Universitat Autònoma de Barcelona

MASTER THESIS

1-Loop Anomalous Dimensions of 4-Quark Operators

Dirk Hornung

Institut de Fisica d'Altes Energies
Universitat Autònoma de Barcelona
Barcelona

September 2015

First proofreader: Prof. Dr. Matthias Jamin

Contents

Chapter 1

Introduction

The pertubative expansion in QCD is known to lead to a divergent series, which is at best asymptotic. The asymptotic behavior of the pertubative series manifest itself in the appearance of singularities for its Borel transform. Those singularities connected with renormalization are termed renormalons.

The presence of infrared (IR) renormalons, on the positive real Borel axis, lead to ambiguities. They appear due to a splitting, caused by the operator product expansion (OPE) of the function into an pertubative and and operator correction part. The splitting contains ambiguities, which should cancel, because the full function cannot have ambiguities in its definitions to be a physical quantity. In the framework of the OPE higher dimensional operators corrections appear, which contain the associated ambiguities, such that the full function is unambiguous. Those operators are the so called QCD condensates. The operators that display renormalon ambiguities are a subset of those that arise in the framework of the OPE.

Limiting ourselves to correlation functions of vector or axialvector currents with respect to the QCD vacuum, the lowest-lying IR renormalon pole is associated to the vacuum matrix element of one dimension-4 operator, the gluon condensate. The next-closest singularity then is found to correspond to the dimension-6 triple gluon condensate and a set of dimension-6 4-quark condensates. It is these latter 4-quark condensates that we intend to investigate in more detail in the present work.

In the upcoming chapter we want to provide the fundamentals of renormalization in QCD. Our main issue will be to present one-loop calculations of anomalous-dimension matrices γ for four-quark operators. In addition to the computations of the anomalous dimension matrix we want to give the basics of the renormalisation group equation (RGE) and test it for the from us calculated anomalous dimension matrix.

Chapter 2

Fundamental Concepts

Within this chapter we want to introduce the fundamentals needed for our calculations . Starting with the field of Quantum Chromodynamics (QCD) we introduce elemental interaction methods and yield the needed propagators. Further on we will discuss the for this thesis important process of Renormalisation with the help of the Dimensional Regularization's (DR). Finally we want to give some basic Renormalisation Group Equation (RGE). In the explanations down below we want to follow the lecture [4].

2.1 QCD Basics

The fundamental degrees of freedom of Quantum Chromodynamics (QCD) are the matter fields (quarks) and the fields inter-mediating strong interactions (gluons). The classical Lagrangian density is very similar to the the one given by Quantum electrodynamics (QED)

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2} \bar{q}^a(x) \gamma^\mu \overleftrightarrow{D}_\mu q^a(x) - m_A \bar{q}^A(x) q^A(x) \right] \tag{2.1}$$

We now want to explain the different parts of the above Lagrangian density. Starting with the field strength tensor $G^a_{\mu\nu}$ corresponding to the gluon field $B^a_\mu(x)$ and defined as

$$G^a_{\mu\nu}(x) \equiv \partial_\mu B^a_\mu(x) - \partial_\mu B^a_\mu(x) + g f^{abc} B^b_\mu(x) B^x_\nu(x). \tag{2.2}$$

We are using Greek letters for the Lorentz indices, running from 0 to 3 and Latin letters $a, b, c = 1, \ldots, 8$ for the colour indices in the adjoint representation of the colour group SU(3). In addition f^{abc} represents the structure constant of the colour group SU(3) satisfying

$$[t^a, t^b] = if^{abc}t^c, \tag{2.3}$$

with t^a being the generator of SU(3). The fundamental representation of the generators t^a can be given in terms of the hermitian, traceless, 3x3 Gell-Mann matrices λ^a

$$t^a = \frac{\lambda^a}{2} \qquad \text{with} \qquad Tr[\lambda^a, \lambda^b] = 2\delta^{ab}. \tag{2.4}$$

Furthermore g is the coupling constant, quantifying the interaction strength between the quarks and gluons and also the gluon self-coupling, which arises due to the non-abelian nature of the gauge group.

Regarding the quark fields $q^A_{\alpha,i}(x)$, we used the capitalized letter A as flavour index for the six different flavours up, down, strange, charm, bottom and top. Moreover we use (α) as spinor and (i) as colour indice. Both indices will be written in matrix notation to suppress further indices. Next we need to introduce the Clifford algebra for the γ-matrices in four space-time dimensions

$$\{\gamma_\mu \gamma_\nu\} \equiv \gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\nu = 2g_{\mu\nu}\mathbb{1}, \tag{2.5}$$

where $g_{\mu\nu} \equiv diag(1, -1, -1, -1)$ denotes the Minkowski-metric of the flat space covariant derivative D_μ given by

$$D_\mu = \partial_\mu \mathbb{1} - ig\frac{\lambda^a}{2}B^a_\mu. \tag{2.6}$$

The bidirectional array $\overleftrightarrow{D}$ implies that the derivatives act to the right, as well as with a minus sign to the left side in eq. (2.1). Finally m_A is the mass term for a flavour of type A.

The QCD Lagrangian was constructed to maintain local SU(3) colour gauge transformations. Employing the Euler-Lagrange equations for the quark and gluon fields yield the classical equations of motion.

$$[i\gamma^\mu D_\mu - m_A]q^A(x) = 0 \tag{2.7}$$

$$[D^\mu, G^a_{\mu\nu}(x)] = -\frac{i}{2}g\sum_A \bar{q}^A(x)\gamma_\nu\lambda^a q^A(x) \tag{2.8}$$

The first equations corresponds to the Dirac-equations, whereas the second describes the coupling of the gauge field to a matter source. So the QCD Lagrangian describes the free Dirac spinors and the non-abelian gauge field with their interactions and self-interactions gluon field.

Now we want to perform the quantisation, yielding the needed free propagators (Green's functions) for the quark and the gluon field. If $\Psi(x)$ denotes a generic spinor field the quantum field can be written in terms of creation and annihilation operators

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3 wE(\vec{p})} \sum_{\lambda} [u(\vec{p},\lambda)a(\vec{p},\lambda)e^{-ipx} + v(\vec{p},\lambda)b^{\dagger}(\vec{p},\lambda)e^{ipx}], \quad (2.9)$$

where the integration ranges over the positive sheet of the mass hyperboloid $\Omega_+(m) = \{p|p^2 = m^2, p^0 > 0\}$ with m being the mass of the quark. The four spinors $u(\vec{p},\lambda)$ and $v(\vec{p},\lambda)$ are solutions to the free Dirac equation in the momentum space

$$[\not{p} - m]u(\vec{p},\lambda) = 0 \qquad \text{and} \qquad [\not{p} + m]v(\vec{p},\lambda) = 0 \quad (2.10)$$

with λ representing the helicity state of the spinors. Combining the commutation relations, the creation and annihilation operators and the above equations gives us the propagator for the free spinor field

$$iS^{(0)AB}_{ij}(x-y) \equiv \langle 0|T\{q^A_i(x)\bar{q}^B_j(y)\}|0\rangle \equiv q^A_i(x)\bar{q}^B_j(y) = \delta_{ij}\delta^{AB} iS^{(0)}(x-y). \quad (2.11)$$

The T operator implies time-ordering of the fields in the curly brackets. The connection line (connecting the two quark fields in the third expression) denotes a contraction (Wick's theorem).

The quantisation of the gauge fields will be much more cumbersome as the previous one. Applying the canonical quantisation procedure will lead to a loss of covariance of the theory. Consequently we have to add a gauge-fixing term

$$\mathcal{L}_{gf} = -\frac{1}{2a}[\partial^\mu B^a_\mu(x)][\partial^\nu B^a_\nu(x)] \quad (2.12)$$

Proceeding with the canonical quantisation, the Fock-space of states has an indefinite metric in QCD. In order to restore unitary in the physical subspace of the gluons we need a set of massless non-physical fields the so-called Faddeev-Popov ghosts $c^a(x)$ corresponding to

$$\mathcal{L}_{ghost} = -[\partial^\mu \bar{c}^a(x)]\partial_\mu c^a(x) + gf^{abc}[\partial^\mu \bar{c}^a(x)]c^b(x)B^c_\mu(x) \quad (2.13)$$

Following now the steps of the normal quantisation procedure yields the desired propagators

$$
\begin{aligned}
iD^{(0)ab}_{\mu\nu}(x-y) &\equiv \langle 0|T\{B^a_\mu(x)B^b_\nu(y)\}|0\rangle \equiv \overline{B^a_\mu(x)B^b_\nu}(y) \equiv i\delta^{ab}\int\frac{d^4k}{(2\pi)^4}D^{(0)}_{\mu\nu}(k)e^{-ik(x-y)} \\
&= i\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\eta}\left[-g_{\mu\nu}+(1-a)\frac{k_\mu k_\nu}{k^2+i\eta}\right]e^{-k(x-y)}
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
i\widetilde{D}^{(0)ab}(x-y) &\equiv \langle 0\ T\{c^a(x)c^b(y)\}|0\rangle \equiv \overline{c^a(x)c}^b(y) \equiv i\delta^{ab}\int\frac{d^4}{(2\pi)^4}\widetilde{D}^{(}0)(q)e^{-q(x-y)} . \\
&= i\delta^{ab}\int\frac{d^4q}{(2\pi)^4}\frac{-1}{(q^2+i0)}e^{-iq(x-y)}
\end{aligned}
\tag{2.15}
$$

2.2 Dimensional Regularization

As we want to renormalize multi-loop Feynman diagrams, we need a prescription to address the problem of regulating the divergencies, i.e. in our case manifesting them in form of $1/\epsilon$ poles. For our prospect we want to use the method of dimensional regularization (DR). The main advantages of DR is, that it preservers gauge invariance and is the method implemented most easily.

Using DR we want to perform our calculations in an arbitrary space-time dimension D, instead of the normal space-time dimension 4. The dimension D shall be an integer for us, but can also be non-integer. Using this prescription the divergencies will appear as poles of the resulting expression when we perform the limes of $D \to 4$. For future calculations we want to define our arbitrary dimension D as

$$D = 4 - 2\epsilon \tag{2.16}$$

Working into D-dimensions we also have to employ a different Dirac-algebra for arbitrary D dimension. The defining equation has the same form as in 4 dimensions

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu = 2g_{\mu\nu}\mathbb{1}, \tag{2.17}$$

but with the space-time indices μ and ν running from $0, 1, 2, \ldots, D-1$. Furthermore we can easily check for the trivial relations

$$g_{\mu\nu}g^{\mu\nu} = D \qquad \text{and} \qquad \gamma_\mu\gamma^\mu = D\mathbb{1} \tag{2.18}$$

to be true. The first one clearly is just the trace over the $D \otimes D$ sized metric tensor product and the second can be easily shown by combining the metric tensor product with the defining equation eq. (2.17). In addition to the trivial relations we need to implement some anti-commutation rules for the γ-matrices.

$$\gamma_\mu \gamma_\nu \gamma^\mu = (2g_{\mu\nu} - \gamma_\nu \gamma_\mu)\gamma^\mu = (2 - D)\gamma_\nu \tag{2.19}$$

$$\gamma_\mu \gamma_\nu \gamma_\lambda \gamma^\mu = 2\gamma_\lambda \gamma_\nu - \gamma_\nu \gamma_\mu \gamma_\lambda \gamma^\mu = 4g_{\nu\lambda}\mathbb{1} + (D - 4)\gamma_\nu \gamma_\lambda \tag{2.20}$$

$$\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho \gamma^\mu = -2\gamma_\rho \gamma_\lambda \gamma_\nu + (4 - D)\gamma_\nu \gamma_\lambda \gamma_\rho \tag{2.21}$$

In general we will perform our Dirac-algebra with a Mathematica package named Tracer.m. The codes we used will be displayed in app.(6).
Next we have to regard strings over γ-matrices. Using the cyclicity of traces, the anti-commutating rule in eq. (2.17) and the fact that odd numbers of γ-matrices in a trace will vanish we get for

$$Tr[\gamma_\mu \gamma_\nu] = Tr[\gamma_\nu \gamma_\mu] = \frac{1}{2} Tr[\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu] = g_{\mu\nu} Tr[\mathbb{1}] = 4g_{\mu\nu}, \tag{2.22}$$

where we arbitrarily set $Tr[\mathbb{1}] \equiv 4$. We also could take another choice as $Tr[\mathbb{1}] \equiv 2^{D/2}$, which is would be the dimension of the irreducible and matches our choice at $D = 4$. Different choices yield different additional constants, which can be absorbed into the renormalisation. Consequently we have to define a renormalization scheme. As the trace with three γ-matrices vanishes (odd number of γ-matrices) we want to regard the trace

$$\begin{aligned} Tr[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\omega] &= 8g_{\mu\nu} g_{\lambda\omega} - Tr[\gamma_\nu \gamma_\mu \gamma_\lambda \gamma_\omega] = 8g_{\mu\nu} g_{\lambda\omega} - 8g_{\mu\lambda} g_{\nu\omega} + Tr[\gamma_\nu \gamma_\lambda \gamma_\mu \gamma_\omega] \\ &= 8g_{\mu\nu} g_{\lambda\omega} - 8g_{\mu\lambda} g_{\nu\omega} + 8g_{\mu\omega} g_{\nu\lambda} - Tr[\lambda_\nu \gamma_\lambda \gamma_\omega \gamma_\mu] \end{aligned} \tag{2.23}$$

Hence for the general trace

$$Tr[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\omega] = 4(g_{\mu\nu} g_{\lambda\omega} - g_{\mu\lambda} g_{\omega\lambda} + g_{\mu\omega} g_{\nu\lambda}). \tag{2.24}$$

For now we only regarded the γ-matrices excluding the chirality operator γ_5, which causes an intricate problem of DS. In four dimension γ_5 is defined as

$$\gamma_5 \equiv \frac{i}{4!} \epsilon_{\mu\nu\lambda\omega} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\omega \tag{2.25}$$

and anti-commutes with all other γ-matrices, while satisfying

$$\{\gamma_5, \gamma_\mu\} = 0, \qquad (\gamma_5)^2 = \mathbb{1} \qquad \text{and} \qquad Tr[\gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\omega] = 4i\epsilon_{\mu\nu\lambda\omega}. \tag{2.26}$$

Now we could define γ_5 anti-commutating with all other γ-matrices for arbitrary dimension D. However this is not consistent with the cyclicity of the trace. In the following we want to show how this inconsistency shows up. To compute the trace over γ_5 we regard

$$Tr[\gamma_5\gamma_\mu\gamma^\mu] = DTr[\gamma_5] = -Tr[\gamma_\mu\gamma_5\gamma^\mu] = -DTr[\gamma_5] \tag{2.27}$$

Thus $DTr[\gamma_5] = 0$ and consequently $Tr[\gamma_5] = 0$ except eventually at $D = 0$. With the same procedure we can derive

$$(D-2)Tr[\gamma_5\gamma_\mu\gamma_\nu] = 0 \qquad \text{and} \qquad (D-4)Tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] = 0. \tag{2.28}$$

Hence from the second of the relations we could conclude that $Tr[\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] = 0$ for all D, which is in conflict with eq. (2.26) for the case $D = 4$. Therefore we either have to give up the property that our expressions can be continued analytically from arbitrary D to 4 or we have to abandon the γ_5 anti-commutation.

To deal with these inconsistencies we want to shortly strike to possible solutions. First is the Hooft and Veltman scheme (HF scheme). They defined γ_5 such that it anti-commutes with the 4-dimensional subset of γ-matrices and commutes with the remaining (D-4) dimensional ones. Second would be the naive scheme (NDR scheme). If we only have even numbers of γ_5 appearing in traces we can simply stitch to the conventional use.

Using DR we will notice that we are confronted with a special type of Integrals, so called *Feynman Integrals*. In the following we want to derive a method of solving those integrals.

Feynman integrals

The Feynman integral of most interest has two denominators, consequently we will label it with a two and define it as

$$\begin{aligned} I_2(q) &\equiv \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(q-p)^2} \\ &\overset{!}{=} \frac{i}{(4\pi)^2} \left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} \Gamma(\epsilon) \\ &= \frac{i}{(4\pi)^2} \frac{1}{\epsilon} + \mathcal{O}(1). \end{aligned} \tag{2.29}$$

We now want to derive the given result in the above equation. Therefore we have to introduce **Feynman's parametrization**.

