Cayley - Hamilton Theorem

Statement : Every Square matrix satisfies its own characteristic equation.

Note : ThiruKural format of Cayley - Hamilton Theorem

Every square matrix satisfies
its own "cha.eqn".

Uses of Cayley - Hamilton Theorem :

To Calculate

i) the positive integral power of A and

ii) the inverse of a non-singular square matrix A

Problems

a) Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$

Sol. Given,

$$2x^2 + 8z^2 + 4xy + 10xz - 2yz$$

$$Q = \begin{bmatrix} \text{coeff.}\,x^2 & \frac{1}{2}\,\text{coeff.}\,xy & \frac{1}{2}\,\text{coeff.}\,xz \\ \frac{1}{2}\,\text{coeff.}\,yx & \text{coeff.}\,y^2 & \frac{1}{2}\,\text{coeff.}\,yz \\ \frac{1}{2}\,\text{coeff.}\,zx & \frac{1}{2}\,\text{coeff.}\,zy & \text{coeff.}\,z^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

b) Prove that the Quadratic form $x^2 + 2y^2 + 3z^2 + 2xy + 2yz - 2xz$ is indefinite.

Sol. Given,

$$x^2 + 2y^2 + 3z^2 + 2xy + 2yz - 2xz$$

$$Q = \begin{bmatrix} \text{coeff.}\,x^2 & \frac{1}{2}\,\text{coeff.}\,xy & \frac{1}{2}\,\text{coeff.}\,xz \\ \frac{1}{2}\,\text{coeff.}\,yx & \text{coeff.}\,y^2 & \frac{1}{2}\,\text{coeff.}\,yz \\ \frac{1}{2}\,\text{coeff.}\,zx & \frac{1}{2}\,\text{coeff.}\,zy & \text{coeff.}\,z^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$D_1 = |1| = 1 \ (+ve)$$

$$D_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 - 1 = 1 \ (+ve)$$

$$D_3 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= 1(6-1) - (3+1) - 1(1+2)$$

$$= 5 - 4 - 3$$

$$= -2 \ (-ve)$$

$\therefore$ The Q.F is indefinite

c) Find the index and signature of the Q.F

$$x_1^2 + 2x_2^2 - 3x_3^2$$

Solution : $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 3x_3^2$

it is already in the canonical form.

Index = Number of positive terms in the C.F

= 2

Signature = Number of positive terms in the C.F − Number of negative terms in the C.F.

= 2 − 1

= 1

d) Identify the nature, index and signature of the quadratic form.

$x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

Solution : Given,

$x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

The matrix of the given Q.F

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The Eigen values are $-2, 3, 6$

Eigen values are both positive and negative.

Therefore the Q.F is indefinite.

Index = Number of positive terms in the C.F

= 2

Signature = Number of positive terms in the C.F - Number of negative terms in the C.F

= 2 - 1

= 1

e) Find A^{-1} if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, using Cayley-Hamilton theorem.

Solution : The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ Where}$$

S_1 = Sum of the main diagonal elements

$$= 4 + 3 + (-3) = 4$$

S_2 = Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$= (-9 + 8) + (-12 + 6) + (12 - 6)$$

$$= -1 - 6 + 6$$

$$= -1$$

$$S_3 = |A| = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} = -4$$

$\therefore$ The characteristic eqn is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

By Cayley-Hamilton Theorem, we get

$$\therefore A^3 - 4A^2 - A + 4I = 0 \quad \ldots (1)$$

To Find A^{-1}

$$(1) \times A^{-1} \Rightarrow A^2 - 4A - I + 4A^{-1} = 0$$

$$4A^{-1} = -A^2 + 4A + I$$

$$A^{-1} = \frac{1}{2}[-A^2 + 4A + I] \quad \ldots (2)$$

$$-A^2+4A+I = \begin{bmatrix} -16 & -18 & -18 \\ -5 & -7 & -6 \\ 5 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 16 & 24 & 24 \\ 4 & 12 & 8 \\ -6 & -16 & -12 \end{bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

$$\therefore (2) \Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

f) The closed interval method

Find the absolute maximum and minimum values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, $[-2, 3]$

