Thinking about $N_d(\lambda)$

I worked out what $N_d(\lambda)$ is in general for a 2 part partition $\lambda = n-k, k$. For more details on computation, see handwritten notes, but here are the results. There are several conditions that matter.

For $n-k \neq k$,

$$N_d(\lambda) = \begin{cases} n+2d & d<k \\ n+k-1 & k \leq d < n-k \\ 2(n-1-d) & n-k \leq d \end{cases}$$

For $n-k=k$, the values differ by a factor of 2 since the two different parts are indistinguishable, so

$$N_d(\lambda) = \begin{cases} k+d & d<k \\ n-1-d & n-k \leq d \end{cases}$$

From the plots that I've generated, I noticed that all of the two part partitions with distinct values had the same total multiplicity when you sum over all defects, so I calculated this total for the two part partitions that I computed as a verification.

$$N(\lambda) = \sum_{d=0}^{n-1} N_d(n-k,k) = n^2+n = 2\binom{n}{2}$$

This got me thinking about these sums in general in terms of labeling of tabloids. Recall that $N(\lambda)$ counts the number of orbits of S_m which have multiplicity partition type λ (with no conditions on defect). This is also the multiplicity of the tabloid representation of shape λ, M_λ, in the module of all preference lists. Let $P_{n,m}$ be the set of preference lists for n spots and m cars. Viewed as an S_m module,

$$P_{n,m} \cong \bigoplus_{\lambda \vdash m} N(\lambda) M_\lambda$$

If $\lambda \vdash m$; $|\lambda|$ is the number of parts of λ; $\mu \vdash |\lambda|$ is the exponents of λ when written in shorthand. Then

$$N(\lambda) = \binom{n}{|\lambda|} \frac{|S_{|\lambda|}|}{|S_\mu|}$$

Which you can see as choosing a set of labels for the rows of the tabloid λ, multiplying by the number of arrangements of these labels, then dividing out by $|S_\mu|$ to not double count the orbits of S_m acting on preference lists.

Both of these results can be verrified with plots that I have generated (also see tablet notes for more details)

More Plots!

I generated more plots (see github)

In working through figuring out $N(\lambda)$, I made some plots which sort the partitions by their total multiplicity in the preference list module, so you can easily see what partitions have the same multiplicity.

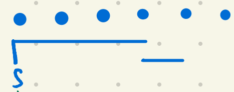
I also was curious to see how many preference lists there were for each of these parts, so I made a histogram which multiplied each orbit of S_n by the size of its orbit, which changes what seems like the most important of the partitions

To Do/Ideas to think about

- Answer questions about maps - also think about if/how the correspondence between PK_n and PC_{n+1} works when $n \neq m$
- Figure out how to generalize choosing a nice basis which is linear combinations of characters - connect to wreath products
- 3 row partitions, $N_d(\lambda)$?
- Write code to figure out distance from uniform distribution for distribution of desired spots?
- Read (continued) - papers from Prof O
- Read - regarding Mackey theory (see Representation Theory of Finite Groups: Algebra and Arithmetic)
- Read and work through examples of constructing bases from a representation (see Group Theoretical Methods and Their Applications by Albert Fassler)

6/15 For Partition $\lambda = 2, n-2$.

$N_\lambda(0) = (n-2) + 2 = n$ (large part second)



$d < k$ $\quad N_\lambda(1) = \sum_{s=1}^{2} \begin{cases} 1 & s=1 \\ n-2 & s=2 \end{cases} + \sum_{s=3}^{s=4} \begin{cases} 1 & s\geq 3 \\ 2 & s=4 \end{cases} = n+2$ for $n>4$

$d \geq k$ $\quad N_\lambda(2) = \underset{s=3}{(n-2)-1} + \sum_{s=4}^{5} \begin{cases} 1 & s=4 \\ 3 & s=5 \end{cases} = n+1$

$\qquad\qquad \hookrightarrow = 2+1$

$N_\lambda(3) = \underset{s=4}{(n-2)-2} + \underset{s=5}{1} + \underset{s=6}{4} = \begin{cases} n+1 & n \geq 6 \\ n-3 = 2 & n=5 \\ 0 & \text{otherwise} \end{cases}$

