Random Variable

A Random Variable is a variable that will have a value. But there is uncertainity as to what value.

Example :

- 3 fair coins are flipped
- Y = no. of "heads" on 3 coins
- Y is a random variable
- $P(Y=0) = 1/8$
- $P(Y=1) = 3/8$
- $P(Y=2) = 3/8$
- $P(Y=3) = 1/8$

Properties of Random Variables :

(i) Probability Mass Function :

$$P(X = a)$$

The relationship b/w values a random variable can take on, and the corresponding probabilities is a function

$X =$ Random Variable

$X = 5$: Event

$P(X=5)$: Probability of Event

$P(X=x)$: Probability Mass Function

(ii) Expectation

The expected values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} x \cdot p(x) \qquad \text{Short hand } \{P(x) = P(X=x)$$

Example:

Roll a 6-sided die. X is outcome of roll

- $p(X=1) = p(X=2) = \cdots = p(X=6) = 1/6$

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$

$$= \frac{7}{2}$$

Properties of Expectation :

Linearity : $E[aX+b] = aE[X] + b$

Expectation of a sum is the sum of expectation

$$E[X+Y] = E[X] + E[Y]$$

Unconscious Statistician :

$$E[g(x)] = \sum_{x} g(x)\, p(x)$$

Exactly K heads in n coin flips :

Probability of exactly K heads :

$$p(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Declare a Random Variable to be Binomial

Random var

num trial

Probability of success on each trial

$$X \sim Bin(n, p)$$

It is distributed as

Binomial

Probability Mass function for a Bionomial

$$P(X=k) = \overbrace{\binom{n}{k} p^k (1-p)^{n-k}}$$

Probability that our variable takes on the value k

This is also called the Bionomial Term

Example : 1000 ads served, each clicked with $p = 0.01$ otherwise ignored.
What is the probability of 10 clicks?

H : no. of clicks

$$H \sim B(n = 1000, p = 0.01)$$

$$p(H = k) = \binom{1000}{k} (0.01)^k (0.99)^{1000-k}$$

$$P(H = 10) = \binom{1000}{10} (0.01)^{10} (0.99)^{990} \approx 0.125$$

If we want to find the probability when
$p(H = 0 \text{ or } H = 10) = p(H=0) + p(H=10)$
Because both of them are mutually exclusive.

Date ______
Page ______

Bernoulli Random Variable:

Experiment result in success or failure

- X is random indicator variable ($1 = \text{Success}$, $0 = \text{Failure}$)
- $P(X=1) = p(1) = p \qquad P(X=0) = p(0) = 1-p$
- X is Bernoulli Random Variable : $X \sim Bern(p)$
- $E[X] = p$

Example :
- Coin flip
- random binary digit

Now we can calculate Expectation of Binomial

Let $X \sim Bin(n, p)$. Let Y_i be 1 if trial I was a success. $Y_i \sim Bern(p)$

$$E[X] = E\left[\sum_{i=1}^{n} Y_i\right] \quad \text{since } X = \sum_{i=1}^{n} Y_i$$

$$= \sum_{i=1}^{n} E[Y_i] \quad \text{Expectation of Sum}$$

$$= \sum p = n.p$$

Variance :

If X is a random variable with mean μ then the variance of X, denoted Var(X), is :

$$Var(X) = E[(X-\mu)^2]$$

Variance is a formal definition of the spread of a random variable.
Variance is square of the standard Deviation

$$Var(X) = E[X^2] - (E[X])^2$$

Poisson Random Variable

X is a Poisson Random Variable : The number of occurance in a fixed interval of time.

$$X \sim \text{Poi}(\lambda)$$

λ is the "rate"
X takes on values 0, 1, 2 ...
has distribution (PMF) :

$$P(X = K) = e^{-\lambda} \frac{\lambda^K}{K!}$$

★ Poisson is great when you have a rate and you care about # of occurances !

Example : Average of 2.79 major earthquakes per year. What is the probability of 3 major earthquakes next year?

$X \sim \text{Poi}(2.79)$

$$P(X = 3) = \frac{e^{-\lambda}\lambda^K}{K!} = \frac{2.79^3 e^{-2.79}}{3!} \approx 0.23$$

$X \sim Poi(\lambda)$ where $\lambda = np$ $(n \to \infty, p \to 0)$

. $E[X] = np = \lambda$

· $Var(X) = np(1-p) = \lambda(1-0) = \lambda$

Geometric Random Variable

X is geometric random variable :

$$X \sim \text{Geo}(p)$$

X is no. of independent trails until first success.
p is probability of success on each trail.
X take on values 1, 2, 3, ... with probability :

$$P(X = n) = (1-p)^{n-1} p$$

$$E[X] = \frac{1}{p}$$

$$Var(X) = \frac{1-p}{p^2}$$

Negative Bionomial Random Variable

X is negative bionomial RV : $X \sim NegBin(r, p)$
X is no. of independent trials until r successes.
p is probability of success on each trial.
X takes on values $r, r+1, r+2 \ldots$ with probability:

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

where $n = r, r+1 \ldots$

$$E[X] = r/p$$

$$Var(X) = \frac{r(1-p)}{p^2}$$

Probability Density Function

The probability density function (PDF) of a continuous random variable represents the relative likelihood of various values.

