$ consisting of elements from $[0,1]$, if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$. Then f is continuous. *True*

4. Let $A, B \in \mathrm{M}_2(\mathbb{Z}/2\mathbb{Z})$ be such that $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\mathrm{tr}(A^2) = \mathrm{tr}(B^2)$. Then A and B have the same eigenvalues. *False*

5. Let v_1, v_2, w_1, w_2 be nonzero vectors in $\mathbb{R}^2$. Then there exists a 2×2 real matrix A such that $Av_1 = v_2$ and $Aw_1 = w_2$. *False*

6. Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ be such that $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$. Assume that $\lim_{m\to\infty} A^m$ exists, and denote it by $B = (b_{ij})$. Then, for all $1 \leq i, j \leq n$, we have $b_{ij} \in \{0, 1\}$. *False*

7. Given any monic polynomial $f(x) \in \mathbb{R}[x]$ of degree n, there exists a matrix $A \in \mathrm{M}_n(\mathbb{R})$ such that its characteristic polynomial equals f. *True*

8. If $A \in \mathrm{M}_4(\mathbb{Q})$ is such that its characteristic polynomial equals x^4+1, then A is diagonalizable in $\mathrm{M}_4(\mathbb{C})$. *True*

9. If $A \in \mathrm{M}_n(\mathbb{R})$ is such that $AB = BA$ for all invertible matrices $B \in \mathrm{M}_n(\mathbb{R})$, then $A = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{R}$. *True*

10. There exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(2x) = 3f(x)$ for all $x \in \mathbb{R}$. *True*

11. There exists a continuous bijection from $[0,1] \times [0,1]$ to $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, which is not a homeomorphism. ***False***

12. Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a nonzero polynomial ($n \geq 1$), and let *True*

$$X = \{z \in \mathbb{C}^n \mid f(z) = 0\}.$$

Then $\mathbb{C}^n \setminus X$ is path connected.

13. A connected metric space with at least two points is uncountable. *True*

14. If A and B are disjoint subsets of a metric space (X, d), then **False**

$$\inf\{d(x, y) \mid x \in A, y \in B\} \neq 0.$$

15. A countably infinite complete metric space has infinitely many isolated points (an element x of a metric space X is said to be an isolated point if $\{x\}$ is an open subset of X). *True*

16. Suppose G and H are two countably infinite abelian groups such that every nontrivial element of $G \times H$ has order 7. Then G is isomorphic to H. *True*

17. There exists a nonabelian group G of order 26 such that every proper subgroup of G is abelian. *True*

18. Let G be a group generated by two elements x and y, each of order 2. Then G is finite. **False**

19. $\mathbb{R}[x]/(x^4 + x^2 + 2023)$ is an integral domain. **False**

20. Every finite group is isomorphic to a subgroup of a finite group generated by two elements. *True*

GS2024 Exam, School of Mathematics, TIFR

Notation and conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- All rings are associative, with a multiplicative identity.
- For any ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. The identity matrix in $\mathrm{M}_n(R)$ will be denoted by Id or by Id_n.
- $\mathrm{M}_n(\mathbb{R})$ will also be viewed as a real vector space, and $\mathrm{M}_n(\mathbb{C})$ as a complex vector space.
- For a ring R, $R[x_1, \dots, x_n]$ denotes the polynomial ring in n variables $x_1, \dots, x_n$ over R.
- If A is a set, $\#A$ stands for the cardinality of A, and equals ∞ if A is infinite.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- Let G be a finite group, and let $S \subset G$. We say that S generates G if no proper subgroup of G contains S.

