current-current two-point functions

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= i \int \mathrm{d}^4 x e^{iqx} \langle 0 | T \left\{ \mathcal{J}^{\mu}_{ij}(x) \mathcal{J}^{\nu\dagger}_{ij}(0) \right\} \rangle \\
&= \left(q_\mu q_\nu - q^2 g_{\mu\nu} \right) \Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij,\mathcal{J}}(q^2)
\end{aligned}
\tag{1}
$$

Inclusive ratio:

$$
R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e]} \tag{2}
$$

$$
R_\tau = 12\pi S_{EW} \int_0^{m_\tau} \frac{\mathrm{d}s}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2} \right) \left[\left(1 + 2\frac{s}{m_\tau^2} \right) \operatorname{Im} \Pi^{(1)}(s) + \operatorname{Im} \Pi^{(0)}(s) \right] \tag{3}
$$

$$
\Pi^{(J)}(s) \equiv \left| V_{uq} \right|^2 \left(\Pi^{(J)}_{ud,V} + \Pi^{(J)}_{ud,A}(s) \right) \tag{4}
$$

OPE:

$$
\Pi^{(1+0)}_{OPE}(s) = \sum_{k=0}^{\infty} \frac{C_{2k}(s)}{(-s)^k} \tag{5}
$$

$$
\mathcal{O}_{4,V/A} = \frac{1}{12} \left[1 - \frac{11}{18} a_s \right] \langle a_s GG \rangle + \left[1 + \frac{\pm 36 - 23}{27} a_s \right] \langle (m_u + m_d) \bar{q} q \rangle \tag{6}
$$

Dimension 6

$$
\gamma^{(1)}_{Q_+} = \left(\begin{array}{ccccccccc}
-\frac{3}{N_C} & \frac{3C_F}{2N_C} & -\frac{1}{3N_C} & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{N_f}{3} - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & \frac{3C_F}{2N_C} & \frac{2}{3} & 0 & 0 & 0 \\
\frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & \frac{3N_c}{4} + \frac{3}{2} - \frac{11}{6N_c} & -\frac{3N_c}{4} + \frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & -\frac{3}{4} - \frac{3}{4N_c} & -\frac{3}{4} - \frac{3}{4N_c} & \frac{3C_F}{4N_c} & \frac{3C_F}{4N_c} \\
0 & 0 & \frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{N_f}{3} + 1 - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & 0 & \frac{3C_F}{2N_c} \\
0 & 0 & 0 & 0 & 0 & \frac{3N_c}{4} - \frac{10}{3N_c} & -\frac{3N_c}{4} & \frac{3C_F}{2N_c} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{11}{3} & 0 & 0 & 0
\end{array} \right) \tag{7}
$$

Countour integral:

$$\int_{s_{th}}^{s_0} \frac{\mathrm{d}s}{s_0} \omega(s)\,\mathrm{Im}\,\Pi_{V/A}(s) = \frac{i}{2}\oint_{|s|=s_0} \frac{\mathrm{d}s}{s_0}\omega(s)\Pi_{V/A}(s) \tag{8}$$

adler function

$$D(s) \equiv -s\frac{\mathrm{d}\Pi^{PT}}{\mathrm{d}s} = \frac{1}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{nk} \ln\left(\frac{-s}{\mu^2}\right) \tag{9}$$

contour integral in terms of adler

$$A^{\omega,PT} = \frac{i}{2s_0}\oint_{|s|=s_o} \frac{\mathrm{d}s}{s}\left[W(s) - W(s_0)\right] D(s) \tag{10}$$

spectral function:

$$\rho(s) \equiv \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \tag{11}$$

aleph:

$$\begin{aligned}
v_1(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2\,S_{EW}} \frac{B(\tau^- \to V^-\nu_\tau)}{B(\tau^- \to e^-\bar{\nu}_e\nu_\tau)} \frac{\mathrm{d}N_V}{N_V\,\mathrm{d}s}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \\
a_1(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2\,S_{EW}} \frac{B(\tau^- \to A^-\nu_\tau)}{B(\tau^- \to e^-\bar{\nu}_e\nu_\tau)} \frac{\mathrm{d}N_A}{N_A\,\mathrm{d}s}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \\
a_0(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2\,S_{EW}} \frac{B(\tau^- \to \pi^-\nu_\tau)}{B(\tau^- \to e^-\bar{\nu}_e\nu_\tau)} \frac{\mathrm{d}N_A}{N_A\,\mathrm{d}s}\left(1-\frac{s}{m_\tau^2}\right)^2
\end{aligned} \tag{12}$$

