Summing up: Theoretical Mechanics

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Chapter 1

D'Alembert Principle and Lagrange's Equations

In classical mechanic problems were described by Newtons 3N[1] differential equations[2] of 2nd order

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} \sum_{i \neq j} \mathbf{F}_{ij}, \tag{1.1}$$

where $\mathbf{F}^{(e)}$ denotes the external force and F_{ij} the internal forces acting on the system. As many systems contain a lot of particles the Newton approach gets very unhandy, leading to the easier to handle *D'Alembert's principle* and *Lagrange equations*. First step to develop these new technics is to introduce *constraints*.

1.1 Constraints

Many systems underly forces of constraints, which can be denoted as

$$f(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N, t) = 0. \tag{1.2}$$

Every force of constraints reduces the degress of freedom. A system of N particles has 3N degrees of freedom a system including p forces of constraints then has only

$$S = 3N - p \tag{1.3}$$

degrees of freedom left.

Until now we considered working in the classical way with *cartesian coordinates*. Due to the forces of constraints some of the cartesian coordinates $\mathbf{r}_i$ are now linearly dependent, hence the *equations of motion* are not all independent. To get the full advantage of using constraints we have to introduce *generalized*

[1] Each of the N particles has three degrees in a three dimensional space.
[2] Also called equations of motion.

coordinates. Meaning that we can choose S totally arbitraily coordinates, which just have to be linearly independent to describe our N particle system completly.

The only problem that is left is, that *forces of constraints* are unkown a priori. Consequently we should like to formulate our theory in a way, that the *forces of constraints* disappear automatically, which will be done with *D'Alembert's principle*.

1.2 D'Alembert's principle

To derive *D'Alembert's principle* we have to introduce the virtual displacements $delta\mathbf{r}$. Virtual displacements is an **assumed** infinitesimal change of system coordinates occuring while **time is held constant**[3]. Apart from that virtual displacement are mathematically treated as a normal displacement $d\mathbf{r}$.

Regarding a system equilibrium, we know that the *total force*[4] on each particle vanishes $\mathbf{F}_i = 0$. Furthermore the classical work is given by $dW = \mathbf{F} \cdot d\mathbf{r}$, thus we can define the virtual work as

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r}. \tag{1.4}$$

Consequently, due to the vanishing total force, the sum over the total virtual work must also vanish

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r} = 0. \tag{1.5}$$

To obtain *D'Alemberts principle* we can now split up the total force $\mathbf{F}_i$ into so called *applied forces* $F^{(a)}$ [5] and the *forces of constraint* f

$$\mathbf{F} = \mathbf{F}^{(a)} + \mathbf{f} \quad \Rightarrow \quad \mathbf{F}^{(a)} + \mathbf{f} - \mathbf{F} = 0 \tag{1.6}$$

Thus we can write for the net virtual work eq. (1.5)

$$\sum_i (\mathbf{F}_i^{(a)} - \mathbf{F}_i) \cdot \delta\mathbf{r} + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r} = 0 \tag{1.7}$$

Now we seperated the *forces of constraints* from the applied forces and physical evidence shows that

$$\sum_i \mathbf{f} \cdot \delta\mathbf{r} = 0. \tag{1.8}$$

We cannot prove this fact, but merely have to accept it![6] Hence we are left with

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r} = 0 \quad \text{with} \quad \mathbf{F} = m\ddot{\mathbf{r}} = \dot{\mathbf{p}}, \tag{1.9}$$

[3] Virtual displacements cannot take part in our real world because time is held constant, there is no movement without time. That's why the displacement is called virtual.

[4] External and internal force on a particle.

[5] Simply all forces, but the forces of constraint.

[6] Goldstein restricts himself for systems with a vanishing net virtual work of the forces of constraints. Excluding all other cases or a derivation.

which is referred to as *D'Alembert's princple*.

Having achieved our aim to cause to vanish the forces of constraints we still are left with linearly dependent virtual displacements $\delta\mathbf{r}$[7]. As before we want to replace those by independent generalized coordinates q_i.