Part A — Multiple Choice Questions

(1) What is the number of even positive integers n such that every group of order n is abelian?
(a) 1
$\boxed{\checkmark}$ (b) 2
(c) Greater than 2, but finite
(d) Infinite

(2) Let n be a positive integer, and let

$$S = \{g \in \mathbb{R}[x] \mid g \text{ is a polynomial of degree at most } n\}.$$

For $g \in S$, let $A_g = \{x \in \mathbb{R} \mid e^x = g(x)\} \subset \mathbb{R}$. Let

$$m = \min\{\#A_g \mid g \in S\}, \qquad \text{and} \qquad M = \max\{\#A_g \mid g \in S\}.$$

Then
(a) $m = 0, M = n$
$\boxed{\checkmark}$ (b) $m = 0, M = n + 1$
(c) $m = 1, M = n$
(d) $m = 1, M = n + 1$

(3) Let V, W be nonzero finite dimensional vector spaces over $\mathbb{C}$. Let m be the dimension of the space of $\mathbb{C}$-linear transformations $V \to W$, viewed as a real vector space. Let n be the dimension of the space of $\mathbb{R}$-linear transformations $V \to W$, viewed as a real vector space. Then
(a) $n = m$
(b) $2n = m$
$\boxed{\checkmark}$ (c) $n = 2m$
(d) $4n = m$

(4) Consider the real vector space of infinite sequences of real numbers

$$S = \{(a_0, a_1, a_2, \ldots) \mid a_k \in \mathbb{R}, k = 0, 1, 2, \ldots\}.$$

Let W be the subspace of S consisting of all sequences $(a_0, a_1, a_2, \ldots)$ which satisfy the relation

$$a_{k+2} = 2a_{k+1} + a_k, \quad k = 0, 1, 2, \ldots.$$

What is the dimension of W?
(a) 1
$\boxed{\checkmark}$ (b) 2
(c) 3
(d) ∞

(5) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. If

$$\lim_{n\to\infty} \int_0^1 f(x + n)\, dx = 2,$$

then which of the following statements about the limit

$$\lim_{n\to\infty} \int_0^1 f(nx)\, dx$$

is correct?
(a) The limit exists and equals 0

(b) The limit exists and equals $\frac{1}{2}$

$\checkmark$ (c) The limit exists and equals 2

(d) None of the remaining three options is correct

(6) Let $f : \mathbb{R} \to [0, \infty)$ be a function such that for any finite set $E \subset \mathbb{R}$ we have

$$\sum_{x \in E} f(x) \leq 1.$$

Let

$$C_f = \{x \in \mathbb{R} \mid f(x) > 0\} \subset \mathbb{R}.$$

Then

(a) C_f is finite

(b) C_f is a bounded subset of $\mathbb{R}$

(c) C_f has at most one limit point

$\checkmark$ (d) C_f is a countable set

(7) Let p be a prime. Which of the following statements is true?

(a) There exists a noncommutative ring with exactly p elements

(b) There exists a noncommutative ring with exactly p^2 elements

$\checkmark$ (c) There exists a noncommutative ring with exactly p^3 elements

(d) None of the remaining three statements is correct

(8) Consider the sequence $\{a_n\}$ for $n \geq 1$ defined by

$$a_n = \lim_{N \to \infty} \sum_{k=n}^{N} \frac{1}{k^2}.$$

Which of the following statements about this sequence is true?

(a) $\lim_{n \to \infty} n a_n$ does not exist

(b) $\lim_{n \to \infty} n a_n$ exists and equals 2

$\checkmark$ (c) $\lim_{n \to \infty} n a_n$ exists and equals 1

(d) $\lim_{n \to \infty} n^2 a_n$ exists and equals 1

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that is a solution to the ordinary differential equation

$$f'(t) = \sin^2(f(t)) \, (\forall \, t \in \mathbb{R}), \quad f(0) = 1.$$

Which of the following statements is true?

(a) f is neither bounded nor periodic

(b) f is bounded and periodic

$\checkmark$ (c) f is bounded, but not periodic

(d) None of the remaining three statements is correct

(10) Let B denote the set of invertible upper triangular 2×2 matrices with entries in $\mathbb{C}$, viewed as a group under matrix multiplication. Which of the following subgroups of B is the normalizer of itself in B?

