CSE 301: Mathematical Analysis for CS

Concrete Math

Probability Models

→ Text: Introduction to Probability Models

— Ross (10th edition) / 9th edition

Probability is a $\frac{\text{value}}{\text{real}}$ assigned to (uncertain) events.

is a measure of uncertainty

occurrence of something

Ex Coin toss {head, tail}

→ sample space, S

$P\{head\} = \frac{1}{2}$

$P\{tail\} = \frac{1}{2}$

Two Properties:

1. $0 \leq P\{events\} \leq 1$

2. $P(S) = 1$

$P(E \cup F) = P(E) + P(F)$ provided that E and F are mutually exclusive.

General rule

$P(EF)$ = Probability that E and F both happen

$P(E \cup F) = P(E) + P(F) - P(EF)$

P{Brazil wins the WC} = $\frac{1}{4}$

P{Germany wins the WC} = $\frac{1}{4}$

P{Netherland wins the WC} = $\frac{1}{4}$

P{Argentina wins the WC} = $\frac{1}{4}$

$\Sigma = 1$

P{B and G win WC} = 0

P{B or A win WC} = $P(B) + P(A) - P(AB)$

$P(EF) = P(E) \cdot P(F)$; provided that E and F are 'independent'

$P(E|F) = \frac{P(EF)}{P(F)}$

Conditional Probability

$P(E|F) = P(E)$

E doesn't depend on F

$P(E|F) \neq P(E)$

E and F depend

Dependent

P{G wins} = $\frac{1}{4}$

P{G wins | B is runners-up} = 0

P{A wins | B is runners-up} = $\frac{1}{2}$

P{A wins} = $\frac{1}{4}$

Probability:

Probability is assigned to uncertain experiments.

$\rightarrow$ Sample space

= all possible experiments

Event = $E \subseteq S$

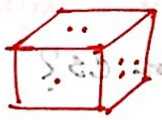
1. $0 \leq P(E) \leq 1$

2. $P(S) = 1$

3. If E_1 and E_2 are mutually exclusive, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Generally,

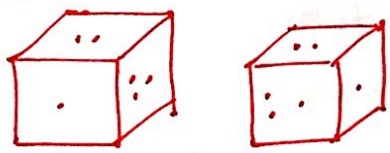
$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i) \text{ if all } E_i\text{'s are ME } [E_i \cap E_j = \emptyset,\ i \neq j]$$



$S = \{1, 2, \ldots, 6\}$

$P\{\text{Getting } 1\} = P(1) = \frac{1}{6}$

$P\{\text{Even}\} = \frac{3}{6} = \frac{1}{2} = P\{2, 4, 6\}$



Two dies

$S = \{(1,1), (1,2), \ldots, (6,6)\}$

#36

$P\{\text{both are odd}\} = \frac{3 \times 3}{36}$

(1,1)
(1,3)
(1,5)
⋮
(5,5)

$P(EF) = P(E \cap F)$

$= P(E) \cdot P(F)$ if E and F are independent.

$P\{\text{both are odd}\} = P\{\text{1st is odd}\} \times P\{\text{2nd is odd}\}$

$= \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

$P(EF) = \underbrace{P(E|F)}_{\text{conditional Probability}} \cdot P(F)$

$P(E|F) = \dfrac{P(EF)}{P(F)}$; if $P(F) > 0$

Q. Student

Economics

A grade $= \frac{1}{2}$

Computer Science

A grade $= \frac{1}{3}$

What is the probability that he gets A in CS?

Let,

A = he gets A grade

C = he chooses CS

E = he chooses Econ.

$P(A|E) = \frac{1}{2}$

$P(A|C) = \frac{1}{3}$

$P(AC) = P(A|C) \cdot P(C)$

$= \frac{1}{3} \cdot \frac{1}{2}$

$= \frac{1}{6}$

E

$EF = E \cap F$

$E = EF \cup EF^c$

EF and EF^c are M.E.

$P(E) = P(EF) + P(EF^c)$

$= P(E|F)\,P(F) + P(E|F^c).\,P(F^c)$

$= P(E|F).P(F) + P(E|F^c).(1-P(F))$

We want to compute:

$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E|F).P(F)}{P(E)}$$

$$= \frac{P(E|F).\,P(F)}{P(E|F).\,P(F) + P(E|F^c).\,P(F^c)}$$ → Bayes formula

Q. Student → K = student know the answer

C = student gives the correct answer

K^c = Student guesses

m → mcq options

$P(K) = p$

$P(C|K) = 1$

$P(C|K^c) = \frac{1}{m}$

What is the probability that, student knew the answer given that he gives the correct answer?

$$P(K|C) = \frac{P(CK)}{P(C)} = \frac{P(C|K).P(K)}{P(C|K)\cdot P(K) + P(C|K^c).P(K^c)}$$

$$= \frac{1.p}{1.p + \frac{1}{m}(1-p)}$$

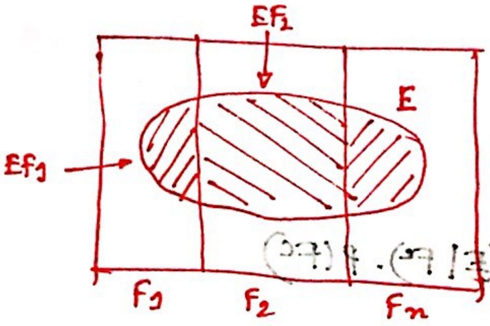
$$= \frac{p}{p + \frac{1}{m}(1-p)}$$

if $m=4$, $p=\frac{1}{2}$

$P(K|C) = \frac{4}{5} = 80\%$

Conditional Probability

$$P(E|F) = \frac{P(EF)}{P(F)}$$



$$E = EF_1 \cup EF_2 \cup \ldots \cup EF_n$$

$$= \bigcup_i EF_i$$

$$P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E|F_i)\,P(F_i)$$

Then, $P(K_j|E) = \dfrac{P(E|F_j).\,P(F_j)}{\sum_{i=1}^{n} P(E|F_i).\,P(F_i)}$ → General Bayes Rule

Ex. 19) Two dies are rolled.

Q1: P{at least one is six}

$= 1 - P\{\text{none are six}\}$

$= 1 - \frac{5}{6}.\frac{5}{6}$

$= 1 - \frac{25}{36}$

$= \frac{11}{36}$

Q2: $P\{\text{at least}^{S} \text{ one six} \mid \underbrace{\text{two faces are different}}_{30}{}^{D}\}$

$= \frac{10}{30}$

$= \frac{1}{3}$

1,6
2,6
⋮
5,6 | 5

$+ = 10$

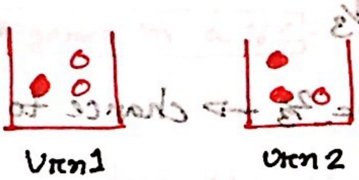
6,1
6,2 | 5
⋮
6,5

$P(S|D) = \frac{P(SD)}{P(D)}$

$= \frac{10/36}{30/36}$

$= \frac{1}{3}$

Ex. 36



Q. One urn is randomly selected and then a marble is picked.

$P\{\text{marble is black}\}$

$= P(B)$

$= P(B|U_1).P(U_1) + P(B|U_2).P(U_2)$

$= \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2}$

$= \frac{1}{2}$

Q. Say the marble is black. What is the probability that the marble was chosen from urn 1?

$P(U_1|B) = \frac{P(B|U_1).P(U_1)}{P(B|U_1).P(U_1) + P(B|U_2).P(U_2)}$

$= \frac{1/3 \cdot 1/2}{1/2}$

$= 1/3$

Ex. 46

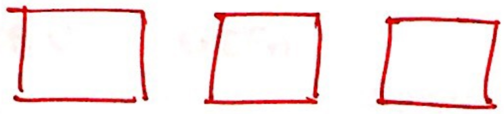
 $\rightarrow$ one will be randomly hanged

A B C

$P(A) = 1/3$

$P(A|\ldots) = 1/2$? $\rightarrow$ see book

Q. Monty Hall Problem

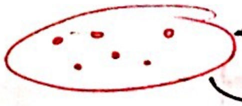


P{choosing correct door} = $1/3$

P{choosing the wrong door} = $2/3$ $\rightarrow$ chance to win if switched

(Check internet)

Random Variable


← Sample Space

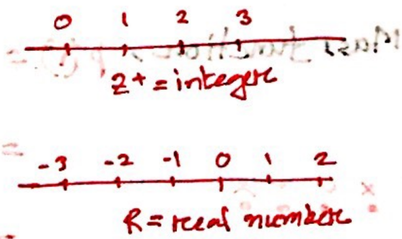
↳ Outcome/event

Coin toss . $S = \{\text{head}, \text{tail}\}$ (head = 1, tail = 0)

coin toss , $S = \{0, 1\}$

possible outcome: just numbers

Event, X ←



Example:

X = # students attending a class ← discrete

X = temperature in a day ← continuous

X = # goals in a WC ← discrete

$\boxed{X = \text{\# games in a WC}}$ → not random

Discrete Random Variables

Standard distributions:

1. Bernoulli (p)

X = outcome of a binary trial → success / Failure (1) (0)

$X = \begin{cases} 1, & \text{if success} \\ 0, & \text{else} \end{cases}$

Probability mass function, $p(x) = P\{X = x\}$ → $[\,p(k) = P(X = k)\,]$

$p(1) = P\{X=1\} = p$; p = success probability

$p(0) = P\{X=0\} = 1-p$; $q = 1-p$ = failure probability

* Brazil looses with probability 0.2. What is the probability, it wins in the next game.

