M374M
Homework #12

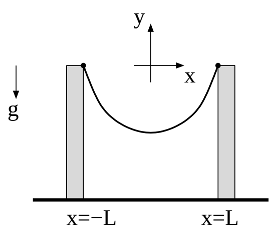
Logan.

Sections 4.4, 4.6: p.252 print [p.435 ut ebook] (#1de). p.271 print [p.469 ut ebook] (#1,5[1]).

[1]Find extremals for $F(y) = \int_0^1 \sqrt{1 + (y')^2}\, dx$ subject to $G(y) = \int_0^1 y\, dx = A$ where $y(0) = 0$ and $y(1) = 0$. For concreteness, assume $A = \pi/16$. What happens if $A > \pi/8$?

Mini-project.

When its ends are attached to two fixed points, a chain of given length and weight will hang to form a suspension curve. The resulting shape can be described as that curve $y(x)$, $x \in [-L, L]$ which minimizes the chain potential energy $E(y)$ subject to the length condition $G(y) = \ell$ where



$$E(y) = \int_{-L}^{L} \rho g y(x) \sqrt{1 + [y'(x)]^2}\, dx, \quad G(y) = \int_{-L}^{L} \sqrt{1 + [y'(x)]^2}\, dx.$$

In the above, ℓ is the chain length, ρ is the chain mass per unit length and g is gravitational acceleration. Here we find candidates for local minimizers in the C^2-norm in the space $V = \{y \in C^2[-L, L] \mid y(-L) = 0,\ y(L) = 0\}$. We assume ρ, g, ℓ and L are given positive constants.

(a) Write out the boundary-value problem that any candidate curve $y \in V$ must satisfy, and find the general solution of the differential equation.

(b) By relabeling constants as necessary, show that the conditions $y(-L) = 0$, $y(L) = 0$ and $G(y) = \ell$ can be reduced to two equations for two unknown constants b and c; specifically

$$\frac{bc}{L} = \cosh(c), \quad \frac{\ell c}{2L} = \sinh(c). \tag{1}$$

(c) Show that (1) has two solutions for the pair b, c if $\ell/(2L) > 1$, and no solution if $0 < \ell/(2L) < 1$. Hence we have two or no candidate curves depending on the ratio $\ell/(2L)$. What happens if $\ell/(2L) = 1$? What is the only possible shape of the suspension curve in this case? What is the physical reason there can be no solution if $0 < \ell/(2L) < 1$?

(d) In the case when $\ell/(2L) > 1$ and there are two candidate curves, it can be shown that one is a local minimizer and the other a local maximizer. Find and make plots of these curves for the case of a chain of length $\ell = 0.5$m of mass $m = 0.005$kg (so $\rho = m/\ell$) suspended between points with separation $2L = 0.2$m, where $g = 9.8\text{m/s}^2$. Indicate which is the minimizer and maximizer; this should be clear. What is the middle sag-depth $q = |y(0)|$ for the energy-minimizing shape? [Verify your results by direct experiment with an open necklace! You should be able to compute and verify the sag-depth at the middle and a few other locations.]

Turn in: responses to (a)–(d).