Final Documentation

1.

1)

i) $P(\text{Success} \mid A) = \alpha \qquad P(A) = 0.5$

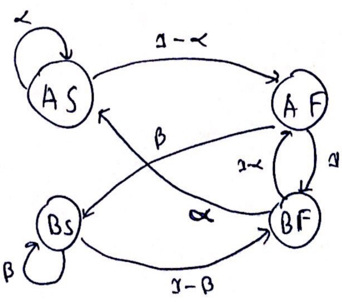
$P(\text{Success} \mid B) = \beta \qquad P(B) = 0.5$

$$P(\text{Success}) = P_R = P(\text{Success} \mid A) \cdot P(A) + P(\text{Success} \mid B) \cdot P(B)$$

$$= \alpha \cdot \frac{1}{2} + \beta \cdot \frac{1}{2}$$

$$\boxed{P_R = \frac{\alpha + \beta}{2}}$$

	AS	AF	BS	BF
AS	α	$1-\alpha$	0	0
AF	0	0	β	$1-\beta$
BS	0	0	β	$1-\beta$
BF	α	$1-\alpha$	0	0



At steady state or equilibrium :-

$\pi T = \pi$ where T = transition matrix

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \beta & 1-\beta \\ 0 & 0 & \beta & 1-\beta \\ \alpha & 1-\alpha & 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}$$

$$\alpha \pi_1 + \alpha \pi_4 = \pi_1$$

$$(1-\alpha)\pi_1 + (1-\alpha)\pi_4 = \pi_2$$

$$\beta \pi_2 + \beta \pi_3 = \pi_3$$

$$(1-\beta)\pi_2 + (1-\beta)\pi_3 = \pi_4$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$\Rightarrow$ $\pi_2 = \pi_4$

$$\pi_1 = \frac{\alpha}{1-\alpha}\pi_4$$

$$\pi_3 = \frac{\beta}{1-\beta}\pi_2$$

$\Rightarrow$ $\frac{\alpha}{1-\alpha}\pi_4 + \pi_4 + \frac{\beta}{1-\beta}\pi_4 + \pi_4 = 1$

$$\pi_4 \left[2 + \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} \right] = 1$$

$$\pi_4 \left[\frac{2-\alpha-\beta}{1-\alpha-\beta+\alpha\beta} \right] = 1$$

$$\pi_4 = \frac{1-\alpha-\beta+\alpha\beta}{2-\alpha-\beta} = \frac{(1-\alpha)(1-\beta)}{2-\alpha-\beta}$$

$$\pi_1 = \frac{\alpha(1-\beta)}{2-\alpha-\beta} \quad , \quad \pi_3 = \frac{\beta(1-\alpha)}{2-\alpha-\beta}$$

As π_1, π_3 correspond to ~~P_T~~ AS, BS

$$P_T = \pi_1 + \pi_3 = \frac{\alpha(1-\beta)}{2-\alpha-\beta} + \frac{\beta(1-\alpha)}{2-\alpha-\beta}$$

$$\boxed{P_T = \frac{\alpha+\beta-2\alpha\beta}{2-\alpha-\beta}}$$

$$\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 = \left[\frac{\alpha(1-\beta)}{2-\alpha-\beta} \quad \frac{(1-\alpha)(1-\beta)}{2-\alpha-\beta} \quad \frac{\beta(1-\alpha)}{2-\alpha-\beta} \quad \frac{(1-\alpha)(1-\beta)}{2-\alpha-\beta} \right]$$

Implementation

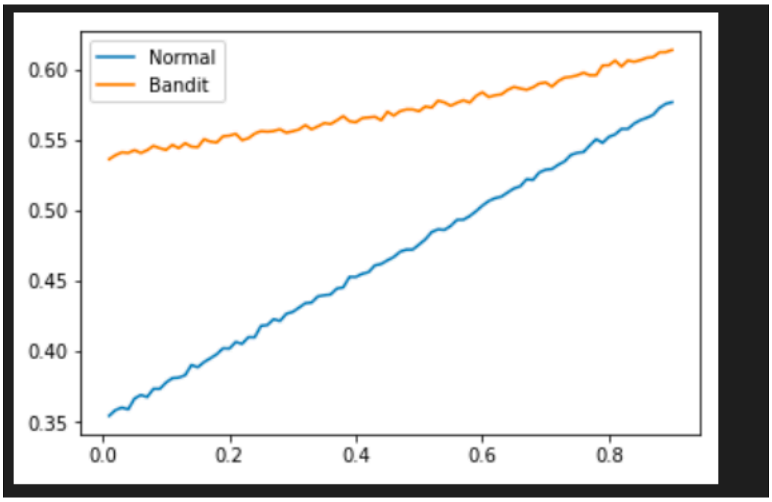
- In the first cell of the code I keep the value of alpha constant and change beta from 0.01 to 0.99. At every beta value I find the probability of success for both strategies using 10000 trials. Then I plot the probability of success **Pr** and **Pt** against the current **beta.**
- In the second cell I merely plot the **Pr** and **Pt** using the formulas obtained above for varying values of beta and keeping alpha fixed.