Droppign the superscript of $\mathbf{F}_i^{(a)} \rightarrow \mathbf{F}_i$ in eq. (1.9) we now want to evaluate the terms $\mathbf{F}_i \cdot \delta\mathbf{r}$ and $-\dot{\mathbf{p}}_i \cdot \delta\mathbf{r}$ yielding the *d'Alembert principle* in form of the generalized coordinates $\mathbf{q}_i$. Starting with the first one we need to make use of the total derivative of the virtual displacement $\delta\mathbf{r}$[8] given by

$$\delta\mathbf{r}_i = \sum_j \frac{\partial\mathbf{r}_i}{\partial q_j}\delta q_i \tag{1.10}$$

Hence for $\sum_i \delta F_i \cdot \delta\mathbf{r}$ we get

$$\begin{aligned}\sum_i \mathbf{F}_i \cdot \delta\mathbf{r} &= \sum_i \sum_j \mathbf{F}_i \cdot \frac{\partial\mathbf{r}_i}{\partial q_j}\delta q_j \\ &= \sum_i Q_j \delta q_j,\end{aligned} \tag{1.11}$$

where we defined the *generalized force* Q_j

$$\mathbf{Q}_j = \sum_i \mathbf{F}_i \frac{\partial\mathbf{r}_i}{\partial q_j}. \tag{1.12}$$

Before evaluating the second term, we need to introduce to idendities, which we are not going to derive here[9] :

$$\frac{\mathbf{r}_i}{q_j} = \frac{\dot{\mathbf{r}}}{\dot{q_j}} \tag{1.13}$$

$$\frac{d}{dt}\frac{\mathbf{r}_i}{q_j} = \frac{\dot{\mathbf{r}_i}}{q_j} \tag{1.14}$$

[7] The cartesian coordinates $\mathbf{r}$ have been linearly dependent, due to the forces of constraint, and so are $\delta\mathbf{r}$.

[8] $\delta\mathbf{r}$ is only dependent on the generalized coordinates $\mathbf{q}_i$ and not on the time t.

[9] We already derived these in our first exercise sheet.

With these two identities we are able to evaluate the second expression

$$\begin{aligned}
\sum_i \dot{\mathbf{p}}_i \cdot \delta\mathbf{r} &= \sum_i \sum_j m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\
&= \sum_i \sum_j m_i \left[\frac{d}{dt} \left(\dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - \dot{\mathbf{r}}_i \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j \\
&= \sum_i \sum_j m_i \left[\frac{d}{dt} \left(\dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\dot{q}_j} \right) - \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right] \delta q_j \\
&= \sum_i \sum_j m_i \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{\mathbf{r}}_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\frac{1}{2} \dot{\mathbf{r}}_i^2 \right) \right\} \delta q_j \\
&= \sum_j \left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\}
\end{aligned} \tag{1.15}$$

Plugging now our first term eq. (1.11) and our second term eq. (1.15) in *d'Alembert's principle* eq. (1.9) yields

$$\sum_j \left\{ \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j \;=\; 0 \tag{1.16}$$

From this point we will derive within a few steps the desired *Lagrange's equations*.

1.3 Lagrange's equations

To finally get *Lagrange's equations* we need to regard a *conservative system* with *holonomic constraints*[10].

Holonomic constraints lead to independent *generalized coordinates* q_j. Thus we can choose δq_j arbitrarily, so that not the sum, but every single coefficient has to be zero. Hence the sum from *d'Alembert's principle* eq. (1.16) vanishes

$$\left\{ \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{q_j} \right] - Q_j \right\} \delta q_j \;=\; 0 \tag{1.17}$$

For a *conservative system* we can define a potential $V(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$ depending only on radi vectors (not on velocities $\dot{\mathbf{r}}_i$), which is responsible for our *applied force*

$$\mathbf{F}_i^{(a)} \;=\; -\nabla_i V. \tag{1.18}$$

Consequently we can write for our *generalized Force* eq. (1.12)

$$Q_j \;=\; \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \;=\; -\sum_i \nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} \;=\; -\frac{V}{q_j}, \tag{1.19}$$

[10] Constraints of the form defined in eq. (1.2).

because $\nabla_i \cdot \partial \mathbf{r}_i$ cancel out.