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^\beta - 1}{[ax + b(1-x)]^{\alpha+\beta}} \tag{2.30}$$

In our case the variables are given by

$$a = (q-p)^2, \qquad b = p^2, \qquad \alpha = 1, \qquad \beta = 1. \tag{2.31}$$

Therefore Eq. (2.30) can be written as

$$\begin{aligned}\hat{I}_2(q) &= \frac{1}{p^2(q-p)^2} \\ &= \int_0^1 dx \frac{1}{[(q-p)^2 x + p^2(1-x)]^2} \\ &= \int_0^1 dx \frac{1}{[q^2 - 2pqx + p^2]^2} \\ &= \int_0^1 dx \frac{1}{[(p-qx)^2 + q^2 x - q^2 x^2]^2},\end{aligned} \tag{2.32}$$

where we completed the square in the last line. Substituting

$$k = p - qx \qquad p = k + qx \qquad \frac{dp}{dk} = 1 \tag{2.33}$$

keeps the equation simple. Inserting the new expression in our integral yields

$$\begin{aligned}I_2(q) &= \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{1}{[k^2 + x(1-x)q^2]^2} \\ &= \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{1}{[k-a]^2},\end{aligned} \tag{2.34}$$

where we substitute $a = -x(1-x)q^2$. Now we want to perform a **Wick rotation**, which rotates the time axis k^0 into an imaginary time direction ik^D. The idea is to get the same sign for our Minkowski metric. Hence

$$\begin{aligned}k^0 \to ik^D \Rightarrow \frac{dk^D}{dk^0} = i \\ k^2 = (k^0)^2 - \vec{k}^2 \to -(k^D)^2 - \vec{k}^2,\end{aligned} \tag{2.35}$$

and for our integral

$$\begin{aligned}I_2(q) &= -\mu^{2\epsilon} \int_0^1 dx \int \frac{d^0 k d^1 k \cdots d^{D-1} k}{(2\pi)^D} \frac{1}{[(k^0)^2 - \sum_{i=1}^D (k^i)^2 - a^2]^2} \\ &\to i\mu^{2\epsilon} \int_0^1 dx \int \frac{d^1 k d^2 k \cdots d^D k}{(2\pi)^D} \frac{1}{[k^2 + a^2]^2} \\ &= i\mu^{2\epsilon} \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + a^2]^2} \\ &= i\mu^{2\epsilon} \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + a^2]^2}.\end{aligned} \tag{2.36}$$

Being done with the Wick rotation, we now can solve the integral over k with a spherical integration. Splitting up the integral into a spherical and a radial part yields

$$I_2(q) = i\mu^{2\epsilon} \int_0^1 dx \int_0^\infty dk \int d\Omega \frac{k^{D-1}}{[k^2 + a^2]^2}, \tag{2.37}$$

whereas the spherical integration is given by

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \tag{2.38}$$

After solving the spherical part we will us **Euler's beta-function**

$$B(u,v) \equiv \int_0^1 z^{u-1}(1-z)^{v-1} dy = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{2.39}$$

to compute the radial one. Substituting

$$\begin{aligned} k^2 = a^2 \frac{(1-z)}{z} &\Rightarrow k = \sqrt{a^2 \frac{(1-z)}{z}} \\ \frac{dk}{dz} = \sqrt{a^2} \frac{1}{2} \sqrt{\frac{z}{(1-z)}} \left(\frac{-1}{z^2}\right) &= -\sqrt{\frac{a^2}{2}} \sqrt{\frac{1}{(1-z)z^4}} \end{aligned} \tag{2.40}$$

gives us the needed expressions for Euler's beta-function

$$\begin{aligned} k^{D-1} &= \left(\frac{a^2(1-z)}{z}\right)^{\frac{D}{2}-\frac{1}{2}} \\ \frac{1}{[k^2+a^2]^2} &= \frac{1}{\left[\frac{a^2(1-z)}{z} + a^2\right]^2} = \left(\frac{z}{a^2}\right)^2. \end{aligned} \tag{2.41}$$

Plugging everything together gives us

$$\begin{aligned} I_2(q) &= \frac{i2\pi^{\frac{D}{2}}\mu^{2\epsilon}}{(2\pi)^D}(-a^2)^{\frac{D}{2}-2} \int_0^1 dx \frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2})}{\Gamma(2)\Gamma(\frac{D}{2})} \\ &= \frac{i\mu^{2\epsilon}}{4^{\frac{D}{2}}\pi^{\frac{D}{2}}}(-a)^{\frac{D}{2}-2} \int_0^1 dx \Gamma(2-\frac{D}{2}) \\ &= \frac{i}{(4\pi)^2}\left(\frac{\mu^2 4\pi}{q^2}\right)^\epsilon \Gamma(\epsilon), \end{aligned} \tag{2.42}$$

where we used $D = 4 - 2\epsilon$. Now we are left with the integral over x, which also can easily be computed using Euler's beta-function

$$\int_0^1 dx \frac{1}{a^2} = -\int_0^1 x^{-\epsilon+1-1}(1-x)^{-\epsilon+1-1}\frac{1}{q^2} = -\frac{q}{q^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}. \tag{2.43}$$

Consequently we are left with our final result

$$I_2(q) = \frac{i}{(4\pi)^2}\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon}\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}. \tag{2.44}$$

Moreover we have to deal with integrals having one (I_1) or three (I_3) factors in their denominators. The first takes the form

$$I_1(q) = \mu^{2\epsilon}\int \frac{d^D p}{(2\pi)^D}\frac{1}{(q-p)^2} = 0 \tag{2.45}$$

and is a scaleless integral, which are zero. The second is given by

$$I_3(q,r) \equiv \mu^{2\epsilon}\int \frac{d^D p}{(2\pi)^2}\frac{1}{p^2(q-p)^2(r-p)^2} = \mathcal{O}(1) \tag{2.46}$$

and does not contain any divergencies.

Additionally we have to deal with $I_2(q)$ multiplied by a Lorentz momentum

$$I_2^\mu(q) \equiv \mu^{2\epsilon}\int \frac{d^D p}{(2\pi)^D}\frac{p_\mu}{p^2(q-p)^2} \stackrel{!}{=} \frac{p_\mu}{2}I_2(q). \tag{2.47}$$

To solve this integral we can use a trick, which will be useful in future computations as well. The result of the above integral must be dependent on q and transform under Lorentz invariance. Hence

$$I_2^\mu(q) \stackrel{!}{=} A(p^2)p_\mu, \tag{2.48}$$

where A is just a variable. Multiplying with $2p^\mu$ will give us three already known integral and consequently the final result. Thus the integral is given by

$$\begin{aligned}
2p^\mu I_2^\mu(q) &= \mu^{2\epsilon}\int \frac{d^D p}{(2\pi)^D}\frac{2(p\cdot q)}{p^2(q-p)^2} \\
&= \mu^{2\epsilon}\int [\frac{d^D p}{(2\pi)^D}\frac{1}{(q-p)^2} + \frac{q^2}{p^2(q-p)^2} - \frac{1}{p^2}] \\
&= I_1(q) + q^2 I_2(q) - I(0) \\
&= q^2 I_2(q),
\end{aligned} \tag{2.49}$$

where we used $2p \cdot q = p^2 + q^2 - (q-p)^2$. Consequently dividing through $2p^\mu$ we get our desired result

$$I_2^\mu(q) = \frac{q_\mu}{2} I_2(q). \tag{2.50}$$

2.3 Renormalisation

Up to now we regularized so called ultraviolet (UV) divergencies. We did so in calculating Green's function, while using the DR regularization. Now knowing the divergencies we need to renormalize, i.e. to eliminate all divergent parts. Leading to finite results for Green's functions in the limit of $\epsilon \to 0$.

The general description to eliminate the divergent parts is to add counter-terms to the QCD Lagrangian eq. (2.1) corresponding to each superficially divergent diagram. Hence we substitute $\mathcal{L}_0$ by $\mathcal{L}_0 + \mathcal{L}_{0,C}(x)$. The introduced counter-terms C_i, understood as a power series in g^2 are all the extra terms needed to remove the UV divergencies.

All new terms in $\mathcal{L}_C(x)$ can be thought of as new pertubative terms in the Lagrangian density with Feynman rules immediately derivable. Thus we must add the new resulting amplitudes to the amplitudes from our classical Lagrangian. Then we have to choose C_i in such a way, that our final amplitude is finite.

To simplify the improved Lagrangian density we want to introduce the renormalization constant Z, defined as

$$Z_i \equiv 1 - C_i. \tag{2.51}$$

With the help of the renormalization constant we can define a relations for so-called bare field, bare coupling constants as well as the bare mass and gauge parameter according to the definitions

$$\begin{aligned} B^\mu_{aO} &\equiv Z_{3YM}^{1/2} B^\mu_a(x), & q_0^A(x) &\equiv Z_{2F}^{1/2} q_\alpha^A(x) \\ \phi_{aO}(x) &\equiv \tilde{Z}_3^{1/2} \phi_a(x), & \bar{\phi}_{a0}(x) &\equiv \tilde{Z}_3^{1/2} \bar{\phi}_a(x) \\ g_{OYM} &\equiv Z_{1YM} Z_{3YM}^{-3/2} g, & \tilde{g}_O &\equiv \tilde{Z}_1 \tilde{Z}_3^{-1} Z_{3YM}^{-1/2} g \\ g_{OF} &\equiv Z_{1F} Z_{3YM}^{-1/2} Z_{2F}^{-1} g, & g_{O5} &\equiv Z_5^{1/2} Z_{3YM}^{-1} g \\ m_{AO} &\equiv Z_4 Z_{2F}^{-1} m_A, & a_O &\equiv Z_6^{-1} Z_{3YM} a \end{aligned} \tag{2.52}$$

where we followed the notation of [7] the right hand side is renormalized and finite. Hence the renormalized Lagrangian density can be written as

$$\mathcal{L}_R(x) \equiv \mathcal{L}_0(x) + \mathcal{L}_{0,C}(x) \tag{2.53}$$

Having simplified the QCD Lagrangian density we now have to discuss the choosing of the added constants C_i, i.e. to discuss different renormalization schemes. As we will get to know in the renormalization group chapter that different renormalization schemes must not change physical quantities. Defining the dimensionless coupling

$$\alpha \equiv \frac{(g\mu^\epsilon)^2}{4\pi} \tag{2.54}$$

we can write all the C_i's as power series expansion in α of the type

$$C_i = \sum_{j=1}^{\infty} \sum_{k=1}^{j} C_{i,k}^{(2j)} \frac{1}{\epsilon^k} \left(\frac{\alpha}{\pi}\right)^j, \tag{2.55}$$

where the coefficients are single, double, etc. poles in $1/\epsilon$. The simplest way to renormalize our theory is now to choose the constants C_i in a way that they exactly cancel the $1/\epsilon$ divergencies, yielding a finite Green's function. This method is called minimal subtraction scheme (MS scheme). Noticing that the $1/\epsilon$ dependencies always appear in combination with $-\ln 4\pi + \gamma$ we can define a modified subtraction scheme ($\bar{M}S$ scheme), i.e. defining Z_i in such a way, that this combination is eliminated from the renormalized Green's function. Hence substituting

$$\frac{1}{\epsilon} \rightarrow \frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} - \ln 4\pi + \gamma \tag{2.56}$$

Having found a way to renormalize our divergent Green's functions we now have the tool to find the renormalization constants used in the renormalization group analysis.

2.4 Renormalisation Group Equation

As mentioned before we know that physical observable must be independent of the renormalisation scheme, which is in general called renormalisation invariance and denominated under the renormalisation group.

Considering a physical quantity $R(q, a_s, m)$, where q stands for the external momenta, $a_s = \alpha_s/\pi$, as well as m denote the renormalized QCD coupling and quark mass. As being a physical quantity $R(q, a_s, m)$ cannot depend on the renormalisation scale μ. Hence the total derivative of $R(q, a_s, m)$ has to be zero

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left\{ \mu \frac{\partial}{\partial \mu} + \mu \frac{d a_s}{d\mu} \frac{\partial}{\partial a_s} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} \right\} R(q, a_s, m) = 0. \tag{2.57}$$

From here we can define the renormalisation group functions $\beta(a_s)$ and $\gamma(a_s)$

$$\beta(a_s) \equiv -\mu \frac{da_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \qquad \beta\text{-function} \tag{2.58}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m}\frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \dots \quad \text{mass anomalous dimension.} \tag{2.59}$$

The coefficients of both, the β-function and the anomalous dimension function, are known up to the 4-loop coefficients β_4 and γ_4. Employing these functions in our general renormalization group equation (RGE) eq. (2.57) we can write

$$\left\{\mu\frac{\partial}{\partial\mu} - \beta(a_s)\frac{\partial}{\partial a_s} - \gamma(a_s) m \frac{\partial}{\partial m}\right\} R(q, a_s, m) = 0 \tag{2.60}$$

Running quark mass

The scale dependence of the quark mass, characterized by the mass anomalous dimension, is given by

$$\gamma(a_s) \equiv -\frac{\mu}{m}\frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \gamma_3 a_s^3 + \gamma_4 a_s^4 + \dots. \tag{2.61}$$

Until today the pertubative coefficients of the mass anomalous dimension are know up to the 4-loop coefficient γ_4.

The RGE of the quark mass can be integrated directly by the separation of variables

$$\int_{m(\mu_1)}^{m(\mu_2)} \frac{dm}{m} = \ln\frac{m(\mu_2)}{m(\mu_1)} = -\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu}\gamma(a_s) = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}. \tag{2.62}$$

Thus we obtain

$$m(\mu_2) = m(\mu_1)\exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}\right) \tag{2.63}$$

Explicitly calculating the exponential at one-loop order yields

$$\exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}\right) = \exp\left(\frac{\gamma_1}{\beta_1}\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{a_s}\right) = \exp\left(\frac{\gamma_1}{\beta_1}\ln\frac{a_s(\mu_2)}{a_s(\mu_1)}\right) = \left(\frac{a_s(\mu_2)}{a_s(\mu_1)}\right)^{\gamma_1/\beta_1} \tag{2.64}$$

Thus we obtain for the running mass

$$m(\mu_2) = m(\mu_1)\left(\frac{a_s(\mu_2)}{a_s(\mu_1)}\right)^{\gamma_1/\beta_1}\left[1 + \mathcal{O}(a_s)\right] \tag{2.65}$$

Chapter 3

Non-singlet Vector and Axialvector Correlators

3.1 Introduction

In this chapter we want to investigate the dimension-6 operator product expansion (OPE) contributions to the two-point correlation functions $\Pi^{V/A}_{\mu\nu}(q)$ of non-singlet vector and axialvector currents $j^V_\mu(x) = (\bar{u}\gamma_\mu d)(x)$ and $j^A_\mu(x) = (\bar{u}\gamma_\mu\gamma_5 d)(x)$. For simplicity we will only assume massless light quarks (u,d and s) in which case the correlators take the form

$$\Pi^{V/A}_{\mu\nu}(q) \equiv i\int \mathrm{d}x\, \mathrm{e}^{iqx}\, \langle\Omega|T\{j^{V/A}_\mu(x) j^{V/A}_\nu(0)^\dagger\}|\Omega\rangle \;=\; (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi^{V/A}(q^2)\,. \tag{3.1}$$

Here $|\Omega\rangle$ denotes the full QCD vacuum and the second identity holds since the vector current is conserved, i.e. $\partial^\mu j_\mu(x) = 0$ and thus $q^\mu \Pi_{\mu\nu}(q^2)$ has to vanish, which implies the Lorentz-structure on the right.

In the framework of the OPE the scalar function $\Pi^{V/A}$ then permits an expansion in powers of $1/Q^2$ with $Q^2 \equiv -q^2$

$$\Pi^{V/A}(Q^2) \;=\; C_0(Q^2) + C_4(Q^2)\,\frac{\langle O_4\rangle}{Q^4} + C_6^{V/A}(Q^2)\,\frac{\langle O_6\rangle}{Q^6} + \ldots\,. \tag{3.2}$$

In the OPE only the Wilson coefficients $C_i^{V/A}$ depend on the momentum, the operators O_i are local, but both depend on the renormalisation scale μ.

Our main concern is to find the contributions of the dimension-6 term, which receives contributions from the three-gluon condensate $\langle g^3 f_{abc} G^a_{\mu\nu} G^{b\nu}_\lambda G^{c\lambda\mu}\rangle$ and the quark-condensates. Due to the fact, that the three gluon condensate does not arise at leading order, we will concentrate on four-quark condensates. The contribution of the four-quark condensates has been calculated in [1] and [5]. To present it for the V-A and V+A correlation functions we need to transform it into another basis. In general we have chosen our basis, because in the V-A contribution the penguin diagrams cancel each other and we only have to regard the current-current diagrams. In the lowest order of perturbation theory the contributions of the 4-quark operators are well-known and read [6]

$$\frac{C_6^{i,V} Q_6^i}{g^2} = -\frac{2}{9}(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu t^a q) - 2\bar{u}\gamma_5\gamma_\mu t^a d\bar{d}\gamma_5\gamma_\mu t^a u \quad (3.3)$$

for the vector correlator and

$$\frac{C_6^{i,A} Q_6^i}{g^2} = -\frac{2}{9}(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu t^a q) - 2\bar{u}\gamma_\mu t^a d\bar{d}\gamma_\mu t^a u \quad (3.4)$$

for the axial vector one. To shorten the equations we want to give the from us used basis of four-quark operators, given by

$$Q_V^o = (\bar{u}\gamma_\mu t^a d \bar{d}\gamma^\mu t^a u)\,, \quad Q_A^o = (\bar{u}\gamma_\mu\gamma_5 t^a d \bar{d}\gamma^\mu\gamma_5 t^a u)\,, \tag{3.5}$$

$$Q_V^s = (\bar{u}\gamma_\mu d \bar{d}\gamma^\mu u)\,, \quad Q_A^s = (\bar{u}\gamma_\mu\gamma_5 d \bar{d}\gamma^\mu\gamma_5 u)\,, \tag{3.6}$$

$$Q_3 \equiv (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu t^a q)\,, \tag{3.7}$$

$$Q_4 \equiv (\bar{u}\gamma_\mu\gamma_5 t^a u + \bar{d}\gamma_\mu\gamma_5 t^a d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu\gamma_5 t^a q)\,, \tag{3.8}$$

$$Q_5 \equiv (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu q)\,, \tag{3.9}$$

$$Q_6 \equiv (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu\gamma_5 q)\,, \tag{3.10}$$

$$Q_7 \equiv \sum_{q=u,d,s} (\bar{q}\gamma_\mu t^a q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu t^a q')\,, \tag{3.11}$$

$$Q_8 \equiv \sum_{q=u,d,s} (\bar{q}\gamma_\mu\gamma_5 t^a q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu\gamma_5 t^a q')\,, \tag{3.12}$$

$$Q_9 \equiv \sum_{q=u,d,s} (\bar{q}\gamma_\mu q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu q')\,, \tag{3.13}$$

$$Q_{10} \equiv \sum_{q=u,d,s} (\bar{q}\gamma_\mu\gamma_5 q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu\gamma_5 q')\,. \tag{3.14}$$

where $Q_{V/A}^O$ $Q_{V/A}^S$ are termed current-current operators and Q_3 to Q_{10} penguin operators. In addition we have defined the four current-current operators

$$Q_\pm^O = Q_V^O \pm Q_A^O \qquad \text{and} \qquad Q_\pm^S \equiv Q_V^S \pm Q_A^S. \tag{3.15}$$

The explicit calculation of the one-loop corrections to 4-quark condensates in eq. (3.3) and (3.4) gives [5]

$$
\begin{aligned}
\frac{C^{i,V}Q_6^i}{g^2} &= -\frac{2}{9}\left(1+a_s\left[\frac{95}{72}L+\frac{107}{48}\right]\right)Q_3 \\
&-\frac{1}{3}\left(1+a_s\left[\frac{9}{8}L+\frac{431}{96}\right]\right)O_1^{o,A} \\
&-\frac{a_s}{24\pi}\left[(16L-12)Q_V^s+\left(30L-\frac{45}{2}\right)Q_V^o+\left(\frac{16}{9}L-\frac{8}{27}\right)Q_7\right. \\
&\left.+\left(\frac{16}{9}L+\frac{56}{27}\right)Q_6+\left(\frac{10}{3}L+\frac{35}{9}\right)Q_4\right],
\end{aligned}
\tag{3.16}
$$

where we used the operator $O_1^{o,A}$ defined as

$$O_1^{o,A} = \bar{u}\Gamma^{[3]}t^a d\bar{d}\Gamma^{[3]}t^a u. \tag{3.17}$$

For the axial vector correlator the corrections look exactly the same, except for the matrices displayed below change into

$$
\begin{aligned}
&O_1^{o,A} \to O_1^{o,V} \\
&,Q_V^s \to Q_A^s \\
&,Q_V^o \to Q_a^o.
\end{aligned}
\tag{3.18}
$$

Thus using a short Mathematica script (app. (6.4)) we can transform the one-loop corrections to 4-quark condensates into the from us needed basis

$$C_6^{V-A}(Q^2)\langle O_6\rangle \;=\; 4\pi^2 a_s\left\{\left[2+\left(\tfrac{25}{6}-L\right)a_s\right]\langle Q_-^o\rangle-\left(\tfrac{11}{18}-\tfrac{2}{3}L\right)a_s\langle Q_-^s\rangle\right\}, \tag{3.19}$$

and

$$
\begin{aligned}
C_6^{V+A}(Q^2)\langle O_6\rangle \;=\; &-4\pi^2 a_s\left\{\left[2+\left(\tfrac{155}{24}-\tfrac{7}{2}L\right)a_s\right]\langle Q_+^o\rangle+\left(\tfrac{11}{18}-\tfrac{2}{3}L\right)a_s\langle Q_+^s\rangle+\right. \\
&\left[\tfrac{4}{9}+\left(\tfrac{37}{36}-\tfrac{95}{162}L\right)a_s\right]\langle Q_3\rangle+\left(\tfrac{35}{108}-\tfrac{5}{18}L\right)a_s\langle Q_4\rangle+ \\
&\left.\left(\tfrac{14}{81}-\tfrac{4}{27}L\right)a_s\langle Q_6\rangle-\left(\tfrac{2}{81}+\tfrac{4}{27}L\right)a_s\langle Q_7\rangle\right\},
\end{aligned}
\tag{3.20}
$$

where $a_s=\alpha_s/\pi$ and $L\equiv\ln Q^2/\mu^2$.

