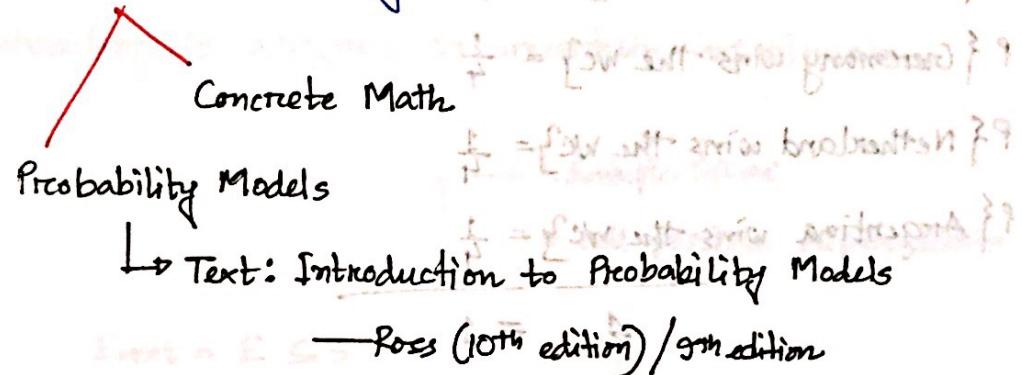


CSE 301: Mathematical Analysis for CS [SW with emphasis]



Probability is a value assigned to (uncertain) events.

is a measure of uncertainty

② coin toss {head, tail}

sample space, S

$$P\{\text{head}\} = \frac{1}{2}$$

$$(H)^2 = (H)(H)$$

$$P\{\text{tail}\} = \frac{1}{2}$$

Two Properties:

1. $0 \leq P\{\text{events}\} \leq 1$

2. $P(S) = 1$

$P(E \cup F) = P(E) + P(F)$ provided that E and F are mutually exclusive.

General rule

$P(EF) =$ Probability that E and F both happen

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

$$P\{\text{Brazil wins the WC}\} = \frac{1}{4}$$

$$P\{\text{Germany wins the WC}\} = \frac{1}{4}$$

$$P\{\text{Netherlands wins the WC}\} = \frac{1}{4}$$

$$P\{\text{Argentina wins the WC}\} = \frac{1}{4}$$

$$\text{Total probability} = \sum P(\text{wins}) = 1$$

$$P\{B \text{ and } G \text{ win WC}\} = 0$$

$$P\{B \text{ or } A \text{ win WC}\} = P(B) + P(A) - P(AB)$$

$$P(EF) = P(E) \cdot P(F); \text{ provided that } E \text{ and } F \text{ are 'independent'}$$

$$P(EIF) = \frac{P(EF)}{P(F)}$$

$$\downarrow \text{E, F are independent}$$

$$\text{Conditional Probability} \quad P(EIF) = P(E)$$

$$\rightarrow E \text{ doesn't depend on } F$$

$$P(EIF) \neq P(E)$$

$$\downarrow \text{E and F depend}$$

$$\text{Dependent} \quad P\{G \text{ wins}\} = \frac{1}{4}$$

$$P\{G \text{ wins} | B \text{ is runners-up}\} = 0$$

$$P\{A \text{ wins} | B \text{ is runners-up}\} = \frac{1}{2}$$

$$P\{A \text{ wins}\} = \frac{1}{4}$$

$$(1/2)^2 - (1/2)^2 + (1/2)^2 = (1/2)^2$$

Probability:

$$(3 \times 2)^9 = 6561$$

Probability is assigned to uncertain experiments.

[the set of sample points] \rightarrow Sample space and total #
= all possible experiments
 $\frac{1}{2} \times \frac{1}{2} =$

Event = $E \subseteq S$

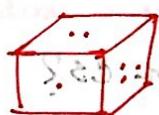
1. $0 \leq P(E) \leq 1$

2. $P(S) = 1$

3. If E_1 and E_2 are mutually exclusive, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Generally,

$$P(\bigcup_i E_i) = \sum_i P(E_i) \text{ if all } E_i \text{'s are ME } [E_i \cap E_j = \emptyset, i \neq j]$$

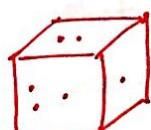
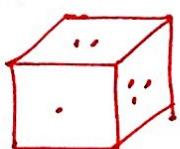


$S = \{1, 2, \dots, 6\}$

$$P\{\text{Getting 1}\} = P(1) = \frac{1}{6}$$

$$P\{\text{Even}\} = \frac{3}{6} = \frac{1}{2} = P\{2, 4, 6\}$$

most work ab = 3



$$S = \{(1, 1), (1, 2), \dots, (6, 6)\}$$

$$\#36 = 6^2 = 36$$

Two dies

$$P\{\text{both are odd}\} = \frac{3 \times 3}{36}$$

$$\left. \begin{array}{l} (1, 1) \\ (1, 3) \\ (1, 5) \\ (3, 1) \\ (3, 3) \\ (3, 5) \\ (5, 1) \\ (5, 3) \\ (5, 5) \end{array} \right\} = 9$$

$$P(EF) = P(E \cap F)$$

$= P(E) \cdot P(F)$ if E and F are independent.

$$P\{\text{both are odd}\} = P\{\text{1st is odd}\} \times P\{\text{2nd is odd}\}$$

referring to diagram above =

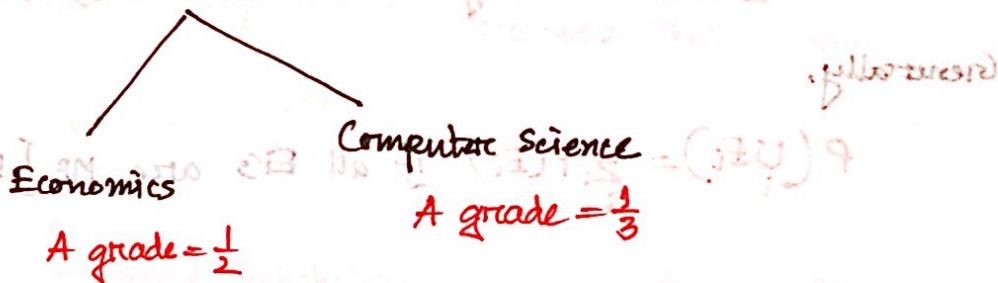
$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

$$P(EF) = \underline{P(E|F) \cdot P(F)}$$

$$P(E|F) = \frac{P(EF)}{P(F)} \rightarrow \text{conditional probability}; \text{ if } P(F) > 0$$

(2) A student chooses yellowware or blue pottery. If he chooses yellowware



What is the probability that he gets A in CS?

$$\text{Let, } \frac{1}{2} = (I)^4 = \text{favourable cases}$$

A = he gets A grade

$$\frac{1}{3} = \text{he chooses CS} = \text{favourable cases}$$

E = he chooses Econ.

$$P(A|E) = \frac{1}{2}$$

$$P(A|C) = \frac{1}{3}$$

$$P(A|C) = P(A|C) \cdot P(C)$$

$$= \frac{1}{3} \cdot \frac{1}{2}$$

$$= \frac{1}{6}$$



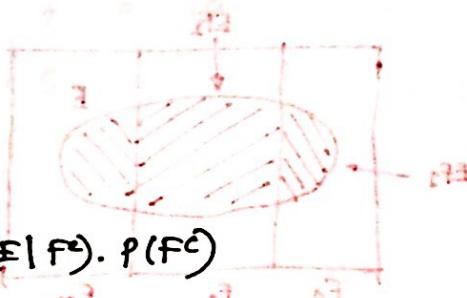
$$\frac{E}{F/F^c}$$

$$EF = E \cap F$$

$$E = EF \cup EF^c$$

$$\frac{G_1 G_2}{G_1 G_2} = G_1 G_2$$

EF and EF^c are M.E.



$$P(E) = P(EF) + P(EF^c)$$

$$= P(E|F) P(F) + P(E|F^c) P(F^c)$$

$$= P(E|F) \cdot P(F) + P(E|F^c) \cdot (1 - P(F))$$

$$P(E) = P(E|F) P(F) + P(E|F^c) P(F^c)$$

We want to compute:

$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E|F) \cdot P(F)}{P(E)}$$

$$= \frac{P(E|F) \cdot P(F)}{P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c)} \rightarrow \text{Bayes formula}$$

Q. Student \rightarrow K = student know the answer

C = student gives the correct answer

K^c = student guesses the answer

$$m \rightarrow \text{no. of options}$$

$$P(K) = p$$

What is the probability that student knew the answer?

$$P(C|K) = 1$$

Given that he gives the correct answer?

$$P(C|K^c) = \frac{1}{m}$$

$$\frac{1}{m}$$

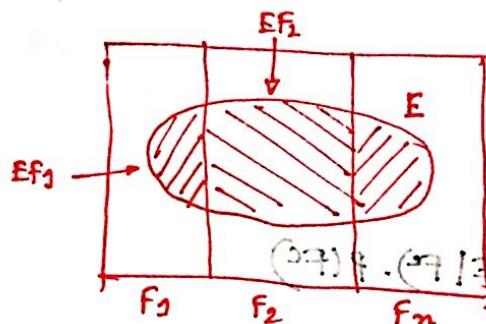
$$\begin{aligned} P(K|C) &= \frac{P(C|K)}{P(C)} = \frac{P(C|K) \cdot P(K)}{P(C|K) \cdot P(K) + P(C|K^c) \cdot P(K^c)} \\ &= \frac{1 \cdot p}{1 \cdot p + \frac{1}{m} (1-p)} \\ &= \frac{p}{p + \frac{1}{m} (1-p)} \end{aligned}$$

If $m=4$, $p=\frac{1}{2}$

$$P(K|C) = \frac{4}{5} = 80\%$$

Conditional Probability

$$P(E|F) = \frac{P(EF)}{P(F)}$$



$$E = EF_1 \cup EF_2 \cup \dots \cup EF_n$$

$$= \bigcup_i EF_i$$

: outcomes of favorable cases

$$P(E) = \sum_{i=1}^n P(EF_i) = \sum_{i=1}^n P(E|F_i)P(F_i)$$

Then, $P(X_j|E) = \frac{P(E|F_j) \cdot P(F_j)}{\sum_{i=1}^n P(E|F_i) \cdot P(F_i)}$ → General Bayes Rule

Probability of at least one six = \rightarrow favorable cases / total cases

Ex. 19 Two dice are rolled. Find the probability of getting at least one six.

Q1: P(at least one six) = ?

$$= 1 - P\{\text{none six}\}$$

$$= 1 - \frac{5}{6} \cdot \frac{5}{6}$$

It is not favorable outcome = 5/6 * 5/6 = 25/36

Number of cases = 36 - 25 = 11

$$= \frac{11}{36}$$

$$\frac{11}{36} = 0.30555555555555554$$

$$\frac{(1/6) \cdot (5/6)}{(1/6) \cdot (5/6) + (5/6) \cdot (1/6)} = \frac{1/36}{1/36 + 1/36} = 1/2 = 0.5$$

$$\frac{1/36}{1/36 + 1/36} = \frac{1}{2} = 0.5$$

Q2: $P\{at \text{ least } 1 \text{ six} | \text{two faces are different}\}$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

1, 1
2, 6
5
5, 6

$$+ = 10$$

6, 1
6, 2
5
6, 5

$$+ = 10$$

$$P(SID) = \frac{P(SD)}{P(D)}$$

$$= \frac{10/36}{30/36}$$

$$= \frac{1}{3}$$



Ex. 36]

0
0

Urn 1

0
0

Urn 2

Explain freedom priors ??