Thus for a *conservative system* with *holonomic constraints* we get

$$\left\{ \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right] + \frac{V}{q_j} \right\} \delta q_j \;=\; \left[\frac{d}{dt} \frac{\partial T}{\dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} \right] \delta q_j \tag{1.20}$$

Now we can define the *Lagrange function*

$$L(q_1, \cdots q_N, \dot{q}_1, \cdots \dot{q}_N, t) = T(q_1, \cdots, q_N, \dot{q}_1, \cdots \dot{q}_N, t) - V(q_1, \cdots, q_N). \tag{1.21}$$

Due to the fact, that $V(q_1, \cdots, q_N)$ does not depend on the velocities $\dot{q}_i$ we can include V in the partial derivative with respect to $\cdot q_j$

$$\left[\frac{d}{dt} \frac{\partial (T-V)}{\dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} \right] \delta q_j \;=\; \left[\frac{d}{dt} \frac{\partial L}{\dot{q}_j} - \frac{\partial L}{\partial q_j} \right] \delta q_j \tag{1.22}$$

The above equations are reffered to as *Lagrange's equations*.

We now obtained a mechanism using energy and work (T - V) instead of momentum and force, as we have been used from the newton mechanic. We now have to deal with S differential equations of 2nd order. Consequently 2S initial editions are needed to solve for the *equations of motions*.

1.4 Exercise

1.4.1 Task

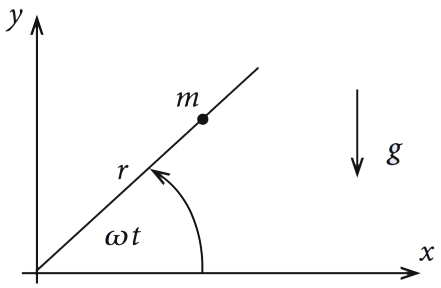

Figure 1.1: Awesome Image

Regard a particle with mass m moving without friction on a wire, which rotates with a constant velocity ω around its origin. The movement of the particle is influenced by the gravitation g.

1. Formulate the constraints.
2. Define the Lagrangian of the particle.

3. Solve for Lagrange's equations and find the general solution for r(t). motion

4. Use the initial conditions

$$r(t=0) \;=\; r_0 \quad \text{and} \dot{r}(t=0) \;=\; 0$$

to get the equations of motion.

5. Solve the problem with classical Newton mechanics.

1.4.2 Solution

1.

The constraints are given by

$$\begin{aligned} z \;&=0 \\ y - x\tan(\omega t) \;&=\; 0 \end{aligned}$$

Consequently we can reduce the degrees of freedom from

$$S \;=\; 3-2 \;=\; 1$$

An apropriate generalized coordinate would be $q(t) \;=\; r(t)$.

2.

To find the Lagrangian we first need a relation between the cartesian coordinates and their polar corespondent

$$\begin{aligned} x \;&=r\cos(\omega t) & y \;&=r\sin(\omega t) \\ \dot{x} \;&=\; -\dot{r}\cos(\omega t) + r\omega\sin(\omega t) & \dot{y} \;&=\dot{y}\sin(\omega t) + r\omega\cos(\omega t) \end{aligned}$$

Then we need to find the kinetical energy

$$T \;=\; \frac{1}{2}m\mathbf{v}^2 \;=\; \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \;=\; \frac{1}{2}m(\dot{r}^2+r^2\omega^2)$$

The potential energy is then obviously given by

$$V \;=\; mgh \;=\; mgr\sin(\omega t)$$

Hence for the Lagrangian

$$L \;=\; T-V \;=\; \frac{1}{2}m(\dot{r}^2+r^2\omega^2) - mgr\sin(\omega t)$$

3.

