Here are some assorted thoughts that I don't think I've written down since they've sort of just been simmering

On a property of the first break point

I was thinking about other ways that you can think about prime parking functions and came up with this combinatorial would-be-proof-if-I-wrote-the-details. The main connection is using information that we already know about the number of prime parking functions to understand more information about the the location of the first break point in an arbitrary parking function.

Note, that we already know that $|PPF_n| = n - 1^{n-1}$ from [1] and from the circular argument that you can apply.

Let $b_1 : PF_n \to \mathbb{Z}$ be a function which returns the location of the first break point in the parking function (also the length of the first prime component of the parking function). For example $b_1(11354) = 2$.

Note that all prime parking functions must have the last car parking in the last spot. Therefore by ignoring the last car of a prime parking function, you get a parking function.

For this second method of enumerating PPF_n, consider taking a parking function $\pi \in PF_{n-1}$ and modifying it to be a prime parking function $\pi' \in PPF_n$ by adding in the final preference. Note that there are $b_1(\pi)$ different choices for the additional preference of the last car which result in a prime parking function π'. Therefore the number of prime paring functions is

$$|PPF_n| = \sum_{\pi \in PF_{n-1}} b_1(\pi)$$

Note this also tells us the coefficient of the trivial character in the character decomposition of $b_1(\pi)$

Counting p-prime parking functions - version 2

Earlier, following from thoughts in my 6/29 research journal, I was trying to figure out how to think of counting the p-prime parking functions using an inclusion exclusion argument to end up with the same expression that you get using abel's identity. It really bugged me that I slipped up on making that argument, so I ended up coming back to that thought because I wanted to understand it.

Here's the scratch work that I used to figure out some patterns, and then an explanation.

P-prime - interpreting the other expression one more time

expression to explain:

$$\sum_{i=0}^{p} \binom{n}{i} (n-i-1)^{n-i-1} (p-i)^{i} (-1)^{i}$$

$\binom{n}{i}$: order cars into parking lot

$(n-i-1)^{n-i-1}$: prime park func n-i

$(p-i)^i$: $T(p-1,i)$ — # ways placing $1 \ldots i$ in $p-1$ boxes such that each box has 1 # & adjacent #s in increasing order

ways to do boxes: (i in p-1)

1:3
2:4

$$\sum_{\lambda \vdash i} \binom{i}{\lambda}$$

$$= \sum_{j=0}^{p-1} \binom{i}{j} T(p-1-1-j,\ i-j)$$

n spots, P bcd

remove contains last element

ex p=2 $\quad \binom{n}{0}(n-1)^{n-1}(2-0)^{0} - \binom{n}{1}(n-2)^{n-2}(2-1)$

ex: p=3 $\quad (n-1)^{n-1} - \binom{n}{1}(n-2)^{n-2}\, 2 + \binom{n}{2}(n-3)^{n-3}\, 1$

$\quad (n-1)^{n-1} - \binom{n}{1}(n-2)^{n-2}\, 3 + \binom{n}{2}(n-3)^{n-3}\, 4 - \binom{n}{3}(n-4)^{n-4}\, 1$

For p=3, n=4

4·4·2

PPF | not

	1 1 1 1	·1	+	
|	1 1 1 2	·4	+	−
|	1 1 1 3	·4	+	−
||	1 1 2 2	·6	+	− − +
|	1 1 2 3	·12	+	−
16 16				

For n=5 For P=3 ... For p=4

		256	5·27·2=270	10·4·1	256	5·27·3=405			10·4·4=160			10
1 1 1 1 1	·1	+			+							
1 1 1 1 2	·5	+			+	|						
1 1 1 1 3	·5	+	|		+		|					
1 1 1 1 4	·5	+	|		+			|				
1 1 1 2 2	·10	+			+	||			|			
1 1 1 2 3	·20	+	|		+	|	|				|	
1 1 1 2 4	·20	+	|		+			|				
1 1 1 3 3	·10	+	||	+	+		||			|		
1 1 1 3 4	·20	+	|		+			|				
1 1 2 2 2	·10	+			+	|||			|||			|
1 1 2 2 3	·30	+	|		+	||	|				||	
1 1 2 2 4	·30	+	|		+			|				
1 1 2 3 3	·30	+	||	+	+		||			|		
1 1 2 3 4	·60	+	|		+			|				
			135 135	1 2		1··	·1·	··1	12·	·12	1·2	123
						2PPF	3PPF	4PPF	2·2s PPF	2·3s PPF	2·1 23 PPF	

