Project - Dirichlet Problems

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Problem 1

Let the boundary temperatures be given by $T(x) = \begin{cases} 100, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0 \end{cases}$. Find the steady state temperature distribution in H.

1. What should the temperature be along the positive part of the imaginary axis? Why?
   The temperature along the positive part of the imaginary axis should be 50. The positive imaginary axis is the half way point between where the boundary temperature is 100 and 0, thus it is being influenced by both temperatures equally.

2. We should expect the temperature to be constant along any ray emanating from the origin. We know on the positive real axis the temperature is 0, and on the negative real axis, the temperature is 100. It seems reasonable that the temperature would vary linearly as the angle changes from $\theta = 0$ to $\theta = \pi$. Find a linear function that describes this situation.
   Temperature is given by $100 * \frac{\theta}{\pi}$

3. We know know the value of the temperature along every ray. Thus, we can write $\psi(x, y)$ as a expression involving $Arg(z)$.

$$\begin{aligned}\psi(x, y) &= \frac{100}{\pi} Arg(z) \\ &= \frac{100}{\pi} Arg(x + iy)\end{aligned}$$

4. We need to show $\psi(x, y)$ is harmonic. Recall that the real and imaginary parts of an analytic function are harmonic. Write down an analytic function $\Omega(z)$ for which $\psi(x, y)$ is the imaginary part. Keep in mind that we only need the function to be analytic where $y > 0$, not on the boundary. Let $\phi(x, y)$ be the real part of $\Omega(z)$.

$$\begin{aligned}\Omega(z) &= \frac{100}{\pi} Ln(z) \\ &= \frac{100}{\pi} (\ln |z| + iArg(z)) \\ \phi(x, y) &= \frac{100}{\pi} \ln \sqrt{x^2 + y^2} \\ \psi(x, y) &= \frac{100}{\pi} Arg(x + iy)\end{aligned}$$

5. Use a computer to plot the level curves of both the real and imaginary parts of $\Omega(z)$. You should get rays and semicircles. The rays are the lines of constant temperature (isotherms) and the semicircles are the lines of heat flux.

