M374M
Homework #6

Logan.

Section 2.6: p.138 print [p.244 ut ebook] (#5[1]). p.143 print [p.253 ut ebook] (#2[2]ab).

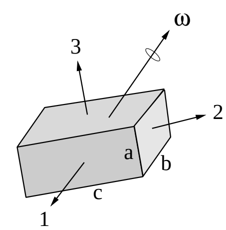
[1]Use the condition $S + I + R = N$ to get a two-variable system for S, I. For concreteness assume $N > b/a$ and $\mu = b/2$; characterize stability of equilibria in terms of the parameter $aN/b > 1$.

[2]Characterize stability of equilibria in terms of the parameters $\lambda > 0$ and $\eta > 0$.

Mini-project.

Consider a uniform, rectangular rigid body of mass m and dimensions a, b and c. In the absence of an applied torque, the rotational motion of the body is described by the system

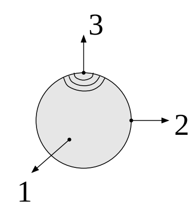
$$\dot{u} = u \times (Ku), \quad u(0) = u_0 \tag{1}$$



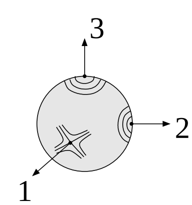
where $u = (u_1, u_2, u_3)$ is the angular momentum vector as seen in the body frame, $K = \mathrm{diag}(\alpha, \beta, \gamma)$ is a diagonal matrix of inertia parameters, and $\times$ is the vector cross product. Here we study the steady states of the above system and their stability. We assume $u_0 \neq 0$; without loss of generality we suppose $|u_0| = 1$. The angular velocity or spin vector is given by $\omega = Ku$ and the parameters are defined by $\alpha = \frac{12}{m(a^2+c^2)}$, $\beta = \frac{12}{m(a^2+b^2)}$ and $\gamma = \frac{12}{m(b^2+c^2)}$. Notice $c > b > a > 0$ implies $\beta > \alpha > \gamma > 0$.

(a) Show that (1) takes the form $\dot{u}_1 = \eta_1 u_2 u_3$, $\dot{u}_2 = \eta_2 u_3 u_1$ and $\dot{u}_3 = \eta_3 u_1 u_2$, where $\eta_1 = \gamma - \beta$, $\eta_2 = \alpha - \gamma$ and $\eta_3 = \beta - \alpha$. Deduce that the only possible steady states are $u_* = (\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Hence the only possible steady spin vectors ω_* are parallel to a coordinate axis.

(b) Show that the functions $E(u) = u_1^2 + u_2^2 + u_3^2$ and $F(u) = \alpha u_1^2 + \beta u_2^2 + \gamma u_3^2$ are constant along every solution of (1). Since $E(u(t)) = E(u_0) = 1$, conclude that every solution evolves on the unit sphere. Moreover, conclude that every solution satisfies $F(u(t)) = F(u_0)$, where $\gamma \leq F(u_0) \leq \beta$.



(c) For any u_0 near the equilibrium $(0, 0, 1)$, we have $F(u_0) \geq \gamma$, say $F(u_0) = \gamma + \delta$ for some small $\delta \geq 0$. Using the equations $E(u) = 1$ and $F(u) = \gamma + \delta$, show that solution curves near $(0, 0, 1)$ are ellipses on the sphere around this point. Hence $(0, 0, 1)$ is a neutrally stable center. [The same result holds for $(0, 0, -1)$ and also $(0, \pm 1, 0)$.]



(d) For any u_0 near the equilibrium $(1, 0, 0)$, the value $F(u_0)$ is near α, but may be larger or smaller, so $F(u_0) = \alpha \pm \delta$ for some small $\delta \geq 0$. Using the equations $E(u) = 1$ and $F(u) = \alpha \pm \delta$, show that solution curves near $(1, 0, 0)$ are hyperbolas on the sphere around this point. Hence $(1, 0, 0)$ is an unstable saddle. [The same result holds for $(-1, 0, 0)$.]

(e) Download the Matlab program file `program6.m` and function file `rigidbody.m` from the course webpage. Use the program to numerically simulate the model equation in (1) and produce solution curves for various u_0. Using $m = 2$kg, $c = 0.24$m, $b = 0.16$m and $a = 0.03$m, illustrate the behavior of the system for initial conditions near the different equilibria. Find a book with dimensions $c > b > a$ (the more different the better) and verify your results by direct experiment! Can you get the book to steadily spin about the axis parallel to edge b? What about a or c?

Turn in: responses to (a)–(e).