07-12-23

Day 62

Q1. Change the order of integration is $\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$ and hence evaluate it.

Soln: Given: $\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$

Rewrite, $\int_0^a \int_x^a (x^2+y^2)\,dy\,dx$.

Given: $dy\,dx$

y varies from x to a

x varies from 0 to a

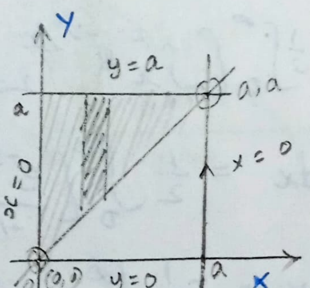
$y = x$; $y = a$

$x = 0$; $x = a$

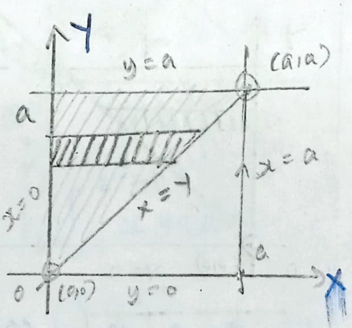
x	0	a
$y=x$	0	a

(0,0)

(a,a)



Change



$\int_0^a \int_0^y (x^2+y^2)\,dx\,dy$

$I = \int_0^a \left[\int_0^y (x^2+y^2)\,dx\right] dy$

$= \int_0^a \left[\left(\frac{x^3}{3}\right)_0^y + y^2 (x)_0^y\right] dy$

$= \int_0^a \left[\left(\frac{y^3}{3} - \frac{0}{3}\right) + y^2(y-0)\right] dy$

$= \int_0^a \left(\frac{y^3}{3} + y^3\right) dy$

$= \int_0^a \left(\frac{y^3 + 3y^3}{3}\right) dy$

$= \frac{1}{3}\int_0^a y^3\,dy + \int_0^a y^3\,dy$

$= \frac{1}{3}\left(\frac{y^4}{4}\right)_0^a + \left(\frac{y^4}{4}\right)_0^a = \frac{a^4}{12} + \frac{a^4}{4} = \frac{a^4 + 3a^4}{12}$

$= 4a^4/12 = a^3/3$

Q2: Change the order of Integration $\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$

Soln: Given :

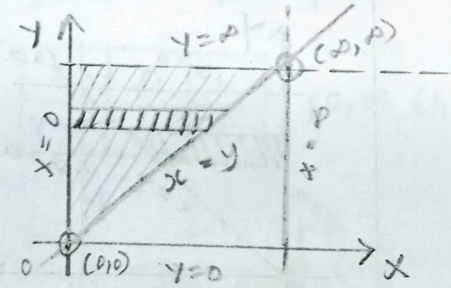
$$\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$$

x varies from 0 to y

y varies from 0 to ∞

$y = 0 \; ; \; y = \infty$

$x = 0 \; ; \; x = y$

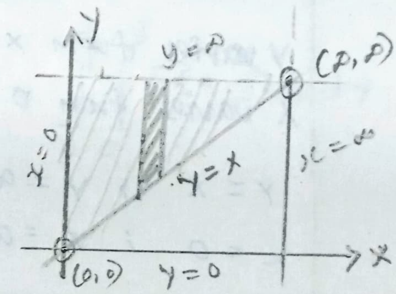


y	0	∞
x=y	0	∞

(0,0)

(∞,∞)

Change :



$$\int_0^\infty \int_{x}^\infty y e^{-y^2/x} \, dy \, dx$$

$$I = \int_0^\infty \int_x^\infty e^{-y^2/x} \, dy dx = \frac{1}{2}\int_0^\infty \left[\int_x^\infty e^{-y^2/x} \, 2 dy \right] dx$$

$$= \frac{1}{2}\int_0^\infty \left[\int_x^\infty e^{-y^2/x} \, d(y^2) \right] dx = \frac{1}{2}\int_0^\infty \left[\frac{e^{-y^2/x}}{-1/x} \right]_x^\infty dx$$

$$= -\frac{1}{2}\int_0^\infty x \left(e^{-\infty} - e^{-x^2/x} \right) dx = -\frac{1}{2}\int_0^\infty x \left(0 - e^{-x} \right) dx$$

$$= \frac{1}{2}\left(-x e^{-x} - 1 \; x e^{-x} + 0 \right)_0^\infty = \frac{1}{2}\left(-e^{-\infty} + e^0 \right)$$

Bernoulli formula

$$\int uv \, dx = uv_1 - u'v_2 - u''v_3 + \cdots$$

$u = x \qquad v = e^{-x}$

$u' = 1 \qquad v_1 = \int e^{-x} dx = \frac{e^{-x}}{-1}$

$u'' = 0 \qquad v_2 = e^{-x}$

Q3. Change the order of Integration $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$.

Soln: Given :

$dy\,dx$

y varies from x^2 to $2-x$

x varies from 0 to 1

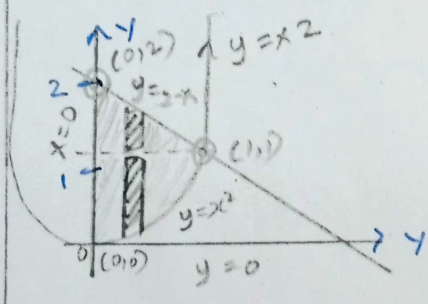
$y = x^2$; $y = 2 - x$

$x = 0$; $x = 1$

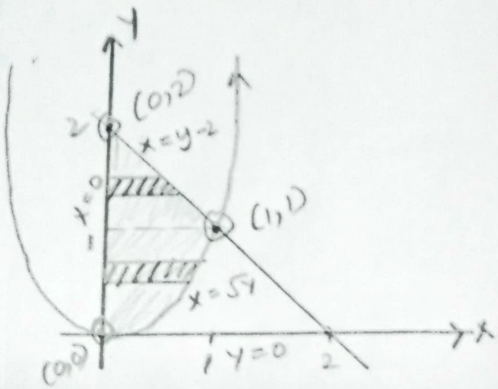
$y=x^2$	0	1
$y=2-x$	2	1

st line → (0,2) (1,1)

Parabola → (0,0) (1,1)



change :



$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$I = \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-x} dy$$

$$= \frac{1}{2}\int_0^1 y(y-0)\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2^2+y^2-4y)\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\,dy.$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\left[4\left(\frac{y^2}{2}\right)_1^2 + \left(\frac{y^4}{4}\right) - 4\left(\frac{y^3}{3}\right)_1^2\right]$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left[2(2^2-1^2) + (2^4-1^4) - \frac{4}{3}(2^2-1^2)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[2(4-1) + (16-1) - \frac{4}{3}(4-1)\right] = \frac{1}{6} + \frac{1}{2}\left[6 + \frac{15}{4} - \frac{4}{3}\times 7\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[6 + \frac{15}{4} - \frac{28}{3}\right] = \frac{1}{6} + \frac{1}{2}\left[\frac{5}{12}\right] = \frac{1}{6} - \frac{15}{24} = \frac{4-5}{24} = \frac{-9}{24}$$