Current explicit expressions for $N_d(\lambda)$ with $n = m$

For 1 row tabloids (as in $\lambda = n$), $N_d(\lambda) = 1$

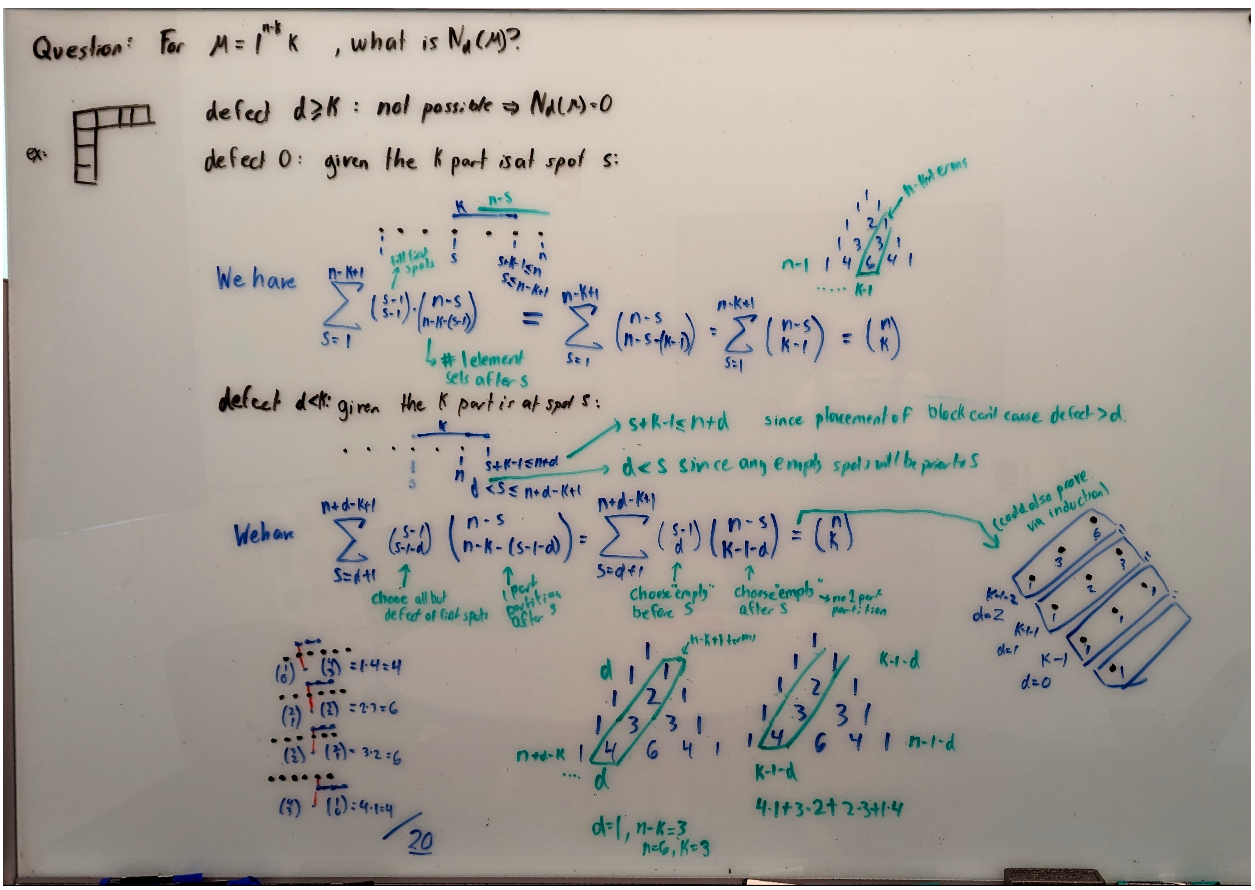
For 2 row tabloids (see calculations on 5/30), I *thought* I found an expression. This expression was wrong.

