L-3, T-2

Date:

3/2

PREVIOUS QUESTION SOLUTION

1st Chapter

Coppy রেডি করা আছে। এসে নিয়ে যান।
তিতুমির হল মনির ভাই।

Sultan Ahmed
CSE-08

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Date:

Buet | 2009-2010

5. (b) consider well-known problem "Tower of Hanoi". There are 3 rods (A, B & C). Find the number of moves made by ith largest ith disk ($1 \le i \le n$) in above algo.

solution:

Algo:
1) Move upper (n-1) disks from rod A to rod B.
2) Move bottom most disk from rod A to rod C.
3) Move (n-1) disks from B to C.

Soln:

In this procedure the largest disk needs to be transferred.

$1 = 2^0$ move

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then (n-1) disks remain.

from this, (n-2) disks must be transferred. to another peg.

the 2nd largest disk needs 2^1 move to be transferred.

Thus we can tell the ith disks needs 2^{i-1} moves to be transferred.

—————— x —————— o —————— x ——

Buet -(2009-2010)

(c) # What is the maximum and minimum numb of regions defined by n instant circles? Just find the recurrence.

tion:

$C_1 = 2$

$C_n = C_{n-1} + 2(n-1)$

Logic

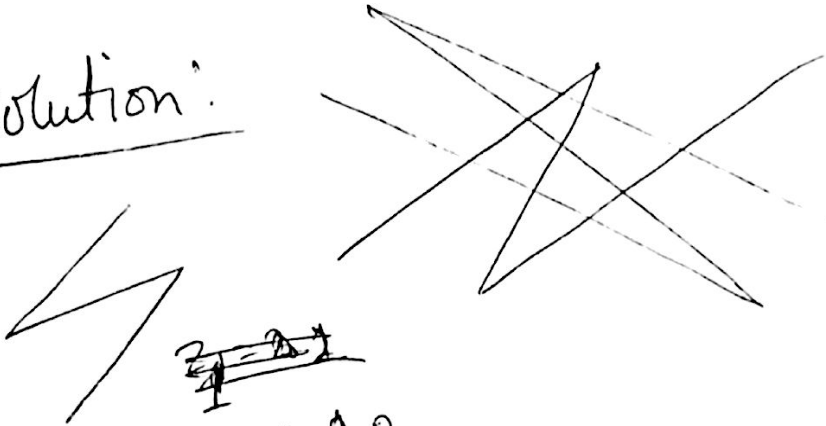
1. n তম circle পূর্ববর্তী সকল circle কে 2টি point এ inte করে।

2. intersect এর ~~point~~ গুলো 2(n সংখ্যক region create হয়।

Buet 2008-2009

5. (a) What is the maximum number of regions definable by n zig-zag lines?

Solution:



$Z_2 =$

$ZZ_1 = 2$

$ZZ_n = ZZ_{n-1} + 9(n-1) + 1$

$\therefore ZZ_n = \sum_{k=1}^{n} (9k-8) + 1$

$= 9\frac{n(n+1)}{2} - 8n + 1$

$= \frac{9n^2 + 9n - 16n + 2}{2}$

$= \frac{9n^2 - 7n + 2}{2}$

$\therefore ZZ_n = \frac{1}{2}(9n^2 - 7n + 2)$

অন্য নিয়মে

$ZZ_n = 2 + \sum_{k=2}^{n} (9k-8)$

$= 2 + \sum_{k=1}^{n-1} (9k+1)$

$= 2 + \frac{9}{2} k(k+1)\Big|_{k=n-1} + (n-1)$

$= 2 + \frac{9}{2}(n-1)n + (n-1)$

$= \frac{9n^2 - 9n + 2n - 2 + 4}{2}$

$= \frac{9n^2 - 7n + 2}{2}$

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Buet 07-08

5(a): Given a ~~k~~ set of $(n-1)$ lines in the pla
explain, how adding a new line can crea
n new regions.

Also explain the case when it would
create fewer than n regions.

write sufficient && necessary conditi
for max^m no. of regions.

Solution:

Given a set of $(n-1)$ lines in the pla
the nth line can intersect $(n-1)$ old
lines in at most $(n-1)$ different points.
So the nth line increases the no.
regions by $(n-1)+1 = n$

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If the nth lines passes through the intersecting point then it would create fewer than n region.

Recurrence:

$L_0 = 1$

$L_n = L_{n-1} + n$

$= L_{n-2} + (n-1) + n$

$= L_{n-3} + (n-2) + (n-1) + n$

$= L_{n-3} + \cdots + (n-1) + n$

$= L_0 + 1 + 2 + 3 + \cdots$

$= 1 + \frac{1}{2} n(n+1)$

$\therefore L_n = \frac{1}{2} n(n+1) + 1$

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Buet 06-07

1. a) Formulate Josephus problem. Deduce the recurrence relations that the soluti[on] satisfies.

solution:

Josephus Problem:

Given n people numbered 1 to n, standing in a circle, if we go around the circle eliminating every 2nd person until only one person is left, what is the survivor number J(n)?

Recurrence Solution:

Let the number of people n in the circle is even i.e. $n = 2l$ for some le[...]

It is noted that all of even number people will be eliminated first:

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$1, 2, 3, 4, \ldots, 2l-2, (2l-1), 2l$

\# After person $2l$ is eliminated we have with l odd numbered people:

$1, 3, 5, \ldots\ldots (2l-3), (2l-1)$

\# The key observation here is that the position of the surviving person out of these l people is given by $J(l)$.

\# Since the number of kth person around the circle with l odd-numbered people is $(2k-1)$, the number of the surviving person out of the $2l$ people is simply $2J(l)-1$

\# Now let $n = 2l+1$ for some $l \in \mathbb{N}$.

\# Again all even numbered people will be eliminated first:

$1, \overset{\times}{2}, 3, \overset{\times}{4}, \ldots\ldots \overset{\times}{(2l-2)}, (2l-1), \overset{\times}{2l}, (2l+1)$

However this time person 1 will be eliminated instead of 3.

After eliminating person 1, we are left with l people 3, 5, ---- $(2l-1)$, $(2l+1)$.

Since the kth persons number in this group of l people is given by (2k+

So the number of surviving person out of the $(2l+1)$ people is simply

$2J(l)+1$

Letting $J(1) = 1$, the recurrent solution is

$$J(n) = \begin{cases} 1 & \text{if } n=1 \\ 2J(l)-1 & \text{if } n=2l \text{ where } l \geq \\ 2J(l)+1 & \text{if } n=2l+1 \text{ where} \end{cases}$$

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Buet 06-07

(b) Argue in favour of solutions to the Josephus Problems where every 2th person is removed from the circle.

utions:

মোঃ মনির হোসেন
প্রোপাইটর
বিসমিল্লাহ ফটোকপি
কেমিক্যাল ডিপার্টমেন্ট এর লুবনা ম্যাডাম এর সকল নোট পাওয়া যায়।
এছাড়া ও অফসেট A4/ লিগ্যাল অফসেট ফটোকপি করা হয়।
তিতুমীর হল বাঁধন অফিসের সামনে।
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Buet 06-07

1.(c)

Find a recurrent relation for P_n, the maximum number ~~the~~ of 3 dimensional regions that can be defined by n differen planes.

Solution:

Let the number of regions created by (n-1) planes is P_{n-1}

the nth plane intersect all ther (n-1) planes.

Two planes intersection produce a lin

So After intersection there are (n-1) new lines

$$\therefore P_n = P_{n-1} + L_{n-1}$$

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recurrence:

$$P_0 = 1$$

$$P_n = P_{n-1} + 2n - 1$$

$$= \underline{P_{n-1}} + \frac{1}{2} n(n-1) + 1$$

$$= 1 + \sum_{k=1}^{n} \left\{ \frac{1}{2} k(k-1) + 1 \right\}$$

$$= 1 + \frac{1}{2} \sum_{k=1}^{n} k^2 - \frac{1}{2} \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 1 + \frac{n(n+1)(2n+1)}{12} - \frac{n(n+1)}{4} + n$$

$$= \frac{12 + n(n+1)(2n+1-3) + 12n}{12}$$

$$= \frac{12 + n(n+1)(2n-1) + 12n}{12}$$

$$= \frac{6 + n(n+1)(n-1) + 6n}{6}$$

$$= \frac{6 + n^3 - n + 6n}{6}$$

$$\therefore P_n = \frac{1}{6}(n^3 + 5n + 6)$$

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1.(a) Formulate multi-peg tower of hanoi problem. Write down the recurrence relation the presumed optimal solution should satisfy. Marks-10

Solution:

Multi-peg Tower Of. Hanoi:

p pegs are fastened to stand a n disks of different radius $D_1, D_2 \ldots$ initially rest on source peg P_1 in a in small to large ordering. What the minimum no. of moves to transfer the tower to destination peg P_p.

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P_1 P_2 P_3 P_p

Recurrent Relation:

$$N(0, p) = 1 \quad \text{for all } p \geq 3$$

$$N(n, 3) = 1 \quad \text{for all } n \geq 0$$

$$N(n, p) = N(n, p-1) + N(n-1, p)$$

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Date:

Buet 2005-2006

1(b):

Given an infinite supply of disks let $N(K,P)$ be the maximum number of disks each of which requires 2^K moves to reach destination in the presumed optimal solution strategy. Deduce the value of $N(K,P)$

Solution:

\# $N(K,P) = N(K-1,P) + N(K,P-1)$

\# We have to find the value of $N(K$

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2t 2005-2006

...s 1) Solve the multi-peg tower of hanoi problem
...ch
...tion $(n, P) = (169, 7)$ by constructing a binary
...arch tree each node of which will
...present a subproblem to be solved.

Marks – 10

...ution: $N(a, b)$

k \ p	3	4	5	6	7	8	9			
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8		
2	1	3	6	10	15	21	28			
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126	210			
5	1	6	21	56	126	252	462			

For $n = 169$ && $P = 7$ $\boxed{k_{max} = 5}$

For $n = 169$

Ist InEqnality:

$N(k_{max}-2, P) + \min\{N(k_{max}-1, P), N_a(k_{max}...$

$...(k_{max}-2, P) < n \leq N'(k_{max}-2, P)$

ue :

$\# N_a(k_{max}) = n - N'(k_{max}-1, P)$

$= n - N(k-1, P+1)$

$= 211 - N(k_{max}-1, P+1)$

$= 211 - N(4, 8)$

$= 211 - 126$

$= 85$

$\# N(k_{max}-1, P) = N(4, 7) = 70$

$\# N'(k_{max}-2, P) = N(k_{max}-2, P+1)$

$= N(3, 8)$

$= 56$

$\cancel{56} \le n_1 \le 56 + \min\{70, 85\}$ ------- (i) $\quad 35 \le n_1 \le 125$

$\Rightarrow 56 \le n_1 \le 126$

nd Inequality

$N'(k_{max}-1, P-1) \le n - n_1 \le N'(k_{max}-1, P-1)$

$+ \min\{N(k_{max}, P-1), N_a(k_{max})\}$

Value :

$\# N'(k_{max}-1, P-1) = N'(4, 6) = N(4, 7) = 70$

$\# N(k_{max}, P-1) = N(5, 6) = 56$

Date:

So

$70 - \cancel{n} \le n - n_1 \le 70 + \min(56, 85)$

or, $70 \le n - n_1 \le 70 + 56$

or, $70 \le 169 - n_1 \le \cancel{70}\ 126$

or, $70 - 169 \le -n_1 \le 126 - 169$

or, $-99 \le -n_1 \le -\cancel{1}43$

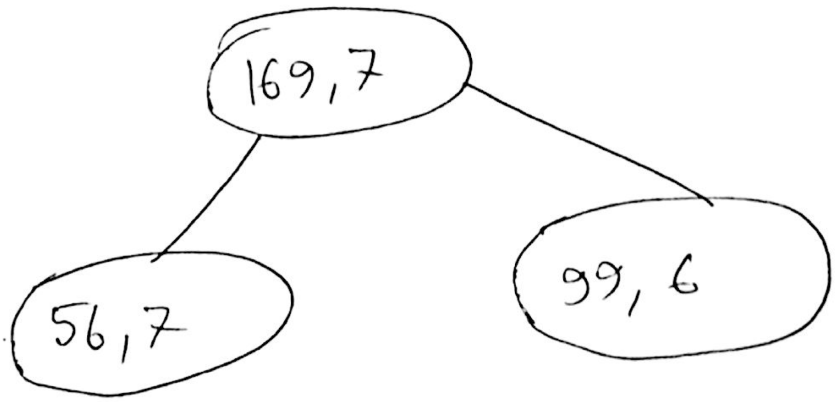
$\therefore 43 \le n_1 \le 99$ .... (2)

from (1) && (2) we get,

$56 \le n_1 \le 99$

Here is the binary tree for solving:

#



This tree is formed according to t
following formula:

$N(k,p) = N(k,p-1) + N(k-1,p)$

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Date:

Buet 2002-2003

1(a)

A double tower of hanoi problem contains $2n$ disks of n different size. How many moves it take to transfer a double tower from one peg to another.

