

Trigonometric Functions - Sum & Difference of Angles

1. $\cos(-x) = \cos x$
2. $\sin(-x) = -\sin x$
3. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
4. $\cos(x - y) = \cos x \cos y + \sin x \sin y$
5. $\cos(\pi/2 - x) = \sin x$
6. $\sin(\pi/2 - x) = \cos x$
7. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
8. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
9. $\cos(\pi/2 + x) = -\sin x$
10. $\sin(\pi/2 + x) = \cos x$
11. $\cos(\pi - x) = -\cos x$
12. $\sin(\pi - x) = \sin x$
13. $\cos(\pi + x) = -\cos x$
14. $\sin(\pi + x) = -\sin x$
15. $\cos(2\pi - x) = \cos x$
16. $\sin(2\pi - x) = -\sin x$
17. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
 x, y and $x + y$ should not be odd multiples of $\pi/2$.
18. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$
19. $\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$
 x, y and $x + y$ should not be multiples of π
20. $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$
21. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = \frac{1 - \tan^2 x}{1 + \tan^2 x}$
22. $\sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$
23. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$
24. $\sin 3x = 3 \sin x - 4 \sin^3 x$
25. $\cos 3x = 4 \cos^3 x - 3 \cos x$
26. $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

$$27. \cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$28. \cos x - \cos y = -2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$29. \sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

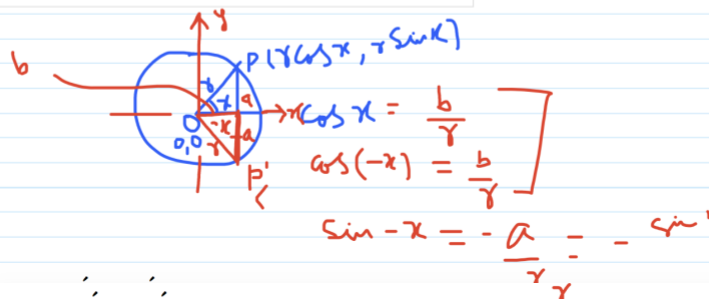
$$30. \sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$31. 2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

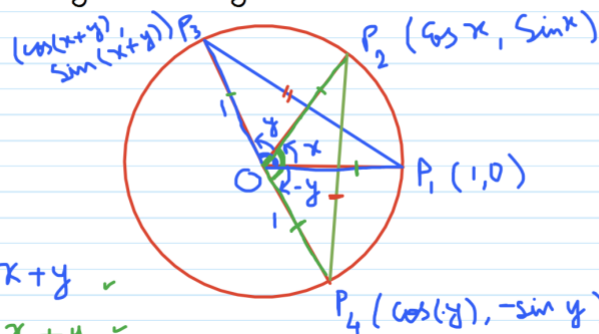
$$-2 \sin A \sin B = \cos(A+B) - \cos(A-B)$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$



$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$



$$\angle P_1 O P_3 = x+y$$

$$\angle P_2 O P_4 = x+y$$

$$\triangle P_1 O P_3 \cong \triangle P_2 O P_4 \rightarrow \text{congruent (SAS congruence)}$$

$$\Rightarrow P_2 P_4^2 = P_1 P_3^2$$

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Diagram illustrating the distance formula. A point $P_1(x_1, y_1)$ is marked on the circle. A point $P_2(x_2, y_2)$ is marked on the circle. The diagram shows the relationship between the coordinates (x_1, y_1) and (x_2, y_2) and the distance d between the points. The diagram also shows the coordinates for the point $P_1(x_1, y_1)$.

$$\Rightarrow (\cos x - \cos y)^2 + (\sin x + \sin y)^2 =$$

$$(\cos(x+y) - 1)^2 + (\sin(x+y))^2$$

$$\Rightarrow \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$

$$= \underbrace{\cos^2(x+y)} + 1 - 2\cos(x+y) + \underbrace{\sin^2(x+y)}$$

$$\sin^2 \theta + \cos^2 \theta = 1 \checkmark$$

$$\Rightarrow 1 + 1 - 2(\cos x \cos y - \sin x \sin y) = 1 + 1 - 2\cos(x+y)$$

$$\Rightarrow \boxed{\cos(x+y) = \cos x \cos y - \sin x \sin y} \star$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \checkmark$$

