Summer Research

Random Sampling from Defect-d Preference Lists

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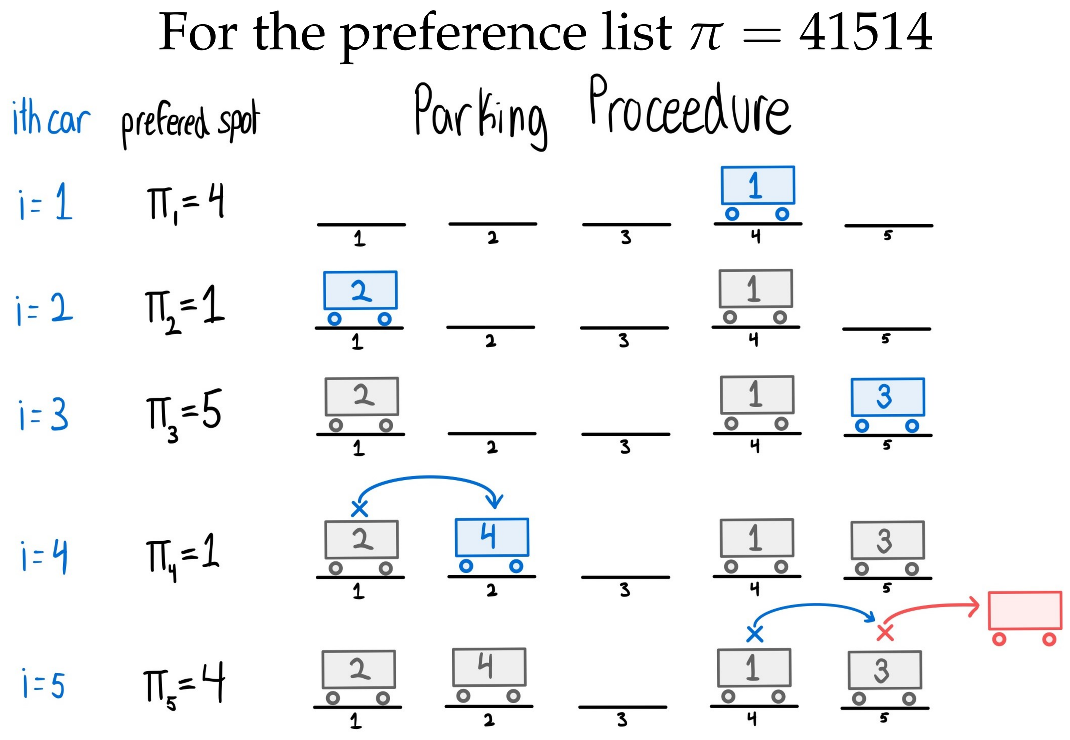
Introduction

Understanding how to sample uniformly often provides a useful starting point for understanding a space, enabling computation and highlighting structure. This poster describes the process of randomly sampling from defect-d preference lists.

Definitions

A **preference list**, $\pi \in [n]^m$, represents the parking preferences for m cars on a one way street with n spots.

For the preference list $\pi = 41514$



The **defect** of a preference list is the number of cars unable to park.

A **parking function**, $\pi \in PF_{n,m}$, is a preference list for n spots and m cars where all cars are able to park.

A **breakpoint** is an occupied spot where no other car attempts to park. For the above example, 2 is a break point.

A **prime parking function**, $\pi \in PPF_n$, is a parking function for n spots and cars with exactly 1 break point.

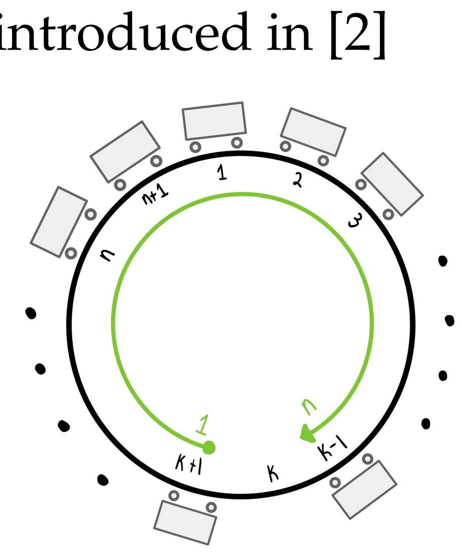
A **shuffle** interweaves two preference lists into a single preference list.
In the example above, $\pi = 41514$ is a shuffle of $\pi_1 = 11$ and $\pi_2 = 454$.