Probability of getting black ball from Urn 1 = $\frac{1}{2}$ (freedom prior)

Q. One urn is randomly selected and then a marble is picked.

Probability of getting black?

$$= P(B)$$

$$= P(B|U_1) \cdot P(U_1) + P(B|U_2) \cdot P(U_2)$$

$$= \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{1}{2}$$

Q. Say the marble is black. What is the probability that the marble was chosen from urn 1?

$$P(U_1|B) = \frac{P(B|U_1) \cdot P(U_1)}{P(B|U_1) \cdot P(U_1) + P(B|U_2) \cdot P(U_2)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{1}{3}$$

Ex. 46

3 doors and one will be randomly changed

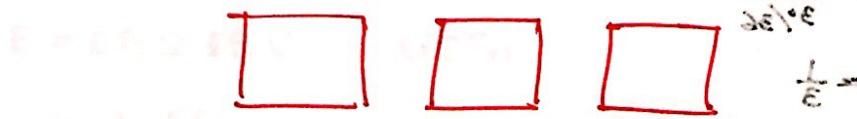
A B C

$$P(A) = \frac{1}{3} \quad \text{so } = + \quad \begin{array}{|c|c|c|} \hline & A & B \\ \hline A & 1,1 & 1,2 \\ \hline B & 2,1 & 2,2 \\ \hline C & 3,1 & 3,2 \\ \hline \end{array}$$

$$P(A \dots) = \frac{1}{2} ? \rightarrow \text{see book}$$

Q. Monty Hall Problem

$$\frac{(1/2)^2}{(2/3)^2} = (1/2)^2$$



$$P\{\text{choosing correct door}\} = \frac{1}{3}$$

$$P\{\text{choosing the wrong door}\} = \frac{2}{3} \rightarrow \text{chance to win if switched}$$

(check internet)

should be different

$$(2)^2 =$$

$$(2/3)^2 \cdot (1/2)^2 + (1/3)^2 \cdot (2/2)^2 =$$

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} =$$

$$\frac{1}{2} =$$

with half probability will be stand at bedroom with you.

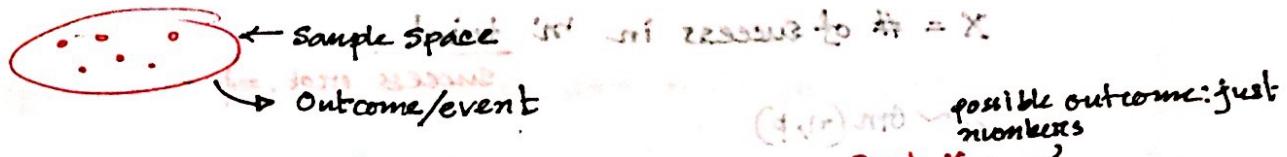
Get your most regards now

$$\frac{(1/2)^2 \cdot (1/2)^2}{(2/3)^2 \cdot (1/2)^2 + (1/3)^2 \cdot (2/2)^2} = \frac{(1/2)^2}{(2/3)^2 + (1/3)^2} =$$

$$\frac{1/4}{13/36} =$$

Random Variable

(4.1) Bernoulli .2



Coin toss. $S = \{\text{head, tail}\}$ \rightarrow $\frac{1}{2}, \frac{0}{2}$ \rightarrow $\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}$ \rightarrow $\frac{0}{2} + \frac{1}{2} + \frac{2}{2} + \frac{3}{2}$ \rightarrow $2^+ = \text{integer}$

Coin toss, $S = \{0, 1\}$

Event X → has outcome if $\exists q = -3, -2, -1, 0, 1, 2, \dots, \infty$ \rightarrow $R = \text{real number}$

Example: $\{1, 10, 100, 200, \dots, \infty\} \rightarrow R = \text{real number}$

$X = \# \text{ students attending a class} \leftarrow \text{discrete}$

$X = \text{temperature in a day} \leftarrow \text{continuous}$

$X = \# \text{ goals in a WC} \leftarrow \text{discrete}$

$X = \# \text{ games in a WC} \rightarrow \text{not random}$

Discrete Random Variables

Standard distributions: $P(X=x) = p(x)$

1. Bernoulli (p)

$X = \text{outcome of a binary trial} \rightarrow$ success/failure

$X = \begin{cases} 1, & \text{if success} \\ 0, & \text{else} \end{cases}$

Probability mass function, $p(x) = P\{X=x\} \rightarrow [p(k) = P(X=k)]$

$p(1) = P\{X=1\} = p$; $p = \text{success probability}$

$p(0) = P\{X=0\} = 1-p$; $q = 1-p = \text{failure probability}$

Persons at home $\exists q =$

* Brazil Loses with probability 0.2. What is the probability, it wins in the next game. $p(0) = 0.2 = P\{X=0\}$

$P\{X=1\} = p(1) = 1 - p(0) = 0.8$

2. Binomial (n, p)

Independent trials

$X = \# \text{ of success in } n \text{ trials}$

$$X \sim \text{Bin}(n, p)$$

→ X is sum

Mass function, $p(i) = P\{X=i\}$

= $P\{i \text{ success out of } n \text{ trials}\}$

$\checkmark \times \checkmark \checkmark \times \checkmark \checkmark \times$
independent

= $P\{i \text{ success and } (n-i) \text{ failure}\}$

$$= \binom{n}{i} p^i (1-p)^{n-i}$$

We can verify, $\sum p(i) = 1$

巴西打 4 场球，赢 3 场的概率是多少？

$$P\{X=3\} = \binom{4}{3} p^3 (1-p)^1$$

(4) illustrated

$$= \binom{4}{3} (0.8)^3 (0.2)^1$$

successes → 3
(3) (1)

failure → 1
(2)

3. Geometric Random Variable (p)

$X = \# \text{ of trials to get the first success} ; [1, \infty)$

$\checkmark \times \checkmark \checkmark \checkmark \checkmark \checkmark$ with

Mass function, $P\{X=i\} = p_x(i)$

= $P\{i \text{th trial is success}\}$

= $P\{(i-1) \text{ failure and } i \text{th trial success}\}$

$$\{ = (1-p)^{i-1} \cdot p$$

$$\Rightarrow p(i) = (1-p)^{i-1} \cdot p$$

We can verify, $\sum_{i=1}^{\infty} p(i) = 1$ as p_i (B) wins \rightarrow
 (comes after 3 losses) probability is

What is the prob. Brazil wins in its 4th game?

$$p(4) = (0.2)^3 \cdot 0.8$$

$$= 0.008 \cdot 0.8 = \frac{8}{1000} \cdot 8 = 0.0064$$

Random Variable

Discrete, Probability mass function, $p_X(i)$

Continuous, Probability density function, $f_X(x)$ if x

Discrete:

Expt 1. Bernoulli (p)

outcomes $\# = x$

2. Binomial (n, p)

outcomes $\# = x$, $\# \text{ wins} = k$

3. Geometric (p)

$$X \sim \text{geom}(p) \Rightarrow p_X(i) = p(1-p)^{i-1}$$

Verify:

$$\sum_{i=1}^{\infty} p_X(i) = 1 \text{ for Geometric, } x \geq 1$$

$$\sum_{i=1}^{\infty} p(i) = \sum_{i=1}^{\infty} p(1-p)^{i-1} = \sum_{i=1}^{\infty} p q^{i-1} = p \sum_{i=1}^{\infty} q^{i-1}$$

$$= p(1+q+q^2+q^3+\dots)$$

$$= p \cdot \frac{1}{1-q}$$

(Q.v) Inverse of to make it easier to calculate

→ back to $p = 0.2$, $1-p = 0.8$

→ back to $p = 0.2$, $1-p = 0.8$

$$\frac{1}{p} = 5 \approx 5.291 \cdot \left(\frac{1}{p}-1\right) = \frac{1}{0.2} - 1 = 4.5$$

4. Poisson (λ), $\lambda = \text{rate}$, $X \sim \text{Poisson}(\lambda)$
 ↳ counting, rare events)

mass function, $p(i) = P\{X=i\} = \frac{e^{-\lambda}\lambda^i}{i!}$

$$\text{verify: } \sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Ex. Typographical mistakes per page is Poisson distribution with rate $\lambda=1$. What is the probability that you see at least one mistake in a certain page?

$$X = \# \text{ mistakes}$$

$$X \sim \text{Poisson}(\lambda), \lambda=1$$

$$\begin{aligned} P\{X \geq 1\} &= 1 - P\{X=0\} \\ &= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} \Big|_{\lambda=1} \\ &= 1 - e^{-1} \end{aligned}$$

$$\text{Probability} = \sum_{i=1}^{\infty} \frac{1}{i!} = 1 - e^{-1} = 0.632$$

Poisson is a version of Binomial (n, p)

$$\begin{aligned} n &\rightarrow \infty \\ p &\rightarrow 0 \rightarrow \text{Poi}(\lambda) \\ \lambda &= np \end{aligned}$$

$$\text{for binomial, } p(i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{n!}{i!(n-i)!} \cdot \frac{\lambda^i}{i!} \left(1 - \frac{\lambda}{n}\right)^{n-i}; np=\lambda \Rightarrow p=\frac{\lambda}{n}$$

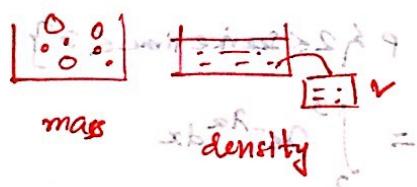
$$\begin{aligned}
 &= \frac{n(n-1)(n-2)\dots(n-i+1)}{i!} \cdot \frac{\lambda^i}{i!} (1-\frac{\lambda}{n})^{n-i} \\
 &= \frac{n(n-1)(n-2)\dots(n-i+1)}{i!} \cdot \frac{\lambda^i}{i!} (1-\frac{\lambda}{n})^{n-i} \\
 \lim_{n \rightarrow \infty} p(i) &= \frac{n(n-1)(n-2)\dots(n-i+1)}{i!} \cdot \frac{\lambda^i}{i!} (1-\frac{\lambda}{n})^{n-i} \quad i \ll n \\
 &\quad \text{1} \qquad \qquad \qquad \text{x} = \lim_{n \rightarrow \infty} (1-\frac{\lambda}{n})^n = e^{-\lambda} \\
 \Rightarrow \lim_{n \rightarrow \infty} p(i) &= e^{-\lambda} \cdot \frac{\lambda^i}{i!}
 \end{aligned}$$

Random Variable

Discrete vs. Continuous
Integer (real)

X , PMF $P\{10 \leq X \leq 15\}$

Prob. mass function $p(x) = P\{X=i\}$
 $p(x) = 0$ for x not integer and negative



Continuous RV:

- Prob. density function, $f_X(x)$

$$P\{a \leq X \leq b\} = \int_a^b f_X(x) dx$$

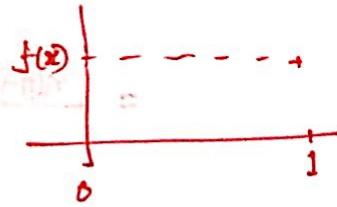
- Prob. distribution function, $F_X(x) = P\{X \leq x\}$

$$\begin{aligned}
 P\{a \leq X \leq b\} &= F_X(b) - F_X(a) \\
 &= \int_a^b f_X(x) dx
 \end{aligned}$$

1. Uniform $\sim U(0,1)$

Density, $f_X(x) = 1$

Distribution, $F_X(x) = P\{X \leq x\}$



$$= \int_0^x 1 \cdot dx$$

$$= x \Big|_0^x$$

$$= x$$

$F_X(x) = x$ with $x \in [0, 1]$

2. Exponential, $\exp(\lambda)$ $\lambda = \text{rate}$

Density, $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Distribution, $F_X(x) = \int_0^x f_X(x) dx = \int_0^x \lambda e^{-\lambda x} dx$

$$= 1 - e^{-\lambda x}$$

Example: time, waiting time, serving time, $X \sim \text{Exp}(0.1)$

$P\{2 \leq \text{service time} \leq 3\}$

$$= \int_2^3 \lambda e^{-\lambda x} dx$$

$$= \int_2^3 0.1 e^{-0.1 x} dx$$

$$= P\{2 \leq X \leq 3\}$$

$$= P\{X \leq 3\} - P\{X \leq 2\}$$

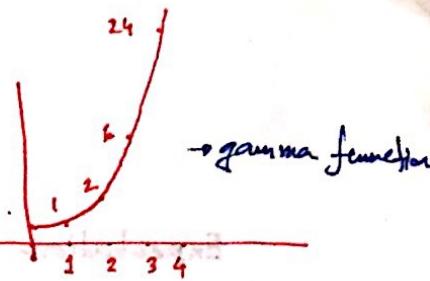
$$= F(3) - F(2)$$

$$= (1 - e^{-0.3}) - (1 - e^{-0.2})$$

(x) \times uniform pricing cost -

$$\times b(x) \times b(x) = \{b(x) \times a\}$$

(x) \times uniform service cost -



3. Gamma (α, λ)

$$\text{Density, } f_X(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$$

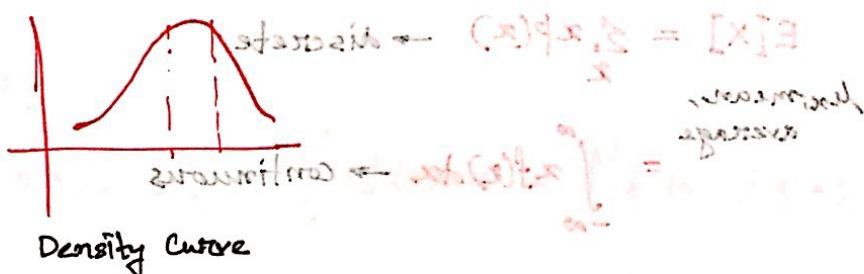
definition: $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$

relationship probability mass \rightarrow

4. Normal distribution, $N(\mu, \sigma^2)$

mean μ , variance σ^2

$$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$



$$q = q_{\cdot i} = (0)q_{\cdot 0} + (1)q_{\cdot 1} = [x]_2 \text{ ; illogical}$$

c.t.

