Summary

Things I got done:

- Made a list of possible histograms that would be interesting to make or helpful to study
- Worked on thinking about the coefficients in front of the tabloid decomposition of the defect d modules for preference lists

Notes from meeting with Prof O - Tabloids and Kostka numbers

The induced representations from the trivial representation of a young subgroup are well understood. Note that if M^λ is module for a tabloid of shape λ, then it is the result of inducing the trivial representation of it's stabilizer (which is the young subgroup). There is already a decomposition of this module into irreducibles.

$$M^\lambda = \bigoplus_{\mu : \mu \trianglerighteq \lambda} \kappa_{\mu\lambda} S^\mu$$

Note that $\mu \trianglerighteq \lambda$ means that μ dominates λ (ie all of the partial sums of μ are larger than all of the partial sums of λ. Also $\kappa_{\mu\lambda}$ are the Kostka numbers which come from counting the possible ways of filling a tabloid of shape μ with contents λ satisfying 2 conditions: weakly increasing across the row, and strictly increasing down the column. If you want more detail - see Sagan 2001 book. Or Hannah's thesis pg30

Distributions of interest

Given that a preference list has defect d, there are several distributions of interest.

Description	Notation
Preference of ith car	π_i
Where did the ith car end up parking	$filling(\pi) \cdot i$
What car is in the jth spot	$parking(\pi) \cdot j = (filling(\pi))^{-1} \cdot j$
Displacement of the ith car	
Number of lucky cars	
Total inclusive displacement (invariant under action of S_n)	
Total exclusive displacement (not invariant)	

Given a tableau with fixed shape and contents, for all labels, what distributions are of interest?

- Defect
- Lucky cars
- Total exclusive displacement (not invariant)

Given an element of $\mathbb{Z}_{n+1}/D$ where D is the diagonal subgroup which corresponds to parking function π, what is the defect of the other elements of the coset?
Observation: For all valid preference lists in the same coset, they have the same number of lucky cars! (and in general cars that get to park with displacement d)

Specifically interesting visuals to compute (just write bad code):

- Make an equivalent of Diaconis and Hicks histogram, but this time for defects. (stack them so you can see how it all adds up :))
- Would be interesting to put some order on partitoins and then make a plot with partitions λ on the x axis and $N_d(\lambda)$ on the y axis. This way if you stacked all of them on top of each other like the extension of the Diaconis Hicks paper, you would end up with the module structure of preference lists

Decomposing Module for defect d

I've decided on notation (for now). $N_d(\lambda)$ is the number of orbits of S_n corresponding to the partition λ with defect d. For reference, from the intro of Rhoades, Stanley's 1997 paper finds that for the decomposition of parking functions, $N_0(\lambda) = NC(\lambda)$. If P_d is the module of preference lists with defect d, then

$$P_d = \bigoplus_{\lambda \vdash n} N_d(\lambda) M^\lambda$$

From todays calculations, I know that for 1 part partitions $N_d(\lambda) = 1$ for all d. For 2 part partitions $\lambda = n-k, k$ with $k \leq 2n$, I found

$$N_d(n-k,k) = \begin{cases} k+d & n-k=k \\ n & k=1 \\ n+2 & k=2 \\ \vdots & \\ n+2d & k>d \end{cases}$$

NOTE: THIS EXPRESSION IS INCORRECT BASED ON LATER CALCULATIONS

For details see whiteboard work. The key reason for the incrementing by 2 part is that for partitions where a lot of the stuff is on one side of the partition, the end of the parking lot gets in the way.