Solution : Given,

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$$

Step 1 : Differentiating $f(x)$

$$f'(x) = 3(4x^3) - 4(3x^2) - 12(2x) + 0$$

$$= 12x^3 - 12x^2 - 24x$$

Step 2 : Put $f'(x) = 0$

$$12x^3 - 12x^2 - 24x = 0$$

$$12x[x^2 - x - 2] = 0$$

$$12x = 0 \quad \Big| \quad x^2 - x - 2 = 0$$

-2

$-2 \quad 1$

$$\boxed{x = 0} \quad \Big| \quad \boxed{x = 2} \quad \boxed{x = -1}$$

$\therefore$ The critical numbers are $-1, 0, 2$

Step 3 : Apply $-1, 0, 2, -2, 3$ in $f(x)$

- $f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1 = 3 + 4 - 12 + 1$
  $= -4$
- $f(0) = 1$
- $f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1 = 48 - 32 - 48 + 1$
  $= -31$
- $f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 48 + 32 - 48 + 1$
  $= 33$
- $f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1$
  $= 3(81) - 4(27) - 12(9) + 1$
  $= 243 - 108 - 108$
  $= 28$

Step 4 : Absolute minimum at $f(2)$

Absolute maximum at $f(-2)$

T3) The Squeeze Theorem (or)
Sandwich Theorem (or)
The Pinching Theorem

If $f(x) \leq g(x) \leq h(x)$

when x is near 'a' and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\boxed{\lim_{x \to a} g(x) = L}$

g) If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$.

Solution : Given, $u = x/y + y/z + z/x$

$x\,\partial u/\partial x + y\,\partial u/\partial y + z\,\partial u/\partial z$

Let $u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^0 u(x, y, z)$$

$\Rightarrow$ u is a homogeneous function of x, y, z in degree 0.

$\therefore$ By Euler's theorem, we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu = 0 \cdot u = 0$$

h) If $x = u(1+v)$ and $y = v(1+u)$, find $\frac{\partial(x, y)}{\partial(u, v)}$

Solution : $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{bmatrix} 1+v & u \\ v & 1+u \end{bmatrix}$$

$$= (1+u)(1+v) - uv = 1 + u + v$$

T4) Maximum value

$f(a,b)$ is a maximum value of $f(x,y)$, if there exists some neighbourhood of the point (a,b) such that for every point $(a+h, b+k)$ of the neighbourhood,

$$f(a,b) > f(a+h, b+k)$$

T5) Minimum value

$f(a,b)$ is a minimum value of $f(x,y)$, if there exists some neighbourhood of the point (a,b) such that for every point $(a+h, b+k)$ of the neighbourhood,

$$f(a,b) < f(a+h, b+k)$$

T6) Extremum value

$f(a,b)$ is said to be an extremum value of $f(x,y)$ if it is either a maximum or a minimum.

T7) Necessary conditions for a maximum or a minimum.

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Notations : $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$,

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

i) Find the domain of each function

$f(x) = \sqrt{3-x} - \sqrt{2+x}$

Solution :

Given : $f(x) = \sqrt{3-x} - \sqrt{2+x}$

$3 - x \geq 0 \Rightarrow x \leq 3$

$2 + x \geq 0 \Rightarrow x \geq -2$

So, the domain is $-2 \leq x \leq 3$ i,e., $[-2, 3]$

j) Find $\lim_{x \to 1} \frac{x^2 - 4x}{x^2 - 3x - 4}$

Solution : $\lim_{x \to 1} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \to 1} \frac{x(x-4)}{(x-4)(x+1)}$

$= \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{1+1} = \frac{1}{2}$

K) Working Rule of Q.F to Canonical Form

Step 1. Write the matrix of the given Q.F

Step 2. To find the characteristic equation.

Step 3. To Solve the characteristic equation

Step 4. To Find the Eigenvector orthogonal to each other.

Step 5. Form Normalised matrix N.

Step 6. Find N^T.

Step 7. Find AN

Step 8. Find $D = N^T AN$

Step 9. Canonical form $[y_1 \; y_2 \; y_3] \, D \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$