30-11-23

Day 57

Lagrange's method.

① A rectangular box open at the top is to have a volume of 32 cc. Find the dimension of the box. That requires least material for its construction:

Soln. Given: Let x, y, z be the length, breadth, height

Surface area $= xy + 2yz + 2zx \rightarrow ①$

volume $\Rightarrow xyz = 32$

$xyz - 32 = 0 \rightarrow ②$

The auxillary function

step 1: $F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - 32)$

$F_x = \partial F/\partial x = (y + 0 + 2z) + \lambda(yz - 0) = y + 2z + \lambda yz$

$F_y = \partial F/\partial y = (x + 2z + 0) + \lambda(xz - 0) = x + 2z + \lambda xz$

$F_z = \partial F/\partial z = 0 + 2y + 2x + \lambda(xy - 0) = 2y + 2x + \lambda xy$

step 2: Put, $F_x = 0$

$y + 2z + \lambda yz$

$y + 2z = -\lambda yz$

$(\div yz)$

$\frac{y}{yz} + \frac{2z}{yz} = \frac{-\lambda yz}{yz}$

$\frac{1}{z} + \frac{2}{y} = -\lambda$ ③

$F_y = 0$

$x + 2z + \lambda xz$

$x + 2z = -\lambda xz$

$(\div xz)$

$\frac{1}{z} + \frac{2}{x} = -\lambda$ ④

$F_z = 0$

$2y + 2x + \lambda xy$

$2y + 2x = -\lambda xy$

$(\div xy)$

$\frac{2}{x} + \frac{2}{y} = -\lambda$ ⑤

From ③ and ④

$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$

$\frac{2}{y} = \frac{2}{x} \Rightarrow y = x$ ⑥

from ④ and ⑤

$\frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y}$

$\frac{1}{z} = \frac{2}{y} \Rightarrow y = 2z$ ⑦

step3: From ⑥ and ⑦

$x = y = 2z$

$xyz = 32$

$(2z)(2z)z = 32$

$4z^3 = 32$

$z^3 = 32/4 = 8$

$z^3 = 2^3 \Rightarrow z = 2$

$x = 2z$

$x = 2\times2$

$x = 4$

$y = 2z$

$y = 2\times2$

$y = 4$

$\therefore$ The dimensions are

$(x, y, z) = (4, 4, 2)$

The cost is minimum when

$x = 4 ; y = 4 ; z = 2$

② A rectangular box open at the top is to have a ~~volume~~ capacity of K. Find the dimensions of the box. This requires least material for its construction.

Soln: Given,

let $x\ y\ z$ be the length, breadth, height / dimen

surface area $= xy + 2yz + 2zx$. → ①

Volume $xyz = K. \Rightarrow xyz - K = 0$ → ②

The auxillary Function.

$F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - k)$

$F_x = \partial F/\partial x = (y + 0 + 2z) + \lambda(yz - 0) = y + 2z + \lambda yz$

$F_y = \partial F/\partial x = x + 2z + \lambda xz$

$F_z = \partial F/\partial z = 2y + 2x + \lambda xy$

Put, $F_x = 0$

$y + 2z + \lambda yz = 0$

$y + 2z = -\lambda yz$

$\frac{1}{z} + \frac{2}{y} = -\lambda$ ③

$F_y = 0$

$x + 2z + \lambda xz = 0$

$x + 2z = -\lambda xz$

$\frac{1}{z} + \frac{2}{x} = -\lambda$ ④

$F_z = 0$

$2y + 2x + \lambda xy = 0$

$2y + 2x = -\lambda xy$

$\frac{2}{x} + \frac{2}{y} = -\lambda$ ⑤

From ③ & ④

$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$

$y = x$ ⑥

from ④ & ⑤

$\frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y}$

$1/z = 2/y$ ⑦ $\Rightarrow y = 2z$

From ⑥ & ⑦

$x = y = 2z$

$xyz = k$

$(2z)(2z)z = k \Rightarrow 4z^3 = k \Rightarrow z^3 = k/4 \Rightarrow z = \left(\frac{k}{4}\right)^{1/3}$

$x = 2z$

$x = 2\left(\frac{k}{4}\right)^{1/3}$

$y = 2z$

$y = 2\left(\frac{k}{4}\right)^{1/3}$

The dimensions are

$(x, y, z) = 2(k/4)^{1/3}, \ 2(k/4)^{1/3}, \ (k/4)^{1/3}$

The cost is minimum.