4th week
7th week) Sunday

eldest son measure to $q_i = x$

$$(0)q_{\cdot 2} = [x]_2$$

$$\sum_{i=1}^{k-1} (i-1)q_{\cdot i} =$$

$$= C^{k-1}_{k-1} + C^{k-1}_{k-2} + \dots + C^{k-1}_{1} (k-1)q_{\cdot 1} = \sum_{i=1}^{k-1} =$$

q. first student in class

$$= C^{k-1}_{k-1} + C^{k-1}_{k-2} + \dots + C^{k-1}_{1} (k-1)q_{\cdot 1} =$$

further 1st, common for all $[x]_2 =$

$$= C^{k-1}_{k-1} + C^{k-1}_{k-2} + \dots + C^{k-1}_{1} (k-1)q_{\cdot 1} =$$

Expectation of random variable

(x, p) known &

$X \rightarrow$ random variable

$$\frac{\text{defn}}{(x, p) \text{ known}} = (x) \times t, \text{ general}$$

↳ has probability distribution

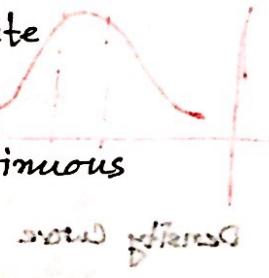
discrete mass function, $p(x) = P\{X=x\}$

continuous density function $f(x) = (x) \times t$

$$E[X] = \sum_x x p(x) \rightarrow \text{discrete}$$

lex, mean, average

$$= \int_{-\infty}^{\infty} x f(x) dx \rightarrow \text{continuous}$$



$$\text{Bernoulli: } E[X] = 1 \cdot p(1) + 0 \cdot p(0) = 1 \cdot p = p$$

Binomial:

$X = \# \text{ of success in } n \text{ trials}$

$$E[X] = \sum_{i=0}^n i p(i)$$

$$= \sum_{i=0}^n i \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^n i \frac{\cancel{n}}{\cancel{i} \cancel{n-i}} p^i (1-p)^{n-i}$$

$$= np \sum_{i=0}^n \frac{\cancel{n-i}}{\cancel{i+1} \cancel{n-i}} p^{i+1} (1-p)^{n-i}$$

$$= np \times (p+1-p)^n$$

$$= np$$

* $E[X]$ is not random, is constant

For 10 fair coins, $E[X] = 10 \times \frac{1}{2}$ Scanned by CamScanner

Geometric: X = # trials to get the first success \rightarrow go to x_0

$X = \# \text{ trials to get the first success}$

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} i p(i) \\ &= \sum_{i=1}^{\infty} i (1-p)^{i-1} \cdot p \\ &= \frac{1}{p} \quad \text{[check in book]} \end{aligned}$$

Initial with i to another $= iX$ (not i)

continuous:

$$\text{Uniform}(0, 1): E[X] = \int_0^1 x f(x) dx = \int_0^1 x dx, f(x) = 1$$
$$= \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

$[x]E + \dots + [x]E + [x]I = [x]E$, each

Exponential: $f(x) = \lambda e^{-\lambda x}, \lambda > 0, \lambda = \text{rate}$

$$E[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \dots$$

$$= \frac{1}{\lambda}$$

λ :  median life

$$\operatorname{operator} \rightarrow E[X^2] = \sum x^2 p(x)$$

$$E[ax+b] = a E[x] + b$$

$$S = [x]E$$

$$E[c] = c ; \text{ constant}$$

but with step increasing with the $E[x] = iX$ (not i)

$$E[g(x)] = \sum g(x) \cdot p(x)$$

$$E[X^3] = \sum x^3 p(x) \quad X \sum x^3 p(x) \quad X \sum x^3 (p(x))^3$$

Linearity of expectation:

$$E[ax + bY] = aE[X] + bE[Y]$$

$$E[X+Y] = E[X] + E[Y]$$

Binomial: : n trials

$X = \# \text{ of success}$

Let, $X_i = \text{outcome of } i\text{th trial}$

$= \begin{cases} 1, & \text{success} \\ 0, & \text{fail} \end{cases}$

$$X = X_1 + X_2 + \dots + X_n$$

$$\text{Now, } E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$

Binomial Bernoulli

$$= p + p + p + \dots + p$$

$$= np$$

Hat Problem:

: n

$X = \# \text{ person who get their hats}$

$$E[X] = ?$$

Let, $X_i = \begin{cases} 1 & \text{if } i\text{th person gets his hat} \\ 0 & \text{else} \end{cases}$

Now, $X = X_1 + X_2 + \dots + X_n = \sum_i X_i$ Summation from i=1 to n

$$\begin{aligned} E[X] &= E\left[\sum_i X_i\right] = \sum_i E[X_i] \\ &\text{since } E[X_i] = \frac{1}{N} + O\left(1 - \frac{1}{N}\right) \\ &= \frac{1}{N} + \frac{1}{N} + \dots + \frac{1}{N} = \frac{1}{N} \\ &= \frac{N}{N}(E[X] - \mu) = (X) \text{ avg} \end{aligned}$$

Significance: $E[X] = E[X]$ is a measure of average

Variance:

$$[(X-\mu)(Y-\mu)] \text{Var}(X) = E[(X - E[X])^2]$$

degree of variation

$$E[X^2] - E[X]^2 = E[(X-\mu)^2]$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

$$\text{Co-variance: } \text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Properties:

$$1. \text{cov}(X, X) = \text{Var}(X)$$

$$2. \text{cov}(aX, Y) = a \text{cov}(X, Y)$$

$$3. \text{cov}(X, Y+Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$$

$$4. \text{cov}(X, Y) = \text{cov}(Y, X)$$

Properties are part of: $(X) \text{avg} + (Y) \text{avg} = (X+Y) \text{avg}$

Expectation and Variance

Expectation, $E[X] = \sum x p(x)$, discrete

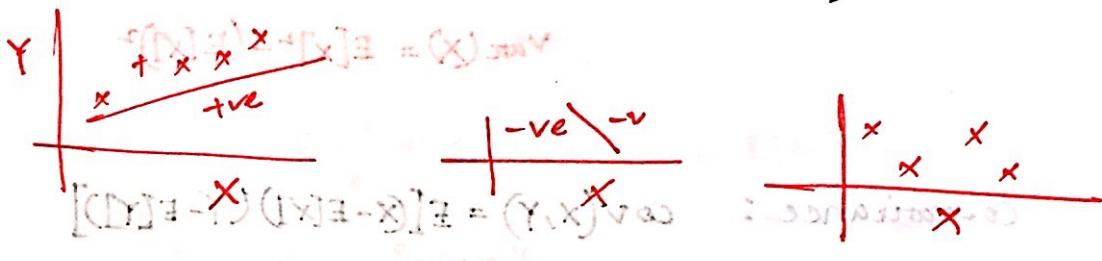
$= \int_{-\infty}^{\infty} x f(x) dx$, continuous

Variance, $\text{Var}(X) = E[(X - E[X])^2]$

Degree of variation, $= E[X^2] - E[X]^2$ - Prove it

Covariance $\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$

$E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$ reduction



$\text{Cov}(X, Y) = 0$, if X and Y are independent

$\Rightarrow E[XY] = E[X]E[Y]$

$$(X \otimes Y) \text{ v.o. } = (X \otimes 0) \text{ v.o.}$$

From

$$(S \otimes X) \text{ v.o.} + (Y \otimes X) \text{ v.o.} = (S + Y \otimes X) \text{ v.o.}$$

$$E[X+Y] = E[X] + E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y); \text{ If they are independent}$$

$$\text{Var}(x+y) = (\text{Var}x + \text{Var}y)$$

$$\text{Var}(x+y) = \text{Var}(x) + \text{Var}(y) + 2\text{Cov}(x,y). \quad [7] \text{ Ex. 1.2}$$

Generally,

$$\text{Var}\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(x_i, x_j)$$

$$\text{Proof: } \text{Cov}(x, x) = \text{Var}(x) \quad [7] \text{ Ex. 1.2}$$

$$\text{Var}\left(\sum x_i\right) = \text{Cov}\left(\sum_{i=1}^n x_i, \sum_{j=1}^n x_j\right) \quad [7] \text{ Ex. 1.2}$$

$$\text{Cov}(x, x) = \text{Cov}(x_1 + x_2 + \dots + x_n, x_1 + x_2 + \dots + x_n) \quad [7] \text{ Ex. 1.2}$$

$$E[x+y] \quad \text{from uniform probability theorem}$$

$$\text{Var}[x+y]$$

$$x \sim P_x(x) \quad E[x] = \mu \quad E[x] = \mu \quad E[x] = \mu$$

$$y \sim P_y(y)$$

$$(x+y) \sim \frac{P_x(x) + P_y(y)}{\text{without sum}} \quad E[x+y] = (\mu_x + \mu_y) \quad \text{from uniform probability theorem}$$

joint distribution

$$[x+y] = (\mu_x + \mu_y)$$

Let, $x_1, x_2, \dots, x_n \sim \text{i.i.d}$ with mean μ , variance σ^2
 identically and independently distributed

Let us define, $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$: illustrates

$$\text{Sample mean, } \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} \quad E[\bar{x}] = \mu$$

$$\text{Var}(\bar{x}) = \frac{\text{Var}(x_1 + x_2 + \dots + x_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \quad \text{because } \text{Var}(x_i) = \sigma^2$$

$$\text{Var}(\bar{x}) = \frac{\text{Var}(x_1 + x_2 + \dots + x_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\star \text{Var}(cx) = c^2 \text{Var}(x)$$

$$1. E[\bar{x}] = ? \quad 2. \text{Var}(\bar{x}) = \text{Var}\left(\frac{1}{n} \sum x_i\right)$$

$$\text{Ansatz: } \bar{x} = \frac{1}{n} (x_1 + x_2 + \dots + x_n) \quad \Rightarrow \quad \frac{1}{n^2} \leq \text{Var}(x_i), \text{ all independent}$$

$$= \frac{1}{n} E[x_i] \quad (x) \text{ var} = (x, x) \text{ var} = \frac{1}{n^2} (n \sigma^2)$$

$$= \frac{1}{n} \cdot (n \mu) \quad (x \leq x_i, i \leq n) \text{ var} = (\bar{x}, \bar{x}) \text{ var} = \frac{\sigma^2}{n}$$

$$(x_1 + \dots + x_n + nx + nx + \dots + nx + nx) \text{ var} = \text{Var}(\bar{x}) \approx 0 \text{ if } n \rightarrow \infty$$

Definition: $E[X] = \int_{-\infty}^{\infty} x f(x) dx$

Moment Generating Function, MGF $E[e^{tX}]$

$E[X]$	$E[X^2]$	$E[X^3]$
1st	2nd	3rd

MGF of X , $\phi_X(t) = E[e^{tx}]$
 \rightarrow function of t (not x)

$$\phi_X(t) = E[e^{tx}]$$

$$\begin{aligned} &= \sum e^{tx} p(x), \text{ discrete} \\ &= \int_{-\infty}^{\infty} e^{tx} f(x) dx, \text{ continuous} \end{aligned}$$

$$\text{Bernoulli: } \phi(t) = e^{t \cdot 1} p + e^{t \cdot 0} (1-p)$$

$$m = E[X] \quad \text{Ansatz: } \sum x p + 1 - p \bar{x}$$

$$\text{Exponential: } \phi(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \int_0^{\infty} \lambda e^{-(\lambda-t)x} dx$$