Before continuing with the calculation of the anomalous dimension matrix $\hat{\gamma}_O$ we want to investigate the scale dependence of a general term R_O in the OPE, corresponding to a set of operators $\vec{O}(\mu)$

$$R_O = \vec{R}^T(\mu)\langle\vec{O}(\mu)\rangle, \tag{3.21}$$

where we displayed the dependency on the scale μ explicitly and the dependence on other potential dimensionful parameters implicitly. Due to the fact, that for the vector and axialvector currents renormalisation scale dependence only arises from pertubative contributions, R_O should not be dependent on the scale μ and we can write

$$\left[\mu\frac{d}{d\mu}\vec{C}^T(\mu)\right]\langle\vec{O}(\mu)\rangle = -C^T(\mu)\left[\mu\frac{d}{d\mu}\langle\vec{O}(\mu)\rangle\right] \tag{3.22}$$

Furthermore the anomalous dimension matrix $\hat{\gamma}_O$ of the operator matrix can be defined by

$$-\mu\frac{d}{d\mu}\langle\vec{O}(\mu)\rangle \equiv \hat{\gamma}_O(a_\mu)\langle\vec{O}(\mu)\rangle \tag{3.23}$$

, where $a_\mu = a_s(\mu)$. Plugging in eq. (3.22) into eq. (3.23) and transposing the matrices one obtains the RGE that has to be satisfied by the coefficient function $\vec{C}(\mu)$.

$$\mu\frac{d}{d\mu}\vec{C}(\mu) = \hat{\gamma}_O^T(a_\mu)\vec{C}(\mu). \tag{3.24}$$

This equation shall be checked for the coefficient functions of the dimension-6 operators in eqs.(3.19) and (3.20).

Hereafter we want to calculate the anomalous dimension matrices for the cases V-A and V+A, before we continue with checking the RGE for each of them.

3.2 4-point Green's Function With Operator Insertion

To show the basic calculation method we want to start by calculating the most accessible diagram. We will use the first two operators of our proposed basis in eq. (3.5), given by

$$Q^S_{V,A} = \left(\bar{q}^A\Gamma_1 q^B\bar{q}^B\Gamma_2 q^A\right)(z) \qquad \text{and} \qquad Q^O_{V,A} = \left(\bar{q}^A t^a\Gamma_1 q^B\bar{q}^B t^a\Gamma_2 q^A\right)(z), \tag{3.25}$$

$$Q_- = Q_{V-A} = Q^{S,O}_V - Q^{S,O}_A \tag{3.26}$$

where A and B denote flavor indices. Additionally $Q^S_{V,A}$ stands for the singlet operator and $Q^O_{V,A}$ for the octet operator (the octet operator contains the color matrix t^a). The indices V, A denote the structure of the operators, which can be either vector (γ_μ) or axialvector ($\gamma_\mu\gamma_5$). Γ_1 and Γ_2 are equal

4x4 matrices and will be replaced later by vector γ_μ or axial-vector $\gamma_\mu\gamma_5$ structure.

To show the principles of the calculation method we will be very explicit within the first computation. Starting with the construction of a 4-point Green's function we need to add four external fields

$$\bar{q}^i_\alpha(x_1)q^j_\beta(x_2)\bar{q}^k_\delta(x_3)q^l_\gamma(x_4), \tag{3.27}$$

where i,j,k,l denote color-indices and $\alpha,\beta,\delta,\gamma$ stand for Dirac-indices. Then the general definition of the 4-point Green's function is given by

$$\begin{aligned}\Gamma = &\int d^Dx_1d^Dx_2d^Dx_3d^Dx_4e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\ &\times \langle 0|T\{\bar{q}^i_\alpha(x_1)q^j_\beta(x_2)\bar{q}^k_\delta(x_3)q^l_\gamma(x_4)e^{i\int d^Dz\mathcal{L}_I(z)}\}|0\rangle,\end{aligned} \tag{3.28}$$

where the interaction term ($e^{i\int d^Dz\mathcal{L}_I(z)}$) is equal to one for the zeroth-order calculation. Now inserting our first operator from eq. (3.25) yields

$$\begin{aligned}\Gamma^0 = &\int d^Dx_1d^Dx_2d^Dx_3d^Dx_4d^Dze^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\ &\times \langle 0|T\{q^i_\alpha(x_1)\bar{q}^j_\beta(x_2)[\bar{q}^At^a\Gamma_1q^B\bar{q}^Bt^a\Gamma_2q^A](z)q^k_\delta(x_3)\bar{q}^l_\gamma(x_4)\}|0\rangle \\ &= (-1)^4\int d^Dx_1d^Dx_2d^Dx_3d^Dx_4d^Dze^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\ &\times \langle 0|T\{q^i_\alpha(x_1)\bar{q}^A(z)t^a\Gamma_1q^B(z)\bar{q}^j_\beta(x_2)q^k_\delta(x_3)\bar{q}^B(z)t^a\Gamma_2q^A(z)\bar{q}^l_\gamma(x_4)\}|0\rangle,\end{aligned}$$

where we had to perform four permutations in the last line. Every permutation yields a factor of -1, due to the anti-commutating characteristics of the Dirac-fields.

$$q_\alpha\bar{q}_\beta = (-1)\bar{q}_\beta q_\alpha \tag{3.29}$$

Using Wick's theorem and the propagator relation eq. 2.11 we now want to contract the fields. Hence regarding only the time-ordered part

$$\begin{aligned}\Gamma^0 = &\int d^Dx_1d^Dx_2d^Dx_3d^Dx_4dze^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\ &\cdot \langle 0|T\{q^i_\alpha(x_1)\bar{q}^A(z)t^a\Gamma_1q^B(z)\bar{q}^j_\beta(x_2)q^k_\delta(x_3)\bar{q}^B(z)t^a\Gamma_2q^A(z)\bar{q}^l_\gamma(x_4)\}|0\rangle \\ &= \delta^{ij}\delta^{kl}(i)^4\int d^Dx_1d^Dx_2d^Dx_3d^Dx_4dze^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\ &\cdot [S^A(x_1-z)t^a\Gamma_1S^B(z-x_2)]_{\alpha\beta}[S^B(x_3-z)t^a\Gamma_2S^A(z-x_4)]_{\delta\gamma},\end{aligned}$$

The two Kronecker deltas (δ^{ij},δ^{kl}) rise from color conservation between the in- and outgoing quarks and the factor $(i)^4$ is obtained from contraction forming the quark-propagator eq. (2.11).

Now we want to perform a Fourier-transformation. As we are not facing a loop, all integrals are supposed to vanish. Transforming all four propagators gives

$$\begin{aligned}
S^A(x_1 - z) &= \int \frac{d^D k_1}{(2\pi)^D} e^{-ik_1(x_1-z)} S^A(k_1) \\
S^B(z - x_2) &= \int \frac{d^D k_2}{(2\pi)^D} e^{-ik_2(z-x_2)} S^B(k_2) \\
S^B(x_3 - z) &= \int \frac{d^D k_3}{(2\pi)^D} e^{-ik_3(x_3-z)} S^B(k_3) \\
S^A(z - x_4) &= \int \frac{d^D k_4}{(2\pi)^D} e^{-ik_4(z-x_4)} S^A(k_4).
\end{aligned} \tag{3.30}$$

Inserting the above relations into Γ_0 yields

$$\begin{aligned}
\Gamma^0 = \delta^{ij}\delta^{kl} \int & d^D x_1 d^D x_2 d^D x_3 d^D x_4 dz \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{d^D k_3}{(2\pi)^D}\frac{d^D k_4}{(2\pi)^D} \\
&\cdot e^{-ik_1(x_1-z)} e^{-ik_2(z-x_2)} e^{-ik_3(x_3-z)} e^{-ik_4(z-x_4)} e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\
&\cdot [S^A(k_1) t^a \Gamma_1 S^B(k_2)]_{\alpha\beta} [S^B(k_3) t^a \Gamma_2 S^A(k_4)]_{\delta\gamma}.
\end{aligned} \tag{3.31}$$

Reordering the exponential function and integrating over dx_1, dx_2, dx_3, dx_4 and $d^D z$ will give us the needed Dirac delta distributions

$$\begin{aligned}
\Gamma_0 &= \delta^{ij}\delta^{kl} \int d^D x_1 d^D x_2 d^D x_3 d^D x_4 dz \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{d^D k_3}{(2\pi)^D}\frac{d^D k_4}{(2\pi)^D} \\
&\cdot e^{ix_1(p_1-k_1)} e^{ix_2(p_2+k_2)} e^{ix_3(p_3-k_3)} e^{ix_4(p_4+k_4)} e^{iz(k_1+k_3-k_2-k_4)} \\
&\cdot [S^A(k_1) t^a \Gamma_1 S^B(k_2)]_{\alpha\beta} [S^B(k_3) t^a \Gamma_2 S^A(k_4)]_{\delta\gamma} \\
&= \delta^{ij}\delta^{kl} \int d^D x_1 d^D x_2 d^D x_3 d^D x_4 dz \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{d^D k_3}{(2\pi)^D}\frac{d^D k_4}{(2\pi)^D} \\
&\cdot \delta^{(D)}(p_1 - k_1)\delta^{(D)}(p_2 + k_2)\delta^{(D)}(p_3 - k_3)\delta^{(D)}(p_4 + k_4)\delta^{(D)}(k_1 + k_3 - k_2 - k_4) \\
&\cdot [S^A(k_1) t^a \Gamma_1 S^B(k_2)]_{\alpha\beta} [S^B(k_3) t^a \Gamma_2 S^A(k_4)]_{\delta\gamma}.
\end{aligned} \tag{3.32}$$

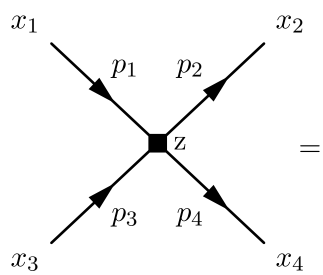
Integrating over k_1, k_2, k_3 and k_4 gives us the following relations due to the Dirac-deltas

$$k_1 = p_1, \qquad k_2 = -p_2, \qquad k_3 = p_3, \qquad k_4 = -p_4, \qquad k_1 + k_3 = k_2 + k_4, \tag{3.33}$$

The last relation has been contributed from the z integration and represents the impulse conservation of the in-going (p_1, p_3) and outgoing (p_2, p_4) impulses. Thus we are left with

$$\Gamma_0 = \delta^{ij}\delta^{kl}[S^A(-p_1)t^a\Gamma_1 S^B(p_2)]_{\alpha\beta}[S^B(-p_3)t^a\Gamma_2 S^A(p_4)]_{\delta\gamma}, \tag{3.34}$$

which can be displayed as a Feynman diagram



$$= q^i_\alpha(x_1)\bar{q}^j_\beta(x_2)[\bar{q}^A\Gamma_1 q^B\bar{q}^B\Gamma_2 q^A](z)q^k_\delta(x_3)\bar{q}^l_\gamma(x_4). \tag{3.35}$$

Amputating the external-propagators will give us the zeroth-order structures. Hence for the singlet operator $Q^S_{V,A}$ we get a contribution of

$$\Gamma^{0,S}_{amp} = \delta^{ij}\delta^{kl}[\Gamma_1]^{AB}_{\alpha\beta}[\Gamma_2]^{BA}_{\delta\gamma} \tag{3.36}$$

and for the octet operator $Q^O_{V,A}$ we get

$$\Gamma^{0,O}_{amp} = (t^a)^{ij}(t^a)^{kl}[\Gamma_1]^{AB}_{\alpha\beta}[\Gamma_2]^{BA}_{\delta\gamma} \tag{3.37}$$

3.3 V-A Current-current Diagram Contributions

Having understood the basic calculation method we now want to deal with first-order diagrams, concentrating on the quark-gluon interactions. Remembering the definition of our four quark Green's function eq. (3.28) we need to expand the interaction term in the exponential function

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \\ e^{i\int d^D z \mathcal{L}_I(z)} &= 1 + g_s B_a^\mu(x) \sum_f \bar{q}_f^\alpha \gamma^\mu (t^a)_{\alpha\beta} q_f^\beta \\ &\quad + \frac{g_s^2}{2} [B_a^\mu(x) \sum_f \bar{q}_f^\alpha \gamma^\mu (t^a)_{\alpha\beta} q_f^\beta][B_b^\nu(x) \sum_f \bar{q}_f^\sigma \gamma^\nu (t^b)_{\sigma\delta} q_f^\delta] + \ldots, \end{aligned} \tag{3.38}$$

which will give us the following possible contractions

$$a) = q_\alpha^i(x_1)\bar{q}_\beta^j(x_2)[\bar{q}^A\Gamma_1 q^B \bar{q}^B \Gamma_2 q^A](z)[B_b^\lambda \bar{q}\gamma_\lambda t^b q](y_1)[B_c^\sigma \bar{q}\gamma_\sigma t^c q](y_2) q_\delta^k(x_3)\bar{q}_\gamma^l(x_4), \tag{3.39}$$

$$b) = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2), \tag{3.40}$$

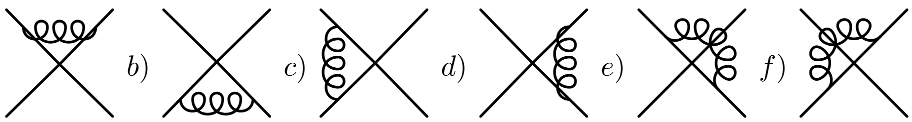
$$c) = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2), \tag{3.41}$$

$$d) = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2), \tag{3.42}$$

$$e) = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2), \tag{3.43}$$

$$f) = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2). \tag{3.44}$$

Notice that we only displayed terms needed for the contractions for all diagrams (except for a). In Addition we can draw the former contraction in forms of diagrams



(3.45)

Until this point we have only regarded current-current diagrams excluding penguin diagrams. As we will see they later the penguin contributions cancel each others for the Q_- operators. Consequently we will now calculate the contributions of the six current-current diagrams and deal with the penguin diagrams to a later point.

Now we want to calculate the contributions of the anomalous dimension matrix of the six diagrams explicitly (labeled by a,. . .,f). Therefore we will compute their divergencies and consequently their renormalisation constants, which will finally lead with an simple relation to the anomalous dimension matrix of our Q_- operator.

Diagram a)

Inserting the expanded e-function into the four-quark Green's function eq. (3.28) yields

$$
\begin{aligned}
\Gamma^a = g_s^2 \int & d^Dx_1 d^Dx_2 d^Dx_3 d^Dx_4 d^Dz d^Dy_1 d^Dy_2 e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} \\
& \cdot \langle 0|T\{q^i_\alpha(x_1)\bar{q}^j_\beta(x_2)[\bar{q}^A\Gamma_1 q^B \bar{q}^B\Gamma_2 q^A](z)[B^\lambda_b \bar{q}\gamma_\lambda t^b q](y_1)[B^\sigma_c \bar{q}\gamma_\sigma t^c q](y_2) q^k_\delta(x_3)\bar{q}^l_\gamma(x_4)|0\rangle \\
& = i g_s^2 [t^b t^c]^{ij}\delta^{kl} \int d^Dx_1 d^Dx_2 d^Dx_3 d^Dx_4 d^Dz d^Dy_1 d^Dy_2 e^{i(x_1p_1+x_2p_2+x_3p_4+x_4p_4)} \\
& \cdot \Big[S^A(x_1-y_1)\gamma^\lambda S^A(y_1-z)\Gamma_1 S^B(z-y_2)\gamma^\sigma S^B(y_2-x_2)\Big]_{\alpha\beta} \Big[S^B(x_3-z)\Gamma_2 S^A(z-x_4)\Big]_{\delta\gamma} \\
& \cdot \delta^{bc} D_{\lambda\sigma}(y_1-y_2)
\end{aligned}
\tag{3.46}
$$

Using the from the gluon contraction arising δ^{bc} combined with the definition of the **Casmir operator**

$$
C_F\delta^{ij} = [t^a t^a]^{ij} = \frac{(N_C^2-1)}{2N_C}\delta^{ij} \tag{3.47}
$$

shortens the color matrix notation

$$[t^b t^c]^{ij} \delta^{bc} = C_F \delta^{ij}. \tag{3.48}$$

Now we want to perform a Fourier-transformation. Regarding the emerging e-function structure

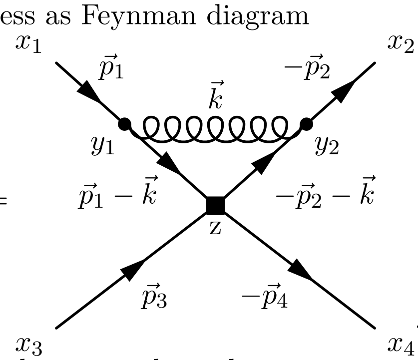
$$\begin{aligned} &e^{i(x_1p_2+x_2p_2+x_3p_3+x_4p_4)-ik_1(x_1-y_1)-ik_2(y_1-z)-ik_3(z-y_2)-ik_4(y_2-x_2)-ik_5(x_3-z)-ik_6(z-x_4)-ik_z(y_1-y_2)} \\ &\quad = e^{ix_1(p_1-k_1)} e^{ix_2(p_2+k_4)} e^{ix_3(p_3-k_5)} e^{ix_4(p_4+k_6)} e^{iy_1(k_1-k_2-k_7)} e^{iy_2(k_3-k_4+k_7)} e^{iz(k_2-k_3+k_5-k_6)}, \end{aligned} \tag{3.49}$$

we can directly read the Dirac functions and, as a result get the relations between the impulses k and p. Thus we are left with

$$\begin{aligned} \Gamma^a = ig_s^2 C_F \delta^{ij} \delta^{kl} \int \frac{d^D k}{(2\pi)^D} \left[S^A(p_1) \gamma^\delta S^A(p_1 - k) \Gamma_1 S^B(-p^2 - k) \gamma^\sigma S^B(-p_2) \right]_{\alpha\beta} \\ \cdot \left[S^B(p_3) \Gamma_2 S^A(-p_4) \right]_{\delta\gamma} D_{\lambda\sigma}(k), \end{aligned} \tag{3.50}$$

which we also can express as Feynman diagram

$\Gamma^a =$



(3.51)

