The Fundamental Theorem of Calculus

If f is continuous on $[a,b]$, then the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \;;\; a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Problems

a) For what values of p in the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent ?

Solution:

If $p \neq 1$, $\lim\limits_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim\limits_{t\to\infty} \int_1^t x^{-p}\,dx$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] = \lim_{t\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{t^{p-1}}\right]$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1, \text{ converges} \\ \infty, & p \le 1, \text{ diverges} \end{cases}$$

b) If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$

Solution: Given : f is continuous, $\int_0^4 f(x)\,dx = 10 \quad \ldots(1)$

Let $2x = t$ $\qquad x \to 0 \Rightarrow t \to 0$

$\Rightarrow 2dx = dt$ $\qquad x \to 2 \Rightarrow t \to 4$

$$\therefore \int_0^2 f(2x)\,dx = \int_0^4 f(t)\frac{dt}{2} = \frac{1}{2}\int_0^4 f(t)\,dt$$

$$= \frac{1}{2}\int_0^4 f(x)\,dx \quad [\because t \text{ is a dummy variable}]$$

$$= \frac{1}{2}[10] \quad \text{by (1)}$$

$$= 5$$

c) Evaluate $\int_1^\infty \frac{\log x}{x}\,dx$

Solution : Given : $\int_1^\infty \frac{\log x}{x}\,dx$

Put $t = \log x$; $\quad dt = \frac{1}{x}\,dx$

$$\therefore \int \frac{\log x}{x}\,dx = \int t\,dt = \frac{t^2}{2} = \frac{(\log x)^2}{2}$$

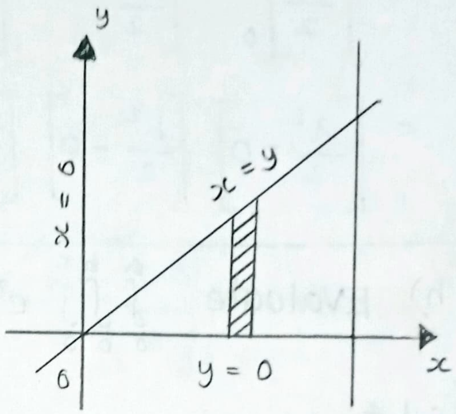
$$\therefore \int_1^\infty \frac{\log x}{x}\,dx = \left[\frac{(\log x)^2}{2}\right]_1^\infty = \infty - 0 = \infty$$

$\therefore \int_1^\infty \frac{\log x}{x}\,dx$ is divergent

e) Sketch roughly the region of intergration for

$$\int_0^1 \int_0^x f(x, y)\, dy\, dx$$

Sol. Given : $\int_0^1 \int_0^x f(x, y)\, dy\, dx$



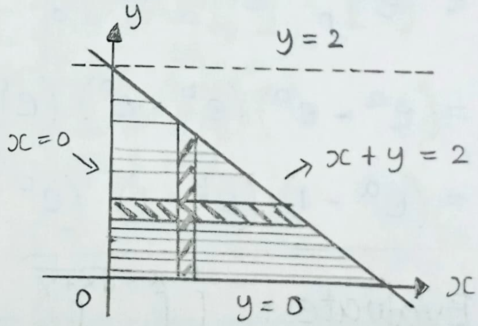
x varies from $x = 0$ to $x = 1$

y varies from $y = 0$ to $y = x$

f) Find the limits of integration $\iint_R f(x, y)\, dx\, dy$ where R is the triangle bounded by $x = 0$, $y = 0$, $x + y = 2$

Solution :

$$\iint_R f(x, y)\, dx\, dy = \int_0^2 \int_0^{2-y} f(x, y)\, dx\, dy$$



g) Evaluate : $\int_0^a \int_0^b \int_0^c xyz\, dz\, dy\, dx$

Solution : Let $I = \int_0^a \int_0^b \int_0^c xyz\, dz\, dy\, dx$

$$= \left[\int_0^a x\, dx\right] \left[\int_0^b y\, dy\right] \left[\int_0^c z\, dz\right]$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \left[\frac{a^2}{2} - 0\right] \left[\frac{b^2}{2} - 0\right] \left[\frac{c^2}{2} = 0\right] = \frac{(abc)^2}{8}$$

h) Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z} \, dz \, dy \, dx$

Solution :

$$I = \int_0^a \int_0^b \int_0^c e^{x+y+z} dz\, dy\, dx = \int_0^a \int_0^b \int_0^c e^x e^y e^z \, dz\, dy\, dx$$

$$= \left[\int_0^a e^x dx\right] \left[\int_0^b e^y dy\right] \left[\int_0^c e^z dz\right]$$

$$= \left[e^x\right]_0^a \left[e^y\right]_0^b \left[e^z\right]_0^c$$

$$= (e^a - e^0)(e^b - e^0)(e^c - e^0)$$

$$= (e^a - 1)(e^b - 1)(e^c - 1)$$

i) Evaluate : $\int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z \, dz \, dy \, dx$

