CSE 301

09.02.14

Yousuf Sir Class-01

Probability models ⇒ Conditional Expectation

Markov chain

Ref: Probability models
Sheldon Ross
10^{th} ed.

Probability is the study of something

"outcome uncertain"

⇓

"quantifying uncertinity"

Applications:
- "Simulation"
- "Algorithm analysis"

Sample Space: (S.)

set of possible outcomes

1. Toss of a ~~chai~~ coin, S = {head, tail}
2. Roll of a dice, S = {1, 2, 3 ... 6}
3. ~~Pick~~ card from deck, S = {(H, 2) (D, 2) ... } : 52

hearts, diamonds

2 color

13 rank: 2, 3, ~ 10, J, K, Q, A

4. Level- term of a BUET stud.

$S = \{(1,1), (1,2) \text{---} (5,2) \ldots\} : 8$

5. Lifetime of a bulb, $S = [0, \infty)$

$a \leq x \leq b$ $[a,b] \Rightarrow$ close

$a < x < b$ $(a,b) \Rightarrow$ open

6. # of students in a class,

$S = [0, 60]$

Event : E

subset of sample space,

1. Head appers, $E = \{head\}$ $1/2$
2. 6 comes in dice, $E = \{6\}$ $1/6$

$E = \{\text{greater than } 3\}$

$= \{4,5,6\} = \frac{3}{6} = \frac{1}{2}$

3. $E\{clubs\} = \{2, \text{-} 10, J, K, Q, A\} : 13$

$E\{Ace\} = \{H, D, C, S\}$

5. $E = \{\geq 10 \text{ hours}\} = [10, \infty)$

$= \{\leq 10 \text{ hours}\} = [0, 10]$

4. $E = \{4\text{th Level}\}$

$= \{(4,1), \underbrace{(4,2)}_{\text{tuple}}\}$

Event E, F

$E = \{1, 2, 3\}$

$F = \{4, 3\}$

$E \cup F = \cancel{1, 2, 3, 4}, \{1, 2, 3, 4\}$

$E \cap F = \{3\}$

$\Downarrow$

$EF = \{3\}$

$E = \{1, 2\}$ $\quad$ $EF = \phi$.

$F = \{4, 5\}$ $\quad$ $\Rightarrow E$ & F are mutually exclusive

Probability :

probability, P, of an event

$P(E) =$ prob that E occurs

P is a function from sample space to $[0, 1]$

$$P : 2^{|S|} \longrightarrow [0, 1]$$

probability :

$\boxed{p(\phi) = 0}$

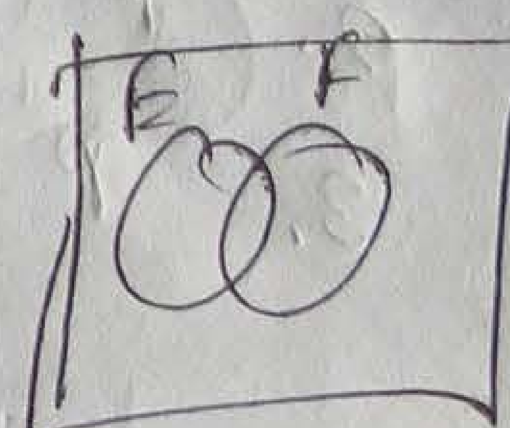
why? self study

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, E_2, E_3, \ldots E_n$ which are mutually exclusive

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

$$P(E \cup F) = P(E) + P(F)$$

$$P(E) = \frac{\text{sample favouring Event } E}{\text{Total sample}}$$



$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

CSE 301
Yousuf Sir cls-2

10.02.14

Probability Models

S = Sample space, {outcome}
E = event, subset of S

$P(E)$ = Probability that event E happened

1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. $P(E \cup F) = P(E) + P(F)$
   when E, F mutualy exclusive
   $E \cap F = \phi$ / $EF = \phi$.

$P(E \cap F) \rightarrow$ probability that both E & F happen

✱ Conditional Probability

$P(E|F)$ = Prob that E happen given F has been occured.

$P(E) \le P(E|F)$

1. $P(E) = P(E|F)$ হলে, E & F ~~and~~ are independent

By definition,

$$\boxed{P(E|F) = \frac{P(EF)}{P(F)}} \Rightarrow \boxed{\begin{aligned} P(EF) &= P(E|F) * P(F) \\ &= P(F|E) * P(E) \end{aligned}}$$

conditioning

1

if E & F are ind.,

$$P(E) = P(E|F)$$

then

$$P(EF) = P(E) * P(F) \quad [\text{① এ } P(E|F) \text{ এর value বসিয়ে}]$$

but

$P(EF) = P(E) * P(F)$ হলে E & F ind. বলা হয় থাকে

#

$$P(EF) = P(E|F)\, P(F) = P(F|E)\, P(E)$$

$$\therefore \cancel{P(E|F) =} \frac{P(EF)}{P(F)}$$

$$P(E|F) = \frac{P(F|E)\, P(E)}{P(F)} \quad \text{Bayes Rule}$$

Example:

Ben taken computer (CSE) or chemistry (ChE).