put y as $-y$ in $\cos(x+y)$

$$\cos x \cos(-y) - \sin x \sin(-y)$$

$$= \cos x \cos y + \sin x (\sin y)$$

$$= \cos x \cos y + \sin x \sin y \checkmark$$

$$\checkmark \cos(\pi/2 - x) = \sin x \checkmark$$

$$= \cancel{\cos \frac{\pi}{2}} \cos x + \cancel{\sin \frac{\pi}{2}} \sin x$$

$$= \sin x$$

$$\checkmark \sin(\frac{\pi}{2} - x) = \cos x$$

$$= \cos(\frac{\pi}{2} - \frac{\pi}{2} + x)$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x + y)$$

$$= \cos(\pi/2 - (x + y))$$

$$= \cos(\pi/2 - x - y)$$

$$= \cos((\pi/2 - x) - y)$$

$$= \cos(\pi/2 - x) \cos(y) + \sin(\pi/2 - x) \sin y$$

$$= \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\sin x \cos(-y) + \cos x \sin(-y)$$

$$= \sin x \cos y - \cos x \sin y$$

$$\cos(\pi/2 + x) = -\sin x$$

$$\cos(\pi/2 + x)$$

$$= \cos(\pi/2) \cdot \cos x - \sin(\pi/2) \sin x$$

$$= 0 - \sin x$$

$$= -\sin x$$

$$\sin(\pi/2 + x)$$

$$= \sin(\pi/2) \cdot \cos x + \cos(\pi/2) \cdot \sin x$$

$$= \cos x + 0$$

$$= \cos x$$

$$\cos(\pi - x)$$

$$= \cos \pi \cos x + \sin \pi \sin x$$

$$= -\cos x + 0$$

$$= -\cos x$$

$$\sin(\pi - x)$$

$$= \sin \pi \cos x - \cos \pi \sin x$$

$$= 0 + \sin x$$

$$= \sin x$$

$$\begin{aligned}
 \cos(\pi + x) &= \cos \pi \cos x - \sin \pi \sin x \\
 &= -\cos x - 0 \\
 &= -\cos x
 \end{aligned}$$

$$\begin{aligned}
 \sin(\pi + x) &= \sin \pi \cos x + \cos \pi \sin x \\
 &= 0 + -\sin x \\
 &= -\sin x
 \end{aligned}$$

$$\begin{aligned}
 \cos(2\pi - x) &= \cos(2\pi) \cos x + \sin(2\pi) \sin x \\
 &= \cos x + 0 \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \sin(2\pi - x) &= \sin(2\pi) \cos(x) - \cos(2\pi) \sin x \\
 &= 0 - \sin x \\
 &= -\sin x
 \end{aligned}$$

$$\begin{aligned}
 \tan(x + y) &= \sin(x+y)/\cos(x+y) \\
 &= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} \\
 &= \text{divide by } \cos x \cos y \\
 &= \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y - \left(\frac{\sin x}{\cos x}\right) \sin y}{\cos x \cos y}} \\
 &= \frac{\tan x + \tan y}{1 - \left(\frac{\sin x}{\cos x}\right) \left(\frac{\sin y}{\cos y}\right)} \\
 &= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \tan(x+y)
 \end{aligned}$$

$$\begin{aligned}
 & \tan(x-y) \\
 &= \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} \\
 &= \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \tan(-y) &= \frac{\sin(-y)}{\cos(-y)} \\
 &= -\frac{\sin y}{\cos y} \\
 &= -\tan y
 \end{aligned}$$

$$\begin{aligned}
 & \cot(x+y) \\
 &= \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} \\
 &= \frac{\left(\frac{\cos x}{\sin x}\right) \frac{\cos y}{\sin y} - 1}{\frac{\cos y}{\sin y} + \frac{\cos x}{\sin x}}
 \end{aligned}$$

[Dividing N^r & D^r by $\sin x \sin y$]

$$= \frac{\cot x \cot y - 1}{\cot x + \cot y} \quad \cot(x+y)$$

$$\begin{aligned}
 \cot(-x) &= \frac{\cos x}{-\sin x} \\
 &= -\cot x
 \end{aligned}$$

$$\cot(x-y) = \frac{\cot x \cot(-y) - 1}{\cot x + \cot(-y)}$$

$$= \frac{-\cot x \cot y - 1}{\cot x - \cot y}$$

$$= \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

$$\begin{aligned}
 \cos 2x &= \cos (x+x) \\
 &= \cos^2 x - \sin^2 x \quad \text{--- (1)} \\
 &= \cos^2 x - (1 - \cos^2 x) \\
 &= \cos^2 x - 1 + \cos^2 x \\
 &= 2\cos^2 x - 1 \quad \text{--- (2)} \\
 &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \\
 &= \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\sin^2 x}{\cos^2 x}} \rightarrow \left(\frac{\sin x}{\cos x} \right)^2 = \tan^2 x \\
 &= \frac{1 - \tan^2 x}{1 + \tan^2 x} \quad \text{--- (3)}
 \end{aligned}$$

