Random Walks

There are two properties that could potentially be helpful in working through how to compute results after a random walk: functions that are constant under reordering or functions that are constant under the action of the diagonal subgroup (since this is a whole family of functions).

If we focus on an initial distribution d, a random walk w, and statistic s, then after t steps, the distribution that we want to be able to say something about is $(d * w^{*t}) \cdot s$. Note that if we focus on the expected value of the statistic on that distribution, this can be written as $\frac{1}{n^m}\langle d * w^{*t}, s\rangle$ (where the inner product doesn't include division by the order of the group).

Now, the same logic that Ian's thesis and Hultman uses for studying functions which are constant on conjugacy classes can also be helpful in this case. Let $\mathcal{O}$ be a partition of G. For a statistic $s \in \mathbb{C}G$, let $\overline{s}^{\mathcal{O}}$ be the result of averaging s over each of the parts in $\mathcal{O}$. So

$$\overline{s}^{\mathcal{O}}(g) = \frac{1}{|O_g|} \sum_{h \in O_g} s(h) \qquad \text{Where } g \in O_g \text{ and } O_g \in \mathcal{O}$$

Then in general, you can use the same argument that Hultman uses to show that for any $a, b \in \mathbb{C}G$ $\langle \overline{a}^{\mathcal{O}}, b\rangle = \langle \overline{a}^{\mathcal{O}}, \overline{b}^{\mathcal{O}}\rangle$. For chain of equality see pg25 of Ian's thesis - same argument.

By slight abuse of notation, in the context of preference lists let $\overline{a}^{S_m}$ be the result of averaging over orbits of S_m. By similar (but different) abuse of notation, let $\overline{a}^{D}$ be the result of averaging over cosets of the diagonal subgroup.

Now in the context of the random walks that we care about, let us take d to be a starting distribution which is invariant under reordering, w to be a random walk that uses the same distribution for each component, and s to be a statistic that is constant under the action of the diagonal subgroup.

Since d and w are both constant under reordering, their fourier coefficients are indexed by group elements which are constant under reordering, so convolution will maintain this property, so $d * w^{*t}$ is constant under reorodering. In other words $d * w^{*t} = \overline{d * w^{*t}}^{S_m}$.

Similarly $s = \overline{s}^{D}$

This means that we can apply the property above to rewrite the inner product as follows:

$$\langle d * w^{*t}, s\rangle = \langle d * w^{*t}, \overline{s}^{S_m}\rangle = \langle \overline{d * w^{*t}}^{D}, s\rangle$$

Averaging a function which is constant on orbits of S_m over cosets of D will remain constant on orbits of S_m. This hinges on the fact that the action of S_m commutes with the action of D. This means that we can take a second step using the property about averages and inner products to get the following:

$$\langle d * w^{*t}, s\rangle = \langle \overline{d * w^{*t}}^{D}, \overline{s}^{S_m}\rangle$$

Looking at $\overline{s}^{S_m}$

I was wondering if you could easily calculate $\overline{s}^{S_m}$ in the Fourier domain by averaging over orbits of S_m acting on the subscripts. First I checked this experimentally with some computations, and it was true. Here's one example

```
In [141]: stats = IterateStats(3,3, True) #3 spots, 3 cars, linear probing
. . . loading from file . . .

In [142]: stat = CnmStat(3,3, stats.disp_i[0]) #This is the lucky function

In [143]: stat.print_by_value() #Prints fourier coefficients by their values
6
-9.+0.j            : (2, 1, 0)(1, 2, 0)
-4.5-2.598076j     : (1, 0, 2)(0, 1, 2)
-4.5+2.598076j     : (2, 0, 1)(0, 2, 1)
4.5-2.598076j      : (2, 2, 2)
4.5+2.598076j      : (1, 1, 1)
54.+0.j            : (0, 0, 0)
9

In [144]: stat.average_Sm()

In [145]: stat.print_by_value() #Prints fourier coefficients by their values
4
-6.+0.j            : (2, 1, 0)(1, 2, 0)(2, 0, 1)(0, 2, 1)(1, 0, 2)(0, 1, 2)
4.5-2.598076j      : (2, 2, 2)
4.5+2.598076j      : (1, 1, 1)
54.+0.j            : (0, 0, 0)
9
```

So here's a proof outline.