Solution:

Let

$$A(n) = \text{no. of moves}$$

$$A(1) = 2$$

$$A(n) = A(n-1) + 2 + A(n-1)$$

$$= 2A(n-1) + 2$$

$$= 2^2 A(n-2) + 2^2 + 2$$

$$= 2^n + 2^{n-1} + \cdots + 2^2 + 2 + 1 - 1$$

$$= 2^{n+1} - 1 - 1$$

$$\therefore A(n) = 2^{n+1} - 2$$

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Date:

et 2002-2003

(b):
What is the maximum number of pieces to which a single thick ~~of~~ piece of cheese an be devided by n plane cuts?

olution:

$\# \; P_1 = 2$

$\# \; P_{n+1} = P_n + 2n$

$= P_{n-1} + 2n-1$

$\therefore P_n = P_{n-1} + \frac{1}{2} n(n-1) + 1$

$= P_{n-1}$

$\therefore P_n = 1 + \sum_{k=1}^{n} \left\{ \frac{1}{2} k(k-1) + 1 \right\}$

$= 1 + \frac{1}{2} \left\{ \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 2n \right\}$

$= 1 + \frac{1}{2} \frac{n(n+1)(\cancel{n}\,n-1) + 6n}{6}$

$\therefore P_n = \frac{1}{6}(n^3 + 5n + 6)$

Ans

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Buet 2002 - 2003

1(c)
Deduce recurrence relations that solutio of Josephus Problem satisfy.

Solution:

Let the no. of people n in the circle is even i.e. $n = 2l$ for some $l \in N$.

It is noted that all of even numbered people will be eliminated first.

before: $1, 2, 3, 4, \cdots (2l-2), (2l-1), 2l$

after: $1, 3, 5, 7, \cdots\cdots (2l-3), (2l-1)$

the key observation here is that the position of the surviving person out of the l people is given by $J(l)$

Since the number of kth person around the circle with l numbered people is $(2k-1)$ the no. of surviving person out of the $2l$ is simply $2J(l) - 1$.

$J(2l) = 2J(l) - 1$ # $J(2n) = 2J(n) - 1$

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$\boxed{\text{for odd no. } n = 2l+1}$

Again all even numbered people will be eliminated first:

$\Rightarrow$ before: $1, 2, 3, 4, 5 \cdots (2l-1), 2l, (2l+1)$

after: $3, 5, 7, \cdots (2l-1), (2l+1)$

This time person 1 will be eliminated

After eliminating person 1, we are let with l people $3, 5, 7, \cdots, (2l-1), (2l+1)$

Since the kth person numbered in this group of l people is given by $(2k+1)$

So the number of surviving person out of the $(2l+1)$ people is simply $2J(l)+1$

$$\Rightarrow J(2l+1) = 2J(l) + 1$$

So According to Josephus Problem

$$\left.\begin{aligned} J(1) &= 1 \\ J(2n) &= 2J(n) - 1 \\ J(2n+1) &= 2J(n) + 1 \end{aligned}\right\} \text{solutions} \rightarrow$$

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Date:

ACM UVA - 254

Tower of Hanoi:

Problem statement:

Given the number of move n, find the disks on A, B, C.

solution:

1. ~~ta~~ Let the no. of disk is 2 and the no. of move is m.
2. convert m into n bit binary number.

   n = 5, m = 3

   number = 00011

3. d[0] = d[1] = d[2] = 0
4. beg = 0, aux = 1, dest = 2
5. for bit 0 to (n-1)

   6. if bit = 0

      d[beg] = d[beg] + 1

      swap (aux, dest)

   7. else if bit = 1

      d[dest] = d[dest] + 1

      swap (aux, ~~des~~ beg)

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8. if even number of disk

print d[0] d[1] d[2]

9. else

10 print d[0] d[2] d[1]

2nd Solution:

We can make use of the fact that the position at powers of two is easily known

Time	Heights
	$\{0, 0, T\}$
$2^T - 1$	$\{0, T-1, T\}$
2^{T-1}	$\{1, T-1, 0\}$
$2^{T-1} - 1$	$\{1, 1, T-2\}$
2^{T-2}	$\{2, 0, T-2\}$
$2^{T-2} - 1$	$\{2, T-3, 1\}$
2^{T-3}	$\{3, T-3, 0\}$
$2^{T-3} - 1$	
⋮	⋮
0	$\{T, 0, 0\}$

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3rd Solution:

Let us look at the first few moves of the puzzle:

1-0	8-3	# Disk 0 moved every 2 turns starting on turn 1.
2-1	9-0	
3-0	10-1	
4-2	11-0	# Disk 1 moved every 4 turns starting on turn 2.
5-0	12 2	
6-1	13-0	
7-0	14-1	
	15-0	

Disk i is moved on every 2^{i+1} turns starting on turn 2^i.

So in constant time we can determine how many times a given disk has removed given m

$$\text{moves} = \frac{m+2^i}{2^{i+1}}$$

odd numbered disks move to the left each time

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Problems

Solutions of Questions of Prev Year

2nd Chapter

Sultan Ah
CSE-08

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Date:

Buet 2007-2008

5(b)

Prove by re-arranging terms that

$$\sum_{1\le j\le k\le n} (a_j b_k - a_k b_j)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

Solution:

Let

$$S_{j,k} = (a_j b_k - a_k b_j)(A_j B_k - A_k B_j)$$

$$\therefore S_{j,k} = S_{k,j}$$

$$\sum_{1\le j,k\le n} S_{j,k} = \sum_{1\le j<k\le n} S_{j,k} + \sum_{1\le j\le k\le n} S_{j,k}$$

$$+ \sum_{1\le k<j\le n} S_{k,j}$$

$$= \sum_{1\le j\le k\le n} S_{j,k} + \sum_{1\le k<j\le n} S_{k,j}$$

$$S_{j,k}$$

$$\therefore \sum_{1\le j,k\le n} S_{j,k} = 2\sum_{1\le j<k\le n}$$

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$$S_{j,k} = (a_j b_k - a_k b_j)(A_j B_k - A_k B_j)$$

$$= a_j b_k A_j B_k - a_j A_k b_k B_j - a_k A_j b_j b_k$$

$$+ a_k A_k b_j B_j$$

For 1st term: ~~Let $S_{j,k} = a_j^2 b_k^2$~~

$$\sum_{1 \le j,k \le n} S_{j,k} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j A_j B_k b_k$$

$$= \left(\sum_{j=1}^{n} a_j A_j \right) \left(\sum_{k=1}^{n} B_k b_k \right)$$

Putting all together,

$$\sum_{1 \le j < k \le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j)$$

$$= \left(\sum_{k=1}^{n} a_k A_k \right) \left(\sum_{k=1}^{n} b_k B_k \right) - \left(\sum_{k=1}^{n} a_k B_k \right) \left(\sum_{k=1}^{n} A_k b_k \right)$$

If $a_k = A_k$ and $b_k = B_k$

$$\left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \left(\sum_{k=1}^{n} a_k b_k \right)^2$$

$$= \sum_{1 \le j < k \le n} (a_j b_k - a_k b_j)^2$$

[Showed]

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Buet 2007-2008

5(c)

Sum the series using perturbation method

$$\sum_{1 \le k \le 2n} (-1)^k k$$

$\boxed{\text{Formula}}$: $S_n + a_{n+1} = a_0 + \sum_{0 \le k \le n} a_{k+1}$

Solution:

Let $S_n = \sum_{k=1}^{2n} (-1)^k k$

$$S_n + (-1)^{2n+1}(2n+1) = (-1)^1 \times 1 + \sum_{k=1}^{2n} (-1)^{k+1}(k+1)$$

$$= -1 - 1 \times \sum_{k=1}^{n} (-1)^k k + \sum_{k=1}^{2n} (-1)^{k+1}$$

$$= -1 - S_n + 0$$

or, $S_n - (2n+1)$

or, $S_n + S_n = (2n+1) - 1$

$$= 2n + 1 - 1$$

or, $2S_n = 2n$

$\therefore S_n = 2n/2$

$$= n$$

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বিকল্পের পদ্ধতি: (Check Method)

N.B : # The following procedure is not perturbation method.

$$\sum_{k=1}^{2n} (-1)^k k$$

$$= -1+2-3+4-\cdots+2n$$

$$= 2(1+2+3+\cdots+n) - (1+3+5+\cdots+(2n-1))$$

$$= 2\frac{n(n+1)}{2} - n^2$$

$$= n(n+1) - n^2$$

$$= n^2+n-n^2$$

$$= n$$

$$\therefore \sum_{k=1}^{2n} (-1)^k k = n$$

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Problem: upper limit 2n ছাড়া solution পাওয়া যায়না। তাই upper limit 2n দিয়ে solve করা হল।

$$\sum_{k=0}^{2n} k^2 (-1)^k = \sum (2k)^2 - \sum (2k-1)^2$$

$$= \sum (4k^2 - 4k^2 + 4k - 1)$$

$$= \sum (4k-1)$$

$$= 4\sum k - \sum 1$$

$$= 4\,\tfrac{1}{2}n(n+1) - n$$

$$= 2(n+1)n - n$$

$$\therefore \sum_{k=0}^{2n} k^2 (-1)^k = n(2n+1)$$

আমাদের approach হল: n হয় odd or even হবে। এখানে n এর even value এর জন্য compute করা হল। তাহলে তোমার হোমওয়ার্ক! compute the value of sum for odd value of n.

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In prev math we ~~show~~ have showed

$$\sum_{k=1}^{2n} k(-1)^k = n$$

$$S_n + a_{n+1} = a_0 + \sum_{0 \le k < 2n} a_{k+1}$$

$$= \sum_{0}^{2n} (-1)^{k+1} (k+1)^3$$

$$= \sum_{0}^{2n} (-1)^{k+1} (k^3 + 3k^2 + 3k + 1)$$

$$= -\sum_{0}^{2n} (-1)^k k^3 - 3\sum_{0}^{2n} (-1)^k k^2 - 3\sum_{0}^{2n} (-1)^k k - \sum_{0}^{2n} (-1$$

$$= -S_n - 3k(2k+1) - k$$

$$\Rightarrow 2S_n + (-1)^{n+1}(n+1)^3 = -3k(2k+1) - k$$

$$\Rightarrow 2S_n = -(-1)^{n+1}(n+1)^3 - 3k(2k+1) - k$$

$$= (-1)^n (n+1)^3 - 3k(2k+1) - k$$

$$= (-1)^n (n+1)^3 - 2k(3k+1)$$

$$\therefore S_n = \frac{1}{2}\left\{(-1)^n (n+1)^3 - 2k(3k+1)\right\}$$

Date:

Buet 2008-2009

6(a)

Use repertoire method to solve the following recurrence relation to a close form:

$$r_0 = 1$$
$$r_n = r_{n-1} + (3n + 5)$$

solution:

$$r_0 = \alpha \quad \cdots (1)$$
$$r_n = r_{n-1} + (\beta + \gamma n) \quad \cdots (2)$$

$$r_n = A(n)\alpha + B(n)\beta + C(n)\gamma \quad \cdots (3)$$

step 1: Let $r_n = 1$

$\therefore \alpha = 1$

(2) হতে, $1 = 1 + \beta + \gamma n$

$\Rightarrow 0 = \beta + \gamma n$

$\Rightarrow \beta = 0, \gamma = 0$

$\therefore (\alpha, \beta, \gamma) = (1, 0, 0)$

(3) হতে,

$1 = A(n) \times 1$

$\therefore A(n) = 1$

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Step 2:

Let, $r_n = n$

$\Rightarrow r_0 = 0$

$\Rightarrow \alpha = 0$

(2) ধরি

$n = n-1 + \beta + \gamma n$

or, $1 = \beta + \gamma n$

$\Rightarrow \beta = 1, \gamma = 0$

(3) ধরি,

$n = B(n) \times 1$

$\Rightarrow B(n) = n$

step-3

Let $r_n = n^2$

$\Rightarrow r_0 = 0$

$\Rightarrow \alpha = 0$

(2) ধরি,

$n^2 = (n-1)^2 + (\beta + \gamma n)$

$\Rightarrow 2n - 1 = \beta + \gamma n$

$\Rightarrow \gamma = 2, \beta = -1$

(3) ধরি,

$n^2 = \cancel{2} - B(n) + 2C(n)$

$= -n + 2C(n)$

or, $2C(n) = n^2 + n$

or, $2C(n) = n(n+1)$

$\therefore C(n) = \frac{n(n+1)}{2}$

$\therefore r_n = \alpha + n\beta + \frac{n(n+1)}{2}\gamma$

$= 1 + 5n + \frac{n(n+1)}{2} 3$

$= 1 + 5n + \frac{3}{2} n(n+1)$

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7(c)

Using induction prove that

$$\sum_{0 \le k < n} kH_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$$

solution:

$\sum_{0 \le k < n} kH_k$ # ~~for~~ $\int x \ln x \, dx$ # $\sum x H_x$

$\sum u \Delta v = uv - \sum E v(\Delta u)$ with $E f(x) = f(x+1)$

$u = H_x$, $\Delta v(x) = x = x^{\underline{1}}$

$v(x) = \frac{x^{\underline{2}}}{2}$

$\Delta u(x) = u(x) - u(x-1)$

$= H_x - H_{x-1}$

$= \frac{1}{x}$

$= x^{\underline{-1}}$

$E v(x) = \frac{(x+1)^{\underline{2}}}{2}$

$\sum x H_x \delta x = \frac{x^{\underline{2}}}{2} H_x - \sum \frac{(x+1)^{\underline{2}}}{2} \times x^{\underline{-1}} \delta x$ $\left[x^{\underline{m+n}} = x^{\underline{m}} (x-m)^{\underline{n}} \right]$

