01-12-23

Day 58

Euler's theorem

If 'u' is a Homogeneous Function of degree 'n' is x & y then.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

① If $u = x/y + y/z + z/x$, then find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z\, \partial u/\partial z$

soln: Given : $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

Substitution.

$x = t_x$;

$y = t_y$

$z = t_z$

$$u = \frac{(t_x)}{t_y} + \frac{t_y}{t_z} + \frac{t_z}{t_x}$$

$$= t^0 \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$$

$$n = 0$$

Euler's theorem

$$\frac{x\,\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0 \times u = 0$$

2. If $u = \sin^{-1} \frac{(x^3 - y^3)}{x+y}$ then P.T $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2\tan u$.

Soln: Given:

$$u = \sin^{-1} (x^3 - y^3) / x + y$$

$$f(x,y) = \sin u = x^3 - y^3 / x + y$$

$$f(t_x, t_y) = (t_x)^3 - (t_y)^3 / t_x + t_y$$

$$= t^3 x^3 - t^3 y^3 / t(x + y)$$

$= t^3 (x^3 - y^3) / t(x+y)$

$f(tx, ty) = t^2 (x^3 - y^3 / x + y)$

$n = 2$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$

$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = 2 \sin u$

$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y}$

$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = 2 \sin u$

$x \cos u \cdot \frac{\partial u}{\partial x} + y \cos u \; u \cdot \frac{\partial u}{\partial y} = 2 \sin u$

$\cos \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = 2 \sin u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \sin u / \cos u = 2 \tan u.$

3. If $u = \tan^{-1} \frac{(x^3 + y^3)}{x - y}$, than P.T $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} =$ $\sin 2u$.

soln. Given :

$u = \tan^{-1} \left(\frac{x^3 + y^3}{x + y} \right)$

$f(x, y) = \tan u = x^3 + y^3 / x - y$

$f(tx, ty) = (tx)^3 + (ty)^3 / tx - ty$

$= t^3 x^3 + t^3 y^3 / t(x - y) = \frac{t^3 (x^3 + y^3)}{t(x - y)}$

$f(tx, ty) = t^2 \left(\frac{x^3 + y^3}{x - y} \right)$

$n = 2$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$

$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$

$x \frac{\partial}{\partial x} (\tan u) = y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$

$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$

$\sec^2 u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = 2 \tan u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u / \sec^2 u$

$= (2\sin u / \cos u) \times \cos^2 u$

$= 2\sin u \cos u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$ proved.

4. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

Soln:

Given,

$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$f(x,y) = \cos u = x+y / \sqrt{x} + \sqrt{y}$

$f(tx, ty) = \cos u = tx + ty / \sqrt{tx} + \sqrt{ty}$

$= t(x+y) / \sqrt{t}(\sqrt{x} + \sqrt{y})$

$= t(x+y) / t^{1/2}(\sqrt{x} + \sqrt{y})$

$= t^1 . t^{-1/2} (x+y) / \sqrt{x} + \sqrt{y}$

$= t^{1-1/2} (x+y) / \sqrt{x} + \sqrt{y}$

$= t^{1/2} (x+y)/(\sqrt{x} + \sqrt{y})$

$n = 1/2$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = 1/2 \cos u$

$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$

$-\sin u\left[x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] = 1/2 \cos u = \frac{-1}{2}\frac{\cos u}{\sin u}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$