Relating defect and total displacement - proofs edition

I can explain at least some of the relation between s_k the genralized defect function and d the total displacement function. The key trick (maybe a technique at this point) is to think about the action of the diagonal subgroup on both s_k and on characters.

Let $e = (1 \cdots 1)$ be the generator of the diagonal subgroup. Note that contrary to the normal notation, multiplicative notation is used for the C_m^n (maybe it doensn't make sense to do that, but we are here for now).

First, consider the action of e^d on s_k. Note that if you increase everyone's preference by i, then the number of cars that passes the kth spot becomes the number of cars that pass the $i+k$th spot. Therefore,

$$e^d \cdot s_k = s_{k+d}.$$

Now consider the action of e^d on a character χ_g for $g \in C_n^m$. Recall that χ_g is defined as the following where ω is an nth root of unity:

$$\chi_g(h) = \omega^{\sum_{i=0}^m g_i h_i}$$

Therefore,

$$\begin{aligned}
e^d \cdot \chi_g(h) &= \chi_g(e^{-d}h) \\
&= \omega^{\sum_{i=0}^m g_i (e^{-d}h)_i} \\
&= \omega^{\sum_{i=0}^m g_i (-d+h_i)} \\
&= \omega^{\sum_{i=0}^m g_i (-d+h_i)} \\
&= \omega^{-d\sum_{i=0}^m g_i} \omega^{\sum_{i=0}^m g_i h_i} \\
&= \omega^{-d\sum_{i=0}^m g_i} \chi_g(h)
\end{aligned}$$

This is helpful since it turns a group action into scalar multiplication

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These insights provide a path to seeing how s_k and total displacement relate. Note that by their definitions, adding up the number of cars that pass the kth spot for all spots gives total displacement, so

$$d = \sum_{k=1}^n s_k$$

Recall that total displacement is constant on cosets of the diagonal subgroup, so all of it's nonzero fourier coefficients are indexed by group elements which satisfy $\sum_{i=1}^m g_i = 0$.

Note that the properties described above give another way of showing that for total displacement, all of the nonzero fourier coefficients are indexed by group elements which satisfy $\sum_{i=1}^m g_i = 0$. The overall strategy is to rewrite d in terms of s_1 and the action of the diagonal subgroup then leverage information about the sum of roots of unity. This strategy will also provide an explicit connection between s_k and d.

Here is the computation:

$$\begin{aligned}
d &= \sum_{k=1}^{n} s_k \\
&= \sum_{k=1}^{n} e^{k-1} \cdot s_1 && \text{Action of diagonal subgroup on } s_k \\
&= \sum_{k=1}^{n} e^{k-1} \cdot \sum_{g \in G} \widehat{s_1}(g)\chi_g && \text{Character decomposition} \\
&= \sum_{k=1}^{n} \sum_{g \in G} \widehat{s_1}(g) \left(e^{k-1} \cdot \chi_g\right) \\
&= \sum_{k=1}^{n} \sum_{g \in G} \widehat{s_1}(g)\omega^{-(k-1)\sum_{i=0}^{m} g_i}\chi_g && \text{Action of diagonal subgroup on } \chi_g
\end{aligned}$$

Note that in this final expression, if $\sum_{i=1}^{m} g_i \neq 0$, then the summation from 1 to n will be a constant times a sum over all of the roots of unity, which is 0. Therefore, the only part of this summation that doesn't cancel out is when $\sum_{i=1}^{m} g_i = 0$. Let G_0 be the set of group elements where $\sum_{i=1}^{m} g_i = 0$. So,

$$\begin{aligned}
d &= \sum_{k=1}^{n} \sum_{g \in G_0} \widehat{s_1}(g)\chi_g && \text{Roots of unity cancel} \\
&= n \sum_{g \in G_0} \widehat{s_1}(g)\chi_g
\end{aligned}$$

This shows that the only non-zero fourier coefficients of d are those that sum to 0 (and would also apply to any other function with values constant on cosets of the diagonal subgroup).

Additionally, the final expression gives a way of explicitly relating the fourier coefficients of d to the Fourier coefficients for s_1 for $g \in G_0$. We have

$$\frac{d}{n} = \sum_{g \in G_0} \widehat{s_1}(g)\chi_g$$

Which can also be expressed as

$$\widehat{s_1}(g) = \frac{\widehat{d}(g)}{n} \qquad \text{For all } g \in G_0.$$

Finally, note that given the coefficients for s_1, we can compute the coefficients for s_k using the action of the diagonal subgroup. For convenience, let G_j be the set of group elements where $\sum_{i=1}^m g_i \equiv j (\mathrm{mod}\ n)$. Then

$$\begin{aligned}
s_k &= e^{k-1} \cdot s_1 \\
&= e^{k-1} \cdot \sum_{g \in G} \widehat{s_1}(g) \chi_g && \text{Fourier decomposition} \\
&= e^{k-1} \cdot \sum_{i=0}^{n-1} \sum_{g \in G_i} \widehat{s_1}(g) \chi_g && \text{Partitioning by } \sum_{i=1}^m g_i \\
&= \sum_{i=0}^{n-1} \sum_{g \in G_i} \widehat{s_1}(g)(e^{k-1} \cdot \chi_g) \\
&= \sum_{i=0}^{n-1} \sum_{g \in G_i} \widehat{s_1}(g) \omega^{-i(k-1)} \chi_g && \text{Action of diagonal subgroup on } \chi_g
\end{aligned}$$

Given this equation for s_k, we can reinterpret the equation as a formulation for calculating $\widehat{s_k}$ using $\widehat{s_1}$.

$$\widehat{s_k}(g) = \omega^{-(k-1)\sum_{i=0}^m g_i}\ \widehat{s_1}(g)$$

There remains one key question to understanding the Fourier transforms of all of the s_ks, which is what about $\widehat{s_k}(g)$ for $g \notin G_0$? (Well... kind of 2, the second being how do you explicitly compute $\widehat{d}(g)$?)

From computational observations for $n = m$ yesterday, I had the conjecture that the rest of the non-zero fourier coefficients can be described as the following:

$$\widehat{s_n}(S_m \cdot je_1) = -\frac{n^{m-1}}{2} + \frac{n^{m-1}}{2\tan\left(\pi\left(\frac{j}{n} - \frac{1}{2}\right)\right)} i$$

And for $n = m+1$, it seems that s_n might also have a well behaved relationship with $d_{n,n}/n^2$ which helps define a broader set of coefficients. (see screenshots from yesterday).

This also gives another little conjecture that the fourier transform of $d_{n,n-1}$ should maybe just be $\frac{1}{n}$ times the fourier transform of $d_{n,n}$ restricted to characters indexed by group elements containing at least 1 zero, so that this 0 can be dropped, with the fourier coefficient of the trivial character modified.

The point of trying to describe the patterns for $n \neq m$ is also to point out that to show the nice equation for $\widehat{s_n}$ from above, somehow it would have to lean on the fact that $n = m$