The Incomplete Codex of Basic Mathematics for Computer Scientists

From Programmers to Hackers: Mathematical Basis to Computer Science

None(@n0n3x1573n7), jh05013(@jhznktrkstlhknm)

December 27, 2018

Chapter 1

Introduction

Contents

Part I

Mathematical Preliminaries

Chapter 2

Logic

Chapter 3

Algebraic Structures

3.1 Algebraic Structures

3.1.1 Sets

Definition 1 (Set)
A set is a collection of distinct objects.

To see some traits on sets, we literally start from nothing:

Axiom 2 (Empty Set Axiom)
There is a set containing no members, that is:

$$\exists B \text{ such that } \forall x, (x \notin B)$$

We call this set the empty set, and denote it by the symbol $\emptyset$.

We now have $\emptyset$; we now write down a few rules for how to manipulate sets.

Axiom 3 (Axiom of Extensionality)
Two sets are equal if and only if they share the same elements, that is:

$$\forall A, B[\forall z, ((z \in A) \Leftrightarrow (z \in B)) \Rightarrow (A = B)]$$

Axiom 4 (Axiom of Pairing)
Given any two sets A and B, there is a set which have the members just A and B, that is:

$$\forall A, B \exists C \forall x[x \in C \Leftrightarrow ((x = A) \vee (x = B))]$$

If A and B are distinct sets, we write this set C as $\{A, B\}$; if $A = B$, we write it as $\{A\}$.

Axiom 5 (Axiom of Union, simple version)
Given any two sets A and B, there is a set whose members are those sets belonging to either A or B, that is:

$$\forall A, B \exists C \forall x[x \in C \Leftrightarrow ((x \in A) \vee (x \in B))]$$

We write this set C as $A \cup B$.

In the simplified version of Axiom of Union, we take union of only two things, but we sometimes we want to take unions of more than two things or even more than finitely many things. This is given by the full version of the axiom:

Axiom 6 (Axiom of Union, full version)
Given any set A, there is a set C whose elements are exactly the members of the members of A, that is:

$$\forall A \exists C[x \in C \Leftrightarrow (\exists A'(A' \in A) \wedge (x \in A'))]$$

We denote this set C as

$$\bigcup_{A' \in A} A'$$

Axiom 7 (Axiom of Intersection, simple version)
Given any two sets A and B, there is a set whose members are member of both A and B, that is:

$$\forall A, B \exists C \forall x[(x \in C) \Leftrightarrow ((x \in A) \wedge (x \in B))]$$

Sometimes as union, we would want to take intersection of more than finitely many things. This is given by the full version of the axiom:

Axiom 8 (Axiom of Intersection, full version)
Given any set A, there is a set C whose elements are exactly the members of all members of A, that is:

$$\forall A \exists C \forall x[(x \in C) \Leftrightarrow (\forall A'((A' \in A) \Rightarrow (x \in A')))]$$

We denote this set C as

$$\bigcap_{A' \in A} A'$$

Axiom 9 (Axiom of Subset)
For any two sets A and B, we say that $B \subset A$ if and only if every member of B is a member of A, that is:

$$(B \subseteq A) \Leftrightarrow (\forall x(x \in B) \Rightarrow (x \in A))$$

By the Axiom of Subset we can define the power set of an any given set:

Definition 10 (Power Set)
For any set A, the power set of the set A, denoted $P(A)$, whose members are precisely the collection of all possible subsets of A, that is:

$$\forall A \exists P(A) \forall B((B \subseteq A) \Leftrightarrow (B \in P(A)))$$

Definition 11 (Equivalence Relation)
Let S be a set. An Equivalence Relation on S is a relation, denoted by ~, with the following properties, $\forall a, b, c \in S$:

- **Reflexivity** $a \sim a$
- **Symmetry** $a \sim b \Leftrightarrow b \sim a$
- **Transitivity** $(a \sim b) \wedge (b \sim c) \Rightarrow (a \sim c)$

Definition 12 (Setoid)
A setoid is a set in which an equivalence relation is defined, denoted $(S, \sim)$.

Definition 13 (Equivalence Class)
The equivalence class of $a \in S$ under ~, denoted $[a]$, is defined as $[a] = \{b \in S | a \sim b\}$.

Definition 14 (Order)
Let S be a set. An order on S is a relation, denoted by $<$, with the following properties:

- If $x \in S$ and $y \in S$ then one and only one of the following statements is true:

$$x < y, x = y, y < x$$

- For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Remark

- It is possible to write $x > y$ in place of $y < x$
- The notation $x \leq y$ indicates that $x < y$ or $x = y$.

Definition 15 (Ordered Set)
An ordered set is a set in which an order is defined, denoted $(S, <)$.

Definition 16 (Bound)
Suppose S is an ordered set, and $E \subset S$.
If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E. If there exists $\alpha \in S$ such that $x \geq \alpha$ for every $x \in E$, we say that E is bounded below, and call α a lower bound of E.

Definition 17 (Least Upper Bound)
Suppose that S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ with the following properties:

- β is an upper bound of E
- If $\gamma < \beta$, then γ is not an upper bound of E

Then β is called the Least Upper Bound of E or the supremum of E, denoted

$$\beta = sup(E)$$

Definition 18 (Greatest Lower Bound)
Suppose that S is an ordered set, and $E \subset S$. If there exists a $\alpha \in S$ with the following properties:

- α is a lower bound of E
- If $\gamma < \alpha$, then γ is not an lower bound of E

Then α is called the Greatest Lower Bound of E or the infimum of E, denoted

$$\beta = inf(E)$$

Definition 19 (least-upper-bound property)
An ordered set S is said to have the least-upper-bound property if the following is true:
if $E \subset S$, E is not empty, and E is bounded above, then $sup(E)$ exists in S.

Definition 20 (greatest-lower-bound property)
An ordered set S is said to have the greatest-lower-bound property if the following is true:
if $E \subset S$, E is not empty, and E is bounded below, then $inf(E)$ exists in S.

Theorem 21
Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below.
Let L be the set of all lower bounds of B. Then

$$\alpha = sup(L)$$

exists in S, and $\alpha = inf(B)$.