$p(0) = 0.2 = P\{X=0\}$

$P\{X=1\} = p(1) = 1 - p(0) = 0.8$

2. Binomial (n, p)

$X = \#$ of success in 'n' trial

success prob. $= p$

$X \sim Bin(n, p)$

Mass function, $p(i) = P\{X = i\}$

$= P\{i$ success out of 'n' trials$\}$

$= P\{i$ success and $(n-i)$ failure$\}$

independent

$= \binom{n}{i} p^i (1-p)^{n-i}$

We can verify, $\sum_{i=0}^{n} p(i) = 1$

Brazil plays 4 games. What is the prob. that it wins in 3?

$P\{X=3\} = \binom{4}{3} p^3 (1-p)^1$

$= \binom{4}{3} (0.8)^3 (0.2)^1$

3. Geometric Random variable (p)

$X = \#$ of trials to get the first success ; $[1, \infty)$

$\leftarrow$ ith

Mass function, $P\{X=i\} = p_X(i)$

$= P\{$ ith trial is success$\}$

$= P\{(i-1)$ failure and ith trial success$\}$

$= (1-p)^{i-1} . p$

$\Rightarrow p(i) = (1-p)^{i-1} . p$

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We can verify, $\sum_{i=1}^{\infty} p(i) = 1$

Ex. What is the prob. Brazil wins in its 4th game?

$$p(4) = (0.2)^3 \cdot (0.8)$$

Random Variable

- Discrete, Probability mass function, $p_X(i)$
- Continuous, Probability density function, $f_X(x)$

Discrete:

1. Bernoulli (p)
2. Binomial (n, p)
3. Geometric (p)

$$X \sim geom(p) \Rightarrow p_X(i) = p(1-p)^{i-1}$$

Verify:

$\sum_{i=1}^{\infty} p_X(i) = 1$ for Geometric, X

$$\sum_{i=1}^{\infty} p(i) = \sum_{i=1}^{\infty} p(1-p)^{i-1} = \sum_{i=1}^{\infty} p q^{i-1} = p \sum_{i=1}^{\infty} q^{i-1}$$

$$= p(1+q+q^2+q^3+\cdots)$$

$$= p \cdot \frac{1}{1-q}$$

4. Poisson (λ) , λ= rate $\quad X[0,\infty)$

↳ counting, (rare events)

mass function, $p(i) = P\{X=i\} = \frac{e^{-\lambda}\lambda^i}{i!}$

verify: $\sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} \cdot e^{\lambda} = 1$

Ex. Tpyographical mistakes per page is Poisson distribution with rate $\lambda=1$. What is the probability that you see at least one mistake in a certain page?

X = # mistakes

$X \sim$ Poisson (λ) , $\lambda = 1$

$$P\{X \geq 1\} = 1 - P\{X=0\}$$
$$= 1 - e^{-\lambda} \cdot \frac{\lambda^i}{i!}\Big|_{i=0}$$
$$= 1 - e^{-\lambda}$$
$$= 1 - e^{-1}$$

Poisson is a version of Bionomial (n,p)

$$\begin{matrix} n \to \infty \\ p \to 0 \end{matrix} \longrightarrow \text{Poi}(\lambda)$$
$$\lambda = np$$

for bionomial, $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$

$$= \frac{n!}{i!(n-i)!} \cdot \frac{\lambda^i}{n^i} \left(1 - \frac{\lambda}{n}\right)^{n-i} \quad ; np = \lambda \Rightarrow p = \frac{\lambda}{n}$$

$$= \frac{n(n-1)(n-2)\ldots(n-i+1)}{i!} \cdot \frac{\lambda^i}{n^i} \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)(n-2)\ldots(n-i+1)}{n^i} \cdot \frac{\lambda^i}{i!} \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$\lim_{n\to\infty} p(i) = \underbrace{\frac{n(n-1)(n-2)\ldots(n-i+1)}{n^i}}_{1} \cdot \frac{\lambda^i}{i!} \underbrace{\left(1-\frac{\lambda}{n}\right)^{n-i}}_{} \qquad i \ll n$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\Rightarrow \lim_{n\to\infty} p(i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Random Variable

Discrete (integer) vs. Continuous (real)

X, PMF

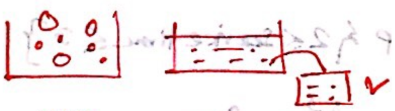
Prob. mass function

$p(i) = P\{X=i\}$

$X \rightarrow P\{10 \le X \le 15\}$

$P\{X=i\}=0$

no mass



mass

density

Continuous RV:

– Prob. density function, $f_X(x)$

$$P\{a \le X \le b\} = \int_a^b f_X(x)\,dx$$

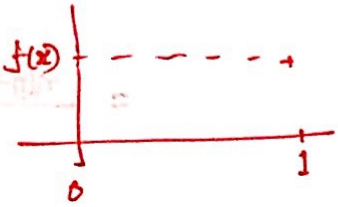
$$P\{-\infty \le X \le \infty\} = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

– Prob. distribution function, $F_X(x) = P\{X \le x\}$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

1. Uniform $U(0,1)$

Density, $f_X(x) = 1$



Distribution, $F_X(x) = P\{X \le x\}$

$$= \int_0^x 1 \cdot dx$$

$$= x \Big|_0^x$$

$$= x$$

2. Exponential, $\exp(\lambda)$ λ = rate

Density, $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

Distribution, $F_X(x) = \int_0^x f_X(x)dx = \int_0^x \lambda e^{-\lambda x} dx$

$$= 1 - e^{-\lambda x}$$

Example: time, waiting time, serving time, $X \sim Exp(0.1)$

$P\{2 \leq \text{service time} \leq 3\}$

$$= \int_2^3 \lambda e^{-\lambda x} dx$$

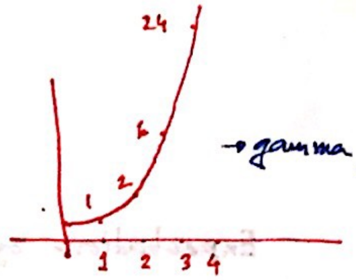
$$= \int_2^3 0.1 e^{-0.1x} dx$$

* $P\{2 \leq X \leq 3\}$

$= P\{X \leq 3\} - P\{X \leq 2\}$

$= F(3) - F(2)$

$= (1 - e^{-\lambda \cdot 3}) - (1 - e^{-\lambda \cdot 2})$



→ gamma function

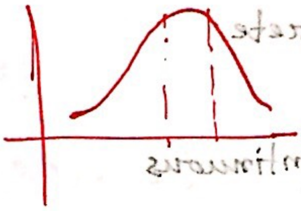
3. Gamma (α, λ)

Density, $f_X(x) = \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}$ $\quad \Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\infty} dx$

4. Normal distribution, $N(\mu, \sigma^2)$ variance

mean

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < X < +\infty$$



Density Curve

C.T.