computer নিলে A পাওয়ার chance

$P(A \text{ grade}) = \frac{1}{2}$

$P(A|CSE) = \frac{1}{2}$

chemistry নিলে A পাওয়ার chance

$P(A \text{ grade}) = \frac{1}{3}$

$P(A|ChE) = \frac{1}{3}$

Q. what is the probability that ben gets A in ChE

Ans

$$P(A\,ChE) = P(A \mid ChE) \times P(ChE)$$

$$= \frac{1}{3} \times \frac{1}{2}$$

$$= \frac{1}{6}$$

Also,

$$P(A\,CSE) = \frac{1}{2} \times \frac{1}{2}$$

$$P(A \text{ grade}) = P(A\,CSE) + P(A\,ChE)$$

$$= P(CSE)\,P(A \mid CSE) + P(ChE)\cdot P(A \mid ChE)$$

$$= \cancel{\tfrac{1}{2}} \cdot \frac{1}{4} + \frac{1}{6}$$

CSE	ChE

A

F' = complement of F.

$$P(F \mid E) = \frac{P(E \mid F)\,P(F)}{P(E)}$$

$$= \frac{P(E \mid F)\,P(F)}{\underbrace{P(E \mid F)\,P(F)}_{\text{F ঘটেছে এই respect এ F E ঘটে}} + \underbrace{P(E \mid F')\,P(F')}_{\text{F ঘটে নাই এই respect এ E ঘটে}}}$$

in MCQ exam,

Q. A student knows answer or makes a guess.

$P(\text{Know}) = p$

$P(\text{Guess}) = 1-p$

$\underbrace{P(\text{Correct} \mid \text{Knows})}_{\text{correct ans. given that he knows}} = 1$

$P(C \mid K^c) = P(C \mid G) = \frac{1}{m}$

[no. of options m o o o o o o]

Find Prob that student 'knows' given that he gave the 'correct ans'?

$$P(K \mid C) = \frac{P(C \mid K) \cdot P(K)}{P(C)}$$

$$= \frac{1 \cdot p}{P(C \mid K)\, P(K) + P(C \mid G)\, P(G)}$$

$$= \frac{p}{1 \cdot p + \frac{1}{m}(1-p)}$$

if $p = \frac{1}{2}$;

$$P(K|C) = \frac{1/2}{1/2 + \frac{1}{m}\frac{1}{2}} = \frac{m}{m+1}$$

if $p = \frac{1}{3}$;

$$P(K|C) = \frac{1/3}{1/3 + \frac{1}{m}\frac{2}{3}} = \frac{m}{m+2}$$

if $m = 4$,

$$P(K|C) = \frac{4}{5}$$

$\frac{1}{2}$ $\frac{2}{3}$

Random Variable

Yusuf sir - 3

22/02/14

Sample space

event , p (event) =

$$0 \le p(E) \le 1$$

$$\sum_S p(E) = 1$$

$\boxed{X : \text{Event} \to R}$ যেকোন event কে real number এ represent করে

1. Toss of a coin, {head, tail}
   $\{1, 0\}$ $\quad p\{X=1\} = \frac{1}{2}$
2. Roll of a dice $\{1, 2, 3, \ldots 6\}$ $\quad p\{X=3\} = \frac{1}{6}$ / $p(3) = \frac{1}{6}$
3. pick of a card, $X = \{1, 2, \ldots, 52\}$
4. Life of a bulb $0 \longmapsto \infty$

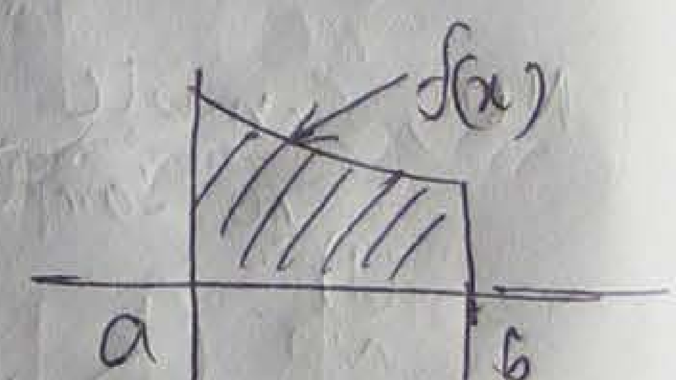
Now, Sample space is nothing but Real Line / Number Line

1 $\frac{1}{10}$ sample space

0 1

Random Variable → Discrete
Random Variable → Continuous

Discrete	Continuous
· X takes a discrete value	X takes a continuous val
① $p(i) = P\{X=i\}$ (from def.) ↑ PMF (Probability Mass function)	① $p(i) = P\{X=i\} = 0$ $\underbrace{a \quad i \quad b}_{I = \text{interval}}$
② $\sum_{i \in X} p(i) = 1$	① $P(X \in I) = P(a < x < b)$ $= \int_a^b f(x)\,dx$ ↑ probability density function
"p" mass	"f" density



② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Discrete Random Variable

1. Bernoulli: $X = \{1, 0\}$ Result of a "binary trial".
(p)

Bernoulli(p) $X = \begin{cases} 1, & \text{if success} \\ 0, & \text{if fail} \end{cases}$

mass, $p(1) = P\{X=1\} = p$ success probability
$p(0) = P\{X=0\} = 1-p$ failure "

2. Binomial: # of success in n trials. $\{[0,n]\}$
(n, p) $p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$
bin(n, p)

✓ × × ✓ × ✓

Verify: (HW) $\sum_{i=0}^{n} P(i) = 1$.