2 1··· 3 ·1·· 4 ··1· 5 ···1

12·· 22 ·12· 33 ··12 44

123· 222 ·123 333

1·2· 2·1· 23 ·1·2 ·2·1 34 1··2 2··1 24

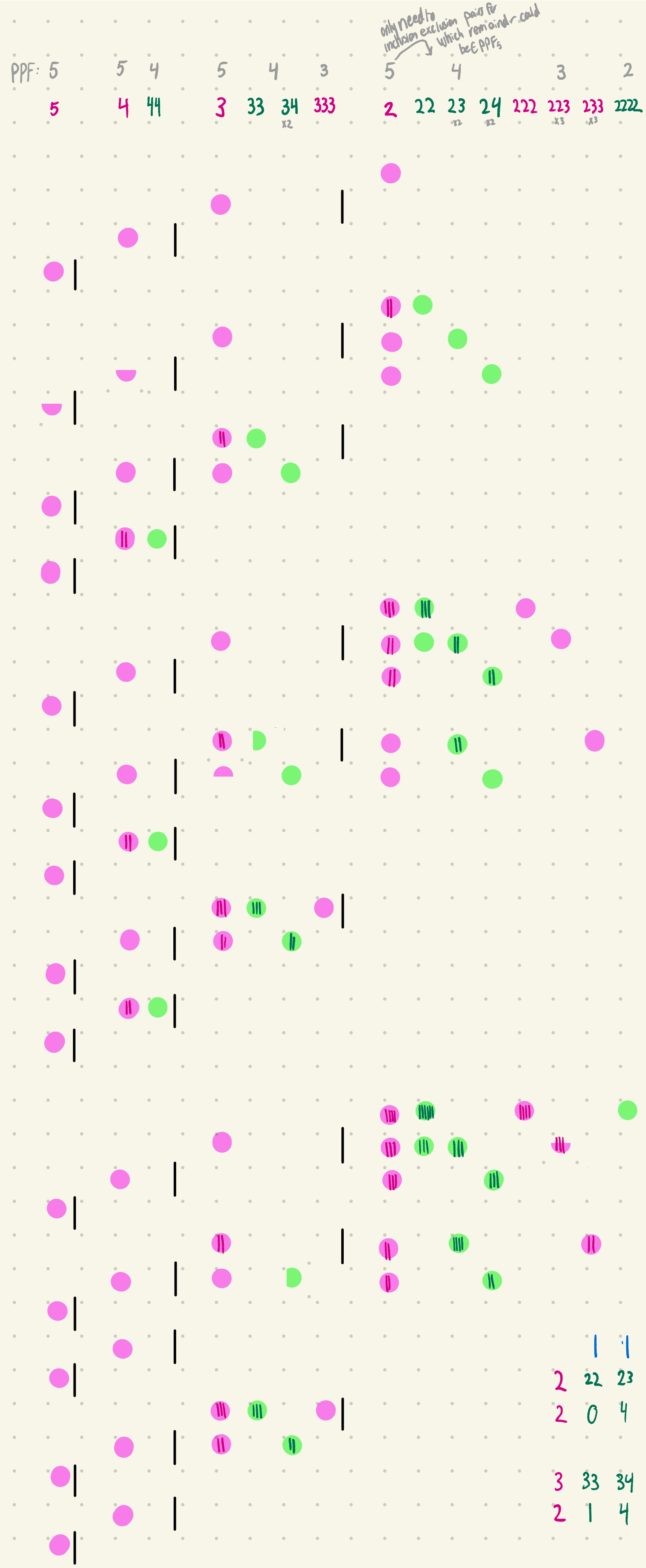
12·3 223 13·2 23·1

233

For $n=6$ $|PPF_6| = 5^5$

1 1 1 1 1 1
1 1 1 1 1 2
1 1 1 1 1 3
1 1 1 1 1 4
1 1 1 1 1 5
1 1 1 1 2 2
1 1 1 1 2 3
1 1 1 1 2 4
1 1 1 1 2 5
1 1 1 1 3 3
1 1 1 1 3 4
1 1 1 1 3 5
1 1 1 1 4 4
1 1 1 1 4 5
1 1 1 2 2 2
1 1 1 2 2 3
1 1 1 2 2 4
1 1 1 2 2 5
1 1 1 2 3 3
1 1 1 2 3 4
1 1 1 2 3 5
1 1 1 2 4 4
1 1 1 2 4 5
1 1 1 3 3 3
1 1 1 3 3 4
1 1 1 3 3 5
1 1 1 3 4 4
1 1 1 3 4 5