4th week
7th week } Sunday

Expectation of random variable

$X \rightarrow$ random variable
$\hookrightarrow$ has probability distribution

discrete mass function, $p(x) = P\{X = x\}$

continuous density function $f(x)$

$E[X] = \sum_x x\,p(x) \rightarrow$ discrete

μ_X, mean, average

$= \int_{-\infty}^{\infty} x f(x)\,dx \rightarrow$ continuous

Bernoulli: $E[X] = 1 \cdot p(1) + 0 \cdot p(0) = 1 \cdot p = p$

Binomial:

$X = \#$ of success in n trials

$$E[X] = \sum_{i=0}^{n} i\,p(i)$$

$$= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} i \frac{n!}{i!\,(n-i)!} p^i (1-p)^{n-i}$$

$$= np \sum_{i=\cancel{0}\,1}^{n} \frac{(n-1)!}{(i-1)!\,(n-i)!} p^{i-1} (1-p)^{n-i}$$

$$= np \times (p + 1 - p)^n$$

$$= np$$

* $E[X]$ is not random, is constant

For 10 fair coins, $E[X] = 10 \times \frac{1}{2} = 5$

Geometric:

X = #trials to get the first success

$$E[X] = \sum_{i=1}^{\infty} i\,p(i)$$

$$= \sum_{i=1}^{\infty} i\,(1-p)^{i-1}\cdot p$$

$$= \frac{1}{p}$$ [check in book]

continuous:

Uniform(0, 1): $E[X] = \int_0^1 x f(x)\,dx = \int_0^1 x\,dx$, $f(x) = 1$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Exponential: $f(x) = \lambda e^{-\lambda x}$, $x > 0$, λ = rate

$$E[X] = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$= \ldots\ldots$$

$$= \frac{1}{\lambda}$$

operator

$$E[X^2] = \sum x^2 p(x)$$

$$E[aX+b] = a\,E[X]+b$$

$$E[c] = c \text{ ; constant}$$

$$E[g(x)] = \sum g(x)\cdot p(x)$$

$$E[x^3] = \sum x^3 p(x) \quad \times \sum x^3 p(x^3) \quad \times \sum x^3 (p(x))^3$$

Linearity of expectation:

$$E[aX + bY] = aE[X] + bE[Y]$$

$$E[X+Y] = E[X] + E[Y]$$

Binomial: $\overset{0}{\circ}\ \overset{1}{\circ}\ \overset{1}{\circ}\ \circ\ \circ \cdots\cdots \overset{0}{\circ}$: n trials

X = # of success

Let, X_i = outcome of ith trial

$$= \begin{cases} 1, & \text{success} \\ 0, & \text{fail} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

Now, $\underset{\text{Bionomial}}{E[X]} = \underset{\text{Bernoulli}}{E[X_1]} + E[X_2] + \cdots + E[X_n]$

$$= p + p + p + \cdots + p$$

$$= np$$

Hat Problem:  : N

X = # person who get their hats

$E[X] = ?$

let, $X_i = \begin{cases} 1 & \text{; if } i\text{th person gets his hat} \\ 0 & \text{; else} \end{cases}$

Now, $X = X_1 + X_2 + \dots + X_n = \sum_i X_i$

$$E[X] = E\left[\sum_i X_i\right] = \sum_{i=1}^{n} E[X_i]$$

$$\underbrace{E[X_i]}_{= 1 \cdot \frac{1}{N} + 0 \cdot \left(1 - \frac{1}{N}\right) = \frac{1}{N}}$$

$$= \frac{1}{N} + \frac{1}{N} + \dots + \frac{1}{N}$$

$$= \frac{N}{N} = 1$$

Variance:

$$Var(X) = E\left[(X - E[X])^2\right]$$

degree of variation

$$= E\left[(X - \mu)^2\right]$$

$$Var(X) = E[X]^2 - (E[X])^2$$

Co-variance: $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$

Properties:

1. $Cov(X,X) = Var(X)$

Proof

2. $cov(aX,Y) = a\,cov(X,Y)$

3. $cov(X, Y+Z) = cov(X,Y) + cov(X,Z)$

4. $cov(X,Y) = cov(Y,X)$

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Expectation and Variance

Expectation, $E[X] = \sum x p(x)$, discrete

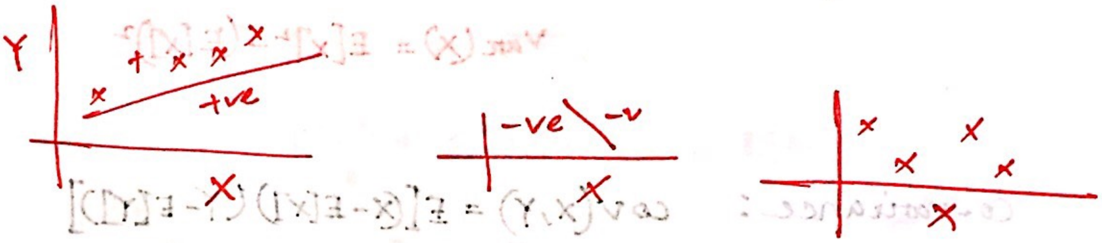
$$= \int_{-\infty}^{\infty} x f(x)\,dx \text{, continuous}$$

Variance, $Var(X) = E[(X - E[X])^2]$

Degree of variation, $= E[X^2] - E[X]^2$ – Prove it

Covariance, $Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$

$$= E[XY] - E[X].E[Y]$$ reduction



$Cov(X,Y) = 0$, if X and Y are independent

$\Rightarrow E[XY] = E[X].E[Y]$

linearity

$E[X+Y] = E[X] + E[Y]$

$Var(X+Y) = Var(X) + Var(Y)$; if they are independent

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Generally, $Var\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(x_i, x_j)$

Proof : $Cov(X,X) = Var(X)$

$$Var(\Sigma X_i) = Cov\left(\sum_{i=1}^{n} X_i , \sum_{j=1}^{n} X_j\right)$$

$$= Cov(X_1 + X_2 + \dots + X_n , X_1 + X_2 + \dots + X_n)$$

$E[X+Y]$

$Var[X+Y]$

$X \sim P_X(x)$

$Y \sim P_Y(y$

$X+Y \nsim \underline{P_X(x) + P_Y(y)}$

↳ joint distribution

Let, $X_1, X_2, \dots, X_n \sim$ i.i.d with mean μ, variance σ^2

identically independently distributed

Let us define,

Sample mean, $\bar{X} = \dfrac{X_1 + X_2 + \dots + X_n}{n}$

$E[X_i] = \mu$

$Var(X_i) = \sigma^2$

$* \; Var(cX) = c^2 Var(X)$

1. $E[\bar{x}] = ?$

$= E\left[\frac{1}{n}(x_1 + x_2 + \dots + x_n)\right]$

$= \frac{1}{n} E[X_i]$

$= \frac{1}{n} \cdot (n\mu)$

$= \mu$

2. $Var(\bar{x}) = Var\left(\frac{1}{n} \sum x_i\right)$

$= \frac{1}{n^2} \sum Var(x_i)$, all independent

$= \frac{1}{n^2}(n\sigma^2)$

$= \frac{\sigma^2}{n}$

$Var(\bar{x}) \sim 0$ if $n \to \infty$

Moment Generating Function, MGF

$E[X]$	$E[X^2]$	$E[X^3]$
1st	2nd	3rd

MGF of X, $\phi_X(t) = \underline{E[e^{tX}]}$

$\hookrightarrow$ function of t (not x)

$\phi_X(t) = E[e^{tX}]$

$= \sum e^{tx} p(x)$, discrete

$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$, continuous

Bernoulli: $\phi(t) = e^{t \cdot 1} p + e^{t \cdot 0} (1-p)$

$= e^t p + 1 - p$

Exponential: $\phi(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$

$= \int_0^{\infty} \lambda e^{-(\lambda - t)x} \dots = \frac{\lambda}{\lambda - t}$

Binomial : $X = X_1 + X_2 + \dots + X_n$

$\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$

$\Downarrow$

$E[e^{t(X+Y)}]$

$= E[e^{tX} \cdot e^{tY}]$

$= E[e^{tX}] \cdot E[e^{tY}]$

$= \phi_X(t) \cdot \phi_Y(t)$

$\phi_X(t) = (pe^t + 1 - p)^n$

$\phi(t) = E[e^{tX}]$

$\phi'(t) = E\left[\frac{d}{dt} e^{tX}\right] = E[Xe^{tX}]$, $\phi'(0) = E[X]$

$\phi''(t) = E[X^2 e^{tX}]$ $\qquad \phi''(0) = E[X^2]$

$\phi'''(t) = E[X^3 e^{tX}]$ $\qquad \phi'''(0) = E[X^3]$

3rd moment : $E[X^3] = \int_{-\infty}^{\infty} x^3 f(x)\, dx$

$= \int_{-\infty}^{\infty} x^3 \lambda e^{-\lambda x}\, dx \longrightarrow$ harder to calculate