Binomial: $x = x_1 + x_2 + \dots + x_n$

unbiased linear

$$\phi_{x+y}(t) = \phi_x(t) \cdot \phi_y(t)$$



$$E[e^{tx+n}]$$

$$= E[e^{tx} \cdot e^{tn}]$$

$$= E[e^{tx}] \cdot E[e^{tn}]$$

$$= \phi_x(t) \cdot \phi_y(t)$$

$$\phi_x(t) = (pe^t + 1-p)^n$$

(x)

t

$$[x]_{\text{new}}$$

unbiased Bi-variate $\Rightarrow E[X] = E[X]$

$$\therefore E[X] = E[X] \quad \text{or } E[X] = E[X]$$

tilde moment

$$\frac{E[X^2]}{\infty} \geq \text{tilde moment} \geq X \text{ unadjusted mean or mean}$$

$$\phi(t) = E[e^{tx}] \leq [e^{tb(x)t}] \leq [e^{b(x)t}] = [X]^2$$

$$\phi'(t) = E\left[\frac{d}{dt} e^{tx}\right] = E[x e^{tx}]^2, \quad \phi(0) = E[X]$$

$$\phi''(t) = E[X^2 e^{tx}]$$

$$\phi''(0) = E[X^2]$$

$$\phi'''(t) = E[X^3 e^{tx}]$$

$$\phi'''(0) = E[X^3]$$

$$\frac{E[X^3]}{\infty} \geq \text{tilde moment} \geq$$

3rd moment: $E[X^3] = \int_{-\infty}^{\infty} x^3 f(x) dx$

$$= \int_{-\infty}^{\infty} x^3 \lambda e^{-\lambda x} dx \rightarrow \text{harder to calculate}$$

Exponential Distribution

$\phi'''(0) = (X)_{\text{new}}$

$$f(x) = \lambda e^{-\lambda x}, x > 0 \rightarrow \text{density function}$$

$$\phi(t) = \frac{1}{1-\lambda t} \quad t < \lambda \rightarrow \text{MGF: mean: } \mu = \lambda$$

$$\frac{1}{1-\lambda t} \geq \text{tilde moment}$$

Limit Theorems

$$x \rightarrow \begin{cases} p(x) \\ f(x) \\ E[x] \\ \text{Var}[x] \end{cases}$$

$x = \# \text{ waiting customers}$

$$E[x] = 10 \quad P\{x \geq 15\} = ?$$

Markov inequality

for a non-negative rv X , $P\{X \geq a\} \leq \frac{E[X]}{a}$

$$E[X] = \int_0^\infty xf(x)dx \geq \int_a^\infty xf(x)dx \geq \int_a^\infty af(x)dx = (a)E[X]$$

$$[X \geq a]_E = [X \geq a]_E = \left[X \geq \frac{a}{a} \right]_E = (a)E[X]$$

$$E[X] = (a)E[X]$$

$$\text{so, } E[X] \geq aP\{X \geq a\}$$

$$\Rightarrow P\{X \geq a\} \leq \frac{E[X]}{a}$$

$$\text{so, } P\{X \geq 15\} \leq \frac{10}{15} = \frac{2}{3} \quad E[X] = 10 : \text{mean line}$$

$$X \rightarrow E[X] = \mu \quad \int_{-\infty}^{\infty} x f(x) dx = \mu$$

$$\text{Var}(X) = \sigma^2$$

standard deviation about μ

Chebychev's Inequality: $\forall x, |x - \mu| \geq t \Rightarrow P\{|x - \mu| \geq t\} \leq \frac{\sigma^2}{t^2}$

$$\text{For any random variable } X, P\{|x - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

$$\Rightarrow P\{|x - \mu| \geq k\} \leq \frac{1}{k^2}$$

Proof:

Let, $Y = (X-\mu)^2$

$$E[Y] = E[(X-\mu)^2] = \sigma^2$$

Due to Markov, Y is non-negative

$$P\{Y \geq k^2\} \leq \frac{E[Y]}{k^2}$$

$$\Rightarrow P\{(X-\mu)^2 \geq k^2\} \leq \frac{E[Y]}{k^2}$$

$$\Rightarrow P\{|X-\mu| \geq k\} \leq \frac{E[Y]}{k^2}$$



Q. $E[X] = 10$, $\text{Var}(X) = 3$

$$P\{8 \leq X \leq 12\}$$

$$= P\{|X-10| \leq 2\}$$

$$= 1 - P\{|X-10| \geq 2\}$$

So, $P(|X-10| \geq 2) \leq \frac{3}{4}$

Now, $\frac{\sum (x_i - \mu)^2}{n} = \bar{x}^2$

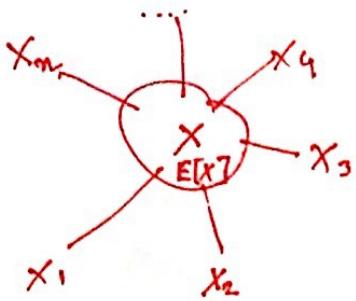
$$\therefore \bar{x}^2 = [E(X)]^2 + \dots + [X]^2 + [X]^2 = [E(X)]^2$$

Now, $\bar{x}^2 = (\bar{x})^2$

Now to understand this with an example

After sufficient trials we get the result as follows

Mean = 10
Variance = 3
Standard deviation = 1.732



\bar{x}

if x_i

x_1, x_2, \dots, x_n have common distribution.

$$\mathbb{E}[x_i] = \mu$$

then mean of \bar{x} is μ and

Sample mean; $\bar{X} = \frac{x_1 + x_2 + \dots + x_n}{n}$

strong law of numbers

if $n \rightarrow \infty$, $\bar{x} \rightarrow \mu$ with probability 1.

$$\frac{\bar{x} - \mu}{\sigma}$$

$$\bar{x} \rightarrow \mu$$



$$\bar{x} = (\bar{x}) \text{Var}(\bar{x})^{-0.5} = [\bar{x}]^{\frac{1}{2}}$$

Central Limit Theorem

x_1, x_2, \dots, x_n i.i.d

$$\mathbb{E}[x_i] = \mu$$

$$\text{Var}(x_i) = \sigma^2$$

$$S_n = x_1 + x_2 + \dots + x_n$$

$$\bar{x} = \frac{S_n}{n}$$

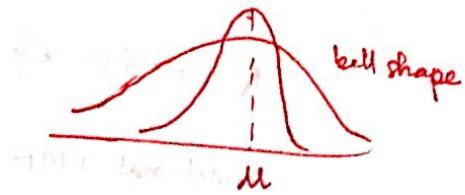
$$\mathbb{E}[S_n] = \mathbb{E}[x_1] + \mathbb{E}[x_2] + \dots + \mathbb{E}[x_n] = n\mu.$$

$$\text{Var}(S_n) = n\sigma^2$$

What is the distribution of S_n ?

By CLT: If $n \rightarrow \infty$, then S_n takes a Normal distribution with
mean = $n\mu$
variance = $n\sigma^2$

$$\boxed{\text{Q}} \quad \lim_{n \rightarrow \infty} S_n \sim N(\mu, \sigma^2)$$



Normal: Density function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Standard normal ($\mu=0, \sigma^2=1$)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\boxed{\text{Q}} \quad \lim_{n \rightarrow \infty} \frac{S_n}{\sigma} \stackrel{D}{\sim} N(0, 1)$$

$$\lim_{n \rightarrow \infty} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \stackrel{D}{\sim} N(0, 1)$$

Let X be a random variable

Conditional Expectation

Two events E and F (E is dependent on F)

$$\text{conditional probability } P(E|F) = \frac{P(EF)}{P(F)}$$

$x=a$ is a random event

$= P\{X=a\}$ is given by PMF

$P\{X=a|X=b\}$ is given by Conditional PMF

$$P_X(x) = P\{X=x\}$$

$$\text{Conditional PMF } P_{X|Y}(x|y) = P\{X=x | Y=y\}$$

$$= \frac{P\{X=x, Y=y\}}{P\{Y=y\}}$$

$$= \frac{P(x,y)}{P_Y(y)} \quad \leftarrow \text{joint mass function}$$

We have,

$$\begin{aligned} P_Y(y) &= P\{Y=y\} = \sum_x P\{X=x, Y=y\} \\ &= \sum_x P(x,y) = 1 \end{aligned}$$

$$P_X(x) = \sum_y P(x,y)$$

Example 3.1: Joint mass function of X and Y

$$P(1,1) = 0.5$$

$$P(2,1) = 0.1$$

$$P(1,2) = 0.1$$

$$P(2,2) = 0.3$$

Conditional PMF of X given $Y=1$

$$P_{X|Y}(1|1) = P\{X=1 | Y=1\}$$

$$= \frac{P\{X=1, Y=1\}}{P\{Y=1\}}$$

$$= \frac{0.5}{P(1,1) + P(2,1)}$$

$$= \frac{0.5}{0.5 + 0.1} = \frac{5}{6}$$

$$P_{X|Y}(2|1) = \frac{P(2,1)}{P_Y(1)} = \frac{0.1}{0.6} = \frac{1}{6}$$

For continuous Random Variable,

$$(Y)_{\mathbb{B}} = [Y/X]_{\mathbb{B}}$$

density, $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ $\left[\begin{array}{l} \text{joint density} \\ f(x,y) = [Y/X]_{\mathbb{B}} \end{array} \right]$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx$$

$$\text{if } Y \neq X \text{ then } f_Y(y) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y) dx}$$

Example 3.6: $f(x,y) = \begin{cases} 6xy(2-x-y), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{else} \end{cases}$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$\text{Now, } f_Y(y) = \int_0^1 6xy(2-x-y) dx$$

$$= y(4-3y)$$

$$\therefore f_{X|Y}(x|y) = \frac{6xy(2-x-y)}{y(4-3y)}$$

$$= -\frac{6x(2-x-y)}{4-3y}$$

Conditional Expectation

$$E[X] = \sum x p(x)$$

$$\text{const} \quad E[X|Y=y] = \sum x P_{X|Y}(x|y)$$

function of y

$$= \sum x \frac{P(x,y)}{P_Y(y)}$$

$E[X|Y=y]$ is a function of y

$E[X|Y]$ is a function of Y

Random Variable

$$E[X|Y] = g(Y)$$

$$E[E[X|Y]] = E[X]$$

const proof:

$$E[E[X|Y]]$$

$$= E[g(Y)]$$

$$= \sum_y g(y) p(y)$$

$$= \sum_y E[X|Y=y] p(y)$$

$$= \sum_y E[X|Y=y] p\{Y=y\}$$

$$= \sum_y \sum_x x p\{X=x|Y=y\} p\{Y=y\}$$

$$= \sum_y \sum_x \frac{p\{X=x, Y=y\}}{p\{Y=y\}} \cdot p\{Y=y\}$$

$$= \sum_y \sum_x x p\{X=x, Y=y\}$$

$$= \sum_x x \sum_y p\{X=x, Y=y\}$$

$$= \sum_x x p\{X=x\}$$

$$= E[X]$$

$$E[X] = E[E[X|Y]] \rightarrow \text{"conditioning"}$$

\rightarrow Y to irrelevant in $E[Y|X]\mathbb{E}$

\rightarrow X to irrelevant in $E[X|X]\mathbb{E}$

Conditional Expectation

Expectation, $E[x] = \sum \text{prob}(x) \text{ of event } \# = x$

(1) $\text{margin } x$

$$\frac{1}{\#} = [x]E$$

conditional expectation, $E[x] = E[E[x|Y]]$

$$= \sum_y E[x|Y=y] P\{Y=y\}$$

Want to do to find out x

Example:

NM.