3. Geometric: # of trials to have first success.
(p) ← probability of success

Geo(p) $X = [1, \infty)$
$p(i) = P\{X=i\} = P(i-1 \text{ fail}, i\text{th success})$
$= (1-p)^{i-1} \cdot p = p(1-p)^{i-1}$

× × × × × ✓
(i-1) ith

Verify: (HW) $\sum_{i=1}^{\infty} p(i) = 1$ (HW)

Ex: There are 10 MCQ (4 options) — Yousuf Sir-4 — 23/02/14

Q: Prob that a std gets 4 correct answer by guessing

Sol: $X \rightarrow$ # correct ans out of 10

$X \sim bin(n, p)$ $n = 10$

p = succ. probab. $= \frac{1}{4}$

$P(X=4) = \binom{10}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^6$

Q: prob that he gets more than 50%

$P(X>5) = P(X=6) + \cdots + P(X=10)$

Poison: a limiting case of binomial, n is very large, p is small

$\lambda \Rightarrow$ rate

$X \sim bin(n, p)$ $np = \lambda$, constant

poision is calculate quantity (std error)

$P(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$

$= \frac{\lfloor n}{\lfloor i \lfloor n-i} \cdot \frac{\lambda^i}{n^i}\left(1-\frac{\lambda}{n}\right)^{n-i}$ $\left[\because p = \frac{\lambda}{n}\right]$

$= \frac{\lfloor n}{\lfloor n-i \cdot n^i} \frac{\lambda^i}{\lfloor i} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$

i terms

$$= \frac{n(n-1)(n-2)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{\lfloor i} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

$n \to \infty$

$$= e^{-\lambda} \frac{\lambda^i}{i!} \qquad \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\boxed{p(i) = p(X=i) = e^{-\lambda} \frac{\lambda^i}{\lfloor i}}$$

sum করলে 1 হবে

self study.

Ex: The # errors/spelling mistakes in a textbook page in poisson distribution with $\lambda = 10$

Q: Prob there is at least one error

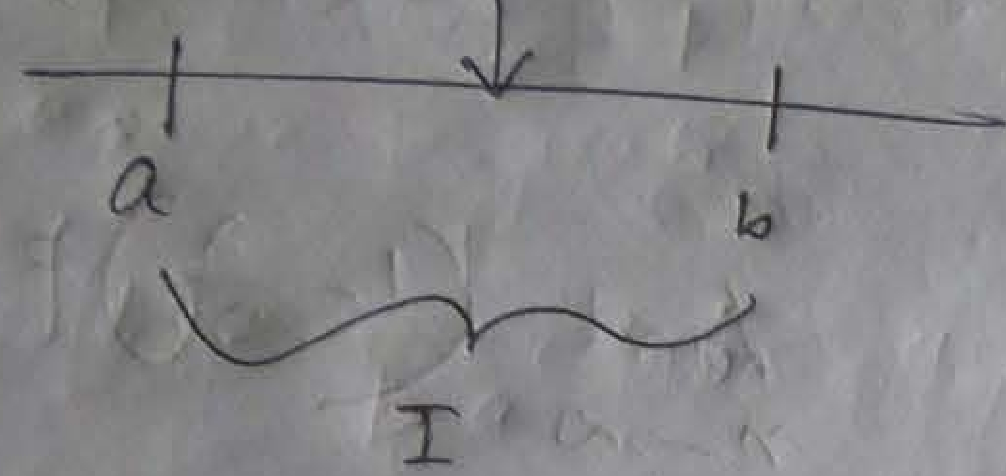
Sol: $X \to$ # of error $\quad X \sim \text{Poisson}(\lambda)$

$$p(X>0) = 1 - p(X=0)$$

$$= 1 - e^{-\lambda} \frac{10^0}{0!}$$

$$= 1 - e^{-10}$$

Continuous Random Variable



prob. density function

$$P(X \in I) = P(a < X < b) = \int_a^b f(x)\,dx$$

$$F(a) = P(X \le a) = P(-\infty \le X \le a)$$

CDF $= \int_{-\infty}^{\infty} f(x)\,dx$

Cummulitive distribution function

$$F(a) = \int_{-\infty}^{a} f(x)\,dx$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} f(u)\,du$$

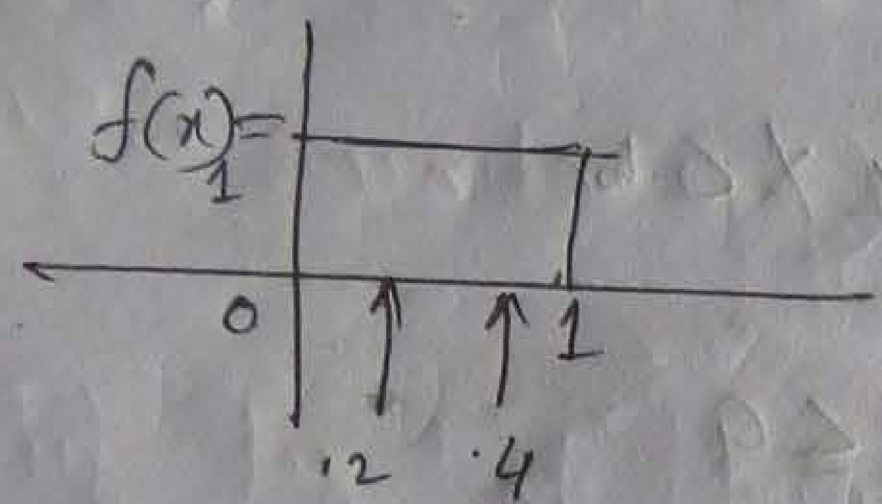
$$F(y) = \int_{-\infty}^{y} f(x)\,dx$$

Uniform (0, 1)