Exponential Distribution

$f(x) = \lambda e^{-\lambda x}, \; x > 0 \longrightarrow$ density function

$\phi(t) = \frac{\lambda}{\lambda - t}, \; t < \lambda \longrightarrow$ MGF

Limit Theorems

X — $p(x)$
$f(x)$

$E[X]$

$Var[X]$

X = # waiting customer

$E[X] = 10$ $\quad P\{x \geq 15\} = ?$

Markov inequility

For a non-negative rv X, $P\{x \geq a\} \leq \frac{E[X]}{a}$

$$E[X] = \int_0^\infty x f(x)\,dx \geq \int_a^\infty x f(x)\,dx \geq \int_a^\infty a f(x)\,dx$$

$$= a\int_a^\infty f(x)\,dx$$

$$= a P\{X \geq a\}$$

So, $E[X] \geq a P\{X \geq a\}$

$$\Rightarrow P\{x \geq a\} \leq \frac{E[X]}{a}$$

So, $P\{x \geq 15\} \leq \frac{10}{15} = \frac{2}{3}$

$X \longrightarrow E[X] = \mu$

$Var(X) = \sigma^2$

Chebyshev's Inequality

For any random variable X, $P\{|x-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

$$\Rightarrow P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

Proof:

Let, $Y = (X-\mu)^2$

$E[Y] = E[(X-\mu)^2] = \sigma^2$

Due to Markov, Y is non -ve

$P\{Y \geq k^2\} \leq \frac{E[Y]}{k^2}$

$\Rightarrow P\{(x-\mu)^2 \geq k^2\} \leq \frac{E[Y]}{k^2}$

$\Rightarrow P\{|x-\mu| \geq k\} \leq \frac{E[Y]}{k^2}$

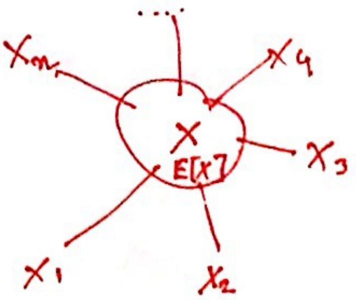
Q. $E[X] = 10$, $Var(X) = 3$

$P\{8 \leq X \leq 12\}$

$= P\{|X-10| \leq 2\}$

$= 1 - P\{|X-10| \geq 2\}$

So, $P(|X-10| \geq 2) \leq \frac{3}{4}$



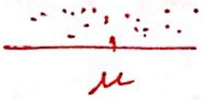
$X_1, X_2, \ldots, X_n$ have common distribution.

$E[X_i] = \mu$

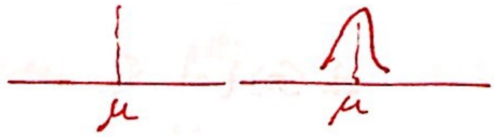
Sample mean; $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$

Strong law of number

if $n \to \infty$, $\bar{x} \to \mu$ with probability 1.



$\bar{x} \to \mu$



Central Limit Theorem

$X_1, X_2, \ldots\ldots, X_n$ i.i.d $\quad E[X_i] = \mu$

$\quad Var[X_i] = \sigma^2$

$S_n = X_1 + X_2 + \cdots + X_n \qquad \bar{X} = \frac{S_n}{n}$

$E[S_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = n\mu.$

$Var(S) = n\sigma^2$

What is the distribution of S_n?

CLT: If $n \to \infty$, then S_n takes a Normal distribution with

mean $= n\mu$

variance $= n\sigma^2$

$\boxed{}$ $\lim_{n\to\infty} S_n \sim N(n, \mu, n\sigma^2)$

Normal: Density function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(μ, σ^2)

Standard normal $(\mu = 0, \sigma^2 = 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} = e^{-\frac{x^2}{2}}$$

$\boxed{}$ $\lim_{n\to\infty} \frac{S_n}{n} = \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\lim_{n\to\infty} \frac{\bar{x} - N}{\sigma\sqrt{n}} \sim N(0,1)$

Conditional Expectation

Two events E and F (E is dependent on F)

conditional probability $P(E|F) = \frac{P(EF)}{P(F)}$

$X=a$ is a random event

$= P\{X=a\}$ is given by PMF

$P\{X=a | X=b\}$ is given by Conditional PMF

$P_X(x) = P\{X = x\}$

Conditional PMF $P_{X|Y}(x|y) = P\{X=x \mid Y=y\}$

$$= \frac{P\{X=x, Y=y\}}{P\{Y=y\}}$$

$$= \frac{P(x,y)}{P_Y(y)}$$ joint mass function

We have,

$$P_Y(y) = P\{Y=y\} = \sum_x P\{X=x, Y=y\}$$

$$= \sum_x P(x,y)$$

$$P_X(x) = \sum_y P(x,y)$$

Example 3.1: Joint mass function of X and Y

$P(1,1) = 0.5$ $\quad$ $P(2,1) = 0.1$

$P(1,2) = 0.1$ $\quad$ $P(2,2) = 0.3$

Conditional PMF of X given Y=1

$$P_{X|1}(1|1) = P\{X=1 \mid Y=1\}$$

$$= \frac{P\{X=1, Y=1\}}{P\{Y=1\}}$$

$$= \frac{0.5}{P(1,1)+P(2,1)}$$

$$= \frac{0.5}{0.5+0.1} = \frac{5}{6}$$

$$P_{X|1}(2|1) = \frac{P(2,1)}{P_Y(1)} = \frac{0.1}{0.6} = \frac{1}{6}$$

For continuous Random Variable,

density, $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ joint density

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

So, $f_{X|Y}(x|y) = \dfrac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx}$

Example 3.6: $f(x,y) = \begin{cases} 6xy(2-x-y), & 0<x<1,\ 0<y<1 \\ 0, & \text{else} \end{cases}$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Now, $f_Y(y) = \int_0^1 6xy(2-x-y)\,dx$

$= y(4-3y)$

$\therefore f_{X|Y}(x|y) = \dfrac{6xy(2-x-y)}{y(4-3y)}$

Conditional Expectation

const $\quad E[X] = \sum x\,p(x)$

$E[X|Y=y] = \sum x\, p_{X|Y}(x|y)$

function of y $\quad = \sum x \dfrac{p(x,y)}{p_Y(y)}$

$E[X|Y=y]$ is a function of y

$\underbrace{E[X|Y]}_{\text{Random Variable}}$, is a function of Y

$E[X|Y] = g(Y)$

$E[E[X|Y]] = E[X]$

const

Proof:

$E[E[X|Y]]$

$= E[g(Y)]$

$= \sum_y g(y)\, p(y)$

$= \sum_y E[X|Y=y]\, p(y)$

$= \sum_y E[X|Y=y]\, P\{Y=y\}$

$= \sum_y \sum_x x\, P\{X=x|Y=y\}\, P\{Y=y\}$

$= \sum_y \sum_x x \frac{P\{X=x, Y=y\}}{P\{Y=y\}} \cdot P\{Y=y\}$

$= \sum_y \sum_x x\, P\{X=x, Y=y\}$

$= \sum_x x \sum P\{X=x, Y=y\}$

$= \sum_x x\, P\{X=x\}$

$= E[X]$

$E[X|Y=y] = \sum x\, P_{X|Y}(x|y)$

$= \sum_x x\, P\{X=x|Y=y\}$

$E[X] = E[E[X|Y]] \longrightarrow$ "Conditioning"

Conditional Expectation

Expectation, $E[X] = \sum x p(x)$

Conditional expectation, $E[X] = E[E[X|Y]]$

$$= \sum_y E[X|Y=y]\, P\{Y=y\}$$

Example:

X = Delay between New Market and shamoly

$E[X] = ?$

Y = Day of travel.