$= \frac{x^{\underline{2}}}{2} H_x - \frac{1}{2} \sum x^{\underline{1}} \delta x$

$= \frac{x^{\underline{2}}}{2} H_x - \frac{x^{\underline{2}}}{4}$

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$$= \frac{x(x-1)}{2} H_x - \frac{x(x-1)}{4} + C$$

$$\therefore \sum_{0 \le k < n} k H_k = \sum_0^n x H_x \delta x$$

$$= \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$$

$\boxed{\text{N.B}}$ $\boxed{\text{\# Just Solution Procedure}}$ $\boxed{\text{যেভাবে করা হয়}}$

<u>Proof by Induction</u>

<u>Base case:</u>

when $n = 2$ $\sum_{0 \le k < 2} = H_1 = 1$

$$\text{RHS} = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$$

$$= \frac{2 \times 1}{2} H_2 - \frac{2 \times 1}{4}$$

$$= (1 + 1/2) - 1/2$$

$$= 1$$

For $n = 2$

$$\sum_{0 \le k < n} k H_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$$

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Induction step:

Let the statement is true for $n=(m-1)$ Then

$$S_m = \sum_{0 \le k \le m-1} k H_k = \frac{m(m-1)}{2} H_m - \frac{n(n-1)}{4}$$

$S_m + (m+1) H_m$

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Buet 2006-2007

2(b)

Using $x^{\overline{m+n}} = x^{\overline{m}}(x-m)^{\overline{n}}$ define $x^{\overline{-n}}$. Show that the following formulas can be used to convert between rising and falling factorial powers, for all integers m:

i) $x^{\overline{m}} = (-1)^m (-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \dfrac{1}{(x-1)^{\underline{-m}}}$

ii) $x^{\underline{m}} = (-1)^m (-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \dfrac{1}{(x+1)^{\overline{-m}}}$

Solution: Proof of 1st part

$$x^{\overline{m}}(x-1)^{\underline{-m}} = x(x+1)(x+2)\cdots(x+m-1) \cdot \frac{1}{x(x+1)(x+2)\cdots(x}$$

$$= 1$$

$$\Rightarrow x^{\overline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$

$$(-1)^m(-x)^{\underline{m}} = (-1)^m (-x)(-x-1)(-x-2)\cdots(-x-m+$$
$$(x+m$$

$$= (-1)^{2m} x(x+1)(x+2)\cdots$$

$$= x^{\overline{m}}$$

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$$(x+m-1)^{\underline{m}} = (x+m-1)(x+m-2)\cdots\cdots(x-2)(x-1)x$$

$$= x(x-1)(x-2)\cdots\cdots(x+m-1)$$

$$= x^{\overline{m}}$$

$$\therefore x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$

Proof of 2nd Part

$$x^{\underline{m}}(x+1)^{\overline{-m}} = x(x-1)(x-2)\cdots(x-m+1)\frac{1}{x(x-1)(x-2)\cdots(x-m+1)}$$

$$= 1$$

$$\therefore x^{\underline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

$$(-1)^m(-x)^{\overline{m}} = (-1)^m(-x)(-x+1)(-x+2)\cdots(-x+m-1)$$

$$= (-1)^{2m}x(x-1)(x-2)\cdots\cdots(x-m+1)$$

$$= x^{\underline{m}}$$

$$(x-m+1)^{\overline{m}} = (x-m+1)(x-m+2)\cdots\cdots(x-m+(m-1))(x-m+m)$$

$$= x(x-1)(x-2)\cdots\cdots(x-m+1)$$

$$= x^{\underline{m}}$$

$$\therefore x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

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Buet 2005-2006

2(a) Average Number of comparision steps made by quick-sort when it is applied to n items in random order satisfies

$$C_0 = 0$$
$$C_n = (n+1) + \frac{2}{n}\sum_{k=0}^{n-1} C_k \quad \text{for } n>0$$

solve this recurrence by using summation factors.

Solution:

1. $C_0 = 0$
2. $C_n = (n+1) + 2/n \sum_{k=0}^{n-1} C_k$

Now

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k \quad \text{------ (1)}$$

$$(n-1)C_n = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

$$= n^2 - 2n + 1 + n - 1 + 2\sum_{k=0}^{n-2} C_k$$

$$\therefore (n-1)C_{n-1} = (n^2 - n) + 2\sum_{k=0}^{n-2} C_k \quad \text{------ (}$$

Date:

subtracting (2) from (1)

$n C_n - (n-1) C_{n-1} = 2n + 2 C_{n-1}$

$\therefore n C_n = (n+1) C_{n-1} + 2n \quad \cdots (3)$

Now summation factor $= \dfrac{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+1)\, n\, (n-1)\cdots 3\cdot 2}$

$\therefore S_n = \dfrac{1}{n(n+1)}$

Multiplying (3) by S_n

$$\frac{C_n}{n+1} = \frac{C_{n-1}}{n} + \frac{2}{n+1}$$

$\therefore S_n = S_{n-1} + a_n$

Now $S_0 = 0$

$S_n = a_{n-1} + a_n$

$$\therefore S_n = \sum_{k=1}^{n} a_n$$

$$= \sum_{k=1}^{n} \frac{2}{k+1}$$

$$= 2 \sum_{k=1}^{n} \frac{1}{k+1}$$

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Buet 05-06

2(b)

Solomon Golomb's "self-describing sequence $(f(1), f(2), f(3), \dots)$ is the only non-decreasing sequence of ~~k~~ ~~for~~ ~~each k~~ positive integers with the property that it contains exactly $f(k)$ occurrences of k for each k. The sequences must begin as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	2	2	3	3	4	4	4	5	5	5	6

Let $g(n)$ be the largest integer m such that $f(m) = n$

Show that $g(n) = \sum_{k=1}^{n} k f(k)$

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Solution:

Given that

$g(n)$ = largest integer m such that $f(m) = n$

$g(n-1)$ = largest integer m' such tha $f(m') = n-1$

So from this definition we see that

$$g(n) - g(n-1) = f(n)$$

$$\Rightarrow g(n) = g(n-1) + f(n)$$

Recurrence solution:

$$g(0) = f(0) = 0$$

$$g(n) = g(n-1) + f(n)$$

The solution of this recurren

$$g(n) = \sum_{k=1}^{n} f(k)$$

Ans

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Date:

Illustration of Problem

এই অংশটা বুঝার জন্য

Not part of solution

$g(4) = ?$

$f(6) = 4$, $f(7) = 4$, $f(8) = 4$

যে Max^m m এর জন্য $f(m) = 4$ হয়

তাই $g(4)$ এর মান।

$g(4) = 8$

$g(4) = \sum_{i=1}^{4} f(4)$

$= f(1) + f(2) + f(3) + f(4)$

$= 1 + 2 + 2 + 3$

$= 8$

$\therefore g(n) = \sum_{K=1}^{n} f(K)$

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Buet 2004-2005

1(a)

Solve the following recurrence relation using repertoire method:

$$R_0 = \alpha$$
$$R_n = R_{n-1} + \beta + \gamma n + \delta n^2$$

Use the solution found for the recurrence above, find a closed form expression for $\sum_{k=0}^{n} k^2$.

solution:

$$R_0 = \alpha \quad \dots (1)$$
$$R_n = R_{n-1} + \beta + \gamma n + \delta n^2 \quad \dots (2)$$

$\longrightarrow$ $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta \quad \dots (3)$

Step 1

$R_n = 1$ হলে

(1) হতে, $\alpha = 1$

(2) হতে, $1 = 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow \gamma n + \beta + \delta n^2 = 0$

$\Rightarrow \beta = 0, \gamma = 0, \delta = 0$

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(3) হতে
$1 = A(n) \times 1 + 0 + 0 + 0$

$\Rightarrow A(n) = 1$

Step 2

Let $R_n = n$

(1) হতে $a = \alpha$

(2) হতে $n = n-1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 1 = \beta + \gamma n + \delta n^2$

$\Rightarrow \beta = \cancel{0}1, \gamma = 0, \delta = 0$

(3) হতে
$n = A(n)\alpha\cancel{0} + B(n)$

$\Rightarrow n = 0 + B(n)$

$\therefore B(n) = (n\cancel{-0}) = n$

Step 3

Let $R_n = n^2$

(1) হতে $a = \alpha$

(2) হতে $n^2 = (n-1)^2 + \beta + \gamma n + \delta n^2$

বা, $n^2 = n^2 - 2n + 1 + \beta + \gamma n + \delta n^2$

বা, $2n - 1 = \beta + \gamma n + \delta n^2$

বা, $\gamma = 2, \delta = 0$

অ, $\beta = -1,$

$\therefore (\alpha, \beta, \gamma, \delta) = (0, -1, 2, 0)$

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(3) হতে,

$$n^2 = -B(n) + 2C(n)$$

$$= -n + 2C(n)$$

বা, $C(n) = \dfrac{n(n+1)}{2}$

Step 4

Let $R_n = n^3$

(1) হতে $0 = \alpha$

(2) হতে $n^3 = (n-1)^3 + \beta + \gamma n + \delta n^2$

$$= (n-1)^3 + \beta + \gamma n + \delta n^2$$

বা, $n^3 = n^3 - 3n^2 + 3n - 1 + \beta + \gamma n + \delta n^2$

বা, $3n^2 - 3n + 1 = \beta + \gamma n + \delta n^2$

বা, $\beta = 1, \gamma = -3, \delta = 3$

Now

$$n^3 = B(n) - 3C(n) + 3D(n)$$

$$\Rightarrow n^3 = n - \tfrac{3}{2} n(n+1) + 3D(n)$$

$$\Rightarrow D(n) = \tfrac{n}{6}(2n^2 + 3n + 1) = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore R_n = \alpha + n\beta + \frac{n(n+1)}{2}\gamma + \frac{n}{6}(2n^2 + 3n + 1)\delta$$

$\boxed{\sum_{k=0}^{n} k^2}$: For this sum, $\alpha = 0$, $\beta = 0$, $\gamma = 0$,

$$R_n = \frac{n(n+1)(2n+1)}{6}$$

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Date:

Buet 2004-2005

2(b)
Using summation by parts find a closed form expression for sum: $\sum_{0 \le k < n} kH_k$

Solution:

$u(x) = H_x$

$\Delta u_x = H_{x+1} - H_x$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x+1}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}\right)$

$= \frac{1}{x+1}$

$= x^{\underline{-1}}$

$\Delta v(x) = x = x^{\underline{1}}$

$v(x) = \frac{1}{2} x^{\underline{2}}$

$= \frac{1}{2} x(x+1)$

$Ev = v(x+1)$

$= \frac{1}{2} x(x+1) = \frac{1}{2}(x+1)^{\underline{2}}$

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$\cancel{\sum k H_k}$

We know

$$\sum u \Delta v = uv - \sum E v \Delta u$$

$$\therefore \sum k H_k = \frac{1}{2} k^2 H_k - \sum \frac{1}{2} (k+1)^{\underline{2}} k^{\underline{-1}}$$

$$= \frac{1}{2} k^2 H_k - \frac{1}{2} \sum k^{\underline{1}}$$

$$= \frac{1}{2} k^2 H_k - \frac{1}{2} \times \frac{1}{2} k^{\underline{2}}$$

$$= \frac{1}{2} k^2 H_k - \frac{k^{\underline{2}}}{4} + c$$

$$\therefore \sum_{0 \le k < n} k H_k = \frac{1}{2} k^2 H_k - \frac{k^{\underline{2}}}{4} \Bigg|_0^n$$

$$= \left(\frac{1}{2} n^2 H_n - \frac{n^{\underline{2}}}{4} \right) - 0$$

$$= \frac{1}{2} n^2 \left(H_n - \frac{1}{2} \right)$$

Ans

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3(c)

Use summation factor to solve recurrence:

$$T_0 = 5$$

$$2T_n = nT_{n-1} + 3n! \quad \cdots (2)$$

solution:

$a_n = 2, \quad b_n = n$

$$s_n = \frac{a_{n-1}\, a_{n-2}\, a_{n-3} \cdots a_2 a_1}{b_n\, b_{n-1}\, b_{n-2} \quad b_3 b_2}$$

$$= \frac{2\cdot2\cdot2 \cdots\cdots 2\cdot2}{n(n-1)(n-2) \cdots 3\cdot2\cdot1}$$

$$= \frac{2^{n-1}}{n!}$$

By Multiplying eqn (2) by s_n

$$\frac{2^n}{n!} T_n = \frac{2^{n-1}}{(n-1)!} T_{n-1} + 3\cdot2^{n-1}$$

$$S_n = S_{n-1} + 3\cdot2^{n-1}$$

$$S_0 = 5$$

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Date:

Now

$s_0 = 5$

$s_n = s_{n-1} + 3 \cdot 2^{n-1}$

Solution of recurrence:

$$s_n = 5 + \sum_{i=1}^{n} 3 \cdot 2^{n-1}$$

$$= 5 + 3 \sum_{i=1}^{n} 2^{n-1}$$

$$= 5 + 3(2^n - 1)$$

$$= 3 \cdot 2^n + 2$$

or, $\frac{2^n}{n!} T_n = 3 \cdot 2^n + 2$

$\therefore T_n = 3n! + \frac{n!}{2^{n-1}}$

[Ans]

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Date:

Buet 2004-2005

4(b)

Consider following recurrence relation:

$$T_0 = 0$$

$$T_1 = 1$$

$$T_n = 5T_{n-1} - 6T_{n-2} + 6$$

Find a closed form generating function for sequence $\langle T_0, T_1, T_2, \ldots \rangle$ From it find a closed form expression for T_n.

