0.1 Mathematical Structures

0.1.1 Sets

[Set] A *set* is a collection of distinct objects.

0.1.2 Group

[Group] A *group* is a set G with a binary operation $\bullet$, denoted $(G, \bullet)$, which satisfies the following conditions:

- **Closure**: $\forall a, b \in G, a \bullet b \in G$
- **Associativity**: $\forall a, b, c \in G, (a \bullet b) \bullet c = a \bullet (b \bullet c)$
- **Identity**: $\exists e \in G, \forall a \in G, a \bullet e = e \bullet a = a$
- **Inverse**: $\forall a \in G, \exists a^{-1} \in G, a \bullet a^{-1} = a^{-1} \bullet a = e$

[Semigroup] A *semigroup* is $(G, \bullet)$, which satisfies Closure and Associativity. [Monoid] A *monoid* is a semigroup $(G, \bullet)$ which also has identity. [Abelian Group] An *Abelian Group* or *Commutative Group* is a group $(G, \bullet)$ with the following property:

- **Commutativity**: $\forall a, b \in G, a \bullet b = b \bullet a$

0.1.3 Ring

[Ring] A *Ring* is a set R with two binary operations $+$ and $\cdot$, denoted $(R, +, \cdot)$, which satisfies the following conditions:

- $(R, +)$ is an abelian group
- $(R, \cdot)$ is a semigroup
- **Distribution** $\cdot$ is distributive with respect to $+$, that is, $\forall a, b, c \in R$:
  [label=-]$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

0.1.4 Field