11-12-23

Day 65

Integral Calculus

Partial Fraction (Type 1)

Q1. Evaluate $\int \frac{1}{(x-1)(x-2)}\,dx$

soln:- Given:

$$\int \frac{1}{(x-1)(x-2)}\,dx = \frac{A}{(x-1)} + \frac{B}{(x-2)} \rightarrow ①$$

$$\frac{1}{(x-1)(x-2)} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

$$1 = A(x-2) + B(x-1)$$

Put $x = 2$

$$1 = A(2-2) + B(2-1) = B(1) \Rightarrow B = 1$$

Put $x = 1$

$$1 = A(1-2) + B(1-1) = A(-1) \Rightarrow A = -1$$

From ①

$$\frac{1}{(x-1)(x-2)} = \frac{-1}{(x-1)} + \frac{1}{(x-2)}$$

$$\int \frac{1}{(x-1)(x-2)} = \int \frac{-1}{(x-1)} + \int \frac{1}{(x-2)}$$

$$= -\log(x-1) + \log(x-2)$$

$$\int \frac{1}{(x-1)(x-2)} = \log\left(\frac{x-2}{x-1}\right)$$

Q2. Evaluate $\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$ (Type II)

Soln: Given:

$$\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$$

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{(x-3)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \rightarrow ①$$

$$= A(x-2) + B(x-3)(x-2) + C(x-3) / (x-3)(x-2)^2$$

$$x^2+1 = A(x-2) + B(x-3)(x-2) + C(x-3) \rightarrow ②$$

Put $x=3$

$3^2+1 = A(3-2)^2 + B(3-3)(3-2) + C(3-3)$.

$10 = A(1)^2 \Rightarrow A = 10$

Put $x=2$

$2^2+1 = A(2-2)^2 + B(2-3)(2-2) + C(2-3)$

$5 = C(-1) \Rightarrow C = -5$

From ②

Equating x^2, we get

$1 = A + B \Rightarrow 1 = 10 + B \Rightarrow 1 - 10 = B$

$\Rightarrow -9 = B$

① $\Rightarrow \dfrac{x^2+1}{(x-3)(x-2)^2} = \dfrac{10}{(x-3)} - \dfrac{9}{(x-2)} - \dfrac{5}{(x-2)^2}$

$\int \dfrac{x^2+1}{(x-3)(x-2)} dx = 10\int \dfrac{1}{(x-3)} dx - 9\int \dfrac{1}{(x-2)} dx - 5\int \dfrac{1}{(x-2)^2} dx$

$= 10 \log(x-3) - 9\log(x-2) - 5\int (x-2)^{2} dx$

$= 10\log(x-3) - 9\log(x-2) - 5\left(\dfrac{(x-2)^{-2+1}}{-2+1}\right)$

$I = 10\log(x-3) - 9\log(x-2) + 5/(x-2)$

Q3. Evaluate $\int \dfrac{10}{(x-1)(x^2-9)} dx$ (Type (II))

Soln: $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{(x^2+9)} \rightarrow$ ①

$\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A(x^2+9) + (Bx+C)(x-1)}{(x-1)(x^2+9)}$

$10 = A(x^2+9) + (Bx+C)(x-1) \rightarrow$ ②

- Put $x=1$

$10 = A(1+9) \Rightarrow 10 = A(10) \Rightarrow 10/10 = A \Rightarrow A = 1$

- Put $x=0$

$10 = 1(0^2+9) + (B(0)+C)(0-1) \Rightarrow 10 = 9 + C(-1)$

$\Rightarrow 10 - 9 = -C \Rightarrow C = -1$

Coeffectient of x^2, in ②

$10 = A + B \Rightarrow 10 = 1 + B \Rightarrow B = 9$

From ①

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{9x-1}{x^2+9}$$

$$\int \frac{10}{(x-1)(x^2+9)} = \int \frac{1}{x-1}\,dx + \int \frac{(9x-1)}{x^2+9}\,dx$$

$$= \log(x-1) + \int \frac{9x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$I = \log(x-1) + \frac{9}{2}\int \frac{2x}{x^2+9}\,dx - \int \frac{1}{x^2+3^2}\,dx$$

$$I = \log(x-1) + \frac{9}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

Q4. Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3+x-x+1}\,dx$ (Type iv) Improper fraction

Soln:

$$\begin{array}{r|l}
 & x + 1 \\
\hline
x^3 - x^2 + 1 & x^4 - 0x^3 - 2x^2 + 4x + 1 \\
 & x^4 - x^3 - x^2 + x \\
\hline
 & x^3 - x^2 + 3x + 1 \\
 & x^3 - x^2 - x + 1 \\
\hline
 & 4x
\end{array}$$

$$\int \frac{x^4-2x^2+4x+1}{x^3+x-x+1}\,dx = \int (x+1)\,dx + \int \frac{4x}{x^3-x^2+1}\,dx$$

$$I = \frac{x^2}{2} + x + 4\int \frac{x}{(x+1)(x-1)^2}\,dx \rightarrow ①$$

$$\frac{x}{(x+1)(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \rightarrow ②$$

$$\frac{x}{(x+1)(x-1)^2} = \frac{A(x+1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