For partitions of the form $\mu = 1^{n-k}k$, I found

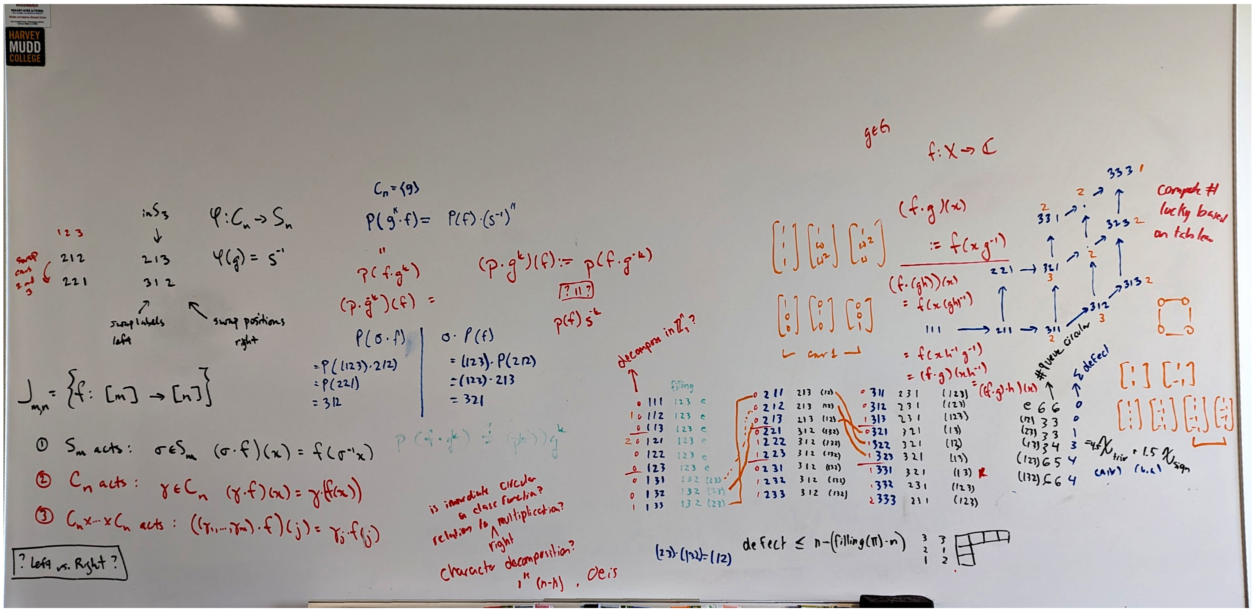
$$N_0(\mu) = \sum_{s=1}^{n-k+1} \binom{n-s}{k-1} = \binom{n}{k}$$

Then more generally,

$$N_d(\mu) = \begin{cases} 0 & d \geq k \\ \displaystyle\sum_{s=d+1}^{n+d-k+1} \binom{s-1}{d}\binom{n-s}{k-1-d} = \binom{n}{k} & d < k \end{cases}$$



Conversations with Prof O 6/6 and 6/7



From 6/6

- We thought a little bit about what happens if the parking lot is circular rather than linear. In this sense, every car will end up parked somewhere, so when $n = m$, you will always get a permutation. In terms of applications, this parallels the original application of linear probing. On the board, computed permutations for $n = 3, m = 3$.
- One choice is where the cars get back in line - do cars return to the end of the line or continue to park immediately? Or some intermediate (think Ontario airport pick up)
- Used these counts to get a permutation statistic on S_n. It turns out that for $n = m = 3$ and linear probing as a procedure (every car parks on its turn), the number of preference lists which give rise to a certain permutation is a class function (which can then be decomposed into characters).
- This conversation also led towards thinking of data on a cube or on a circular cube, which begs the question of how does a function decompose in $\mathbb{Z}_3^n$, and is it high or low frequency? (seems like defect would be low frequency on a cube/hypercube)
- My writing in red are my questions after 6/6
- A small result: for a preference list π with defect d, $d \leq n - (filling(\pi) \cdot n)$ this is $filling(\pi) \cdot n$ is the car occupying the last spot. Before this car parks, every other car must have been able to park.

