current-current two-point functions

$$
\begin{aligned}
\Pi_{\mu\nu}(q) &= i\int \mathrm{d}^4\, x e^{iqx} \langle 0|T\left\{\mathcal{J}_{ij}^{\mu}(x)\mathcal{J}_{ij}^{\nu\dagger}(0)\right\}\rangle \\
&= \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi_{ij,\mathcal{J}}^{(1)}(q^2) + q^\mu q^\nu \Pi_{ij,\mathcal{J}}^{(0)}(q^2)
\end{aligned}
\tag{1}
$$

Inclusive ratio:

$$
R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e]} \tag{2}
$$

$$
R_\tau = 12\pi S_{EW}\int_0^{m_\tau} \frac{\mathrm{d}s}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right] \tag{3}
$$

$$
\Pi^{(J)}(s) \equiv \left|V_{uq}\right|^2 \left(\Pi_{ud,V}^{(J)} + \Pi_{ud,A}^{(J)}(s)\right) \tag{4}
$$

OPE:

$$
\Pi_{OPE}^{(1+0)}(s) = \sum_{k=0}^{\infty} \frac{C_{2k}(s)}{(-s)^k} \tag{5}
$$

$$
\mathcal{O}_{4,V/A} = \frac{1}{12}\left[1-\frac{11}{18}a_s\right]\langle a_s GG\rangle + \left[1+\frac{\pm 36-23}{27}a_s\right]\langle (m_u+m_d)\bar{q}q\rangle \tag{6}
$$

D=6: Anomalous Dimension Matrix (V-A):

$$
\hat{\gamma}_{Q_-}^{(1)} = \begin{pmatrix} -\frac{3N_C}{2}+\frac{3}{N_C} & -\frac{3C_F}{2N_C} \\ -3 & 0 \end{pmatrix} \tag{7}
$$

Anomalous Dimension Matrix (V+A):

$$\gamma^{(1)}_{Q_+} = \begin{pmatrix} -\frac{3}{N_C} & \frac{3C_F}{2N_C} & -\frac{1}{3N_C} & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N_f}{3} - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & \frac{3C_F}{2N_C} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & \frac{3N_c}{4} + \frac{3}{2} - \frac{11}{6N_c} & -\frac{3N_c}{4} + \frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & -\frac{3}{4} - \frac{3}{4N_c} & -\frac{3}{4} - \frac{3}{4N_c} & \frac{3C_F}{4N_c} & \frac{3C_F}{4N_c} \\ 0 & 0 & \frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{N_f}{3} + 1 - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & 0 & \frac{3C_F}{2N_c} \\ 0 & 0 & 0 & 0 & 0 & \frac{3N_c}{4} - \frac{10}{3N_c} & -\frac{3N_c}{4} & \frac{3C_F}{2N_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{11}{3} & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

non-diagonal V/A current

$$j^V_\mu(x) = (\bar{u}\gamma_\mu d)(x) \tag{9}$$

$$j^A_\mu(x) = (\bar{u}\gamma_\mu\gamma_5 d)(x) \tag{10}$$

Countour integral:

$$\int_{s_{th}}^{s_0} \frac{\mathrm{d}s}{s_0}\omega(s)\,\mathrm{Im}\,\Pi_{V/A}(s) = \frac{i}{2}\oint_{|s|=s_0} \frac{\mathrm{d}s}{s_0}\omega(s)\Pi_{V/A}(s) \tag{11}$$

adler function

$$D(s) \equiv -s\frac{\mathrm{d}\Pi^{PT}}{\mathrm{d}s} = \frac{1}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{nk} \ln\left(\frac{-s}{\mu^2}\right) \tag{12}$$

contour integral in terms of adler

$$A^{\omega,PT} = \frac{i}{2s_0}\oint_{|s|=s_o} \frac{\mathrm{d}s}{s}\left[W(s) - W(s_0)\right] D(s) \tag{13}$$

spectral function:

$$\rho(s) \equiv \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \tag{14}$$

normalized invariant mass-squared distribution

$$\left(\frac{1}{N_{V/A}}\right)\left(\frac{\mathrm{d}N_{V/A}}{\mathrm{d}s}\right) \tag{15}$$

aleph:

$$\begin{aligned}
v_1(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2\,S_{EW}}\,\frac{B(\tau^-\to V^-\nu_\tau)}{B(\tau^-\to e^-\bar{\nu}_e\nu_\tau)}\,\frac{\mathrm{d}N_V}{N_V\,\mathrm{d}s}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1}\\
a_1(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2\,S_{EW}}\,\frac{B(\tau^-\to A^-\nu_\tau)}{B(\tau^-\to e^-\bar{\nu}_e\nu_\tau)}\,\frac{\mathrm{d}N_A}{N_A\,\mathrm{d}s}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1}\\
a_0(s) &\equiv \frac{m_\tau^2}{6|V_{ud}|^2\,S_{EW}}\,\frac{B(\tau^-\to \pi^-\nu_\tau)}{B(\tau^-\to e^-\bar{\nu}_e\nu_\tau)}\,\frac{\mathrm{d}N_A}{N_A\,\mathrm{d}s}\left(1-\frac{s}{m_\tau^2}\right)^2
\end{aligned} \tag{16}$$

$$\begin{aligned}
\mathrm{Im}\,\Pi^{(1)}_{\bar{u}d,V}(s) &= \frac{1}{2\pi}v_1(s)\\
\mathrm{Im}\,\Pi^{(1)}_{\bar{u}d,A}(s) &= \frac{1}{2\pi}a_1(s)\\
\mathrm{Im}\,\Pi^{(0)}_{\bar{u}d,A}(s) &= \frac{1}{2\pi}a_0(s)
\end{aligned} \tag{17}$$