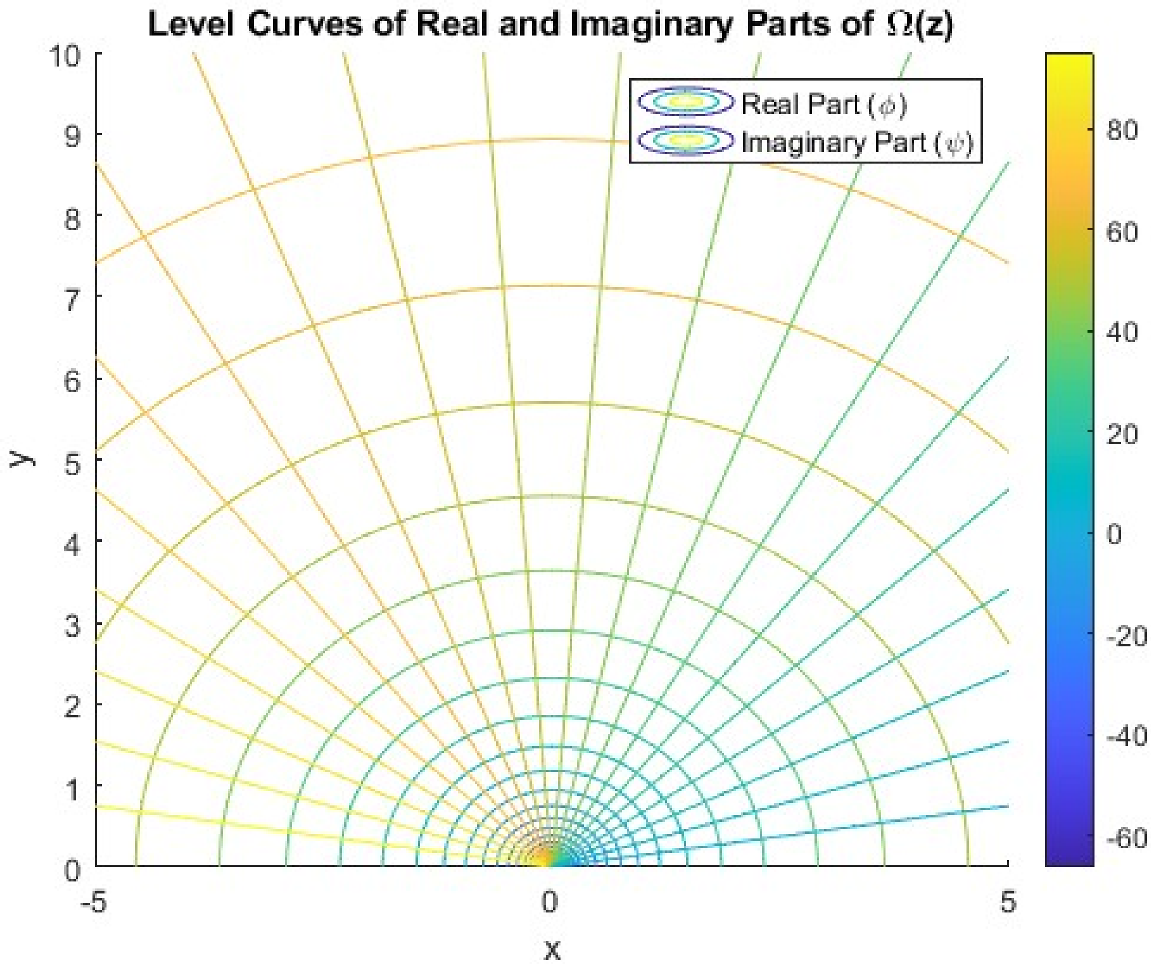

Figure 1: Problem 1 - level curves of $\Omega(z)$

Problem 2

We wish to generalize the solution in Problem 1 to allow for any two temperatures, and to have the discontinuity occur at any point on the real axis.

1. Modify your work from Problem 1 so that the temperature varies from T_0 on the positive part of the real axis to T_1 on the negative part of the real axis.

$$T(x) = \begin{cases} T_1, & \text{if } x < 0, \\ T_0, & \text{if } x \geq 0 \end{cases}$$

   The positive imaginary axis has a temperature of $\frac{T_1+T_0}{2}$
   Thus, the temperature is given by $(T_1 - T_0)\frac{\theta}{\pi} + T_0$

$$\begin{aligned} \psi(x,y) &= Arg(z)\frac{T_1 - T_0}{\pi} + T_0 \\ &= Arg(x+iy)\frac{T_1 - T_0}{\pi} + T_0 \\ \Omega(z) &= Ln(z)\frac{T_1 - T_0}{\pi} + T_0 i \\ &= (ln|z| + iArg(z))\frac{T_1 - T_0}{\pi} + T_0 i \\ \phi(x,y) &= \frac{T_1 - T_0}{\pi} \ln \sqrt{x^2 + y^2} \\ \psi(x,y) &= Arg(x+iy)\frac{T_1 - T_0}{\pi} + T_0 \end{aligned}$$

2. Next, modify the answer to the previous step by allowing the jump in the temperature to occur at x_1 on the real axis. This is easily accomplished by simply shifting the z to $z - x_1$ in your answer to part 1.
   $T(x) = \{ {}^{T_1,\ x<x_1}_{T_0,\ x \geq x_1}$

$$\begin{aligned} \Omega(z - x_1) &= Ln(z - x_1)\frac{T_1 - T_0}{\pi} + T_0 i \\ &= (ln|z - x_1| + iArg(z - x_1))\frac{T_1 - T_0}{\pi} + T_0 i \\ \phi(x,y) &= \frac{T_1 - T_0}{\pi} \ln \sqrt{(x - x_1)^2 + y^2} \\ \psi(x,y) &= Arg((x - x_1) + iy)\frac{T_1 - T_0}{\pi} + T_0 \end{aligned}$$

3. Finally, obtain the analytic function $\Omega(z)$ for which the imaginary part is the desired harmonic function.

$$\begin{aligned} \Omega(z) &= Ln(z - x_1)\frac{T_1 - T_0}{\pi} + T_0 i \\ &= (ln|z - x_1| + iArg(z - x_1))\frac{T_1 - T_0}{\pi} + T_0 i \end{aligned}$$

Problem 3

From the previous problem, it should be clear that shifted argument functions play a role in matching the boundary conditions. This problem explores what happens if we have more than two temperatures along the boundary.