Solution

syllabus বহির্ভূত

7th chapter এ আছে।

Buet 2003-2004

1(a)

Solve the following recurrence relation using repertoire method:

$R_0 = \alpha$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

From general solution find

$T_n = \sum_{k=0}^{n} (-1)^n (2k-3)$

Using the ~~cto~~ result found for T_n, find a closed form for $S_n = \sum_{k=0}^{n} (-1)^k (k+1)^2$ by perturbation method.

Solution:

1. $R_0 = \alpha$ ...(1)
2. $R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$ ....(2)

Step 1:

~~$R_n = 1 \Rightarrow \alpha = 1$~~

(2) $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$

Step 1

Let $R_n = 1 \Rightarrow \alpha = 1$

② হতে
$1 = 1 + (-1)^n (\beta + \gamma n + \delta n^2)$

$\Rightarrow \beta = 0, \gamma = 0, \delta = 0$

(3) হতে $1 = A(n)$

Step 2

Let $R_n = n \Rightarrow \alpha = 0$

② হতে, $n = (n-1) + (-1)^n (\beta + \gamma n + \delta n^2)$

$\Rightarrow 1 = (-1)^n (\beta + \gamma n + \delta n^2)$

$\Rightarrow (-1)^n = \beta + \gamma n + \delta n^2$

$\Rightarrow \beta = (-1)^n, \gamma = 0, \delta = 0$

③ হতে,
$n = (-1)^n B(n) \Rightarrow B(n) = (-1)^n n$

Step 3:

Let $R_n = n^2 \Rightarrow \alpha = 0$

(2) হতে, $n^2 = (n-1)^2 + (-1)^n (\beta + \gamma n + \delta n^2)$

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$\Rightarrow (2n-1) = (-1)^n (\beta + \gamma n + \delta n^2)$

$\Rightarrow (-1)^n (2n-1) = \beta + \gamma n + \delta n^2$

$\Rightarrow \beta = \cancel{1}-(-1)^n, \quad \gamma = 2(-1)^n, \quad \delta = 0$

(3) হতে পাই,

$$n^2 = -\beta(n)\,(-1)^n + 2(-1)^n c(n)$$
$$= -(-1)^n n\,(-1)^n + 2(-1)^n c(n)$$
$$= -n + 2(-1)^n c(n)$$

$$\therefore c(n) = \frac{(-1)^n}{2}\{n(n+1)\}$$

Step 4:

Let $R_n = n^3 \Rightarrow \alpha = 0$

(2) হতে পাই,

$$n^3 = (n-1)^3 + (-1)^n (\beta + \gamma n + \delta n^2)$$

বা, $3n^2 - 3n + 1 = (-1)^n (\beta + \gamma n + \delta n^2)$

বা, $(3n^2 - 3n + 1)(-1)^n = \beta + \gamma n + \delta n^2$

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Date:

~~$\Rightarrow \beta = (-1)$~~

$\Rightarrow \beta = (+1)(-1)^n, \quad \gamma = -3(-1)^n, \quad \gamma = 3(-1)^n$

so from (3)

$$n^3 = B(n)\,(-1)^n - 3C(n)(-1)^n + 3(-1)^n D_n$$

$$= (-1)^n \left(B(n) - 3C(n) + 3D(n) \right)$$

$$= (-1)^n \left\{ (-1)^n n - 3\frac{(-1)^n n(n+1)}{2} + 3D(n) \right\}$$

$$\Rightarrow D(n) = (-1)^n \frac{n(n+1)(2n+1)}{6}$$

~~$\therefore R_n = (-1)^n \{ \alpha + n\beta + \frac{n}{} \}$~~

~~$\therefore R_n = \alpha + (-1)^n \{ 1 + n\beta + \frac{n(n+1)(2n+1)}{} \}$~~

$$R_n = \alpha + (-1)^n \left\{ \beta + \frac{n(n+1)}{2}\gamma + \frac{n(n+1)(2n+1)}{6}\delta \right\}$$

$\boxed{T_n = \sum_{k=0}^{n} (-1)^n (2k-3)}$: $\gamma = 2, \ \delta = 0, \ \beta = -3, \ \alpha = -3$

$$\therefore T_n = (-3) + (-1)^n \left\{ (-3) + \frac{n(n+1)}{2} 2 \right\}$$

$$= (-3) + (-1)^n (n^2 + n - 3)$$

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Buet 2003-2004

1(b)

Find the law of expressions for falling rising factorial power. From the la[w] define $x^{-\overline{n}}$ and $x^{\underline{-n}}$

Prove the following relations:

$$x^{\overline{m}} = (-1)^m (-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$

Solution:

$$x^{\overline{m}} (x-1)^{\underline{-m}} = x(x+1)(x+2)\cdots(x+m-1) \times \frac{1}{x(x+1)(x+2)\cdots}$$

$$= 1$$

$$\therefore x^{\overline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$

$$(-1)^m (-x)^{\underline{m}} = (-1)^m (-x)(-x-1)(-x-2)\cdots(-x-m\ldots$$

$$= (-1)^{2m} x(x+1)(x+2)(x+3)\cdots(x\ldots$$

$$= x^{\overline{m}}$$

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$(x+m-1)^{(m)} = (x+m-1)(x+m-2)\cdots\cdots(x-2)(x-1)x$

$= x(x-1)(x-2)\cdots\cdots(x+m-2)(x+m-1)$

$= \dfrac{1}{(x-1)^{(m)}}$

[showed]

Date:

Buet 2003-2004

2(c)

Use summation by parts to find the value

$\sum_{0\le k<n} H_k$, [H_k = kth Harmonic number]

solution:

$u(x) = H_x \qquad \Delta u(x) = x^{\underline{-1}}$

$v(x) = x$

$\Delta v(x) = 1$

$E(v) = x+1$

we know

$\cancel{\sum k H_k}$ $\sum u \Delta v = uv - \sum E(v)\Delta u$

$\therefore \sum H_x = xH_x - \sum (x+1) x^{\underline{-1}}$

$= xH_x - \sum (x+1)^{\underline{1}} x^{\underline{-1}}$

$= xH_x - \sum x^{\underline{0}}$

$= xH_x - \sum 1$

$= xH_x - x$

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Date:

Buet 2003-2004

4(a)

Consider the following recurrence relation

$$T_n = \begin{cases} 1 & n=0 \\ 2 & n=1 \\ 2T_{n-1}+3T_{n-2} & n\geq 2 \end{cases}$$

syllabus [illegible] | 7th chap [illegible]

Let $f(x)$ be generating function for the sequence $\langle T_0, T_1, T_2, T_3, \dots \rangle$

i) Give a generating function in terms of $f(x)$ for sequence $\langle 1, 2, 2T_1+3T_0, 2T_2+3T_1, 2T_3+3T_2, \dots \rangle$

2) Use result found in (i) to find a closed form generating function for $f(x)$.

3) Expand closed form for $f(x)$ using partial fraction and find a closed form ~~that~~ for T_n.

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Date:

Buet 2002-2003

2(a)

Prove that $x^{\underline{m}}/(x-n)^{\underline{m}} = x^{\underline{n}}/(x-m)^{\underline{n}}$

Solution:

$$x^{\underline{m}}(x-m)^{\underline{n}} = x^{\underline{m+n}}$$

$$= x^{\underline{n}}(x-n)^{\underline{m}}$$

$$\therefore \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

[Proved]

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2(b)

Use a summation factor to solve the recurrence:

$$T_0 = 5$$

$$2T_n = nT_{n-1} + 3n!$$

Solution:

$$T_0 = 0 \quad \dots (1)$$

$$2T_n = nT_{n-1} + 3n! \quad \dots (2)$$

$$s_n = \frac{2 \cdot 2 \cdots (n-1) \text{ সংখ্যক } 2}{n(n-1)(n-2)\cdots 3 \cdot 2}$$

$$= \frac{2^{n-1}}{n!}$$

Multiplying (2) by s_n

$$\frac{2^n}{n!} T_n = \frac{2^{n-1}}{(n-1)!} T_{n-1} + 3 \frac{2^{n-1}}{\cancel{(n-1)!}\, 1}$$

$$\Rightarrow S_n = S_{n-1} + 3 \frac{2^{n-1}}{\cancel{(n-1)!}\, 1} = S_{n-1} + 3 \cdot 2^{n-1}$$

$$S_0 = 5$$

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This reduces to

$$s_n = \sum_{k=0}^{n} 3 \cdot 2^{n-1}$$

$$= 3 \sum_{k=0}^{n} 2^{n-1}$$

$$= 3 \frac{\frac{1}{2}(1-2^n)}{1-2}$$

$$= \frac{3}{2}(2^n - 1)$$

$$\Rightarrow \frac{2^n}{n!} T_n = \frac{3}{2}(2^n - 1)$$

$$\therefore T_n = \frac{n!}{2^n} \frac{3}{2}(2^n - 1)$$

$$\therefore T_n = \frac{3}{2} n! \left(1 - \frac{1}{2^n}\right)$$

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Date:

Chapter 4

Sol^n Of Questions Of Prev Years

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Date:

Buet - 2009-2010

5. (b) Suppose $\mathcal{E}_n(m!)$ denotes the maximum power of n (say x) such that n^x devides m!. Find $\mathcal{E}_2(100!)$ and $\mathcal{E}_8(100!)$

Solution:

$$\mathcal{E}_2 = \left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{2^2}\right\rfloor + \left\lfloor\frac{100}{2^3}\right\rfloor + \left\lfloor\frac{100}{2^4}\right\rfloor + \left\lfloor\frac{100}{2^5}\right\rfloor + \left\lfloor\frac{100}{2^6}\right\rfloor$$

$$= 50 + 25 + 12 + 6 + 3 + 1$$

$$= 97$$

$$\mathcal{E}_8(100!) = \mathcal{E}_{2^3}(100!)$$

$$= \left\lfloor\frac{97}{3}\right\rfloor$$

$$= 32$$

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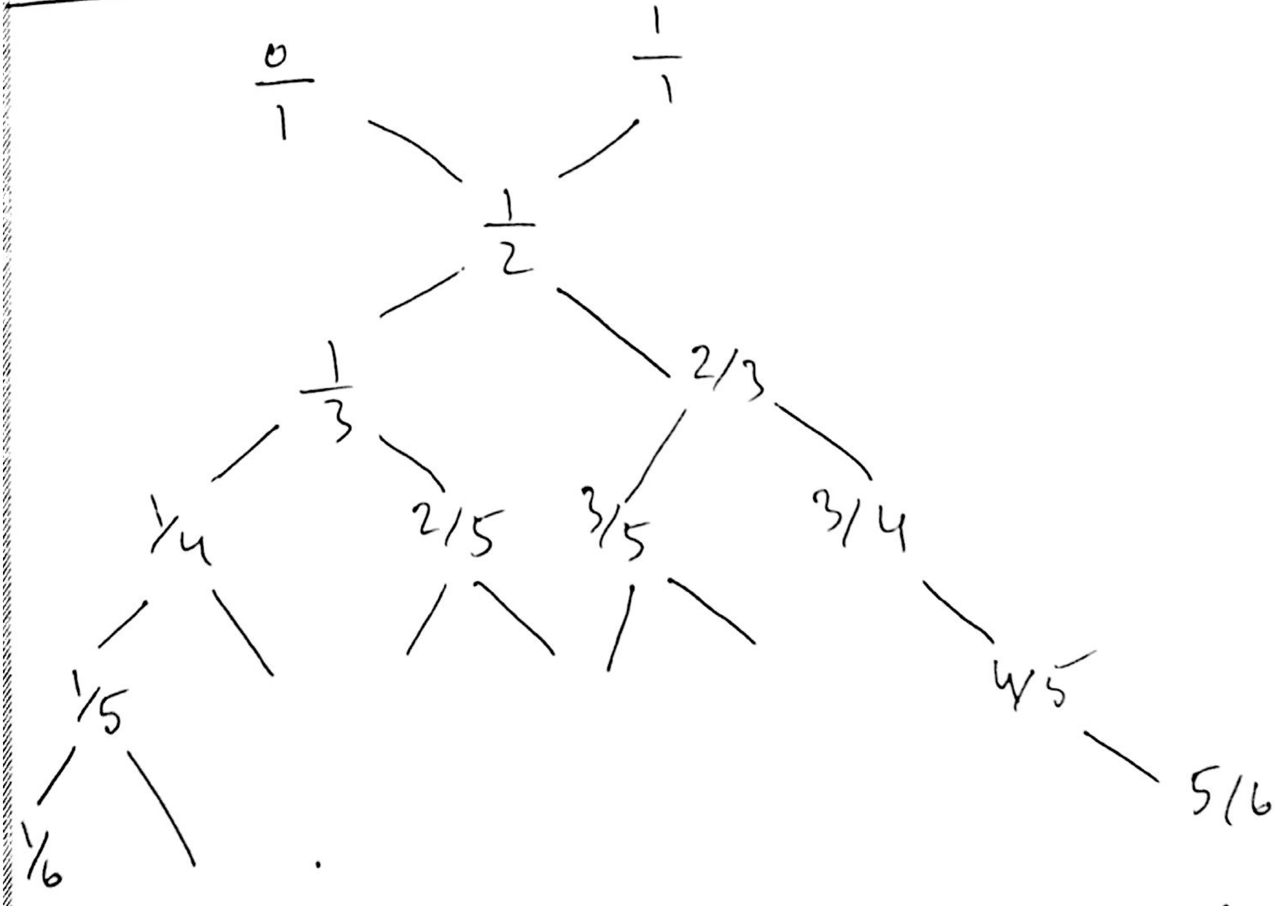
Buet 2009-2010

5(c):

Farey series of order n is the seq of completely reduced fraction between 0 and 1 where numerator and denominator are atmost n. ~~For~~ Find farey series of order 6.

solution:



$\therefore$ Farey series $= \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \right.$

$\left. 3/4, 4/5, 5/6, \frac{1}{1} \right\}$

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Date:

Buet 08-09

5(c)
What is the largest power of 5 that devides 100! ?