Finally amputating the external quark propagators gives us the needed diagram structure

$$
\begin{aligned}
\Gamma^a_{amp} &= ig_s^2 C_F \delta^{ij}\delta^{kl}\int \frac{d^Dk}{(2\pi)^D}\gamma^\lambda \left[S(p_1-k)\Gamma_1 S(-p_2-k)\gamma^\sigma\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma} D_{\lambda\sigma}(k) \\
&= ig_s^2 C_F \delta^{ij}\delta^{kl}\int \frac{d^Dk}{(2\pi)^D}\frac{[\gamma^\lambda(\not{p}_1-\not{k})\Gamma_1(-\not{p}_2-\not{k})\gamma^\sigma]^{AB}_{\alpha\beta}}{k^2(p_1-k)^2(-p_2-k)^2}\left[-g_{\lambda\sigma}+(1-a)\frac{k_\lambda k_\sigma}{k^2}\right]\left[\Gamma_2\right]^{BA}_{\delta\gamma} \\
&= ig_s^2 C_F \delta^{ij}\delta^{kl}\int \frac{d^Dk}{(2\pi)^D}\frac{(p_1-k)_\mu(-p_2-k)_\nu[\gamma^\lambda\gamma^\mu\Gamma_1\gamma^\nu\gamma^\sigma]^{AB}_{\alpha\beta}}{k^2(p_1-k)^2(-p_2-k)^2}\left[-g_{\lambda\sigma}+(1-a)\frac{k_\lambda k_\sigma}{k^2}\right]\left[\Gamma_2\right]^{BA}_{\delta\gamma} \\
&= ig_s^2 C_F \delta^{ij}\delta^{kl}\int \frac{d^Dk}{(2\pi)^D}\frac{[(p_1)_\mu(-p_2)_\nu-(p_1)_\mu k_\nu - k_\mu(p_2)_\nu + k_\mu k_\nu][\gamma^\lambda\gamma^\mu\Gamma_1\gamma^\nu\gamma^\sigma]^{AB}_{\alpha\beta}}{k^2(p_1-k)^2(-p_2-k)^2} \\
&\cdot \left[-g_{\lambda\sigma}+(1-a)\frac{k_\lambda k_\sigma}{k^2}\right]\left[\Gamma_2\right]^{BA}_{\delta\gamma}\,,
\end{aligned}
\tag{3.52}
$$

where we have substituted $p_1 = p$ and $-p_2 = s$. In addition one should notice that we are working in the masseless case, while plugging in the quark-propagators.
As $g_{\lambda\sigma}$ contracts the gamma matrices γ^λ and γ^σ, we are left with three types of integrals

$$I_i = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(p-k)^2(s-k)^2}, \tag{3.53}$$

$$I_{ii} = \int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu}{k^2(p-k)^2(s-k)^2}, \tag{3.54}$$

$$I_{iii} = \int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{k^2(p-k)^2(s-k)^2}. \tag{3.55}$$

Checking for the dimensions, one can easily see, that only the last of the three integrals contains a ultraviolet divergency. A sloppy explanation for the first two integrals being finite is, that the integral adds a factor k^4, in the four dimensional Minkowski space. Regarding the nominator of the integrals we see that they are of order k^6. Consequently only a diagram with a nominator of order two or bigger can be ultraviolet divergent. Thus we are left with

$$
\begin{aligned}
\Gamma_{int} &= ig_s^2 C_F \delta^{ij}\delta^{kl} \\
&\cdot\left\{\int \frac{d^Dk}{(2\pi)^D}\frac{-k_\mu k_\nu}{k^2(p-k)^2(s-k)^2}\left[\gamma_\sigma\gamma^\mu\Gamma_1\gamma^\nu\gamma^\sigma\right]^{AB}_{\alpha\beta} + \int\frac{d^Dk}{(2\pi)^D}\frac{(1-a)\left[\Gamma_1\right]^{AB}_{\alpha\beta}}{(p-k)^2(s-k)^2}\right\}\left[\Gamma_2\right]^{BA}_{\alpha\beta}\,,
\end{aligned}
\tag{3.56}
$$

We know want to evaluate the leftover integral, which will be also play a central role in the computation of the other current-diagrams. The momentum integral can be decomposed according to the general structure

$$\begin{aligned}\Gamma^a_{int} &= \mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{k^2(p-k)^2(s-k)^2}\\ &= \underbrace{g_{\mu\nu}}_{i}A + \underbrace{[p_\mu p_\nu + s_\mu s_\nu]}_{ii}B + \underbrace{[p_\mu s_\nu + p_\nu s_\mu]}_{iii}C,\end{aligned} \tag{3.57}$$

where that we are dealing with three factors A, B and C which have to be calculated. Multiplying both sides of the equation with each of those factors will give us a solvable equation system, one equation for each factor, which we indexed with i, ii and iii. In the following we want to compute the left and the right-hand side of each of them. In general this methods leads us to known Feynman integrals with which we have dealt in sec. (2.2).

i) We start by multiplying the left-hand side of eq. (3.57) with the metric tensor $g^{\mu\nu}$ yields

$$\begin{aligned}i) &= g^{\mu\nu}\Gamma^a_{int}\\ &= \mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(p-k)^2(s-k)^2}\\ &= \mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{u^2(p-s-k)^2},\end{aligned} \tag{3.58}$$

where we have used the substitution $u = k - s$ in the last line. Now we obtained the first type of integrals defined as in eq. (2.29)

$$I_2(p) \equiv \mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(p-k)^2}, \tag{3.59}$$

where I is classified by having two denominators and therefore carries an indice two. We have derived every needed Feynman diagram in sec. (2.2). Hence for the left-hand side our first integral i) we get

$$g^{\mu\nu}\mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{k^2(p-k)^2(s-k)^2} = I_2(p-s) \tag{3.60}$$

and for the right hand side

$$g^{\mu\nu}\left\{g_{\mu\nu}A + [p_\mu p_\nu + s_\mu s_\nu]B + [p_\mu s_\nu + p_\nu s_\mu]C\right\} = DA + [p^2+s^2]B + 2(p\cdot s)C. \tag{3.61}$$

The contractions of the right-hand sides of all three integrals has been also calculated with a Mathematica script and can be found in lst. (6).

ii) Now we want to evaluate the second equation. Thus starting again with the left-hand side we find

$$ii) = [p^\mu p^\nu + s^\mu s^\nu]\mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{k^2(p-k)^2(s-k)^2}$$
$$= \mu^{2\epsilon} \int \frac{(p\cdot k)^2 (s\cdot k)^2}{k^2(p-k)^2(s-k)^2}$$
$$= \mu^{2\epsilon} \int \underbrace{\frac{\left(\frac{1}{2}[p^2+k^2-(p-k)^2]\right)^2}{k^2(p-k)^2(s-k)^2}}_{\alpha} + \underbrace{\frac{\left(\frac{1}{2}[s^2+k^2-(s-k)^2]\right)^2}{k^2(p-k)^2(s-k)^2}}_{\beta}, \tag{3.62}$$

where we used $(p\cdot k) = \frac{1}{2}[p^2+k^2-(p-k)^2]$ in the last line, splitting up the diagram in two parts (α and β). Hence for the α part

$$\alpha) = \frac{\mu^2}{4} \int \frac{d^D k}{(2\pi)^D} \frac{[p^2+k^2-(p-k)^2]^2}{k^2(p-k)^2(s-k)^2}$$
$$= \frac{\mu^{2\epsilon}}{4} \int \frac{d^D k}{(2pi)^D} \frac{\overbrace{p^4}^{a} + \overbrace{2p^2k^2}^{b} + \overbrace{[-2p^2(p-k)^2]}^{c} + \overbrace{k^4}^{d} + \overbrace{[-2k^2(p-k)^2]}^{e} + \overbrace{(p-k)^4}^{f}}{k^2(p-k)^2(s-k)^2}. \tag{3.63}$$

Yielding another six integrals to compute, which we denoted by the Latin alphabet a)-f).

a)
$$\frac{\mu^{2\epsilon}}{4} \int \frac{d^D k}{(2\pi)^D} \frac{p^4}{k^2(p-k)^2(s-k)^2} = \frac{(p\cdot p)^2}{4} I_3(p,s), \tag{3.64}$$

with $I_3(p,s)$ defined as in eq. (2.46)

$$I_3(p,s) \equiv \frac{\mu^{2\epsilon}}{4} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(p-k)^2(s-k)^2} = \text{finite}. \tag{3.65}$$

$I_3(p,s)$ represents a finite integral and will be consequently neglected in our further calculations.

b)
$$\frac{\mu^{2\epsilon}}{4} \int \frac{d^D k}{(2\pi)^D} \frac{2p^2}{(p-k)^2(s-k)^2} = \frac{2p^2}{4} I_2(p-s) \tag{3.66}$$

c)

$$\frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{-2p^2}{k^2(s-k)^2}=\frac{-2p^2}{4}I_2(s) \tag{3.67}$$

d) The fourth integral can be solved by substituting $u=k-s$

$$\begin{aligned}\frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{k^2}{(p-k)^2(s-k)^2} &= \frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{(u+s)^2}{u^2(p-s-u)^2}\\ \frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{u^2+s^2+2(u\cdot s)}{u^2(p-s-u)^2} &= \frac{1}{4}[s^2I_2(p-s)+s\cdot(p-s)I_2(p-s)]\\ &= \frac{1}{4}[(p\cdot s)I_2(p-s)]\end{aligned} \tag{3.68}$$

e)

$$\frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{-2}{(s-k)^2}=\frac{-2}{4}I_1(s)=0, \tag{3.69}$$

where I_1 is defined as in eq. (2.45)

$$I_1(p)\equiv\frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(p-k)^2}=0, \tag{3.70}$$

f)

$$\begin{aligned}\frac{\mu^{2\epsilon}}{4}\int\frac{d^dk}{(2\pi)^D}\frac{(p-k)^2}{k^2(s-k)^2} &= \frac{\mu^{2\epsilon}}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{p^2+k^2-2pk}{k^2(s-k)^2}\\ &= \frac{1}{4}[pI_2(s)+I_1(s)-(p\cdot s)I_2(s)]\end{aligned} \tag{3.71}$$

Combining all the integrals contained in the α part yields

$$\begin{aligned}\alpha=\frac{1}{4}[&(p\cdot p)^2I_3(p,s)+2p^2I_2(p-s)-2p^2I_2(s)\\ &+(s\cdot p)I_2(p-s)-2I_1(s)+pI_2(s)+I_1(s)-(p\cdot s)I_2(s)]\end{aligned} \tag{3.72}$$

Now we want to repeat the same calculation for the β) integral, where p is exchanged with s.

$$\beta) = \frac{\mu^{2\epsilon}}{4}\int \frac{d^Dk}{(2\pi)^D}\frac{[s^2+k^2-(s-k)^2]^2}{k^2(p-k)^2(s-k)^2}$$
$$= \frac{\mu^{2\epsilon}}{4}\int \frac{d^Dk}{(2\pi)^D}\frac{s^4+2s^2k^2-2(s-k)^2+k^4-2k^2(s-k)^2+(s-k)^4}{k^2(p-k)^2(s-k)^2}$$
$$= \frac{1}{4}[2s^2I_2(p-s)-2s^2I_2(p)+(s\cdot p)I_2(p-s)+s^2I_2(p)-(s\cdot p)I_2(p)] \tag{3.73}$$

Adding now the final terms for $\alpha)$ and $\beta)$ yields

$$\alpha)+\beta) = \frac{1}{4}[(p^4+s^4)I_3(p,s)+2(p^2+s^2+p\cdot s)I_2(p-s)-(s^2+p\cdot s)I_2(p)-(p^2+p\cdot s)I_2(s)] \tag{3.74}$$

Before getting to the last equation iii) we still have to evaluate the right-hand side of ii). Hence

$$\begin{aligned}&[p^\mu p^\nu + s^\mu s^\nu]\left\{g_{\mu\nu}A+[p_\mu p_\nu+s_\mu s_\nu]B+[p_\mu s_\nu+p_\nu s_\mu]C\right\}\\ &=[p^2+s^2]A+[p^4+2(p\cdot s)^2+s^4]B+[2(p\cdot s)(p^2+s^2)]C.\end{aligned} \tag{3.75}$$

iii) The final equation iii) will be equally calculated as the former ones i) and ii). Starting once again with the left-hand side gives us

$$iii) = [p^\mu s^\nu p^\nu s^\mu]\mu^{2\epsilon}\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{k^2(p-k)^2(s-k)^2}$$
$$= \mu^{2(\epsilon)}\int \frac{d^Dk}{(2\pi)^D}\frac{2(p\cdot k)(s\cdot k)}{k^2(p-k)^2(s-k)^2}$$
$$= \frac{\mu^{2\epsilon}}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{p^2s^2+p^2k^2-p^2(s-k)^2+k^2s^2+k^4-k^2(s-k)^2-s^2(p-k)^2}{k^2(p-k)^2(s-k)^2}$$
$$+ \frac{-k^2(p-k)^2+(p-k)^2(s-k)^2}{k^2(p-k)^2(s-k)^2}$$
$$= \frac{1}{2}[p^2s^2I_3(p,s)+p^2I_2(p-s)-p^2I_2(p)+s^2I_2(p-s)+(s\cdot p)I_2(p-s)$$
$$- s^2I_2(s)-I_1(s)+I_1(0)]$$
$$= \frac{1}{2}\left\{p^2s^2I_3(p,s)+[p^2+s^2+(s\cdot p)]I_2(p-s)-p^2I_2(p)-s^2I_2(s)\right\}. \tag{3.76}$$

And finally for the right-hand side we get

$$\begin{aligned}&[p^\mu s^n u + p^\nu s^\mu]\left\{g_{\mu\nu}A+[p_\mu p_\nu+s_\mu s_\nu]B+[p_\mu s_\nu p_\nu p_\mu]C\right\}\\ &=(2p\cdot s)A+2[p^2(p\cdot s)+s^2(p\cdot s)]B+2[p^2s^2+(p\cdot s)^2]C.\end{aligned} \tag{3.77}$$

With the help of Mathematica Lst. (6) we can easily solve the equation system and get our desired result for the integral

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{k^2(p-k)^2(s-k)^2} = \frac{i}{(4\pi)^2} \frac{1}{4\epsilon} g_{\mu\nu} + \mathcal{O}(1), \tag{3.78}$$

with

$$A = \frac{i}{(4\pi)^2} \frac{1}{4\epsilon} + \mathcal{O}(1), \qquad B = \mathcal{O}(1), \qquad C = \mathcal{O}(1). \tag{3.79}$$

Having found a solution for the integral we now just have to plug it in and calculate the vector γ_μ and axialvector $\gamma_\mu\gamma_5$ contribution of the first current-current diagram. Hence inserting the above expression in our Green's function eq. (3.56) yields

$$\Gamma_{amp} = \frac{g_s^2}{(4\pi)^2} C_F \delta^{ij} \delta^{kl} \left\{ \frac{1}{4\epsilon} \left[\gamma_\sigma \gamma^\lambda \Gamma_1 \gamma_\lambda \gamma^\sigma\right]^{AB}_{\alpha\beta} - (1-a)\frac{1}{\epsilon} \left[\Gamma_1\right]^{AB}_{\alpha\beta} \right\} \left[\Gamma_2\right]^{BA}_{\delta\gamma} \tag{3.80}$$

Notice, that the second integral in eq. (3.56) is the same integral as in eq. (3.58). To get the vector and axialvector contributions we are left with the two substitution cases

$$\Gamma_1 = \Gamma_2 = \gamma_\mu \qquad \text{and} \qquad \Gamma_1 = \Gamma_2 = \gamma_\mu \gamma_5 \tag{3.81}$$

Regarding only important terms and using $\Gamma_1 = \gamma_\mu$ gives us

$$\gamma_\sigma \gamma^\lambda \gamma_\mu \gamma_\lambda \gamma^\sigma = (2-D)\gamma_\sigma \gamma_\mu \gamma^\sigma = (2-D)^2 \gamma_\mu = (2-4+2\epsilon)^2 \gamma_\mu \xrightarrow{\epsilon\to 0} 4\gamma_\mu = 4\Gamma_1. \tag{3.82}$$

Repeating the calculation for the axialvector substitution $\Gamma_1 = \gamma_\mu \gamma_5$

$$\gamma_\sigma \gamma^\lambda \gamma_\mu \gamma_5 \gamma_\lambda \gamma^\sigma \xrightarrow{\epsilon\to 0} 4\Gamma_1. \tag{3.83}$$

Therefore we are left in both cases with the applied Γ_1 structure

$$\begin{aligned} \Gamma^a_{amp} &= \frac{g_s^2}{(4\pi)^2} C_F \delta^{ij} \delta^{kl} \left\{ \frac{1}{4\epsilon} \left[4\Gamma_1\right]^{AB}_{\alpha\beta} - (1-a)\frac{1}{\epsilon} \left[\Gamma_1\right]^{AB}_{\alpha\beta} \right\} \left[\Gamma_1\right]^{BA}_{\delta\gamma} \\ &= \frac{a\alpha_s}{(4\pi)} C_F \delta^{ij} \delta^{kl} \frac{1}{\epsilon} \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma}, \end{aligned} \tag{3.84}$$

where we used the coupling constant

$$\alpha_s = \frac{g^2}{4\pi} \tag{3.85}$$

Hence our final contribution for the first diagram is given by

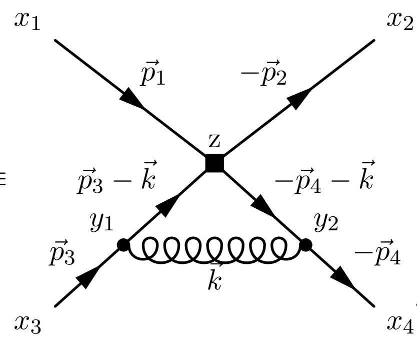
$$\Gamma^a_{amp} = \frac{\alpha_s}{4\pi} C_F \delta^{ij}\delta^{kl}\frac{a}{\epsilon}\left[\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma}\,, \tag{3.86}$$

One can easily see the gauge dependence of the above term. As we are working in a gauge-independent theory the contribution of the a-diagram (for both cases: $\Gamma_1 = \gamma_\mu$ and $\Gamma_1 = \gamma_\mu\gamma_5$) to our striven divergency is zero.

Having calculated our first loop contribution we now want to continue with the second diagram, which is basically the same as the first.