1. Assume $x_1 < x_2$. Consider the function $Arg(Z - x_1) + Arg(z - x_2)$. Explain what this function gives for the three cases $x < x_1$, $x_1 < x < x_2$, and $x_2 < x$
   For $x < x_1$, x is to the left of both x_1 and x_2. On the real axis,
   $Arg(Z - x_1) + Arg(z - x_2) = \pi + \pi = 2\pi$.
   For $x_1 < x < x_2$, x is to the right of x_1 and to the left of x_2. On the real axis,
   $Arg(Z - x_1) + Arg(z - x_2) = 0 + \pi = \pi$.
   For $x_2 < x$, x is to the right of both x_1 and x_2. On the real axis,
   $Arg(Z - x_1) + Arg(z - x_2) = 0 + 0 = 0$.

2. If we modify the expression in part 1 to be $c_0 + c_1 Arg(z - x_1) + c_2 Arg(z - x_2)$, then explain how to determine a system of equation to find the coefficients that satisfy the boundary conditions. You should also be able to find a general solution that gives c_0, c_1, and c_2 in terms of the three temperatures.

$$T(x) = \begin{cases} T_2, & \text{if } x < x_1, \\ T_1, & \text{if } x_1 < x < x_2, \\ T_0, & \text{if } x_2 < x \end{cases}$$

$$\begin{aligned}
&T_2 = c_0 + c_1\pi + c_2\pi = c_0 + \pi(c_1 + c_2) \\
&T_1 = c_0 + c_1(0) + c_2(\pi) = c_0 + c_2(\pi) \\
&T_0 = c_0 + c_1(0) + c_2(0) = c_0 \\
&c_0 = T_0 \\
&c_2 = \frac{T_1 - c_0}{\pi} = \frac{T_1 - T_0}{\pi} \\
&c_1 = \frac{T_2 - c_0}{\pi} - c_2 = \frac{T_2 - T_0}{\pi} - \frac{T_1 - T_0}{\pi} = \frac{T_2 - T_1}{\pi}
\end{aligned}$$

   Thus $c_0 = T_2$, $c_1 = \frac{T_2 - T_1}{\pi}$, and $c_2 = \frac{T_1 - T_0}{\pi}$

3. Use the formula you found to solve the Dirichlet problem $\nabla^2\phi = 0$ in H with boundary conditions given by

$$T(x) = \begin{cases} 4, & \text{if } x < -2, \\ 7, & \text{if } -2 < x < 3, \\ 1, & \text{if } x > 3 \end{cases}$$

$$\begin{aligned}
\phi(z) &= c_0 + c_1 Arg(z - x_1) + c_2 Arg(z - x_2) = T_0 + \frac{T_2 - T_1}{\pi} Arg(z - x_1) + \frac{T_1 - T_0}{\pi} Arg(z - x_2) \\
&= 1 + \frac{4 - 7}{\pi} Arg(z - (-2)) + \frac{7 - 1}{\pi} Arg(z - 3) \\
&= 1 - \frac{3}{\pi} Arg(z + 2) + \frac{6}{\pi} Arg(z - 3)
\end{aligned}$$

4. Construct an analytic function $\Omega(z)$ for which the harmonic function found in the previous step is the imaginary

part.

$$\Omega(z) = c_0 i + c_1 Ln(z - x_1) + c_2 Ln(z - x_2)$$
$$= i - \frac{3}{\pi} Ln(z+2) + \frac{6}{\pi} Ln(z-3)$$