1 1 2 2 2 2
1 1 2 2 2 3
1 1 2 2 2 4
1 1 2 2 2 5
1 1 2 2 3 3
1 1 2 2 3 4
1 1 2 2 3 5
1 1 2 2 4 4
1 1 2 2 4 5
1 1 2 3 3 3
1 1 2 3 3 4
1 1 2 3 3 5
1 1 2 3 4 4
1 1 2 3 4 5



	1	1	2	2	3	3	4	4	
2	22	23	24	223	224	233	234	244	2344
2	0	4	4				8	2	4

3 33 34 334 344 3344
2 1 4 2 2 1 /0

4 44
2 1

From 6/29:

$$|PPF_{n,p}| = \sum_{i=0}^{p} \binom{n}{i} (n-i-1)^{n-i-1}(p-i)^i(-1)^i$$

For this inclusion exclusion argument, I need to clarify some definitions and ways of thinking about shuffles in terms of collections (with repetition allowed).

The idea of shuffling comes from thinking about combining two preference lists into one by interweaving them. So $sh(\pi_1, \pi_2)$ is the collection of all preferece lists which result from choosing which cars will have preferences determined by π_1. Therefore the number of elements of $sh(\pi_1, \pi_2)$ is always $\binom{|\pi_1|+|\pi_2|}{|\pi_1|}$ where $|\pi|$ means the number of cars in the preference list.

In addition to acting on single preference lists, it is helpful to let shuffles act on sets of of preference lists. So $sh(A, B) = \sum_{a\in A}\sum_{b\in B} sh(a, b)$ where the summation is list concatenation. When all of the elements of A and all of the elements of B have the same number of cars, then $|sh(A, B)| = |A||B|\binom{|a|+|b|}{|b|}$ for some $a \in A$ and $b \in B$

Question. First, let us think about a simpler question: How many prime parking functions have the property that thier largest preference is $n-p$? In other words, what is $|PPF_{n,p} - PPF_{n,p+1}|$?

Answer. Let us define A_i as the shuffle of prime parking functions for $n-i$ and lists of length i which come from the range $n-p$ to $n-i$ and contain the element $n-p$. So

$$A_i = sh(X, PPF_{n-i}) \text{ where } X = \{x \in \{n-p, \cdots, n-i\}^i | n-p \in x\}$$

Then

$$|\{\pi \in PPF_n | \text{max\_pref}(\pi) = n-p\}| = |A_1| - |A_2| + |A_3| - \cdots |A_p|$$

Explanation. To show that this strategy provides an accurate count, first note that any preference list involved in any of the A_is is a prime parking function since each A_i begins with a prime parking function for $n-i$ and adds in additional preferences less than or equal to $n-i$. Among prime parking functions, we can focus on three different categories based on the largest preference for any car: the latest preference is less than $n-p$, the largest preference is exactly $n-p$, and the latest preference is larger than $n-p$.

In the first case, for prime parking functions π where the largest preference is less than $n-p$, the parking function should be excluded from the count. Note that a preference list whose latest preference is less than $n-p$ will never be included in any collection A_i, so it will be excluded, as desired.

In the second case, for prime parking functions π where the largest preference is exactly $n-p$, the alternating summation acts as an inclusion exclusion argument based on the number of preferences for $n-p$. For this subset of prime parking functions, A_i counts the number of times that the preference $n-p$ appears. Note that for A_i the only option for x which leaves $n-p$ as the largest preference is i copies of $n-p$. Additionally, when you remove the latest preference from a prime parking function, the resulting parking function is still prime. This means that the restriction to prime parking functions for the second half of the shuffle doesn't affect this inclusion exclusion argument. To see why, recall that to verify that a preference list is a prime parking function, you can check that the increasing rearrangement each preference is strictly less than its index. By removing the largest preference, you don't change the alignment of the increasing rearrangement, so the result is still a prime parking function for fewer cars. Since A_i counts the number of times that the preference $n-p$ appears, the alternating summation is an inclusion exclusion argument to check if $n-p$ is in the prime parking function, which will contribute 1 to the count, as desired.