$$\begin{aligned}
\mathrm{Im}\,\Pi^{(1)}_{\bar{u}d,V}(s) &= \frac{1}{2\pi}v_1(s) \\
\mathrm{Im}\,\Pi^{(1)}_{\bar{u}d,A}(s) &= \frac{1}{2\pi}a_1(s) \\
\mathrm{Im}\,\Pi^{(0)}_{\bar{u}d,A}(s) &= \frac{1}{2\pi}a_0(s)
\end{aligned} \tag{13}$$

chisquared:

$$\chi^2(\alpha) = (I_i^{exp} - I_i^{th}(\alpha))C_{ij}^{-1}(I_j^{exp} - I_j^{th}(\alpha)) \tag{14}$$

$$\begin{aligned}
I^{exp}_{i=kl}(s_k,\omega_l) &= \int_{s_{th}}^{s_k}\frac{\mathrm{d}s}{s_k}\omega_l(s)\,\mathrm{Im}\,\Pi_{V/A}(s) \\
I^{th}_{i=kl}(s_k,\omega_l) &= \frac{i}{2s_k}\oint_{|s|=s_k}\frac{\mathrm{d}s}{s}\left[W_l(s) - W_l(s_k)\right] D(s)
\end{aligned} \tag{15}$$

used weights

$$\omega_{kl}(s) = \left(1 - \frac{s}{m_\tau^2}\right)^{2+k} \left(\frac{s}{m_\tau^2}\right)^l \left(1 + \frac{2s}{m_\tau^2}\right) \tag{16}$$

$(k,l) = (0,0), (1,0), (1,1), (1,2), (1,3)$

weight to OPE:

$$\frac{1}{2\pi i s_0} \oint_{|s|=s_0} \mathrm{d}s \left(\frac{s}{s_0}\right)^n \frac{C_{2k}}{(-s)^k} = (-1)^{n+1} \frac{C_{2(n+1)}}{s_0^{n+1}} \delta_{k,n+1} \tag{17}$$

implying that an n-th degree monomial in the weight $\omega(s/s_0)$ selects the $D = 2k = 2(n+1)$ term in the OPE.

$$\frac{s}{s_0} \quad + \quad \left(\frac{s}{s_0}\right)^2 \quad + \quad \left(\frac{s}{s_0}\right)^3 \quad + \quad \cdots \tag{18}$$

$$\begin{aligned}
A^{ALEPH}_{00,V/A} &= A^{ALEPH}_{00,V/A}(a_s, \mathcal{O}_{6,V/A}) \\
A^{ALEPH}_{10,V/A} &= A^{ALEPH}_{10,V/A}(a_s, \langle a_s GG \rangle, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}) \\
A^{ALEPH}_{11,V/A} &= A^{ALEPH}_{11,V/A}(a_s, \langle a_s GG \rangle, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A} \mathcal{O}_{12,V/A}) \\
A^{ALEPH}_{12,V/A} &= A^{ALEPH}_{12,V/A}(a_s, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}, \mathcal{O}_{12,V/A}, \mathcal{O}_{14,V/A}) \\
A^{ALEPH}_{13,V/A} &= A^{ALEPH}_{13,V/A}(a_s, \mathcal{O}_{10,V/A}, \mathcal{O}_{12,V/A}, \mathcal{O}_{14,V/A}, \mathcal{O}_{16,V/A})
\end{aligned} \tag{19}$$

duality violations

$$\Delta\rho^{DV}_{V/A}(s) = e^{-\delta_{V/A} + \gamma_{V/A} s)} \sin(\alpha_{V/A} + \beta_{V/A} s) \tag{20}$$

$$\Delta A^{\omega,DV}_{V/A}(s_0) \equiv \frac{i}{2} \oint_{|s_0|=s_0} \frac{\mathrm{d}s}{s_0} \omega(s) \left\{ \Pi_{V/A}(s) - \Pi^{OPE}_{V/A}(s) \right\} = -\pi \int_{s_0}^{\infty} \frac{\mathrm{d}s}{s_0} \omega(s) \Delta\rho^{DV}_{V/A}(s) \tag{21}$$

α_s

Channel	$\alpha_s(m_\tau^2)$	$\langle a_s GG \rangle$
V+A (FOPT)	$0.319^{+0.010}_{-0.009}$	-3
V+A (CIPT)	0.339	-16

OPE