$$x = A(x+1)^2 + B(x+1)(x-1) + C(x+1)$$

- Put $x = 1$

$1 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$

$1 = 2C \Rightarrow C = 1/2$

- Put $x = -1$

$-1 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$

$-1 = A(-2)^2 \Rightarrow -1 = 4A \Rightarrow A = -1/4$

- Equating coeffecients of x^2, we get

$0 = A + B \Rightarrow 0 = -1/4 + B \Rightarrow B = 1/4$

From ②

$$\int \frac{x}{(x+1)(x-1)^2} dx = \int \frac{-1/4}{x+1} dx + \int \frac{1/4 \, dx}{(x-1)} + \int \frac{1/2}{(x-1)^2} dx$$

$$= \frac{-1}{4} \log(x+1) + \frac{1}{4} \log(x-1) + \frac{1}{2} \int (x-1)^{-2} dx$$

$$= \frac{1}{4} \left[\log(x-1) - \log(x+1)\right] + \frac{1}{2} \left(\frac{(x-1)^{-1}}{-1}\right)$$

$$= \frac{1}{4} \left(\log \frac{(x-1)}{(x+1)}\right) - \frac{1}{2} \frac{1}{(x-1)}$$

From ①

$$I = \frac{x^2}{2} + x + 4\left[\frac{1}{4} \log\left(\frac{x-1}{x+1}\right) - \frac{1}{2} \frac{1}{x-1}\right]$$

$$I = \frac{x^2}{2} + x + \log\left(\frac{x-1}{x+1}\right) - \frac{2}{x-1}$$

Rational Algebraic function

- $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$

- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$

Q6. Evaluate $\int \frac{1}{3+2x+x^2} dx$

x^2+2x+3

$\Rightarrow 2ab = 2x$ where, $a = x$, $b = ?$

$(2)(x) \cdot b = 2x \Rightarrow b = 1$

$x^2+2x+3 = x^2+2x+1^2-1^2+3 = (x^2+2x+1)+2$

$= (x+1)^2+2 = (x+1)^2+(\sqrt{2})^2$

$\int \frac{1}{3+2x+x^2} dx = \int \frac{1}{(x+1)^2+(\sqrt{2})^2}$

$I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

Q7. Evaluate $\int \frac{dx}{3x^2-4x-5}$

Soln ::

$= \frac{1}{3} \int \frac{dx}{3x^2-(4x/3)-5/3}$

Consider,

$x^2 - 4x/3 - 5/3$

$2ab = 4/3\, x \Rightarrow 2 \times x \times b = 4/3\, x \Rightarrow b = \frac{2}{3}$

$\left(\left(x^2 - \frac{4}{3} + \frac{4}{9}\right) - \frac{4}{9} - \frac{5}{3}\right)$

$(x-2/3)^2 - 4-15/9 \Rightarrow \left(x-\frac{2}{3}\right)^2 - 19/9$

$(x-2/3)^2 - (\sqrt{19}/9)^2$

$I = \int \frac{dx}{(x-2/3)^2-(\sqrt{19}/9)^2}$

$= \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$

$$= \frac{1}{2 \times \sqrt{19}/3} \log \left(\frac{(x-2/3) - \sqrt{19}/3}{(x-2/3) + \sqrt{19}/3} \right)$$

$$I = \frac{3}{2\sqrt{19}} \log \left(\frac{(x-2/3) - (\sqrt{19}/3)}{(x-2/3) + \sqrt{19}/3} \right)$$

Q8: $\int \frac{px+q}{ax^2+bx+c} dx$ (Type II)

$Nr = A \frac{d}{dx}(Dr) + B$

Evaluate

$$\int \frac{2x+3}{x^2+2x+5} dx$$

Soln: Given :

$$\int \frac{2x+3}{x^2+2x+5} dx$$

It is of the form $Nr = A \, d/dx \, (Dr) + B$

$2x+3 = A \frac{d}{dx} x^2+2x+5 + B$

$2x+3 = A(2x+2) + B \rightarrow ①$

- Equating 'x'

$2 = 2A \Rightarrow A = 1$

- Equating constant

$3 = 2A + B \Rightarrow 3 = 2 + B \Rightarrow B = 1$

From ①

$2x+3 = 1(2x+2) + 1$

$$\int \frac{2x+3}{x^2+2x+5} = \int \frac{2x+2}{x^2+2x+5} dx + \int \frac{1}{x^2+2x+5}$$

$$= \log(x^2+2x+5) + \int \frac{1}{(x^2+2x+1)-1+5}$$

$$= \log(x^2+2x+5) + \int 1/.(x+1)^2+4$$

$$= \log(x^2+2x+5) + \int \frac{1}{(x+1)^2+2^2}\,dx$$

$$I = \log(x^2+2x+5) + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$$

Convergent & Divergent

$\uparrow$ finite $\qquad \uparrow$ ∞

Q9. Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent ~~and~~ or divergent.

Soln: Given: $\int_1^\infty \frac{1}{x}\,dx$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}(\log x)_1^t$$

$$= \lim_{t\to\infty}(\log t - \log 1) = \lim_{t\to\infty}(\log t) = \infty$$

The given $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

Q10. Determine whether the integral $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent or divergent.

Soln: Given: $\int_1^\infty \frac{1}{x^2}\,dx$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(\frac{x^{2-1}}{-1}\right)_1^t$$

$$= \lim_{t\to\infty}\left(\frac{-1}{x}\right)_1^t = \lim_{t\to\infty}\left(\frac{1}{t}-\frac{1}{1}\right) = \lim_{t\to\infty}\left(\frac{1}{t}-1\right)$$

$$= -\left(\frac{1}{\infty}-1\right) = -(0-1) = 1$$

The given $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.