5. Create a plot that shows the level curves of the real and imaginary parts of Ω

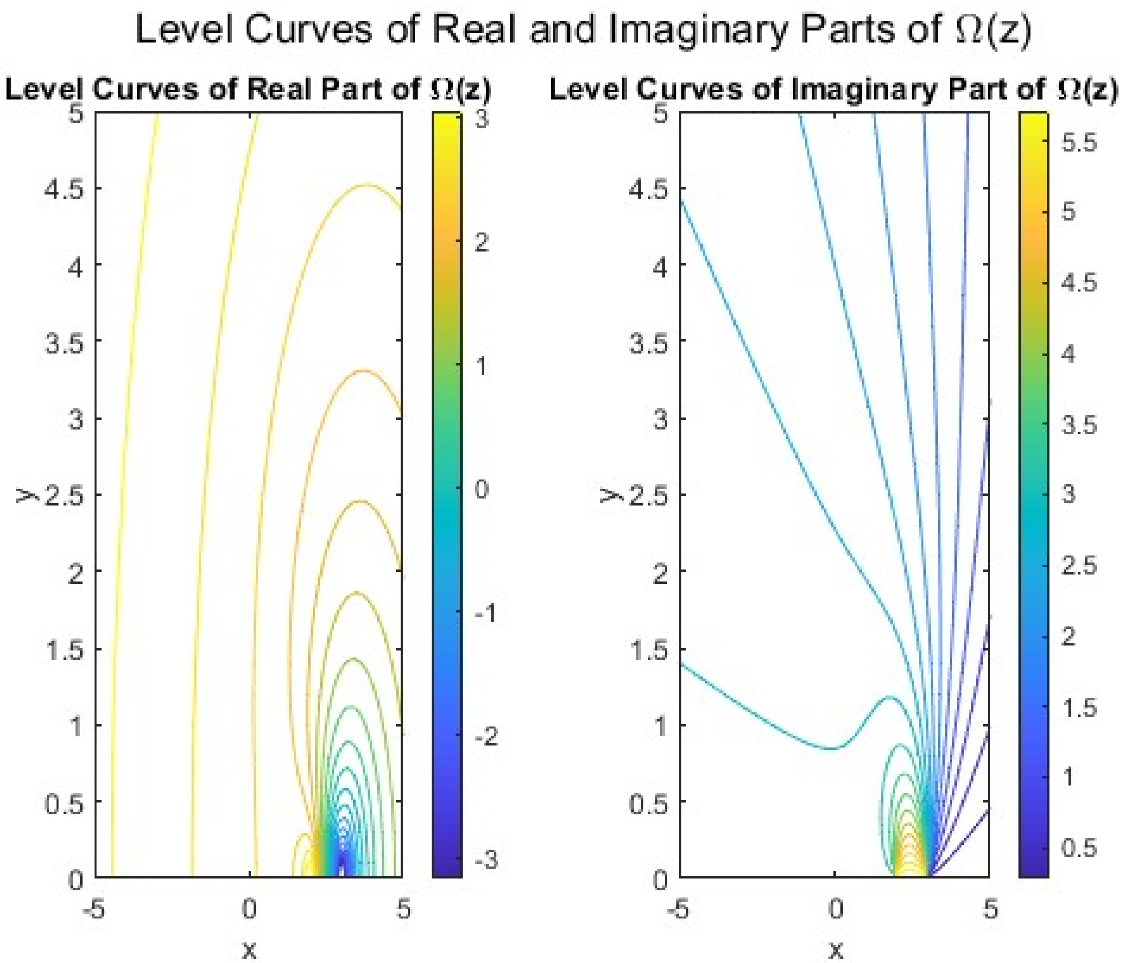

Figure 2: Question 3 - Level curves of $\Omega(z)$

6. Use the analytic function found in the previous step and a computer or calculator to determine the temperature

in H at the points $1+i$, $2i$, and $-1+i$

$$\begin{aligned}
\Omega(1+i) &= 0.4375 + 5.8072i \\
T(1+i) &= 5.8072 \\
\Omega(2i) &= 1.4565 + 5.1270i \\
T(2i) &= 5.1270 \\
\Omega(-1+i) &= 2.3746 + 5.7821i \\
T(-1+i) &= 5.7821
\end{aligned}$$

The temperature at $1+i$ is 5.8072 degrees. The temperature at $2i$ is 5.1270 degrees. The temperature at $-1+i$ is 5.7821 degrees.

Problem 4

It is possible to solve the Dirichlet problem in regions other than H. For example. suppose we wanted to solve the Dirichlet problem in the 1^{st} quadrant. Now, the boundary consists of the positive parts of the real and imaginary axes.

1. Suppose the temperature distribution on the boundary is given as follows: $T(x) = 0$ for $x > 2$; $T(yi) = 10$ for $y > 3$, and the temperature is 5 for bounds on the boundary between 2 and 39. Sketch the region and label the boundary with these temperatures.