The Lagrange equation is given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0.$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} \quad \text{and} \quad \frac{\partial L}{\partial r} = mr\omega^2 - mg\sin(\omega t).$$

Combined we get the following differential equation

$$\ddot{r} - r\omega^2 + g\sin(\omega t) = 0.$$

To find the general solution we start with the homogenous equation

$$\ddot{r} - r\omega^2 = 0.$$

Which can be solved with the ansatz

$$r_0(t) = \alpha e^{\omega t} + \beta e^{-\omega t}.$$

To get the general solution we now have to find a special solution and add it to the homogeneous solution. The following ansatz will serve us well

$$r_s(t) = \gamma\sin(\omega t) \quad \Rightarrow \quad \ddot{r}_s(t) = -\gamma\omega^2\sin(\omega t).$$

Plugging the above relations into our Lagrange equation yields

$$-\gamma\omega^2\sin(\omega t) - \gamma\omega^2\sin(\omega t) + g\sin(\omega t) = 0 \quad \Rightarrow \quad \gamma = \frac{g}{2\omega^2}$$

And thus for our general solution (adding homogeneous and special solution)

$$r(t) = \alpha e^{\omega t} + \beta e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$$

4.

Pluggin the initial conditions

$$r(0) = r_0 \quad \text{and} \quad \dot{r}(0) = 0$$

into our general solution

$$\begin{aligned} r(t) &= \alpha e^{\omega t} + \beta e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t) \\ \dot{r}(t) &= \alpha\omega e^{\omega t} - \beta\omega e^{-\omega t} + \frac{g}{2\omega}\cos\omega t \end{aligned}$$

yields

$$\begin{aligned} \alpha + \beta &= r_0 \\ \alpha - \beta + \frac{g}{2\omega} &= 0 \quad \Rightarrow \quad \alpha = \beta - \frac{g}{2\omega^2} \end{aligned}$$

Hence plugin the α from the second line into the first yields

$$2\beta - \frac{g}{2\omega^2} = r_0 \quad \Rightarrow \quad \beta = \frac{r_0}{2} + \frac{g}{4\omega^2}$$

and

$$\alpha = \frac{r_0}{2} - \frac{g}{4\omega^2}.$$

Pluggin α and β in our general solution we get

$$r(t) = \frac{r_0}{2}\left(e^{\omega t} + e^{-\omega t}\right) - \frac{g}{4\omega^2}\left(e^{\omega t} - e^{-\omega t}\right) + \frac{g}{2\omega^2}\sin(\omega t)$$

5.

To solve the system in the Newtonian way we first need to formulate the forces of constraints, which is not that simple. Anyway we are left with the gravitational force $\mathbf{F}_g$ and the zentrifugal force $\mathbf{F}_z$

$$m\ddot{r} = -F_g + \mathbf{F}_z.$$

The gravitational force is simply given by

$$\mathbf{F}_g = mg\sin(\omega t)$$

whereas the zentrifugal force is defined as

$$\mathbf{F}_z = -m(\omega \times (\omega \times \mathbf{r})) = -m(\omega(\omega \cdot \mathbf{r}) - \mathbf{r}(\omega \cdot \omega)) = mr\omega^2$$

where we used the triple cross product identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot b)$, and the fact that ω is orthogonal to $\mathbf{r}$. Hence

$$m\ddot{r} - mr\omega^2 + mg\sin(\omega t) = 0,$$

which is exactly the same as our Lagrangian equation in the 3rd part.

Chapter 2

Variational Principle and Lagrange's Equations

In the last chapter we got to know a new principle of classical mechanics, which equivalently to *Newton mechanic* led us to the equations of motion. In general the laws of classical mechanics can be described within two **Variational principles**.

- Differential principle (D'Alembert)
- Integral principle (Hamilton)

Within the *d'Alembert principle* we compare the instanteneous state of a system with its infinitesimal virtual displacements and get the equations of motion as a result. Within the *Hamilton principle* we consider the entire *motion of the system*[1] between times t_1 and t_2 and small virtual variations of this motion, leading aswell to the equations of motion.