Solution:

$$E_p(100!) = \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{5^2} \right\rfloor + \left\lfloor \frac{100}{5^3} \right\rfloor$$

$$= 20 + 4 + 0$$

$$= 24$$

et 08-09

7(b)
Prove that if $a \perp b$ and $a > b$ then $\gcd(a^m - b^m, a^n - b^n) = a^{\gcd(m,n)} - b^{\gcd(m,n)}$

$0 \leq m < n$

Here all variables are integer and $\perp$ denotes relatively prime.

Solution:

Let $d \mid (a^n - b^n)$ and $d \mid (a^m - b^m)$

then $\gcd(a^n - b^n, a^m - b^m) = d$

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we know from euclid ~~po~~ theorm

$$mx+ny = gcd(m,n)$$

so from this we know

$$a^m \equiv b^m \pmod d$$

$$a^n \equiv b^n \pmod d$$

so we now write

$$a^{gcd(m,n)} = (a^m)^x (a^n)^y \pmod d$$

$$\equiv (b^m)^x (b^n)^y \pmod d$$

$$= b^{gcd(m,n)} \pmod d$$

$$\Rightarrow a^{gcd(m,n)} - b^{gcd(m,n)} \equiv 0 \pmod d$$

$\therefore$ ~~d~~

$$\therefore gcd(a^m - b^n, a^m - b^m) = a^{gcd(m,n)} - b^{gcd}$$

[Proved

Alternate Solution:

Date:

Buet 2007-2008

7) Give recursive definition of Euclid's numbers. Give an example which shows that the euclids numbers are not always prime. Proe that two euclid numbers are relatively prime.

From this prove, there are infinite prime.

solution:

defn:

The recursive defn of euclids numbers can be given as:

$$e_1 = 2$$

$$e_n = e_1 e_2 e_3 \cdots e_{n-1} + 1$$

Example

$$e_5 = 2 \cdot 3 \cdot 7 \cdot 43 + 1$$

$$= 1807$$

$$= 13 \cdot 139$$

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To prove this we need to prove $\gcd(e_m, e_n) = 1$ when $m \neq n$

Proof:

From euclid algo we know $e_m \bmod e_n = 1$

we also know that when $n > m$

$\cancel{\gcd(e_m, e_n)} = \gcd(m, n) = \gcd(n \bmod m, m)$

$= \gcd(e_m \bmod e_n, e_m)$

Now $\gcd(e_m, e_n) = \gcd(e_m \bmod e_n, e_m)$

$= \gcd(1, e_m) = \gcd(e_m \bmod 1,$

$= \gcd(0, 1)$

$\therefore \gcd(e_m, e_n) = 1$ [Proved]

Prime Numbers are infinite:

Let us suppose there are finitely many primes. k of them are $2, 3, 5, \ldots P_k$.

Then according to euclid number

$M = 2 \cdot 3 \cdot 5 \cdots P_k + 1$

None of k primes can devide M beacause each devides $(M-1)$. Thus there must be some prime that devides M perhaps M itself

$\therefore$ M is a primes.

$\therefore$ This contradicts our assumption [Proved]

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6(c)

Let $f_n = 2^{2^n}+1$. Prove that $f_m \perp f_n$ if $m<n$. Here $\perp$ denotes relatively prime.

Solution:

Here

$f_n = f_0 f_1 f_2 \cdots f_{n-1} + 2$

$= 2^{2^n}+1$

$f_m \bmod f_n = 2$

$\gcd(f_n, f_m) = \gcd(f_m \bmod f_n, f_n)$

$= \gcd(2, f_n)$

$= \gcd(f_n \bmod 2, 2)$

$= \gcd(0,2) = 1$

$\therefore f_n \perp f_m$

[proved]

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Buet 2007-2008

7(b) Explain that the largest power of p that devides n! is

$\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$

From this equation find the largest power of 3 that devides 100!.

solution:

n/p marks the numbers that contribute to calculate $\varepsilon_p(n!)$ to a power of

n/p^2 marks the numbers that [illegible] to calculate $\varepsilon_p(n!)$ [illegible]

This goes on [illegible]

$\varepsilon_p(n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$

Calculations:

[illegible]

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Buet 2005-2006

3(a). calculate multiplicity of 360 in 1000!.

solution:

$$360 = 2^3 \times 3^2 \times 5^1$$

$$\varepsilon_2(1000!) = \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{2^2} \right\rfloor + \left\lfloor \frac{1000}{2^3} \right\rfloor + \left\lfloor \frac{1000}{2^4} \right\rfloor$$

$$+ \left\lfloor \frac{1000}{2^5} \right\rfloor + \left\lfloor \frac{1000}{2^6} \right\rfloor + \left\lfloor \frac{1000}{2^7} \right\rfloor + \left\lfloor \frac{1000}{2^8} \right\rfloor$$

$$+ \left\lfloor \frac{1000}{2^9} \right\rfloor + \left\lfloor \frac{1000}{2^{10}} \right\rfloor$$

$$= 500 + 250 + 125 + 62 + 31 + 15$$

$$+ 7 + 3 + 1$$

$$= 994$$

$$\varepsilon_{2^3}(1000!) = \left\lfloor \frac{994}{3} \right\rfloor = 331$$

$$\varepsilon_{3}(1000!) = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{3^2} \right\rfloor + \left\lfloor \frac{1000}{3^3} \right\rfloor$$

$$+ \left\lfloor \frac{1000}{3^4} \right\rfloor + \left\lfloor \frac{1000}{3^5} \right\rfloor + \left\lfloor \frac{1000}{3^6} \right\rfloor$$

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Date:

$$= 333 + 111 + 37 + 12 + 4 + 1$$

$$= 498$$

$$\varepsilon_9(1000!) = \left\lfloor \frac{498}{2} \right\rfloor = 249$$

$$\varepsilon_5(1000!) = \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{5^2} \right\rfloor + \left\lfloor \frac{1000}{5^3} \right\rfloor + \left\lfloor \frac{1000}{5^4} \right\rfloor$$

$$= 200 + 40 + 8 + 1$$

$$= 249$$

$$\varepsilon_{360}(1000!) = \min(331, 249, 249)$$

$$= 249$$

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Bnet 2004-2005

3(b)

Let m and n be two positive integer. How could you find the largest power of m that devides n! ?

Solution:

- First m has to be factorized.

$$m = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_{k'}^{k_{k'}}$$

- find $\mathcal{E}_{p_i}(n!)$ for $i = 1$ to k' using the formula

$$\mathcal{E}_p(n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$$

- Let the values are $m_1, m_2, \cdots m_{k'}$

- $\mathcal{E}_M(n!) = \min(m_1, m_2, \cdots m_{k'})$

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Date:

Buet 2003-2004

2(d)

Answer the following questions with regard to Stern-Brocot tree:

i) Why does each fraction turn out to be in lowest terms when it appears in tree.

ii) Why do all possible fractions occur exactly once?

iii) Why can't a particular fraction occur twice or more or not at all?

solution:

Proof of part i

if m/n and m'/n' are consecutive fractions at any stage of constructions

$$m'n - mn' = 1 \quad \text{----}(1)$$

If we insert a new mediant $\frac{m+m'}{n+n'}$ cases that need to be checked

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if $m/n < m'/n'$ and if all values ~~that~~ are non-negative, it's easy to verify that

$$m/n < (m+m')/(n+n') < m'/n'$$

A mediant fraction isn't halfway between its progenitors, but it does lies somewhere in between.

Therefore the construction preserves order.

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Proof of part (iii)

As from part (ii), the fraction in stern-brocot tree preserves order so we ~~cannot~~ couldn't possibly get the same fraction in two different places.

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Date:

Buet 2003-2004

3(a)

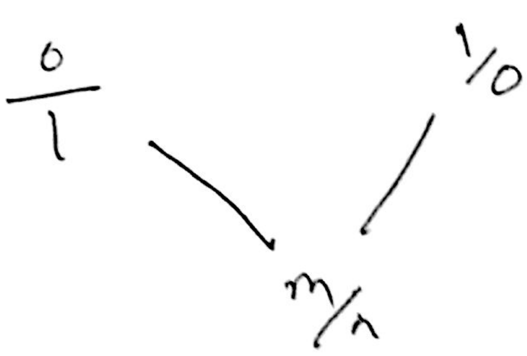
Prove that in stern-Brocot tree for any fraction m/n, two fractions $\frac{m}{m+n}$ and $\frac{m+n}{n}$ are located just one row below that of $\frac{m}{n}$.

Solutions:

\# To build a stern-Brocot tree we follow the following principle:

\# "Insert $\frac{m+m'}{n+n'}$ between two adjacent fractions $\frac{m}{n}$ and $\frac{m'}{n'}$"

\# Let at some point the picture of tree is as follows:



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Date:

Then we will contue to build up the tree by inserting $\frac{m+0}{n+1}$ between $\frac{0}{1}$ and m/n.

and by inserting $\frac{m+1}{n+0}$ between m/n and $1/0$

∴ For any fractions, two fractions $\frac{m}{m+n}$ and $\frac{m+n}{n}$ are located just ~~below~~ one row below that of m/n.

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Date:

Buet 2003-2004

3(b)

We say that simplicity of an irreducible fraction $\frac{m}{n}$ is the number $w\frac{m}{n} = \frac{1}{mn}$ using this fact stated in Q. 3(a). prove that the sum of all fractions in any row of the stern-Brocot tree equals ....

question is incomplete.

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Date

3 (c) — Buet 2003-2004

Express $\epsilon_p(n!)$ in terms of v_p the sum of the digits in the radix prepresen- tation of n

Solution:

:: $\epsilon_p(n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \lfloor n/p^4 \rfloor + \cdots + \lfloor n/p^{a-1} \rfloor$

:: Let n =

Date:

Buet 2003-2004

3(d)

Using the properties of farey series prove that whenever $m \perp n$ we can find integers a and b such that

(Marks- 06)

$ma - nb = 1$

Solution:

farey properties:

If m/n and m'/n' are consecutive fractions at any stage of construction we have

$$m'n - mn' = 1$$

Proof:

Let b/a be the fraction that preceeds m/n in F_n.

Hence according to this property

$$ma - nb = 1$$

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Date:

Buet 2002-2003

3(a)

Let $E_m(n!)$ be multiplicity of m in the $n!$. find its value when $m = 675$ and $n = 1,00,000$

Solution:

$$\begin{array}{r|l} 5 & 675 \\ \hline 5 & 115 \\ \hline & 23 \end{array}$$

$$\begin{array}{r|l} 5 & 100,000 \\ \hline 5 & 20,000 \\ \hline 5 & 4000 \\ \hline 5 & 800 \\ \hline 5 & 160 \\ \hline 5 & 32 \\ \hline 5 & 6 \\ \hline & 1 \end{array}$$

$$\begin{array}{r|l} 23 & 100,000 \\ \hline 23 & 4347 \\ \hline 23 & 189 \\ \hline & 8 \end{array}$$

* প্রত্যেক বার devision করা হচ্ছে

No of 5 in $(100,000)!$ is $= 20,000 + 4000 + 800 + 160 + 32 + 7$

$= 24,999$

No of 23 in $(100000)!$ is $= 4347 + 189 + 8$

$= 4544$

No of 25 in $(100,000)!$ is $= 12499$

No of 675 in $(100,000)!$ is $= \min(12499, 4544)$

$= 4544$

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Date:

Buet 2008-2009

8(b) Any positive integer n can be written as a product of primes, $n = \prod_{k=1}^{n} P_k$, $P_1 \le P_2 \le \cdots \le P_m$

show that expansion of n in such a form is unique.