Average delay

Week days = 1   70 min = $E[X|\text{weekday}]$

Week ends = 2   45 min = $E[X|\text{weekend}]$

$\Downarrow$

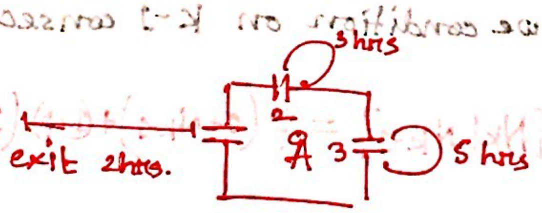
$E[X|Y=1]$

$E[X|Y=2]$

$$E[X] = E[X|Y=1]\, P\{Y=1\} + E[X|Y=2]\, P\{Y=2\}$$

$$= 70 \times \frac{5}{7} + 45 \times \frac{2}{7}$$

$\boxed{3.B}$ (Trapped Miner)



X = Time to exit , Y = Door

$E[X|Y=1] = 2$

$E[X|Y=2] = 3 + E[X]$

$E[X|Y=3] = 5 + E[X]$

$E[X] = 2 \times \frac{1}{3} + (3 + E[X]) \times \frac{1}{3} + (5 + E[X]) \times \frac{1}{3}$

Solve for $E[X] \Rightarrow$ $E[X] = 10$

* exercise: 40, 41

Q. What if the miner can remember the gate? — Think

$X \sim Geom(p)$

$E[X] = \frac{1}{p}$ $\quad$ X = # trial to get the first success

by conditioning:

Y = the result of first trial

$E[X|Y=1] = 1 \longrightarrow p$

$E[X|Y=0] = 1 + E[X] \longrightarrow 1-p$

$E[X] = p \cdot 1 + (1-p)\,[1 + E[X]]$

$\Rightarrow E[X] = \frac{1}{p}$

22

N_k = # trials to get k consecutive success

$E[N_k] = M_k$

Now we condition on $k-1$ consecutive success,

$$E[N_k | N_{k-1}] = p(1 + N_{k-1}) + (1-p)(1 + N_{k-1} + E[N_k])$$

$\underbrace{- - - - - . .}$ | $k-1$ success

$\uparrow$

trial

Success: $N_{k-1} + 1$

Fail: $N_{k-1} + 1 + E[N_k]$

$E[N_k] \Rightarrow E[N_k | N_{k-1}] = 1 + N_{k-1} + (1-p)E[N_k]$

$\Rightarrow E[N_k] = 1 + E[N_{k-1}] + (1-p)E[N_k]$

$\Rightarrow M_k = 1 + M_{k-1} + (1-p)M_k$

$\Rightarrow M_k = \frac{1}{p} + \frac{1}{p} M_{k-1}$

$M_1 = \frac{1}{p}$, $M_2 = \ldots$

$M_k = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} = \frac{(\frac{1}{p})^k - 1}{\frac{1}{p} - 1}$

Exercise: 22, 23, 27

Example:

shop

$\boxed{1} \longrightarrow \boxed{2} \longrightarrow \boxed{3}$

p = shop has the book

X_i = time to search the book in shop i

Y = time to purchase the book

$Y = X_1 + X_2 + X_3 + \dots + X_N$

N = #visit to get the book

$$E[Y] = E\left[\sum_{i=1}^{N} X_i\right] \neq \sum_{i=1}^{N} [E[X_i]]$$

N itself is a random variable

if N was constant then linearity of expectation would be applicable

$$E\left[\sum_{i=1}^{N} X_i \mid N=n\right] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\,E[X]$$

$$E\left[\sum_{i=1}^{N} X_i \mid N\right] = N\,E[X]$$

$$\Rightarrow E\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right] = E[N\,E[X]]$$

$$\Rightarrow E\left[\sum_{i=1}^{N} X_i\right] = E[X] \cdot E[N]$$

$$\Rightarrow E[Y] = E[N] \cdot E[X]$$

$E\left[\sum_{i=1}^{n} X_i\right] = n\,E[X]$

$E\left[\sum_{i=1}^{N} X_i\right] = E[N]\,E[X]$ → "Compounding"

if $p = 0.2$, $E[X] = 20$ min

$$E[Y] = \frac{1}{0.2} \times 20$$

$=$

* exercise: 34

Conditioning

$\mathrm{Var}(X) = E[(X-\mu)^2]$

$= E[X^2] - E[X]^2$

- $E[X] = E[E[X|Y]] = \sum_y E[X|Y=y]\,P\{Y=y\}$
- $\mathrm{var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)] \rightarrow$ Proof
- Compounding, $S = \sum_{i=1}^{N} x_i$
  - $E[S] = E[N]\cdot E[X]$
  - $\mathrm{Var}(S) = \sigma^2 E[N] + \mu^2 \mathrm{Var}(N) \rightarrow$ Prove it

Exercise: 44, 56, 77

Computing Probabilities by Conditioning

$$P(E) = \sum_y P(E|Y=y)\cdot P\{Y=y\}$$

Example:

X_1 Rickshow $\quad X_1 \sim \exp(\lambda_1)$

X_2 CNG $\quad X_2 \sim \exp(\lambda_2)$

$P\{$You get a rickshow$\}$

$= P\{X_1 < X_2\}$

$= \int_0^\infty P\{X_1 < X_2 | X_2 = y\}\, f_{X_2}(y)\,dy$

$= \int_0^\infty P\{X_1 < y\}\, f_{X_2}(y)\,dy$

$= \int_0^\infty F_{X_1}(y)\cdot f_{X_2}(y)\,dy$

$P\{X_1 < a\} = F(a)$ ← constant

for exponential, $f(x) = \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

$$\int_0^\infty (1 - e^{-\lambda_1 y}) \cdot (\lambda_2 e^{-\lambda_2 y})\, dy$$

$$= \underbrace{\int_0^\infty \lambda_2 e^{-\lambda_2 y}\, dy}_{\text{whole density function}} - \int_0^\infty e^{-(\lambda_1 + \lambda_2) y}\, dy$$

$$= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

So, $P\{X_1 < X_2\} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

Example:

$X \sim Bin(n, p) \to P\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k} \to p$ is constant here

$p \sim uniform(0, 1) \to$ if p in not constant

$$P\{X = k\} = \int_0^1 P\{X = k \mid p\} f(p)\, dp$$

$$= \int_0^1 \binom{n}{k} p^k (1-p)^k \cdot 1 \cdot dp$$

$$= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k}\, dp$$

from literature

$$= \frac{n!}{k!(n-k)!} \, \frac{k!(n-k)!}{(n+1)!}$$

$$= \frac{1}{n+1}$$ $\to$ All values are equally likely

Ex: 53

(Best Price Problem) $\overbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}^{n}$ ↑ max best

Strategy: $\underbrace{\circ\ \circ\ \circ \cdots \circ}_{k}\ \circ\ \circ \cdots \circ$ ↑ return the first value > any value within k.

$\underbrace{2\ 5\ 8\ 7}_{k}\quad 4\ 6\ 11\ 12\ 21$

$\underbrace{2\ 5\ 11\ 7}_{k}\quad 4\ 6\ 12\ 21$

$\underbrace{2\ 5\ 12\ 7}_{k}\quad 4\ 6\ 11\ 21 \rightarrow \otimes$

$$P_k(\text{best}) = \sum_{i=1}^{n} P(\text{best} \mid x=i)\cdot P\{x=i\}$$

x = position of the best file.

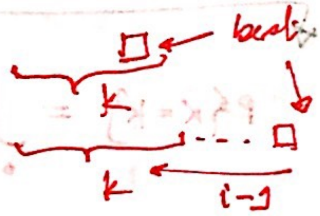
$$\text{Best value} = \frac{1}{n}\sum_{i=1}^{n} P_k(\text{best} \mid x_i)$$

$P_k(\text{best} \mid x_i) = 0$ if $i \le k$

$= P\{$2nd best within i is within $k\}$

$= P\{$best among $i-1$ lies within $k\}$

$= \dfrac{k}{i-1}$



$$P_k(\text{best}) = \frac{1}{n}\sum_{i=k+1}^{n} \frac{k}{i-1}$$

$$= \frac{k}{n}\sum_{i=k}^{n-1}\frac{1}{i} \approx \frac{k}{n}\int_{k}^{n-1}\frac{1}{x}\,dx$$

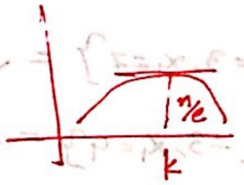
$$\approx \frac{k}{n}\int_{k}^{n-1} \frac{1}{x}\,dx$$

$$= \frac{k}{n} \log x \Big|_{k}^{n-1}$$

$$= \frac{k}{n} \log\left(\frac{n-1}{k}\right)$$

$$\approx \frac{k}{n} \log\left(\frac{n}{k}\right)$$ → what value of k maximizes this?