$0 < X < 1$

density $f(x) = 1, \; ; 0 < x < 1$

$\quad\quad\quad\quad = 0 \; ; \text{ else}$



$$P(0.2 < X < 0.4) = \int_{0.2}^{0.4} f(x)\,dx = \int_{0.2}^{0.4} dx$$

$$= x \Big|_{0.2}^{0.4} = 0.4 - 0.2 = 0.2$$

CDF,
$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{0}^{x} 1\,dt = t \Big|_{0}^{x} = x$$

$\therefore$ CDF; $F(x) = x$

$$F(0.7) = P(X \leq 0.7) = 0.7$$

$U\,(a,b)$

pdf, $f(x) = \dfrac{1}{b-a}$; $a < X < b$

cdf, $F(x) = \dfrac{x-a}{b-a}$; $a < X < b$

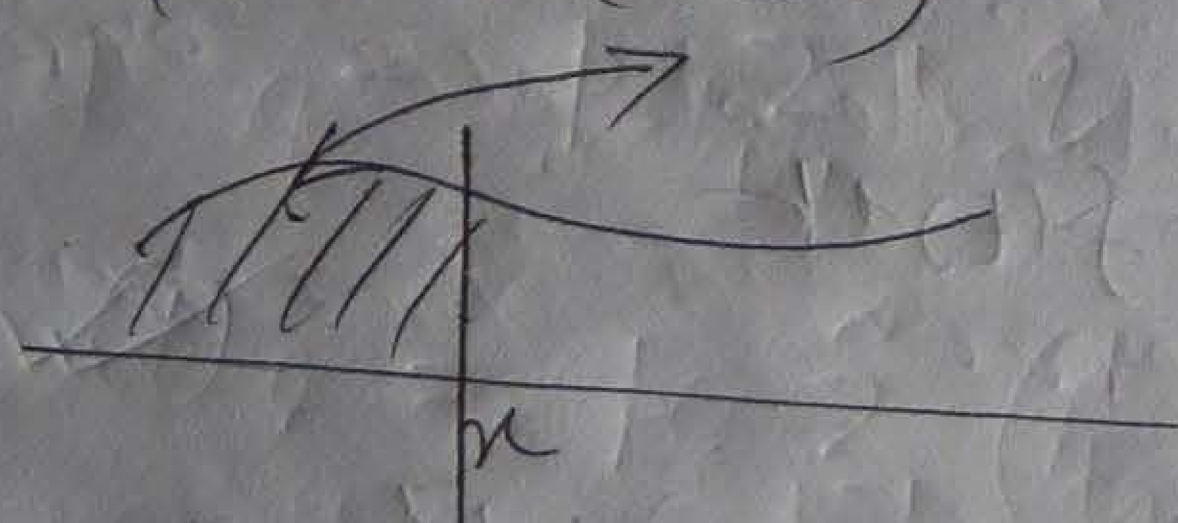
$\quad = 0 \qquad X \leq a$

$\quad = 1 \qquad X \geq b$

$\underline{F(x) = \text{monotonically increasing}}$

$F(x) = P(X \leq x)$

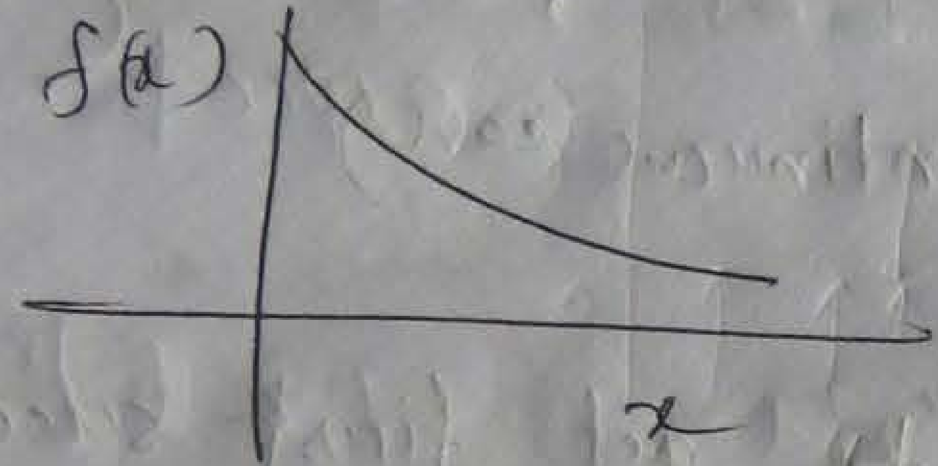
area always increasing for increasing x.



