26/10/23 Day 31

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Q1 If $y = (1-x^2)^{10}$, find dy/dx

Soln: Given,

$$y = (1-x^2)^{10}$$

- let $u = 1-x^2$ | $y = u^{10}$

$$\frac{du}{dx} = -2x \quad | \quad \frac{dy}{du} = 10u^9$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow = (10u^9)(-2x) = -20u^9 x$$

$$= -20(1-x^2)^9 x$$

Q2. If $y = \frac{1}{(t^4+1)^3}$ find $\frac{dy}{dt}$

Soln: Given,

$$y = \frac{1}{(t^4+1)^3}$$

- let $u = t^4 + 1$ | $y = \frac{1}{u^3} \Rightarrow y = u^{-3}$

$$\frac{du}{dt} = 4t^3 \quad | \quad \frac{dy}{du} = -3u^{-3-1} = -3u^{-4}$$

$$\frac{dy}{du} = \frac{-3}{u^4}$$

$$\frac{dy}{dt} = \frac{-12t^3}{u^4} \Rightarrow \frac{dy}{dt} = \frac{-12t^3}{(t^4+1)^4}$$

Q3. If $y = \sqrt{\cos(\sqrt{x})}$ find $\frac{dy}{dx}$

Soln:

Given, $y = \sqrt{\cos(\sqrt{x})}$

- Let $u = \sqrt{x} \Rightarrow u = x^{1/2}$ • $y = \sqrt{\cos u} = (\cos u)^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} \quad \frac{dy}{du} = -\frac{1}{2}\cos u^{-1/2} \sin u$$

$$\frac{dy}{dx} = -\frac{1}{2}\cos u^{-1/2} \sin u \times \frac{1}{2} x^{-1/2}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{\cos u}} \sin u \times \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{-1}{4\sqrt{x}} \frac{\sin\sqrt{x}}{\sqrt{\cos\sqrt{x}}}$$

Q4. H.W

$y = \sqrt{\sin(\sqrt{x})}$

Soln:

Given: $y = \sqrt{\sin(\sqrt{x})}$

- let $u = \sqrt{x} \Rightarrow u = x^{1/2}$ • $y = \sqrt{\sin u} = (\sin u)^{1/2}$

$$\frac{du}{dx} = \frac{1}{2} x^{-1/2} \qquad \frac{dy}{du} = \frac{1}{2} \sin u^{-1/2} \cos u.$$

$$\frac{dy}{dx} = \frac{1}{2} \sin u^{-1/2} \cos u \times \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{\sin u}} \cos u \times \frac{1}{2}\frac{1}{\sqrt{x}}$$

$$= \frac{1}{4\sqrt{x}} \frac{\cos u}{\sqrt{\sin u}} = \frac{1}{4\sqrt{x}} \frac{\cos\sqrt{x}}{\sqrt{\sin\sqrt{x}}}$$

* Implicit Function

Q1. If $xy = c^2$ find $\frac{dy}{dx}$

Soln: Given,

$$xy = c^2$$

Diff w.r.to 'x'

$$x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx} = 0$$

$$x\frac{dy}{dx} + y = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

Q2. If $y^2 = 4ax$, find $\frac{dy}{dx}$

Soln: Given,

$$y^2 = 4ax$$

Diff w.r.to 'x'

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Q3. HW. Find $\frac{dy}{dx}$ if $y = \log(x^3+1)$

Soln: Given, $y = \log(x^3+1)$

- Let $u = x^3+1$, $\frac{du}{dx} = 3x^2$
- $y = \log u$, $\frac{dy}{du} = \frac{1}{u}$
- $\frac{dy}{dx} = \frac{1}{u} \times 3x^2 = \frac{3x^2}{x^3+1}$

Q4. Find $\frac{dy}{dx}$ if $y = \cot x + \sin x$

Soln: Given, $y = \cot x - \sin x$

Diff w.r.to 'x'

