current-current two-point functions

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= i\int \mathrm{d}^4 x e^{iqx}\langle 0|T\left\{\mathcal{J}^{\mu}_{ij}(x)\mathcal{J}^{\nu\dagger}_{ij}(0)\right\}\rangle \\
&= \left(q_\mu q_\nu = q^2 g_{\mu\nu}\right)\Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij,\mathcal{J}}(q^2) \\
&= \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi^{(1+0)}(q^2) + q^2 g_{\mu\nu}\Pi^{(0)}(q^2)
\end{aligned} \tag{1}
$$

Inclusive ratio:

$$
R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau \mathrm{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e]} \tag{2}
$$

$$
R_\tau = 12\pi S_{EW}\int_0^{m_\tau}\frac{\mathrm{d}s}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\left[\left(1+2\frac{s}{m_\tau^2}\right)\operatorname{Im}\Pi^{(1)}(s) + \operatorname{Im}\Pi^{(0)}(s)\right] \tag{3}
$$

$$
\Pi^{(J)}(s) \equiv \left|V_{uq}\right|^2\left(\Pi^{(J)}_{ud,V} + \Pi^{(J)}_{ud,A}(s)\right) \tag{4}
$$

OPE:

$$
\Pi^{(1+0)}_{OPE}(s) = \sum_{k=0}^{\infty}\frac{C_{2k}(s)}{(-s)^k} \tag{5}
$$

$$
\mathcal{O}_{4,V/A} = \frac{1}{12}\left[1-\frac{11}{18}a_s\right]\langle a_s GG\rangle + \left[1+\frac{\pm 36-23}{27}a_s\right]\langle (m_u+m_d)\bar{q}q\rangle \tag{6}
$$

Countour integral:

$$
\int_{s_{th}}^{s_0}\frac{\mathrm{d}s}{s_0}\omega(s)\operatorname{Im}\Pi_{V/A}(s) = \frac{i}{2}\oint_{|s|=s_0}\frac{\mathrm{d}s}{s_0}\omega(s)\Pi_{V/A}(s) \tag{7}
$$

adler function

$$
D(s) \equiv -s\frac{\mathrm{d}\Pi^{PT}}{\mathrm{d}s} = \frac{1}{12\pi^2}\sum_{n=0}^{\infty}a_\mu^n\sum_{k=1}^{n+1}kc_{nk}\ln\left(\frac{-s}{\mu^2}\right) \tag{8}
$$

contour integral in terms of adler

$$
A^{\omega,PT} = \frac{i}{2s_0}\oint_{|s|=s_o}\frac{\mathrm{d}s}{s}\left[W(s)-W(s_0)\right]D(s) \tag{9}
$$

aleph:

$$
\begin{aligned}
v_1(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2 S_{EW}}\frac{B(\tau^-\to V^-\nu_\tau)}{B(\tau^-\to e^-\bar{\nu}_e\nu_\tau)}\frac{\mathrm{d}N_V}{N_V\,\mathrm{d}s}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \\
a_1(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2 S_{EW}}\frac{B(\tau^-\to A^-\nu_\tau)}{B(\tau^-\to e^-\bar{\nu}_e\nu_\tau)}\frac{\mathrm{d}N_A}{N_A\,\mathrm{d}s}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \\
a_0(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2 S_{EW}}\frac{B(\tau^-\to \pi^-\nu_\tau)}{B(\tau^-\to e^-\bar{\nu}_e\nu_\tau)}\frac{\mathrm{d}N_A}{N_A\,\mathrm{d}s}\left(1-\frac{s}{m_\tau^2}\right)^2
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\mathrm{Im}\,\Pi^{(1)}_{\bar{u}d,V}(s) &= \frac{1}{2\pi} v_1(s) \\
\mathrm{Im}\,\Pi^{(1)}_{\bar{u}d,A}(s) &= \frac{1}{2\pi} a_1(s) \\
\mathrm{Im}\,\Pi^{(0)}_{\bar{u}d,A}(s) &= \frac{1}{2\pi} a_0(s)
\end{aligned}
\tag{11}
$$

chisquared:

$$
\chi^2(\alpha) = (I_i^{exp} - I_i^{th}(\alpha)) C_{ij}^{-1} (I_j^{exp} - I_j^{th}(\alpha)) \tag{12}
$$

used weights

$$
\omega_{wk}(s) = \left(1 - \frac{s}{m_\tau^2}\right)^{2+k} \left(\frac{s}{m_\tau^2}\right)^l \left(1 + \frac{2s}{m_\tau^2}\right) \tag{13}
$$

$(k, l) = (0, 0), (1, 0), (1, 1), (1, 2), (1, 3)$
weight to OPE:

$$
\frac{1}{2\pi i s_0} \oint_{|s|=s_0} \mathrm{d}s \left(\frac{s}{s_0}\right)^n \frac{C_{2k}}{(-s)^k} = (-1)^{n+1} \frac{C_{2(n+1)}}{s_0^{n+1}} \delta_{k,n+1} \tag{14}
$$

implying that an n-th degree monomial in the weight $\omega(s/s_0)$ selects the $D = 2k = 2(n+1)$ term in the OPE.

$$
\begin{aligned}
A^{ALEPH}_{00,V/A} &= A^{ALEPH}_{00,V/A}(a_s, \mathcal{O}_{6,V/A}) \\
A^{ALEPH}_{10,V/A} &= A^{ALEPH}_{10,V/A}(a_s, \langle a_s GG\rangle, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}) \\
A^{ALEPH}_{11,V/A} &= A^{ALEPH}_{11,V/A}(a_s, \langle a_s GG\rangle, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}\mathcal{O}_{12,V/A}) \\
A^{ALEPH}_{12,V/A} &= A^{ALEPH}_{12,V/A}(a_s, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}, \mathcal{O}_{12,V/A}, \mathcal{O}_{14,V/A}) \\
A^{ALEPH}_{13,V/A} &= A^{ALEPH}_{13,V/A}(a_s, \mathcal{O}_{10,V/A}, \mathcal{O}_{12,V/A}, \mathcal{O}_{14,V/A}, \mathcal{O}_{16,V/A})
\end{aligned}
\tag{15}
$$

α_s

Channel	$\alpha_s(m_\tau^2)$	$\langle a_s GG\rangle$
V+A (FOPT)	$0.319^{+0.010}_{-0.009}$	-3
V+A (CIPT)	0.339	-16