15/11/23

UNIT - 3

Day 44

Functions of Several Variables

#1 Jacobian (Partial differentiation)

* $$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

* $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

* $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \phi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \phi \end{vmatrix}$$

Q1 : If $x = r\cos\theta$, $y = r\sin\theta$

find $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$

Soln: Given,

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = (1)\cos\theta$

$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = (1)\sin\theta$

$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

Q2. If $x = u(1+v)$ and $y = v(1+u)$ find $\frac{\partial(x,y)}{\partial(u,\theta)}$

Soln: Given:

$x = u(1+v)$

$\frac{\partial x}{\partial u} = (1)(1+v)$

$\frac{\partial x}{\partial v} = 0 + u(1)$

$y = v(1+u)$

$\frac{\partial y}{\partial u} = v(1+u) = 0 + v(1)$

$\frac{\partial y}{\partial v} = 1 + u(1)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - vu$$

$$= 1 + v + u + uv - uv$$

$$= 1 + u + v$$

Q3: If $u = y^2/x$, $v = x^2/y$, find $\partial(u,v)/\partial(x,y)$

Soln: Given:

$u = y^2/x \quad \therefore -\frac{1}{x^2}$

$\frac{\partial u}{\partial x} = y^2\left(-\frac{1}{x^2}\right) = -\frac{y^2}{x^2}$

$\frac{\partial u}{\partial y} = \frac{1}{x}(2y) = \frac{2y}{x}$

$v = x^2/y$

$\frac{\partial v}{\partial x} = \frac{1}{y}(2x) = \frac{2x}{y}$

$\frac{\partial v}{\partial y} = x^2\left(-\frac{1}{y^2}\right) = -\frac{x^2}{y^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix} = \frac{y^2}{x^2} \times \frac{x^2}{y^2} - \frac{4xy}{xy}$$

$$= 1 - 4 = -3$$

Q4. Find the jacobian of $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

soln: Given,

$x = r\sin\theta\cos\phi$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$

$z = r\cos\theta$

$\frac{\partial z}{\partial r} = \cos\theta$