A note on explaining break points to my dad:

The thing that was most helpful to clearly communicate the idea of a break point was describing a break point as a place where you could put a cement wall and if all of the cars tried to park with the cars with a preference before the cement wall starting out before the cement wall and the cars with a preference after the cement wall starting out after the cement wall, then no car will encounter a problem with the cement wall. For parking functions with $n = m$, this turns out to be equivalent to the number of cars which want to park in spot i or before is exactly i.

Some how the cement wall description worked much better than saying a break point is an occupied spot where where no car tries to park in that spot and has to move on to the next spot. Which is nearly the same, but I guess cement walls are a more visceral starting point, and the details of which index that break point counts as and how empty spots count/don't count can come later

Filling in a gap - Sampling from prime parking functions

Recall that Stanley's exercise 5.49f (solution on page 141, problem around pg 98?ish) gives a way of sampling from prime parking functions at random that my method of sampling from defective parking functions relies on [1]. Recall that to sample from a prime parking function, you take an element of C_{n-1}^n at random, and there must be exactly one element of the coset of the diagonal subgroup which is a prime parking function. Earlier I didn't prove this, so here's a way of thinking through why this must be true.

Note that a prime parking function can never have a car with the preference for the last spot (since $n-1$ would be a break point), so we are looking for an element of C_{n-1}^n to interpret as a preference list.

Now why is there only 1 element of each coset of D which is a prime parking function? Consider the parking outcome of the first $n-1$ cars in a circular parking lot with $n-1$ spots. At this point, since we have a circular parking lot, all $n-1$ of the cars will have parked. Additionally, there will be at least one break point since the spot in which the last car parked must be a break point.

Note that when an element of the diagonal subgroup acts, the set of break points will also change by the action of the corresponding group element to the action of the diagonal subgroup. Ie if $(g, \cdots, g) \in C_{n-1}^n$ acts on a preference list and the set of break points used to be $S = \{s_1, s_2, \cdots, s_k\}$, then the new set of break points will be $g \cdot S = \{gs_1, gs_2, \cdots, gs_k\}$ (requires some justification, but is the same argument as 7/12 write up for the second kind of indicator function)

Now, let us return to the classical parking context to show that the coset of the diagonal subgroup has exactly one prime parking function for all n cars parking.
As noted before, the location where the last car parks must be a break point, so for a prime parking function, $filling(\pi) \cdot n = n$, and this is the only break point. Note that this means that the first $n-1$ cars must be a parking function for the first $n-1$ spots. This means that for the element of D selected, for the first $n-1$ cars parking in a circular lot with $n-1$ spots, $(g, \cdots, g) \cdot \pi$ must have a break point at $n-1$.

Additionally, note that after the final car parks, the only break point must be n. Therefore, the final car must attempt to park in all of the spots in the set S of break points for the first $n-1$ cars parking. If s_1 is the first break point after π_n and s_0 is the first break point before π_n, then the only element of the diagonal subgroup for which $(g, \cdots, g) \cdot \pi$ will be a prime parking function is the element which satisfies $g \cdot s_0 = n-1$ since for any other element of the diagonal coset will either not be a parking function, or the final car will not pass s_0, so the parking function will not be prime.

I don't know if this is particularly written in my favorite order right now... Here's a list of the pieces that come together in case I want to reorder things:

- for prime parking function, no car's preference can be n
- the location where the last car parks is always a break point
- action of diagonal subgroup on set of break points for circular parking
- prime parking function must have exactly 1 break point at n
- To be a prime parking function where the last car parks in the last spot, the parking of the first $n-1$ cars must leave a break point at $n-1$
- Every break point must be passed by the last car, which leaves only one option

---

Observation relating lucky to functions on the symmetric group/partial permutations

You can think of the lucky function as the "pull-back" of the fixed point function on permutations across either the $filling$ map or the $parking$ map. This is because displacement of the ith car in line is $filling(\pi) \cdot i - \pi_i$. From the parking lots perspective, displacement of the car in the jth spot is $j - \pi_{parking(\pi) \cdot j}$ for all occupied spots j

I wonder if you could take advantage of the fixed point function on permutations being 1 local? Specifically, if you can answer the question what happens to indicator functions $\mathbf{1}_{i,i}$, when they are pulled back to their pre-images on preference lists, then you should be able to understand the lucky function. Or if you can understand the similar pull back of $\mathbf{1}_{i,i+k}$, then you should be able to understand the displaced exactly k function. Note that this should work for either circular parking or linear parking but the way that the pull back function works would be different (working with the symmetric group vs partial permutations), and the $i+k$ for the second indicator function would be modular addition for the circular version.

References

[1] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.