2. exponential Random Variable

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

Verify, $\int_0^\infty f(x) = 1$



$$F(x) = 1 - e^{-\lambda x}$$

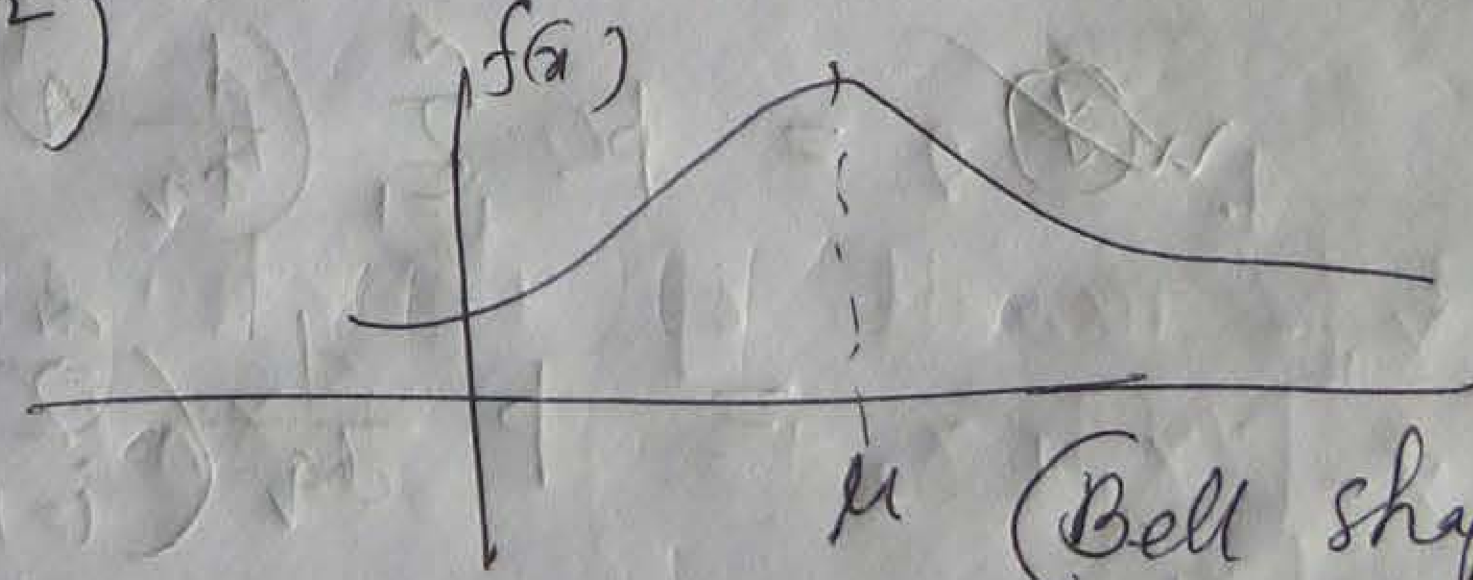
যেকোনো time measurement এ exponential হয় না। এর ও [illegible] std [illegible] [illegible] std [illegible]

3. <u>Gamma:</u> $f(x) = \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, \quad x > 0$

4. <u>Normal:</u> $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$

(avg height of tree)
(" height ~~age~~ of people of a certain age)

gaussian (μ, σ^2)



(Bell shape)

③ ও ④ এদের detail এ যাব না ✓

Expectation   Yousuf Sir-5   24/02/14

~~Exception~~: of a random variable

mean, average

expected value

$E[X] = \sum_i i\, p(i)$ (in discrete ~~continuous~~ case)

$= \int_{-\infty}^{\infty} x f(x) dx$ (" continuous case)

Ex: $X \sim geom(p)$. X = # trial to get first success

$E[X] = \sum_{i=1}^{\infty} i\, p(i)$

$= \sum_{i=1}^{\infty} i\, p(1-p)^{i-1}$

$= p \sum_{i=1}^{\infty} i\, q^{i-1}$ $[q = 1-p]$

$= p \sum_{i=1}^{\infty} \left(\frac{d}{dq} q^i\right)$

$= p \frac{d}{dq}\left(\sum_{i=1}^{\infty} q^i\right)$

$= p \frac{d}{dq}(q + q^2 + \cdots)$

$= p \frac{d}{dq}\left[q(1+q+q^2+\cdots)\right]$

$= p \frac{d}{dq} q \frac{1}{1-q}$

$$= p \frac{1}{(1-q)^2}$$

$$= p \frac{1}{p^2} = \frac{1}{p}$$

$$\boxed{E[X] = \frac{1}{p}}$$

a team's winning probability $= \frac{1}{4}$
(in a knockout series)

$\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{3}{4}$
✓ ✓ ✓ ✗

So, $E[X] = \frac{1}{3/4}$

Name	$E[X]$
Bernoulli	p
Binomial	np
Geometric	$\frac{1}{p}$
Poison	λ
Uniform $(0, 1)$	$\frac{1}{2}$
Normal	μ
exponential (λ)	$\frac{1}{\lambda}$

Theory (proof) লাগবে না।

Poison: $\boxed{E[X] = \sum_{i=0}^{\infty} i e^{-\lambda} \frac{\lambda^i}{\lfloor i} = \lambda}$

এই ২টো proof দেখতে হবে

exponential: $\boxed{E[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}}$

Identity:

1. $E[aX+b]$
$= aE[X] + b$

constant

$E[c] = c$

2. $E[X_1+X_2] = E[X_1] + E[X_2]$; linearity of expectation

3. $E[aX_1 + bX_2 + \cdots zX_n]$
$= aE[X_1] + bE[X_2] + \cdots + zE[X_n]$

Binomial:

$$E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np$$

$X \to$ # of trials in n trials.

1. 0 0 1. 0 1

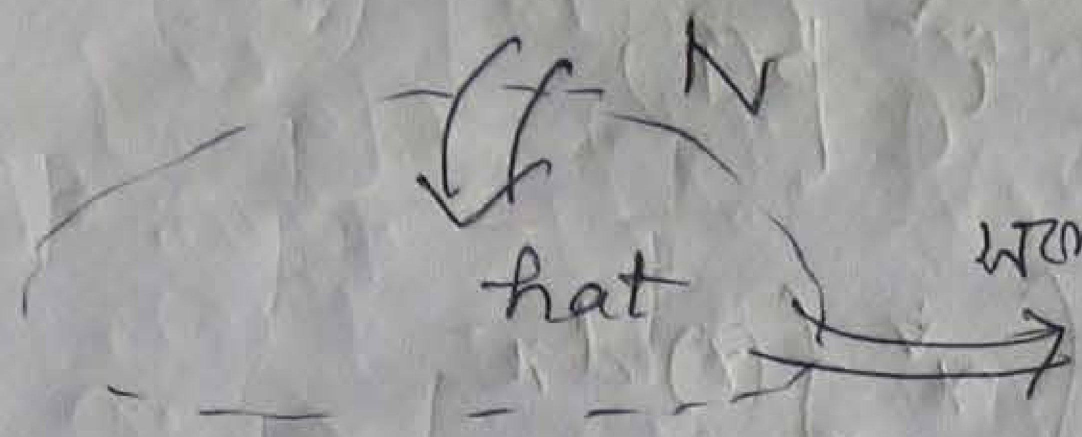
X_i = result of ith trial.

