M374M
Homework #8

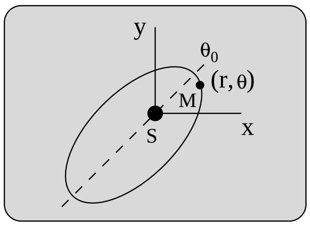
Logan.

Sections 3.1, 3.2: p.165 print [p.289 ut ebook] (#8a). p.178 print [p.310 ut ebook] (#1[1]ac).

[1]Find leading-order term in expansion of each root.

Mini-project.

One of the first successes of the theory of relativity was to predict the anomalous behavior of the orbit of Mercury (M) around the Sun (S). According to the relativistic theory, the orbital curve $r(\theta)$ satisfies



$$\begin{aligned} &\frac{d^2u}{d\theta^2} + u = \frac{1}{\rho} + \frac{\gamma u^2}{\rho}, \quad \theta \geq \theta_0 \\ &\frac{du}{d\theta}(\theta_0) = 0, \quad u(\theta_0) = \frac{1+c}{\rho} \end{aligned} \tag{1}$$

where $u(\theta) = 1/r(\theta)$ is the inverse of the radius, ρ is a constant determined by the angular momentum of the system, γ is a constant which quantifies relativistic effects, and c is a constant which defines the initial radius. Here we study (1) in the case corresponding to Newton's theory ($\gamma = 0$) and the case corresponding to Einstein's theory ($0 < \gamma \ll \rho^2$). Relevant physical dimensions are $[\rho] = L$, $[\gamma] = L^2$ and $[c] = 1$. We assume $\rho > 0$ and $0 < c < 1$.

(a) Introduce the dimensionless variable $v = \rho u$ and shifted angle $\phi = \theta - \theta_0$ and show that (1) can be written in the following form for an appropriate parameter ε:

$$\frac{d^2v}{d\phi^2} + v = 1 + \varepsilon v^2, \quad \phi \geq 0, \qquad \frac{dv}{d\phi}(0) = 0, \quad v(0) = 1 + c. \tag{2}$$

(b) Solve (2) in the Newtonian case when $\varepsilon = 0$. Using the fact that $r(\theta) = \rho/v(\theta - \theta_0)$, show that the smallest value of r (the perihelion of the orbit) occurs at the angle $\theta_\mathrm{p} = \theta_0 + 2n\pi$, and that the largest value of r (the aphelion of the orbit) occurs at the angle $\theta_\mathrm{a} = \theta_\mathrm{p} + \pi$. Does Newton's theory predict that the location of the perihelion/aphelion change with each cycle of the orbit?

(c) Approximate (2) in the relativistic case when $0 < \varepsilon \ll 1$. Use the Poincaré-Lindsted method to develop a two-term approximation of the form

$$v(\phi) = v_0(s) + \varepsilon v_1(s), \qquad s = (\omega_0 + \varepsilon\omega_1)\phi.$$

You need only determine v_0, ω_0 and ω_1. Using the approximation $r(\theta) = \rho/v_0(s)$, where $s = (\omega_0 + \varepsilon\omega_1)\phi$ and $\phi = \theta - \theta_0$, show that the smallest value of r now occurs at $\theta_\mathrm{p} = \theta_0 + 2n\pi(1 + \frac{\gamma}{\rho^2-\gamma})$, and that the largest value of r occurs at $\theta_\mathrm{a} = \theta_\mathrm{p} + \pi(1 + \frac{\gamma}{\rho^2-\gamma})$. Does Einstein's theory predict that the location of the perihelion/aphelion change with each cycle of the orbit?

(d) Download the Matlab files `program8.m` and `mercury.m` from the course webpage and simulate the equations in (1) for $\theta_0 = \frac{\pi}{4}$, $\rho = 1$ and $c = 0.7$. For each of the cases $\gamma = 0$ and $\gamma = 0.01$, make a plot of the orbital curve $(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$ for $\theta \in [\theta_0, \theta_0 + 2n\pi]$ for about $n = 10$ cycles. Do the simulations agree with your analysis in (b) and (c)? Astronomical observations show that the perihelion/aphelion of Mercury advance with each cycle of the orbit in agreement with the relativistic theory.

Turn in: responses to (a)–(d).