Diagram b)

$\Gamma^b \equiv$



. (3.87)

Hence

$$\Gamma^b = i g_s^2 C_F \delta^{ij}\delta^{kl}\int\frac{d^Dk}{(2\pi)^D}\left[S(p_1)\Gamma_1 S(p_2)\right]^{AB}_{\alpha\beta}\left[S(p_3)\gamma^\lambda S(p_3-k)\Gamma_2 S(p_4-k)\gamma^\sigma\right]^{BA}_{\delta\gamma}\,, \tag{3.88}$$

where we simply followed the in and outgoing impulses. The prefactor $(ig_s^2 C_F)$ comes from the fact that we still have to deal with seven propagators (i), the quark-gluon coupling (g_s^2), the gluon color structure (C_F) and the color structure of our external quark fields $(\delta^{ij}\delta^{kl})$. As we are regarding a loop, we have to face the integral $\int d^Dk/(2\pi)^D$. Starting now to follow the impulses, from the left upper side, yields a factor $S(p_1)$ followed by the crossing of the four-quark operator, giving and additional factor Γ_1. Because of the outgoing impulse p_2 we have to add another factor $S(p_2)$. The flavor and Dirac structure can easily be checked by our a-diagram computation. Now we are leftover with the lower part of the diagram. Repeating the same steps, as mentioned above, we only have to deal with the vertices, given by the gluon propagator, yielding instead of a Γ while crossing a factor γ_λ. Finally the flavor and Dirac structure can easily be checked with our calculations of the ultimately calculated diagram. Amputating the external fields will give us

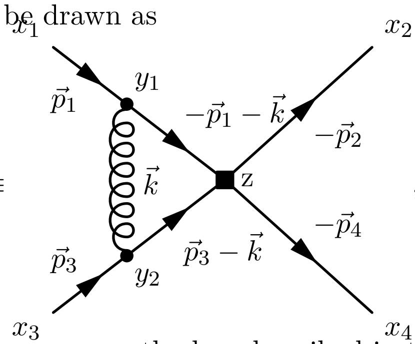
$$
\begin{aligned}
\Gamma^b &= ig_s^2 C_F \delta^{ij}\delta^{kl} \left[\Gamma_1\right]^{AB}_{\alpha\beta} \int \frac{d^Dk}{(2\pi)^D} \left[\gamma^\lambda S(p_3-k)\Gamma_2 S(p_4-k)\gamma^\sigma\right]^{BA}_{\delta\gamma} \frac{1}{k^2}\left[-g_{\lambda\sigma}+(1-a)\frac{k_\lambda k_\sigma}{k^2}\right] \\
&= ig_s^2 C_F \delta^{ij}\delta^{kl} \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left\{\int \frac{d^Dk}{(2\pi)^D} \frac{-k_\rho k_\sigma}{k^2(p-k)^2(s-k)^2} \left[\gamma_\lambda \gamma^\rho \Gamma_2 \gamma^\sigma \gamma^\lambda\right]^{BA}_{\delta\gamma} + (1-a)\int \frac{d^Dk}{(2\pi)^D}\right\} \\
&= \frac{\alpha_s}{4\pi}\frac{a}{\epsilon} C_F \delta^{ij}\delta^{kl} \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma}
\end{aligned}
\tag{3.89}
$$

As one can see the contribution of the second diagram also vanishes in the Landau gauge (a=0). For the further diagrams c to f, we will now face non-vanishing divergencies.

Diagram c)

The third diagram can be drawn as

$$\Gamma_c \equiv$$



, (3.90)

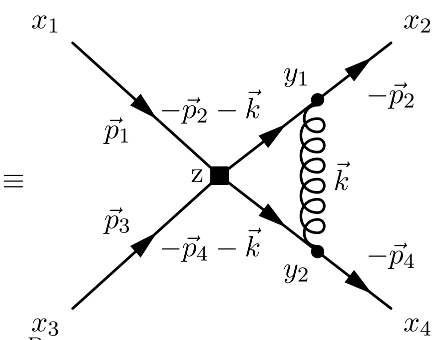
yielding, while using the same method as described in the former diagram,

$$
\begin{aligned}
\Gamma_c &= ig_s^2 (t^b)^{ij}(t^b)^{kl} \int \frac{d^Dk}{(2\pi)^D} \left[\gamma^\lambda S(p_1-k)\Gamma_1\right]^{AB}_{\alpha\beta} \left[\gamma^\sigma S(p_3+k)\Gamma_2\right]^{BA}_{\delta\gamma} D_{\lambda\sigma}(k) \\
&= ig_s^2 (t^b)^{ij}(t^b)^{kl} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2(p_1-k)^2(p_3+k)} \left[-\gamma^\lambda \not{k}\Gamma_1\right]^{AB}_{\alpha\beta} \left[\gamma^\sigma \not{k}\Gamma_2\right]^{AB}_{\delta\gamma} \left[-g_{\lambda\sigma}+(1-a)\frac{k_\lambda k_\sigma}{k^2}\right] \\
&= ig_s^2 (t^b)^{ij}(t^b)^{kl} \left\{\int \frac{d^Dk}{(2\pi)^D} \frac{k_\omega k_\nu}{k^2(p_1-k)^2(p_3+k)} \left[\lambda_\sigma \lambda^\omega \Gamma_1\right]^{AB}_{\alpha\beta} \left[\gamma^\sigma \gamma^\nu \Gamma_2\right]^{BA}_{\delta\gamma}\right. \\
&\left. - \int \frac{d^Dk}{(2\pi)^D} \frac{(1-a)}{k^4(p_1-k)^2(p_3+k)^2} \left[k^2\Gamma_1\right]^{AB}_{\alpha\beta} \left[k^2\Gamma_2\right]^{BA}_{\delta\gamma}\right\} \\
&= -\frac{g_s^2}{(4\pi)^2}(t^b)^{ij}(t^b)^{kl}\frac{1}{\epsilon}\left\{\frac{g_{\omega\nu}}{4}\left[\gamma_\sigma\gamma^\omega\Gamma_1\right]^{AB}_{\alpha\beta}\left[\gamma^\sigma\gamma^\nu\Gamma_2\right]^{BA}_{\delta\gamma} - (1-a)\left[\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma}\right\} \\
&= -\frac{g_s^2}{(4\pi)^2}(t^b)^{ij}(t^b)^{kl}\frac{1}{\epsilon}\left\{\frac{1}{4}\left[\gamma_\sigma\gamma_\omega\Gamma_1\right]^{AB}_{\alpha\beta}\left[\gamma^\sigma\gamma^\omega\Gamma_2\right]^{BA}_{\delta\gamma} - (1-a)\left[\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma}\right\}
\end{aligned}
\tag{3.91}
$$

As the methodology should be known by now, we will just give the results for the last diagrams.

Diagram d)

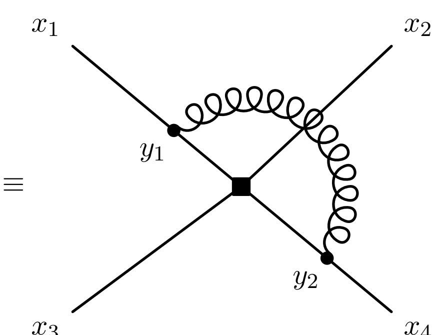
$$\Gamma_d \equiv \tag{3.92}$$



$$\begin{aligned}\Gamma_d &= ig_s^2(t^b)^{ij}(t^b)^{kl}\int\frac{d^Dk}{(2\pi)^D}\left[\Gamma_1 S(p_2+k)\gamma^\lambda\right]^{AB}_{\alpha\beta}\left[\Gamma_2 S(p_4-k)\gamma^\sigma\right]^{BA}_{\delta\gamma} D_{\lambda\sigma}(k)\\ &= ig_s^2(t^b)^{ij}(t^b)^{kl}\left\{\int\frac{d^Dk}{(2\pi)^D}\frac{k_\omega k_\nu}{k^2(p_2+k)^2(p_4-k)^2}\left[\Gamma_1\gamma_\sigma\gamma^\lambda\right]^{AB}_{\alpha\beta}\left[\Gamma_2\gamma^\nu\gamma^\sigma\right]^{BA}_{\delta\gamma}\right.\\ &\left.-(1-a)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(p_2+k)^2(p_4-k)^2}\left[\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma}\right\}\\ &= a_s(t^b)^{ij}(t^b)^{kl}\frac{1}{\epsilon}\left\{-\frac{1}{4}\left[\Gamma_1\gamma_\sigma\gamma_\omega\right]^{AB}_{\alpha\beta}\left[\Gamma_2\gamma^\omega\gamma^\sigma\right]^{BA}_{\delta\gamma}+(1-a)\left[\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma}\right\},\end{aligned} \tag{3.93}$$

Diagram e)

$$\Gamma_e \equiv \tag{3.94}$$

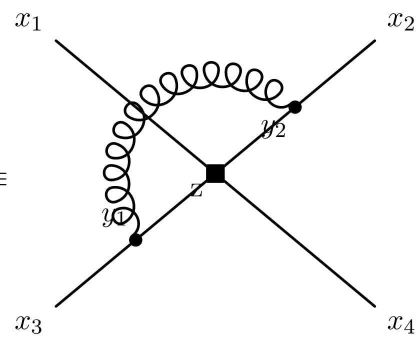


To get a better overview we will from now on skip the impulse labeling.

$$\begin{aligned}\Gamma_e &= ig_s^2(t^b)^{ij}(t^b)^{kl}\int\frac{d^Dk}{(2\pi)^D}\left[\gamma^\lambda S(p_1-k)\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2 S(p_4-k)\gamma^\sigma\right]^{BA}_{\delta\gamma} D_{\lambda\sigma}(k)\\ &= a_s(t^b)^{ij}(t^b)^{kl}\frac{1}{\epsilon}\left\{\frac{1}{4}\left[\gamma_\sigma\gamma_\omega\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\gamma^\omega\gamma^\sigma\right]^{BA}_{\delta\gamma}-(1-a)\left[\Gamma_1\right]^{AB}_{\alpha\beta}\left[\Gamma_2\right]^{BA}_{\delta\gamma}\right\}\end{aligned} \tag{3.95}$$

Diagram f)

$\Gamma_f \equiv$



(3.96)

$$\Gamma_f = ig_s^2 (t^b)^{ij}(t^b)^{kl} \int \frac{d^D k}{(2\pi)^D} \left[\Gamma_1 S(p_2 - k)\gamma^\sigma\right]^{AB}_{\alpha\beta} \left[\gamma^\lambda S(p_3 - k)\Gamma_2\right]^{BA}_{\delta\gamma} D_{\lambda\sigma}(k)$$
$$= a_s (t^b)^{ij}(t^b)^{kl} \frac{1}{\epsilon} \left\{ \frac{1}{4} \left[\Gamma_1 \gamma_\omega \gamma_\sigma\right]^{AB}_{\alpha\beta} \left[\gamma^\sigma \gamma^\omega \Gamma_2\right]^{BA}_{\delta\gamma} - (1-a) \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma} \right\} \tag{3.97}$$

We now have calculated the contributions of all appearing current-current diagrams. To get an overview we want to sum up the calculated results. Hence

$$\Gamma^a = \frac{\alpha_s}{4\pi} C_F \frac{a}{\epsilon} \delta^{ij}\delta^{kl} [\Gamma_1]^{AB}_{\alpha\beta} [\Gamma_2]^{BA}_{\delta\gamma} \tag{3.98}$$
$$\Gamma^b = \frac{\alpha_s}{4\pi} C_F \frac{a}{\epsilon} \delta^{ij}\delta^{kl} [\Gamma_1]^{AB}_{\alpha\beta} [\Gamma_2]^{BA}_{\delta\gamma} \tag{3.99}$$
$$\Gamma^c = \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (t^b)^{ij}(t^b)^{kl} \left\{ -\frac{1}{4} \left[\gamma_\sigma \gamma_\omega \Gamma_1\right]^{AB}_{\alpha\beta} \left[\gamma^\sigma \gamma^\nu \Gamma_2\right]^{BA}_{\delta\gamma} + (1-a) \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma} \right\} \tag{3.100}$$
$$\Gamma_d = \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (t^b)^{ij}(t^b)^{kl} \left\{ -\frac{1}{4} \left[\Gamma_1 \gamma_\sigma \gamma_\omega\right]^{AB}_{\alpha\beta} \left[\Gamma_2 \gamma^\omega \gamma^\sigma\right]^{BA}_{\delta\gamma} + (1-a) \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma} \right\} \tag{3.101}$$
$$\Gamma^e = \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (t^b)^{ij}(t^b)^{kl} \left\{ \frac{1}{4} \left[\gamma_\sigma \gamma_\omega \Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2 \gamma^\omega \gamma^\sigma\right]^{BA}_{\delta\gamma} - (1-a) \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma} \right\} \tag{3.102}$$
$$\Gamma^f = \frac{\alpha_s}{4pi} \frac{1}{\epsilon} (t^b)^{ij}(t^b)^{kl} \left\{ \frac{1}{4} \left[\Gamma_1 \gamma_\omega \gamma_\sigma\right]^{AB}_{\alpha\beta} \left[\gamma^\sigma \gamma^\omega \Gamma_2\right]^{BA}_{\delta\gamma} - (1-a) \left[\Gamma_1\right]^{AB}_{\alpha\beta} \left[\Gamma_2\right]^{BA}_{\delta\gamma} \right\} \tag{3.103}$$

While summing over all diagrams in the Landau gauge (a=0, so that Γ_a and Γ_b cancel) one can see that the gauge dependent terms have an alternating sign and cancel. Thus for the case $\Gamma_1 = \Gamma_2 = \gamma_\mu$ we are left with

$$c) + d) + e) + f) = \frac{a_s}{\epsilon}(t^b)^{ij}(t^b)^{kl}\left(-\frac{3}{2}\right)[\gamma_5\gamma_\mu]^{AB}_{\alpha\beta}\,[\gamma^\mu\gamma_5]^{BA}_{\delta\gamma} = \frac{a_s}{\epsilon}\left(-\frac{3}{2}\right)Q^o_A \tag{3.104}$$

and for $\Gamma_1 = \Gamma_2 = \gamma_\mu\gamma_5$

$$c) + d) + e) + f) = \frac{a_s}{\epsilon}(t^b)^{ij}(t^b)^{kl}\left(-\frac{3}{2}\right)[\gamma_\mu]^{AB}_{\alpha\beta}\,[\gamma^\mu]^{BA}_{\delta\gamma} = \frac{a_s}{\epsilon}\left(-\frac{3}{2}\right)Q^o_V, \tag{3.105}$$

where we made use of Mathematica (Lst. 6.2) to calculate the tensor products.
To this point one should notice that Q^S_V mixes into Q^O_A and Q^S_A mixes into Q^O_V.

$$\begin{aligned} Q^S_V &\to Q^O_A \\ Q^S_A &\to Q^O_V \end{aligned} \tag{3.106}$$

Hence Q^S_- mixes into Q^O_-

$$Q^S_- \equiv Q^S_V - Q^S_A \to Q^O_A - Q^O_V = -Q^O_-. \tag{3.107}$$

Consequently our singlet operator Q^S_- mixes into

$$\Sigma^S_{a-f} = \frac{a_s}{\epsilon}\frac{3}{2}Q^O_-. \tag{3.108}$$

With this result we are now able to directly read off the needed renormalization constant. But before we need the relation between the divergence of the Green's function and $\hat{Z}_0$. Starting with a general set of bare operators $\vec{O}^B$

$$\vec{O}^B = \hat{Z}_0\vec{O}^R \qquad \Rightarrow \qquad \vec{O}^R = \hat{Z}^{-1}\vec{O}^B \tag{3.109}$$

and being aware that the expectation value of the Green's function should be finite

$$\Gamma = \langle q\bar{q}\vec{O}^R q\bar{q}\rangle \stackrel{!}{=} \text{finite}, \tag{3.110}$$

while using the series expansion of $\hat{Z}_0$

$$\hat{Z}_0 = \mathbb{1} + \hat{Z}_0^{(1)}\frac{\alpha_s}{\epsilon} + \mathcal{O}(\alpha_s), \tag{3.111}$$

we can combine the above equations to get a criterion for obtaining the renormalization constants.

$$\Gamma = \langle q\bar{q}\vec{O}^R q\bar{q}\rangle \stackrel{!}{=} finite \Rightarrow \langle q\bar{q}(Z^{-1}\vec{O}^B)q\bar{q}\rangle = \langle q\bar{q}(\mathbb{1} - Z_0^{(1)}\frac{\alpha_s}{\epsilon})\vec{O}^R q\bar{q}\rangle. \tag{3.112}$$

Hence for the Q_- current-current diagram renormalization constants we get

$$(\hat{Z}_0^{(1)})_{21} = \frac{3}{2} \qquad \text{and} \qquad (\hat{Z}_0^{(1)})_{22} = 0, \tag{3.113}$$

where the first number of the indice signifies, that we are using the singlet operator as input and the second number denotes that we are not mixing into singlet (Z_{21}) or octet (Z_{22}) operator. Hence we are mixing into the octet operator with a factor of 3/2, but not into the singlet operator.

Having dealt with the renormalization constants of the two singlet operators we are now one step before obtaining the first row of the anomalous dimension matrix, but want to dedicate our self to the divergencies appearing from the octet operators $Q^O_{V,A}$ (with V,A meaning either Q^O_V or Q^O_A) first.