SM

$$E = [0 = Y|x]E$$

$x = \text{Delay between NM and SM}$

$$E[x] = ?$$

$Y = \text{Day of travel.}$

Average delay

Week days = 1

Week ends = 2

$$70 \text{ min} = E[x|\text{weekday}]$$

$$45 \text{ min} = E[x|\text{weekend}]$$

↓

$$E[x|Y=1]$$

$$E[x|Y=2]$$

$$E[x] = E[x|Y=1] P\{Y=1\} + E[x|Y=2] P\{Y=2\}$$

$$= 70 \times \frac{5}{7} + 45 \times \frac{2}{7}$$

$$AM = [AM]E$$

3.B (Trapped Miner)

Miner 1 →



$X = \text{Time to exit}$

$$E[x|Y=1] = 2$$

$$E[x|Y=2] = 3 + E[x]$$

$$E[x|Y=3] = 5 + E[x]$$

$$E[x] = (AM)E(1) + (AM)E(2) + (AM)E(3) = AME$$

$$E[x] = 2 \times \frac{1}{3} + (3 + E[x]) \times \frac{1}{3} + (5 + E[x]) \times \frac{1}{3}$$

$$\text{Solve for } E[x] \Rightarrow E[x] = 10$$

$$* \text{exercise: } 40, 9, 10$$

Q. What if the miner can remember the state?

Think

$X \sim \text{Geom}(p)$

no failures (first success)

$$E[X] = \frac{1}{p} \quad x = \# \text{ trial to get the first success}$$

by conditioning: $E[X|Y=1] = E[X]$, no failure (first success)

$$\{Y=1\} \cap \{x \in \mathbb{N}\} =$$

$Y = \text{the result of first trial}$

$$E[X|Y=1] = 1 \rightarrow p$$

$$E[X|Y=0] = 1 + E[X] \rightarrow 1-p$$

where $\text{first trial with success} = n$

$$E[X] = p \cdot 1 + (1-p)[1 + E[X]]$$

$$S = E[X]$$

$$\Rightarrow E[X] = \frac{1}{p}$$

$$\Downarrow$$

$$\{Y=1\} \cap \{x \in \mathbb{N}\}$$

$$\{S=Y\} \cap \{X \in \mathbb{N}\} + \{S=Y\} \cap \{X \in \mathbb{N}\} = E[X]$$

22

$N_k = \# \text{ trials to get } k \text{ consecutive success}$

$$E[N_k] = M_k$$

Now we condition on $k-1$ consecutive success,

$$E[N_k | N_{k-1}] = p(1 + N_{k-1}) + (1-p)(1 + N_{k-1} + E[N_k])$$

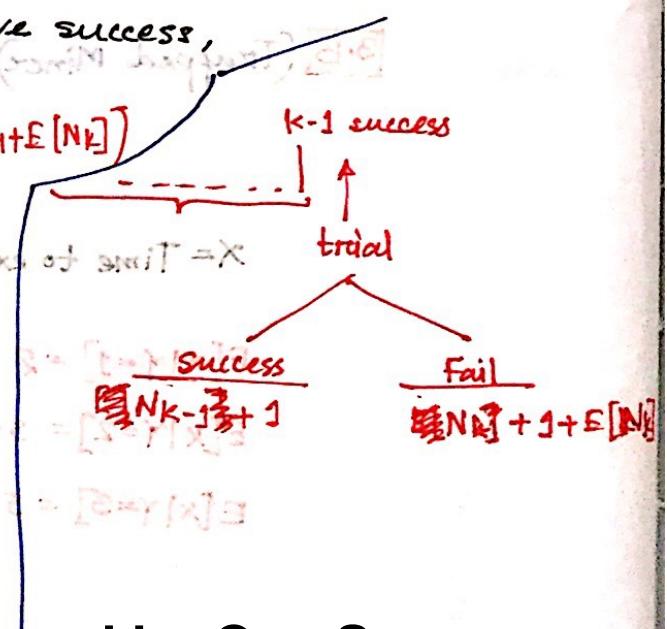
$$E[N_k] \Rightarrow E[N_k | N_{k-1}] = 1 + N_{k-1} + (1-p)E[N_k]$$

$$\Rightarrow E[N_k] = 1 + E[N_{k-1}] + (1-p)E[N_k]$$

$$\Rightarrow M_k = 1 + M_{k-1} + (1-p)M_k$$

$$\Rightarrow M_k = \frac{1}{p} + \frac{1}{p}M_{k-1}$$

$$M_1 = \frac{1}{p}, M_2 = \dots$$



$$M_k = \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^k} = \frac{(1/p)^k - 1}{1 - 1/p} = \frac{(1/p)^k - 1}{p/(p-1)}$$

Example:

Probability

shop

1

$$P(\text{book}) = P(\text{book} \cap \text{not book}) = [P(\text{book})] \cdot [1 - P(\text{book})]$$

2

$$P(\text{book}) = P(\text{book} \cap \text{not book}) + P(\text{book}) \cdot P(\text{not book}) = P(\text{book}) \cdot 2$$

3

$$P(\text{book}) = P(\text{book} \cap \text{not book}) + P(\text{book}) \cdot P(\text{not book}) + P(\text{book}) \cdot P(\text{not book}) \cdot P(\text{not book}) = P(\text{book}) \cdot 3$$

$p = \text{shop has the book}$

$X_i = \text{time to search the book in shop } i$

$Y = \text{time to purchase the book}$

$$Y = X_1 + X_2 + X_3 + \dots + X_N$$

$N = \# \text{ visit to get the book}$

$$E[Y] = E\left[\sum_{i=1}^N X_i\right] \neq \sum_{i=1}^N [E[X_i]]$$

N itself is a random variable

If N was constant then

linearity of expectation would

be applicable

$$E\left[\sum_{i=1}^N X_i \mid N=n\right] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = n E[X]$$

$$E\left[\sum_{i=1}^N X_i \mid N\right] = N E[X]$$

$$\Rightarrow E\left[E\left[\sum_{i=1}^N X_i \mid N\right]\right] = E[N E[X]]$$

$$\Rightarrow E\left[\sum_{i=1}^N X_i\right] = E[X] \cdot E[N]$$

$$E[X] = \int x p(x) dx$$

$$\Rightarrow E[Y] = E[N] \cdot E[X]$$

If $p = 0.2$, $E[X] = 20 \text{ min}$

$$E[Y] = \frac{1}{0.2} \times 20$$

=

$$E\left[\sum_{i=1}^N X_i\right] = n E[X]$$

$$E\left[\sum_{i=1}^N X_i\right] = E[N] E[X]$$

\Rightarrow "compounding"

* exercise: 34

Conditioning

$$- E[X] = E[E[X|Y]] = \sum_y E[X|Y=y] P\{Y=y\}$$

$$- \text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}(X|Y)] \rightarrow \text{Proof}$$

$$- \text{Compounding}, S = \sum_{i=1}^N X_i \quad \text{Add with end game} \approx Y$$

$$- E[S] = E[N] \cdot E[X] \quad \text{Add with base of sum} \approx Y$$

$$- \text{Var}(S) = \sigma^2 E[N] + \mu^2 \text{Var}(N) \rightarrow \text{Prove it}$$

$$\text{Var}(X) = E[(X-\mu)^2]$$

$$= E[X^2] - E[X]^2$$

Exercise: 44, 56, 77

Add with top of sum = N

N random variables

with Computing Probabilities by Conditioning

variables to dimensions

$$\text{P}(E) = \sum_{[x|Y=y]} P(E|Y=y) \cdot P\{Y=y\}$$

Example:

$$\begin{array}{c} \xrightarrow{x_1} \text{rickshaw} \\ \xleftarrow{x_2} \text{cng} \end{array} \quad x_1 \sim \exp(\lambda_1) \quad x_2 \sim \exp(\lambda_2)$$

P{You get a rickshaw}

$$= P\{x_1 < x_2\}$$

$$= \int_0^\infty P\{x_1 < x_2 | x_2 = y\} f_{x_2}(y) dy$$

$$= \int_0^\infty P\{x_1 < y\} f_{x_2}(y) dy$$

$$= \int_0^\infty F_{x_1}(y) \cdot f_{x_2}(y) dy$$

$$\begin{aligned} & [x] \cdot [x] = \left[\frac{1}{2} x^2 \right]_0^{\infty} \\ & [x] \cdot [x] = [Y] \end{aligned}$$

$$P\{x < a\} = F(a)$$

constant

$$\cos \frac{1}{s \cdot o} = [Y]$$

for exponential, $f(x) = \lambda e^{-\lambda x}$ (constant with λ)
 $f(x) = 1 - e^{-\lambda x}$

$$\int_0^\infty (1 - e^{-\lambda_1 y}) \cdot (\lambda_2 e^{-\lambda_2 y}) dy$$

under y-axis

$$= \int_0^\infty \lambda_2 e^{-\lambda_1 y} dy - \int_0^\infty e^{-(\lambda_1 + \lambda_2)y} dy$$

whole density function

$$= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$\otimes \leftarrow i = 11 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$

So, $P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$ \rightarrow $(\text{ex}|\text{head}) \underset{i=1}{\overset{8}{\sum}} = (\text{head})$

Example:

$$X \sim \text{Bin}(n, p) \rightarrow P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k} \rightarrow p \text{ is constant here}$$

$p \sim \text{uniform}(0, 1) \rightarrow$ if p is not constant

$$P\{X=k\} = \int_0^1 P\{X=k|p\} f(p) dp$$

{if variable}

$$= \int_0^1 \binom{n}{k} p^k (1-p)^{n-k} \cdot 1 dp$$

{if variable}

$$= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k} dp$$

from literature

$$= \frac{n!}{k!(n-k)!} \cdot \frac{k! (n-k)!}{(n+1)!}$$

$$= \frac{1}{n+1} \rightarrow$$

(Best Price Problem)



Strategy:

$\underbrace{\circ \circ \circ \dots \circ}_{k} \text{ } \circ \circ \dots \circ$

return the first
value > any value

$$\begin{array}{c} 2 \ 5 \ 8 \ 7 \\ \hline k \end{array} \quad \begin{array}{c} 4 \ 6 \ 11 \ 12 \ 2 \ 1 \\ \hline \end{array}$$

$\xrightarrow{\text{if } x_i > x_k}$

$$\begin{array}{c} 2 \ 5 \ 11 \ 7 \\ \hline k \end{array} \quad \begin{array}{c} 4 \ 6 \ 12 \ 2 \\ \hline \end{array}$$

$\xrightarrow{\text{return } x_1}$

$$\begin{array}{c} 2 \ 5 \ 12 \ 7 \\ \hline k \end{array} \quad \begin{array}{c} 4 \ 6 \ 11 \ 2 \ 1 \\ \hline \end{array} \rightarrow \text{⊗}$$

$\xrightarrow{\frac{1}{n-k+1}}$

$$P_k(\text{best}) = \sum_{i=1}^n P(\text{best} | x=i) \cdot P\{x=i\} = \{x > x_i\}$$

x = position of the best file.

: signified

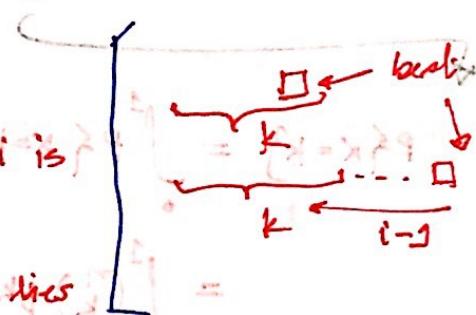
$$\text{Best value} = \frac{1}{n} \sum_{i=1}^n P_k(\text{best} | x=i)$$

→ uniform for all i if $k \gg n$

$$P_k(\text{best} | x=i) = 0 \text{ if } i \leq k$$

$$= P\{2nd \text{ best within } [i, n] \text{ is } \\ \text{within } k\}$$

$$= P\{ \text{best among } i \text{ files} \\ \text{within } k\}$$



$$= \frac{k}{n} \cdot \frac{(k+1)(k+2)\dots(n)}{(n+1)n} =$$

$$P_k(\text{best}) = \frac{1}{n} \sum_{i=k+1}^n \frac{k}{i-1} \cdot \frac{(i-1)(i-2)\dots(1)}{(n+1)n} =$$

$$\text{that is } = \frac{k}{n} \sum_{i=k}^{n-1} \frac{1}{i} \approx \frac{1}{k} \sum_{i=k}^{n-1} 1 =$$

$$\approx \frac{k}{n} \int_k^{n+1} \frac{1}{x} dx$$

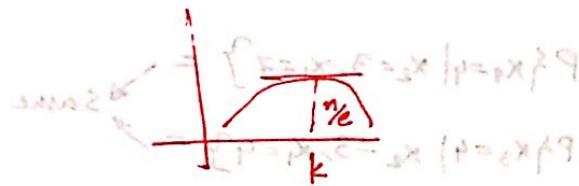
$$= \frac{k}{n} \log x \Big|_k^{n+1}$$

$$= \frac{k}{n} \log \left(\frac{n+1}{k} \right)$$

$$\approx \frac{k}{n} \log \left(\frac{n}{k} \right) \rightarrow \text{what value of } k \text{ maximizes this?}$$

VS evaluating $\frac{n}{k}$ no longer satisfies $e^{\frac{n}{k}} = e \rightarrow k = \frac{n}{e}$ being maximum

so $k = n/e$

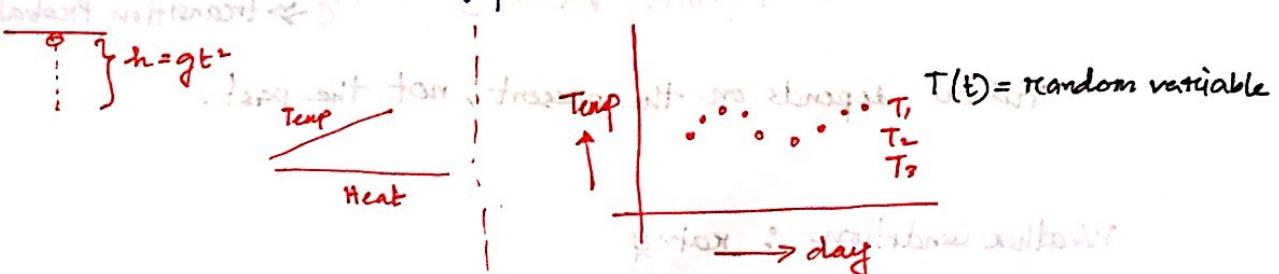


Markov Chain \rightarrow Stochastic state of system, with transition

Markov Chain

Markov chain is a stochastic process that has Markovian property.