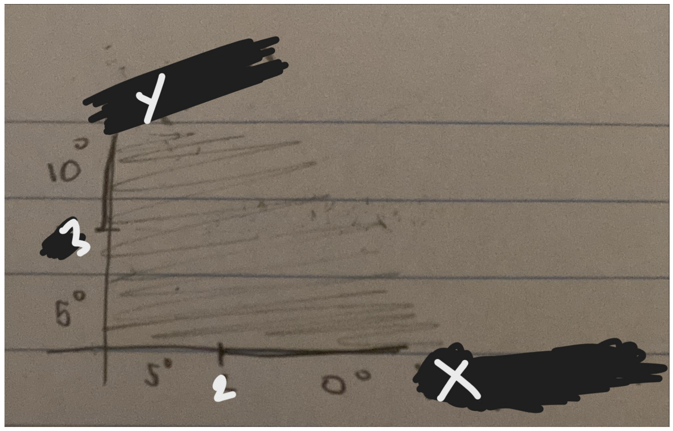

Figure 3: Problem 4 - Region Sketch

2. The key to solving this type of problem is to find an analytic function g(z) that maps our region of interest to the half plane H (and maps the boundary of our region to the real axis). In this example, explain why $g(z) = z^2$ maps the 1^{st} quadrant onto the upper half plane.

$$\begin{aligned} g(z) &= z^2 \\ &= (x + iy)^2 \\ &= x^2 - y^2 + 2ixy \end{aligned}$$

The imaginary part of $g(z)$ is given by $2ixy$ and since z is in the 1^{st} quadrant so $x, y > 0$ which makes $2ixy > 0$ which is the upper half plane.
The real part of $g(z)$ is given by $x^2 - y^2$ which is negative if $y > x$ and positive if $x > y$.
Thus $g(z) = z^2$ maps the first quadrant onto the upper half plane.

3. In the original problem, the points of discontinuity for the temperature along the boundary were 2 and $3i$. Where does g send these points.

$$\begin{aligned} g(2 + 0i) &= (2)^2 = 4 + 0i \\ g(3i) &= (3i)^2 = -9 + 0i \end{aligned}$$

4. using earlier work from this project, solve the transformed problem on H to obtain $\Omega(z)$

$$T(x) = \begin{cases} 10, & \text{if } x < -9, \\ 5, & \text{if } -9 < x < 4, \\ 0, & \text{if } x > 4 \end{cases}$$

$$
\begin{aligned}
\Omega(z) &= c_0 i + c_1 Ln(z - x_1) + c_2 Ln(z - x_2) = T_0 + \frac{T_2 - T_1}{\pi} Ln(z - x_1) + \frac{T_1 - T_0}{\pi} Ln(z - x_2) \\
&= 0i + \frac{10 - 5}{\pi} Ln(z - (-9)) + \frac{5 - 0}{\pi} Ln(z - 4) \\
&= 0i + \frac{5}{\pi} Ln(z + 9) + \frac{5}{\pi} Ln(z - 4)
\end{aligned}
$$

5. Using your solution from the previous problem, the solution to the original problem in the 1^{st} quadrant is obtained by composing Ω and g. We obtain our solution as $\Omega_1(z) = \Omega(g(z))$. Find an expression for this solution and then plot the contours of the real and imaginary parts in the 1^{st} quadrant. Make sure the discontinuities on the boundary are where they should be.