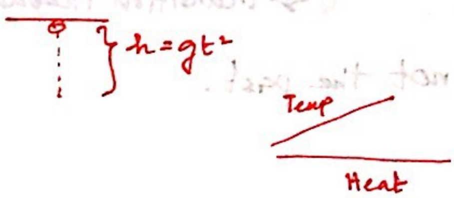
$$\frac{n}{k} = e \Rightarrow k = \frac{n}{e}$$

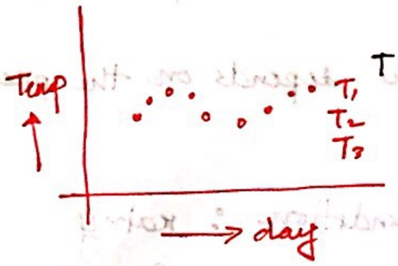


Markov Chain

Markov chain is a stochastic process that has Markovian property.

⇓ probabilistic





$T(t)$ = random variable

A stochastic process is a series/collection of RVs.

discrete $X_n = \{0,1,2,\dots\}$ (statespace)

$$\{X(t),\ t\in T\} \Rightarrow \{X_n,\ n=0,1,2,\dots\} \rightarrow \text{discrete}$$

$$\{X(t),\ t \geq 0\} \rightarrow \text{continuous}$$

Value / State

Index → Continuous / Discrete

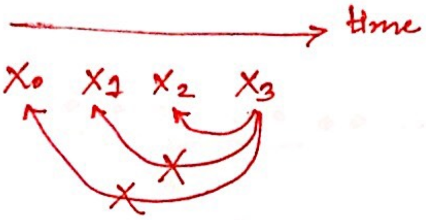
customer# served in every hour dis. dis.

of vehicles along time Dis. cont

temp. in every day Cont. Discrete

[illegible] Cont. Cont.

Markov chain:



~~make~~
markovian property: Every RV is dependant on its previous RV

$$P\{X_3=4 \mid X_2=3, X_1=7\} = \searrow$$
same
$$P\{X_3=4 \mid X_2=3, X_1=4\} = \nearrow$$

* Discrete time, discrete state space

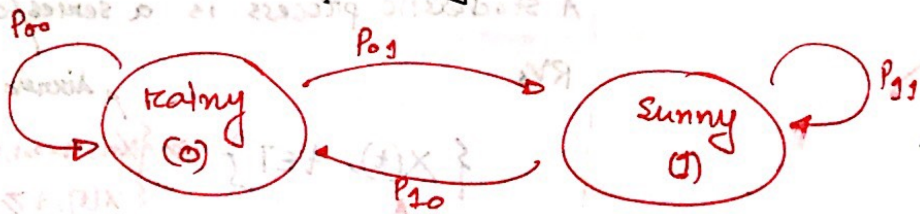
A SP is Markovian if

$$P\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, X_{n-2}=i_{n-2}, \cdots, X_0=i_0\}$$

$$= P\{X_{n+1}=j \mid X_n=i\} = P_{ij}$$ ⇒ transition Probability from $i \to j$

"Future depends on the present, not the past.

Weather Condition : rainy
Sunny



→ Transition Diagram

P_{00} = P{tomorrow is rainy | today is ~~rainy~~ rainy} = 0.7
P_{01} = P{Sunny | rainy} = 0.3
P_{10} = P{rainy | sunny} = 0.2
P_{11} = P{sunny | sunny} = 0.8

Transition Matrix : $\bar{P} = \begin{Vmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{Vmatrix}$

$$\begin{array}{cccc|l} 0 & 1 & 2 & & \\ & & & & \end{array}$$

$$\bar{P} = \begin{array}{c} \\ \begin{Vmatrix} 0.2 & 0.5 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.0 & 0.8 \end{Vmatrix} \end{array} \begin{array}{ll} 0 & \text{rainy} \\ 1 & \text{sunny} \\ 2 & \text{cloudy} \end{array}$$

$P\{\text{cloudy} \mid \text{sunny}\}$

Two properties:

① $P_{ij} \geq 0$, $\forall i, j$

② $\sum_j P_{ij} = 1$

Markov chain: ① State space $\{0, 1, 2\}$

② State transition probability $P_{ij}, \forall i, j$ presented by $\bar{P}$

Markov chain is a stochastic process

discrete time $X_0, X_1, X_2, \dots, X_n, X_{n+1}, \dots$

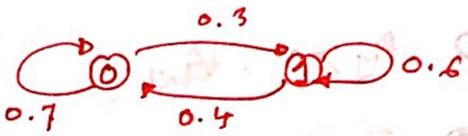
$\downarrow$
discrete

- discrete state space, $S = \{0, 1\}$ e.g.

$\uparrow$ rainy $\nwarrow$ sunny

- transitional probability, $P_{ij} = P\{X_{n+1} = j \mid X_n = i\}$

$$\bar{P} = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$$



Q. $P\{$Wed is sunny $\mid$ Tues is rainy$\} = 0.3$

1-step prob.

Q. $P\{$Thursday is sunny $\mid$ wednesday is rainy$\} = ?$

$= P\{X_2 = 1 \mid X_0 = 0\}$ → Condition on X_1

$= P\{X_2 = 1 \mid X_1 = i, X_0 = 0\}, P\{X_1 = i \mid X_0 = 0\} = \sum P_{0i} \times P_{i1}$

$= P_{01}^2 = P_{00} + P_{01} + P_{01} \times P_{11}$

2-step trans. prob $\bar{P}^{(2)} = \bar{P} \cdot \bar{P} = (\bar{P})^2$

Gen. n-step trans. prob. $\bar{P}^{(n)} = \bar{P} \cdot \bar{P} \dots \bar{P} = (\bar{P})^n$

Q. Given that sunday is raining, what is the prob. that it will be raining 4 concecutive days?

P_{00}^{1}

P_{00}^{2}

P_{00}^{3}

$P_{00}^{4} = P_{00} \times P_{00}^{3} + P_{01} P_{10}^{3}$

Chapman-kolmogorov Equation

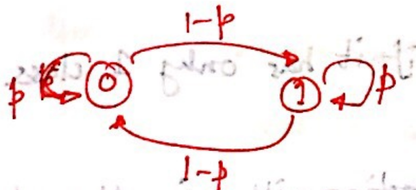
$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{n}$$

$$\bar{P}(n+m) = (\bar{P})^{n} \cdot (\bar{P})^{m} = (\bar{P})^{n+m}$$

Ex. (comm. system) $\underset{0}{X_n} \longrightarrow \boxed{\quad} \longrightarrow \underset{0/1}{X_{n+1}}$

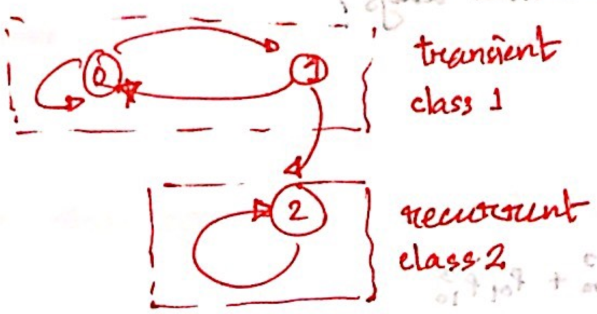
p = prob that bit remain unchanged

This is a 2-step markov chain



$$\bar{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

Markov Graph



state j is accessible from state i,

$i \to j$ if $P_{ij}^{n} > 0$ for some n

state i communicable with j, if $i \to j$ and $j \to i$: $i \leftrightarrow j$

$i \leftrightarrow j$: This relation has the following properties:

Property of a class

i) Reflexive : $i \leftrightarrow i$

ii) Symmetric : $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

iii) Transitive : $i \leftrightarrow j$, $j \leftrightarrow k$ then $i \leftrightarrow k$

* Markov chain is irreducible if it has only 1 class.

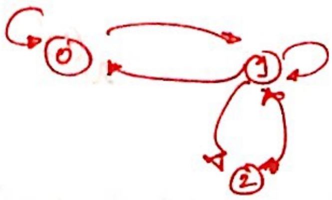
* State i is recurrent if starting with i, the chain can re-enter i infinitely many times.