Proof. Note that $\forall x \in L, y \in B, x \leq y$.
L is nonempty as B is bounded below.
L is bounded above since $\forall x \in S \backslash L, \forall y \in L, x > y$.
Since S has the least-upper-bound property and $L \subset S$, $\exists \alpha = sup(L)$.
The followings hold:

- α is a lower bound of B.
  $(\because)$ $\forall \gamma \in B, \gamma > \alpha$
- β with $\beta > \alpha$ is not a lower bound of B
  $(\because)$ Since α is an upper bound of L, $\beta \notin L$.

Hence $\alpha = inf(B)$. $\square$

Corollary 22
For all ordered sets, the Least Upper Bound property and the Greatest Lower Bound Porperty are equivalent.

3.1.2 Group

Definition 23 (Group)
A group is a set G with a binary operation $\cdot$, denoted $(G, \cdot)$, which satisfies the following conditions:

- **Closure:** $\forall a, b \in G, a \cdot b \in G$
- **Associativity:** $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Identity:** $\exists e \in G, \forall a \in G, a \cdot e = e \cdot a = a$
- **Inverse:** $\forall a \in G, \exists a^{-1} \in G, a \cdot a^{-1} = a^{-1} \cdot a = e$

Definition 24 (Semigroup)
A semigroup is $(G, \cdot)$, which satisfies Closure and Associativity.

Definition 25 (Monoid)
A monoid is a semigroup $(G, \cdot)$ which also has identity.

Definition 26 (Abelian Group)
An Abelian Group or Commutative Group is a group $(G, \cdot)$ with the following property:

- **Commutativity:** $\forall a, b \in G, a \cdot b = b \cdot a$

3.1.3 Ring

Definition 27 (Ring)
A Ring is a set R with two binary operations $+$ and $\cdot$, often called the addition and multiplication of the ring, denoted $(R, +, \cdot)$, which satisfies the following conditions:

- $(R, +)$ is an abelian group
- $(R, \cdot)$ is a semigroup
- **Distribution:** $\cdot$ is distributive with respect to $+$, that is, $\forall a, b, c \in R$:
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
  - $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

The identity element of $+$ is often noted 0.

Definition 28 (Ring with identity(1))
A Ring with identity is a ring $(R,+,\cdot)$ of which $(R,\cdot)$ is a monoid. The identity element of $\cdot$ is often noted 1.

Definition 29 (Commutative Ring)
A commutative ring is a ring $(R,+,\cdot)$ of which $\cdot$ is commutative.

Definition 30 (Zero Divisor)
For a ring $(R,+,\cdot)$, let 0 be the identity of $+$.
$a,b \in R$, $a \neq 0$ and $b \neq 0$, if $a \cdot b = 0$, a,b are called the zero divisors of the ring.

Definition 31 (Integral Domain)
An integral domain is a commutative ring $(R,+,\cdot)$ with 1 which does not have zero divisors.

3.1.4 Field

Definition 32 (Field)
A Field is a set F with two binary operations $+$ and $\cdot$, often called the addition and multiplication of the field, denoted $(R,+,\cdot)$, which satisfies the following conditions:

- $(F,+,\cdot)$ is a ring
- $(F\backslash\{0\},\cdot)$ is a group

Alternatively, a Field may be defined with a set of Field Axioms listed below:

(A) **Axioms for Addition**

- (A1) **Closed under Addition**
  $\forall a,b \in F, a+b \in F$
- (A2) **Addition is Commutative**
  $\forall a,b \in F, a+b = b+a$
- (A3) **Addition is Associative**
  $\forall a,b,c \in F, (a+b)+c = a+(b+c)$
- (A4) **Identity of Addition**
  $\exists 0 \in F, \forall a \in F, 0+a = a$
- (A5) **Inverse of Addition**
  $\forall a \in F, \exists -a \in F, a+(-a) = 0$

(M) **Axioms for Multiplication**

- (M1) **Closed under Multiplication**
  $\forall a,b \in F, a \cdot b \in F$
- (M2) **Multiplication is Commutative**
  $\forall a,b \in F, a \cdot b = b \cdot a$
- (M3) **Multiplication is Associative**
  $\forall a,b,c \in F, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (M4) **Identity of Multiplication**
  $\exists 1 \in F, \forall a \in F, 1 \cdot a = a$

(M5) **Inverse of Multiplication**
$\forall a \in F\backslash\{0\}, \exists a^{-1} \in F, a \cdot a^{-1} = 1$

(D) **Distributive Law**
$\forall a, b, c \in F, (a+b) \cdot c = a \cdot c + b \cdot c$
where $\cdot$ takes precedence over $+$.

Definition 33 (Ordered Field)
An ordered field is a field F which is an ordered set, such that the order is compatible with the field operations, that is:

- $x + y < x + z$ if $x, y, z \in F$ and $y < z$
- $xy > 0$ if $x, y \in F$, $x > 0$ and $y > 0$

3.1.5 Polynomial Ring

Definition 34 (Polynomial over a Ring)
A polynomial $f(x)$ over the ring $(R, +, \cdot)$ is defined as

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x^1 + \cdots, a_i \in R$$

where $a_i = 0$ for all but finitely many values of i.
The degree of the polynomial $\deg(f)$ is defined as $\deg(f) = \max\{n | n \in \mathbb{N}, a_n \neq 0\}$.
The leading coefficient of the polynomial is defined as $a_{\deg(f)}$.

Definition 35 (Addition and Multiplication of Polynomials)
Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, $a_i, b_i \in R$ be a polynomial over the ring $(R, +, \cdot)$. Define:

$$f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

$$f(x)g(x) = \sum_{k=0}^{\infty} (c_k) x^k \text{ where } c_k = \sum_{i+j=k} a_i b_j$$

Definition 36 (Polynomial Ring)
The set of polynomials over the ring $(R, +, \cdot)$, $R[x] = \{f(x) | f(x) \text{ is a polynomial over } R\}$ is called the Polynomial Ring(or Polynomials) over R.

Theorem 37 (Degree of Polynomial on Addition and Multiplication)
Let $f(x), g(x) \in R[x]$ with $\deg(f) = n$, $\deg(g) = m$.