$$
\Omega_1(z) = \frac{5}{\pi} Ln(z^2 + 9) + \frac{5}{\pi} Ln(z^2 - 4)
$$

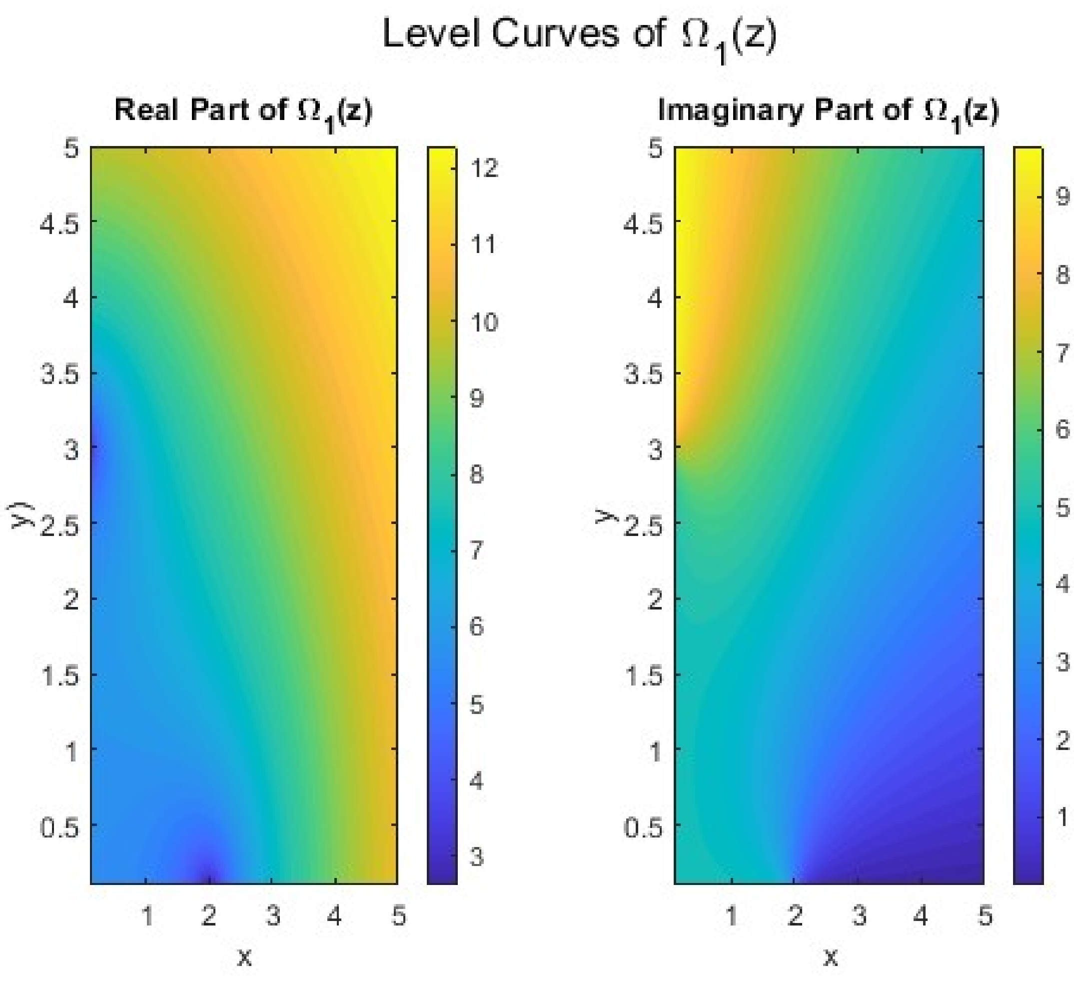

Figure 4: Problem 5 - Contours

Appendix

Code for Problem 1 step 5 from ChatGPT.

```
% Set up the grid for plotting
[x, y] = meshgrid(-5:0.1:5, 0:0.1:10); % Create a grid from -5 to 5 in both x and y
    directions
z = x + 1i * y; % Define complex number z = x + iy

% Calculate the real and imaginary parts of Omega(z)
phi = (100 / pi) * log(abs(z)); % Real part: phi = (100 / pi) * ln(sqrt(x^2 + y^2))
psi = (100 / pi) * angle(z); % Imaginary part: psi = (100 / pi) * arg(z)

% Plot both the real and imaginary parts of Omega(z) on the same graph
figure;
hold on;

% Plot level curves for the real part (phi)
contour(x, y, phi, 20, 'ShowText', 'off'); % Contour plot for real part with 20 levels

% Plot level curves for the imaginary part (psi)
contour(x, y, psi, 20, 'ShowText', 'off'); % Contour plot for imaginary part with 20
    levels

% Labels and title
title('Level␣Curves␣of␣Real␣and␣Imaginary␣Parts␣of␣\Omega(z)');
xlabel('x');
ylabel('y');
colorbar;

% Add a legend for clarity
legend('Real␣Part␣(\phi)', 'Imaginary␣Part␣(\psi)');

% Hold off to stop adding to the current figure
hold off;
```

