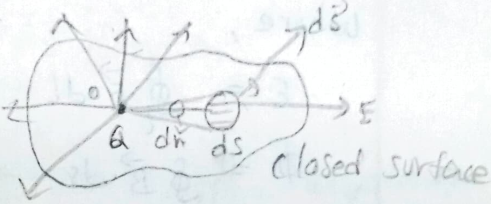
14/10/23

Day 24

Derivations of Maxwell's Equation

1. Maxwell's 1st eqn from electric Gauss law

MD1 $\oint_S \vec{E} \cdot ds = \frac{Q}{\varepsilon_0}$ (Gauss law for electric field)



* $\oint_S \varepsilon_0 \vec{E} \cdot ds = Q$ (1)

$[\varepsilon = \varepsilon_0 \varepsilon_r \Rightarrow \varepsilon = \varepsilon_0 1$

$\Rightarrow \varepsilon = \varepsilon_0]$ (2)

* $\oint_S \varepsilon \vec{E} \cdot ds = Q$ (3) $\therefore \vec{D} = \varepsilon \vec{E}$

* $\oint_S \vec{D} \cdot ds = Q$ (4)

* $\oint_V \rho \cdot dv = Q$ (5) $\therefore$ Eqns 5 = 6

* $\oint_S \vec{D} \cdot ds = \oint_V \rho \cdot dv$ (6)

MI1 Gauss divergence theorem

$\oint_S \vec{D} \cdot ds = \oint_V \vec{\nabla} \cdot \vec{D} \cdot dv$ (7)

Subs' 7 in 6

$\oint_V \vec{\nabla} \cdot \vec{D} \cdot dv = \oint_V \rho \cdot dv$ (8) $\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho$ (9)

2. Maxwell's 2nd eqn from magnetic Gauss law

MI2 $\oint_S \vec{B} \cdot ds = \phi = 0$

From $\phi = 0 \Rightarrow \oint \vec{B} \cdot ds \Rightarrow \oint_S \vec{B} \cdot ds = \phi$

* $\phi = \oint_S \vec{B} \cdot ds = 0 \rightarrow$ (10)

D2 * Gauss divergence theorem

$\oint_V \vec{\nabla} \cdot \vec{B} \, dv = 0$ (11) $\Rightarrow \vec{\nabla} \cdot \vec{B} = 0 \rightarrow$ (12)

3. Maxwell's third equation from law of Magnetic Induction (Faraday's law):

MI3

* $\mathcal{E} = -d\phi / dt \rightarrow$ 13

Where,

$\mathcal{E} = \oint_l \vec{E} \cdot dl \rightarrow$ 14 (line)

$\phi = \oint_S \vec{B} \cdot ds \rightarrow$ 15 (surface)

Substitute 15 and 14 in 13

$\oint_l \vec{E} \cdot dl = -d(\oint_s \vec{B} \cdot ds) / dt$

$\oint_L \vec{E} \cdot dl = -\oint_S \frac{d\vec{B}}{dt} ds$

* Gauss divergence theorem

MD3 $\oint_s \vec{\nabla} \times \vec{E} \, ds = -\oint_S \frac{d\vec{B}}{dt} ds$

$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{dt} \rightarrow$ 16

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4. Maxell's fourth eqn from Ampere's Circuital law.

* $\oint_L \vec{H} \cdot d\vec{l} = I$ 17 $\qquad \oint_S \vec{J} \cdot d\vec{s} = I$ 18

Hence 17 = 18

$\oint_L \vec{H} \cdot d\vec{l} = \oint_S \vec{J} \cdot d\vec{s}$

- Line $\rightarrow$ Surface
  Stoke's theorem
- Surface $\rightarrow$ Volume
  Gauss's diver

$\oint_s \vec{\nabla} \times \vec{H} \, ds = \oint_s \vec{J} \cdot d\vec{s}$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} \rightarrow$ 19

Multiply by $\vec{\nabla}$.

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$

$$0 = \vec{\nabla} \cdot \vec{J} \rightarrow (20)$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \rightarrow (21)$$ (adding charge density)

$$\vec{\nabla} \cdot \vec{J} = -\partial \rho / \partial t$$

$$0 = \partial \rho / \partial t \rightarrow (22)$$

Rewriting 19 eqn

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d \rightarrow (23)$$

$\vec{J}_d$ - displacem current densi

Multi by $\vec{\nabla}$.

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot (\vec{J} + \vec{J}_d)$$

$$0 = \vec{\nabla} \cdot (\vec{J} + \vec{J}_d)$$

$$0 = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} + \vec{J}_d \rightarrow (24)$$

Sub $-\partial \rho / \partial t$ in 24 eqn

$$0 = -\partial \rho / \partial t + \vec{\nabla} + \vec{J}_d$$

$$\vec{\nabla} \cdot \vec{J}_d = \partial \rho / \partial t \rightarrow (25)$$

$$\vec{\nabla} \cdot \vec{J}_d = \frac{\partial (\vec{\nabla} \cdot \vec{D})}{\partial t} \rightarrow (26)$$ (Maxwell's 1st eqn)

$$\vec{\nabla} \cdot \vec{J}_d = \vec{\nabla} \cdot \partial \vec{D} / \partial t$$

$$\vec{J}_d = \partial \vec{D} / \partial t \rightarrow (27)$$

From (23)

MD4 $$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (28)$$

MI4 $$\oint_l \vec{H} \cdot d\vec{l} = \oint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) ds$$