$X = X_1 + X_2 + \cdots + X_n$

$E[X_i] = 1.p + 0.(1-p) = p.$

$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = np$

Example :



সবাই একইসময় hat নিলো

$X \rightsquigarrow$ # people who gets his own hat.

$E[X] = ?$

X_i = bernoulli trial $\begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$

$X = X_1 + X_2 + \cdots + X_n$

$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n]$

$E[X] = \frac{1}{N} \cdot N = 1.$

$E[X_i] = \frac{1}{N} + 0\left(1 - \frac{1}{N}\right)$

$E(\text{constant}) = \text{constant}$

Variance

$\text{Var}(X) = E\left[(X - E[X])^2\right]$

[Standard form]

$= E\left[X^2 - 2X\,E[X] + E[X]^2\right]$

$= E[X^2] - 2E[X]\,E[X] + E[X]^2$

$\therefore \boxed{\text{Var}(X) = E[X^2] - E[X]^2}$

$$E[X^2] = \sum_i i^2 p(i)$$

$$= \int x^2 f(x)\, dx$$

$$E[5X] = 5\, E[\boxed{X}]$$

$$E[\ln X] = \int \ln x \; f(x)\, dx$$

$$= \sum_i \ln i \;\; p(i)$$

So,

$$E[g(X)] = \sum_i g(i)\, p(i)$$

$$= \int g(x)\, f(x)\, dx$$

$E[X^2]$
$E[X^3]$
$E[X^4]$

moments

এগুলো produce করে যে moment generating function হয়,

Moment Generating Function

$$\phi(t) = E\left[e^{tX}\right]$$

$$\phi(t) = \int_0^\infty e^{tx} \underbrace{\lambda e^{-\lambda x}}\, dx$$

$$= \int_0^\infty \lambda \, e^{-(\lambda - t)x}\, dt$$

$$= \frac{\lambda}{\lambda - t}$$

$$\phi'(t) = E\left[\frac{d}{dt} e^{tX}\right]$$

$$= E\left[X e^{tX}\right]$$

$$\phi''(t) = E\left[X^2 e^{tX}\right]$$

$$\phi'(0) = E[X]$$

$$\phi''(0) = E[X^2]$$

$$\phi'''(0) = E[X^3]$$

MGF Moment Generating Function 01/03/14

MGF's property:

$\phi(t) = E[e^{tx}]$

$E[X] = \phi'(0)$; $E[X^2] = \phi''(0)$

$E[X^3] = \phi'''(0)$;

① $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$; X and Y are independent

② MGF uniquely identify's a distribution

$Z \longrightarrow \phi_Z(t) = \frac{5}{5-t}$

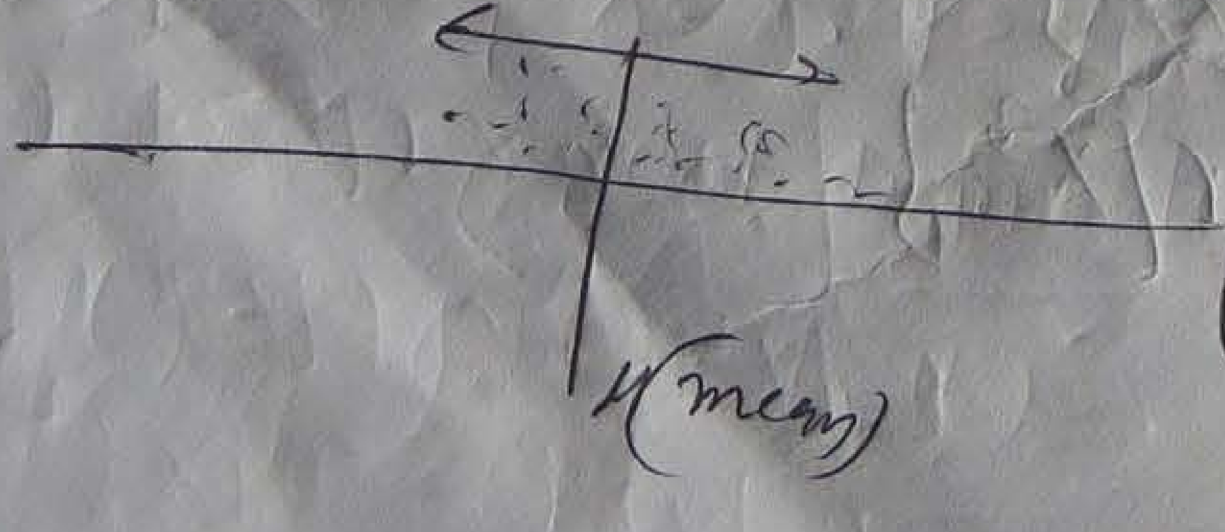
we know, if ~~poisson~~ exponent. (λ), $\phi(t) = \frac{\lambda}{\lambda - t}$

?? (mean)

So, Z is exponential with $\lambda = 5$.