For Partition $\lambda = n-k, k$ $\quad$ note blue & green color code swapped

$N_\lambda(0) = \underset{n-k \text{ first}}{(n-k)} + \underset{k \text{ first}}{k} = n$

$N_\lambda(1) = \underset{s=1}{1} + \underset{s=2}{n-k} + \underset{s'=1}{1} + \underset{s'=2}{k} = n+2$

$d<k : n+2d$

$N_\lambda(2) = \underset{s=1}{1} + \underset{s=2}{1} + \underset{s=3}{n-k} + \underset{s'=1}{1} + \underset{s'=2}{1} + \underset{s'=3}{k} = n+4$

$k \leq d$

$N_\lambda(k) = \underset{s=n+1-(n-k)=k+1}{n-k-1} + \overbrace{\underset{s=(k+1)+1}{1} + \underset{(k+1)+2}{1} + \dots + \underset{(k+1)+(k-1)}{1}}^{k-1} + \underset{(k+1)+k}{(k+1)} = (n-k+1) + 2k = n+k-1$

$d<n-k$ $\quad N_\lambda(d) = \underset{s=d+1}{n-d-1} + \overbrace{\underset{(d+1)+1}{1} + \underset{(d+1)+2}{1} + \dots + \underset{d+1+(k-1)}{1}}^{k-1} + \underset{d+1+k}{d+1} = n+k-1$ for $n \geq d+1+k$

$n-k \leq d$ $\quad N_\lambda(d) = \underset{s=d+1}{n-d-1} + \underset{s'=d+1}{n-d-1} = 2(n-d-1)$

For 2 part partitions $\lambda = n-k, k$, want to know $N_\lambda(d)$.
What conditions matter? $\quad$ Yep! (matches data)

For $k \neq n-k$, 3 cases:

$d<k$	$n+2d$
$k \leq d < n-k$	$n+k-1$
$n-k \leq d$	$2n-2-2d$

For $k = n-k$, 2 cases

$d<k$	$\frac{n+2d}{2} = k+d$
$k \leq d$	$n-(d+1)$

note: $\displaystyle\sum_{d=0}^{n-1} N_d(n-k,k) = \sum_{d=0}^{k-1} n+2d + \sum_{d=k}^{n-k-1} n+k-1 + \sum_{d=n-k}^{n-1} 2n-2-2d$

$= k\cdot n + 2\cancel{\binom{k-1}{2}} + (n-2k)(n+k-1) + k(2n-2-2(n-k)) - 2\cancel{\binom{k-1}{2}}$

$= \cancel{nk} + n^2 + \cancel{nk} + n - \cancel{2kn} - \cancel{2k^2} + 2k + k(-2+\cancel{2k}) = n^2+n = (n+1)n = 2\binom{n}{2}$

If $\lambda \vdash m$

$\mathrm{len}(\lambda)$ is # parts

$\mu \vdash \mathrm{len}(\lambda)$ is the exponents

example

$\lambda = 3^2 1^3$

$\mathrm{len}(\lambda) = 5$

$\mu = 32$

Then in the S_m module of all preference lists, let $N(\lambda)$ be the multiplicity of S_λ, we can compute $N(\lambda)$ by counting the unique orbits of S_n which result from labeling a tabloid of shape λ and fixed contents with elements of $[n]$.

These can be counted as follows:

$$N(\lambda) = \sum_{d=0}^{m-1} N_d(\lambda) = \prod_{i=1}^{\mathrm{len}\,\mu} \binom{n - \sum_{j=1}^{i-1} \mu_j}{\mu_i} = \binom{n}{\mathrm{len}\,\lambda} \cdot \binom{\mathrm{len}\,\lambda}{\mu_1} \binom{\mathrm{len}\,\lambda - \mu_1}{\mu_2} \cdots$$

$$= \frac{n!}{(n - \mathrm{len}(\lambda))!} \cdot \frac{1}{\mu_1!\,\mu_2! \cdots} = \binom{n}{\mathrm{len}(\lambda)} \frac{|S_{\mathrm{len}\,\lambda}|}{|S_\mu|}$$