\downarrow probabilistic



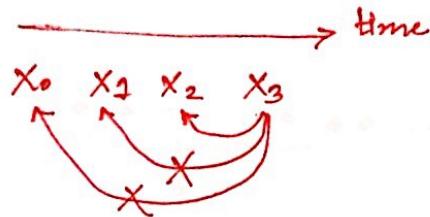
A stochastic process is a series/collection of RVs.
discrete $X_t = \{0, 1, 2, \dots\}$ (State space)

$\{X(t), t \in T\} \Rightarrow \{X_n, n = 0, 1, 2, \dots\} \rightarrow$ discrete
 $\{X(t), t \geq 0\} \rightarrow$ continuous

Value Index \rightarrow Continuous
State Discrete

* customer served in every hour dis. disc. # temp. in every day cont. Discrete
* # of vehicles along time dis. cont

Markov chain:



~~markovian~~ markovian property: Every RV is dependant on its previous RV

$$P\{X_3=4 | X_2=3, X_1=7\} = \text{same}$$
$$P\{X_3=4 | X_2=3, X_1=4\} = \text{same}$$

* Discrete time, discrete state space

A SP is Markovian if

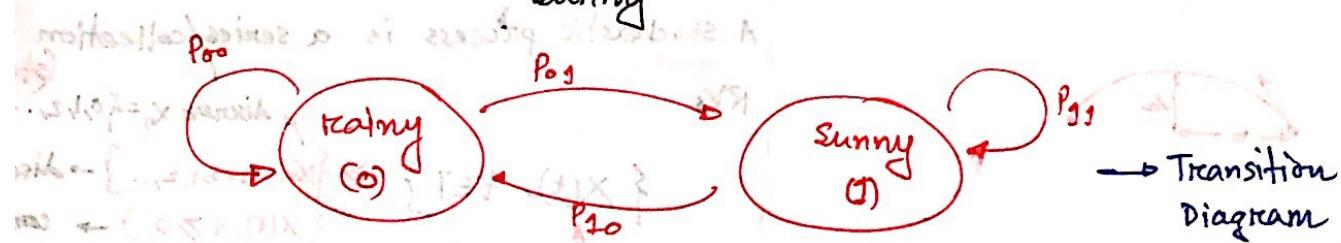
$$P\{X_{n+1}=j | X_n=i, X_{n-1}=i_{n-1}, X_{n-2}=i_{n-2}, \dots, X_0=i_0\}$$

$$= P\{X_{n+1}=j | X_n=i\} = p_{ij} \Rightarrow \text{transition Probability from } i \rightarrow j$$

"future depends on the present, not the past."

Weather Condition : rainy

sunny



$$P_{00} = P\{\text{tomorrow is rainy} | \text{today is rainy}\} = 0.7$$

$$P_{01} = P\{\text{sunny} | \text{rainy}\} = 0.3$$

$$P_{10} = P\{\text{rainy} | \text{sunny}\} = 0.2$$

$$P_{11} = P\{\text{sunny} | \text{sunny}\} = 0.8$$

Transition Matrix: $\bar{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$ label to 2x2 matrix

$$\bar{P} = \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.0 & 0.8 \end{bmatrix} \quad \begin{array}{l} 0 \text{ rainy} \\ 1 \text{ sunny} \\ 2 \text{ cloudy} \end{array}$$

row 1: P{rainy | rainy}, P{rainy | sunny}, P{rainy | cloudy}

row 2: P{sunny | rainy}, P{sunny | sunny}, P{sunny | cloudy}

row 3: P{cloudy | rainy}, P{cloudy | sunny}, P{cloudy | cloudy}

Two properties:

$$\textcircled{1} \quad P_{ij} \geq 0, \forall i, j \quad \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} = 0$$

$$\textcircled{2} \quad \sum_j P_{ij} = 1 \quad \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} = 1$$

Markov chain: ① State space $\{0, 1, 2\}$ q12-2

Ex: $\{0, 1, 2\}$ q12-2 ② State transition probability $P_{ij}, \forall i, j$ presented by \bar{P}

$$\begin{aligned} \text{Ex: } & P_{00} = \frac{\text{no. of transitions from state 0 to 0}}{\text{no. of transitions from state 0}} = \\ & = \frac{0.2 \times 100}{100} = 0.2 \\ & P_{01} = \frac{\text{no. of transitions from state 0 to 1}}{\text{no. of transitions from state 0}} = \\ & = \frac{0.5 \times 100}{100} = 0.5 \\ & P_{02} = \frac{\text{no. of transitions from state 0 to 2}}{\text{no. of transitions from state 0}} = \\ & = \frac{0.3 \times 100}{100} = 0.3 \end{aligned}$$

$$P_{10} = \frac{0.1 \times 100}{100} = 0.1 \quad \text{daily rainy prob.}$$

$$P_{11} = \frac{0.6 \times 100}{100} = 0.6 \quad \text{daily sunny prob.}$$

$$P_{12} = \frac{0.3 \times 100}{100} = 0.3 \quad \text{daily cloudy prob.}$$

Markov chain is a stochastic process

discrete time

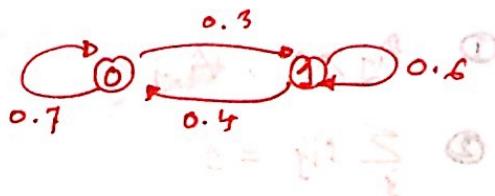
$$x_0, x_1, x_2, \dots, x_n, x_{n+1}, \dots$$

↓
discrete

- discrete state space, $S = \{0, 1\}$ e.g. rainy sunny

- transitional probability, $P_{ij} = P\{x_{n+1}=j | x_n=i\}$

$$\bar{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$



Q. $P\{\text{Wed is sunny} | \text{Yues is rainy}\} = 0.3$

1-step prob.

Q. $P\{\text{Thursday is sunny} | \text{Wednesday is rainy}\} = ?$

$$= P\{x_2=1 | x_0=0\} \rightarrow \text{Condition on } x_1$$

$$= P\{x_2=1 | x_1=i, x_0=0\}, P\{x_1=i | x_0=0\} = \sum P_{0i} \times P_{i1}$$

$$= P_{01}^2 = P_{00} + P_{01} + P_{01} \times P_{11}$$

2-step trans. prob $\bar{P}^{(2)} = \bar{P} \cdot \bar{P} = (\bar{P})^2$

Gen. n-step trans. prob. $\bar{P}^{(n)} = \bar{P} \cdot \bar{P} \cdots \bar{P} = (\bar{P})^n$

Q. Given that Sunday is raining, what is the prob. that it will be raining 4 consecutive days?

Poo

P₀₀²

p_{oo}³

P⁴
oo

Digitized by srujanika@gmail.com

Chapman-Kolmogorov Equation

$$p_{ij}^{(n+m)} = \sum_{k=0}^{\infty} p_{ik}^n p_{kj}^m$$

$\bar{P}^m \cdot \bar{P}^n = \bar{P}^{m+n}$ is also true.

zur Abreisezeit gefüllt mit und verhindert zint: $\Sigma \rightarrow$

Ex. (Comm. System) x_n → $\boxed{\text{?}}$ → x_{n+1}

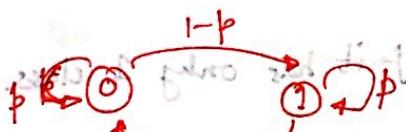
p = prob that bit remain unchanged

to a) exist of one foss : existent (iii 1)

This is a 2-step matkow chain

This is a 2-step matkow chain

1-4

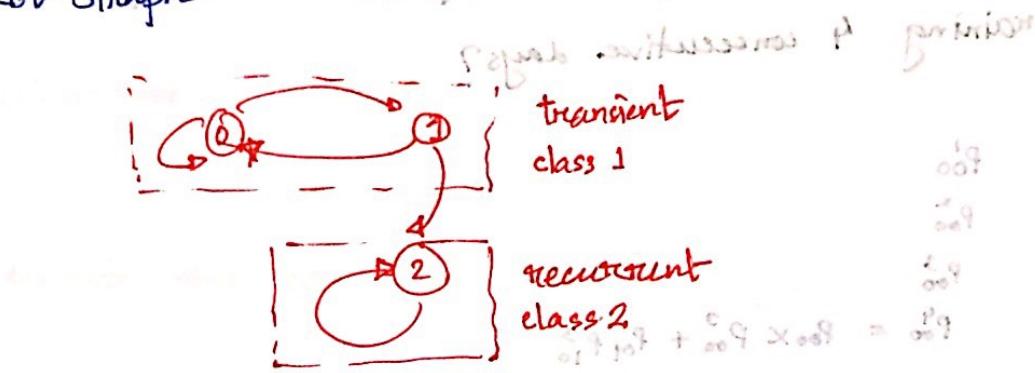


$$\bar{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

$\infty = \lim_{n \rightarrow \infty} \sum_{i=1}^n f_i$ from where it is evident that the sum is finite.

202539 2 G. Fischer

11.6. Markov Graph



state j is accessible from state i , ~~irreducible - reversible~~

$i \rightarrow j$ if $P_{ij}^n > 0$ for some n

state i communicable with j if $i \rightarrow j$ and $j \rightarrow i$: $i \leftrightarrow j$

$i \leftrightarrow j$: This relation has the following properties

- Property of a class
- i) Reflexive: $i \leftrightarrow i$. (natural moral)
 - ii) Symmetric: $i \leftrightarrow j \Rightarrow j \leftrightarrow i$
 - iii) Transitive: $i \leftrightarrow j$, $j \leftrightarrow k$ then $i \leftrightarrow k$

* Markov chain is irreducible if it has only 1 class.

* State i is recurrent if starting with i , the chain can re-enter i infinitely many times.

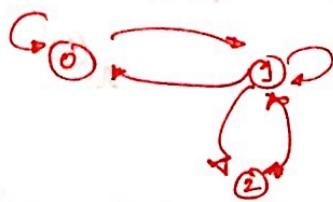
It can be proved, state i is recurrent if $\sum_{n=0}^{\infty} P_{ii}^n = \infty$

transient if $\sum_{n=0}^{\infty} P_{ii}^n < \infty$

In a finite state Markov chain all states cannot be transient.



↳ going nowhere means remaining in the state



irreducible
recurrent
finite state

"ergodic"

$P_{ij}^{(1)}, P_{ij}^{(2)}, P_{ij}^{(3)}, \dots, P_{ij}^{(n)}$, what happens when $n \rightarrow \infty$?

$P_{ij}^{(n)}$ converges and it does not depend on i .