$$\begin{aligned}
 \sin 2x &= \sin (x+x) \\
 &= \sin x \cdot \cos x + \cos x \sin x \\
 &= 2 \sin x \cos x \quad \text{--- (1)} \\
 &= \frac{2 \sin x \cos x}{\sin^2 x + \cos^2 x} \\
 &= \frac{2 \sin x \cdot \cancel{\cos x}}{\frac{\sin^2 x}{\cancel{\cos^2 x}} + \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}} \cdot 1} = \frac{2 \tan x}{1 + \tan^2 x} \quad \text{--- (2)}
 \end{aligned}$$

$$\begin{aligned}
 & \tan 2x \\
 &= \tan(x+x) \\
 &= \frac{\tan x + \tan x}{1 - \tan x \cdot \tan x} \\
 &= \frac{2 \tan x}{1 - \tan^2 x} \quad \checkmark
 \end{aligned}$$

sin 3x

$$\begin{aligned}
 &= \sin(2x+x) \\
 &= \sin 2x \cos x + \cos 2x \sin x \\
 &= \underbrace{2 \sin x \cos x}_{\cos 2x} \cos x + (\cos^2 x - \sin^2 x) \sin x \\
 &= 2 \sin x \cos^2 x + (1 - \sin^2 x - \sin^2 x) \sin x \\
 &= 2 \sin x (1 - \sin^2 x) + \sin x (1 - 2 \sin^2 x) \\
 &= \underline{2 \sin x} - \underline{2 \sin^3 x} + \underline{\sin x} - \underline{2 \sin^3 x} \\
 &= \underline{3 \sin x - 4 \sin^3 x}
 \end{aligned}$$

cos 3x

$$\begin{aligned}
 &= \cos(2x+x) \\
 &= \cos 2x \cdot \cos x - \sin 2x \cdot \sin x \\
 &= (2 \cos^2 x - 1) \cos x - \underbrace{2 \sin x \cos x}_{\sin 2x} \cdot \sin x \\
 &= 2 \cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x \\
 &= \underline{2 \cos^3 x} - \cos x - 2 \cos x + \underline{2 \cos^3 x} \\
 &= \underline{4 \cos^3 x - 3 \cos x}
 \end{aligned}$$

$$\tan 3x$$

$$= \tan(2x + x)$$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \cdot \tan x}$$

$$= \frac{\frac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2 \tan x}{1 - \tan^2 x} \cdot \tan x}$$

$$= \frac{2 \tan x + \tan x (1 - \tan^2 x)}{1 - \tan^2 x - 2 \tan^2 x}$$

$$= \frac{2 \tan x + \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$= \left[\frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} \right]$$

$$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) \quad \checkmark$$

let. $A = \frac{x+y}{2} \quad \checkmark$

$B = \frac{x-y}{2} \quad ; \text{ then}$

$x = A + B$

$y = A - B$

$\left[\begin{array}{l} - A + B = x \\ A - B = y \end{array} \right]$

$$\begin{aligned}
 & \cos(A+B) + \cos(A-B) \\
 = & \cos A \cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B} \\
 = & 2 \cos A \cos B \\
 = & 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)
 \end{aligned}$$

$$\cos x - \cos y$$

$$\begin{aligned}
 \Rightarrow & \cos(A+B) - \cos(A-B) \\
 = & \cos A \cos B - \sin A \sin B - (\cos A \cos B + \sin A \sin B) \\
 = & \cancel{\cos A \cos B} - \sin A \sin B - \cancel{\cos A \cos B} - \sin A \sin B \\
 = & -2 \sin A \sin B \\
 = & -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \quad \checkmark
 \end{aligned}$$

$$\sin x + \sin y$$

$$\begin{aligned}
 = & \sin(A+B) + \sin(A-B) \\
 = & \sin A \cos B + \cancel{\cos A \sin B} + \sin A \cos B - \cancel{\cos A \sin B} \\
 = & 2 \sin A \cos B \\
 = & 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)
 \end{aligned}$$

$$\sin x - \sin y$$

$$\begin{aligned}
 = & \sin(A+B) - \sin(A-B) \\
 = & \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B \\
 = & 2 \cos A \sin B \\
 = & 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \quad \checkmark
 \end{aligned}$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

*

$$\begin{aligned} & \Downarrow \\ & 2 \cos \left(\frac{A+B + A-B}{2} \right) \cos \left(\frac{A+B - (A+B)}{2} \right) \\ & = 2 \cos(A) \cos B \end{aligned}$$