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x - \cos x$$

Q5. If $e^{x/y} = x - y$ find $\frac{dy}{dx}$

Soln: Given, $e^{x/y} = x - y$

diff w.r.t 'x'

$$e^{x/y}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) = 1 - \frac{dy}{dx}$$

$$e^{x/y}\left(\frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx}\right) = 1 - \frac{dy}{dx}$$

$$e^{x/y}\left(\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right) = 1 - \frac{dy}{dx}$$

$$\frac{1}{y}e^{x/y} - \frac{x}{y^2}e^{x/y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{1}{y}e^{x/y} - 1 = -\frac{dy}{dx} + \frac{x}{y^2}e^{x/y}\frac{dy}{dx}$$

$$\frac{1}{y}e^{x/y} - 1 = \frac{dy}{dx}\left(e^{x/y}\frac{x}{y^2} - 1\right)$$

$$\frac{\left(\frac{1}{y}e^{x/y} - 1\right)}{\left(e^{x/y}\frac{x}{y^2} - 1\right)} = \frac{dy}{dx}$$

Q6. Find $\frac{dy}{dx}$, If $y = x^{\sqrt{x}}$

Soln:

Given, $y = x^{\sqrt{x}}$

Taking log on both sides.

$$\log y = \log x^{\sqrt{x}}$$

$$\log y = \sqrt{x} \log x$$

Diff w.r.to 'x'

$$\frac{1}{y}\frac{dy}{dx} = x^{1/2}\frac{1}{x} + \log x\left(\frac{1}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = y\left[\frac{\sqrt{x}}{x} + \frac{\log x}{2\sqrt{x}}\right]$$

$$\frac{dy}{dx} = x^{\sqrt{x}}\left[\frac{\sqrt{x}}{x} + \frac{\log x}{2\sqrt{x}}\right]$$

Q7. Find y' if $y = x^{x^{x^{\cdots\cdots\infty}}}$

Soln:- Given,

$$y = x^{x^{x^{\cdots\cdots\infty}}}$$

$$y = x^{y}$$

Taking log on both sides

$$\log y = \log x^{y}$$

$$\log y = y \log x$$

Diff w.r.to 'x'

$$\frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x\left(\frac{dy}{dx}\right)$$

$$\frac{1}{y}\frac{dy}{dx} - \frac{dy}{dx}\log x = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x}\cdot\frac{1}{\left(\frac{1}{y} - \log x\right)}$$

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{1}{\left(\frac{1-y\log x}{y}\right)} \qquad y' = \frac{y^2}{x(1-y\log x)}$$

Q8. Find y' if $x^y = y^x$

Soln:

Given, $x^y = y^x$

Taking log on both sides:

$\log x^y = \log y^x$

$y \log x = x \log y$

Diff. w.r. to 'x'.

$\frac{dy}{dx} \cdot \frac{1}{x} = \frac{1}{y}$

$y \cdot \frac{1}{x} + \log x \frac{dy}{dx} = x \frac{1}{y} \frac{dy}{dx} \log y$

$\frac{y}{x} + \log x \frac{dy}{dx} = \frac{x}{y} \frac{dy}{dx} \log y$

$\log x + \frac{y}{x} \frac{dy}{dx} = \log y - \frac{x}{y}$

$dy/dx = \frac{\log y - y/x}{\log x - x/y}$

Q9. Find y' if $y = (\sin x)^{\cos x}$

Soln. Given, $y = (\sin x)^{\cos x}$

Taking log on both sides.

$\log y = \log (\sin x)^{\cos x}$

$\log y = \cos x \log (\cos x)$

Diff w.r.to 'x'.

$\frac{1}{y} \frac{dy}{dx} = \cos x \left(\frac{1}{\sin x} \cos x\right) + \log(\sin x)(-\sin x)$

$\frac{1}{y} \frac{dy}{dx} = \frac{\cos^2 x}{\sin x} - \sin x \log (\sin x)$

$dy/dx = y \left[\frac{\cos^2 x}{\sin x} - \sin x \log (\sin x)\right]$

Inflection point:

Working rule:

Step 1: To find critical point ($f''(x) = 0$)

Step 2: To find increasing & decreasing function.

sign of f'	function
+	Increasing
−	decreasing

Step 3: To find local maxima & local minima

$f'(c)$ = +ve (local maxima)

$f(c)$ = -ve (local minima)

Step 4: To find concave up & concave down

$f''(x) = 0$

sign of f''	function
+	concave up
−	concave down

Step 5: To find inflection point.