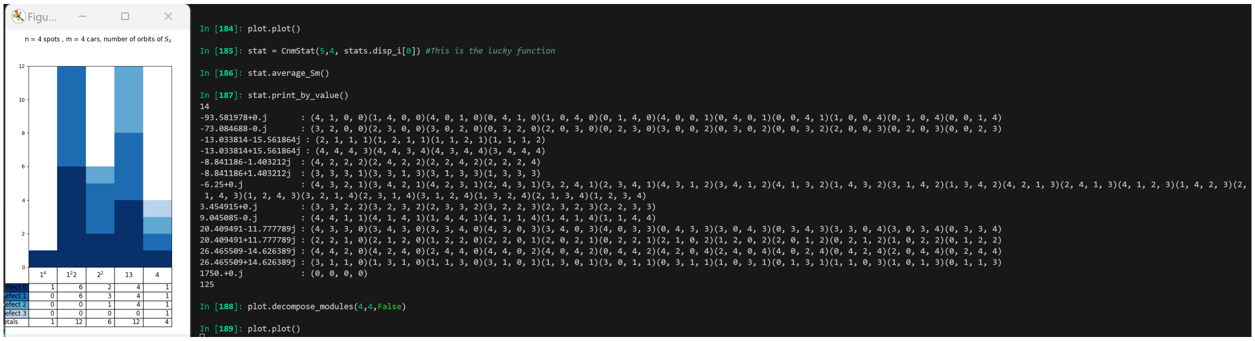
One way that you could achieve averaging over orbits of S_m is the following

$$
\begin{aligned}
\overline{s}^{S_m} &= \frac{1}{|S_m|}\sum_{\sigma\in S_m}\sum_{g\in G} s(g)(\sigma\cdot g) && \text{Averaging over } S_m \\
&= \frac{1}{|S_m|}\sum_{\sigma\in S_m} \sigma\cdot s && \text{Averaging over } S_m \\
&= \frac{1}{|S_m|}\sum_{\sigma\in S_m} \sigma\cdot\sum_{g\in G}\hat{s}(g)\chi_g \\
&= \frac{1}{|S_m|}\sum_{\sigma\in S_m}\left(\sum_{g\in G}\hat{s}(g)(\sigma\cdot\chi_g)\right) \\
&= \frac{1}{|S_m|}\sum_{\sigma\in S_m}\left(\sum_{g\in G}\hat{s}(g)\chi_{\sigma\cdot g}\right)
\end{aligned}
$$

Given that the averaging over the orbits of S_m gives the same result in both the Fourier domain and not, this gives an interesting connection for functions which are constant on cosets of D: The number of orbits of S_m corresponding to characters indexed by elements summing to n is the same as the number of orbits of S_mD for preference lists.

This is interesting because orbits of D for preference lists with $n = m+1$ have one representative which is a parking function for m cars in m spots. So this means that the number of irreducible modules in the S_m module decomposition of these parking functions is the same as the number of orbits of S_m for elements of C_n^m where the sum of the elements is 0 mod n.

This relates $\sum_{\lambda \dashv m} N_0(\lambda)$ to the number of distinct non-zero values in the Fourier domain for a statistic which is constant on cosets of $S_m D$ for $n = m + 1$. The partition types λ in the module decomposition also correspond to the partition types of the corresponding to the cosets of $S_m D$ (see computational evidence - this works because the circular relation between the two objects doesn't change the partition type)



This leaves me with a couple of questions. If $n \neq m + 1$, can this be related to a different module decomposition in a similar way? Or with a twist?

Can you compute the values of these coefficients with a nice expression?

Given information about $\overline{s}^{S_m}$, how does this help with computing random walks? Is convolution in the fourier domain (ie pointwise multiplication) with $\overline{d * w^{*t}}^D$ easier? What about convolution in the fourier domain with $d * w^{*t}$?

Assorted notes on my handwritten notes

I also noticed that you can use prime components to break apart some statistics to make computing them for averages of orbits of $S_m D$ easier

I was also trying to organize the coset representatives for $S_m D$ by their corresponding sums to see if that structure would provide a way to work with a tensor product of sin distributions and computing inner products. This path wasn't super fruitful, but there were many lattices, and some familiar counts.