In order to treat the insertions of $Q^O_{V,A}$, we have to decompose the appearing colour structures. Therefore we shortly want to introduce some important colour relations. Let t^a ($a = 1, 2, \ldots, 3^2 - 1$) be the SU(3) generators. The basic relation between the Gell-Mann matrix and t^a, their commutator and anti-commutator are given by

$$t^a = \frac{\lambda_a}{2}, \qquad [\lambda_a, \lambda_b] = i2f_{abc}\lambda_c, \qquad \{\lambda_a, \lambda_c\} = \frac{4}{N}\delta_{ab}\mathbb{1} + 2d_{abc}\lambda_c, \tag{3.114}$$

where f_{abc} is a real and total anti-symmetric matrix, normalized in such a way that

$$f_{abc}f_{dbc} = N\delta_{ad} \tag{3.115}$$

and d_{abc} is the real and total symmetric correspondent, normalized as

$$d_{abc}d_{dbc} = \left(N - \frac{N}{4}\right)\delta_{ad}. \tag{3.116}$$

Combining the commutator and anti-commutator of the group generators yields

$$\begin{aligned}\lambda_a\lambda_b &= [\lambda_a, \lambda_b] + \lambda_b\lambda_a \\ &= [\lambda_a, \lambda_b] + \{\lambda_a, \lambda_b\} - \lambda_a\lambda_b \\ &= if_{abc}\lambda_c + \frac{2}{N}\lambda_{ab}\mathbb{1} + d_{abc}\lambda_c.\end{aligned} \tag{3.117}$$

With the above relation we are now able to calculate the for the octet operator occurring colour structure. Hence for our first needed tensor product

$t^a t^b \otimes t^a t^b$ we get

$$\begin{aligned}
t^a t^b \otimes t^a t^b &= \frac{1}{16}[\lambda^a \lambda^b \otimes \lambda^a \lambda^b] \\
&= \frac{1}{16}\left[\frac{2}{N_c}\delta^{ab}\mathbb{1} + d^{abc}\lambda_c + if^{abc}\lambda^c\right] \otimes \left[\frac{2}{N_c}\delta^{ab}\mathbb{1} + d^{abc}\lambda_c + if^{abc}\lambda^c\right] \\
&= \frac{1}{16}\left[\frac{4}{N_c^2}(N_c^2-1)\mathbb{1}\otimes\mathbb{1} + d^{abc}d^{abc}\lambda^c\otimes\lambda^c - f^{abc}f^{abc}\lambda^c\otimes\lambda^c\right] \\
&= \frac{1}{16}\left[\frac{8C_F}{N_c}\mathbb{1}\otimes\mathbb{1} + \left(N_c - \frac{4}{N_c} - N_c\right)\lambda^c\otimes\lambda^c\right] \\
&= \frac{C_F}{2N_C}\mathbb{1}\otimes\mathbb{1} - \frac{1}{N_c}t^a\otimes t^a,
\end{aligned} \tag{3.118}$$

where we used once again the Casmir operator and substituted the Gell-Mann matrices λ^a into the group generators t^a. For the second tensor product $t^a t^b \otimes t^b t^a$ we get a slightly different result.

$$\begin{aligned}
t^a t^b \otimes t^b t^a &= \frac{1}{16}[\lambda^a \lambda^b \otimes \lambda^b \lambda^a] \\
&= \frac{1}{16}\left[\frac{2}{N_c}\delta^{ab}\mathbb{1} + d^{abc}\lambda^c + if^{abc}\lambda^c\right] \otimes \left[\frac{2}{N_c}\delta^{ba}\mathbb{1} + d^{bac}\lambda^c + if^{bac}\lambda^c\right] \\
&= \frac{1}{16}\left[\frac{4}{N_c^2}(N_c^2-1)\mathbb{1}\otimes\mathbb{1} + d^{abc}d^{abc}\lambda^c\otimes\lambda^c + f^{abc}f^{abc}\lambda^c\otimes\lambda^c\right] \\
&= \frac{1}{16}\left[\frac{8C_F}{N_c}\mathbb{1}\otimes\mathbb{1} + \left(N_c - \frac{4}{N_c} + N_c\right)\lambda^c\otimes\lambda^c\right]\lambda^c\otimes\lambda^c \\
&= \frac{C_F}{2N_C}\mathbb{1}\otimes\mathbb{1} + \left(\frac{N_c}{2} - \frac{1}{N_c}\right)t^a\otimes t^a
\end{aligned} \tag{3.119}$$

Finally a product of three group generators $t^a t^b t^a$ will appear within the first two current-current diagrams a and b, which is given by

$$t^a t^b t^a = \frac{1}{2N_C}t^a. \tag{3.120}$$

Having obtained all needed colour relations we now can evaluate the octet operators Q_-^Oa.

Hence using the operators

$$Q_-^O = Q_V^O - Q_A^O = qt^a\gamma_\mu\bar{q}qt^a\gamma_\mu - qt^a\gamma_\mu\gamma_5\bar{q}qt^a\gamma_\mu\gamma_5\bar{q}, \tag{3.121}$$

we get the same structures as appearing in Q_-^S but with a different color structure.

$$\begin{aligned}\Gamma_a^O &= [\gamma^\lambda S(p_1-k)\Gamma_1 S(p_2-k)\gamma^\sigma t^b t^a t^b]_{\alpha\beta}^{AB}[\Gamma_2 t^a]_{\delta\gamma}^{BA} \\ &= \frac{1}{2N_c}\frac{a_s}{4}\frac{a}{\epsilon}Q_{V,A}^O\end{aligned} \tag{3.122}$$

$$\begin{aligned}\Gamma_b^O &= [\gamma^\lambda S(p_1-k)\Gamma_1 S(p_2-k)\gamma^\sigma]_{\alpha\beta}^{AB}[\Gamma_2 t^b t^a t^b]_{\delta\gamma}^{BA} \\ &= \frac{1}{2N_c}\frac{a_s}{4}\frac{a}{\epsilon}Q_{V,A}^O\end{aligned} \tag{3.123}$$

$$\begin{aligned}\Gamma_c^O &= \left[\gamma^\lambda t^a S(p_1-k)t^b\Gamma_1\right]_{\alpha\beta}^{AB}\left[\gamma^\sigma t^a S(p_3+k)\Gamma_2 t^b\right]_{\delta\gamma}^{BA} D(k) \\ &= [t^a t^b\otimes t^a t^b]\left[-\frac{5}{8}Q_{V,A}-\frac{3}{8}Q_{A,V}+(1-a)Q_{V,A}\right]\end{aligned} \tag{3.124}$$

$$\begin{aligned}\Gamma_d^O &= \left[\Gamma_1 t^b S(p_2+k)\gamma^\lambda t^a\right]_{\alpha\beta}^{AB}\left[\Gamma_2 t^b S(p_4-k)\gamma^\sigma t^a\right]_{\delta\gamma}^{BA} \\ &= [t^a t^b\otimes t^a t^b]\left[-\frac{5}{8}Q_{V,A}-\frac{3}{8}Q_{A,V}+(1-a)Q_{V,A}\right]\end{aligned} \tag{3.125}$$

$$\begin{aligned}\Gamma_e^O &= \left[\Gamma_1 t^b S(p_2-k)\gamma^\sigma t^a\right]_{\alpha\beta}^{AB}\left[\gamma^\lambda t^a S(p_3-k)\Gamma_2 t^b\right]_{\delta\gamma}^{BA} \\ &= \left[t^a t^b\otimes t^b t^a\right]\left[\frac{5}{8}Q_{V,A}-\frac{3}{8}Q_{A,V}-(1-a)Q_{V,A}\right]\end{aligned} \tag{3.126}$$

$$\begin{aligned}\Gamma_f^O &= \left[\gamma^\lambda t^a S(p_1-k)\Gamma_1 t^b\right]_{\alpha\beta}^{AB}\left[\Gamma_2 t^b S(p_4-k)\gamma^\sigma t^a\right]_{\delta\gamma}^{BA} \\ &= \left[t^a t^b\otimes t^b t^a\right]\left[\frac{5}{8}Q_{V,A}-\frac{3}{8}Q_{A,V}-(1-a)Q_{V,A}\right],\end{aligned} \tag{3.127}$$

where we used the shortly before introduced colour relations (eq. (3.118), eq. (3.119) and eq. (3.120)). Thus adding the current-current diagrams with the same colour structure leads to

$$
\begin{aligned}
\Gamma^O_{a+b} &= \frac{a_s}{\epsilon}\left(-\frac{a}{4N_c}\right)Q^O_{V,A} \\
\Gamma^O_{c+d} &= \frac{a_s}{\epsilon}\left[\frac{C_F}{2N_c}\mathbb{1}\otimes\mathbb{1} - \frac{1}{N_c}t^a\otimes t^a\right]\left[-\frac{5}{4}Q_{V,A} - \frac{3}{4}Q_{A,V} + \frac{1}{2}(1-a)Q_{V,A}\right] \\
&= \frac{a_s}{\epsilon}\left[\frac{C_F}{2N_c}\mathbb{1}\otimes\mathbb{1} - \frac{1}{N_c}t^a\otimes t^a\right]\left[\left(-\frac{3}{4}-\frac{a}{2}\right)Q_{V,A} - \frac{3}{4}Q_{A,V}\right]
\end{aligned} \tag{3.128}
$$

$$
\begin{aligned}
\Gamma^O_{e+f} &= \frac{a_s}{\epsilon}\left[\frac{C_F}{2N_c}\mathbb{1}\otimes\mathbb{1} + \left(\frac{N_c}{2}-\frac{1}{N_c}\right)t^a\otimes t^a\right]\left[\frac{5}{4}Q_{V,A} - \frac{3}{4}Q_{A,V} - \frac{1}{2}(1-a)Q_{V,A}\right] \\
&= \frac{a_s}{\epsilon}\left[\frac{C_F}{2N_c}\mathbb{1}\otimes\mathbb{1} + \left(\frac{N_c}{2}-\frac{1}{N_c}\right)t^a\otimes t^a\right]\left[\left(\frac{3}{4}+\frac{a}{2}\right)Q_{V,A} - \frac{3}{4}Q_{A,V}\right].
\end{aligned} \tag{3.129}
$$

justing over all diagrams yields

$$
\begin{aligned}
\sum_{i=a,\ldots,f}\Gamma^O_i &= \frac{a_s}{\epsilon} - \frac{a}{4N_c}Q^O_{V,A} - \frac{3}{2}\frac{C_F}{4N_c}Q^S_{A,V} + \frac{N_c}{2}\left(\frac{3}{4}+\frac{a}{2}\right)Q^O_{V,A} - \frac{3}{4}\frac{N_c}{2}Q^O_{A,V} \\
&= \frac{a_s}{\epsilon}\left[\left(\frac{3}{8}N_c - \frac{a}{4}C_F\right)Q^O_{V,A} - \frac{3}{8}\frac{C_F}{N_c}Q^S_{A,V} + \frac{3}{2}\left(\frac{1}{N_c}-\frac{1}{4}N_c\right)Q^O_{A,V}\right],
\end{aligned} \tag{3.130}
$$

which turns written in the Landau gauge, while inserting the Q^O_{V-A} operator into

$$
\sum_{i=a,\ldots,f}\Gamma^O_i = \frac{a_s}{\epsilon}\left(\frac{3N_c}{4} - \frac{3}{2N_c}\right)Q^O_{V-A} + \frac{3C_F}{4N_C}Q^S_{V-A}. \tag{3.131}
$$

Having computed the divergent part of the octet operator Q^O_- we can once more directly read off the renormalization constant using eq. (3.112)

$$
(\hat{Z}^{(1)}_0)_{12} = \frac{3N_C}{4} - \frac{3}{2N_C} \qquad \text{and} \qquad (\hat{Z}^{(1)}_0)_{12} = \frac{3C_F}{4N_C}. \tag{3.132}
$$

Now we require the relation between the renormalisation matrix and the anomalous dimension, which is defined by

$$
\begin{aligned}
\hat{\gamma}_O(a_\mu) &\equiv Z^{(1)}_O(\mu)\mu\frac{d}{d\mu}\hat{Z}_O(\mu) \\
&= Z^{-1}_O(\mu)\mu\frac{da_\mu}{d\mu}\frac{d\hat{Z}_O(\mu)}{da_\mu} \\
&= -\beta(a_\mu)Z^{-1}_O(\mu)\frac{d\hat{Z}_O(\mu)}{da_\mu}.
\end{aligned} \tag{3.133}
$$

As we are only interested in the leading order we can plug in the series expansion of the anomalous dimension matrix $\hat{\gamma}_O$, the beta function β and the renormalization constant $\hat{Z}_O$.

$$\gamma_O^{(1)} a_s + \cdots = -(2\epsilon a_s + \beta^{(1)} a_s^2) \frac{\hat{Z}_O^{(1)}}{\mathbb{1} + \cdots} \tag{3.134}$$

Comparing the factors of the first order yields the required relation

$$\hat{\gamma}_0^{(1)}(a_\mu) = -2\hat{Z}_0^{(1)}. \tag{3.135}$$

3.3.1 V-A Anomalous Dimension Matrix

Hence the anomalous dimension matrix for the operators Q_- is given by

$$\hat{\gamma}_{O_{V-A}}^{(1)} = \begin{pmatrix} -\frac{3N_C}{2} + \frac{3}{N_C} & -\frac{3C_F}{2N_C} \\ -3 & 0 \end{pmatrix} \tag{3.136}$$

Until know we excluded the possible penguin diagram contractions, under the assumption, that they cancel each others in the V-A case (which is actually the reason for the choose of our operator basis). For the next chapter we now will have to deal with the V+A case. Consequently we will have to calculate the penguin contributions and see, why they cancel in the V-A case.

3.4 V+A Contribution

For the operators appearing in the V+A case, at leading order, we can choose a closed set as $Q_+ = (Q_+^O, Q_-^O, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10})$. In general, also the operator Q5 of the basis presented above arises. However, in four dimensions, one operator in the full set is redundant and can be expressed through the others by means of Fierz transformations. In the further chapter we want to discuss the mixing of each operator of our closed set. At this point we also have to regard appearing penguin diagrams and will see why they cancelled in the V-A case.

Q1 mixing

Our first operator is given by

$$Q_+^O = Q_V^O + Q_A^O = \bar{u}\gamma_\mu t^a d\bar{d}\gamma^\mu u + \bar{u}\gamma_\mu\gamma_5 t^a \bar{d}\gamma^\mu\gamma_5 t^a u \tag{3.137}$$

and will give us the first row of the desired anomalous dimension matrix. As we already dealt with most of the calculation we just have to use our previous results. Noticing that we already calculated the current-current diagrams and that we haven't to deal in the Q1-mixing with an octet operator we can simply use our former result for the current-current diagrams of eq. (3.130)

$$
\begin{aligned}
\sum_{a,\dots,f} \Gamma_i^{Q1} &= \frac{a_s}{\epsilon}\left[\frac{3N_c}{8}Q^O_{V,A} - \frac{3C_F}{4N_c}Q^S_{A,V} - \frac{3}{4}\left(\frac{N_c}{2} - 2N_c\right)Q^O_{A,V}\right] \\
&\overset{Q_1}{=} \frac{a_s}{\epsilon}\left[\frac{3N_c}{8}Q_1 - \frac{3C_F}{4N_c}Q_2 - \frac{3}{4}\left(\frac{N_c}{2} - 2N_c\right)Q_1\right] \\
&\overset{Q_1}{=} \frac{a_s}{\epsilon}\left[\frac{3}{2N_c}Q_1 - \frac{3C_F}{4N_c}Q_2\right].
\end{aligned} \tag{3.138}
$$

From here we easily read off the renormalizations constants

$$
Z_{11} = \frac{3}{2N_c} \qquad \text{and} \qquad Z_{12} = -\frac{3C_F}{4N_c}. \tag{3.139}
$$

Which lead to the anomalous dimension matrix entries

$$
\gamma_{11} = -\frac{3}{N_c} \qquad \text{and} \qquad \gamma_{12} = \frac{3C_F}{2N_c}. \tag{3.140}
$$

Having finished the discussion of the current-current diagrams we now have to treat a non-vanishing penguin diagram contributions (see fig. (3.1)), which is caused by the following contraction

$$
\Gamma^{(1)}_{pen} = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2), \tag{3.141}
$$

where pen stands for penguin and we added an indice (1). Furthermore as we were only interested in the contraction we did only wrote down the needed fields. As we will see there exists another possible penguin contraction, which we want to introduce to a later point of time. To get an useful, more general, relation we want to start computing the penguin diagram appearing from the singlet vector operator Q^S_V. Later we will see, that the singlet axialvector, the octet vector and the octet axial contributions can be easily constructed from the singlet vector penguin contribution. Hence for

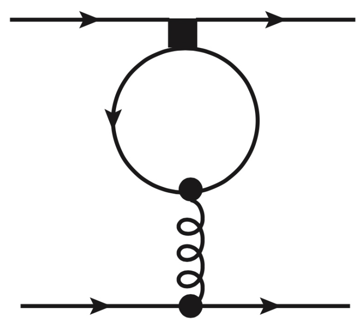


Figure 3.1: Exemplary one-loop penguin diagram that have to be calculated within the V+A case.

our desired penguin diagram

$$
\begin{aligned}
\Gamma^{S(1)} &= -g_s^2 \int d^d x_1 d^d x_2 d^d x_3 d^d x_4 d^d y_1 d^d y_2 d^d z e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4)} \\
&\cdot \langle 0 | \{ q^i_\alpha(x_1) \bar{q}^j_\beta(x_2) [\bar{q}^A \gamma_1 q^B \bar{q}^B \gamma_2 q^A](z) \sum_q [\bar{q}\gamma^\lambda t^b b^b_\lambda q](y_1) \sum_q [\bar{q}\gamma^\sigma t^c b^c_\sigma q](y_2) q^k_\delta(x_3) \bar{q}^j_\gamma(x_4) \} | 0 \rangle \\
&= i g_s^2 [S^A(x_1 - z)\gamma_1 S^B(z - y_1)\gamma^\lambda t^b S^B(y_1 - z)\gamma_2 S^A(z - x_2)]_{\alpha\beta} \delta^{bc} D_{\lambda\sigma}(y_1 - y_2) \\
&\cdot \sum_q [S^q(x_3 - y_2)\gamma^\sigma t^c S^q(y_2 - x_4)]_{\delta\gamma} \\
&= i g_s^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} [S^A(p_1)\Gamma_1 S^B(p_1 + k)\gamma^\lambda t^b S^B(-p_2 + k)\Gamma_2 S^A(-p_2)]_{\alpha\beta} \\
&\cdot \sum_q [S^q(p_3)\gamma^\sigma t^b S^q(-p_4)]_{\delta\gamma} D_{\lambda\sigma}(p_1 + p_2).
\end{aligned}
\tag{3.142}
$$

Amputating the external quark-propagators and plugging in the non-external ones yields

$$
\begin{aligned}
\Gamma^{S(1)}_{amp} &= i g_s^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} [\Gamma_1 S^B(p_1 + k)\gamma^\lambda t^b S^B(-p_2 + k)\Gamma_2]^{AB}_{\alpha\beta} \sum_q [\gamma^\sigma t^b]^{qq}_{\delta\gamma} D_{\lambda\sigma(p_1+p_2)} \\
&= -i g_s^2 \mu^{2\epsilon} \int \frac{d^D s}{(2\pi)^D} [\Gamma_1 S^B(p - s)\gamma^\lambda t^b S^B(-s)\Gamma_2]^{AB}_{\alpha\beta} \sum_q [\gamma^\sigma t^b]^{qq} D_{\lambda\sigma}(p) \\
&= i g_s^2 \mu^{2\epsilon} \int \frac{d^D s}{(2\pi)^D} \frac{s_\mu (p - s)_\nu}{s^2 (p - s)^2} [\Gamma_1 \gamma^\nu \gamma^\lambda \gamma^\mu \Gamma_2 t^b]^{AB}_{\alpha\beta} \sum_q [\gamma^\sigma t^b]^{qq}_{\delta\gamma} [-g_{\lambda\sigma} + (1 - a)\frac{p_\lambda p_\sigma}{p^2}] \frac{1}{k^2},
\end{aligned}
\tag{3.143}
$$

where we performed the substitution $p \equiv p_1 + p_2$ and $s \equiv p_2 - k$. The appearing integral is given by

$$\int \frac{d^D s}{(2\pi)^D} \frac{s_\mu (p-s)_\nu}{s^2 (p-s)^2} = \frac{i}{(4\pi)^2} \left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon \frac{\Gamma(2-\epsilon)^2}{\Gamma(4-2\epsilon)} \Gamma(\epsilon) \left[\frac{p^2}{2(1-\epsilon)} g_{\mu\nu} + p_\nu p_\mu\right]. \tag{3.144}$$