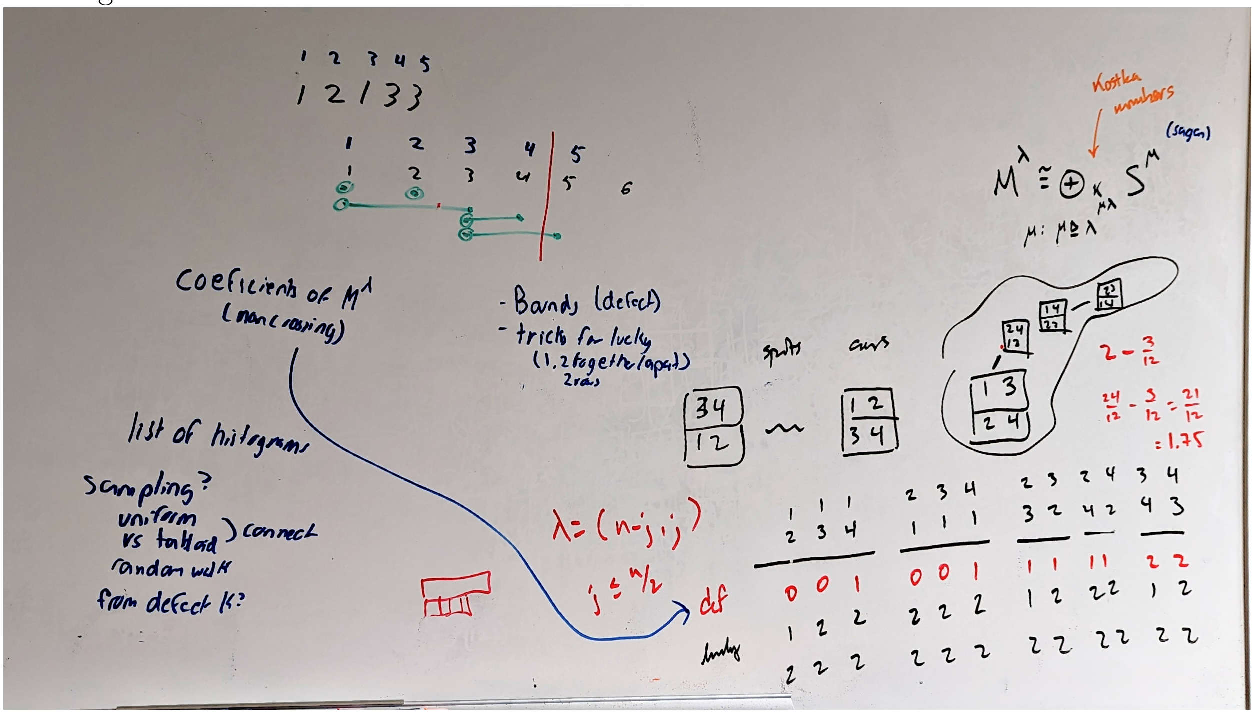
Things for tomorrow:

- Write code. (update code with trick for checking defect, write bad code to make Diaconis and Hicks histogram, write bad code to compute $N_d(\lambda)$ - use notes from 5/24)
- Calculations for tabloids with fixed shape and contents