The development of the second approach will be topic of this chapter.

2.1 Hamilton principle

As we have learned we can describe the state of our system, containing N particles with the *generalized coordinates* $q_1, \cdots, q_N$. The space describing our state in *generalized coordinates* is calles *configuration space*. As time goes on the time dependent *generalized coordinates* $q_1(t), \cdots, q_N(t)$ will change, and so will our point in *configuration space*, describing a curve called *the path of motion of the system*.

$$\mathbf{q}(t) = (q_1(t), \cdots, q_N(t)) \tag{2.1}$$

For the following derivation we will only consider *conservative holonomic systems*. Plugin in the time dependent *generalized coordinates* into our *Lagrangian*

[1] Term will be described in the section of Hamilton's principle.

we get a explicitly time dependent function $\tilde{L}$

$$L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \, = \, \tilde{L}(t) \tag{2.2}$$

Thus we can define a functional[2]

$$S[\mathbf{q}(t)] \, = \, \int_{t_1}^{t_2} \tilde{L}(t), \tag{2.3}$$

which is referred to as *the action* and dependend on $\mathbf{q}(t)$ and the t. For fixed t_1 and t_2 the functional $S(\mathbf{q}(t))$ maps a value to every $q(t)$. Consequently there are many *paths of motion of the system* connecting the intial state $\mathbf{q}(t_1)$ with the final state $\mathbf{q}(t_2)$. Our aim is now to variate the paths and find the optimized *path of motion for the system*. Hence we are dealing with an optimization problem. Introducing virtual displacements $\delta\mathbf{q}(t)$, which are zero at the end points $\delta\mathbf{q}(t_1) \, = \, \delta\mathbf{q}(t_2) \, = \, 0$[3]. We can now define *Hamilton's principle*

$$\delta S = \delta \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt \, = \, 0 \tag{2.4}$$

which states, that for all possible **variations** $(\delta\mathbf{q}(t)_i)$ the system will choose the one leading to a *path of motion* for which *the action* is stationary!

Our next task will be to find a way using *Hamilton's principle*. Therefore we have to have a deeper insight in the *Variational Calculus*.

2.2 Lagrange's Equation from the Action Integral

Our aim is to find an *action* for which the *path of motion* is *stationary*. To simplify the introduction we will consider a one dimensional Lagrangian $L(q, \dot{q}, t)$, implicitly depending on t. Let $q(t)$ represent a transition in the states $q_1 \, = \, q(t_1)$ to $q_2 \, = \, q(t_2)$. We consider now a variation in the path, introducing a small pertubation $\epsilon(t)$, which is zero at the endpoints t_1 and t_2

$$q(t) \longrightarrow q(t) + \epsilon \tag{2.5}$$

Consequently the first order changes in our integral action are given by

$$\begin{aligned} \delta S[q] \, &= S[q + \epsilon] - S[q] \\ &= \int_{t_1}^{t_2} L(q + \epsilon, \dot{q} + \dot{\epsilon}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt. \end{aligned} \tag{2.6}$$

[2] A functional is simply a function depending on a function. In our case the function S depending on the function $\mathbf{q}(t)$.

[3] Every $\mathbf{q}(t)$ represents a state in the *configuration space*. As the initial and the final state are fixed, the variation at the end points $\mathbf{q}(t_1), \mathbf{q}(t_2)$ have to be fixed to.