- $0 \leq \deg(f+g) \leq \max(\deg(f), \deg(g))$
- $\deg(fg) \leq \deg(f) + \deg(g)$.
  If $(R, +, \cdot)$ is an integral domain, $\deg(fg) = \deg(f) + \deg(g)$

Theorem 38 (Relationship between a Ring and its Polynomial Ring)
Let $(R, +, \cdot)$ be a ring and $R[x]$ the polynomials over R.

1. If $(R, +, \cdot)$ is a commutative ring with 1, then $(R[x], +, \cdot)$ is a commutative ring with 1.
2. If $(R, +, \cdot)$ is a integral domain, then $(R[x], +, \cdot)$ is a integral domain.

Theorem 39 (Division Algorithm for Polynomials over a Ring)
Let $(R, +, \cdot)$ be a commutative ring with 1.
Let $f(x), g(x) \in R[x]$, $g(x) \neq 0$ with the leading coefficient of $g(x)$ being invertible.
Then, $\exists! q(x), r(x) \in R[x]$ such that

$$f(x) = q(x)g(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(g)$.

Proof. Use induction on $\deg(f)$.
1. $f(x) = 0$ or $\deg(f) < \deg(g)$: $q(x) = 0, r(x) = f(x)$
2. $\deg(f) = \deg(g) = 0$: $q(x) = f(x) \cdot g(x)^{-1}, r(x) = 0$
3. $\deg(f) \geq \deg(g)$:

1) Existence

Let $\deg(f) = n$, $\deg(g) = m$, $n > m$.
Suppose the theorem holds for $\deg(f) < n$.
Let $f(x) = a_0 + a_1x^1 + \cdots + a_nx^n$, $g(x) = b_0 + b_1x^1 + \cdots + b_mx^m$.
Choose $f_1(x) = f(x) - (a_nb_m^{-1})x^{n-m}g(x) \in R[x]$.
Since $\deg(f_1) < n$, $\exists q(x), r(x) \in R[x]$ so that $f_1(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg(r) < \deg(g)$.
$f_1(x) = f(x) - (a_nb_m^{-1})x^{n-m}g(x) = g(x)q(x) + r(x)$
$f(x) = g(x)((a_nb_m^{-1})x^{n-m} + q(x)) + r(x)$
Hence such pair exists.

2) Uniqueness

Suppose $f(x) = g(x)q_1(x) + r_1(x) = g(x)q_2(x) + r_2(x)$.
$g(x)(q_1(x) - q_2(x)) = r_2(x) - r_1(x)$
If $r_1 \neq r_2$, $\deg(g) > \deg(r_2 - r_1) = \deg(g(q_1 - q_2))$.
Since $\deg(g(q_1 - q_2)) \geq \deg(g)$ if $q_1 - q_2 \neq 0$, $q_1 = q_2$, but if so, $r_1 = r_2$.
If $r_1 = r_2$, trivially $q_1 = q_2$.
Hence they exist uniquely. $\square$

Chapter 4

Number Theory

4.1 Arithmetic

4.1.1 Integer Arithmetic

Theorem 40 (Division Algorithm)

Definition 41 (Divisibility)

Theorem 42 (Euclidean Algorithm)

Theorem 43 (Extended Euclidean Algorithm)

Definition 44 (Linear Diophantine Equation)

Theorem 45 (Solutions for Linear Diophantine Equation)

4.1.2 Modular Arithmetic

Definition 46 (Modulus)

Chapter 5

Analysis

Chapter 6

Linear Algebra

Chapter 7

Calculus

Chapter 8

Statistics

Chapter 9

From $\mathbb{N}$ to $\mathbb{R}$

9.1 $\mathbb{N}$: The set of Natural Numbers

9.1.1 Construction of $\mathbb{N}$

We start from the Axioms of Set[2,3,4,5,6,9], the definition of power set[10], the definition of equivalence relation and class[11,13] and the following definitions:

Definition 47 (Successor)
For any set x, the successor of x, denoted $\sigma(x)$, is defined as the following set:

$$\sigma(x) = x \cup \{x\}$$

Let us define $0 = \emptyset$, $1 = \sigma(\emptyset) = \sigma(0)$. Using the definition of successors, and following the pattern, $2 = \sigma(1)$, $3 = \sigma(2)$, and so on. Basically we can make any finite number using the definition of successor and the Axioms of Set, but actually getting all of the natural numbers at once(or any infinitely large set, since only the empty set is guaranteed to exist by the axioms) is not possible with our axioms. We define the concept of Inductive Sets and make another Axiom for this purpose:

Definition 48 (Inductive Set)
A set A is called inductive if it satisfies the following two properties:

- $\emptyset \in A$
- $(x \in A) \Rightarrow (\sigma(x) \in A)$

Axiom 49 (Axiom of Infinity)
There is an inductive set, that is:

$$\exists A(\emptyset \in A) \wedge (\forall x \in A, \sigma(x) \in A)$$

Theorem 50
Take any two inductive sets, S and T. Then, $S \cap T$ is also an inductive set.

Proof. Let $U = S \cap T$.

1. $\emptyset \in U$

   $\emptyset \in S$ and $\emptyset \in T$ since S and T are both inductive.

2. $(x \in U) \Rightarrow (\sigma(x) \in U)$

   $\forall x \in U, (x \in S) \wedge (x \in T)$.
   Since S and T are both inductive, $(\sigma(x) \in S) \wedge (\sigma(x) \in T)$
   Therefore $\sigma(x) \in U$.

Therefore U is inductive. $\square$

Corollary 51
An intersection of any number of inductive sets is inductive.

Theorem 52
For any inductive set S, define N_S as follows:

$$N_S = \bigcap_{\substack{A \subseteq S \\ A \text{ is inductive}}} A$$

Take any two inductive sets, S and T. Then $N_S = N_T$.

Proof. Suppose not; WLOG, $\exists x$ such that $x \in N_S$ and $x \notin N_T$.
Let $X = N_S \cap N_T$. Then X is inductive, $X \subset N_S$, and $x \notin X$.
Since by the definition of N_S, $N_S = X \cap N_S$, but $x \notin X \cap N_S$ hence the RHS and the LHS are different.
Therefore the assumption is wrong; therefore $N_S = N_T$. $\square$

Using this theorem, we can finally define the set of natural numbers:

Definition 53 (The Set (N) of natural numbers)
Take any inductive set S, and let

$$N = \bigcap_{\substack{A \subseteq S \\ A \text{ is inductive}}} A$$

This set is the natural numbers, which we denote as $\mathbb{N}$.