Markov chain

$$\text{if } p \cdot x_j \cdot \pi + q \cdot x_i \cdot \pi = \pi$$

for an irreducible ergodic Markov chain, $x_j \cdot \pi = \pi$

$\lim_{n \rightarrow \infty} P_{ij}^{(n)}$ exists (after a very long run)

and it does not depend on i .

$$\begin{aligned} \lim_{n \rightarrow \infty} P_{ij}^{(n)} &= \lim_{n \rightarrow \infty} P\{X_n=j | X_0=i\} \\ &= \lim_{n \rightarrow \infty} P\{X_n=j\} = \pi_j \end{aligned}$$

π_j = limiting probability

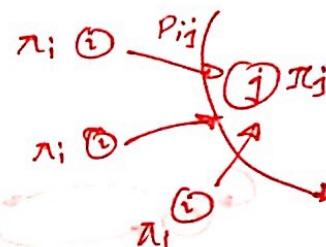
$$\frac{\pi_j}{\pi} = \pi_j$$

= long run proportion of time chain remains at state j
= fraction of time chain is in state j

Theorem: π_{ij} 's are computed from the following:

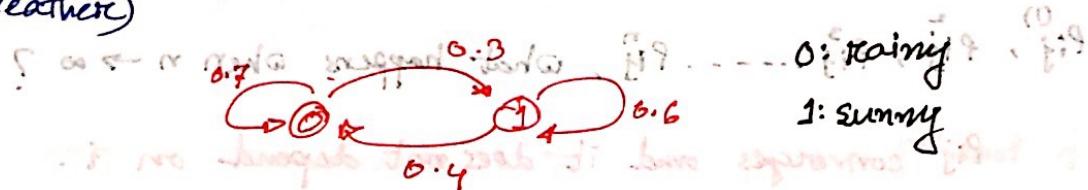
$$\text{state } i \text{ is dominant} \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \text{ for } j \in B$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$



States i & j are said to be in equilibrium if $\pi_i = \pi_j$

* (Weather)



Q. How many days are rainy?

Let's compute π_0 and π_1 .

$$\pi_0 = \pi_0 \times 0.7 + \pi_1 \times 0.4 \quad (1)$$

$$\pi_1 = \pi_0 \times 0.3 + \pi_1 \times 0.6 \quad (2)$$

$$\pi_0 + \pi_1 = 1$$

Solve for π_0 and π_1

$$(1) \Rightarrow 0.3\pi_0 = 0.4\pi_1 ; 3\pi_0 = 4\pi_1$$

$$\Rightarrow 3\pi_0 = 4 - 4\pi_0$$

$$\therefore \pi_0 = \frac{4}{7}$$

$$\pi_1 = \frac{3}{7}$$

probability of getting rain = $\frac{3}{7}$

* Class Mobility:



Upper, Middle, Lower

$$P = \begin{pmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{pmatrix}$$

Q. What fraction of people are in middle class?

$$\pi_0 = \pi_0 \times 0.45 + \pi_1 \times 0.05 + \pi_2 \times 0.01$$

$$\pi_1 = \pi_0 \times 0.48 + \pi_1 \times 0.70 + \pi_2 \times 0.50 = 0.9$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad \frac{1}{\pi} = \frac{1}{1-k} = \frac{1}{0.9} = 1.11$$

$$\text{Solve: } \boxed{\pi_1 = 0.62}$$

$$\frac{1}{1-k} - \pi_0 = 0.07$$

$$\frac{1}{1-k} - \pi_2 = 0.31$$

$$\frac{1}{1-k} = \frac{1}{0.9} = 1.11$$

$$\pi_0 = 0.07$$

$$0.07(1.11) + 0.62(1.11) = 0.7777$$

$$\frac{1}{1-k} + \frac{1}{1-k} + \frac{1}{1-k} =$$

To construct a Markov Chain: steps:

- identify the stochastic variable, X_n chain $\{X_n, n \geq 0\}$

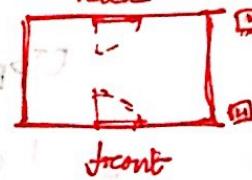
- Possible values of X_n , states

- Compute state transition prob.

[Ex. 33] → for k pair of shoes

Example:

Q. How many days he goes out barefooted?



$X_n = \# \text{ shoe pair in the front door at day } n.$

states = {0, 1}

Since number of shoe always to even then 0, 2

{ $X_n, n \geq 0$ }

$$10.0x_{n+1} + 20.0x_{n+1} + 30.0x_{n+1} = 0.5$$

$$P_{00} = \frac{3}{4}x_{n+1} + 0.5x_{n+1} + 20.0x_{n+1} = 0.5$$

$$P_{01} = P\{RF\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \Rightarrow \bar{p} = \left| \begin{array}{c} \frac{3}{4} \\ \frac{1}{4} \end{array} \right|$$

$$P_{10} = P\{FR\} = \frac{1}{4}$$

$$P_{11} = \frac{3}{4}$$

$$\left| \begin{array}{c} 0 = p \\ \frac{1}{2} = n = \frac{1}{2} \end{array} \right.$$

$$\text{odd number } P_0 \text{ and } P_1, 10.0 = \pi_1 - \frac{1}{2}$$

$$P\{\text{barefoot}\} = P(F) \cdot \pi_0 + P(R) \cdot \pi_1$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{2}$$

with wind will reduce probability of getting out

but for the end result will go to earlier situation

dry different state occurs

P₀₁ = 1

P₁₀ = 0

$$(Exam) \quad P_1 = 0.3 \quad P_2 = 0.6 \quad P_3 = 0.9$$

$$Q_1 \quad Q_2 \quad Q_3$$

reject value M

if student does well, then next exam can be any of type

if student does bad, next exam is type 1.

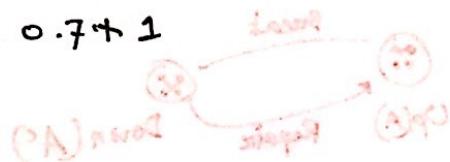
X_n = type of exam of n -th exam

= states $\{1, 2, 3\}$

ps. p ignored

$$P_{11} = P\{\text{next exam is 1} | \text{current exam 1}\}$$

$$= 0.3 \times 0.33 + 0.7 \times 1$$



$$P_{12} = 0.3 \times 0.33$$

Explanation of the system is to start
in state A and move to state B
with probability 0.33.
From state B, move back to state A
with probability 0.3.

Ex. 25, 30, 32, 33, 46, 28, 23

$\{X_n : n \geq 0\}$ is a markov chain of state

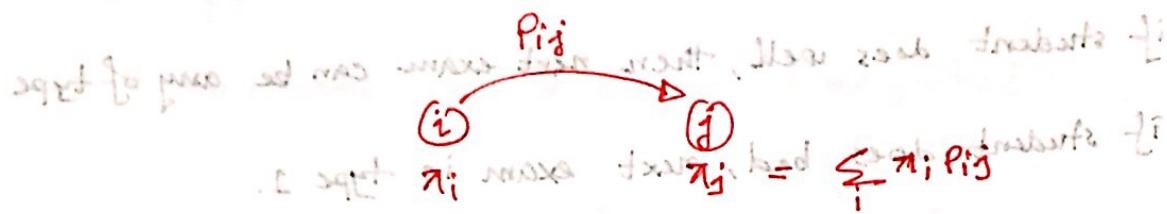
$\{X_n : n \geq 0\}$ is a markov chain of state

* Define markov chain

$\{X_n : n \geq 0\}$ is a markov chain $\left| \begin{array}{l} \text{continuous} \\ \{X(t) : t \geq 0\} \text{ is a markov chain} \\ \text{A process of state} \\ \text{continuous to other} \end{array} \right.$

define X and n

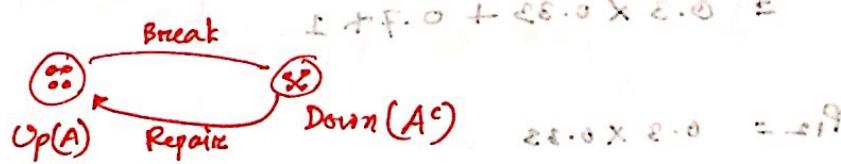
Markov Chain



$\pi_i p_{ij}$ = the rate at which chain transits from i to j

Example 4.24

A production process works in 2 type of states



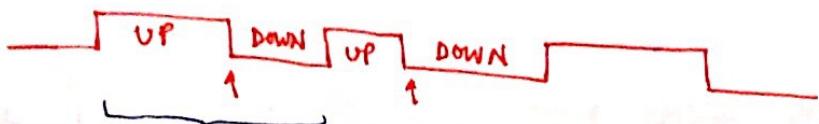
- Q. The rate of breakdowns
- Q. The average length, the process is in UP state.

Ans.

Rate to enter j from i = $\pi_i p_{ij}$

Rate to enter A^c from i = $\sum_{j \in A^c} \pi_i p_{ij}$

Rate to enter A^c from A = $\sum_{i \in A} \sum_{j \in A^c} \pi_i p_{ij}$ — (A)
 ↑
 rate of breakdown



$$\text{Period} = \text{cycle length} = \overline{U} + \overline{D}$$

1 break per $\overline{U} + \overline{D}$ time

$$\text{rate of breakdown} = \frac{1}{\overline{U} + \overline{D}} \quad \text{--- (B)}$$

$$f_{\text{break}} = 90$$

Equation (A) and (B) \Rightarrow

$$\frac{1}{\overline{U} + \overline{D}} = \sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij} \quad \text{--- (I)}$$

π_i = fraction of time in state i

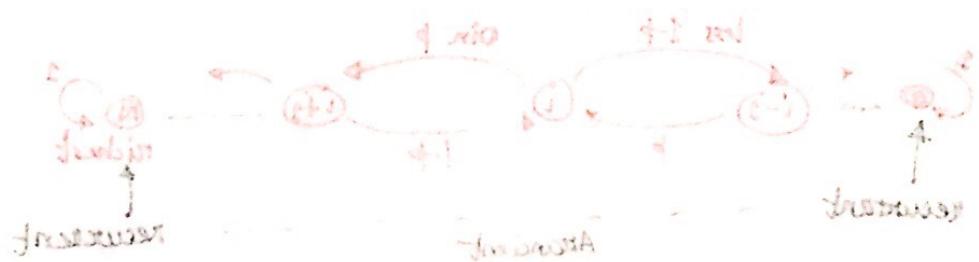
Proportion of the system is in UP state,

$$\frac{\overline{U}}{\overline{U} + \overline{D}} = \sum_{i \in A} \pi_i \quad \text{--- (II)}$$

initial condition

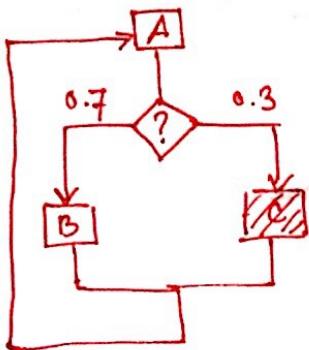
Solve (I), (II) for \overline{U} and \overline{D} starting with $\pi_0 = 1$

initial values of π_i & P_{ij}



1 month in each state required with both λ & μ = 19 days

$$f_{\text{break}} = ?$$



Q. What is the average length of time, the program produces correct output?

$$UP = \{A, B\}$$

$$DOWN = \{C\}$$

$$\text{Fraction of wrong output} = \pi_C$$

$$(1) \rightarrow \frac{\pi_C}{\pi_A + \pi_C} = \frac{1}{2 + 1}$$

$$\text{The rate of producing error output} = \pi_A P_{AC}$$

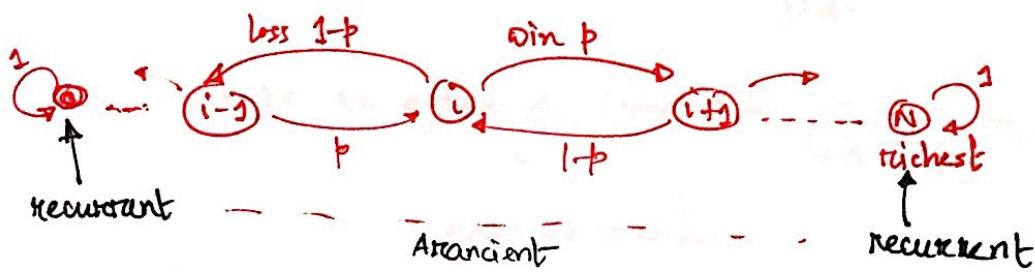
Ex. 41, 42

starts at 0 and moves with probability p to state 1 and with probability $1-p$ to state -1.

Gambler's Ruin

X_n = money the gambler has at n th move

$\{X_n : n \geq 0\}$ is a Markov chain.



