**Theorem** (The Central Limit Theorem)

Let $X_1, \dots, X_n$ be independent random variables with mean $\mu$ and variance $\sigma^2$, for any distribution.

For example, $X_i \sim \text{Binomial}(n, p)$ for each $i$, so $\mu = np$ and $\sigma^2 = np(1-p)$

Then the sum $S_n = X_1 + \dots + X_n = \sum_{i=1}^{n} X_i$ has a distribution that tends to Normal as $n \to \infty$

According to Central limit theorem :-

$$S_n = X_1 + X_2 + \dots + X_n \longrightarrow \text{Normal}(n\mu, n\sigma^2)$$

as $n \to \infty$

**Important points :-**

1) The limit holds for any distribution of $X_1, \dots, X_n$
2) A sufficient condition on $X$ for the Central limit theorem to apply is that $Var(X)$ is finite.
3) The speed of convergence of $S_n$ to the Normal distribution depends upon the distribution of $X$. Skewed distributions converge more slowly than symmetric Normal-like distributions.

<u>Simulations :-</u>

1) Triangular distribution : $f_X(x) = 2x$ for $0 < x < 1$

Find $E(X)$ and $Var(X)$

$$\mu = E(X) = \int_0^1 x f_X(x)\,dx$$
$$= \int_0^1 2x^2\,dx$$
$$= \left[\frac{2x^3}{3}\right]_0^1$$
$$= \frac{2}{3}$$

$$\sigma^2 = Var(X) = E(X^2) - \{E(X)\}^2$$
$$= \int_0^1 x^2 f_X(x)\,dx - \left(\frac{2}{3}\right)^2$$
$$= \left[\frac{2x^4}{4}\right]_0^1 - \frac{4}{9}$$
$$= \frac{1}{18}$$

Let $S_n = X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are independent

Then
$$E(S_n) = E(X_1 + \cdots + X_n) = n\mu = \frac{2n}{3}$$
$$Var(S_n) = Var(X_1 + \cdots + X_n) = n\sigma^2$$
$$\Rightarrow \quad Var(S_n) = \frac{n}{18}$$

So $S_n \sim$ approximate Normal $\left(\frac{2n}{8}, \frac{n}{18}\right)$ for large $n$