Trigonometric Functions - Sum & Difference of Angles

1. $\cos(-x) = \cos x$
2. $\sin(-x) = -\sin x$
3. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
4. $\cos(x - y) = \cos x \cos y + \sin x \sin y$
5. $\cos(\pi/2 - x) = \sin x$
6. $\sin(\pi/2 - x) = \cos x$
7. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
8. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
9. $\cos(\pi/2 + x) = -\sin x$
10. $\sin(\pi/2 + x) = \cos x$
11. $\cos(\pi - x) = -\cos x$
12. $\sin(\pi - x) = \sin x$
13. $\cos(\pi + x) = -\cos x$
14. $\sin(\pi + x) = -\sin x$
15. $\cos(2\pi - x) = \cos x$
16. $\sin(2\pi - x) = -\sin x$
17. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

    x, y and $x+y$ should not be odd multiples of $\pi/2$.
18. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
19. $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

    x, y and $x+y$ should not be multiples of π
20. $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$
21. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$
22. $\sin 2x = 2 \sin x = \dfrac{2 \tan x}{1 + \tan^2 x}$
23. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
24. $\sin 3x = 3 \sin x - 4 \sin^3 x$
25. $\cos 3x = 4 \cos^3 x - 3 \cos x$
26. $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

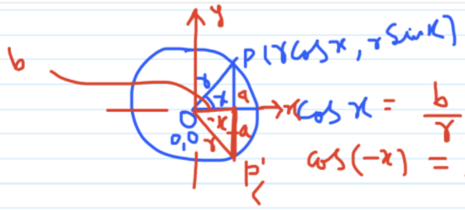
27. $\cos x + \cos y = 2\cos\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$

28. $\cos x - \cos y = -2\sin\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$

29. $\sin x + \sin y = 2\sin\frac{(x+y)}{2}\cos\frac{(x-y)}{2}$

30. $\sin x - \sin y = 2\cos\frac{(x+y)}{2}\sin\frac{(x-y)}{2}$

31. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$
$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

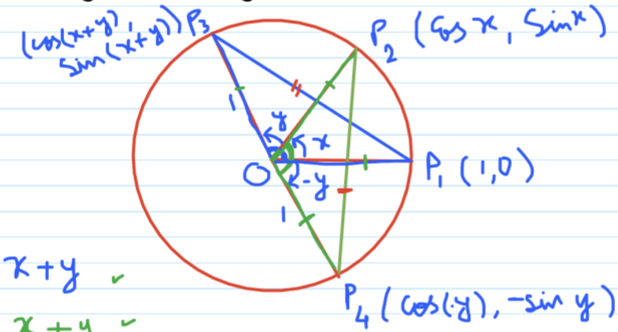


$\cos x = \frac{b}{r}$

$\cos(-x) = \frac{b}{r}$

$\sin -x = -\frac{a}{r} = -\sin$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

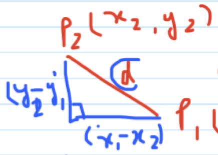


$\angle P_1OP_3 = x+y$

$\angle P_2OP_4 = x+y$

$\Delta P_1OP_3 \triangleq \Delta P_2OP_4$ → congruent (SAS congruence)

$\Rightarrow P_2P_4^2 = P_1P_3^2$



$d^2 = (x_1-x_2)^2 + (y_1-y_2)^2$

$\Rightarrow (\cos x - \cos y)^2 + (\sin x + \sin y)^2 =$

$$(\cos(x+y) - 1)^2 + (\sin(x+y))^2$$

$\Rightarrow \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$

$= \cos^2(x+y) + 1 - 2\cos(x+y) + \sin^2(x+y)$

$\sin^2\theta + \cos^2\theta = 1$ ✓

$\Rightarrow \cancel{1} + \cancel{1} \cancel{-2}(\cos x \cos y - \sin x \sin y) = \cancel{1} + \cancel{1} \cancel{-2} \cos(x+y)$