Units of probability divided by units of X
Integrate it to get probabilities

$$P(a < X < b) = \int_{x=a}^{b} f(X = x)\,dx$$

- Probability density functions articulate relative belief.
- $f(X = x)$ is not probability. Rather, it has "units" of : probability divided by units of X.

Uniform Random Variable

A uniform random variable is equally likely to be any value in an interval.

$$X \sim \text{Uni}(\alpha, \beta)$$

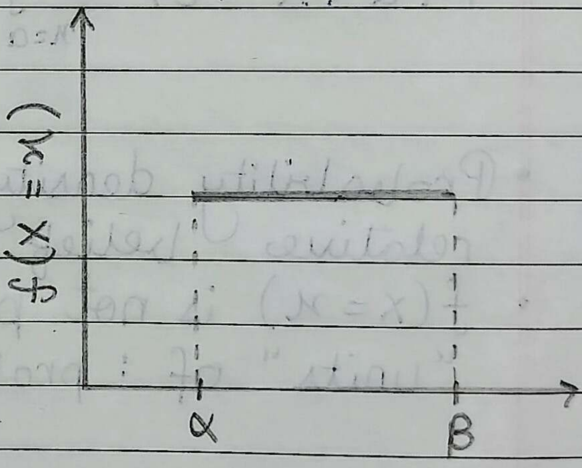
Probability density :

$$f(X = x) = \begin{cases} 1/\beta-\alpha & \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

Properties :

$$E[X] = \frac{\beta - \alpha}{2}$$

$$\text{Var}(X) = \frac{(\beta - \alpha)2}{12}$$



Exponential Random Variable

Consider an experiment that lasts a duration of time until success occurs.

defenition : An exponential random variable X is the amount of time until success.

$$X \sim Exp(\lambda)$$

Example : Time until next earthquake

PDF $\quad f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Expectation $\quad E[X] = 1/\lambda$

Variance $\quad Var(X) = 1/\lambda^2$

Cumulative Density Function

A cumulative density function (CDF) is a "Closed form" equation for the probability that a random variable is less that a given value

$$F(x) = P(X < x)$$

CDF of Exponential => $F_X(x) = 1 - e^{-\lambda x}$

Normal Random Variable

def : An Normal random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Variance

Mean

Other name : Gaussian random variable

PDF $\quad f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$E[X]$ $\quad$ Expectation $= \mu$

Variance $\quad Var(x) = \sigma^2$

The $F(x)$ for a gaussian distribution has no closed form.
So it can not be solved analytically.

However, we can solve for probabilities numerically using function ϕ:

$$F(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

$F(x)$ = CDF of $X \sim N(\mu, \sigma^2)$

Date ____
Page ____

Linear Transformation of Normal is Normal

Let $X \sim N(\mu, \sigma^2)$

if $Y = aX + b$ then Y is also normal

$$E[Y] = E[aX + b] \qquad Var(Y) = Var(aX+b)$$
$$= aE[X] + b \qquad\qquad = a^2 Var(X)$$
$$= a\mu + b \qquad\qquad = a^2\sigma^2$$

$$Y \sim N(a\mu + b,\ a^2\sigma^2)$$

There is a special case of linear transformation for any X:

$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X + \frac{\mu}{\sigma} \Rightarrow a = \frac{1}{\sigma},\ b = \frac{\mu}{\sigma}$$

$$Z \sim N(a\mu + b,\ a^2\sigma^2)$$
$$\sim N\left(\frac{\mu}{\sigma} - \frac{\mu}{\sigma},\ \frac{\sigma^2}{\sigma^2}\right)$$
$$\sim N(0, 1)$$

Let $X \sim N(\mu, \sigma^2) \qquad Z = \frac{X-\mu}{\sigma}$

$$F_X(x) = P(X \le x) = P(X - \mu \le x - \mu)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$\therefore F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

Look up $\phi(z)$ in table

* Symmetry of PDF of Normal RV

$\phi(-z) = 1 - \phi(z)$

$P(c < z < d) = \phi(d) - \phi(c)$

Example : $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$

$\sigma^2 = 16 \Rightarrow \sigma = 4$

1. $P(X > 0) = 1 - P(X < 0)$
$= 1 - \phi\left(\frac{0-3}{4}\right) = 1 - \phi\left(\frac{-3}{4}\right)$

2. $P(2 < X < 5) = P(X < 5) - P(X < 2)$
$= \phi\left(\frac{5-3}{4}\right) - \phi\left(\frac{2-3}{4}\right)$

3. $P(|X-3| > 6) = P(X < -3) + P(X > 9)$
$= F_X(-3) + (1 - F_X(9))$
$= \phi\left(\frac{-3-3}{4}\right) + \left(1 - \phi\left(\frac{9-3}{4}\right)\right)$

Who gets to approximate :

$$X \sim Bin(n, p)$$
$$E[X] = np$$
$$Var(X) = np(1-p)$$

$Y \sim Poi(\lambda)$

$\lambda = np$

Poisson Approximation

n large (> 20),

p small (< 0.05)

$Y \sim N(\mu, \sigma^2)$

$\mu = np$

$\sigma^2 = np(1-p)$

Normal Approximation

n Large (> 20)

variance large (> 10)

1. If there is a choice, either is fine
2. When using Normal to approximate a discrete RV, use a continuity correction.