$\sigma_x^2 = Var(X) = E[(X-\mu)^2] = E[X^2] - E[X]^2$

σ^2 = Varience



[Varience denote কত value গুলো mean হতে কত দূরে আছে]

Only σ = standard deviation

Covarience $(X, Y) = E[(X-\mu_x)(Y-\mu_Y)]$

$Cov(X, Y)$ $= E[XY] - E[X]E[Y]$ — degree of dependency

এখানে $Y = X$ করলে Var এর formula আসে

$Cov(X, Y) = Var(X)$

(X, Y একে অপরের উপর কতটা dependent)

(*) If X & Y are independent,

$Cov(X, Y) = 0$.

$$-1 \leq Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}} \leq +1$$

$\downarrow$

$\boxed{\text{Corelation}}$

$Cor(X, Y) = +1$ হলে

if $X > \mu_x$ then $Y > \mu_y$ হবে

property:

① $Cov(X,Y) = Cov(Y,X)$

② $Cov(X,X) = Var(X)$

proof (cor...) ③ $Cov(X, Y+Z) = Cov(X,Y) + Cov(X,Z)$

④ $Cov(cX, Y) = c\, Cov(X,Y)$

$Var(X+Y) \neq Var(X) + Var(Y)$

$= Var(X) + Var(Y) + 2\, Cov(X,Y)$ gtr,

what we've seen so far

⇒ if X & Y are independent

① $Cov(X,Y) = 0$.

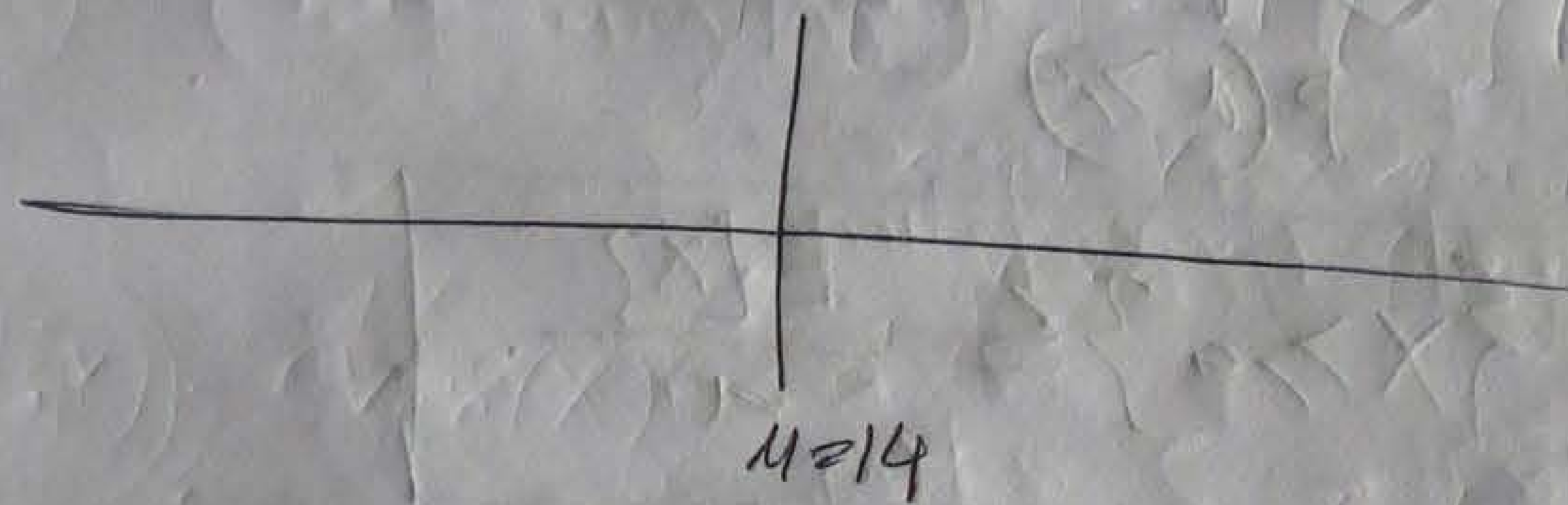
② $Var(X+Y) = Var(X) + Var(Y)$

③ PMF, $p(x,y) = P\{X=x, Y=y\} = P\{X=x\} \cdot P\{Y=y\}$

(joint distribution)

④ MGF $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$

$X, \ \mu = E[X], \ \sigma^2$



$\boxed{P(\text{score} = 18)}$ Ans. বের করা যাবে না

আর, $P(\text{score} \geq 18)$ বের করা যাবে।

(Markov Inequility):

X is a non-negative RV (Random Variable),

for any $a > 0$,

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

So, $$P\{X \geq 18\} \leq \frac{14}{18}$$

Proof: $$E[X] = \int_0^\infty x f(x)\,dx$$

$$= \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx \geq \int_a^\infty a f(x)\,dx$$

$$= a\underbrace{\int_a^\infty f(x)\,dx}_{P(x\geq a)} = a\cdot P(X\geq a)$$

$$\boxed{P\{X\geq a\}\leq \frac{E[X]}{a}.}$$

$$P\{X<a\} = 1 - P\{X\geq 18\}$$

Chebyshev's Inequility

X with mean μ, σ^2

$$P\{|X-\mu|\geq k\}\leq \frac{\sigma^2}{k^2}$$

proof: let, $Y = (X-\mu)^2$, Y is non-neg

$$P\{Y\geq k^2\}\leq \frac{E[Y]}{k^2}$$ [markov inequility]