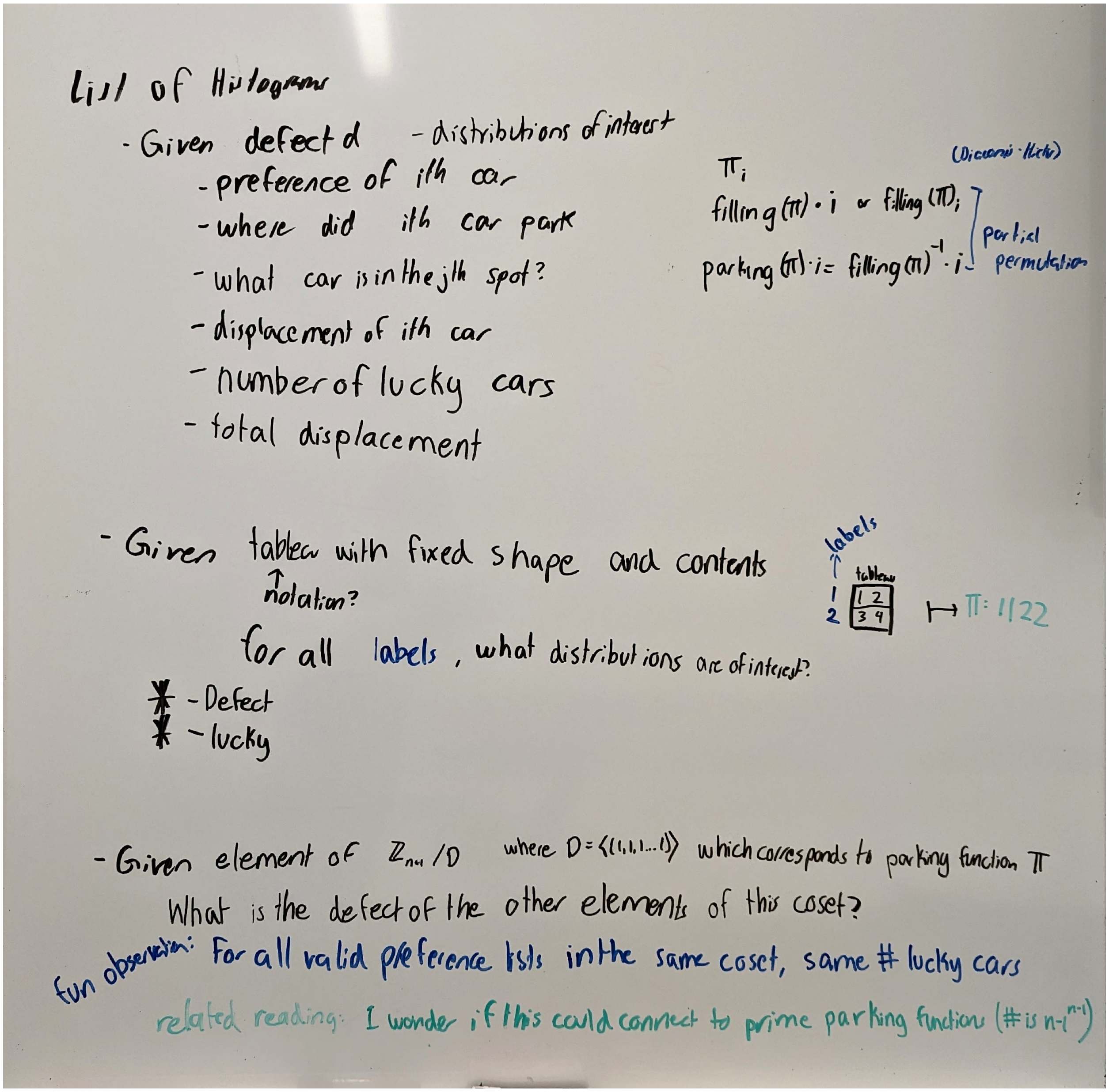
In terms of related reading, it would be good to learn more about noncrossing partitions. Also, the idea of prime parking functions might be helpful at some point

