# PHSY1500J-RC4

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## 1. Momentum and Conservation of Momentum

### 1.1 Basic Concepts

**Definition**

$$\vec{p} = m\vec{v}(Unit : kg \cdot m/s)$$

**Derived Formulas**

- $$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \text{ (Follows from Newton's II law)}$$
- $$p_2 - p_1 = \int_{t_1}^{t_2} F \mathrm{d}t \text{ (Momentum-Impulse Theorem)}$$

**Note:**

1. Momentum is vector quantity, and can be decomposed.
2. There're various ways to change momentum of a system

$\Sigma F \to F$

mass

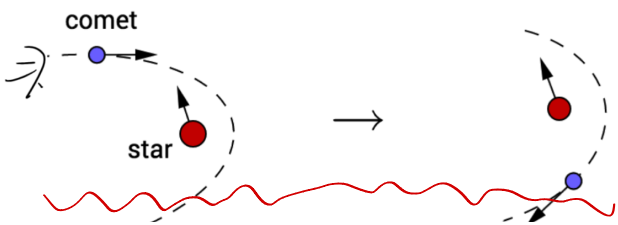
**Conservation of Momentum Law**

If the sum of external forces on a system is equal to zero, then the total momentum of the system is constant (i.e. is conserved).

### 1.2 Collision

Collision – two objects interact (directly or indirectly) over a finite time-interval.

这个也是 collision




Three types $K_{final} = K_{initial} + Q$ with $Q$ $\begin{cases} =0 & \text{elastic} \\ >0 & \text{superelastic} \\ <0 & \text{inelastic} \end{cases}$

$\Delta$

**Below are some special cases:**

(1) two identical ball collide, one of them has initial velocity. Consider the inner product of their final velocity. (not necessarily head–on collision)

$\vec{v}_1$ → ⇒ $\vec{u}_1$ $\vec{u}_2$

$$\begin{cases}\vec{v}_1 = \vec{u}_1 + \vec{u}_2 \\ v_1^2 = u_1^2 + u_2^2\end{cases} \Rightarrow \text{what's } \vec{u}_1 \cdot \vec{u}_2 \text{ ?}$$

$$\vec{v}_1^2 = \vec{u}_1^2 + \vec{u}_2^2 + \underbrace{2\vec{u}_1 \cdot \vec{u}_2}_{0}$$

$v_1$ $v_2$ ($v_2=0$) $m_1 \to m_2 \Longleftrightarrow v_1'$ $v_2'$

(2) head-on collision, important conclusions:

$$\begin{cases} v_1' = \dfrac{(m_1-m_2)v_1 + 2m_2v_2}{m_1+m_2} \\[2ex] v_2' = \dfrac{(m_2-m_1)v_2 + 2m_1v_1}{m_1+m_2} \end{cases}$$

→ highly symmetric

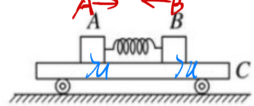
$\boxed{\text{更特别的，} v_2=0 \text{ 时：“大碰小，一起跑；小碰大，要反弹”}}$

$m_1 > m_2$ o o ⇉ $v_1'$ $v_2'$

$m_1 < m_2$ $v_1' \leftarrow$ o o $\rightarrow v_2'$

## 1.3 Exercises

As shown in the figure, the masses of objects $A$ and $B$ satisfy the ratio $m_A : m_B = 1 : 2$. Initially, both objects are at rest on the horizontal platform $C$. There is a massless compressed spring placed between objects $A$ and $B$. The kinetic friction coefficient between both objects and the platform is the same, while the horizontal ground is frictionless.




$f \propto N$

$\mu mg$ $2\mu mg$

$A \rightarrow$ $\leftarrow B$

When the spring is suddenly released, objects $A$ and $B$ are pushed apart (they never slide off the platform during the process). Then ( D ).