Now employing this result for $\Gamma_1 = \Gamma_2 = \gamma_\mu$ yields

$$\begin{aligned} \Gamma^{Q_V^S}_{amp} &= -\frac{g_s^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon \frac{\Gamma(2-\epsilon)^2}{\Gamma(4-2\epsilon)} \Gamma(\epsilon) 4(1-\epsilon) \left\{ [\gamma_\lambda t^a]^{AB}_{\alpha\beta} \sum_q [\gamma_\lambda t^a]^{qq}_{\delta\gamma} - \frac{[\not{p} t^a]^{AB}_{\alpha\beta} \sum_q [\not{p} t^a]^{qq}_{\delta\gamma}}{p^2} \right\} \\ &= -\frac{a_s}{6} \left[\frac{1}{\hat{\epsilon}} - ln\left(\frac{-p^2}{\mu^2}\right) + \frac{2}{3} + \mathcal{O}(\epsilon)\right] \left\{ \left[\gamma^\lambda t^b\right]^{\bar{u}u} \sum_q \left[\gamma_\lambda t^b\right]^{\bar{q}q} + \frac{\left[\not{p} t^b\right]^{\bar{u}u} \left[\not{p} t^b\right]^{\bar{q}q}}{p^2} \right\}, \end{aligned} \tag{3.145}$$

where we used lst. (6.6). For insertion of $\Gamma_1 = \Gamma_2 = \gamma_\mu\gamma_5$ we get the same result. This demonstrates the vanishing penguin diagram contribution of the V-A case! Hence for the V+A case we get twice the contribution. Regarding only the divergent term leads to

$$\Gamma^{Q^S_{V+A}}_{amp} = -\frac{a_s}{3}\frac{1}{\hat{\epsilon}} \left[[\gamma_\lambda t^a]^{\bar{u}u} [\gamma^\lambda t^a]^{\bar{q}q} - [\not{p} t^a]^{\bar{u}u} [\not{p} t^a]^{\bar{q}q} \frac{1}{p^2} \right] + \mathcal{O}(1). \tag{3.146}$$

As we are only interested in the mixing into local operators we will neglect the impulse dependent terms. Thus we can write

$$\Gamma^{Q^S_{V+A}}_{amp}(local) = -\frac{a_s}{3}\frac{1}{\hat{\epsilon}} \left[[\gamma_\lambda t^a]^{\bar{u}u} [\gamma^\lambda t^a]^{\bar{q}q} \right] + \mathcal{O}(1). \tag{3.147}$$

Coming back to the Q1-octet-mixing contribution we just have to adjust the former penguin diagram with a substitution of

$$t^a \to t^a t^b t^a = -\frac{1}{2N_c}, \tag{3.148}$$

simply yielding another prefactor. Hence for our penguin diagram we get

$$\Gamma^{Q^O_{V+A}}_{amp} = \frac{1}{6N_c}\frac{a_s}{\epsilon} \left\{ [\gamma_\mu t^a]^{\bar{u}u} + [\gamma_\mu t^a]^{\bar{d}d} \right\} \sum_q [\gamma^\mu t^a]^{\bar{q}q}. \tag{3.149}$$

In our former penguin diagram calculation we contracted the $\bar{u}u$ quark fields. The $\bar{d}d$ is another possible contraction with the same contribution. Hence its corresponds to a mixing into another operator Q3

$$Q_3 = (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a) \sum_{q=u,d,s} (\bar{q}\gamma^\mu t^a q). \tag{3.150}$$

As before we can now easily read of the renormalization constant

$$Z_{13} = \frac{1}{6N_c}. \tag{3.151}$$

Hence we get for the anomalous dimension matrix the additional entry

$$\gamma_{13} = -\frac{1}{3N_c}. \tag{3.152}$$

and for the total anomalous row

$$\gamma_{Q1} = \begin{pmatrix} -\frac{3}{N_c} & \frac{3C_F}{2N_c} & -\frac{1}{3N_c} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.153}$$

Q2 mixing

For the Q2 operator

$$Q_2 = Q_V^S + Q_A^S = \bar{u}\gamma_\mu d\bar{d}\gamma_\mu u + \bar{u}\gamma_\mu\gamma_5 d\bar{d}\gamma_\mu\gamma_5 u \tag{3.154}$$

we can as in the Q1 mixing use our former results. Noticing that Q2 is Q_-^S alike, we get from the current-current diagrams

$$Q_V^S + Q_A^S \rightarrow Q_A^O + Q_V^O \qquad \Rightarrow \qquad Q_+^S \rightarrow \frac{3}{2}Q_+^O. \tag{3.155}$$

$$\Gamma^{Q^S_{V+A}} = \frac{a_s}{\epsilon}\frac{3}{2}Q_+^O \tag{3.156}$$

Hence for the renormalization constant and the anomalous dimension we get

$$Z_{21} = \frac{3}{2} \qquad \text{and} \qquad \gamma_{21} = -3. \tag{3.157}$$

As for the Q1 mixing we also we also get a penguin contribution

$$\Gamma_{pen}^{Q_+^S} = -\frac{1}{3}\frac{a_s}{\epsilon}. \tag{3.158}$$

Thus the contributions lead to

$$Z_{23} = -\frac{2}{6} \qquad \text{and} \qquad \gamma_{23} = \frac{2}{3}. \tag{3.159}$$

For the total Q2 mixing we get

$$\gamma_{Q2} = \begin{pmatrix} 3 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.160}$$

Q3 mixing

Due to the similarity of the Q3 operator

$$Q_3 = (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a) \sum_{q=u,d,s} (\bar{q}\gamma^\mu t^a q). \tag{3.161}$$

to the singlet vector operator we get a contribution from the current-current diagrams of

$$\sum_{a,\dots,f} \Gamma_i = \frac{a_s}{\epsilon}\left[\frac{3N_c}{8}Q_3 - \frac{3C_F}{4N_c}Q_6 - \frac{3}{4}\left(\frac{N_c}{2} - \frac{2}{N_c}\right)Q_4\right] \tag{3.162}$$

As before we also have to regard the penguin contraction, but besides the penguin contraction we already have been calculating in subsec. (3.4), we also require a different penguin contraction given by

$$\Gamma^{(2)}_{pen} = q(x_1)\bar{q}(x_2)[\bar{q}q\bar{q}q](z)q(x_3)\bar{q}(x_4)[B\bar{q}q](y_1)[B\bar{q}q](y_2), \tag{3.163}$$

where we once again only displayed the fields we wanted to contract. Hence for the second type of penguin diagrams we get the following contribution

$$\begin{aligned}\Gamma^{(2)}_{pen} &= -g_s^2 Tr[t^a t^b]\mu^{2\epsilon}\int \frac{d^D s}{(2\pi)^D} Tr[S(p-s)\gamma^\lambda S(-s)\Gamma_2] D_{\lambda\sigma}(p)[\Gamma_1 t^a]^{\bar{u}u}[\gamma^\sigma]^{\bar{q}q} \\ &= -\frac{i}{2}g_s^2[\Gamma_1 t^a]^{\bar{u}u}[\gamma^\sigma]^{\bar{q}q}\mu^{2\epsilon}\int \frac{d^D s}{(2\pi)^D}\frac{s_\alpha(p-s)_\beta}{s^2(p-s)^2} Tr[\gamma^\beta\gamma^\lambda\gamma^\alpha\Gamma_2][g_{\lambda\sigma} - (1-a)\frac{p_\lambda p_\sigma}{p^2}]\frac{1}{p^2}.\end{aligned} \tag{3.164}$$

For the vector case $\Gamma_1 = \Gamma_2 = \gamma_\mu$ we get the same singularity as for the other penguin contraction eq. (3.145). Only the finite parts are different

$$\begin{aligned}\Gamma^{O}_{V,pen} &= -\frac{g_s^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon \frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}\Gamma(\epsilon)4\left\{[\gamma_\mu t^a]^{\bar{u}u}[\gamma^\mu t^a]^{\bar{q}q} - [\not{p}t^a]^{\bar{u}u}[\not{p}t^a]^{\bar{q}q}\frac{1}{p^2}\right\} \\ &= -\frac{a_s}{6}\left\{\frac{1}{\hat{\epsilon}} - ln\left(-\frac{p^2}{\mu^2}\right) + \frac{5}{3} + \mathcal{O}(\epsilon)\right\}\left\{[\gamma_\mu t^a]^{\bar{u}u}[\gamma^\mu t^a]^{\bar{q}q} - [\not{p}t^a]^{\bar{u}u}[\not{p}t^a]^{\bar{q}q}\frac{1}{p^2}\right\}.\end{aligned} \tag{3.165}$$

Having found the singularities of the two appearing penguin diagrams we still have three different possibilities of contracting the gluon-vertex.

Starting with the first we want to contract one of the gluon vertexes with the sum of $\bar{q}q$

$$\Gamma_{\bar{q}q} = q(x_1)\bar{q}(x_2)[(\bar{u}u + \bar{d}d)\sum \bar{q}q](z)q(x_3)\bar{q}(x_4)[\sum \bar{q}q](y_1)[\sum \bar{q}q](y_2). \tag{3.166}$$

We see that this contraction yields N_f times the contribution of the penguin diagram

$$\Gamma_{\bar{q}q} = -\frac{a_s}{\epsilon}\frac{N_f}{6}Q_3. \tag{3.167}$$

For the second contraction possibility we have a $u\bar{u}$ contraction of the gluon vertex. Hence

$$\Gamma_{\bar{u}u} = q(x_1)\bar{q}(x_2)[(\bar{u}u + \bar{d}d)\sum \bar{q}q](z)q(x_3)\bar{q}(x_4)[\sum \bar{q}q](y_1)[\sum \bar{q}q](y_2). \tag{3.168}$$

As the gluon vertex contraction of $\bar{u}u$ we get the same result contracting $d\bar{d}$

$$\Gamma_{\bar{d}d} = -\frac{a_s}{\epsilon}\frac{1}{3}\sum_q(\bar{q}\gamma_\mu t^a q)\sum_{q'}(\bar{q}'\gamma^\mu t^a q') = -\frac{a_s}{\epsilon}\frac{1}{3}Q_7. \tag{3.169}$$

The final contribution are two cross contractions

$$\Gamma_{cross} = q(x_1)\bar{q}(x_2)[(\bar{u}u+\bar{d}d)(\bar{u}u+\bar{d}d+\bar{s}s)](z)q(x_3)\bar{q}(x_4)[\sum \bar{q}q](y_1)[\sum \bar{q}q](y_2), \tag{3.170}$$

where we also could cross contract the $\bar{d}d$. Hence the contribution is denoted by

$$\bar{u}\gamma_\mu t^a u t^b \gamma^\lambda \bar{u}\gamma^\mu t^a u + \bar{d}\gamma_\mu t^a d t^b \gamma^\lambda \bar{d}\gamma^\mu t^a d, \tag{3.171}$$

yielding

$$\Gamma_{cross} = -\frac{a_s}{\epsilon}\frac{1}{N_c}\frac{1}{6}Q_3. \tag{3.172}$$

Multiplying every piece by -2 yields the third row of the anomalous dimension matrix $\hat{\gamma}^{(1)}$ entries (the Q_3 mixing)

$$\gamma_{Q3} = \begin{pmatrix} 0 & 0 & -\frac{3N_c}{4} + \frac{N_f}{3} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & \frac{3C_F}{2N_c} & \frac{2}{3} & 0 & 0 & 0. \end{pmatrix} \tag{3.173}$$

Q4 Mixing

As the procedure of finding the entries for the anomalous dimension matrix is going to repeat with ever mixing we will only go back into details if they are needed. For the Q4 mixing

$$Q_4 = (\bar{u}\gamma_\mu\gamma_5 t^a u + \bar{d}\gamma_\mu\gamma_5 d)\sum_{q=u,d,s}(\bar{q}\gamma^\mu\gamma_5 t^a q) \tag{3.174}$$

we get the current-current diagram contribution

$$\sum_{i=a,\ldots,f}\Gamma_i = \frac{a_s}{\epsilon}\left[\frac{3N_c}{8}Q_4 - \frac{3C_F}{4N_c}Q_5 - \frac{3}{4}\left(\frac{N_c}{2} - \frac{2}{N_c}\right)Q_3\right]. \tag{3.175}$$

Noticing that we have a mixing into the operator Q5. Q5 is in general an basic operator. However, in four dimensions, one operator in the full set is redundant and can be expressed through the others by means of Fierz transformations yielding the relation

$$Q_2 = \frac{\left(2Q_1 + 2Q_3 + 2Q_4 - \left(1 - \frac{1}{N_c}\right)(Q_5 + Q_6) - Q_7 - Q_8 - \left(1 - \frac{1}{N_c}\right)\left(\frac{Q_9 + Q_{10}}{2}\right)\right)}{1 - \frac{1}{N_c}}. \tag{3.176}$$

Using Mathematica lst. (6.3) we can solve for Q5 and substitute C_F through its N_c dependencies, giving us the input of the normal diagrams. Adding as well the contribution of the penguin diagrams

$$\Gamma_{cross} = \frac{a_s}{\epsilon}\frac{1}{6N_c}Q_3 \tag{3.177}$$

will give us the total input of the divergencies

$$\begin{array}{c|c|c} \Gamma_{Q_1} = -\frac{3}{4N_c} - \frac{3}{4} & \Gamma_{Q_2} = \frac{3C_F}{4N_c}, & \Gamma_{Q_3} = -\frac{3}{4} - \frac{3N_c}{8} + \frac{11}{12N_c}, \\ \hline \Gamma_{Q_4} = -\frac{3}{4N_c} - \frac{3}{4} + \frac{3N_c}{8}, & \Gamma_{Q_6} = \frac{3C_F}{4N_c}, & \Gamma_{Q_7} = \frac{3}{8N_c} + \frac{3}{8}, \\ \hline \Gamma_{Q_8} = \frac{3}{8N_c} + \frac{3}{8}, & \Gamma_{Q_9} = \frac{3C_F}{8N_c}, & \Gamma_{Q_{10}} = \frac{3C_F}{8N_c}. \end{array} \tag{3.178}$$

Hence for the anomalous dimension we get

$$\gamma_{Q4} = \begin{pmatrix} 0 & 0 & \frac{N_f}{3} - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & -\frac{3C_F}{2N_c} & -\frac{3}{4} - \frac{3}{4N_c} & -\frac{3}{4} - \frac{3}{4N_c} & \frac{3C_F}{4N_c} & \frac{3C_F}{4N_c} \end{pmatrix} \tag{3.179}$$

As we substituted the redundant operator Q_5 we will directly continue with Q_6.

Q6 Mixing

The Q6 operator is given by

$$Q_6 = (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d)\sum(\bar{q}\gamma^\mu\gamma_5 q). \tag{3.180}$$

Using former results we get

$$\sum_{i=a,\dots,f}\Gamma_i = -\frac{a_s}{\epsilon}\frac{3}{2}Q_3 \tag{3.181}$$

for the current-current diagrams. For the penguin contribution we only regard the first type of of penguin contractions. The Q6 operator has no octet structure and consequently the trace $Tr[t^a t^b t^a]$ in eq. (3.164) is going to vanish. Hence we only get an additional contribution of

$$\Gamma_{pen} = -\frac{a_s}{\epsilon}\frac{1}{3}Q_3 \tag{3.182}$$

and in total for the $\hat{\gamma}$ matrix

$$\gamma_{Q_6} = \begin{pmatrix}0 & 0 & \frac{11}{3}Q_3 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \tag{3.183}$$

Q7 Mixing

The Q_7 operator is given by

$$\sum_{q=u,d,s}(\bar{q}\gamma_\mu t^a q)\sum_{q'=u,d,s}(\bar{q}'\gamma^\mu t^a q'), \tag{3.184}$$

yielding the contributions

$$\frac{a_s}{\epsilon}\left[\frac{3N_c}{8}Q_7 - \frac{3C_F}{4N_c}Q_{10} - \frac{3}{4}\left(\frac{N_c}{2} - \frac{2}{N_c}\right)\right] \tag{3.185}$$

and

$$\Gamma^{Q_7}_{q\bar{q}} = \frac{a_s}{\epsilon}\frac{N_f}{6}Q_7, \qquad \Gamma^{Q_7}_{\bar{u}u} = -\frac{a_s}{\epsilon}\frac{1}{2}Q_7, \qquad \Gamma^{Q_7}_{cross} = \frac{a_s}{\epsilon}\frac{1}{6N_c}Q_7. \tag{3.186}$$

Consequently we get

$$\gamma_{Q7} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & \frac{N_f}{3} + 1 - \frac{3N_c}{4} - \frac{1}{3N_c} & -\frac{3}{4} - \frac{3}{4N_c} & \frac{3C_F}{4N_c} & \frac{3C_F}{4N_c}\end{pmatrix} \tag{3.187}$$

Q8 Mixing

Q8 is given by

$$Q_8 = \sum_{q=u,d,s} (\bar{q}\gamma_\mu \gamma_5 t^a q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu \gamma_5 t^a q). \tag{3.188}$$

Hence we get the following contributions

$$\sum_{i=a,\dots,f} \Gamma_i = \frac{a_s}{\epsilon} \left[\frac{3N_c}{8} Q_8 - \frac{3C_F}{4N_c} Q_9 - \frac{3}{4} \left(\frac{N_c}{2} - \frac{2}{N_c} \right) Q_7 \right] \tag{3.189}$$

for the current-current diagrams and

$$\Gamma^{Q_8}_{cross} = \frac{a_s}{\epsilon} \frac{1}{6N_c} Q_7 \tag{3.190}$$

for the penguin ones. The two additional contributions from the $\bar{q}q$ and $\bar{u}u$ contraction we applied do not exist. The additional γ_5 matrix let the trace appearing in eq. (3.164) disappear for an odd number of γ matrices. Hence in total we get for the anomalous dimension a contribution of

$$\gamma_{Q8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \left(\frac{3N_c}{4} - \frac{10}{3N_c} \right) & -\frac{3N_c}{4} & +\frac{3C_F}{2N_c} \end{pmatrix} \tag{3.191}$$

Q9 Mixing

The Q9 operator is given by

$$Q_9 = \sum_{q=u,d,s} (\bar{q}\gamma_\mu q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu q') \tag{3.192}$$

and gives us the contributions

$$\sum_{i=a,\dots,f} \Gamma_i = -\frac{a_s}{\epsilon} \frac{3}{2} Q_8 \qquad \text{and} \qquad cross_{Q_9} - \frac{a_s}{\epsilon} \frac{1}{3} Q_7 \tag{3.193}$$

Hence in total

$$\gamma_{Q9} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 3 & 0 \end{pmatrix}. \tag{3.194}$$

Q10 Mixing

Our final operator is given by

$$Q_{10} = \sum_{q=u,d,s} (\bar{q}\gamma_\mu \gamma_5 q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu \gamma_5 q') \tag{3.195}$$

contributing with the factors

$$\sum_{i=a,\dots,f} \Gamma_i = -\frac{a_s}{\epsilon}\frac{3}{2}Q_7 \qquad \text{and} \qquad \Gamma_{pen}^{Q_{10}} = -\frac{a_s}{\epsilon}\frac{1}{3}Q_7. \tag{3.196}$$

Hence for our anomalous dimension matrix we get

$$\gamma_{Q10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{11}{3} & 0 & 0 & 0 \end{pmatrix} \tag{3.197}$$

3.4.1 V+A Anomalous Dimension Matrix

Putting all former factors from the operator mixing together we get our desired matrix for the anomalous dimension.

$$\gamma_{Q_+}^{(1)} = \begin{pmatrix} -\frac{3}{N_C} & \frac{3C_F}{2N_C} & -\frac{1}{3N_C} & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N_f}{3} - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & \frac{3C_F}{2N_C} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & \frac{3N_c}{4} + \frac{3}{2} - \frac{11}{6N_c} & -\frac{3N_c}{4} + \frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & -\frac{3}{4} - \frac{3}{4N_c} & -\frac{3}{4} - \frac{3}{4N_c} & \frac{3C_F}{4N_c} & \frac{3C_F}{4N_c} \\ 0 & 0 & \frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{N_f}{3} + 1 - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & 0 & \frac{3C_F}{2N_c} \\ 0 & 0 & 0 & 0 & 0 & \frac{3N_c}{4} - \frac{10}{3N_c} & -\frac{3N_c}{4} & \frac{3C_F}{2N_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{11}{3} & 0 & 0 & 0 \end{pmatrix} \tag{3.198}$$

3.5 Confirming RGE

To check the RGE eq. (3.24) we just have to insert C_6^{V-A} eq. (3.19), C_6^{V+A} eq. (3.20) as well as the two former calculated anomalous dimension matrices of the first order, given by eq. (3.136) and eq. (3.198). Furthermore we want to remind that we have to deal with the total derivative of μ which is given in the RGE eq. (2.60) yielding

$$\begin{aligned} \mu\frac{d}{d\mu} &= \mu\frac{\partial}{\partial\mu} - \beta(a_s)\frac{\partial}{\partial a_s} \\ &= \mu\frac{\partial}{\partial\mu} - \beta_1 a_s^2\frac{\partial}{\partial a_s}, \end{aligned} \tag{3.199}$$

where we are only interested in the first order, with a give $\beta_1 = (11N_c - 2N_f)/6$ and neglected the partial derivative of the mass, as we are working in the massless case.