Solution :

$$\text{Let } I = \int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z \, dz\, dy\, dx = \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\, dx$$

$$= \int_0^1 \int_0^x \left[\frac{x+y}{2} - 0\right] dy\, dx = \frac{1}{2} \int_0^1 \int_0^x (x+y)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\left(x^2 + \frac{x^2}{2}\right) - (0+0)\right] dx$$

$$= \frac{1}{2}\int_0^1 \frac{3}{2}x^2\,dx = \frac{3}{4}\int_0^1 x^2 dx = \frac{3}{4}\left[\frac{x^3}{3}\right]_1^1$$

$$= \frac{3}{4}\left[\frac{1}{3} - 0\right] = \frac{1}{4}$$

j) Evaluate $\int_1^2 \int_1^3 xy^2\,dx\,dy$

Solution :

$$\int_1^2 \int_1^3 xy^2\,dx\,dy = \left[\int_1^2 y^2 dy\right]\left[\int_1^3 x\,dx\right]$$

$$= \left[\frac{y^3}{3}\right]_1^2 \left[\frac{x^2}{2}\right]_1^3 = \left[\frac{8}{3} - \frac{1}{3}\right]\left[\frac{9}{2} - \frac{1}{2}\right]$$

$$= \left[\frac{7}{3}\right]\left[\frac{8}{2}\right] = \frac{28}{3}$$

k) Change the order of integration

$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ and hence evaluate it

Solution : HINT : $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y} x\right]_{x=0}^{x=y} dy = \int_0^\infty (e^{-y} - 0)\, dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\left[e^{-y}\right]_0^\infty = -[0-1] = 1$$

1) Integrate w.r.t. x

$$\int \frac{2x+3}{x^2+x+1} dx$$

Solution: Let $2x+3 = A \frac{d}{dx}(x^2+x+1) + B$

$$2x+3 = A(2x+1) + B \quad \ldots (1)$$

Equating the coefficients of x we get

$$2 = 2A$$

$$A = 1$$

Put $x = 0$, we get

$$3 = A + B$$

$$3 = 1 + B$$

$$B = 2$$

$\therefore (1) \Rightarrow 2x + 3 = (2x+1) + 2$

$$\int \frac{2x+3}{x^2+x+1} dx = \int \frac{2x+1}{x^2+x+1} dx + \int \frac{2}{x^2+x+1} dx$$

$$= \log(x^2+x+1) + 2\int \frac{1}{(x+1/2)^2 + 1 - 1/4} dx$$

$$= \log(x^2+x+1) + 2\frac{1}{(\sqrt{3}/2)} \tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + C$$

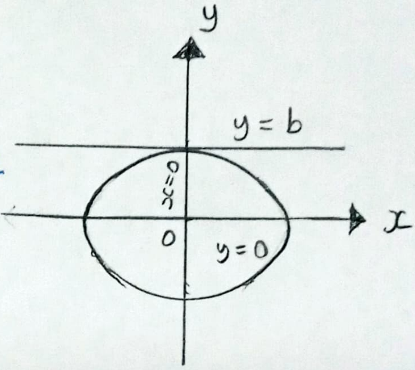
$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}} \tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] + c$$

m) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution :

Area of ellipse = 4 x area of quadrant.

Divide the area into horizontal strips of width δy.



x varies from $x = 0$ to $x = \frac{a}{b}\sqrt{b^2-y^2}$

y varies from $y = 0$ to $y = b$

∴ The required area

$$= 4\int_0^b \int_0^{a/b\sqrt{b^2-y^2}} dx\, dy$$

$$= 4\int_0^b [x]_0^{a/b\sqrt{b^2-y^2}} dy = 4\int_0^b \left[\frac{a}{b}\sqrt{b^2-y^2} - 0\right] dy$$

$$= \frac{4a}{b}\int_0^b \sqrt{b^2-y^2}\, dy = \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{y}{b} + \frac{y}{2}\sqrt{b^2-y^2}\right]_0^b$$

$$= \frac{4a}{b}\left[\left(\frac{b^2}{2}\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{4a}{b}\frac{b^2}{2}\frac{\pi}{2}$$

$= \pi ab$ square units.

d) Evaluate : $\int_0^3 \int_0^2 e^{x+y}\, dy\, dx$

Solution : Let $I = \int_0^3 \int_0^2 e^{x+y}\, dy\, dx = \int_0^3 \int_0^2 e^x e^y\, dy\, dx$

$$= \left[\int_0^3 e^x dx\right]\left[\int_0^2 e^y dy\right] = [e^x]_0^3 [e^y]_0^2$$

$$= [e^3 - e^0][e^2 - e^0] = [e^3-1][e^2-1]$$