Solution:

Base: $n=1$. This prove the statement

Now let $n > 1$ and that all smaller numbers factor uniquely.

suppose we have two factorization;

$$n = P_1 P_2 \cdots P_m \qquad P_1 \le P_2 \le \cdots \le P_m$$

$$= q_1 q_2 \cdots q_k \qquad q_1 \le q_2 \le \cdots q_k$$

Here P's and q's are all prime. We will prove that $P_1 = q_1$

If not we can assume that $P_1 < q_1$ making P_1 smaller than all of q's.

Since P_1 and q_1 are prime from euclids algo

$$aP_1 + bq_1 = \gcd(P_1, q_1) = 1$$

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~~$\Rightarrow a p_1 p_2 p_3$~~

$\Rightarrow a p_1 q_2 q_3 \cdots q_k + b q_1 q_2 \cdots q_k = q_2 q_3 \cdots q_k$

Now p_1 devides both terms on the left

since $q_1 q_2 q_3 \cdots q_k = n$

Hence p_1 devides the right hand side.

Thus $\dfrac{q_2 q_3 \cdots q_n}{p_1}$ is an integer.

$q_2 q_3 \cdots q_k$ has a prime factorization in

which p_1 occurs

But $q_2 q_3 \cdots q_k < n$, so it has a unique

factorization.

This contradiction shows that p_1 must be

equal to q_1

$\therefore$ The other factors must likewise be

equal.

n has a unique prime factorization.

(showed)

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INTEGER FUNCTION

Solutions of Questions of Pre

Sultan Ahmed Sa
CSE, L-3, T-2, 080
Buet Sultan

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Date:

Buet 2009–2010

6(a)
Let the set spectrum (spec) and overspectrum (ospec) be defined as follows:

$$\operatorname{spec}(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \lfloor 4\alpha \rfloor, \cdots\}$$

$$\operatorname{ospec}(\alpha) = \{\lceil \alpha \rceil, \lceil 2\alpha \rceil, \lceil 3\alpha \rceil, \lceil 4\alpha \rceil, \cdots\}$$

Find out the 1st 3 terms of $\operatorname{spec}(\pi)$, $\operatorname{spec}(-\pi)$, $\operatorname{ospec}(\pi)$, $\operatorname{ospec}(-\pi)$

Solutions:

~~Sol~~^{n} Excercise গুলোর sol^n এ এই প্রশ্নের sol^n দিয়েছেন।

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Date:

Buet CSE - (08-09)

6(b)

Let α and β be positive real numbers. Prove that spec(α) and spec(β) partition the positive integers if and only if α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ (Marks - 14)

solution:

Excercise এর sol^n এগুলোর মধ্যে আছে।

[illegible])

Prove that $\left[\frac{n}{m}\right] = \left[\frac{n+m-1}{m}\right]$ for all integers n and all positive integers m.

(Marks-8)

Solution:

[illegible]

6(a)

Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumtances does each case arise?

(Marks-15)

Solution:

Excercise এর sol^n গুলোর মধ্যে আছে।

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Q 7(c)

Prove or disprove

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor$$

Sol^n:

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1(c)

Define the spectrum of a real number. Prove that spec($\sqrt{2}$) and spec($2+\sqrt{2}$) are disjoint and partitions of set of integers.

(Marks 15)

Solution:

Spectrum:

The spectrum of a real number α to be an infinite multi-set of integers such that

$$spec(\alpha) = \{ \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \cdots \}$$

Proof of 2nd Part:

To prove this assertion, we will count how many of the elements of Spec($\sqrt{2}$) are $\leq n$

And how many elements of Spec($2+\sqrt{2}$) $\leq$

Date:

If the total is n, for each n, these two spectra do indeed partition the integers.

Let α be positive. The no. of elements in $Spec(\alpha)$ that are $\leq n$ is

$$N(\alpha, n) = \sum_{k>0} [\lfloor k\alpha \rfloor \leq n]$$

$$= \sum_{k>0} [\lfloor k\alpha \rfloor < n+1]$$

$$= \sum_{k>0} [k\alpha < n+1]$$

$$= \sum_{k>0} [0 < k < (n+1)/\alpha]$$

$$= \lceil (n+1)/\alpha \rceil - 1$$

$$N(\sqrt{2}, n) + N(2+\sqrt{2}, n) = \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1$$

$$\Leftrightarrow \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor$$

$$\Leftrightarrow \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\}$$

$$\Leftrightarrow n$$

Hence $\cancel{N}$ spec $(\sqrt{2}, N)$ & $Spec(2+\sqrt{2}, N)$ partition set of integ

2(c)

The Buet Gamers Association has a roulette wheel with two thousands slots numbered 1 to 2000.

If the number n that comes up a spin is divisible by twice the floor of its cubic root that is $2\lfloor\sqrt[3]{n}\rfloor$ then its a winner. And the association pays Tk w.

Otherwise its looser and we must pay taka 1. Find the largest integer w for which we can still make money by playing the game.

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Date:

solution:

No. of winner $= w$

No. of looser $L = 2000 - w$

$$w = \sum_{n} [n \text{ is winner}]$$

$$= \sum_{n=1}^{1000} \left[2\lfloor \sqrt[3]{n} \rfloor \backslash n \right]$$

$$= 2 \sum_{k,m,n} \left[k = \lfloor \sqrt[3]{n} \rfloor \right] [n = km] [1 \le k \le 2000]$$

$$= 2 \sum_{k,m,n} \left[k^3 \le n < (k+1)^3 \right] [n = km] [1 \le k \le 2000]$$

$$= 2 \sum_{k,m,n} \left[k^3 \le km < (k+1)^3 \right] [1 \le k \le 11]$$

$$= 2 + 2 \sum_{k,m} \left[m \in [k^2 \ldots (k+1)^3/k) \right] [1 \le k \le 11]$$

$$= 2 + 2 \sum_{k,m} \left(\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil \right)$$

$$= 2 + 2 \sum_{k=1}^{11}$$

$$= 2 + 2 \sum_{k=1}^{11} (3k + 4)$$

$\cancel{2}$

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$= 2 + 2 \times \frac{3}{2} \times 11 \times 12 + 4 \times 11$

$= 2 + 396 + 44$

$\therefore W = 442$

But average winning money $= \frac{5W-2\ldots}{200\ldots}$

$= \frac{6W-2000}{2000}$

So $6W - 2000 = 0$

$\Rightarrow W = \lceil 2000/6 \rceil$

$= 333$

So we can expect to make money by playing this game.

End

play.

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Date:

Buet CSE 2004-2005

1(b)

Find out the number of integers the interval $[\alpha \cdots \beta]$ contains where α and β are arbitary real numbers.

Use this result to find the sum of all multiplies of x in that interval when $x>0$.

সমাধান

$x \le n$

$\Rightarrow \lceil x \rceil \le n$

$x \geq n$

$\Rightarrow \lfloor x \rfloor \geq n$

solution:

$[\alpha, \cdots; \beta]$:

$\alpha \le n \le \beta$

$\Rightarrow \lceil \alpha \rceil \le n \le \lfloor \beta \rfloor$

no. of integers

$= (\lfloor \beta \rfloor - \lceil \alpha \rceil + 1)$

Buet CSE 2004-2005

Date:

2(a)

Prove the following rule: (Marks-5)

$x \le n \iff \lceil x \rceil \le n$ integer n

solution:

$x \le n$

since $\lceil x \rceil \ge x$

$\therefore \lceil x \rceil \le n$

Again $\lceil x \rceil \le n$

$\Rightarrow x \le n$

$\therefore$ ~~$\lceil x \rceil$~~

$\therefore x \le n \iff \lceil x \rceil . \le n$

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Buet CSE 2003-2004

1(c)

Prove that $\lceil \log(n+1) \rceil$ bits are requir[ed]
to write n in binary.

(Marks-03)

Solution:

It takes m bits to write each num[ber]
such that

$$2^{m-1}-1 < n \leq 2^{m}-1$$

$$\Rightarrow 2^{m-1} < n+1 \leq 2^{m}$$

$$\Rightarrow (m-1) < \lg(n+1) \leq m$$

$$\therefore \lceil \lg(n+1) \rceil = m$$

$\therefore \lceil \lg(n+1) \rceil$ bits are required to wr[ite]
n in binary.

[Proved]

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Date:

Buet CSE 2003-2004

2(a)

Let f(x) be any continuous, monotonically increasing function with the property that

$$f(x) = \text{integer} \Rightarrow x = \text{integer}.$$

Then prove that $\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$ whenever $f(x)$, $f(\lfloor x \rfloor)$ are defined. (Marks-8)

Solution:

1. if $x = \lfloor x \rfloor$ then ~~we~~ the proof is done
2. otherwise let, $x > \lfloor x \rfloor$

$\Rightarrow f(x) > f(\lfloor x \rfloor)$

$\Rightarrow \lfloor f(x) \rfloor > \lfloor f(\lfloor x \rfloor) \rfloor$ $\left[\text{let } f(y) = \lceil f(x) \rceil \right]$

$\Rightarrow f(y) > f(\lfloor x \rfloor)$

3. Here $\cancel{x \leq y < \lceil x \rceil}$ $\lfloor x \rfloor < y \leq x$ and y is a integer
4. There is no integer between $\lfloor x \rfloor$ and x
5. Hence $x = \lfloor x \rfloor$
6. ~~Here~~ $\therefore \lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$

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Buet CSE 2003-2004:

2(b)

Find the no. of integers the open inte

$(\alpha \cdots \beta)$ contains.

Use your result to find the sum of multiples of x in that interval.

(Marks - 6)

Solution:

$\Rightarrow (\alpha, \cdots, \beta)$

$\Rightarrow \alpha < n < \beta$

$\Rightarrow \lfloor \alpha \rfloor < n < \lceil \beta \rceil$

So (α, β) contains exactly $(\lceil \beta \rceil - \lfloor \alpha \rfloor - 1)$ integers

$$\sum_{k=\lfloor \alpha \rfloor}^{\lceil \beta \rceil}$$

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Date:

Buet CSE 2002-2003

2(c)

The concrete math club has a casino in which there is a roulette wheel with one thousands slot numbered 1 to 1000.

If a number n that comes up on a spin is divisible by the floor of its cubic root then it is winner and the house pays us \$5.

otherwise its a looser and we must pay \$1. can we expect to make money if we play this game?

Solution:

No. of winner = W

No. of looser L = 1000 - W

$$\text{Average Winning money} = \frac{5W - L}{1000} = \frac{6W - 1000}{1000}$$

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Date:

So we will play this game if

$6W - 1000 = 0 \Rightarrow W = \lceil 1000/6 \rceil = 67$

$$W = \sum_{n=1}^{1000} [n \text{ is a winner}]$$

$$= \sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \backslash n \right]$$

$$= \sum_{k,n,m} [k = \lfloor \sqrt[3]{n} \rfloor][n = km][1 \le n \le 1000]$$

$$= \sum_{k,n,m} [k \le \sqrt[3]{n} < k+1][n = km][1 \le n \le 1000]$$

$$= \sum_{k,n,m} [k^3 \le mk < (k+1)^3][1 \le k < 10]$$

$$= 1 + \sum_{k,m} [m \in [k^2, \dots (k+1)^3/k)][1 \le k \le 10]$$

$$= 1 + \sum_{k,m} [\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil]$$

$$= 1 + \sum_{k=1}^{10} (3k + 4)$$

$$= 1 + \frac{3}{2} \times 9 \times 10 + 4 \times 9$$

$$= 1 + 135 + 36$$

$$= 172$$

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Buet (2008-2009)

Prove that if $a \perp b$ and $a > b$

then
$$\gcd(a^m - b^m, a^n - b^n) = a^{\gcd(m,n)} - b^{\gcd(m,n)}$$

Solution:

$$a^n - b^n = (a^m - b^m)\left(a^{n-m} b^0 + a^{n-2m} b^m + \cdots + a^{n \bmod m} b^{n-m-n \bmod m}\right) + b^{m\lfloor n/m \rfloor}\left(a^{n \bmod m} - b^{n \bmod m}\right)$$

$\Rightarrow$ a from euclid ~~number~~ algo we write

$$\gcd(a^n - b^n, a^m - b^m) = \gcd\left(a^m - b^m, b^{m\lfloor n/m \rfloor}\left(a^{n \bmod m} - b^{n \bmod m}\right)\right)$$

$\Rightarrow$ $b^{m\lfloor n/m \rfloor}$ is relatively prime to $(a^n - b^n)$ since it devides the 2nd term. and is releatively prime to 1st term

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Date:

Therefore it will be relatively prime to gcd that is being computed.

we might as well remove the factor from 2nd argument of gcd

This gives us

$$gcd(a^n - b^m, a^m - b^m) = gcd(a^m - b^m, a^{n \bmod m} - b^{n \bmod m})$$

$gcd(a^n - b^m, a^m - b^m) = a^{gcd(m,n)} - b^{gcd(m,n)}$;

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Sol^n of Question of prev years

6th Chapter

Sultan Ahmed S
0805099, CSE, B

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Date:

Buet 2002-2003

4(e) Establish the following identity ~~in~~ with involvement of stirling number.

$$x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}} \quad n \geq 0$$

Solution:

Basis:

The eqn. is true for $n=1$.