$\mu mg \leftarrow$ A B

**A.** The linear momentum of the $A$ and $B$ system is conserved. ✗

**B.** The total mechanical energy of the entire system consisting of $A$, $B$, $C$, and the spring is conserved. ✗

~~**C.**~~ The cart $C$ first moves to the left and then to the right.

C $\rightarrow \mu mg$

**D.** The cart $C$ keeps moving to the right until it comes to rest.

A× B× friction negative work C. $F_{net}$ D √

As is show in the figure, a person pulls a car filled with coal in a constant force $F$. Since there is a leak hole in the bottom of the car, the coal is leaking out of the car with a constant speed of $q$(kg/s). Assume that the car is originally stationary, and the mass is $m_0$ (together with the coal). The person begins pulling the car at $t = 0$. The coal begin leaking at the same time. Try to find the speed of the car at time t.

$(m-qdt)(v+dv)$

$qdt \cdot v$

$dv$ $p$ $Fdt$

Since according to the Momentum-Impulse Theorem,

$$Fdt = dp = (m - qdt)(v + dv) + qdt \cdot v - mv \approx mdv$$

Therefore,

$$F = m\frac{dv}{dt} = (m_0 - qt)\frac{dv}{dt}$$

Therefore,

$$\frac{dt}{m_0 - qt} = \frac{dv}{F}$$

By integration,

$$\frac{1}{q}ln(\frac{m_0}{m_0 - qt}) = \frac{v}{F}$$

Finally,

$$v = \frac{F}{q}ln(\frac{m_0}{m_0 - qt})$$ √

A soft, inextensible rope with uniform mass distribution falls vertically from a height onto the floor. Let the total length of the rope be $l$, and the mass per unit length be $\sigma$. Find the speed of the rope and the force it exerts on the floor when the remaining length of the rope still in the air is $x$. Assume that the rope starts from rest and its lower end is initially at a height $h$ above the floor.

$a$ $l$ $h$ $x$ $dx$ $l-x$

$\cancel{\frac{1}{2}gt_0^2=h}$

$\cancel{v_0=gt_0}$

$(x-dx)\lambda g\,dt = (x-dx)\lambda\,dv$

$x\lambda g\,dx = x\lambda v\,dv$ $\quad \frac{dv}{dt}=g$

$v^2-0^2=2g(h+l-x)$ √

$V \leftrightarrow x$ $\quad a=g$ $\quad N=G$

$v^2-v_0^2=2g\Delta h$

Then analyze the pare on the ground:

$$\begin{cases} Vdm = [N-(l-x)\lambda g]\,dt \\ dm = \lambda dx \\ v^2 = 2g(h+l-x) \end{cases} \Rightarrow N = \lambda g[2h+3(l-x)]$$

## 2. Center of mass

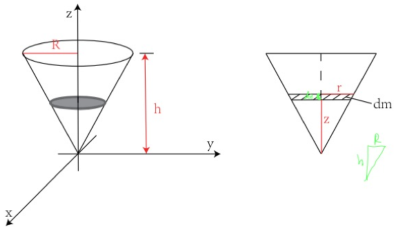
### 2.1 Some concepts

$$\boxed{\bar{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \bar{r}_i}{\sum_{i=1}^{N} m_i}} \rightarrow \text{discrete distribution of mass}$$

*Note.* The sum is replaced by an integral for a continuous distribution of mass.

$$\boxed{M\bar{v}_{cm} = \sum_{i=1}^{N} \bar{p}_i = \bar{p}} \qquad \text{where } M = \sum_{i=1}^{N} m_i$$

Find the center of mass of a non-uniform cone with $\varrho(\bar{r}) = \alpha z$.




From symmetry: $x_{cm} = 0$, $y_{cm} = 0$. To find $z_{cm}$, divide the cone into infinitesimal slices of mass $dm = \varrho(z)\pi r^2\, dz$.