$\Rightarrow \boxed{\cos(x+y) = \cos x \cos y - \sin x \sin y}$ ☆

$\cos(x - y) = \cos x \cos y + \sin x \sin y$ ✓

put y as $-y$ in $\cos(x+y)$

$\cos x \cos(-y) - \sin x \sin(-y)$

$= \cos x \cos y + \sin x (+\sin y)$

$= \cos x \cos y + \sin x \sin y$ ✓

✓ $\cos(\underbrace{\pi/2}_{x} - \underbrace{x}_{y}) = \sin x$ ✓

$= \cancel{\cos \frac{\pi}{2}}^{0} \cos x + \cancel{\sin \frac{\pi}{2}}^{1} \sin x$

$= \sin x$

✓ $\sin(\underbrace{\frac{\pi}{2} - x}_{\alpha}) = \cos x$

$= \cos\left(\cancel{\frac{\pi}{2}} - \cancel{\frac{\pi}{2}} + x\right)$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x+y)$

$= \cos(\pi/2 - (x + y))$

$= \cos(\pi/2 - x - y)$

$= \cos((\pi/2 - x) - y)$

$= \cos(\pi/2 - x)\cos(y) + \sin(\pi/2 - x)\sin y$

$= \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\sin x \cos(-y) + \cos x \sin(-y)$

$= \sin x \cos y - \cos x \sin y$

$\cos(\pi/2 + x) = -\sin x$

$\cos(\pi/2 + x)$

$= \cos(\pi/2).\cos x - \sin(\pi/2) \sin x$

$= 0 - \sin x$

$= -\sin x$

$\sin(\pi/2 + x)$

$= \sin(\pi/2).\cos x + \cos(\pi/2).\sin x$

$= \cos x + 0$

$= \cos x$

$\cos(\pi - x)$

$= \cos\pi \cos x + \sin \pi \sin x$

$= -\cos x + 0$

$= -\cos x$

$\sin(\pi - x)$

$= \sin \pi \cos x - \cos \pi \sin x$

$= 0 + \sin x$

$= \sin x$

$\cos(\pi + x)$
$= \cos\pi \cos x - \sin\pi \sin x$
$= -\cos x - 0$
$= -\cos x$

$\sin(\pi + x)$
$= \sin\pi \cos x + \cos\pi \sin x$
$= 0 + -\sin x$
$= -\sin x$

$\cos(2\pi - x)$
$= \cos(2\pi)\cos x + \sin(2\pi)\sin x$
$= \cos x + 0$
$= \cos x$

$\sin(2\pi - x)$
$= \sin(2\pi)\cos(x) - \cos(2\pi).\sin x$
$= 0 - \sin x$
$= -\sin x$

$\tan(x + y)$
$= \sin(x+y)/\cos(x+y)$

$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

$=$ divide by $\cos x \cos y$

$$= \frac{\dfrac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \dfrac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cos y}}{\dfrac{\cancel{\cos x \cos y}}{\cancel{\cos x \cos y}} - \left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{\sin y}{\cos y}\right)}$$

$$= \frac{\tan x + \tan y}{1 - \left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{\sin y}{\cos y}\right)}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \underline{\tan(x+y)}$$

$$\tan(-y) = \frac{\sin(-y)}{\cos(-y)} = -\left(\frac{\sin y}{\cos y}\right) = -\tan y$$

$$\tan(x-y)$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)}$$

$$= \frac{\tan x - \tan y}{1 + \tan x \tan y} \checkmark$$

$$\cot(x+y)$$

$$= \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

$$= \frac{\frac{\cos x}{\sin x} \cdot \frac{\cos y}{\sin y} - 1}{\frac{\cos y}{\sin y} + \frac{\cos x}{\sin x}}$$

[Dividing N^r & D^r by $\sin x \sin y$]

$$= \frac{\cot x \cot y - 1}{\cot x + \cot y} \quad \cot(x+y)$$

$$\cot(-x) = \frac{\cos x}{-\sin x} = -\cot x$$

$$\cot(x-y) = \frac{\cot x \cot(-y) - 1}{\cot x + \cot(-y)}$$

$$= \frac{-\cot x \cot y - 1}{\cot x - \cot y}$$

$$= \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

$$\cos 2x = \cos(x+x)$$

$$= \underline{\cos^2 x - \sin^2 x} \quad \text{———} \quad (1)$$

$$= \cos^2 x - (1 - \cos^2 x)$$

$$= \cos^2 x - 1 + \cos^2 x$$

$$= \underline{2\cos^2 x - 1} \quad \text{———} \quad (2)$$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$$

