M374M
Homework #10

Logan.

Sections 4.2, 4.3: p.234 print [p.405 ut ebook] (#3[1],13[2]). p.243 print [p.419 ut ebook] (#2[3]ab,4).

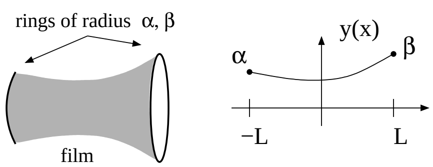
[1]Find first variation at y_0 for arbitrary h as described, and show result vanishes. The functional is $J(y) = \int_0^1 (1+x)(y')^2\, dx$.

[2]For arbitrary y and h consider $J(y+\epsilon h)$. Find an expression for the second variation $\frac{d^2}{d\epsilon^2} J(y+\epsilon h)|_{\epsilon=0}$.

[3]Assume $[a,b] = [1,2]$ and $y(1) = 0$ and $y(2) = 1$.

Mini-project.

When stretched between two symmetrically placed rings, a thin soap film will form a curved surface of revolution. If we denote the profile curve by $y(x)$ where $x \in [-L, L]$, then the surface area is given by



$$A(y) = \int_{-L}^{L} 2\pi y(x)\sqrt{1+[y'(x)]^2}\, dx.$$

The specific shape adopted by the film can be described as that which minimizes the functional $A(y)$ in a given function space. Here we find candidates for local minimizers in the C^2-norm in the space $V = \{y \in C^2[-L, L] \mid y(-L) = \alpha,\ y(L) = \beta,\ y(x) > 0\}$. For simplicity, we assume $\alpha = \beta > 0$.

(a) After a change of variable $x \to x/L$ and $y \to y/L$, show that any local minimizer $y_* \in V$ must satisfy the following boundary-value problem, where $\gamma = \alpha/L$:

$$yy'' - (y')^2 = 1, \quad -1 \le x \le 1. \qquad y(-1) = \gamma, \quad y(1) = \gamma. \tag{1}$$

(b) Consider the smooth function $y_*(x) = \frac{1}{c}\cosh(cx + d)$, where $c > 0$ and d are arbitrary constants. Show that y_* satisfies the differential equation. Moreover, show that y_* will satisfy the boundary conditions and hence will be in the space V only if $d = 0$ and $c > 0$ satisfies the equation

$$\cosh(c) = \gamma c. \tag{2}$$

(c) Show that (2) has two solutions for c if $\gamma > \gamma_\#$, one solution if $\gamma = \gamma_\#$, and no solution if $0 < \gamma < \gamma_\#$, where $\gamma_\#$ is an appropriate constant which you should find. Hence we have two, one or no candidates for a local minimizer $y_* \in V$ depending on the value of γ.

(d) In the case when $\gamma > \gamma_\#$ and there are two candidate curves y_*, it can be shown that the candidate with the smaller c is indeed a local minimizer whereas the other is not. Find and make plots of these curves on $[-1, 1]$ for the case $\gamma = 2$ and indicate which is the local minimizer. When $\gamma \le \gamma_\#$, it can be shown that there are no local minimizers of A in V; in this case, a surface of minimum area no longer has a class C^2 profile graph. What do you think might happen to the soap film in this case? [Try it by direct experiment with some bubble solution and wire rings! Note that decreasing γ is equivalent to increasing L with fixed α.]

Turn in: responses to (a)–(d).