$LHS = x^1 = x$ && $RHS = \sum_k \left\{ {1 \atop k} \right\} x^{\underline{k}}$

$= x^{\underline{1}} = x$

Induction:

Let the eqn holds for $(n-1)$. Thus the following statement is true.

$$x^{n-1} = \sum_k \left\{ {n-1 \atop k} \right\} x^{\underline{k}}$$

We now show that $x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}$

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Date:

$\cancel{x \cdot x \sum_k {n-1 \brace k}}$

For this proof the following identity is useful:

$$x \cdot x^{\underline{k}} = x^{\underline{k+1}} + k \cdot x^{\underline{k}}$$

$$x^n = x \cdot x^{n-1} = x \sum_k {n-1 \brace k} x^{\underline{k}} = \sum_k {n-1 \brace k} \cdot x \cdot x^{\underline{k}}$$

$$= \sum_k {n-1 \brace k} x^{\underline{k+1}} + \sum_k {n-1 \brace k} k \cdot x^{\underline{k}}$$

$$= \sum_k {n-1 \brace k-1} x^{\underline{k}} + \sum_k {n-1 \brace k} k x^{\underline{k}}$$

$$= \sum_k \left({n-1 \brace k-1} + k {n-1 \brace k} \right) x^{\underline{k}}$$

$$= \sum_k {n \brace k} x^{\underline{k}}$$

$$\therefore x^n = \sum_k {n \brace k} x^{\underline{k}}$$

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Date:

Buet ~~2003~~ 2004 -2005

A 1(c)

Lets use the letters L and R to stand for going down to the left or right branch. Given a string of L's and R's how could you find the ~~function~~ fraction that corresponds to that string?

Solution:

Algo:

1. $S = I$
2. while $\frac{m}{n} \neq f(S)$
3. if $\left(\frac{m}{n} < f(S)\right)$
   output L
   $S = SL$
4. else output R
   $S = SR$

$$f(S) = \frac{2+3}{4+3} = 5/7$$

corresponds to string

LRRL we can find

$$M(LRRL) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

Date:.............

Buet 2005-2006

4.(b)

Introduce stirling no. of both kinds. Show the recurrence relations they must satisfy.

(10)

Solution:

Stirling no. of 1st kind:

The no. of ways to arrange n objects into k cycles is represented by stirling no. of 1st kind.

It is denoted by $\left[{n \atop k}\right]$

Recurrence Relations:

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$$

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Date:

Proof:

Every arrangement of n objects into k cycles either

1. puts the last object into a cycle by itself. ($\left[{n-1 \atop k-1}\right]$ ways)

OR

2. puts the last object into one of the $\left[{n-1 \atop k}\right]$ cycles arrangements of $(n-1)$ objects ($(n-1)\left[{n-1 \atop k}\right]$ ways)

$$\therefore \left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]$$

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Date:

Buet 2005-2006

4(c)

Given n cards and a table deduce the largest possible overhang that can be achieved by stacking the cards over the table's edge, subject to the law of gravity. (10)

Solution:

see Questions of Buet (06-

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Date:

Buet 2006-2007

4(a)

Introduce Stirling numbers of both kinds. Deduce recurrence relations they satisfy. (10)

solution:

Stirling No. of 2nd kind:

8' Represent the no. of ways to partition n elements set into k non-empty subset.

Recurrence Relations:

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$

Recurrence Relations: Proof:

We are given a set of $n>0$ objects to be partitioned into k non-empty parts.

Date:

we either put the last object into a class by it-self:

↳ The no. of ways is $\left\{ {n-1 \atop k-1} \right\}$

OR

we put it together with some non-empty subset of first $(n-1)$ objects.

↳ There are $k\left\{ {n-1 \atop k} \right\}$ possibilities in the later case.

$$\# \left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$

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Buet 2006-2007

4(b)

Define euler numbers. Deduce recurrence relations they satisfy. (10)

Solution:

Euler Numbers:

Euler number is represents the number of permutations $\pi_1 \pi_2 \cdots \pi_n$ of $\{1, 2, \cdots n\}$ that have k ascents namely k places where $\pi_i < \pi_{i+1}$.
It is represented by $\left\langle {n \atop k} \right\rangle$.

Recurrence Relations:

Each permutations $p = p_1 p_2 p_3 \cdots p_{n-1}$ of $\{1, 2, 3, \cdots, n-1\}$ leads to n permutation of $\{1, 2, 3, \cdots, n\}$ if we insert new element in all possible ways.

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Date:

suppose we put ~~in~~ n in position j, obtaining the following permutations

$\pi = p_1 p_2 \cdots p_{j-1} \; n \; p_j \; p_{j+1} \cdots p_{n-1}$

The no. of ascents in π is same as the number in p if $j=1$ or $p_{j-1} < p_j$

its greater than the numer in p if $p_{j-1} > p_j$ or if $j=n$

Therefore π has k ascents in a total of $(k+1)\left\langle {n-1 \atop k} \right\rangle$ ways from permutations p that have $(k-1)$ ascents.

plus a total of $\big((n-2)-(k+1)+1\big)\left\langle {n-1 \atop k-1} \right\rangle$ ways from permutations p that $(k-1)$ ascents.

$$\therefore \left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$$

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4(c)

Define Harmonic numbers. Prove that it is possible to create an infinite overhang by stacking the cards up over an tables edge subject to laws of gravity.

Solution:

Harmonic Number:

The harmonic numbers is defined

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

H_n is discrete analogue of the natural algorithm.

Sol^n of 2nd Part:

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Date:

The center of gravity of first k cards must be above the (k+1)st card.

Here table plays the role of (n+1)th card

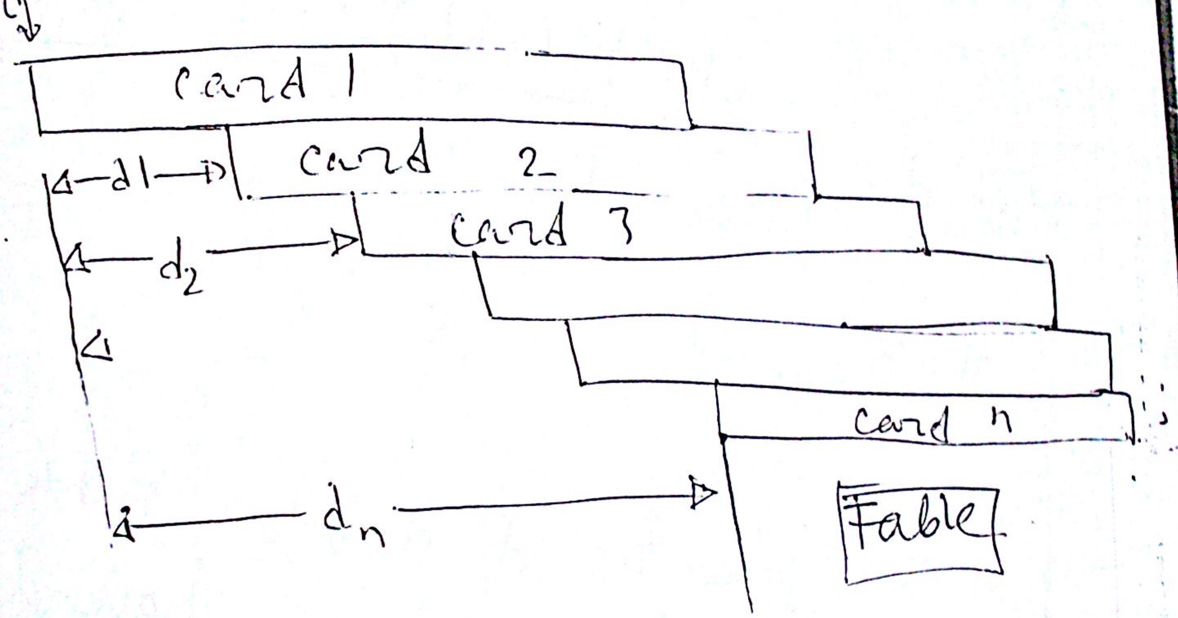

Figure No. 01

In order to compute this center of gravity put $x=0$ at the end of 1st card

Date: ....

The center of gravity of first k cards is

$$\frac{(1+d_0)+(1+d_1)+\cdots+(1+d_k)}{k}$$

$$\therefore d_k \le \frac{(1+d_0)+(1+d_1)+\cdots+(1+d_k)}{k} \quad \cdots($$

Here $d_0 = 0$.

The greater $d_0, d_1, d_2, \ldots d_k$ ar[e] the greater ~~tap~~ the upper bound fo[r] d_k given by eqn (1).

It follows that : d_n is great[est] precisely when equality holds fo[r]

$$d_k = \frac{(1+d_0)+(1+d_1)+\cdots+(1+d_k)}{k}$$

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Date:

$$\Rightarrow k d_k = k + (d_0 + d_1 + \cdots + d_{k-1}) \quad \cdots (2)$$

$$\therefore (k-1) d_{k-1} = (k-1) + (d_0 + d_1 + \cdots + d_{k-2}) \quad \cdots (3)$$

Substracting (2) from (3) we get

$$k d_k - (k-1) d_{k-1} = 1 + d_{k-1}$$

$$\Rightarrow k d_k = k d_{k-1} + 1$$

$$\Rightarrow d_k = d_{k-1} + \frac{1}{k}$$

So we get

$$d_0 = 0$$

$$d_k = d_{k-1} + \frac{1}{k}$$

$$\therefore d_k = \sum_{k=1}^{n} \frac{1}{k}$$

$$= H_k$$

Since $H_n \to \infty$ as $n \to \infty$, so we can get arbitarily ~~long~~ large overhang

$H_4 > 2$, so with 4 cards we can get the top card to be clear on table

But $H_{1000000} = 14.39$ so to get overhang of 7 to card lengths we need nearly 1000000 cards.

——— x ——— o ——— x ———

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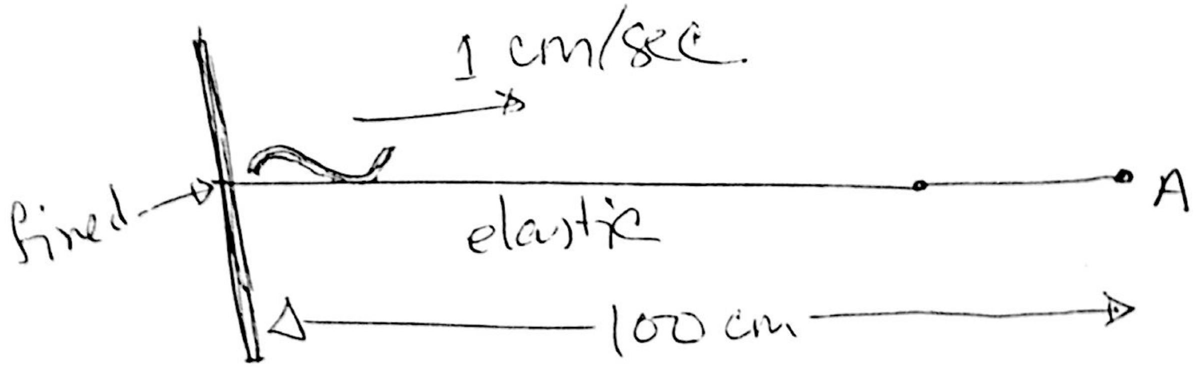
Date:

Example 7 :

Consider a warm on elastic band. The worm starts at one end and crawls towards the other at 1 cm per second. The elastic band initially has 100cm length. but after each second it is stretched by 100 cm. Does worm ever reach end?



Solution :

= after 1 second $\frac{1}{100}$th journey is over

Then the elastic is stretched to 200 cm long.

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But worm remains $\frac{1}{100}$ th way along since the stretching is uniform

In 2nd second, worm completes further $\frac{1}{200}$'th of journey.

So in total $\left(\frac{1}{100} + \frac{1}{200}\right)$ has been done

This remains true after stretching to 300 cm

In 3rd sec $\frac{1}{300}$ th journey is done

After n sec the fraction of journey completed is

$$\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \cdots + \frac{1}{100n}$$

$$= \frac{H_n}{100}$$

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Date: ....

~~# Since $H_n \to \infty$~~ ~~So the answer to the question is yes.~~

~~# We can~~

Since H_n is unbounded, the worm will definitely reach the end.

