Representation theory of wreath products

I got to a point in working through trying to understand the representations of $\mathbb{Z}_2 \wr S_3$ where I felt somewhat stuck, so I worked on reading Ceccherini-Silberstein, Scarabotti, and Tolli's textbook on the representation theory of wreath products. [1] What I was picking up on yesterday (green notes next to table) that somehow normalizers will be relevant was captured by the idea of an inertia group

Some notes on what I read:

Given a normal subgroup $N \trianglelefteq G$ and an irreducible representation $\rho \in \widehat{N}$, a new representation can be found by conjugating by $g \in G$:

$$^{g}\rho(n) = \rho(g^{-1}ng)$$

This is the same thing that we worked with on the second exam for class (see problem 2, 3, and also random extra proofs). From the second exam, we know that sometimes $^{g}\rho$ is equivalent to ρ (like for $g \in N$). And sometimes not (example on exam, $G = S_3, N = \langle (123) \rangle, g = (12)$).

This leads to what the inertia subgroup is.

$$I_G(\rho) = \{g \in G | \, ^{g}\rho \sim \rho\}$$

In order to better understand the coset representatives of N, these coset representatives are split up based on how they interact with $I_G(\rho)$.

Q is a set of left coset representatives for N in $I_G(\rho)$ $(1 \in Q)$

R is a set of left coset representatives for $I_G(\rho)$ $(1 \in R)$

Then we can think of a full system of left coset representatives for N in I_G as RQ.

Useful pages:

- 42-43 on Inertia Groups
- 95 conclusions of what the irreducible representations of $H \wr G$ are. Also, table with general setting vs wreath product setting notation
- 99 summarizing irreducible representations of $\mathbb{Z}_2 \wr G$ (more examples follow)

In understanding theory - should finish constructing irreducibles of $\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$ and of $\mathbb{Z}_2 \wr S_3$. It would be fun to look a little closely at the way in which these irreducible representations play out in terms of the actual matrices that diagonalized the induced representations that I've been using to try to understand things

To do for tomorrow:

Today was very wreath product-y (and finally computer data recovery worked!) which was fun, but it would be good to have tomorrow split up a bit more between thinking about parking functions and thinking about wreath products.

- Wreath products - use the theory that I learned a bit more about today on the example $\mathbb{Z}_2 \wr S_3$ (help from page 99)
- Think more about how to count/sample from defect 1 parking functions (reading, dyck paths, catalan numbers)
- Calculate characters for small cases of S_n acting on parking functions and d-defect parking functions (also decomposition into irreducible)
- More reading (make sure to skim most recent set of papers from Prof O)
- Organization thing: actually update your reference list so it's easy to cite things silly
- (probably won't do this tomorrow, but it's going in the list) Finish exercises from Stanley, Enumerative combinatorics. Volume 2. Pg 95. 5.49. There are more parts I haven't looked at and also still would be good to write things down. or not - they can stay whiteboard notes from 5.22.2023 write up

References

[1] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. *Wreath products of finite groups and their representation theory.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2014.