Dynamical Systems: Project

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Introduction

The Rössler system, introduced by Otto Rössler in 1976, is a three-dimensional nonlinear dynamical system designed as a simplified model of chaotic behavior. The system's equations describe feedback processes with coupling among variables, leading to diverse behaviors, including periodic oscillations and chaotic trajectories, depending on parameter values. Originally, Rössler used $\alpha = 0.2$, $\beta = 0.2$, and $\gamma = 5.7$, but the parameter set $\alpha = 0.1$, $\beta = 0.1$, and $\gamma = 14$ is now more commonly used in the literature.

Applications of the Rössler system span various fields, such as modeling chemical reactions, studying turbulence in fluid dynamics, and analyzing biological oscillations. A notable feature of the system is the Rössler attractor, a chaotic attractor that arises from its solutions under certain conditions.

This report examines the Rössler system analytically and numerically. The system's fixed points are determined by solving the equations for nullclines and applying the quadratic formula, revealing the existence of zero to two fixed points depending on parameter values. A bifurcation is observed at $\gamma = 1$, marking a critical transition in the system's dynamics.

The numerical analysis is performed using Python, with parameters $\alpha = 0.25$, $\beta = 1$, and $\gamma \in [0, 7]$. Solutions are visualized for $\gamma = 1$ and $\gamma = 300$, illustrating changes in the system's behavior. The analysis includes computing fixed points, the Jacobian matrix, eigenvalues, and eigenvectors to provide a comprehensive understanding of the system's dynamics.

Problem Statement

The system is given by

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + \alpha y\\ \dot{z} &= \beta + z(x - \gamma)\end{aligned}$$

where α, β, and γ are constants.
There is nonlinearity on $\dot{z}$ from the zx term
There is no symmetry.
The nullclines occur at:

$$\begin{aligned}\dot{x} &= -y - z = 0 \text{ which occurs when } y = -z\\ \dot{y} &= x + \alpha y = 0 \text{ which occurs when } x = -\alpha y\\ \dot{z} &= \beta + z(x - \gamma) = 0\end{aligned}$$

Putting all terms in terms of x gets:

$$\begin{aligned}y &= \frac{-x}{\alpha}\\ z &= -y = \frac{x}{\alpha}\end{aligned}$$

Plugging these into $\dot{z}$ gets:

$$\begin{aligned}\dot{z} &= \beta + (-y)(x - \gamma) = \beta + \frac{-(-x)}{\alpha}(x - \gamma) = \beta + \frac{x}{\alpha}(x - \gamma)\\ &= \beta + \frac{x^2}{\alpha} - \frac{\gamma}{\alpha}x = x^2 - \gamma x + \beta\alpha = 0\end{aligned}$$

Solving for x gets the solutions for x^*:

$$x^* = \frac{\gamma \pm \sqrt{\gamma^2 - 4\beta\alpha}}{2}$$

The number of fixed points depend on $\gamma^2 - 4\alpha\beta$:
If $\gamma^2 > 4\beta\alpha$, then there are two real fixed points

$$\text{These occur at } (x^*, y^*, z^*) = (\frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2}, -\frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha}, \frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha})$$
$$\text{and } (x^*, y^*, z^*) = (\frac{\gamma - \sqrt{\gamma^2 - 4\beta\alpha}}{2}, -\frac{\gamma - \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha}, \frac{\gamma - \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha})$$

If $\gamma^2 = 4\beta\alpha$, there there is one fixed point when $x^* = \frac{\gamma}{2}$

$$\text{This occurs at } (x^*, y^*, z^*) = (\frac{\gamma}{2}, -\frac{\gamma}{2\alpha}, \frac{\gamma}{2\alpha})$$

If $\gamma < 4\beta\alpha$, there are no real fixed points.