Results

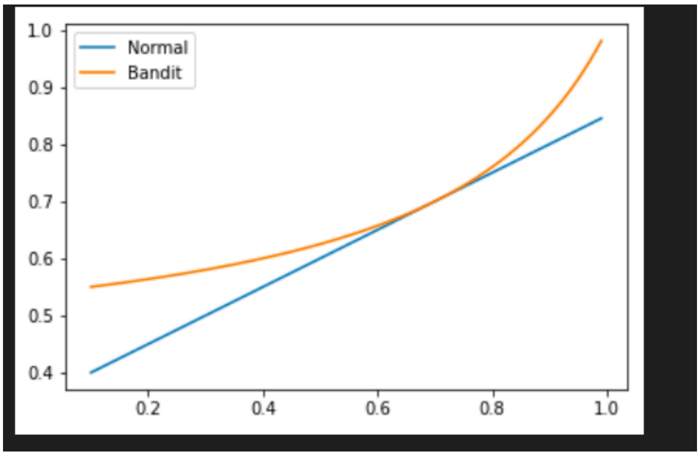
(It takes a while to produce the results)

- From the first graph in which I don't use formulas, there no point at which **Pt** goes below **Pr** hence

$$p_T - p_R \geq 0$$



- from the second graph it can be seen that there's a slight overlap but **Pt** still remains above **Pr** at all possible values



3.

3) $\lambda =$ rate of photons arriving per min

$$P(r|\lambda) = \frac{e^{-\lambda}\lambda^{r}}{r!}$$

At $r = 9$,

$$P(r|\lambda) = \frac{e^{-\lambda}\lambda^{9}}{9!}$$

To estimate λ we take $\frac{\delta}{\delta\lambda} P(r|\lambda) = 0$

$P(r|\lambda) =$ likelihood

Taking log likelihood and equating to 0

$$P(r|\lambda) = \ln\left(\frac{e^{-\lambda}}{9!}\right) + \ln(\lambda^9)$$

$$= \ln(e^{-\lambda}) - \ln(9!) + 9\ln(\lambda)$$

$$= -\lambda - \ln(9!) + 9\ln(\lambda)$$

So $\frac{\delta}{\delta\lambda} = 0 \Rightarrow -1 + \frac{9}{\lambda} = 0$

$$\frac{9}{\lambda} = 1$$

$$\boxed{\lambda = 9}$$

ii) $P(r \mid \lambda+b) = \dfrac{e^{-(\lambda+b)} (\lambda+b)^r}{r!}$

$b = 13$

$P(r \mid \lambda+b) = \dfrac{e^{-(\lambda+13)} (\lambda+13)^r}{r!}$

Taking log of likelihood $_{r=9}$ & equating its derivative to 0 :

$= \ln\left(\dfrac{e^{-(\lambda+13)}}{9!}\right) + \ln(\lambda+13)^9$

$= -(\lambda+13) - \ln(9!) + 9\ln(\lambda+13)$

Takeing derivative & equating to 0

$\Rightarrow \quad -1 + \dfrac{9}{\lambda+13} = 0$

$\lambda + 13 = 9$

$\boxed{\lambda = -4}$ → This is not possible as $\lambda \geq 0$. Therefore, we need to keep λ as minimum as possible to get max likelihood.

$\boxed{\lambda = 0}$ → At which we get max likelihood

Implementation

- In the first cell, the maximum likelihood is calculated for r=9. I take random values between 0 and 100 and compute the log likelihood. The graph and the value at which likelihood is maximum is displayed
- In the second cell, the max likelihood is calculated for a mean **(lambda+13).** Again for random lambda values between 0-100 we find the log likelihood.
- We also know that the restrictions of lambda for exponential distribution is **0<lambda<r** which would be a concave function if we take the log likelihood. But as we needed to apply monte carlo simulations to find lambda I took random values of lambda

Results

We can see that lambda values close to 9 give max likelihood for case 1 and lambda's close to 0 give max likelihood for case 2