9.1.2 Operations on $\mathbb{N}$

We now define two operations on $\mathbb{N}$, addition$(+)$ and multiplication$(\cdot)$.

Definition 54 (Addition and Multiplication on $\mathbb{N}$)
The operation of addition, denoted by $+$, is defined by following two recursive rules:

1. $\forall n \in \mathbb{N}, n + 0 = n$
2. $\forall n, m \in \mathbb{N}, n + \sigma(m) = \sigma(n + m)$

Similarly the operation of multiplication, denoted by $\cdot$, is defined by following two recursive rules:

1. $\forall n \in \mathbb{N}, n \cdot 0 = 0$
2. $\forall n, m \in \mathbb{N}, n \cdot \sigma(m) = n \cdot m + n$

Lemma 55 (Operations on 0)
$\forall x \in \mathbb{N}$

- $x + 0 = 0 + x$
- $x \cdot 0 = 0 \cdot x$

Proposition 56 (Properties of $+$ and $\cdot$)
$\forall x, y, z \in \mathbb{N}$,

- **Associativity of Addition** $x + (y + z) = (x + y) + z$
- **Commutativity of Addition** $x + y = y + x$

- **Associativity of Multiplication** $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- **Commutativity of Multiplication** $x \cdot y = y \cdot x$
- **Distributive Law** $x \cdot (y + z) = x \cdot y + x \cdot z$
- **Cancellation Law for Addition** $x + z = y + z \Rightarrow x = y$

9.1.3 Ordering on $\mathbb{N}$

Definition 57 (Ordering on $\mathbb{N}$)
For $n, m \in \mathbb{N}$, we say that n is less than m, written $n \geq m$, if there exists a $k \in \mathbb{N}$ such that $m = n + k$. We also write $n < m$ if $k \neq 0$.

Theorem 58
$(N, <)$ is an ordered set[15].

Proposition 59
The followings are true:

- If $n \neq 0$, then $0 < n$.
- Let $x, y, z \in \mathbb{N}$. Then the followings are true:
  - $(x \leq y) \wedge (y < z) \Rightarrow (x < z)$
  - $(x < y) \wedge (y \leq z) \Rightarrow (x < z)$
  - $(x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)$
  - $(x < y) \Rightarrow (x + z < y + z)$
  - $(x < y) \Rightarrow (xz < yz)$
- $\forall n \in \mathbb{N}, n \neq n + 1$
- $\forall n, k \in \mathbb{N}, k \neq 0, n \neq n + k$

Definition 60 (Least Element)
Let $S \subset \mathbb{N}$. An element $n \in S$ is called a <u>least element</u> if $\forall m \in S, n \leq m$

Proposition 61 (Uniqueness of the Least Element)
Let $S \subset \mathbb{N}$. Then if S has a least element, then it is unique.

Theorem 62 (Well-Ordering Property)
Let S be a nonempty subset of $\mathbb{N}$. Then S has a least element.

Note
The well-ordering property states that the set of natural numbers $\mathbb{N}$ has the greatest lower bound property[20] and thereby theorem 21, has the least upper bound property[19].

9.1.4 Properties of $\mathbb{N}$

Many of the mathematics book defines the set of Natural Numbers as the set satisfying the <u>Peano Axioms</u>.

Proposition 63 (Peano Axioms)

1. 0, which we defined as the empty set $\emptyset$, is a natural number.
2. There exist a distinguished set map $\sigma : \mathbb{N} \to \mathbb{N}$
3. σ is injective

4. There does not exist an element $n \in \mathbb{N}$ such that $\sigma(n) = 0$
5. (Principle of Induction) If $S \in N$ is inductive, then $S = N$.

Proposition 64
Suppose that a is a natural number, and that $b \in a$. Then $b \subseteq a$, $a \nsubseteq b$.

Proposition 65
For any two natural numbers $a, b \in \mathbb{N}$, if $\sigma(a) = \sigma(b)$, then $a = b$.

Lemma 66
If $n \in \mathbb{N}$ and $n \neq 0$, then there exists $m \in \mathbb{N}$ such that $\sigma(m) = n$.

9.2 $\mathbb{Z}$: The set of Integers

9.2.1 Construction of $\mathbb{Z}$

We now have the set of natural numbers, and starting there, we construct the set of integers.

Proposition 67
Define a relation $\equiv$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \equiv (c, d)$ iff $a + d = b + c$. This relation is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Let $\mathbb{Z}$ be the set of equivalence classes under this relation, and the equivalence class containing (a, b) be denoted by $[a, b]$.

9.2.2 Operations on $\mathbb{Z}$

Definition 68 (Addition and Multiplication on $\mathbb{Z}$)
Addition and multiplication on $\mathbb{Z}$ are defined by:

- $[a, b] + [c, d] = [a + c, b + d]$
- $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$

Definition 69 (Subtraction on $\mathbb{Z}$)
Subtraction on $\mathbb{Z}$ is defined by:

$$[a, b] - [c, d] = [a, b] + [d, c]$$

9.2.3 Ordering on $\mathbb{Z}$

Definition 70 (Ordering on $\mathbb{Z}$)
Let $[a, b], [c, d] \in \mathbb{Z}$. $[a, b] < [c, d]$ iff $a + d < b + c$.

9.2.4 Property of $\mathbb{Z}$

Theorem 71 (Arithmetic Properties of $\mathbb{Z}$)

1. Addition and multiplication are well-defined.
2. Addition and multiplication have identity elements $[n, n]$ and $[n, n + 1]$, respectively.
3. Addition and multiplication are commutative and associative.
4. The distributive law holds.
5. Each element $[a, b]$ has an additive inverse $[b, a]$.

We can treat $\mathbb{N}$ to be a subset of $\mathbb{Z}$ by identifying the number n with the class $[0,n]$. Since $[0,a]+[0,b]=[0,a+b]$ and $[0,a]\cdot[0,b]=[0,ab]$, these operations mirror the corresponding operation in $\mathbb{N}$.

