M374M
Homework #11

Logan.

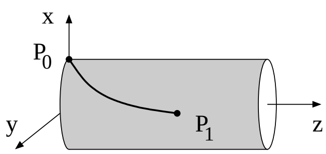
Sec 4.4: p.252 print [p.435 ut ebook] (#5bc, 6[1], 7, 1a[2]).

[1]Determine the boundary-value problem that a local minimizer or maximizer must satisfy.

[2]For simplicity, change the second boundary condition on the second function to $y_2(\pi/4) = 0$.

Mini-project.

A basic problem in geometry is to find the shortest curve along a surface between two points; such a curve is called a geodesic. For a cylinder of radius r, any curve along the surface is defined by $x = r\cos\theta(t)$, $y = r\sin\theta(t)$ and $z = z(t)$, where $\theta(t)$ and $z(t)$ are functions of $t \in [0, 1]$, and the length of the curve (integral of the speed) is



$$F(\theta, z) = \int_0^1 \sqrt{r^2[\theta'(t)]^2 + [z'(t)]^2}\, dt.$$

We suppose the points P_0 and P_1 on the surface are $(r, 0, 0)$ and $(0, r, h)$. The shortest curve between them is that curve which minimizes the functional $F(\theta, z)$ in a given function space. Here we find candidates for local minimizers in the C^2-norm in the space $V = \{\theta, z \in C^2[0, 1] \mid (r\cos\theta, r\sin\theta, z)_{t=0} = P_0,\ (r\cos\theta, r\sin\theta, z)_{t=1} = P_1\}$.

(a) Write out the boundary-value problem that any candidate pair $(\theta, z) \in V$ must satisfy. Using the differential equations, show that $z'(t)/\theta'(t)$ or $\theta'(t)/z'(t)$ must be constant. From this deduce that every solution pair $(\theta(t), z(t))$ must trace out a line in the θ, z-coordinate plane, and any such line can be described by $\theta(t) = At + B$, $z(t) = Ct + D$, where A, B, C and D are arbitrary constants.

(b) By considering the boundary conditions, show that there is a family of solutions to the boundary-value problem. Make and superimpose plots (in xyz-space) of the candidate curves $(x, y, z) = (r\cos\theta(t), r\sin\theta(t), z(t))$, $t \in [0, 1]$, for a few different choices of the constants in the family; include positive and negative values where possible; for plotting purposes take $r = 1$ and $h = 1.5$.

(c) The shortest among the candidate curves can be shown to be a local minimizer of F in V. By inspection, guided by the plots in (b), identify this geodesic curve for the given points P_0 and P_1. If the point P_1 were moved to $(-r, 0, h)$, what would be the geodesic curve? Would there be only one in this case? [Try to visualize the curves by pulling a piece of string tightly over a cylindrical tube! You may need to pin the string at the first point, and then pull it along the surface to the second point.]

Turn in: responses to (a)–(c).

Afternote: geodesics for a few other surfaces can be found by hand, for example cones and spheres; numerical methods are required for general surfaces.