The Jacobian is given by $J(x, y, z) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ z & 0 & x - \gamma \end{bmatrix}$

$J - \lambda I = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ z & 0 & x - \gamma \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & \alpha - \lambda & 0 \\ z & 0 & x - \gamma - \lambda \end{bmatrix}$

By wolfgram alpha:

$$\tau = 0 + \alpha + (x - \gamma) = x + \alpha - \gamma$$
$$\Delta = x + \alpha z - \gamma$$
$$\text{The characteristic is given by } \lambda^3 - \lambda^2(\alpha + x - \gamma) + \lambda(\alpha x + 1 + z - \alpha\gamma) + \gamma - x - \alpha z = 0$$

For $(x^*, y^*, z^*) = (\frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2}, -\frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha}, \frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha})$:

$$\begin{aligned} \Delta = x + \alpha z - \gamma &= \frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2} + \alpha\frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha} - \gamma \\ &= \frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2} + \alpha\frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2} - \gamma = \gamma + \sqrt{\gamma^2 - 4\beta\alpha} - \gamma \\ &= \sqrt{\gamma^2 - 4\beta\alpha} > 0 \\ \tau &= x + \alpha - \gamma \\ &= \frac{\gamma + \sqrt{\gamma^2 - 4\beta\alpha}}{2} + \alpha - \gamma = \frac{\sqrt{\gamma^2 - 4\beta\alpha} - \gamma + 2\alpha}{2} \\ \tau &> 0 \text{ if } \gamma < 2\alpha + \sqrt{\gamma^2 - 4\beta\alpha} \\ \tau &< 0 \text{ if } \gamma > 2\alpha + \sqrt{\gamma^2 - 4\beta\alpha} \\ \tau &= 0 \text{ if } \gamma = 2\alpha + \sqrt{\gamma^2 - 4\beta\alpha} \end{aligned}$$

For $(x^*, y^*, z^*) = (\frac{\gamma - \sqrt{\gamma^2 - 4\beta\alpha}}{2}, -\frac{\gamma - \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha}, \frac{\gamma - \sqrt{\gamma^2 - 4\beta\alpha}}{2\alpha})$:

$$\Delta = -\sqrt{\gamma^2 - 4\beta\alpha} < 0$$
$$\tau = \frac{2\alpha - \sqrt{\gamma^2 - 4\beta\alpha} - \gamma}{2}$$
$$\tau > 0 \text{ if } 2\alpha > \gamma + \sqrt{\gamma^2 - 4\beta\alpha}$$
$$\tau < 0 \text{ if } 2\alpha < \gamma + \sqrt{\gamma^2 - 4\beta\alpha}$$
$$\tau = 0 \text{ if } 2\alpha = \gamma + \sqrt{\gamma^2 - 4\beta\alpha}$$

For $(x^*, y^*, z^*) = (\frac{\gamma}{2}, -\frac{\gamma}{2\alpha}, \frac{\gamma}{2\alpha})$:

$$\begin{aligned}
\Delta &= x + \alpha z - \gamma = \frac{\gamma}{2} + \alpha\frac{\gamma}{2\alpha} - \gamma \\
&= \frac{\gamma}{2} + \frac{\gamma}{2} - \gamma \\
&= \gamma - \gamma = 0 \\
\tau &= x + \alpha - \gamma = \frac{\gamma}{2} + \alpha - \gamma \\
&= \frac{\gamma}{2} + \alpha - \gamma \\
&= \alpha - \frac{\gamma}{2} \\
&\tau > 0 \text{ if } 2\alpha > \gamma \\
&\tau = 0 \text{ if } 2\alpha = \gamma \\
&\tau < 0 \text{ if } 2\alpha < \gamma
\end{aligned}$$

A bifurcation occurs at $\gamma^2 = 4\alpha\beta$

Numerical Approaches

The fixed point is at $(0.5, -2, 2)$ for $\alpha = 0.25$, $\beta = 1$, and $\gamma \in [0, 7]$. It has the Jacobian is given by $A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \frac{1}{4} & 0 \\ z & 0 & x-1 \end{bmatrix}$