Using the *multivariable Taylor expansion* on L yields

$$L(q+\epsilon,\dot{q}+\dot{\epsilon},t) \;=\; L(q,\dot{q},t)+\frac{L(q,\dot{q},t)}{q}\epsilon+\frac{L(q,\dot{q},t)}{\dot{q}}\dot{\epsilon}+\mathcal{O}(\epsilon^2). \tag{2.7}$$

Hence our variated action integral can be written as

$$\delta S[q] \;=\; \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q}\epsilon+\frac{\partial L}{\partial \dot{q}}\dot{\epsilon}\right]dt \tag{2.8}$$

while restricting our self to the first order of the previously used *Taylor expansion*. Regarding the second term in the integral we can perform a *integration by parts*

$$\int_{t_1}^{t_2} \;=\; \frac{\partial L}{\partial \dot{q}}\dot{\epsilon} \;=\; \int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}}\frac{\partial \epsilon}{\partial t} \;=\; \frac{\partial L}{\partial q}\epsilon\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\epsilon dt. \tag{2.9}$$

As $\epsilon(t_1) = \epsilon(t_2) = 0$ the first tirm of the right-hand side vanishes and we can rewrite our *action integral* into

$$\delta S[q] \;=\; \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right]\epsilon dt \tag{2.10}$$

Using the *fundamental Lemma* we can directly read out *Lagrange's equations*

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \;=\; 0 \tag{2.11}$$

2.3 Example

A particle with the mass m is located in the gravitational field of the earth. It has an one dimensional movent in z direction. Calculate the action integral

$$S \;=\; \int_{t_1}^{t_2} L(z,\dot{z},t)dt \tag{2.12}$$

for a path

$$z(t) \;=\; -\frac{1}{2}gt^2+f(t). \tag{2.13}$$

The function $f(t)$ shall be an arbitraly, continous function with $f(t_1) = f(t_2) = 0$. Show that the action gets minimal for $f(t) \equiv 0$.

As our action integral contains the Lagrangian we will start by formulating the kinetic and potential energy. The movement is only in z-direction with a gravitational force in the same direction. Hence

$$T \;=\; \frac{1}{2}m\dot{z}^2, \quad V \;=\; mgz \quad \Rightarrow \quad L \;=\; \frac{1}{2}m\dot{z}^2-mgz.$$

In addition the path z(t) is given in the exercise

$$\begin{aligned} z(t) &= -\frac{1}{2}gt^2 + f \\ \dot{z}(t) &= -gt + \dot{f}. \end{aligned}$$

Thus we can write our action integral as

$$\begin{aligned} S &= \int_{t_1}^{t_2} L(z,\dot{z})dt = \qquad\qquad \int_{t_1}^{t_2} \frac{1}{2}m\dot{z}^2 - mgz \\ &= \int_{t_1}^{t_2} dt[\frac{1}{2}m(-gt+\dot{f})^2] - \int_{t_1}^{t_2} dt[mg(-\frac{1}{2}gt^2+f)] \\ &= \int_{t_1}^{t_2} dt(-\frac{m}{2}g^2t^2 + \frac{m}{2}\dot{f}^2 - mgt\dot{f} + \frac{m}{2}g^2t^2 - mgf) \\ &= mg\int_{t_1}^{t_2} dtt^2 + \frac{m}{2}\int_{t_1}^{t_2} dt\dot{f}^2 - mg\int_{t_1}^{t_2} dt(t\dot{f}+f) \end{aligned}$$

Perfoming a partial integration

$$\int_{t_1}^{t_2} dtt\dot{f} = tf|_{t_1}^{t_2} - \int_{t_1}^{t_2} dtf$$

we notice that the $tf|_{t_1}^{t_2}$ vanishes (because $f(t_1) = f(t_2) = 0$) vanishes with the last term of our action integral. Hence we are left with

$$\begin{aligned} S &= mg^2\int_{t_1}^{t_2} dtt^2 + \frac{m}{2}\int_{t_1}^{t_2} dt\dot{f}^2 \\ &= mg\frac{1}{3}(t_2^3t_1^3) + \frac{m}{2}\int_{t_1}^{t_2} dt\dot{f}^2, \end{aligned}$$

which will be our final expression for the action integral.

Now that we have an expression for our action integral we want to show that it gets minimal for $f \equiv 0$. Regarding our action integral we notice that the first term is independent of f and that the second term gets minimal for $\dot{f}(t) = 0$. As f has to fixed points $f(t_1) = f(t_2) = 0$ and its slope is zero for a minimal action we follow that $f(t) = 0$ for a minimal action.