From 6/7:

One of the things that was difficult for the comparison to rook monoids earlier was that we didn't have a module homomorphism, so we lost several tools. This raises the question of if there are particular subgroups of $C_n \wr S_m$ which behave nicely and do give module homomorphisms. On the left of the board, there is some investigation of what subgroups might behave nicely.

If we call P the map which sends preference lists to permutations This raises the question, what is the largest subgroup of $C_n \wr S_m$ which commutes with P?

Not on the board was some of the visualizing of subgroups and statistics that I thought through because drawing in 3d is hard. Here are some fun visualizations

- Orbits of the diagonal subgroup are all diagonal lines which cut through a hypercube at an angle
- I'm fairly sure that orbits of S_m are all contained within hyperplanes which are normal to the lines formed by orbits of the diagonal subgroup. For example - think about the set of all permutations in S_3 as a part of the 3 by 3 hypercube
- Lucky is constant on orbits of the diagonal subgroup
- Defect is constant on orbits of S_m. In particular for $n = m = 3$, defect is 0 on and below the plane formed by the permutations, 1 in the space above and in the middle of this plane, and 2 only at the top corner. One way that you could notate this more clearly is by defining a relation $<$ where $\pi < \rho$ if $\pi_i \leq \rho_i$ for all i and $\pi \neq \rho$. Let $D(\pi)$ be a function which returns the defect of π. Then if $\pi < \rho$ then $D(\pi) \leq D(\rho)$

Ideas to understand better

Ideas and theory which Prof O brought up/mentioned/explained. Mackey theory, double cosets (versus two sided cosets), Gelfand pairs, induced representations relate to tensor products. It would be interesting to investigate if there are two sided cosets of $C_n \wr S_m$ on which some of our functions of interest are constant.

- Two sided cosets relate to indicator functions like in Brendon and Rhoades to define locality: aHb with $a, b \in G$ is a two sided coset of H in G. Two sided cosets always have the same size and cover the space, but do not partition it.
- Double cosets are of the form HxK. Double cosets partition G like cosets, but might be of different sizes. The subgroup of $H \cap K$ which commutes with x plays an important role in figuring out the size of the coset. Note that $max(|H|, |K|) \leq |HxK| \leq |H||K|$
- Gelfand pairs relate to double cosets HxH. Apparently, the set of functions which are constant on 2 sided cosets are closed under convolution, and Gelfand pairs correspond to that algebra being commutative. Also to induced representations being multiplicity free
- In terms of thinking about the module decomposition - you could relate things to the trivial representation of S_m induced to $C_n \wr S_m$ then restricted to S_m again. (haha should've taken better notes silly)

- Another way to think of inducing representations is to take the tensor product $\mathbb{C}G \otimes_{\mathbb{C}H} M$.

Number of circular preference lists is NOT a class function

In my discussions with prof O and testing for 3 element lists, this permutation statistic was a class function for $n = m = 3$. It turns out that it is not a class function for $n = m = 4$. This makes sense because one of the defining features of this function was that it was constant on cosets of cyclic subgroup of S_m generated by the shift operator. This happens to give a class function for S_3, but doesn't play nicely with conjugacy classes in general.

```
In [72]: print_by_conjugacy_class()
Partition = 1,1,1,1
24

Partition = 2,1,1
12 8 8 8
4 4

Partition = 3,1
8 8 12 8
8 12 8 8

Partition = 2,2
4 4 24

Partition: 4
24 24
8 8
12 8
```

```
In [82]: print_by_cosets_of_cyclic_group()
coset representative: 1,2,3,4.
e, (1234), (13)(24), (1432)
24 24 24 24

coset representative: 1,2,4,3
(34), (124), (1423), (132)
8 8 8 8

coset representative: 1,3,2,4
(23), (134), (1243), (142)
8 8 8 8

coset representative: 1,3,4,2
(234), (1324), (143), (12)
12 12 12 12

coset representative: 1,4,2,3
(243), (14), (123), (1342)
8 8 8 8

coset representative: 1,4,3,2
(24), (14)(23), (13), (12)(34)
4 4 4 4
```