The solved Jacobian is given by $J0 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \frac{1}{4} & 0 \\ 2 & 0 & -0.5 \end{bmatrix}$

The eigenvector is given by $\begin{bmatrix} -0.1741 & 0.1326 & 0.1326 \\ 0.6963 & 0.3204 & 0.3204 \\ -0.6963 & 0.7071 & 0.7071 \end{bmatrix}$

The diagonal is given by $\begin{bmatrix} 2.679 & 0 & 0 \\ 0 & -0.125 & 0 \\ 0 & 0 & -0.125 \end{bmatrix}$

Our code gets $\Delta = 0$ and $\tau = \frac{-1}{4}$ which means the bifurcation at $\gamma = 1$ is a hopf bifurcation.
A new fixed point is created when $\gamma > 1$, so our hopf bifurcation turns into a saddle node and our new fixed point is a stable spiral.
At $\gamma = 2$ the saddle node persists and the stable spiral becomes unstable. This behavior continues and grows bigger as γ increases.
We can see how tight the system is right next to the fixed point and it immediately exploding out into a saddle node bifurcation.
A similar bifurcation still occurs extremely close to the origin of R^3 and the latter bifurcation location grows and grows as its directly related to the value of γ which is rising. The unstable spiral does creep ever closer to a center as the determinant grows much faster than the trace

Results/Examples

Numerically we see $(\frac{\gamma}{2}, -\frac{\gamma}{2\alpha}, \frac{\gamma}{2\alpha})$ is an isolated fixed point and $(\frac{\gamma-\sqrt{\gamma^2-4\beta\alpha}}{2}, -\frac{\gamma-\sqrt{\gamma^2-4\beta\alpha}}{2\alpha}, \frac{\gamma-\sqrt{\gamma^2-4\beta\alpha}}{2\alpha})$ is a saddle node. While $(\frac{\gamma+\sqrt{\gamma^2-4\beta\alpha}}{2}, -\frac{\gamma+\sqrt{\gamma^2-4\beta\alpha}}{2\alpha}, \frac{\gamma+\sqrt{\gamma^2-4\beta\alpha}}{2\alpha})$ depends on $\frac{\sqrt{\gamma^2-4\beta\alpha}-\gamma+2\alpha}{2}$.