Given $n\in\mathbb{N}$, we write $-n$ for $[n,0]$, 0 for $[n,n]$, and 1 for $[n,n+1]$. By the fifth arithmetic property of $\mathbb{Z}$[71], this defines $-n$ to be the additive inverse of n. We also use the minus sign for subtraction; it is therefore natural to write $[a,b]$ as $b-a$.

Proposition 72
For $a,b\in\mathbb{N}$, let $-b$, a, and b be defined in $\mathbb{Z}$ as above. Then

$$a-b=a+(-b) \text{ and } -(-b)=b$$

9.3 $\mathbb{Q}$: The set of Rational Numbers

We construct the set of rational numbers from the set of integers as follows:

9.3.1 Construction of $\mathbb{Q}$

Proposition 73
Define a relation $\equiv$ on $\mathbb{Z}\times(\mathbb{Z}\backslash\{0\})$ by $(a,b)\equiv(c,d)$ iff $ad=bc$. This relation is an equivalence relation on $\mathbb{Z}\times(\mathbb{Z}\backslash\{0\})$.

Let $\mathbb{Q}$ be the set of equivalence classes under this relation, and the equivalence class containing (a,b) is denoted by a/b or $\frac{a}{b}$, and $\frac{a}{b}=\frac{c}{d}$ mean that (a,b) and (c,d) belong to the same equivalence class. Especially we write 0 and 1 to denote $\frac{0}{1}$ and $\frac{1}{1}$, respectively.

9.3.2 Operations on $\mathbb{Q}$

Definition 74 (Addition and Multiplication on $\mathbb{Q}$)
The <u>sum</u> and <u>product</u> of $\frac{a}{b},\frac{c}{d}\in\mathbb{Q}$ are defined by

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \text{ and } \frac{a}{b}\frac{c}{d}=\frac{ac}{bd}$$

Definition 75 (Subtraction on $\mathbb{Q}$)
Subtraction on $\mathbb{Z}$ is defined by:

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

Definition 76 (Division on $\mathbb{Q}$)
Division on $\mathbb{Z}$ is defined by:

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$$

9.3.3 Ordering on $\mathbb{Q}$

Definition 77 (Ordering on $\mathbb{Q}$)
Let $\frac{a}{b},\frac{c}{d}\in\mathbb{Q}$. $\frac{a}{b}<\frac{c}{d}$ iff $(bd>0\wedge ad<bc)\vee(bd<0\wedge ad>bc)$.

9.3.4 Property of $\mathbb{Q}$

Theorem 78 (Arithmetic Properties of $\mathbb{Q}$)

1. Addition and multiplication are well-defined.

2. Addition and multiplication have identity elements 0 and 1, respectively.

3. Addition and multiplication are commutative and associative.

4. The distributive law holds.

5. Each element $\frac{a}{b}$ has an additive inverse $\frac{b}{a}$.

Theorem 79
$(\mathbb{Q},+,\cdot)$ forms an ordered field.

9.4 $\mathbb{R}$: The set of Real Numbers

9.4.1 Construction of $\mathbb{R}$

One simple way to construct $\mathbb{R}$ is by proving the following theorem:

Theorem 80 (Existence of $\mathbb{R}$)
There exists an ordered field $\mathbb{R}$ containing $\mathbb{Q}$ as a subfield which has the least-upper-bound property.

But where's the fun in that? We will be constructing the field of real numbers using Cauchy sequences[**??**], starting with the following proposition:

Theorem 81
Define a relation $\equiv$ on the set S of Cauchy sequences of rational numbers as follows:

$$\{a_n\} \equiv \{b_n\} \text{ iff } (a_n - b_n) \to 0$$

This relation is an equivalence relation.

Now let us define $\mathbb{R}$ as the set of equivalence classes of S under the relation $\equiv$.

9.4.2 Operations on $\mathbb{R}$

Before the definition of operations on $\mathbb{R}$, we need to find out whether if the Cauchy sequences of rational numbers are closed under addition and multiplication, and it turns out they do, as stated in the following proposition:

Proposition 82
The set S of Cauchy sequences of rational numbers is closed under addition, multiplication, and scalar multiplication, that is:

1. If $\{a_n\} \in S$ and $\{b_n\} \in S$, then $\{a_n + b_n\} \in S$

2. If $\{a_n\} \in S$ and $\{b_n\} \in S$, then $\{a_n b_n\} \in S$

3. If $\{a_n\} \in S$ and $c \in \mathbb{Q}$, then $\{ca_n\} \in S$

We can finally go on to defining the operations on $\mathbb{R}$.

Definition 83 (Addition and Multiplication on $\mathbb{R}$)
Let $\{a_n\}$ and $\{b_n\}$ be sequences contained in the real numbers α, β, respectively. Then the <u>sum</u> and <u>product</u> of α and β are defined by:

$$\alpha + \beta = \{a_n + b_n\} \text{ and } \alpha\beta = \{\alpha\beta\}$$

We can define subtraction and division on $\mathbb{R}$ similar to addition and multiplication, by term-by-term calculation on each term of the Cauchy sequence.

9.4.3 Ordering on $\mathbb{R}$

Definition 84 (Ordering on $\mathbb{R}$)
Let $\alpha = \{a_n\}, \beta = \{b_n\} \in \mathbb{R}$. $\alpha < \beta$ iff $\exists N \in \mathbb{N}, \forall n \geq N, a_n < b_n$.

9.4.4 Property of $\mathbb{R}$

Theorem 85 (Arithmetic Properties of $\mathbb{R}$)

1. Addition and multiplication are well-defined.
2. Addition and multiplication have identity elements $\{0\}$ and $\{1\}$, respectively.
3. Addition and multiplication are commutative and associative.
4. The distributive law holds.
5. Each element $\{a_n\}$ has an additive inverse $\{-a_n\}$.

Theorem 86
$(\mathbb{R}, +, \cdot)$ forms an ordered field.

We now define an extension to $\mathbb{R}$ as follows:

Definition 87 (Extended Real Number System)
The <u>extended real number system</u>, denoted $\mathbb{R}^+$, $[-\infty, \infty]$, or $\mathbb{R} \cup \{-\infty, \infty\}$, consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define $\forall x \in \mathbb{R}$,

$$-\infty < x < \infty$$

Remark
The extended real number system does not form a field.