$$= \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\sin^2 x}{\cos^2 x}} \rightarrow \left(\frac{\sin x}{\cos x}\right)^2 = \tan^2 x$$

$$= \frac{1 - \tan^2 x}{1 + \tan^2 x} \quad \text{———} \quad (3)$$

$$\sin \underline{2x}$$

$$= \sin(x+x)$$

$$= \sin x \cdot \cos x + \cos x \sin x$$

$$= 2 \underline{\sin x} \, \underline{\cos x} \quad \text{———} \quad (1)$$

$$= \text{divide by } 1 \; [\sin^2 x + \cos^2 x]$$

$$\frac{2 \sin x \cos x}{\sin^2 x + \cos^2 x} \leftarrow$$

$$= \frac{\frac{2\sin x \cdot \cancel{\cos x}}{\cancel{\cos^2} x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}} \nearrow 1} = \frac{2\tan x}{1 + \tan^2 x} \quad \text{———} \quad (2)$$

$\tan 2x$

$= \tan(x+x)$

$= \dfrac{\tan x + \tan x}{1 - \tan x \cdot \tan x}$

$= \dfrac{2\tan x}{1-\tan^2 x}$ ✓

$\sin 3x$

$= \sin(2x+x)$

$= \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin x \cos x \cos x + (\cos^2 x - \sin^2 x)\sin x$

$= 2\sin x \cos^2 x + (1 - \sin^2 x - \sin^2 x)\sin x$

$= 2\sin x(1-\sin^2 x) + \sin x(1 - 2\sin^2 x)$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= \boxed{3\sin x - 4\sin^3 x}$

$\cos 3x$

$= \cos(2x+x)$

$= \cos 2x \cdot \cos x - \sin 2x \cdot \sin x$

$= (2\cos^2 - 1)\cos x - 2\sin x \cos x \cdot \sin x$

$= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$= \boxed{4\cos^3 x - 3\cos x}$

$$\tan 3x$$

$$= \tan(2x + x)$$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \cdot \tan x}$$

$$= \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x}{1-\tan^2 x}\tan x}$$

$$= \frac{2\tan x + \tan x(1-\tan^2 x)}{1 - \tan^2 x - 2\tan^2 x}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$= \left[\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right]$$

$$\cos x + \cos y = 2\cos\left(\overset{A}{\frac{x+y}{2}}\right)\cos\left(\overset{B}{\frac{x-y}{2}}\right) \quad \checkmark$$

$$A + B = x$$
$$A - B = y$$

Let.

$$A = \frac{x+y}{2} \quad \checkmark$$

$$B = \frac{x-y}{2}; \text{ then}$$

$$x = A + B$$
$$y = A - B$$

$\cos(A+B) \quad + \cos(A-B)$

$= \cos A \cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B}$

$= 2 \cos A \cos B$

$= 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$

$\cos x - \cos y$

$\Rightarrow \cos(A+B) - \cos(A-B)$

$= \cos A \cos B - \sin A \sin B \quad - (\cos A \cos B + \sin A \sin B)$

$= \cancel{\cos A \cos B} - \sin A \sin B - \cancel{\cos A \cos B} - \sin A \sin B$

$= -2 \sin A \sin B$

$= -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$ ✓

$\sin x + \sin y$

$= \sin(A+B) + \sin(A-B)$

$= \sin A \cos B + \cancel{\cos A \sin B} + \sin A \cos B - \cancel{\cos A \sin B}$

$= 2 \sin A \cos B$

$= 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$

$\sin x - \sin y$

$= \sin(A+B) - \sin(A-B)$

$= \cancel{\sin A \cos B} + \cos A \sin B - \cancel{\sin A \cos B} + \cos A \sin B$

$= 2 \cos A \sin B$

$= 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$ ✓

$$2\cos A \cos B = \underbrace{\cos(\overset{x}{A+B}) + \cos(\overset{y}{A-B})}$$

$$\Downarrow$$

$$2\cos\left(\frac{A+\cancel{B}+A-\cancel{B}}{2}\right)\cos\left(\frac{\cancel{A}+B-\cancel{A}+B}{2}\right)$$

$$= \underbrace{2\cos(A)\cos B}$$