Existing Methods

Parking Functions

Key mathematical insight:
Every coset of the diagonal subgroup, $D = \langle (1, \cdots, 1) \rangle$, of C_{n+1}^m contains exactly 1 parking function.
This is Pollak's argument introduced in [2]
To sample:

- Pick $\pi \in C_{n+1}^m$.
- Pick $i \in [n-m+1]$.
- Let k be the ith empty spot from circular parking.
- Let $\pi_i' = \pi_i - k$.



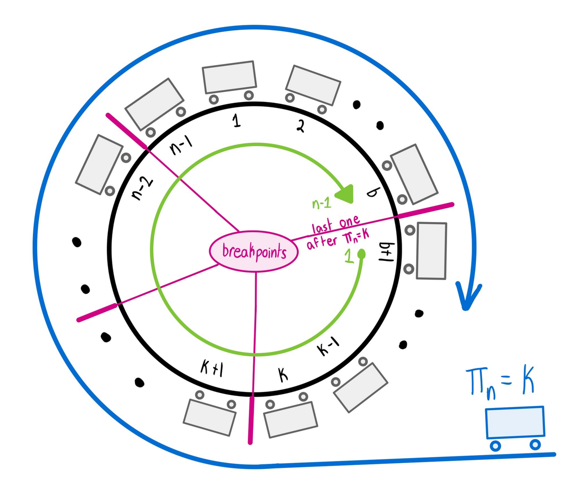
Enumeration:

$$|PF_{n,m}| = (n-m+1)(n+1)^{m-1}$$

Prime Parking Functions

Key mathematical insight:
Every coset of the diagonal subgroup of C_{n-1}^n contains exactly 1 prime parking function.
First seen in [3].
To sample:

- Pick $\pi \in C_{n-1}^n$.
- Let b be the last breakpoint after π_n.
- Let $\pi_i' = \pi_i - b$.



Enumeration:

$$|PPF_n| = (n-1)^{n-1}$$

Sampling from Defect d

To sample from defect d preference lists, focus on s, the number of spots occupied after the last breakpoint. To sample:

- Sample s from the appropriate distribution given n and d.
- Generate a parking function from $PF_{n-s,m-s-d}$.
- Generate a prime parking function from PPF_{s+d} with no preferences for the last d spots. Let us call this a **d-prime parking function**, denoted $PPF_{n,d}$.
- Shuffle the two preference lists.

Distribution for s

Let $s(\pi)$ be the number of occupied spots after the last point and $d(\pi)$ the defect of π. Then the following generating function is computed by counting possible shuffles:

$$F(x,y) = \sum_{\pi \in [n]^m} x^{s(\pi)} y^{d(\pi)}$$

$$F(x,y) = \sum_{d=0}^{m-1} \sum_{s=1}^{m-d} \binom{m}{s+d} |PF_{n-s,m-s-d}||PPF_{s+d,d}| x^s y^d + (n-m)n^{m-1}x^0 d^0 .$$

Note that this generating function provides all of the necessary information to sample from the distribution for s given n and p. Additionally, it provides a different method to enumerate defect d parking functions than [1].

p-prime Parking Function

Method 1

- Sample from $[s]^n$ where $s = n - p$ until you get a prime parking function.

Removing the preference lists which have an empty first spot and those with their first break point at i gives the following count:

$$|PPF_{n,p}| = s^n - (s-1)^n - \sum_{i=1}^{s-1} \binom{n}{i} (i-1)^{i-1}(s-i)^{n-i}.$$

Method 2

- Repeatedly sample from prime parking functions until you get a p-prime parking function.

Following an inclusion-exclusion based argument to remove prime parking functions which are not p-prime gives the following:

$$|PPF_{n,p}| = \sum_{i=0}^{p} \binom{n}{i} (n-i-1)^{n-i-1}(p-i)^i(-1)^i .$$

Bridging the Two

- Abel's identity gives a computational way to show the two counts are equal.
- When $p < n - (n-1)^{\frac{n-1}{n}}$, method 2 is more likely to succeed on the first attempted sample. Otherwise the reverse is true.

Conclusions

- Break points and prime segments are basic building blocks of preference lists.
- If you understand an attribute of interest for parking functions and p-prime parking functions, the distribution for s provides a method of extending this result to defect d preference lists.
- Defect 1 preference lists can be understood with only prime parking functions and parking functions.

References

[1] Peter J. Cameron, Daniel Johannsen, Thomas Prellberg, and Pascal Schweitzer. Counting defective parking functions, 2008.

[2] Dominique Foata and John Riordan. Mappings of acyclic and parking functions. *Aequationes mathematicae*, 10:10–22, 1974.

[3] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.

Acknowledgments

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