9.5 $\mathbb{C}$: The set of Complex Numbers

We construct the set of complex numbers from $\mathbb{R}$. Unlike the previous constructions, we do not construct it using equivalence class. Instead the construction is done by considering the quotient ring of polynomial ring over $\mathbb{R}$ modulo $i^2 + 1$.

Definition 88
Complex number is defined as the quotient ring $\mathbb{R}[i]/(i^2 + 1)$, with operations defined as normal.

Theorem 89
$(\mathbb{C}, +, \cdot)$ forms a field.

Part II

Applications to Computer Science

Chapter 10

Automata

Chapter 11

Complexity Theory

11.1 Turing Machine and Complexity

(TODO: Move this to Automata.) (TODO: Before giving the definition of Turing Machine, I have to give some intuition here.)

Definition 90 (Turing machine)
A Turing machine is a tuple $M = (\Gamma, Q, \delta)$, where:

- Q is the set of states, which contains the starting state q_0 and the halting state q_F.
- Γ is the set of symbols, which contains the blank symbol $\square$, and two numbers 0 and 1. Γ is called the alphabet of M.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the decision function.

The definition of a Turing Machine is not unique. Some definitions use multiple tapes, using one of them as the input tape that can't be modified and another as the output tape. Some has more than one halting states. Some include the "starting symbol" in the alphabet. But in general, a Turing machine starts from one state, follows the decision function every step, and halts at the halting state.

In fact, the different definitions of a Turing machine turns out to be the same, in the sense that a function $f : \{0,1\}^* \to \{0,1\}$ is computable using one definition of a Turing machine iff it is computable using another definition of a Turing Machine.

(TODO: Write something about asymptotic notation here)

Definition 91 (Asymptotic notation)
Let f and g be two functions from $\mathbb{N}$ to $\mathbb{N}$. Then we say:

- $f = O(g)$ if there is a constant c such that $f(n) \leq c \cdot g(n)$ for every sufficiently large n. That is, $n > N$ for some N.
- $f = \Omega(g)$ if $g = O(f)$.
- $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.
- $f = o(g)$ if for every constant $c > 0$, $f(n) < c \cdot g(n)$ for every sufficiently large n.
- $f = \omega(g)$ if $g = o(f)$.

11.2 Complexity Classes

Definition 92 (P)
P is the set of boolean function computable in time $O(n^c)$ for some constant $c > 0$.

(TODO: Non-deterministic Turing Machine)
(TODO: NP)
(TODO: EXP)

11.3 Reduction

Is there a polynomial-time algorithm for a given decision problem? Computer scientists are interested in this question because if there is one, it is usually a small-degree polynomial like $O(n^2)$ or $O(n^5)$. Some problems have a special property that if the problem has a polynomial-time algorithm, then several other problems do.

Definition 93 (Polynomial-time Karp reduction)
A problem $A \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to $B \subseteq \{0,1\}^*$, denoted $A \leq_p B$, if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $x \in A$ iff $f(x) \in B$.

The intuitive meaning is that a problem of A can be "reduced" to a problem of B, and if we can solve B in polynomial-time, then we can solve A in polynomial-time too.

Definition 94 (NP-complete)
A problem A is NP-hard if every problem in **NP** is polynomial-time reducible to A, and NP-complete if A is NP-hard and NP.

Theorem 95

1. If $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$.
2. An NP-complete problem A is in **P** iff $\mathbf{P} = \mathbf{NP}$.
3. If $A \leq_p B$ and A is NP-hard, then B is NP-hard.

Proof. (1) Let f be a reduction from A to B with polynomial time $p(n)$, and g from B to C with $q(n)$. Then $g \circ f$ is a reduction from A to C with polynomial time $q(p(n))$.

(2) Suppose A is NP-complete and in **P**. Then any problem B in **NP** can be polynomial-time reduced to A, so transitivity implies that B is polynomial-time computable. The converse is trivial.

(3) Any problem C in **NP** can be polynomial-time reduced to A. Transitivity implies that C can be polynomial-time reduced to B. □

Now the obvious question is, does such a strong problem actually exist? The answer is yes, and a lot of important problems are NP-complete.

(TODO: SAT)

Having proven that SAT is NP-hard, more problems can be proven NP-hard if we can reduce SAT to those problems in polynomial-time. Here are only a tiny fraction of the NP-complete problems:

Definition 96 (NP-complete problems)

- The 3-SAT problem is a SAT problem where each clause contains exactly 3 variables.

- Given a graph G and an integer $0 \leq k \leq |V(G)|$, the clique problem asks whether there is a complete induced subgraph of G with size at least k.
- The independent set problem asks whether there is a subset S of $V(G)$ with size at least k such that no two vertices in S are adjacent, and 0 otherwise.
- The vertex cover problem asks whether there is a subset S of $V(G)$ with size at most k such that each edge is adjacent to at least one vertex in S.
- The chromatic number problem asks whether G is 3-colorable.
- Given a set S of integers and an integer k, the subset sum problem asks whether there is a subset of S whose sum of elements equals k.
- Given an $n \times m$ matrix A and an $n \times 1$ matrix b of integers, the integer programming problem asks whether there is an $m \times 1$ matrix x of integers such that each element of $Ax+b$ is non-negative.

Theorem 97
All problems in 96 are NP-complete.

Proof. 123 □

Chapter 12

Graph Theory

12.1 Basic Graph Definitions

12.2 Degrees

12.3 Trees

12.4 Planar Graphs

12.5 Coloring

Definition 98 (Coloring)
A k-coloring of a graph G is a function $c : V(G) \to \{1, 2, \cdots, k\}$ such that if u and v are adjacent vertices, then $c(u) \neq c(v)$. G is k-colorable if there is a k-coloring of G. The chromatic number $\chi(G)$ of G is the smallest integer k such that G is k-colorable.

Perhaps the most famous theorem about graph coloring is the four-color theorem. (TODO: write something)

Theorem 99 (Four-color Theorem)
If G is planar, then $\chi(G) \leq 4$.

Unfortunately, the proof is too long and complicated to contain in the codex. We prove a weaker result:

Theorem 100 (Five-color Theorem)
If G is planar, then $\chi(G) \leq 5$.

Proof. Induction on $|V(G)|$. For $|V(G)| \leq 5$, the theorem is trivial.