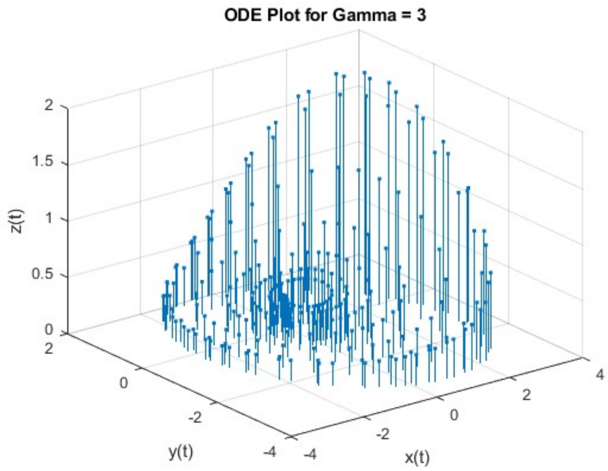

Figure 1: ODE Plot for Gamma = 3

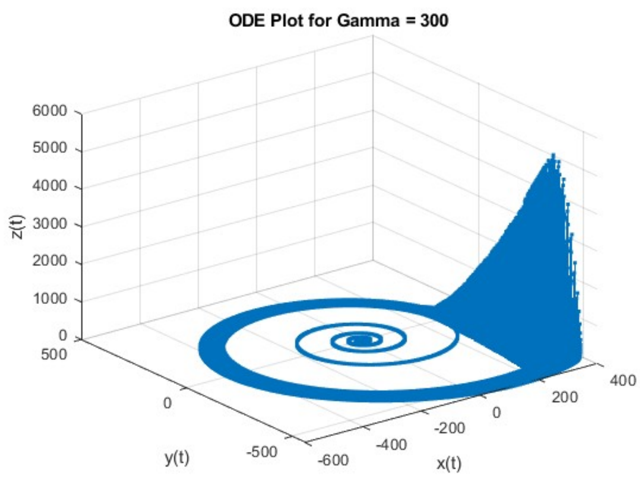

Figure 2: ODE Plot for Gamma = 300

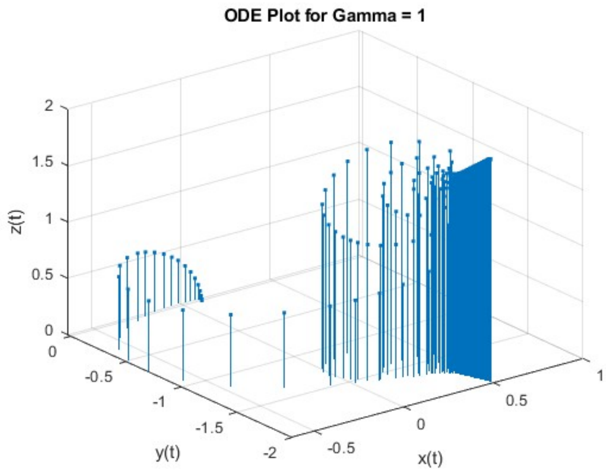

Figure 3: ODE Plot for Gamma = 1

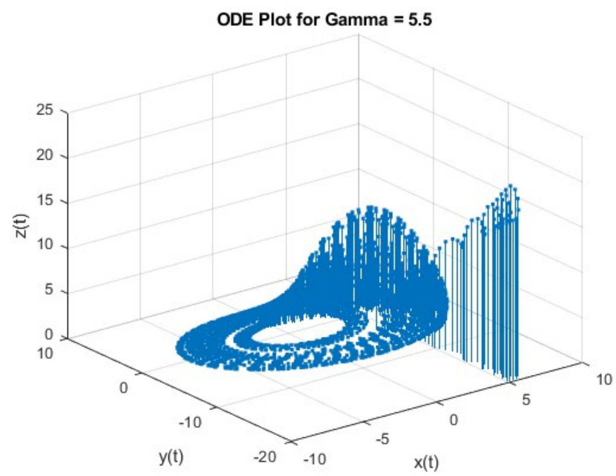

Figure 4: ODE Plot for Gamma = 5.5

This is the code used to generate figure 1.

```
a = 0.25;              %alpha
b = 1;                 %beta
gamma = 5;             %gamma

x0 = 0.1;              %intial values for the plot
y0 = 0;
z0 = 0;

function dxdt = func(t,x,a,b,gamma)
%function that defines the 3d system
  dxdt = zeros(3,1);
  dxdt(1) = (-x(2)-x(3));
  dxdt(2) = (x(1)+a*x(2));
  dxdt(3) = (b + x(3)*(x(1)-gamma));
```

```
end

tspan = [0 500];
%span of time to plot
[t,x] = ode23(@(t,x)func(t,x,a,b,gamma),tspan,[x0;y0;z0]);
%solves the 3d system with respect to t
xsol = x(:,1);
%solution array of x, y, and z vals
ysol = x(:,2);
zsol = x(:,3);

figure
stem3(xsol,ysol, zsol, '.');
grid on
xlabel('x(t)')
ylabel('y(t)')
zlabel('z(t)')
title(['ODE Plot for Gamma = ', num2str(gamma)]);
%creates a 3d figure to plot 3 components
```