In the third case, for prime parking functions π where the largest preference is $n-k > n-p$, the result is also a consequence of an inclusion exclusion style computation. **For this third case, I don't have a nice proof - this current section goes something like *I know what I'm over counting and undercounting from doing it, so have some observations* which is not a proof.** One key property (from observation) that is maintained through the course of the summation is that after the A_ith term, any prime parking functions whose largest preference is $n-i$ will be excluded and not modified thereafter. For example, after the first term A_1, any preference lists with the largest preference of $n-1$ will be excluded since x must be $n-p$ and an element of PPF_{n-1} cannot contain an preference for $n-1$. The set of values for X for the term A_i doesn't include the preference $n-i+1$ because the preference $n-i+1$ wouldn't have contributed to overcounting because A_{i-1} was a shuffle of X with prime parking functions which didn't include the preference $n-i+1$. Also, any prime parking function with only k preferences that are $n-p$ or larger will be excluded after the A_kth term of the summation.

Now that we have answered a simpler question, we can put the pieces together to construct the count of $PPF_{n,p}$ by subtracting off the prime parking functions whose largest preference is too large.

Theorem Let B_i be the following collection

$$B_i = sh(\{n-p+1, \cdots, n-i\}^i, PPF_{n-i})$$

Then

$$|PPF_{n,p}| = |PPF_n| - |B_1| + |B_2| - |B_3| \cdots |B_{p-1}|$$

Proof. This is a consequence of some regrouping based on expressions that we already have. Note that we can subtract out all of the prime parking functions whose largest preference is larger than $n-p+1$, so

$$
\begin{aligned}
|PPF_{n,p}| &= |PPF_n| - \sum_{i=0}^{p-1} |\{\pi \in PPF_n \mid \text{max\_pref}(\pi) = n-i\}| \qquad \text{by definition} \\
&= |PPF_n| - \sum_{i=1}^{p-1} |\{\pi \in PPF_n \mid \text{max\_pref}(\pi) = n-i\}| \qquad \text{dropping a term which is 0} \\
&= |PPF_n| - \sum_{i=1}^{p-1} \sum_{j=1}^{i} (-1)^{j+1} \left| sh(\{x \in \{n-i, \cdots, n-j\}^j \mid n-i \in x\}, PPF_{n-j}) \right| \qquad \text{previous theorem} \\
&= |PPF_n| + \sum_{j=1}^{p-1} (-1)^j \sum_{i=j}^{p-1} \left| sh(\{x \in \{n-i, \cdots, n-j\}^j \mid n-i \in x\}, PPF_{n-j}) \right| \qquad \text{swap summations} \\
&= |PPF_n| + \sum_{j=1}^{p-1} (-1)^j \left| sh(\{n-p+1, \cdots, n-j\}^j, PPF_{n-j}) \right| \qquad \text{combining summation} \\
&= |PPF_n| + \sum_{j=1}^{p-1} (-1)^j |B_i| \qquad \text{by definition}
\end{aligned}
$$

On Sampling From p-prime parking functions

One thing worth noting is which method of sampling is more efficient for which values of p.

Method 1 - sample using $[n-p]^n$
Method 2 - sample using prime parking functions

For each of these sampling methods, there are the same number of p-prime parking functions, but a different number of total options, so to find the crossing point where they will have the same probability of success upon sampling, just examine these counts.

Method 1 - $(n-p)^n$ total options
Method 2 - $(n-1)^{n-1}$ total options

So the crossing point of interest occurs at

$$\begin{aligned}(n-p)^n &= (n-1)^{n-1}\\ n-p &= (n-1)^{(n-1)/n}\\ p &= n-(n-1)^{(n-1)/n}\end{aligned}$$

It is certainly true that as n goes to infinity, p/n goes to 0, which means that as n gets larger, in most cases it is better to use Method 1. However, I'm not sure how to compute the limit of p as n goes to infinity, though computationally from desmos and wolfram alpha, it continues to grow slowly, meaning that for small p and large n, it is still an important strategy. This likely means that the worst sampling situation continues to get worse and worse since it seems that for method 1, the distribution of probability of sampling successfully as a function of p/n seems to approach a single distribution, while the crossing point approaches 0. See this plot

References

[1] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.