It can be proved, state i is recurrent if $\sum_{n=0}^{\infty} P_{ii}^{n} = \infty$

trainsient if $\sum_{n=0}^{\infty} P_{ii}^{n} < \infty$

In a finite state Markov chain all states cannot be transient.

going no where means remaining in the state



irreducible
recurrent
finite state

$\Big\}$ "ergodic"

$P_{ij}^{(1)}, P_{ij}^{2}, P_{ij}^{3} \ldots\ldots P_{ij}^{n}$, what happens when $n \to \infty$?

P_{ij}^{n} converges and it does not depend on i.

Markov chain

For an irreducible ergodic Markov chain,

$\lim_{n\to\infty} P_{ij}^{n}$ exists (after a very long-run)

and it does not depend on i.

$$\lim_{n\to\infty} P_{ij}^{n} = \lim_{n\to\infty} P\{X_n = j \mid X_0 = i\}$$

$$= \lim_{n\to\infty} P\{X_n = j\} = \pi_j$$

π_j = limiting probability

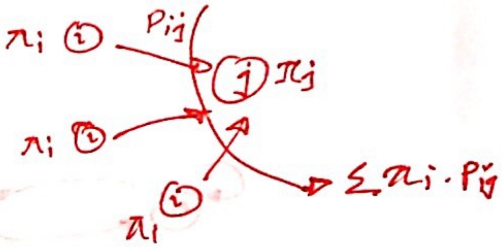
= long run-proportion of time chain remains at state 'j'

= fraction of time chain remains at state 'j'

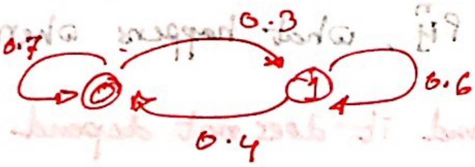
Theorem: π_j's are computed from the following:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$



* (Weather)



0: rainy
1: sunny

Q. How many days are rainy?

Let's compute π_0 and π_1

$\pi_0 = \pi_0 \times 0.7 + \pi_1 \times 0.4$ — (I)

$\pi_1 = \pi_0 \times 0.3 + \pi_1 \times 0.6$ — (II)

$$\bar{P} = \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix}$$

$\pi_0 + \pi_1 = 1$

Solve for π_0 and π_1

(I) ⇒ $0.3\pi_0 = 0.4\pi_1$; $3\pi_0 = 4\pi_1$

⇒ $3\pi_0 = 4 - 4\pi_0$

$\therefore \pi_0 = \frac{4}{7}$

$\pi_1 = \frac{3}{7}$

* Class Mobility:

Upper, Middle, Lower

$$\bar{P} = \begin{vmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{vmatrix}$$

Q. What fraction of people are in middle class?

$$\pi_0 = \pi_0 \times 0.45 + \pi_1 \times 0.05 + \pi_2 \times 0.01$$

$$\pi_1 = \pi_0 \times 0.48 + \pi_1 \times 0.70 + \pi_2 \times 0.50$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solve: $\boxed{\pi_1 = 0.62}$

$\pi_0 = 0.07$

$\pi_2 = 0.31$

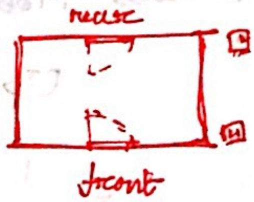
To construct a Markov Chain: steps:

- identify the stochastic variable, X_n Chain $\{X_n, n \geq 0\}$
- Possible values of X_n. states
- Compute state transition prob.

$\boxed{\text{Ex. 33}}$ → for k pair of shoes

Example:

Q. How many days he goes out bare footed?



X_n = # shoe pair in the front door at day n.

States = $\{0, 1\}$

$\{X_n, n \geq 0\}$

$P_{00} = 3/4$

$P_{01} = P\{RF\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\Rightarrow P = \begin{vmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix}$

$P_{10} = P\{FR\} = \frac{1}{4}$

$P_{11} = 3/4$

$\pi_0 = \frac{1}{2}$

$\pi_1 = \frac{1}{2}$

$P\{\text{barefoot}\} = P(F).\pi_0 + P(R).\pi_1$

$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{2}$

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$p_1 = 0.3$ $p_2 = 0.6$ $p_3 = 0.9$

(Exam) Q_1 Q_2 Q_3

if student does well, then next exam can be any of type

if student does bad, next exam is type 1.

X_n = type of exam of n-th exam.

= states $\{1,2,3\}$

$P_{11} = P\{\text{next exam is } 1 \mid \text{current exam } 1\}$

$= 0.3 \times 0.33 + 0.7 \times 1$

$P_{12} = 0.3 \times 0.33$

$\vdots$

Ex. 25, 30, 32, 33, 46, 28, 23

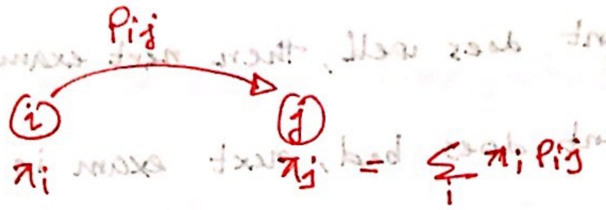
* Define markov chain

$\{X_n : n \geq 0\}$ is a Markov chain | $\{X(t) : t \geq 0\}$ is a Markov chain

continuous

define X and n

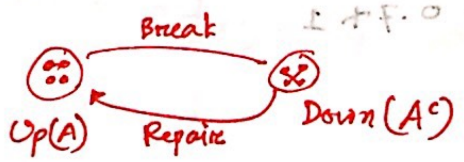
Markov Chain



$\pi_i P_{ij}$ = the rate at which chain transits from i to j

Example 4.24

A production process works in 2 type of states



Q. The rate of breakdowns

Q. The average length, the process is in UP state.

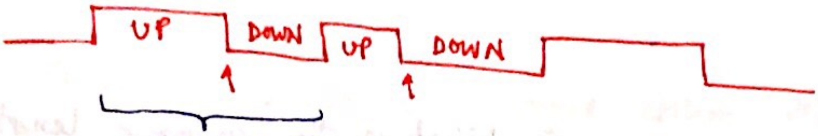
Ans.

Rate to enter j from i $= \pi_i P_{ij}$

Rate to enter A^c from i $= \sum_{j \in A^c} \pi_i P_{ij}$

Rate to enter A^c from A $= \sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij}$ --- (A)

rate of breakdown



Period = cycle length = $\bar{U} + \bar{D}$

Up time   Down time

1 break per $\bar{U}+\bar{D}$ time

rate of breakdown = $\dfrac{1}{\bar{U}+\bar{D}}$ - - - (B)

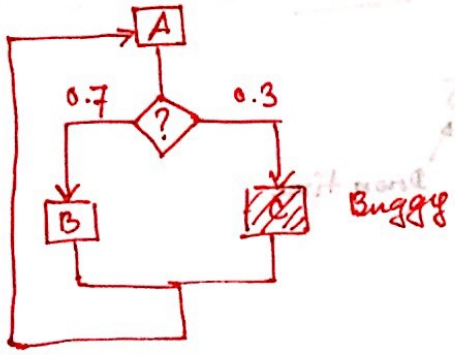
Equation (A) and (B) $\Rightarrow$

$$\frac{1}{\bar{U}+\bar{D}} = \sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij} \quad \text{—(I)}$$

Proportion of the system is is UP state,

$$\frac{\bar{U}}{\bar{U}+\bar{D}} = \sum_{i \in A} \pi_i \quad \text{—(II)}$$

Solve (I), (II) for $\bar{U}$ and $\bar{D}$



Q. What is the average length of time, the program produces correct output?

$UP = \{A, B\}$

$DOWN = \{C\}$

Fraction of wrong output $= \pi_C$

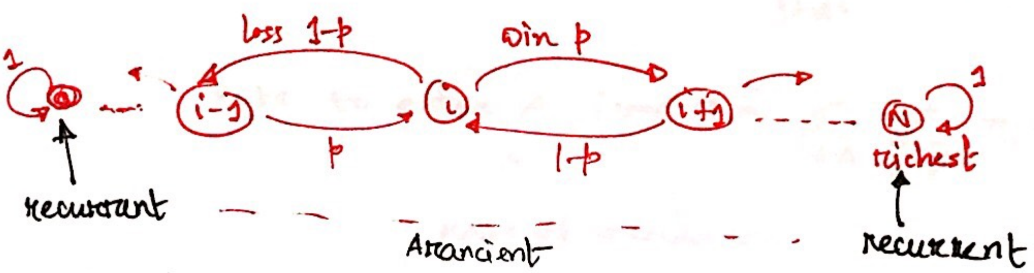
The rate of producing error output $= \pi_A P_{AC}$

$\boxed{\text{Ex. } 41, 42}$

Gambler's Ruin

X_n = money the gambler has at nth move

$\{X_n : n \geq 0\}$ is a Markov chain.