Compute P_i = Prob. that the Gambler reaches N from i

$$= P\{i \rightarrow N\}$$

Condition on the next move, either win or loss.

$$= P\{i+1 \rightarrow N\} \cdot P(\text{win}) + P\{i+1 \rightarrow L\} \cdot P(\text{loss})$$

$$\therefore p_i = p_{i+1} p + p_{i-1} q \quad i, q = 1-p$$

$$\Rightarrow (p+q)p_i = p_{i+1} p + p_{i-1} q$$

\therefore The answer should be in terms of p .

$$\Rightarrow p_{i+1} - p_i = \frac{q}{p} (p_i - p_{i-1})$$

$$p_0 = 0, p_N = 1$$

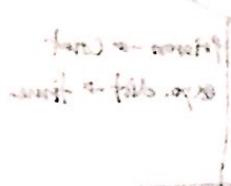
$$p_i = \begin{cases} \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

$\therefore p = \frac{1}{2}$ to start with condition
to prove

Number of steps till absorption $\sim (3)$ are not

Random Walk is a MC with infinite states.

i.e. state n is transient with time ∞



is quite undesirable A



returning to origin : 0 : state

transient not absorbing : L

transient in absorbing : S

absorbing states : K

(K) transient $\sim X$

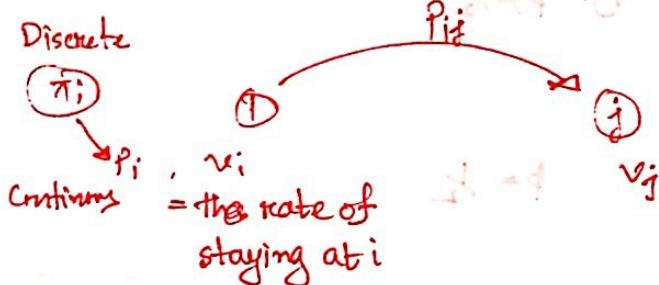
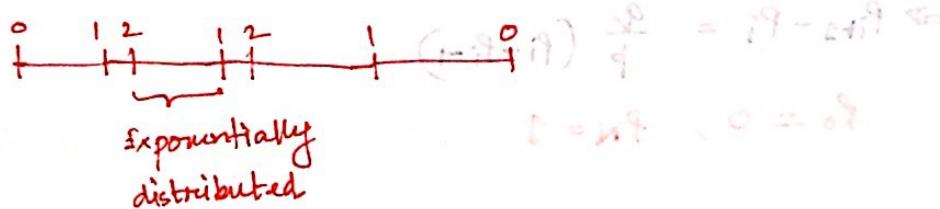
Discrete time : $\{X_n : n \geq 0\}$

continuous time : $\{X_t : t \geq 0\}$

$$\{X_t : t \geq 0\}$$

$$P_{0,1} + P_{1,2} = 0.6 + 0.4 = 1.0$$

$X(t)$ = # customers in a bank queue at t .

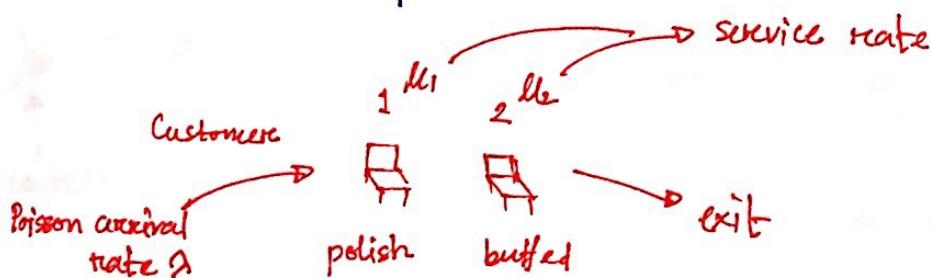


$$P_{0,1} = e^{-\lambda} = 0.6$$

$$\lambda = v_0 = 1.0$$

$T_i \sim \exp(v_i) \rightarrow$ exponentially distributed with rate v_i ;
 T_i = the amount of time the process is in state i

A Shoeshine Shop:

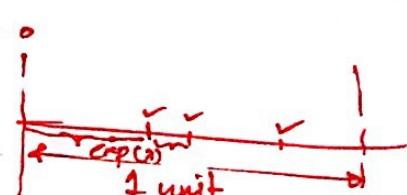


process \rightarrow count
 expo. dist \rightarrow time

state : 0 : empty, no customer

1 : customer is in chair 1

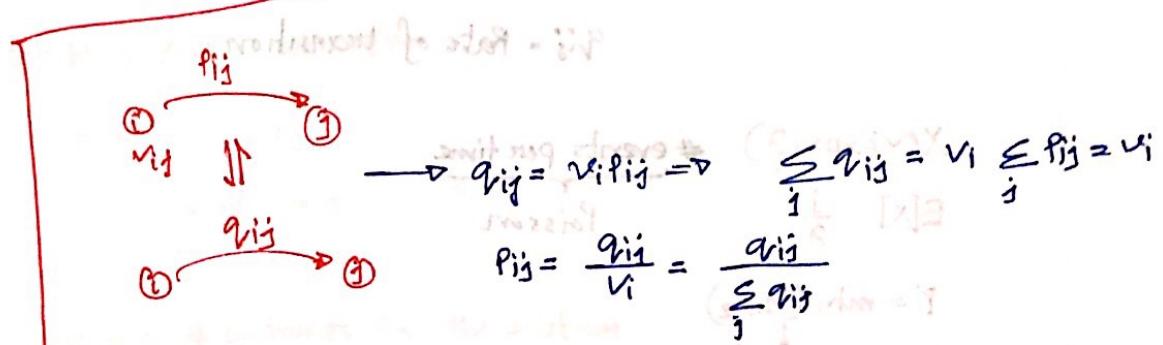
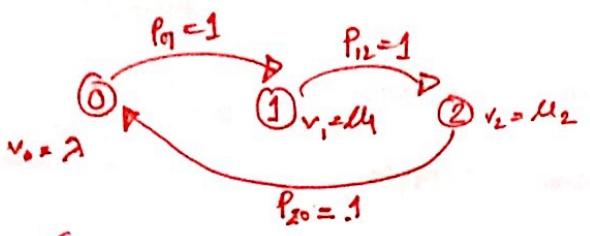
2 : customer is in chair 2



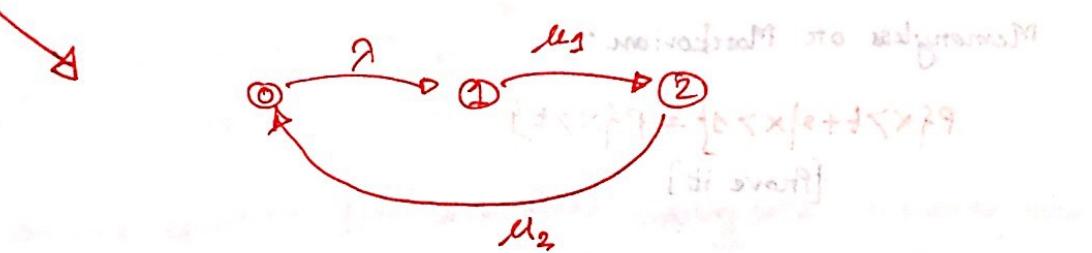
$\lambda = \# \text{ events per unit time}$

$X \sim \text{Poisson}(\lambda)$

Scanned by CamScanner



q_{ij} - the rate at which the process transits from i to j



returning to initial state

from unit to idle waiting = $(1) \times$



(other busy) $\lambda = \lambda$: normal waiting
(other erroneous) $\lambda = \lambda$

$$\lambda = \lambda$$

$R = \mu \lambda$: normal
 $\lambda = \mu \lambda$: failure

$R = \mu \lambda$: repair standard
 $\lambda = \mu \lambda$: repair slow

Chap - 6 | First section
 Chap - 8 | 8.1, 8.2

q_{ij} - Rate of transition

$X \sim \text{Expo}(\lambda)$ # events per time

$$E[X] = \frac{1}{\lambda} \quad \text{Poisson}$$

$$Y = \min(X_1, X_2)$$

\downarrow If i is the first event occurring w.r.t. which the other one = j

$$Y = \text{expo}(\lambda_1 + \lambda_2)$$

[Prove it!]

$$f_Y(x) = (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)x}$$

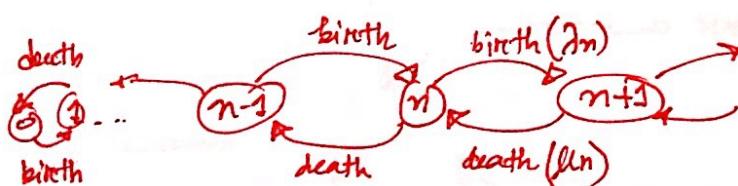
Memoryless or Markovian:

$$P\{X > t+s | X > s\} = P\{X > t\}$$

[Prove it]

Birth and Death Process

$X(t)$ = Population size at time t



Poisson:
arrival

$$\lambda_n = \lambda$$

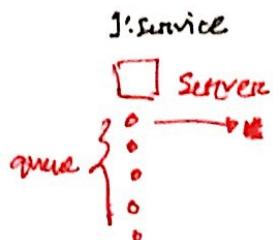
$$\mu_n = 0$$

M/M/1 Queue: $\lambda = \lambda$ (arrival rate)
 $\mu_n = \mu$ (service rate)

Pure birth Process:
Yule Process

$$\lambda_n = n\lambda$$

$$\mu_n = 0$$



arrival is memoryless exponential

M/M/1 Queue; [if M/M/S → for s no. of service]

M/M/1 can be modeled as CTMC.

$$\{x(t) : t \geq 0\}$$

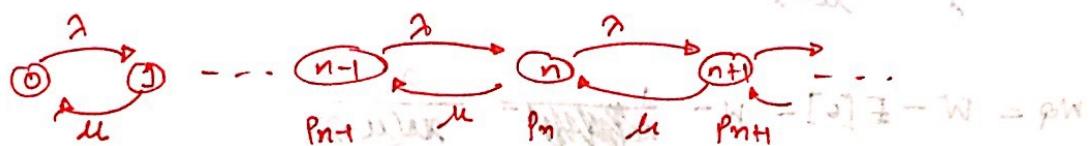
$x(t)$ = # of customers in the system.

L = average # customers in the system

L_q = average # customers in the queue

w = Average wait time in the system

w_q = average wait time in the queue



Balance equations. for each state, leaving rate = entering rate

$$'0' \rightarrow P_0 \lambda = P_1 \mu$$

$$n \rightarrow P_n (\lambda + \mu) = P_{n+1} \mu + P_{n-1} \lambda$$

$$\Rightarrow P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1} \right)$$

$$P_0 = ?$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_m = \left(\frac{\lambda}{\mu}\right)^m P_0$$

We have, $\sum_{n=0}^{\infty} p_n = 1$
 condition of transition

$$\Rightarrow p_0 + \frac{\lambda}{\mu} p_0 + \left(\frac{\lambda}{\mu}\right)^2 p_0 + \dots = 1$$

$$\Rightarrow p_0 \cdot \frac{1}{1 - \frac{\lambda}{\mu}} = 1 \quad \text{if } \frac{\lambda}{\mu} < 1 \Rightarrow \mu > \lambda$$

$$\Rightarrow p_0 = 1 - \frac{\lambda}{\mu}$$

Cost of $G(X)$

Waiting cost or queueing cost $\Rightarrow W = \lambda X$

$$L = \sum_{n=0}^{\infty} n p_n$$

Average waiting cost or queueing cost $= L$

$$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)^{m-n} \sum_{n=0}^{\infty} n x^n = \frac{\lambda x}{(1-x)^2}$$

$$= \frac{\lambda}{\mu - \lambda} \quad \text{Average waiting cost or queueing cost} = W$$

Suppose with infinite arrival rate $\lambda = 1 \Rightarrow W = \rho W$

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$$

$$WQ = W - E[\epsilon] = W - \frac{1}{\mu - \lambda} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$LQ = \lambda \cdot WQ = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$1 - P_0 = \frac{\lambda}{\mu} = \text{utilization}$$



$$? = \alpha$$

$$\alpha = \frac{8}{10} = 0.8$$

$$0.8 \times \left(\frac{1}{9}\right) = 0.8 \times 0.11 = 0.088$$

$$0.8 \times \left(\frac{1}{9}\right) = 0.088$$