Code for Problem 3 step 5 from ChatGPT

```
% Define the grid for the complex plane
[x, y] = meshgrid(-5:0.1:5, -5:0.1:5); % Create a grid from -5 to 5 in both x and y
    directions
z = x + 1i * y; % Define complex number z = x + iy

% Define the boundary values
x1 = -2;  % Boundary point 1
x2 = 3;   % Boundary point 2

% Calculate the real and imaginary parts of Omega(z)
phi = 1 - (3/pi)*log(abs(z + x1)) + (6/pi)*log(abs(z - x2)); % Real part of Omega(z)
psi = -(3/pi)*angle(z + x1) + (6/pi)*angle(z - x2); % Imaginary part of Omega(z)

% Create the figure
figure;

% Plot level curves for the real part (phi)
subplot(1, 2, 1); % Plot on the left side of the figure
contour(x, y, phi, 20, 'ShowText', 'on'); % Contour plot for real part with 20 levels
title('Level␣Curves␣of␣Real␣Part␣of␣\Omega(z)');
xlabel('x');
ylabel('y');
```

```
colorbar;

% Plot level curves for the imaginary part (psi)
subplot(1, 2, 2); % Plot on the right side of the figure
contour(x, y, psi, 20, 'ShowText', 'on'); % Contour plot for imaginary part with 20
    levels
title('Level Curves of Imaginary Part of \Omega(z)');
xlabel('x');
ylabel('y');
colorbar;

% Add a super title for both plots
sgtitle('Level Curves of Real and Imaginary Parts of \Omega(z)');
```

Code for Problem 3 Step 6 made by me.

```
% Define the temperature for the boundar
c0 = 1;
c1 = -3 / pi;
c2 = 6 / pi;

% Define the three points
points = [1 + 1i, 2i, -1 + 1i];

% Boundary points
x1 = -2;
x2 = 3;

% Loop over each point to calculate Omega(z)
for k = 1:length(points)
    z = points(k);

    % Calculate the terms for (z - x1) and (z - x2)
    z_x1 = z - x1;
    z_x2 = z - x2;

    % Calculate the modulus and argument for each point
    mod_z_x1 = abs(z_x1); % |z - x1|
    arg_z_x1 = angle(z_x1); % Arg(z - x1)

    mod_z_x2 = abs(z_x2); % |z - x2|
    arg_z_x2 = angle(z_x2); % Arg(z - x2)

    % Calculate the logarithmic parts
    log_z_x1 = log(mod_z_x1) + 1i * arg_z_x1; % ln(z - x1)
    log_z_x2 = log(mod_z_x2) + 1i * arg_z_x2; % ln(z - x2)

    % Calculate Omega(z)
    Omega_z = c0 * 1i + c1 * log_z_x1 + c2 * log_z_x2;

    % Display the result for the current point
    disp(['Omega(' num2str(z) ') = ']);
    disp(Omega_z);
end
```

Code for Problem 4 step 5 from ChatGPT

```
% Define the function Omega_1(z)
Omega_1 = @(z) 0i + (5 / pi) * log(z.^2 + 9) + (5 / pi) * log(z.^2 - 4);

```

```
% Create a grid of complex numbers in the first quadrant
x_vals = linspace(0.1, 5, 50);  % Real part
y_vals = linspace(0.1, 5, 50);  % Imaginary part
[X, Y] = meshgrid(x_vals, y_vals);
Z = X + 1i * Y;  % Combine real and imaginary parts to form complex grid

% Evaluate Omega_1(z) on the grid
Omega_values = Omega_1(Z);

% Extract the real and imaginary parts of Omega_1(z)
real_Omega = real(Omega_values);
imag_Omega = imag(Omega_values);

% Create contour plots for both real and imaginary parts
figure;

% Plot the real part of Omega_1(z)
subplot(1, 2, 1);
contourf(X, Y, real_Omega, 50, 'LineColor', 'none');
colorbar;
title('Real Part of \Omega_1(z)');
xlabel('x');
ylabel('y)');

% Plot the imaginary part of Omega_1(z)
subplot(1, 2, 2);
contourf(X, Y, imag_Omega, 50, 'LineColor', 'none');
colorbar;
title('Imaginary Part of \Omega_1(z)');
xlabel('x');
ylabel('y');

% Adjust layout for better display
sgtitle('Level Curves of \Omega_1(z)');
```