Random walks

d - distribution
w - random walk
S - statistic of interest

$$(d * w^{*t}) \cdot S$$

Using d: starting at 0^m w/ prob 1 $\Rightarrow \hat{d} = \sum_{g \in [n]^m} \chi_g$
w: $\cos\left(\frac{2k\pi}{n}\right)$ for each component. Specifically for $k=1$
S: total displacement

$|g| \to$ # nonzero entries.

$$\hat{w} = \left(\tfrac{1}{2}\chi_{n-1} + \chi_0 + \tfrac{1}{2}\chi_1\right)^{\otimes m} = \sum_{g \in \{n-1,0,1\}^m} \frac{1}{2}^{|g|} \chi_g$$

so...
$$\widehat{w^{*t}} = \sum_{g \in \{n-1,0,1\}^m} \frac{1}{2}^{|g|t} \chi_g$$

note: these functions are constant under reordering
since the random walk doesn't pick out any particular car.

Let
$$\hat{S} = \sum_{g \in [n]^m} \langle S, \chi_g \rangle \chi_g$$

right now, I know $\langle S, \chi_g \rangle = 0$ for all g where $\sum_i g_i \equiv 0 \bmod n$
and $\langle S, \chi_g \rangle$ is constant under the action of S_m. ie for $\sigma \in S_m$ $\langle S, \chi_g \rangle = \langle S, \chi_{\sigma \cdot g} \rangle$

Then
$$\widehat{(d * w^{*t}) \cdot S} = \left(\sum_{g \in \{n-1,0,1\}^m} \frac{1}{2}^{|g|t} \chi_g\right) * \sum_{g \in [n]^m} \langle S, \chi_g \rangle \chi_g$$
$$= \sum_{g \in [n]^m} c(g) \chi_g$$

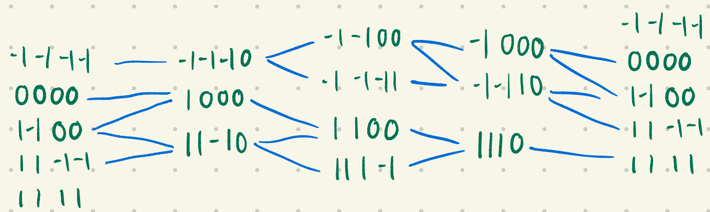
Since both $(d * w^{*t})$ and S are constant under reordering, the resulting function is also constant under reordering

Additionally note that $d * w^{*t}$ takes on $\min(2i+1, n)$ distinct values for $\sum_i g_i$, so
there will be $\min(2i+1, n) \cdot n^{m-1}$ non-zero coefficients.

Entertain this possibility for a moment.

Let $\widehat{d*w^{*t}}\big|_G$ be a restricted version of $\widehat{d*w^{*t}}$ to only the characters $\chi_g \mid \sum_i g_i = 0$. Let us call this subgraph G

Question: Is there a good way of writing $\widehat{d*w^{*t}}$ in terms of $d*w^{*t}\big|_G$ in this case? not really



Similar logic to Hullman

Let $\mathcal{O}$ partition G, so $\bigcup_{O \in \mathcal{O}} O = G$ and if $O_1, O_2 \in \mathcal{O}$, then they are disjoint. Let $d, s \in \mathbb{C}G$.

Let $\bar{d}^{\mathcal{O}}$ be the average, so $\bar{d}^{\mathcal{O}}(g) = \frac{1}{|O_g|} \sum_{h \in O_g} d(h)$ where $g \in O_g$ and $O_g \in \mathcal{O}$

Then $\langle d, \bar{s} \rangle = \langle \bar{d}, \bar{s} \rangle$ (same argument as pg 25 of Ian's thesis)

For d, s as a distribution & statistic on preference lists,
s constant under action of D
d constant on orbits of S_m

Then
$$\langle d, s \rangle = \langle \bar{d}^{D}, s \rangle = \langle d, \bar{s}^{S_m} \rangle$$
$$= \langle \bar{d}^{D}, \bar{s}^{S_m} \rangle$$

since $\bar{s}^{S_m}$ still constant on cosets of D
or similarly $\bar{d}^{D}$ still constant on cosets of S_m

ex stats $\bar{s}^{S_m}$

Question: is average over S_m same in fourier domain?