$\checkmark$ (a) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{C} \setminus \{0\} \right\}$

(b) $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{C} \setminus \{0\} \right\}$

(c) $\left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\}$

(d) $\left\{ \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{C} \setminus \{0\}, c \in \mathbb{C} \right\}$

(11) What is the least positive integer $n > 1$ such that x^n and x are conjugate, for every $x \in S_{11}$? Here, S_{11} denotes the symmetric group on 11 letters.
(a) 10
(b) 11
(c) 12
$\boxed{\checkmark}$ (d) 13

(12) Consider the following statements:
(A) Let G be a group and let $H \subset G$ be a subgroup of index 2. Then $[G, G] \subseteq H$.
(B) Let G be a group and let $H \subset G$ be a subgroup that contains the commutator subgroup $[G, G]$ of G. Then H is a normal subgroup of G.
Which of the following statements is correct?
$\boxed{\checkmark}$ (a) (A) and (B) are both true
(b) (A) and (B) are both false
(c) (A) is true and (B) is false
(d) (A) is false and (B) is true

(13) For any symmetric real matrix A, let $\lambda(A)$ denote the largest eigenvalue of A. Let S be the set of positive definite symmetric 3×3 real matrices. Which of the following assertions is correct?
(a) There exist $A, B \in S$ such that $\lambda(A + B) < \max(\lambda(A), \lambda(B))$
$\boxed{\checkmark}$ (b) For all $A, B \in S$, $\lambda(A + B) > \max(\lambda(A), \lambda(B))$
(c) There exist $A, B \in S$ such that $\lambda(A + B) = \max(\lambda(A), \lambda(B))$
(d) None of the remaining three assertions is correct

(14) Let $\theta \in (0, \pi/2)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which sends a vector v to its reflection with respect to the line through $(0, 0)$ and $(\cos\theta, \sin\theta)$. Then the matrix of T with respect to the standard basis of $\mathbb{R}^2$ is given by

$\boxed{\checkmark}$ (a) $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
(b) $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$
(c) $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$
(d) $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

(15) For a polynomial $f(x, y) \in \mathbb{R}[x, y]$, let $X_f = \{(a, b) \in \mathbb{R}^2 \mid f(a, b) = 1\} \subset \mathbb{R}^2$. Which of the following statements is correct?
(a) If $f(x, y) = x^2 + 4xy + 3y^2$, then X_f is compact
$\boxed{\checkmark}$ (b) If $f(x, y) = x^2 - 3xy + 3y^2$, then X_f is compact
(c) If $f(x, y) = x^2 - 4xy - y^2$, then X_f is compact
(d) None of the remaining three statements is correct

(16) What is the number of distinct subfields of $\mathbb{C}$ isomorphic to $\mathbb{Q}[\sqrt[3]{2}]$?
(a) 1
(b) 2
$\boxed{\checkmark}$ (c) 3
(d) Infinite

(17) Let $\mathbb{F}_3$ denote the finite field with 3 elements. What is the number of one dimensional vector subspaces of the vector space $\mathbb{F}_3^5$ over $\mathbb{F}_3$?
(a) 5
✓ (b) 121
(c) 81
(d) None of the remaining three options

(18) For a positive integer n, let a_n, b_n, c_n, d_n be the real numbers such that
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$
Which of the following numbers equals $\lim_{n\to\infty} a_n/b_n$?
(a) 1
(b) e
(c) 3/2
✓ (d) None of the remaining three options

(19) Consider the complex vector space
$$V = \{f \in \mathbb{C}[x] \mid f \text{ has degree at most 50, and } f(ix) = -f(x) \text{ for all } x \in \mathbb{C}\}.$$
Then the dimension of V equals
(a) 50
(b) 25
✓ (c) 13
(d) 47

(20) Let S denote the set of sequences $a = (a_1, a_2, \dots)$ of real numbers such that a_k equals 0 or 1 for each k. Then the function $f : S \to \mathbb{R}$ defined by
$$f((a_1, a_2, \dots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \dots$$
is
✓ (a) injective but not surjective
(b) surjective but not injective
(c) bijective
(d) neither injective nor surjective