$$P\{(X-\mu)^2\geq k^2\}\leq \frac{E(Y)}{k^2}$$

$$\therefore \{|X-\mu|\geq k\}\leq \frac{\sigma^2}{k^2}$$

Cond

2/3/14

Conditional probability & expectation

$$P(E|F) = \frac{P(EF)}{P(F)} \quad \text{for } P(F) > 0.$$

If E & F are independent,

$$P(E|F) = P(E)$$

$$P(EF) = P(E)\,P(F)$$

Conditional Mass function

$$p_X(x) = P\{X = x\}$$

$$p_{X|Y}(x|y) = P\{X = x \mid Y = y\}$$

$$= \frac{P\{X = x, Y = y\}}{P\{Y = y\}}$$

$$= \frac{p(x,y)}{p_Y(y)}$$

$$f_{X|Y}(x|y) = f(x \mid Y = y)$$

Conditional CDF, $F_{X|Y}(x|y) = P\{X \leq x \mid Y = y\}$

$$= \sum_{a \leq x} P_{X|Y}(a|y)$$

(discrete)

$$F_{X|Y}(x|y) = \int_0^x f_{X|Y}(t|y)\, dt$$
(continuous case)

Conditional Expectation

$$E[X] = \sum_x x\, p(x)$$

$$E[X|Y=y] = \sum_x x\, p_{x|y}(x|y)$$

Ex: $X = 1, 2$ $\quad Y = 1, 2$

$p(1,1) = 0.5$ $\quad p(2,1) = 0.1$

$p(1,2) = 0.1$ $\quad p(2,2) = 0.3$

mass function $\quad p_{x|y}(x|Y=1) = P\{X = x \mid Y = 1\}$

$$= \frac{P\{X = x, Y = 1\}}{P\{Y = 1\}}$$

$$= \frac{p(x,1)}{p_Y(1)}$$

$$= \frac{p(x,1)}{0.6}$$

Now,

$$p_Y(1) = p\{X=1, Y=1\} + p\{X=2, Y=1\}$$

$$= p(1,1) + p(2,1)$$

$$= 0.5 + 0.1$$

$$= 0.6$$

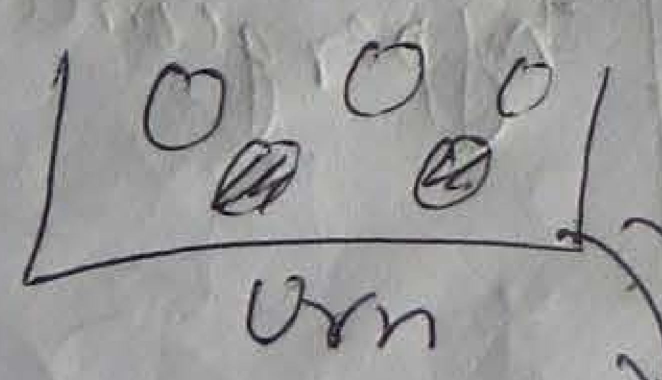
$$p_{X|Y}(1|1) = \frac{p(1,1)}{0.6} = \frac{0.5}{0.6}$$

$$p_{X|Y}(2|1) = \frac{p(2,1)}{0.6} = \frac{0.1}{0.6}$$

(f.)

Ans

Ex:



o: red (n_1)

⊗: blue (n_2)

pick m balls ($m < n_1 + n_2$)

X = # red balls in these m balls.

Q: What is the distribution of X?

Y = # blue balls in m balls.

$$P\{X=k \mid X+Y=m\}$$

$$= \frac{P\{X=k,\ X+Y=m\}}{P\{X+Y=m\}}$$

$$= \frac{P\{X=k,\ Y=m-k\}}{P\{X+Y=m\}}$$

$$= \frac{P\{X=k\}\ P\{Y=m-k\}}{P\{X+Y=m\}}$$

$$= \frac{\binom{n}{k} p^{k} (1-p)^{n_1-k} \binom{n_2}{m-k} (1-p)^{m-k} p^{n_2-m+k}}{P\{X+Y=m\}}$$

$$= \frac{\binom{n_1}{k}\binom{n_2}{m-k}}{\binom{n_1+n_2}{m}}$$

hypergeometric

$$P(X+Y=m) = \sum P(X=k,\ X+Y=m)$$

$$= \sum_k P(X=k,\ Y=m-k)$$

$$= \sum_k \underbrace{P(X=k)}_{\text{binomial}}\ \underbrace{P(Y=m-k)}_{\text{binomial}}$$

$$= \sum_k \binom{n_1}{k}\binom{n_2}{m-k}$$

p a pour [illegible]

$$= \binom{n_1+n_2}{m}$$

$$P\{X=k \mid X+Y=m\}$$

bin (n,p) bin (n,p)

poisson Poisson

exp (λ)

$E[X] =$ value

$E[X \mid Y=y] =$ value

$E[X \mid Y] =$ a function of Y

<u>Important!</u>

proof

$$E[\underbrace{E[X \mid Y]}_{g(Y)}] = \sum_y g(y) \cdot p(y)$$

$$= \sum_y E[X \mid Y=y] \, P(Y=y)$$

$$= \sum_y \sum_x x \cdot \underline{P(X=x \mid Y=y) \cdot P(Y=y)}$$

$\underline{P(ef) = P(e \mid f) \, P(f)}$

$$= \sum_y \sum_x x \, P\{X=x, Y=y\}$$

$$= \sum_x x \underbrace{\sum_y P\{X=x, Y=y\}}$$

$$= \sum x \, P\{X=x\}$$

$$= E[X]$$