From [TODO: which theorem?], G has a vertex v of degree at most 5. If $deg_G(v) < 5$, then inductively find a 5-coloring of $G - v$, and color v by some color in $\{1, 2, 3, 4, 5\}$ not appearing in the neighbors of v. If $deg_G(v) = 5$ and not all colors are used in the neighbors of v, then the same argument applies.

Now suppose all 5 colors are used. Denote the neighbors of v as u_1, u_2, u_3, u_4, u_5, in clockwise order. Without loss of generality, we will assume that $c(u_i) = i$.

The main idea of the rest of the proof is that we want to change the color of one of the neighbors, say change $c(u_i)$ to k. This is impossible if u_i has a neighbor of color k, in which case we want to also change the color of that neighbor, to k'. But then that neighbor might have yet another neighbor of color k', and this continues to form a chain. Hence we introduce the Kempe chain, named after Alfred Kempe.

Let V_{ij} be the set of vertices w in G such that there is a path from u_i to w consisting of vertices of color i or j. Note that if we switch the colors of the vertices in V_{ij} (i.e. change i to j and j to i), and leave everything else the same, then the result is still a coloring.

If V_{13} does not contain u_3, then switch the colors of the vertices in V_{13} and color v by 1. Otherwise, V_{24} does not contain u_4; switch the colors of the vertices in V_{24} and color v by 2. This gives a 5-coloring of G.

(TODO: picture) $\square$

Fun fact: In 1879, the Kempe chain method was used to "prove" the four-color theorem by Alfred Kempe. No one noticed that this "proof" had an error until eleven years later when Percy Heawood found the error. What we saw above is the modification of the incorrect proof to prove the weaker theorem. The correct proof of four-color theorem was completed in 1976 by Kenneth Appel and Wolfgang Haken.

Chapter 13

Cryptosystem

13.1 Basic Terminology

Definition 101 (Basic Terminology on Cryptosystems)

- **Plaintext:** The text before encryption
- **Ciphertext:** The text after encryption
- **Cryptosystems:** Encryption and decryption algorithms, see definition below for more

  Encryption: Using some sort of algorithm to change the content of a message so that it is unrecognizable.

  Decryption: Processing the encrypted message to change it back to the message.
- **Key:** A value required to encrypt or decrypt.

  Encryption Key: The key for encryption.

  Decryption Key: The key for decryption.
- **Cryptanalysis:** Decrypting the ciphertext without any prior knowledge(i.e. key).

Definition 102 (Cryptosystem)
A cryptosystem is defined as a set of three algorithms, (G, E, D);

G The key generation algorithm, sometimes abbreviated as KeyGen, chooses the encryption key k_1 and the decryption key k_2 from the set of possible keys. The set of possible keys is called the key space. Usually each key from the key space is chosen at uniformly random probability.

E The Encryption Algorithm, sometimes abbreviated as Enc, uses the encryption key k_1, takes the plaintext m as an input, and produces the ciphertext c. This is usually denoted as follows:

$$E_{k_1}(m) = c$$

D The Decryption Algorithm, sometimes abbreviated as Dec, uses the decryption key k_2, takes the ciphertext c as an input, and gains the plaintext m. This is usually denoted as follows:

$$D_{k_2}(c) = m$$

For a cryptosystem to be valid, by encrypting the plaintext m and decrypting the ciphertext, we must be able to get m, that is;

$$D_{k_2}(E_{k_1}(m)) = m$$

A cryptosystem is classified into two categories; if the encryption key is the same as the decryption key, it is called a Symmetric Key Algorithm; if not, it is called an Asymmetric Key Algorithm or a Public Key Algorithm. A symmetric key algorithm is again classified into two categories; Block Cipher and Stream Cipher.

Definition 103 (Kerckhoffs' Principle)
Kerckhoffs' Principle states that a cryptosystem must be secure even if everything about the cryptosystem except for the key is exposed.

Kerckhoffs' Principle says that the cryptosystem's security must depend only on the secrecy of the key. Its core comes from the idea that "The enemy knows the system". In some, "Security through obscurity"(i.e. hiding the cryptosystem itself) holds but Kerckhoffs' Principle has its value for the following reasons:

1. Storing a smaller sized key is easier than hiding the entire cryptosystem. Also the cryptosystem is not safe from reverse engineering, but keys are, as they are usually a random number.
2. If the key is exposed, it is easier to change only the key, not the entire cryptosystem.
3. A cryptosystem is often used for many users, and everybody using the same cryptosystem allows for more efficient usage of space.
4. If the cryptosystem itself is kept a secret, if a problem arises(i.e. reverse engineering) to expose the cryptosystem, then the entire thing must be redesigned. This takes a lot of knowledge and time.
5. A cryptosystem is made weak by a small mistake; these mistakes are not found before the cryptosystems are analyzed fully, which is most easily done by making the system public. If they are indeed made public, the cryptosystem can be checked for security, allowing for a more secure system.

13.2 Encryption of Arbitrary Length Message

13.2.1 Padding

When using a block cipher, we need the length of the message to be an exact multiple of the length of the block used in the block cipher. If not, we use padding to make the message longer to make it an exact multiple. There are many ways to do so, but:

- Zero Padding, otherwise known as Null Padding

  Pad the message with zero(00) bytes to make the length be an exact multiple of the cipher block length. This may cause a problem if the last bytes of the message are 00.

- Bit Padding

  Pad the message with $10|00^n$, so that we can know the start of padding. We must pad the message even if its length is a multiple of the cipher block length.

- Byte Padding

  Same as zero padding, except the last byte is equal to the length of padding, that is; if we require four more bytes, the padding is $00|00|00|04$. We must pad the message even if its length is a multiple of the cipher block length.

- PKCS#7 Padding

  Similar to byte padding, except every byte of the padding is equal to the length of padding, that is; if we require four more bytes, the padding is $04|04|04|04$.

13.2.2 Modes of Operation

Sometimes we are required to encrypt a longer message than the length of the block. The plaintext are first padded using one of the techniques above, and the padded plaintext P is separated into blocks of padding length, $P_1, P_2, \cdots, P_N$. They are then encrypted using the key K, sometimes with the help of the initialization vector IV, and produces the ciphertexts $C_1, C_2, \cdots, C_N$. There are five major ways to do this; ECB, CBC, CFB, OFB, and CTR.