(Ans)

—— x —— 0 ——

BF
বিসমিল্লাহ ফটোকপি
মোঃ মনির হোসেন
প্রোপাইটর
কেমিক্যাল ডিপার্টমেন্ট এর লুবনা ম্যাডাম এর সকল নোট পাওয়া যায়।
এছাড়া ও অফসেট A4/ লিগ্যাল অফসেট ফটোকপি করা হয়।
তিতুমীর হল বাঁধন অফিসের সামনে।
মোবাঃ 01766591575, 01851558474

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5th Year Chapter

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$$\sum \binom{n+r}{r} x^n y^m = \sum \binom{-n}{k} (-x)^k (x+y)^n$$

2006-2007

(05-06)

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Buet 2006-2007

3c

Prove that $g(n) = \sum_k \binom{n}{k} (-1)^k f(k)$

$\Leftrightarrow f(n) = \sum_k \binom{n}{k} (-1)^k g(k)$

Buet(05-06)

Solution:

Let $g(n) = \sum_k \binom{n}{k} (-1)^k f(k)$

$\therefore \sum_k \binom{n}{k} (-1)^k g(k) = \sum_k \binom{n}{k} (-1)^k \sum_j \binom{k}{j} (-1)^j f(j)$

$= \sum_{j=1}^{N} f(j) \sum_{k=1}^{j} \binom{n}{k} (-1)^{k+j} \binom{k}{j}$

$= \sum_{j=1}^{N} f(j) \sum_{k=1}^{j} \binom{n}{j} \binom{n-j}{k-j} (-1)^{k+j}$

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Buet 2005-2006

3 (b)

Prove $\sum\limits_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}$

Solution:

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$

$$= \binom{3}{3} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{2}{3} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{1}{3} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{0}{3} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + ($$

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots + ($$

$$= \binom{n+1}{m+1}$$

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Date:

Buet 2005-2006

4(a)

Prove that

$$(1+z)^n = \sum_k \binom{n}{k} z^k$$

Proof:

From taylor series we know that

$$f(z) = \frac{f(0)}{0!}z^0 + \frac{f^1(0)}{1!}z^1 + \frac{f^2(0)}{2!}z^2 + \cdots$$

$$= \sum_{k\geq 0} \frac{f^k(0)}{k!} z^k$$

Let $f(z) = (1+z)^n$

$$f^k(z) = n(n-1)\cdots(n-k+1)(1+z)^{n-k}$$

$$= n^{\underline{k}}(1+z)^{n-k}$$

$$f^k(0) = n^{\underline{k}}(1)^{n-k}$$

$$= n^{\underline{k}}$$

Date:

$$(1+z)^n = \sum_{k=0} \frac{n^k}{k} z^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} z^k$$

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

[proved]

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Buet 2004-2005

Date:

1) Find closed form expression for

$$\sum_{k=0}^{n} k \binom{m-k-1}{m-n-1} \Big/ \binom{m}{n}$$

Solution:

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

$$\Rightarrow \binom{m}{k} \Big/ \binom{n}{k} = \binom{n-k}{m-k} \Big/ \binom{n}{m}$$

$$\sum_{k\geq 0} \binom{n-k}{m-k} = \sum_{m-k\geq 0} \binom{n-(m-k)}{m-(m-k)}$$

$$= \sum_{k\leq m} \binom{n-m+k}{k}$$

From formula we get

$$\sum_{k\leq m} \binom{n-m+k}{k} = \binom{(n-m)+m+1}{m} = \binom{n+1}{n}$$

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Solution:

(Ex 4.3 Page 190)

$$S = \sum_{k=0}^{n} k \binom{m-k-1}{m-n-1}$$

$$= \sum_{k=0}^{n} \left(m-(m-k) \right) \binom{m-k-1}{m-n-1}$$

$$= \sum_{k=0}^{n} m \binom{m-k-1}{m-n-1} - \sum_{k=0}^{n} (m-k) \binom{m-k-1}{m-n-1}$$

$$= m \sum_{k=0}^{n} \binom{m-k-1}{m-n-1} - \sum_{k=0}^{n} (m-k) \binom{m-k-1}{m-n-1}$$

from absorption identity $(m-k)\binom{m-k-1}{m-n-1} = (m-n)\binom{m-k}{m-n}$

$$= m \sum_{k=0}^{n} \binom{m-k-1}{m-n-1} - \sum_{k=0}^{n} (m-n) \binom{m-k}{m-n}$$

$$= m \sum_{0 \le m-k-1 \le n} \binom{m-k-1}{m-1-n} - (m-n) \sum_{0 \le m-k \le n} \binom{m-k}{m-n}$$

$$= m \sum_{m-n-1 \le k \le m-1} \binom{k}{m-n-1} - (m-n) \sum_{m-n \le k \le m} \binom{k}{m-n}$$

$$= m \sum_{0 \le k \le m-1} \binom{k}{m-n-1} - (m-n) \sum_{0 \le k \le m} \binom{k}{m-n}$$

$$= m \binom{m}{m-n} - (m-n) \binom{m+1}{m-n+1}$$

$$= \frac{n}{m-n+1} \binom{m}{m-n}$$

$$\therefore T = \sum_{k=0}^{n} k \binom{m-k-1}{m-n-1} \Big/ \binom{m}{n} = S \Big/ \binom{m}{n} = \frac{n}{m-n+1}$$

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Date:

Buet 2004-2005

(d) Prove that

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

Solution: $\binom{n}{m}\binom{m}{k} = \dfrac{\lfloor n}{\lfloor m \lfloor n-m} \cdot \dfrac{\lfloor m}{\lfloor k \lfloor m-k}$

$$= \frac{\lfloor n}{\lfloor k \lfloor n-k} \cdot \frac{\lfloor n-k}{\lfloor n-m \lfloor m-k}$$

$$\therefore \binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

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Buet 2004-2005

4(a)

Prove vandermonde's convolution

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

from it ~~&~~ prove that

$$\sum_{k \le q} \binom{q-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{q+m}\binom{s-m+1}{q-m-n}$$

[illegible] - পৃষ্ঠা - 184

~~Proof:~~

Solution:

Let r and s be non-negative integers

So $\binom{r+s}{n}$ is the number of ways to choose n people from among r men and s women.

On the left, each term of the sum is the number of ways to choose k of men men and

Date:........

$(n-k)$ of the women.

Summing over all k counts each possibility exactly once.

Hence

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

[Proved]

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Buet 2003-2004

3(c)
Give a combinatorial proof of

$$\sum_{k<n} \binom{m+k}{k} = \binom{m+n+1}{n}$$

এটা দেখে।

olution:

$$\binom{k}{m} = \binom{n+1}{m+1}$$

e know $\sum_{0 \le k \le n}$

$$\binom{m+k}{k}$$

Hence

$$\sum_{k \le n} \binom{m+k}{k} = \sum_{-m \le k \le n} \binom{m+k}{k}$$

$$= \sum_{-m \le k \le n} \binom{m+k}{m+k-k}$$

$$= \sum_{-m \le k \le n} \binom{m+k}{m}$$

$$= \sum_{0 \le k \le m+n} \binom{k}{m}$$

$$= \binom{m+n+1}{m+1}$$

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$$= \binom{m+n+1}{m+n+1-m-1}$$

$$= \binom{m+n+1}{n}$$

$$\therefore \sum_{k \leq n} \binom{m+k}{k} = \binom{m+n+1}{n}$$

[Proved]

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Date:

Be Buet 2003-2004

4(b)

Prove Vandermondes convolution

$$\sum_{k} \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

from it prove that

$$\sum_{k=1} \binom{l+k}{m}\binom{s}{k+n}(-1)^{k} = (-1)^{l+m}\binom{s-m-1}{l+m+n}$$

হবে [illegible]।

Solution:

let there are r men and s women where $r \geq 0$ and $s \geq 0$.

So $\binom{r+s}{n}$ is the number of ways to choose n people from among r men and s women.

On the left, each term of the sum is the number of ways to choose k [illegible] men and $(n-k)$ of the women.

Date

Summing over all k counts each possibility exactly once

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

Proof of [illegible] theorem

4(d)

Prove the following

$$\sum_{k\le m} \binom{n}{k}\left(\frac{n}{2}-k\right) = \frac{m+1}{2}\binom{n}{m+1}$$

Solution:

base case:

At $m=1$ LHS $= \sum_{k\le 1} \binom{n}{k}\left(\frac{n}{2}-k\right)$

$$= \binom{n}{0}\left(\frac{n}{2}\right) + \binom{n}{1}\left(\frac{n}{2}-1\right)$$

$$= \frac{n}{2} + n\left(\frac{n}{2}-1\right)$$

$$= \frac{n^2 - 2n + n}{2}$$

$$= \frac{1}{2}n(n-1)$$

& RHS $= \frac{1+1}{2}\binom{n}{2} = \frac{1}{2}n(n-1)$

So the formula is true for $m=1$.

Induction:

Let the sentence is true for $m = P$.

$$\sum_{k\le P} \binom{n}{k}\left(\frac{n}{2}-k\right) = \frac{P+1}{2}\binom{n}{P+1} \quad \text{--- (2)}$$

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$$\therefore \sum_{k \le p+1} \binom{n}{k}\left(\frac{n}{2}-k\right) = \sum_{k \le p} \binom{n}{k}\left(\frac{n}{2}-k\right) + \binom{n}{k}\left(\frac{n}{2}-\right.$$

$$= \frac{p+1}{2}\binom{n}{p+1} + \binom{n}{p+1}\left(\frac{n}{2}-(p+1)\right)$$

$$= \binom{n}{p+1}\left(\frac{n}{2}-(p+1)+\frac{p+1}{2}\right)$$

$$= \binom{n}{p+1}\left(\frac{n}{2}-\frac{p+1}{2}\right)$$

$$= \frac{\lfloor n}{\lfloor p+1 \; \lfloor n-p-1} \; \frac{n-p-1}{2}$$

$$= \frac{p+2}{2} \; \frac{\lfloor n}{\lfloor p+2 \; \lfloor n-p-2}$$

$$= \frac{p+2}{2}\binom{n}{p+2}$$

Hence the statement is true for all value of m.

so by induction we can say that:

$$\sum_{k \le m} \binom{n}{k}\left(\frac{n}{2}-k\right) = \frac{m+1}{2}\binom{n}{m+1}$$

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Find closed form for sums

$$\sum_{k=0}^{m} \binom{m}{k} \Big/ \binom{n}{k}$$

Solution:

$$\sum_{k\geq 0} \binom{m}{k} \Big/ \binom{n}{k} = \binom{n-k}{m-k} \Big/ \binom{n}{m}$$

$$\sum_{k=0}^{m} \binom{n-k}{m-k} \Big/ \binom{n}{m} = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k}$$

$$\sum_{k\geq 0} \binom{n-k}{m-k} = \sum_{m-k\geq 0} \binom{n-(m-k)}{m-(m-k)}$$

$$= \sum_{k\leq m} \binom{n-m+k}{k}$$

$$= \binom{(n-m)+m+1}{m}$$

$$= \binom{n+1}{m}$$

$$\therefore \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} = \frac{{}^{n+1}C_m}{{}^{n}C_m}$$

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$$= \frac{n+1}{n+1-m}$$

$$\sum_{k=0}^{m} \binom{m}{k} \Big/ \binom{n}{k} = \frac{n+1}{n+1-m}$$

[Proved]

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3(b)(i)

Prove $(-1)^m \binom{-n-1}{m} = (-1)^m \binom{-m-1}{n}$

Solution:

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Date:

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Buet 2002-2003

3·(b) (ii)

Prove $\sum_k \binom{p}{2}\binom{q}{k} k = q\binom{p+q+1}{p+1}$

Solution:

$$\sum_k \binom{n}{k}\binom{s}{k} k = \sum_k \binom{n}{k}\binom{s-1}{k-1} s$$

$$= s \sum_k \binom{n}{k}\binom{s-1}{k-1}$$

$$= s\binom{n+s-1}{n-1}$$

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4 (b)

In how many ways each of the n football fans does not get his own hat back.

Solution:

Let $h(n,k)$ means exactly k fans get their hat back rightly.

~~and~~ $h(n,k) = \binom{n}{k} h(n-k, 0)$

$$= \binom{n}{k} (n-k)i$$

Here $(n-k)i$ means no fan get their hat rightly back. We will find the value of $n i$ (pronounced as n-subfactorial)

$$n! = \sum_{k=0}^{n} h(n,k) = \sum_{k=0 \text{ to } k=n} \binom{n}{k} h(n-k, 0)$$

$$= \sum_{k=0 \text{ to } k=n} \binom{n}{k} (n-k)i$$

$$= \sum_{n-k=0 \text{ to } n-k=n} \binom{n}{n-k} k i$$

Buet 2002-2003

4 (b)

In how many ways each of the n football fans does not get his own hat back.

solution:

let $h(n,k)$ means exactly k fans get their hat back rightly.

and $h(n,k) = \binom{n}{k} h(n-k, 0)$

$= \binom{n}{k} (n-k)_i$

Here $(n-k)_i$ means no fan get their hat rightly back. We will find the value of n_i (pronaunced as n-subfactorial)

$$n! = \sum_{k=0}^{n} h(n,k) = \sum_{k=0 \text{ to } k=n} \binom{n}{k} h(n-k, 0)$$

$$= \sum_{k=0 \text{ to } k=n} \binom{n}{k} (n-k)_i$$

$$= \sum_{n-k=0 \text{ to } n-k=n} \binom{n}{n-k} k_i$$

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$$= \sum_{\substack{k=n \text{ to} \\ k=0}} \binom{n}{k} k_i$$

$$\Rightarrow n! = \sum_{k=0}^{n} \binom{n}{k} k_i$$

from inversion formula we get

$$n_i = (-1)^n \sum_k \binom{n}{k} (-1)^k k!$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} (-1)^{k+n} k!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^{k+n}}{(n-k)!}$$

$$= n! \sum_{\substack{n-k=0 \\ \text{to } n-k=n}} \frac{(-1)^k}{k!}$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$\therefore n_i \approx \left\lfloor n!/e + \frac{1}{2} \right\rfloor + [n=0]$$

So The no. of ways each of the n football fans does not get his own hat back is $= \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor + [n=0]$

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4. (a)

Mobius inversion formula