M374M
Homework #9

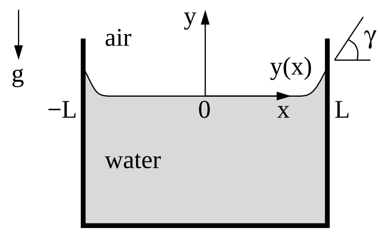
Logan.

Sections 3.3, 3.4: p.189 print [p.328 ut ebook] (#1[1]g). p.199 print [p.348 ut ebook] (#3[1]).

[1] Assume boundary layer is on left side of interval and find leading-order composite approximation.

Mini-project.

Due to surface tension, a liquid-gas interface will rise up or dip down at a solid boundary to form a meniscus, which is responsible for the so-called capillary effect in narrow tubes. In the planar case, the shape of the interface or meniscus curve $y(x)$ is described by



$$\frac{\sigma}{\rho g}\frac{d^2y}{dx^2} = y\Big[1 + \Big(\frac{dy}{dx}\Big)^2\Big]^{3/2}, \quad 0 \le x \le L$$
$$\frac{dy}{dx}(0) = 0, \quad \frac{dy}{dx}(L) = \tan\gamma \tag{1}$$

where ρ is the density of the liquid, g is gravitational acceleration, and σ and γ are the surface tension and wetting angle constants associated with the interface. Here we develop an approximate solution of (1) under the assumption that $0 < \frac{\sigma}{\rho g} \ll L^2$. For this problem we expect a boundary layer at $x = L$, and we will see that $\frac{dy}{dx}$ is regular, but $\frac{d^2y}{dx^2}$ behaves singularly.

(a) Introduce the dimensionless variables $h = y/L$ and $s = x/L$ and show that (1) can be written in the following form for an appropriate parameter $0 < \varepsilon \ll 1$.

$$\varepsilon\frac{d^2h}{ds^2} = h\Big[1 + \Big(\frac{dh}{ds}\Big)^2\Big]^{3/2}, \quad 0 \le s \le 1. \qquad \frac{dh}{ds}(0) = 0, \qquad \frac{dh}{ds}(1) = \tan\gamma. \tag{2}$$

(b) Find the leading-order outer approximation.

(c) For the inner approximation, consider the change of variables $\tau = \varepsilon^{-\alpha}(s-1)$ and $u = \varepsilon^{-\beta}h$. Show that $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ leads to the regular system

$$\frac{d^2u}{d\tau^2} = u\Big[1 + \Big(\frac{du}{d\tau}\Big)^2\Big]^{3/2}, \quad \tau \le 0. \qquad \frac{du}{d\tau}(0) = \tan\gamma. \tag{3}$$

This system is difficult to solve. To simplify it, we expect that $\frac{du}{d\tau}$ varies between zero and $\tan\gamma$ in the boundary layer; hence using an average of $\frac{1}{2}\tan\gamma$, we replace $1 + (\frac{du}{d\tau})^2$ by $1 + \frac{1}{4}\tan^2\gamma$ in the differential equation. Use this simplification to find an inner approximation.

(d) By matching the results from (b) and (c) construct a leading-order composite approximation for $h(s)$, and hence for the meniscus curve $y(x)$. Use the approximation to find an expression for the meniscus height $\ell = y(L)$. Show that $\frac{dy}{dx}$ remains bounded at all points as $\varepsilon \downarrow 0$, but that $\frac{d^2y}{dx^2}$ grows unbounded at $x = L$ as $\varepsilon \downarrow 0$.

(e) Download the Matlab files `program9.m`, `meniscusODE.m` and `meniscusBC.m` from the course webpage and simulate the equations in (1) for $\sigma = 0.05$, $\rho = 1000$, $g = 10$, $L = 0.07$ and $\gamma = \frac{\pi}{3}$ in kg-m-s units. Superimpose plots of the curve $y(x)$ produced by the simulation and by your approximation from (d). Is your prediction of the meniscus height ℓ in (approximate) agreement with the simulation?

Turn in: responses to (a)–(e).