But $\frac{r}{R} = \frac{z}{h}$, hence $r = R\frac{z}{h}$ and $dm = \alpha z \pi R^2 \frac{z^2}{h^2}\, dz$. Hence

$$\int_{\text{cone}} dm = \int_0^h \alpha\pi \frac{R^2}{h^2} z^3\, dz = \frac{1}{4}\alpha\pi R^2 h^2$$

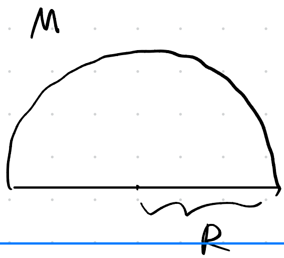
$$\int_{\text{cone}} z\, dm = \int_0^h \alpha\pi \frac{R^2}{h^2} z^4\, dz = \frac{1}{5}\alpha\pi R^2 h^3$$

Eventually

$$\boxed{z_{cm}} = \frac{\int_{\text{cone}} z\, dm}{\int_{\text{cone}} dm} = \boxed{\frac{4}{5}h}$$

This kind of integration is important. Make sure you are capable of doing such calculations.

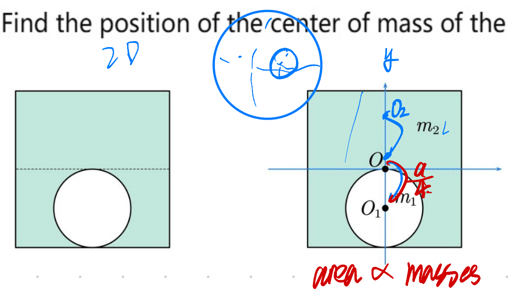
A small practice: find the center of mass of a uniform thin semicircular plate. $M$, $R$, $\sigma = \frac{2M}{\pi R^2}$




$\frac{4R}{3\pi}$

You should also know some tricks… (I think there was one appeared in my final exam last year, similar to the problem below)

A cube of copper with edge length $a$ has a cylindrical hole of radius $a/4$ drilled out from the center of its lower half. Find the position of the center of mass of the remaining copper block.




$$O = \frac{m_1 \cdot \overline{OO_1} + m_2 \overline{OO_2}}{m_1 + m_2}$$

$$m_1 = \pi\left(\frac{a}{4}\right)^2$$

$$m_2 = a^2 - \pi\left(\frac{a}{4}\right)^2$$

$$\frac{|\overline{OO_1}|}{|\overline{OO_2}|} = \frac{m_2}{m_1} = \frac{a^2 - \pi\left(\frac{a}{4}\right)^2}{\pi\left(\frac{a}{4}\right)^2} = \frac{1 - \frac{\pi}{16}}{\frac{\pi}{16}} = \frac{16-\pi}{\pi}$$

$$|\overline{OO_2}| = \frac{a}{4}\frac{\pi}{16-\pi} \qquad \left(0, \frac{a}{4}\frac{\pi}{16-\pi}\right)$$

$$F = \frac{Gm_1m_2}{r^2}$$

There might be some 3 dimensional problem, but the main idea is the same. Later this semester you will know how to derive gravitational force between planets, the idea of mass center will reoccur and function a lot.

## 3. Kinetic Energy of a Rotating Body, Moment of Inertia, Angular Momentum…

Really important!

I strongly suggest you pay full attention to the lectures, or you may get lost easily.

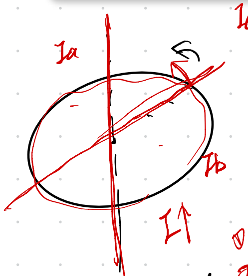
$K$ $\frac{1}{2}mv^2$

The idea is that we start to consider the rotating body. And we should reconsider our definition to kinetic energy. Then torque, which we used to ignore in middle school, is gradually taken into our consideration.

I'll just briefly introduce this part. More will be covered in my next RC.

