27/9/23

Topic - I

Cayley Hamilton Theorem:

Statement:

Every square matrix satisfies its own characteristic equation

Problem 1

Verify Cayley Hamilton theorem, find its inverse and A^4

$$\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = A.$$

Soln:

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \qquad \therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 6$

$$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-2) + (4-1) = 3+2+3 = 8$$

$$S_3 = 2(4-1) + 1(-2+1) + 2(+1-2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6-1-2 = 3$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

$\boxed{\lambda = A}$

$$A^3 - 6A^2 + 8A - 3I = 0$$

Verification: $A^2 = A \times A$

$$= \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (4+1+2) & (-2-2-2) & (4+1+4) \\ (-2-2-1) & (1+4+1) & (-2-2-2) \\ (2+1+2) & (-1-2-2) & (2+1+4) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} (14+6+9) & (-7-12-9) \\ (-10-6+6) & (+5+12+6) \\ (10+5+7) & (-5-10-7) \end{bmatrix}$$

$$\begin{bmatrix} \ldots & \ldots & (14+6+18) \\ \ldots & \ldots & (-10-6-12) \\ \ldots & \ldots & (10+5+14) \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} +$$

$$8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & 8 & 16 \end{bmatrix} -$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 0$$

$$= \begin{bmatrix} (29-42+16-3) & (-28+36-8-0) & (38-54+16-0) \\ (-22+30+16-0) & (23-36+16-3) & (-28+36-8-0) \\ (22-30+8-0) & (-22+30+8-0) & (29-42+16-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ CHT is verified

$$A^4 = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (58+28+76) & (-29-56-38) & (58+28+76) \\ (-44-23-28) & (22+46+28) & (-44-23-56) \\ (44+44+29) & (-22-44-29) & (44+22+58) \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1} $(A^3 - 6A^2 + 8A - 3I = 0)$

$$A^3 \times A^{-1} - 6A^2 A^{-1} + 8A\, A^{-1} - 3IA^{-1} = 0$$

$$A^3 \times \frac{1}{A} - 6A^2 \frac{1}{A} + 8A \frac{1}{A} - 3A^{-1} = 0$$

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$-3A^{-1} = -A^2 + 6A - 8I$$

(Multiply by (-1))

$$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$$

$$A^{-1} = \frac{1}{3}\left[A^2 - 6A + 8I\right]$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$