Diaconis Notes

I kept convincing and then unconvincing myself of theorem 1, so I'm writing it up in my own words.

Theorem 1. $A_{\pi_2,\cdots,\pi_n} = [k]$ if and only if $(\pi_2, \cdots, \pi_n) \in Sh(k-1, n-k)$.

Definitions from Diaconis and Hicks [1]:

- $A_{\pi_2,\cdots,\pi_n}$ is the set of possibilities for the first element k which would make $(k, \pi_1, \cdots, \pi_n)$ a parking function. Note that $[k]$ is shorthand for $\{1, \cdots, k\}$
- Sh$(k-1, n-k)$ is a shuffle. This means that you take a parking function in PF_{k-1} and a parking function in PF_{n-k}. The first parking function describes how the cars park in the first k spots. The second parking function describes how the cars park in the last $n-k$ spots. You can think of this as two lanes merging since the order within each parking function must remain the same, but the two parking functions can be inter-weaved in any way you want

The proof in Diaconis and Hicks is on page 131 - 132. Here's an different way of phrasing things:

When you remove the first car from the preference list to get $\pi' = (\pi_2, \cdots, \pi_n)$. Note that π' is a modified parking function where with $n-1$ cars park in n spots. This will leave 1 spot open. Let this spot be the kth spot.

Note that because the kth spot is open, some subsequence of π' must be a parking function for the spots 1 through $k-1$. Another subsequence of π' must be a parking function for the spots $k+1$ through n. Therefore π' can be written as a shuffle.

Given that the kth spot is open, the last possible value which makes $(i, \pi_2, \cdots \pi_n)$ a parking function must be k. Clearly k works since that spot is currently empty. Additionally, anything after k would not work since the last $n-k$ spots are already occupied.

Related, but not the same ideas that kept coming up in my head (from [2]). These ideas connect nicely to each other, and initially I confused myself because of their slightly different emphasises. Diaconis-Hicks emphasizes the way in which parking functions have an order in order to pick out the first element of the parking function. Meyels, Harris, et al emphasizes the way in which parking functions can be rearranged into a canonical order because they are using that canonical order to streamline their notation (I think that's why? though order does matter for the space that they are looking at unit interval functions, which have displacement at most 1 for every car). Now back to definitions:

- **Break point** - Exactly k cars want to park in the first k spaces. In terms of Diaconis and Hicks, this means that you could write this parking function as a shuffle in the form Sh$(k, n-k)$.
- **Prime parking function** - A prime parking function is a parking function with no break points. If $\pi \in PF_n$ is prime then π cannot be written as the shuffle of two different parking functions.

- **Concatenating** In this paper, order was very often disregarded (and parking functions reordered to be in ascending order). So the equivalent of the shuffle was the concatenate operator | which works the same as shuffle, but doesn't account for reordering.

There is a nice way of connecting Diaconis and Hicks $A_{\pi'}$ function with the idea of prime parking functions and break points. If instead of removing the first element of π to get π', you remove the ith element, you might ask what the possible values of $A_{\pi'}$ are. In other words, given $\pi \in \mathrm{PF}_n$, what is the set $\{k|A_{(\sigma * pi)'} = [k], \sigma \in S_n\}$. In other words, given a parking function π, if you remove an arbitrary element π_i to create a new preference list π', what is the set of possible spots which could be empty?

The answer to this question is the set of break points for the parking function. Which if you write π as a shuffle of prime parking functions is the set of end points for these shuffles.

Also, the result on page 151 is the same as the trick that I worked out 5/22 in the second bullet point "it's easy to see that $I(\pi) = \binom{n+1}{2} - (\pi_1 + \cdots + \pi_n)$". I'm slightly curious if their "obviously" came up with this trick in the same way, or if they see this fact differently (I somewhat ask this because in terms of notation, that's not how I would have written down that piece of information to seem obvious to me)

Next Steps

- Read Aker and Can closely
- More reading of Diaconis
- More skimming of papers
- Write better code (see 6/2 for list)
- Properly look for patterns in the diagrams that you made (maybe do some calculations for $n \neq m$ to get a handle of patterns)
- Read more about non-crossing partitions
- Whiteboard work: work out $N_d(\lambda)$ for 3 row partitions
- Whiteboard work: work out defect and lucky statistics for labeling rows of a fixed tabloid shape. Do this for maybe all tabloids with $n = 4$? (think about how random for tabloids relates to random for an element of n^n)

References

[1] Persi Diaconis and Angela Hicks. Probabilizing parking functions. *Advances in Applied Mathematics*, 89:125–155, 2017.

[2] Lucas Chaves Meyles, Pamela E. Harris, Richter Jordaan, Gordon Rojas Kirby, Sam Sehayek, and Ethan Spingarn. Unit-interval parking functions and the permutohedron, 2023.