对于一个在原地旋转的刚体，怎么定义整体的动能？
——每个质点动能求和

$I_a + I_b = I$




a rotating disk

$$K = \sum_{i=1}^{N} K_i = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \rightarrow \text{discrete}$$

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i \omega^2 r_{i\perp}^2 = \frac{1}{2}\boxed{\left(\sum_{i=1}^{N} m_i r_{i\perp}^2\right)}\omega^2 = \boxed{\frac{1}{2} I \omega^2}$$

① rest → ω

② → rest

moment of inertia

转动惯量

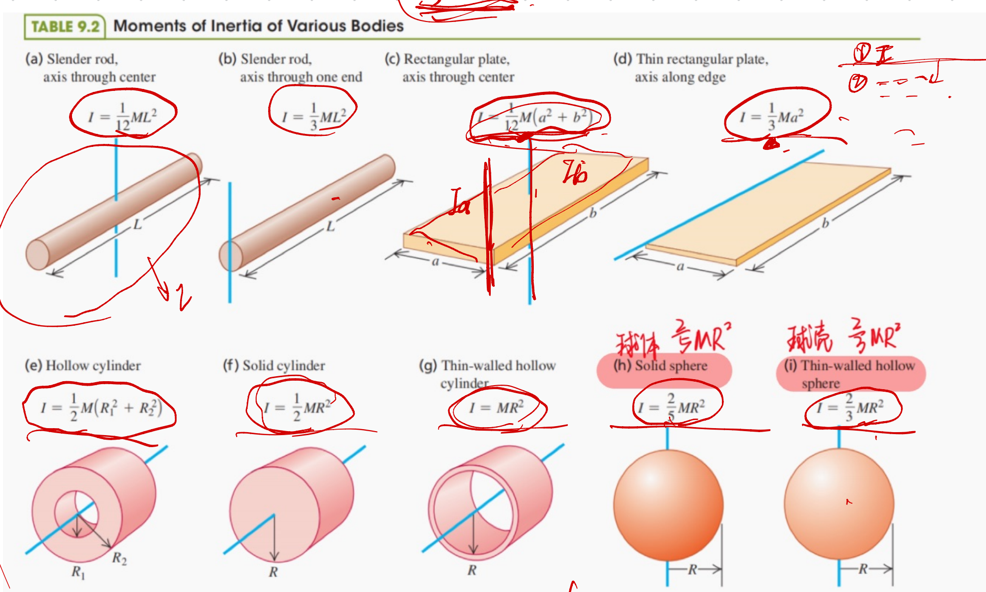
可加性

$$I = \int_{\text{object}} r_\perp^2 \, dm$$

对于一个确定的刚体，只要旋转轴确定了，转动惯量就是固定的，可以理解为这个旋转刚体的一个固有属性（描述物体对于旋转运动的惯性大小，I越大，刚体就越倾向于保持现有的旋转状态）

$\int r^2 dm$

I additive

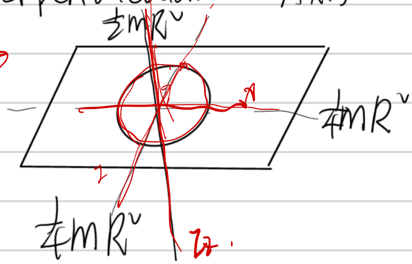



$I \quad \int r^2 dm$

两条定理：垂直轴/平行轴

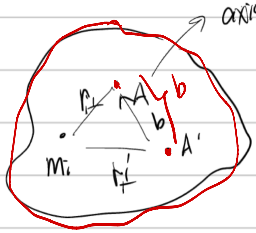
I

Perpendicular Axis Theorem




$I_z = I_x + I_y$

Parallel Axis Theorem (Steiner's Theorem)




axis through center

$\boxed{I_{A'} = I_A + mb^2}$

$I_{A'} = \sum m_i r'^2_{i\perp} = \sum m_i (\vec{r}_{i\perp} + \vec{b})^2 = I_A + mb^2 + 2\underbrace{\sum m_i \vec{r}_{i\perp} \cdot \vec{b}}$

$\Downarrow$

$= m r_{cm\perp} = 0$