Part B — True/False Questions

T (1) If G is a group of order 361, then G has a normal subgroup H such that $H \cong G/H$.
F (2) There exists a metric space X such that the number of open subsets of X is exactly 2024.
F (3) The function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d(x, y) = |e^x - e^y|$ defines a metric on $\mathbb{R}$, and $(\mathbb{R}, d)$ is a complete metric space.
F (4) Let n be a positive integer, and A an $n \times n$ matrix over $\mathbb{R}$ such that $A^3 = \mathrm{Id}$. Then A is diagonalizable in $\mathrm{M}_n(\mathbb{R})$, i.e., there exists $P \in \mathrm{M}_n(\mathbb{R})$ such that P is invertible and PAP^{-1} is a diagonal matrix.
T (5) If $A \in \mathrm{M}_n(\mathbb{Q})$ is such that the characteristic polynomial of A is irreducible over $\mathbb{Q}$, then A is diagonalizable in $\mathrm{M}_n(\mathbb{C})$, i.e., there exists $P \in \mathrm{M}_n(\mathbb{C})$ such that P is invertible and PAP^{-1} is a diagonal matrix.
T (6) The complement of any countable union of lines in $\mathbb{R}^3$ is path connected.

T (7) The subsets $\{(x, y) \in \mathbb{R}^2 \mid (y^2 - x)(y^2 - x - 1) = 0\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y^2 - x^2 = 1\}$ of $\mathbb{R}^2$ (with the induced metric) are homeomorphic.

F (8) $\mathbb{Q} \cap [0, 1]$ is a compact subset of $\mathbb{Q}$.

T (9) Suppose $f : X \to Y$ is a function between metric spaces, such that whenever a sequence $\{x_n\}$ converges to x in X, the sequence $\{f(x_n)\}$ converges in Y (but it is not given that the limit of $\{f(x_n)\}$ is $f(x)$). Then f is continuous.

T (10) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and assume that $|f'(x)| \geq 1$ for all $x \in \mathbb{R}$. Then for each compact set $C \subset \mathbb{R}$, the set $f^{-1}(C)$ is compact.

F (11) There exists a function $f : [0, 1] \to \mathbb{R}$, which is not Riemann integrable and satisfies

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2 < 1,$$

for every choice of a positive integer n and of $0 \leq t_0 < t_1 < t_2 < \cdots < t_n \leq 1$.

F (12) Let $E \subset [0, 1]$ be the subset consisting of numbers that have a decimal expansion which does not contain the digit 8. Then E is dense in $[0, 1]$.

T (13) Let G be a proper subgroup of $(\mathbb{R}, +)$ which is closed as a subset of $\mathbb{R}$. Then G is generated by a single element.

T (14) There exists a unique function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at $x = 0$, and such that for all $x \in \mathbb{R}$

$$f(x) + f\left(\frac{x}{2}\right) = x.$$

F (15) A map $f : V \to W$ between finite dimensional vector spaces over $\mathbb{Q}$ is a linear transformation if and only if $f(x) = f(x - a) + f(x - b) - f(x - a - b)$, for all $x, a, b \in V$.

F (16) Let R be the ring $\mathbb{C}[x]/(x^2)$ obtained as the quotient of the polynomial ring $\mathbb{C}[x]$ by its ideal generated by x^2. Let $R^\times$ be the multiplicative group of units of this ring. Then there is an injective group homomorphism from $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ into $R^\times$.

T (17) Let $A \in \mathrm{M}_2(\mathbb{Z})$ be such that $|A_{ij}(n)| \leq 50$ for all $1 \leq n \leq 10^{50}$ and all $1 \leq i, j \leq 2$, where $A_{ij}(n)$ denotes the (i, j)-th entry of the 2×2 matrix A^n. Then $|A_{ij}(n)| \leq 50$ for all positive integers n.

F (18) Let A, B be subsets of $\{0, \ldots, 9\}$. It is given that, on choosing elements $a \in A$ and $b \in B$ at random, $a + b$ takes each of the values $0, \ldots, 9$ with equal probability. Then one of A or B is singleton.

T (19) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then there exists $M > 0$ such that for all $x \in \mathbb{R} \setminus [-M, M]$, we have $f(x) < x^{100}$.

T (20) If a sequence $\{f_n\}$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ converges uniformly on $(0, 1)$ to a continuous function $f : [0, 1] \to \mathbb{R}$, then $\{f_n\}$ converges uniformly on $[0, 1]$ to f.