This is the code used to the find the fixed point, jacobian, eigenvalues and eigenvectors.

```
a = 0.25;
b = 1;
gamma = 1;
%changed manually to generate different figures and values

syms x y z
Y = vpasolve([-y-z == 0, x+a*y == 0,b+ z*(x-gamma) == 0], [x,y,z]);
%solves each equation for 0
if (gamma >1)
fp = [Y.x(1,1),Y.y(1,1),Y.z(1,1),Y.x(2,1),Y.y(2,1),Y.z(2,1)]
%fixed points
else
fp = [Y.x(1,1),Y.y(1,1),Y.z(1,1)]
end

A = jacobian([-y-z,x+a*y,b+ z*(x-gamma)],[x,y,z])
%Finds the Jacobian Matrix
J0 = subs(A, [x,z], [fp(1),fp(3)])
%Subs the fps into the Jac. Matrix
[V,D] = eig(J0);
%Finds the eigenvectors and diagonal
real(V)
%displays the real eigenvectors
real(D)
%displays the real diagonal
if (gamma > 1)
J1 = subs(A, [x,z], [fp(4),fp(6)])
%Subs the second fps into the Jac. Matrix
[V,D] = eig(J1);
%Finds the eigenvectors and diagonal
real(V)
%displays the real eigenvectors
real(D)
```

```
end
```

This is the code used to find the determinant and trace.

```
d = det(J0)
tr = trace(J0)
```

This is the code used to get Figure 2, Figure 3, and Figure 4 and to analyze the system next to fixed points and the behavior of the system.

```
x0 = 0.1;
y0 = 0;
z0 = 0;
%starting point for data plotting, changed manually to
view at different fixed points

function dxdt = func(t,x,a,b,gamma)
  dxdt = zeros(3,1);
  dxdt(1) = (-x(2)-x(3));
  dxdt(2) = (x(1)+a*x(2));
  dxdt(3) = (b + x(3)*(x(1)-gamma));
end

tspan = [0 500];
[t,x] = ode23(@(t,x)func(t,x,a,b,gamma),tspan,[x0;y0;z0]);
xsol = x(:,1);
ysol = x(:,2);
zsol = x(:,3);

figure
stem3(xsol,ysol, zsol, '.');
grid on
xlabel('x(t)')
ylabel('y(t)')
zlabel('z(t)')
title(['ODE Plot for Gamma = ', num2str(gamma)]);

if (gamma >1)
d = det(J1)
tr = trace(J1)
end
%determinant and trace for the second set of fixed points if they exist
```

Our numerical and analytic results agree. The bifurcations (Hopf and saddle-node) are observed numerically and are consistent with the eigenvalue-based stability analysis. The chaotic behavior predicted for higher γ values is evident in the numerical trajectories, confirming the system's sensitivity to parameter changes. The fixed points, Jacobian, and eigenvalues calculated analytically are validated by the numerical observations near these points. Additionally, as γ increases significantly (e.g., $\gamma > 5.5$), the chaotic attractor grows larger, and the oscillatory behavior becomes more pronounced. This suggests that the system's sensitivity to initial conditions increases with higher γ. The progression from a stable spiral to an unstable one and the emergence of chaos demonstrates the intricate dynamics of the Rössler system.

Conclusion

This report analyzed the dynamics of the Rössler system using both analytical and numerical approaches. The system was found to have zero to two fixed points depending on the parameters α, β, and γ. Stability analysis,

conducted using the Jacobian matrix, revealed the conditions under which these fixed points are stable or unstable, with bifurcations occurring as parameter values change. Notably, a Hopf bifurcation was identified at $\gamma = 1$, marking the transition from stable fixed-point behavior to periodic oscillations. A saddle-node bifurcation was also observed as γ increased, introducing new fixed points that transition from stable to unstable with further parameter changes.
Numerical simulations confirmed the analytical findings, illustrating the system's evolution and the onset of chaotic behavior at higher values of γ. The Rössler attractor demonstrated increasing complexity, reflecting sensitivity to initial conditions, a defining characteristic of chaos.

References

http://www.scholarpedia.org/article/Rossler_attractor
https://www.cfm.brown.edu/people/dobrush/am34/Mathematica/ch3/rossler.html