V-A Case

Starting with the V-A case we used a Mathematica script app. (6.5). Regarding the output the file confirmed the same structure for the left-hand side and right-hand side of eq. (3.24) we wanted to check for. The both sides are thus given by

$$6a_s^2\pi^2\begin{pmatrix} \frac{4}{N_c} - 2N_c \\ -1 + N_c^{-2} \end{pmatrix} \tag{3.200}$$

V+A Case

For the V+A case we are using the same file and get the approval of eq. (3.24) being fulfilled. The structure of the both equation sides are given by

$$a_s^2\pi^2\begin{pmatrix} \frac{24}{N_c} & -6 & \frac{6}{N_c^2} & \frac{88}{27N_c} + \frac{4N_c}{3} - \frac{16N_f}{27} & \frac{16}{3N_c} - \frac{4N_c}{3} & -\frac{4}{3} + \frac{4}{3N_c^2} & -\frac{32}{27} & 0 & 0 & 0 \end{pmatrix} \tag{3.201}$$

Chapter 4

Outlook

Having found an expression for the leading order of the anomalous dimension matrix $\hat{\gamma}^{(1)}$ we should now check for the in the introduction mentioned renormalons. Using the Borel transform to suppress higher orders of the coupling constant a_s we want to show an application of the from us yielded first order anomalous dimension matrix. The in the introduction mentioned singularities lead to ambiguities in the perturbation of higher OPE terms, which should not exist. The anomalous dimension matrix could therefore be one of the needed parameters to fix this ambiguities. Following [2] we want to show application of the anomalous dimension matrix, trying to describe the renormalon poles.

Comparing the energy dependence of a certain term in the OPE to the one of the complex ambiguity of the Borel integral the renormalon singularity that gives rise to this ambiguity can be determined. A generic term in the OPE of the physical Adler function $\hat{D}(s)$, given by

$$D(s) \equiv -s\frac{d}{ds}\Pi(s), \tag{4.1}$$

where $\Pi(s)$ denotes the not physical correlation function, from an operator O_d of dimension d can be written as

$$\hat{C}_{O_d}(A_Q)\frac{\langle\hat{O}_d\rangle}{Q^d} = [a_Q]^{\gamma^{(1)}_{O_d}\beta_1}\left[\hat{C}^{(0)}_{O_d} + \hat{C}^{(1)}_{O_d}a_Q + \hat{C}^(2)_{O_d}a_Q^2 + \cdot\right]\frac{\langle\hat{O}_d\rangle}{Q^d}, \tag{4.2}$$

where we used the scale invariant operator $\hat{O}_d$ defined by

$$\hat{O}_d \equiv O_d(\mu)\exp\left\{-\int\frac{\gamma_{O_d}(a_\mu)}{\beta(a_\mu)}da_\mu\right\}, \tag{4.3}$$

such that higher order coefficients of γ_{O_d} are contained in the Wilson coefficients $\hat{C}^{(k)}_{O_d}$. Employing the RGE for a_Q, the Q-dependence part of eq. (4.2) can be written as

$$\frac{\hat{C}_{O_d}(a_Q)}{Q^d} = const \times \hat{C}_{O_d}(a_Q) e^{-\frac{d}{\beta_1 a_Q}} \left[a_Q\right]^{-d\frac{\beta_2}{\beta_1^2}} \left[1 + b_1 a_Q + b_2 a_Q^2 + \cdots\right], \tag{4.4}$$

where the coefficients b_1 and b_2 are fount to be

$$b_1 = \frac{d}{\beta_1^3}(\beta_2^2 - \beta_1\beta_3), \qquad b_2 = \frac{b_1^2}{2} - \frac{d}{2\beta_1^4}(\beta_2^3 = 2\beta_1\beta_2\beta_3 + \beta_1^2\beta_4). \tag{4.5}$$

To find the Borel transform that matches the Q-dependence of eq. (4.4) one can take the ansatz

$$B[\hat{D}_p^{IR}](u) \equiv \frac{d_p^{IR}}{(p-u)^{1+\tilde{\gamma}}} \left[1 + \mathcal{O}(p-u)\right], \tag{4.6}$$

where we are only interested in the first order. If we wanted to investigate higher orders we needed to calculate the anomalous dimension for higher orders as well. Employing the imaginary ambiguity of $R_p^{IR}(\alpha)$, which has been derived in [2], yields the correspondent imaginary ambiguity corresponding to the Borel integral of $B[\hat{D}_p^{IR}](u)$

$$Im\left[\hat{D}_p^{IR}(a_Q)\right] = const. \times e^{-\frac{2p}{\beta_1 a_Q}} \left[a_Q\right]^{-\tilde{\gamma}} \left[1 + \mathcal{O}(a_Q^2)\right]. \tag{4.7}$$

Comparing eqs. (4.4) and (4.7), one deduces the following parameters

$$p = \frac{d}{2}, \qquad \tilde{\gamma} = 2p\frac{\beta_2}{\beta_1^2} - \frac{\gamma^{(1)}}{\beta_1}. \tag{4.8}$$

Noticing that all parameters except from the from us calculated anomalous dimension matrix $\gamma^{(1)}$ are known, we could now insert their values and compare the results with former computations neglecting the influence of the anomalous dimension matrix. Moreover the anomalous dimension will give us a hint about the strength of the renormalon singularities we are dealing with.

Our results of the leading order anomalous dimension-6 operators could be furthermore employed to improve the Borel model for the pertubative higher order behavior of vector- and axialvector-correlators [2]. This will be discussed in a forthcoming publication not published [3], which will be discussed in a forthcoming publication.

Chapter 5

Acknowledgments

I want to express my profound gratitude and deep regards to the following people for their support, patience and good humor.

- Prof. Dr. Matthias Jamin for his constant strong support and helpfulness.
- My fellow students Kai Keller, Iñigo Morcillo and especially Xabi Llobregat for all the good times physic had to offer.
- My family for being as supportive as ever, especially in my absent in Spain.

Chapter 6

Appendix

Mathematica Code

Listing 6.1: Mathematica: Integral solver

```
<< Tracer.m

(* Lorentz structure *)

L1 = {mu}.{nu};
L2 = p.{mu} p.{nu} + s.{mu} s.{nu};
L3 = p.{mu} s.{nu} + p.{nu} s.{mu};

RHS = L1 A + L2 B + L3 C;

Eq1L = I2[p s];
Eq1R = L1 RHS // Expand;

Eq2L = ((p.p^2 + s.s^2) I3[p,s] + 2(p.p + s.s + p.s)I2[p s]   (s.s + p.s)I2[p]   (p.p + p.s)I2[s])/4;
Eq2R = L2 RHS // Expand;

Eq3L = (p.p s.s I3[p,s] + (p.p + s.s + s.p)I2[p s]   p.p I2[p]   s.s I2[s])/2;
Eq3R = L3 RHS;

Eq1 = Eq1L == Eq1R;
Eq2 = Eq2L == Eq2R;
Eq3 = Eq3L == Eq3R;

Sol = Solve[{Eq1, Eq2, Eq3}, {A, B, C}];

rules = {I2[_] > 1/ep, I3[___] > 0};
Ex1 = Sol[[1]] /. rules;
```

```
Ex2 = Ex1 /. d >  42 ep;

A = Series[ Ex2 [[1,2]],  {ep, 0,  1}  ];
B = Series[ Ex2 [[2,2]],  {ep, 0,  1}  ];
C = Series[ Ex2 [[3,2]],  {ep, 0,  1}  ];
```

Listing 6.2: Mathematica: Tensor product

```
<< Tracer.m

VectorDimension[4];
(* Dirac Structures *)
DSc = G[l1, {la}, {ro}, {mu}] G[l2, {la}, {ro}, {mu}] / 16;
DSd = G[l1, {mu}, {ro}, {la}] G[l2, {mu}, {ro}, {la}] / 16;
DSe = G[l1, {la}, {ro}, {mu}] G[l2, {mu}, {ro}, {la}] / 16;
DSf = G[l1, {mu}, {ro}, {la}] G[l2, {la}, {ro}, {mu}] / 16;

DSG5c = G[l1, {la}, {ro}, {mu}, G5] G[l2, {la}, {ro}, {mu}, G5] / 16;
DSG5d = G[l1, {mu}, G5, {ro}, {la}] G[l2, {mu}, G5, {ro}, {la}] / 16;
DSG5e = G[l1, {la}, {ro}, {mu}, G5] G[l2, {mu}, G5, {ro}, {la}] / 16;
DSG5f = G[l1, {mu}, G5, {ro}, {la}] G[l2, {la}, {ro}, {mu}, G5] / 16;

DSList = {DSc, DSd, DSe, DSf};
Ex1 = ToDiracBasis[DSList] // Expand;
Ex2 = Ex1 /. {la > mu, ro > mu, idx16 > mu, idx19 > mu};
result = Ex2 /. List > Plus;
resultCD = Ex2[[1]] + Ex2[[2]];
resultEF = Ex2[[3]] + Ex2[[4]];

DSListG5 = {DSG5c, DSG5d, DSG5e, DSG5f};
Ex3 = ToDiracBasis[DSListG5] // Expand;
Ex4 = Ex3 /. {la>mu, ro>mu, idx40>mu, idx43>mu};
resultG5 = Ex4 /. List> Plus;
```

Listing 6.3: Mathematica: Q_4 Mixing

```
Ex = Q2 == ( 2 Q1 + 2 Q3 + 2 Q4  (11/Nc)(Q5 + Q6)  Q7  Q8 (11/Nc)(Q9 + Q10)/2
    )/(1  1/Nc);

Ex2 = Solve[Ex, Q5];
Ex3 =  Q5 3/(4 Nc) (Nc^2  1)/(2 Nc) /. Ex2[[1]] // Expand // Simplify;
```

Listing 6.4: Mathematica: Dimension-6 Wilson coefficient

```
(* anticommutating g5 structure *)

O1ACRule = O1AC > (1  as 11/6) 6 OAo + as 25/4 OVo + as 10/3 OVs + as 1/18 Q3;
O2ACRule = O2AC > (1  as 11/6) 6 OVo + as 25/4 OAo + as 10/3 OAs + as 1/18 Q3;
```

```

(* eq. 18 and eq. 19 *)

C6VQ6og2 =  2/9 ( 1 + as (95/72 L + 107/48) ) Q3
      1/3 (1 + as (9/8 L + 431/96 )) O1AC
      as/24 (16 L    12) OVs
      as/24 (30 L    45/2) OVo
      as/24 (16/9 L    8/27) Q7
      as/24 (16/9 L + 56/27) Q6
      as/24 (10/3 L + 35/9) Q4;

C6AQ6og2 =  2/9 ( 1 + as (95/72 L + 107/48) ) Q3
      1/3 (1 + as (9/8 L + 431/96 )) O2AC
      as/24 (16 L    12) OAs
      as/24 (30 L    45/2) OAo
      as/24 (16/9 L    8/27) Q7
      as/24 (16/9 L + 56/27) Q6
      as/24 (10/3 L + 35/9) Q4;

C6VmAQ6og2 = C6VQ6og2  C6AQ6og2;
C6VpAQ6og2 = C6VQ6og2 + C6AQ6og2;

c6VQ6og2 = C6VQ6og2 /. O1ACRule;
c6AQ6og2 = C6AQ6og2 /. O2ACRule;

c6VmAQ6og2 = (c6VQ6og2  c6AQ6og2 // Expand) /. as^2 > 0;
c6VpAQ6og2 = (c6VQ6og2 + c6AQ6og2 // Expand) /. as^2 > 0;
```

Listing 6.5: Mathematica: Check of RGE

```
(* Check of RGE *)

sRule = {Cf > (Nc^2   1)/2/Nc, b1 > 1/6 (11 Nc     2 Nf)}

(* V A, non singlet  *)

gamVmA = {{  3 Nc/2 + 3/Nc,  3 Cf/2/Nc }, {  3,  0 }};

C6VmA0 = 4Pi^2 as { 2, 0 };
C6VmA = 4Pi^2 as { 2 + (25/6  (b1 3Nc/2+3/Nc) LL) as, (* LL *)
                          (11/18      2/3 LL) as  };  (* ok *)

rgeVmARHS = Transpose[ as gamVmA ].C6VmA0 //Expand;
rgeVmALHS =  2 D[ C6VmA, LL ]  b1 as^2 D[ C6VmA0, as ] //Expand;

rgeVmARHSsub = rgeVmARHS /. sRule // Expand;
rgeVmALHSsub = rgeVmARHS /. sRule // Expand;

rgeVmAEqual = rgeVmALHSsub == rgeVmARHSsub;
```

```


(* V+A, nonsinglet *)

gamVpA = { { 3/Nc, 3Cf/2/Nc, 1/3/Nc, 0, 0, 0, 0, 0, 0 },
           { 3, 0, 2/3, 0, 0, 0, 0, 0, 0 },
           { 0, 0, 3 Nc/4+Nf/31/3/Nc, 3Nc/43/Nc, 3Cf/2/Nc, 2/3, 0, 0, 0 },
           { 3/2+3/2/Nc, 3Cf/2/Nc, 3Nc/4+3/211/6/Nc, 3Nc/4+3/2+3/2/Nc,
             3 Cf/2/Nc, 3/43/4/ Nc, 3/43/4/ Nc, 3Cf/4/Nc, 3Cf/4/Nc },
           { 0, 0, 11/3, 0, 0, 0, 0, 0, 0 },
           { 0, 0, 0, 0, 0, 3 Nc/4+Nf/3+11/3/Nc, 3Nc/43/Nc, 0, 3Cf/2/Nc },
           { 0, 0, 0, 0, 0, 3Nc/410/3/Nc, 3 Nc/4, 3Cf/2/Nc, 0 },
           { 0, 0, 0, 0, 0, 2/3, 3, 0, 0 },
           { 0, 0, 0, 0, 0, 11/3, 0, 0, 0 } };

C6VpA0 = 4Pi^2 as { 2, 0, 4/9, 0, 0, 0, 0, 0, 0 };
C6VpA = 4Pi^2 as { 2   + (155/24    (b13/ Nc) LL) as,  (* 7/2 LL *)
                         ( 11/18     2/3 LL ) as,        (* ok *)
                   4/9 + ( 37/36    (13Nc^222)/54/Nc LL ) as,  (* 95/162 LL *)
                         ( 35/108    5/18 LL ) as,       (* ok *)
                         ( 14/81     4/27 LL ) as,       (* ok *)
                         (  2/81  +  4/27 LL ) as,       (* ok *)
                   0,
                   0,
                   0 };

OVpA = { qpo, qps, q3, q4, q6, q7, q8, q9, q10 };

rgeVpARHS = Transpose[ as gamVpA ].C6VpA0 //Expand;
rgeVpALHS = 2 D[ C6VpA, LL ]  b1 as^2 D[ C6VpA0, as ] //Expand;

rgeVpARHSsub = rgeVpARHS /. sRule // Expand;
rgeVpALHSsub = rgeVpARHS /. sRule // Expand;

rgeVpAEqual = rgeVpALHSsub == rgeVpARHSsub;
```

Listing 6.6: Mathematica: Penguin 1

```
<< Tracer.m;

VectorDimension[42ep];

Ex = (1/2/(1ep) {al}.{be} p.p + p.{al} p.{be}) G[l1,{mu},{be},{la},{al},{mu}] G[l2,{
     om}] ({la}.{om}   (1 a) p.{la} p.{om}/p.p) // Expand;
Ex2 = Ex /. om > la // Simplify;
```

Chapter 7

Bibliography

[1] L.E. Adam and K.G. Chetyrkin. Renormalization of four-quark operators and qcd sum rules. *Phys. Lett. B329*, 129, 1994.

[2] Martin Beneke and Matthias Jamin. α s and the τ hadronic width: fixed-order, contour- improved and higher-order perturbation theory. *Journal of High Energy Physics*, 2008(09):044, 2008.

[3] M. Jamin D. Hornung, D Boito. Anomalous dimensions of 4-quark operators and renormalon structure of mesonic 2-point correlators. 2015.

[4] M. Jamin. Qcd and renormalisation group methods. 2006.

[5] V.P. Spiridonov L.V. Lanin and K.G. Chetyrkin. Contribution of four-quark condensates to sum rules for ρ and a_1 mesons. *Yad. Fiz. 44*, 1372, 1986.

[6] A.L Vainshtein M.A. Shiftman and V.I. Zakharov. *Phys. B147*, 385, 448, 519, 1979.

[7] R. Tarrach P. Pascual. *QCD: Renormalization for the Practitioner*. Springer-Verlag, Berlin, 1984.