Whiteboards

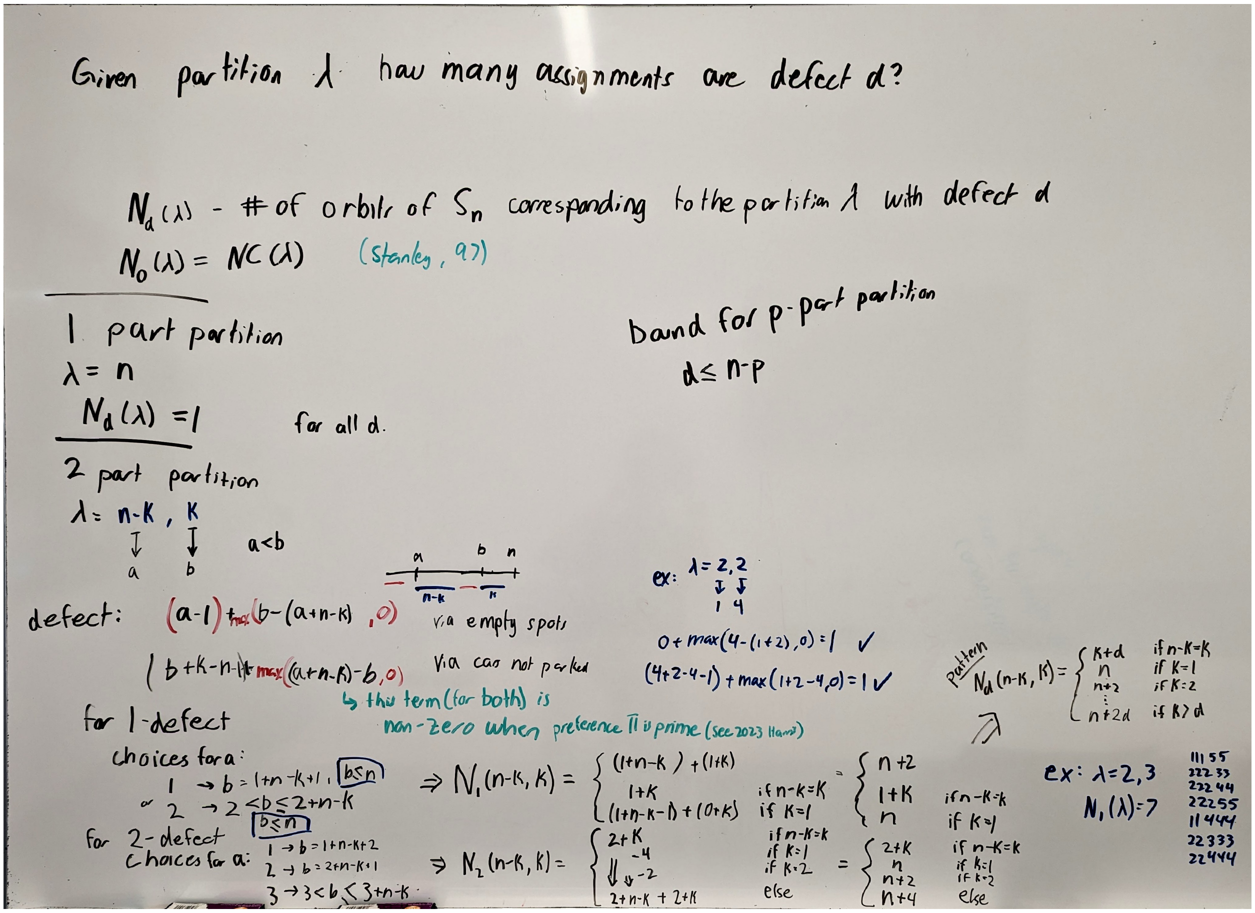
Meeting with Prof O



List of histograms



Calculating coefficients of M^λ for 1 and 2 part partitions



Given partition λ how many assignments are defect d?

$N_d(\lambda)$ - # of orbits of S_n corresponding to the partition λ with defect d

$N_0(\lambda) = NC(\lambda)$ (Stanley, 97)

1 part partition

$\lambda = n$

$N_d(\lambda) = 1$ for all d.

bound for p-part partition

$d \leq n-p$

2 part partition

$\lambda = n-K,\ K$

$a \quad b \qquad a<b$

defect: $(a-1) + \max(b-(a+n-K), 0)$ via empty spots

$(b+K-n-1) + \max((a+n-K)-b, 0)$ via cars not parked

this term (for both) is non-zero when preference Π is prime (see 2023 Ham?)

ex: $\lambda = 2,2 \to 1\ 4$

$0+\max(4-(1+2),0)=1$ ✓

$(4+2-4-1)+\max(1+2-4,0)=1$ ✓

for 1-defect

choices for a:

$1 \to b = 1+n-K+1,\ \boxed{b \leq n}$

or $2 \to 2 < b \leq 2+n-K$ $\boxed{b \leq n}$

$$\Rightarrow N_1(n-K,K) = \begin{cases} (1+n-K)+(1+K) & \\ 1+K & \text{if } n-K=K \\ (1+n-K-1)+(0+K) & \text{if } K=1 \end{cases} = \begin{cases} n+2 & \\ 1+K & \text{if } n-K=K \\ n & \text{if } K=1 \end{cases}$$

for 2-defect

choices for a:

$1 \to b = 1+n-K+2$

$2 \to b = 2+n-K+1$

$3 \to 3 < b \leq 3+n-K$

$$\Rightarrow N_2(n-K,K) = \begin{cases} 2+K & \text{if } n-K=K \\ -4 & \text{if } K=1 \\ -2 & \text{if } K=2 \\ 2+n-K+2+K & \text{else} \end{cases} = \begin{cases} 2+K & \text{if } n-K=K \\ n & \text{if } K=1 \\ n+2 & \text{if } K=2 \\ n+4 & \text{else} \end{cases}$$

pattern

$$N_d(n-K,K) = \begin{cases} K+d & \text{if } n-K=K \\ n & \text{if } K=1 \\ n+2 & \text{if } K=2 \\ \vdots & \\ n+2d & \text{if } K>d \end{cases}$$

ex: $\lambda = 2,3$

$N_1(\lambda) = 7$

11155
22233
22244
22255
11444
22333
22444