Compute P_i = Prob. that the Gambler reaches N from i

$= P\{i \to N\}$

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Condition on the next move, either win or loss.

$$= P\{i+1 \to N\} \cdot P(\text{win}) + P\{i-1 \to N\}\, P(\text{loss})$$

$$\therefore P_i = P_{i+1}\, p + P_{i-1}\, q \quad ; \; q = 1-p$$

$$\Rightarrow (p+q) P_i = P_{i+1}\, p + P_{i-1}\, q$$

$$\vdots$$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p}\,(P_i - P_{i-1})$$

$P_0 = 0, \quad P_N = 1$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \\[2ex] \dfrac{i}{N} & p = \frac{1}{2} \end{cases}$$

Random walk is a MC with infinite states.

* infinite

Discrete time : $\{X_n : n \geq 0\}$

fixed interval

Continuous time :

varying interval

$\{X(t) : t \geq 0\}$

$X(t)$ = # customers in a bank queue at t.

0 1 2 1 2 1 0

exponentially distributed

Discrete

π_i → p_i

Continuous

$i \xrightarrow{P_{ij}} j$

v_i = the rate of staying at i

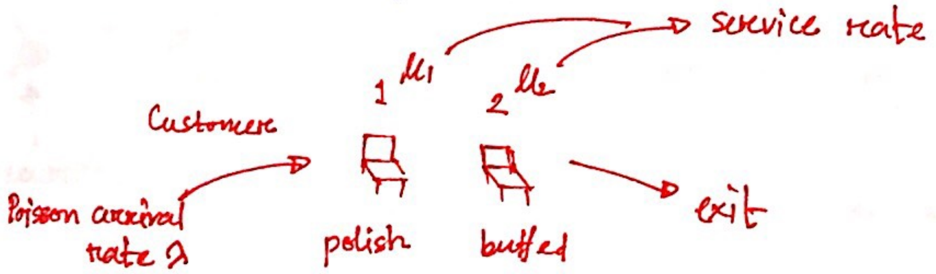
v_j

$T_i \sim \exp(v_i)$ → exponentially distributed with rate v_i

T_i = the amount of time the process is in state i

Poisson → count
expo. dist → time

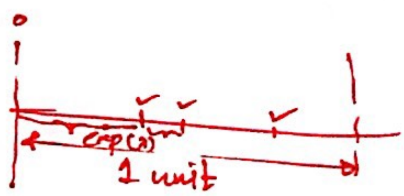
A Shoeshine shop:



state : 0 : empty, no customer

1 : customer is in chair 1

2 : customer is in chair 2



X = # events per unit time

$X \sim$ Poisson (λ)

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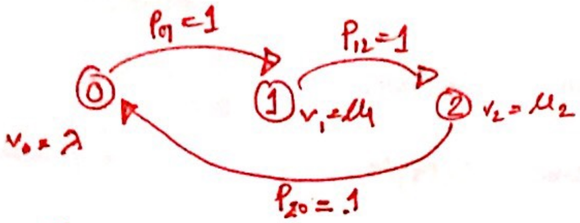
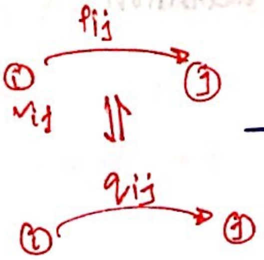
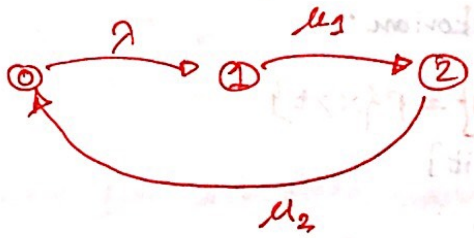




$$\longrightarrow q_{ij} = v_i p_{ij} \Rightarrow \sum_j q_{ij} = v_i \sum_j p_{ij} = v_i$$

$$p_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}$$

q_{ij} = the rate at which the process transits from i to j



Chap-6	First Section
Chap-8	8.1 , 8.2

$i \rightarrow j$

q_{ij} - Rate of transition

$X \sim Expo(\lambda)$ $\underbrace{\text{\# events per time}}_{\text{Poisson}}$

$E[X] = \frac{1}{\lambda}$

$Y = \min(X_1, X_2)$, with $X_1 \to \lambda_1$, $X_2 \to \lambda_2$

$\downarrow$

$Y = expo(\lambda_1 + \lambda_2)$

[Prove it!]

$f_Y(x) = (\lambda + \lambda_2) e^{-(\lambda_1+\lambda_2)x}$

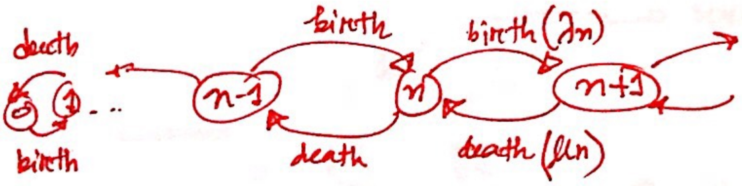
Memoryless or Markovian:

$P\{X > t+s \mid X > s\} = P\{X > t\}$

[Prove it]

Birth and Death Process

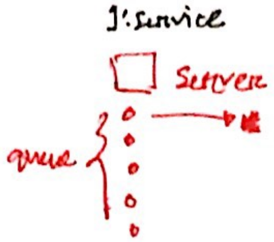
$X(t)$ = Population size at time t



Poisson arrival : $\lambda_n = \lambda$
$\mu_n = 0$

M/M/1 Queue : $\lambda = \lambda$ (arrival rate)
$\mu_n = \mu$ (service rate)

Pure birth Process : $\lambda_n = n\lambda$
Yule Process $\mu_n = 0$



arrival is memoryless exponential

M/M/1 Queue ; [// M/M/S → for s no. of service]

M/M/1 can be modeled as CTMC.

$\{X(t) : t \geq 0\}$

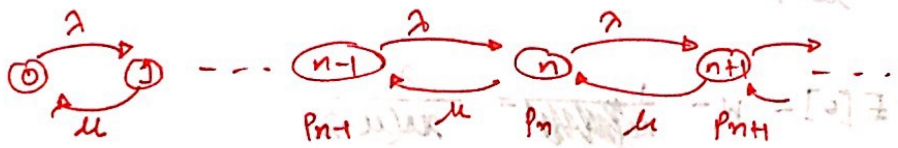
$X(t)$ = # of customers in the system.

L = average # customers in the system

Lq = average # customers in the queue

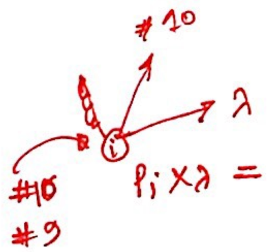
W = Average wait time in the system

Wq = average wait time in the queue



Balance equations. for each state, leaving rate = entering rate

'0' → $\boxed{P_0 \lambda = P_1 \mu}$



"n" → $P_n(\lambda+\mu) = P_{n+1}\mu + P_{n-1}\lambda$

$n \geq 1$

$\Rightarrow P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right)$

$P_0 = ?$

$P_1 = \frac{\lambda}{\mu} P_0$

$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$

$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

We have, $\sum_{n=0}^{\infty} P_n = 1$

$\Rightarrow P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \dots = 1$

$\Rightarrow P_0 \cdot \frac{1}{1-\frac{\lambda}{\mu}} = 1$ if $\frac{\lambda}{\mu} < 1 \Rightarrow \mu > \lambda$

$\Rightarrow P_0 = 1 - \frac{\lambda}{\mu}$

$L = \sum_{n=0}^{\infty} n P_n$

$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$ $\qquad \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$

$= \frac{\lambda}{\mu - \lambda}$

; Little's formula: $L = \lambda W$

$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$

$W_Q = W - E[S] = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$

$L_Q = \lambda \cdot W_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

$P_0 = 1 - \frac{\lambda}{\mu}$

$1 - P_0 = \frac{\lambda}{\mu} = \rho$ utilization