what if:
for walk:
fourier

$\diagdown s$	total disp	$\overline{\text{lucky}}^{S_m}$	$\overline{\text{disp.1}}^{S_m}$	$\overline{\text{disp2}}^{S_m}$	$\overline{\text{disp3}}^{S_m}$	$\overline{\text{maxdisp}}^{S_m}$
$S_mD \cdot 000$	3	1	1	1		2
100	2	$1\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$		$1\frac{2}{3}$
2|00	1	2	1	0		1
210	0	3	0	0		0
$S_mD \cdot 0000$	6	1	1	1	1	3
1000	5	$1\frac{2}{4}$	$\frac{3}{4}$	1	$\frac{3}{4}$	$(n-1)-\frac{1}{n} = 2\frac{3}{4}$
2000	4	$1\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$(n-1)-\frac{2}{n} = 2\frac{1}{2}$
3|000	3	1	1	1	0	2
1100	4	$1\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{2}$	$2\frac{1}{2}$
2100	3	$2\frac{5}{12}$	$\frac{8}{12}$		$\frac{1}{2}$	
3|1|00	2	+1 $\to 2\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	$1\frac{2}{3}$ max of prime components
22|00	2	2	2	0	0	1
32|00	1	3	1	0	0	1
3|2|1|0	0	4	0	0	0	0

all parking func. b/c coset rep of D

coset reps for prime parking functions (of S_mD)

n=m	
1	1
2	1
3	3
4	5
5	14

②

1000 2011 2101
0100 2011 2101
0010 1201 2101
0001 1120 1210

0122

0011 121
0101 2~~0~~2
0110 2101
1010 2101
1001 202
1100 2101

0120 3001 · 6
1002 0102 2110 · 2
0012 1300
0021 211 · 3
0201
2001

Conversation w/ Prof O

permutations get w/ linear probing defect d?

p-prime & ith car lucky

Coset reps. for $S_m D$ by sum

to understand functions constant on 2 sided cosets $S_m D \cdot X \cdot S_m$ in $C_n \wr S_m$
earlier looked at nonzero characters, now coset reps.

How does this relate to decomposition into irreducibles for action of $C_n \wr S_m$?

For $n=m=3$

representatives are reps of $(g+D) \cdot S_m$
so 000: 000, 111, 222
100: 100, 010, 001, 211, 121, 112, 022, 202, 220

0   000   012
1   100
2   200

4

For $n=m=4$

0   0000   1300   2200 $4+2+4=10$ $4\cdot1+2\cdot6+4\cdot12=64$
1   1000   3200 $4+4 = 8$ $4\cdot4+4\cdot12=64$
2   2000   1100   3210 $4+4+1 = 9$ $4\cdot4+4\cdot6+1\cdot24=64$
3   3000   1200 $4+4 = 8$ $4\cdot4+4\cdot12=64$
4   0000   1300   2200

10

recall:

A128562	# distinct values of coef. constant on $S_m D \times S_m$
1	1
1 1	2
1 2 1	4
1 4 4 1	10
1 6 12 6 1	26
1, 10, 29	80
1, 14, 61, 94	246
1 21, 120, 263	810
1 29, 222, 645, 910	2704

oeis A003239 related to rooted trees

For $n=m=5$

0   00000   14000   23000   11300   12200   43210 $5+5+5+5+5+1=26$

26

1   10000   24000   11400   33000   12300
2   20000   11000   34000   12400   33100
3   30000   21000   44000   43100   42200

For $n=m=6$ $6+6+6+6+6+6+6+6+6+6+6+6+6+2 = 6\cdot13+2=80$ ✓

31^30^2 $5^21^20^2$ 4^2310^2

14

0   0^6 510^4 420^4 3^20^4 41^20^3 3210^2 2^30^3 20^35^2 5430^3 $2^21^20^2$ 54210^2 53^210^2 532^20^2 $4^22^20^2$

1   10^5 520^4 51^20^3 430^4 4210^3 430^4 5^230^3 3^210^3 32^20^3 321^20^2 54^20^3 54310^2 542^20^2 53^220^2

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To Do

- Given the way that my code is implemented right now, I think I could extend the structure I have now to generate the nice bases from earlier in the summer. It would be interesting to be able to look at these bases
- Would be helpful to implement S_m and C_n^m acting on CnmStat object.
- Given the ability to sample randomly from defect 1 parking functions (and the ability to iterate through them), what can you say about statistics of interest?
- Speaking of iterating - write an iterator for defect d parking functions. There's a whole class for that wow!
- Think on if connection to $N_d(\lambda)$ works more broadly or only when $n = m + 1$