chisquared:

$$\chi^2(\alpha) = (I_i^{exp} - I_i^{th}(\alpha))C_{ij}^{-1}(I_j^{exp} - I_j^{th}(\alpha)) \tag{18}$$

$$\begin{aligned}
I^{exp}_{i=kl}(s_k,\omega_l) &= \int_{s_{th}}^{s_k}\frac{\mathrm{d}s}{s_k}\omega_l(s)\,\mathrm{Im}\,\Pi_{V/A}(s)\\
I^{th}_{i=kl}(s_k,\omega_l) &= \frac{i}{2s_k}\oint_{|s|=s_k}\frac{\mathrm{d}s}{s}\left[W_l(s)-W_l(s_k)\right]D(s)
\end{aligned} \tag{19}$$

used weights

$$\omega_{kl}(s) = \left(1-\frac{s}{m_\tau^2}\right)^{2+k}\left(\frac{s}{m_\tau^2}\right)^{l}\left(1+\frac{2s}{m_\tau^2}\right) \tag{20}$$

$(k,l) = (0,0),(1,0),(1,1),(1,2),(1,3)$

weight to OPE:

$$\frac{1}{2\pi i s_0}\oint_{|s|=s_0}\mathrm{d}s\left(\frac{s}{s_0}\right)^n\frac{C_{2k}}{(-s)^k} = (-1)^{n+1}\frac{C_{2(n+1)}}{s_0^{n+1}}\delta_{k,n+1} \tag{21}$$

implying that an $n$-th degree monomial in the weight $\omega(s/s_0)$ selects the $D = 2k = 2(n+1)$ term in the OPE.

$$\frac{s}{s_0} \quad + \quad \left(\frac{s}{s_0}\right)^2 \quad + \quad \left(\frac{s}{s_0}\right)^3 \quad + \quad \cdots \tag{22}$$

$$
\begin{aligned}
A^{ALEPH}_{00,V/A} &= A^{ALEPH}_{00,V/A}(a_s, \mathcal{O}_{6,V/A}) \\
A^{ALEPH}_{10,V/A} &= A^{ALEPH}_{10,V/A}(a_s, \langle a_s GG \rangle, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}) \\
A^{ALEPH}_{11,V/A} &= A^{ALEPH}_{11,V/A}(a_s, \langle a_s GG \rangle, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}\mathcal{O}_{12,V/A}) \\
A^{ALEPH}_{12,V/A} &= A^{ALEPH}_{12,V/A}(a_s, \mathcal{O}_{6,V/A}, \mathcal{O}_{10,V/A}, \mathcal{O}_{12,V/A}, \mathcal{O}_{14,V/A}) \\
A^{ALEPH}_{13,V/A} &= A^{ALEPH}_{13,V/A}(a_s, \mathcal{O}_{10,V/A}, \mathcal{O}_{12,V/A}, \mathcal{O}_{14,V/A}, \mathcal{O}_{16,V/A})
\end{aligned} \tag{23}
$$

duality violations

$$
\Delta\rho^{DV}_{V/A}(s) = e^{-\delta_{V/A} + \gamma_{V/A} s} \sin(\alpha_{V/A} + \beta_{V/A} s) \tag{24}
$$

$$
\Delta A^{\omega,DV}_{V/A}(s_0) \equiv \frac{i}{2} \oint_{|s_0|=s_0} \frac{\mathrm{d}s}{s_0} \omega(s) \left\{ \Pi_{V/A}(s) - \Pi^{OPE}_{V/A}(s) \right\} = -\pi \int_{s_0}^{\infty} \frac{\mathrm{d}s}{s_0} \omega(s) \Delta\rho^{DV}_{V/A}(s) \tag{25}
$$

Pich 2016 $\alpha_s$ V+A from five weights

| Channel | $\alpha_s(m_\tau^2)$ | $\langle a_s GG \rangle$ $(10^{-3} GeV^4)$ | $\mathcal{O}_6$ $(10^{-3} GeV^6)$ | $\mathcal{O}_8$ $(10^{-3} GeV^8)$ |
|---|---|---|---|---|
| V+A (FOPT) | $0.319^{+0.010}_{-0.006}$ | $-3^{+6}_{-11}$ | $1.3^{+1.4}_{-0.8}$ | $-0.8^{+0.4}_{-0.7}$ |
| V+A (CIPT) | $0.339^{+0.011}_{-0.009}$ | $-16^{+5}_{-5}$ | $0.9^{+0.3}_{-0.4}$ | $-1.0^{0.5}_{-0.7}$ |

Boito 2015 $\alpha_s$ V+A from three weights, multiple s0 starting by $s_{min}$

| Channel | $s_{min}(GeV^2)$ | $\alpha_s(m_\tau^2)$ | $\mathcal{O}_{6V,A}$ | $\mathcal{O}_{8V,A}$ | $\delta_{V,A}$ | $\gamma_{V,A}$ | $\alpha_{V,A}$ | $\beta_{V,A}$ |
|---|---|---|---|---|---|---|---|---|
| V+A (FOPT) | 1.550 | 0.292(9) | −0.90(13) | 1.57(22) | 3.19(51) | 0.80(30) | −2.65(79) | 4.42(41) |
| | | | −0.63(61) | 3.0(2.2) | 1.53(56) | 1.42(24) | 5.73(84) | 1.84(43) |
| V+A (CIPT) | 1.550 | 0.312(13) | −0.90(13) | 1.48(25) | 3.35(49) | 0.70(29) | −2.28(81) | 4.23(42) |
| | | | 1.59(55) | 1.44(25) | 5.37(89) | 2.03(46) | −0.33(56) | 2.0(1.8) |