M374M
Homework #5

Logan.

Sections 2.4: p.107 print [p.194 ut ebook] (#1e[1],4f[2],8[3]).

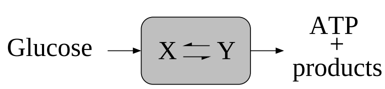
[1]Find all equilibrium solutions; sketch phase diagram in a small region around each.

[2]Assume μ can take any value; find all equilibrium solutions; describe type and stability of each in terms of μ.

[3]Introduce $y = x'$ and rewrite as a first-order system.

Mini-project.

In the biochemical process of glycolysis, living cells obtain energy by breaking down sugar. Various intermediate compounds are involved, and the overall process requires that some of these be interconverted from one type to the other, while experiencing some gains and losses. A simple model for the dynamics of two of these compounds is, in non-dimensional form,



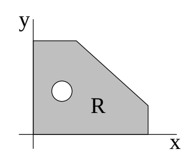
$$\begin{aligned} \dot{x} &= -x + ay + x^2 y, \quad x(0) = x_0 \\ \dot{y} &= \quad b - ay - x^2 y, \quad y(0) = y_0. \end{aligned} \tag{1}$$

Here $x(t)$ is the concentration of compound X (adenosine diphosphate), $y(t)$ is the concentration of compound Y (fructose-6-phosphate), and a and b are positive constants that describe the reaction kinetics. Here we perform a qualitative analysis to understand the behavior of solutions of this system depending on a and b.

(a) Sketch the nullclines and show that the system has only one equilibrium solution (x_*, y_*) for any $a > 0$ and $b > 0$. Explicitly find the equilibrium.

(b) For simplicity, assume $b = 1/2$ is fixed. Determine the stability of the equilibrium in terms of $a > 0$. Show that the equilibrium is unstable for $0 < a < a_\#$, and asymptotically stable for $a > a_\#$, for an appropriate constant $a_\#$.

(c) Consider a shaded region R of the phase plane as shown, where the circular hole is centered at (x_*, y_*). Show that the straight edges of R can be chosen such that the direction field along these edges either points inward or along the edge. Using the result in (b), and the Poincaré-Bendixson Theorem, deduce that the system must contain a closed orbit in R for any $0 < a < a_\#$. Give explicit locations for the vertices of R.



(d) Download the Matlab program file `program5.m` and function file `glycolysis.m` from the course webpage. Use the program to numerically simulate the model equation in (1) and produce solution curves for various (x_0, y_0). Using $b = 1/2$, illustrate the behavior of the system in the cases $0 < a < a_\#$ and $a > a_\#$. Is the closed orbit predicted in (c) a stable limit cycle? What happens to the closed orbit as a is increased toward $a_\#$? Does there appear to be a periodic solution for $a > a_\#$?

Turn in: responses to (a)–(d).