Electronic Code Book(ECB)

ECB mode is the simplest mode of them all. They simply take each blocks and encrypt them separately. In equation:

- **Encryption** $C_i = E_K(P_i)$
- **Decryption** $P_i = D_K(C_i)$

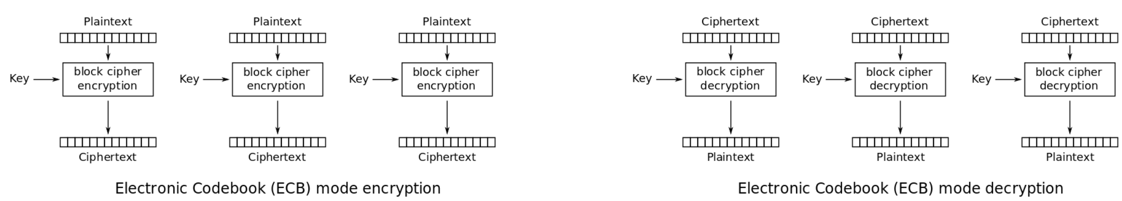


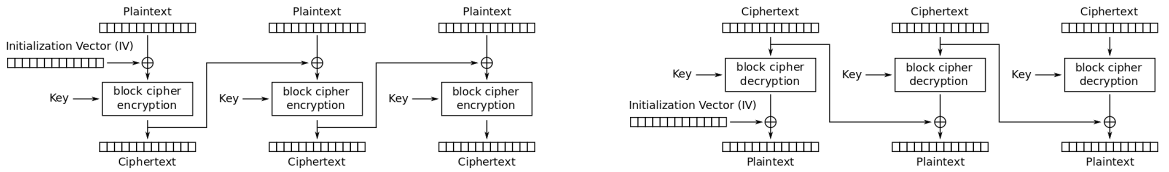
Figure 13.1: ECB Mode

Since same plaintext blocks are encrypted into same ciphertext block, the blocks can be copied, or replayed, to change the message easily. This is called the Block Replay Attack.

Cipher Block Chaining(CBC)

CBC takes the previous ciphertext block and XOR($\bigoplus$) it with the plaintext before encryption. The first block has no previous ciphertext block, hence it is XOR-ed with the IV. In equation:

- **Encryption** $C_0 = IV$, $C_i = E_K(P_i \bigoplus C_{i-1}), i = 1, 2, 3, \cdots, N$
- **Decryption** $C_0 = IV$, $C_i = D_K(C_i) \bigoplus C_{i-1}, i = 1, 2, 3, \cdots, N$



Cipher Block Chaining (CBC) mode encryption

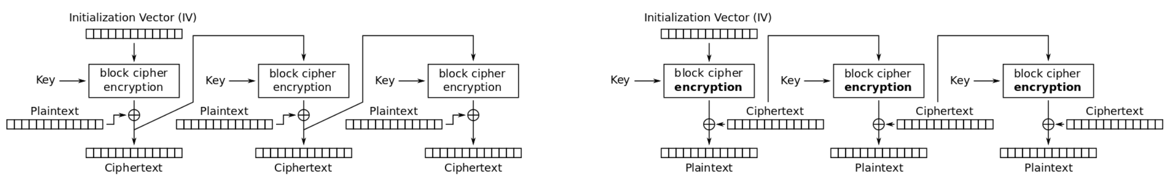
Cipher Block Chaining (CBC) mode decryption

Figure 13.2: CBC Mode

Cipher Feedback(CFB)

CFB can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher. In the diagram given below, original block sizes are used. In equation:

- **Encryption** $C_0 = IV$, $C_i = E_K(P_i \bigoplus C_{i-1}), i = 1, 2, 3, \cdots, N$
- **Decryption** $C_0 = IV$, $C_i = D_K(C_i) \bigoplus C_{i-1}, i = 1, 2, 3, \cdots, N$



Cipher Feedback (CFB) mode encryption

Cipher Feedback (CFB) mode decryption

Figure 13.3: CFB Mode

By altering the equation to the following we have the "stream cipherized" version, where << is the shift operation, $head(a, x)$ is the first x bits of a, and n is the size of the IV:

- **Shift Register** $S_0 = IV$, $S_i = ((S_i << x) + C_i) \mod 2^n$
- **Encryption** $C_i = head(E_K(S_{i-1}), x) \bigoplus P_i$
- **Decryption** $P_i = head(E_K(S_{i-1}), x) \bigoplus C_i$

Output Feedback(OFB)

OFB can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher.

- **Input and Output** $I_0 = IV$, $I_j = E_K(I_{j-1}), j = 1, 2, 3, \cdots, N$
- **Encryption** $C_j = P_j \bigoplus I_j, i = 1, 2, 3, \cdots, N$
- **Decryption** $P_j = C_j \bigoplus I_j, i = 1, 2, 3, \cdots, N$

We can similarly alter the equation as OFB so that it can be used as a stream cipher.

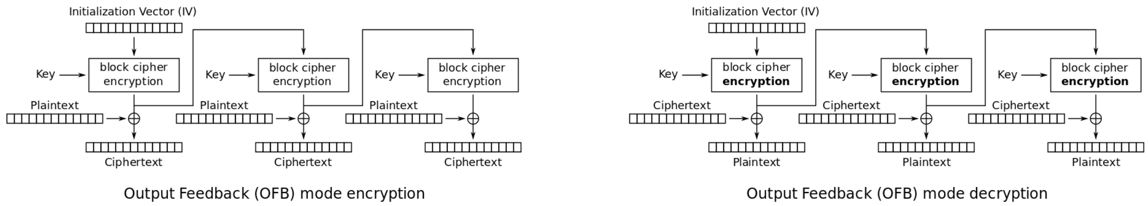


Figure 13.4: OFB Mode

Counter(CTR)

CTR can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher. It encrypts the counter value instead of the plaintext, and XORs the value to gain the ciphertext.

- **Encryption** $C_i = P_i \bigoplus E_K(Counter), i = 1, 2, 3, \cdots, N$
- **Decryption** $P_i = C_i \bigoplus E_K(Counter), i = 1, 2, 3, \cdots, N$

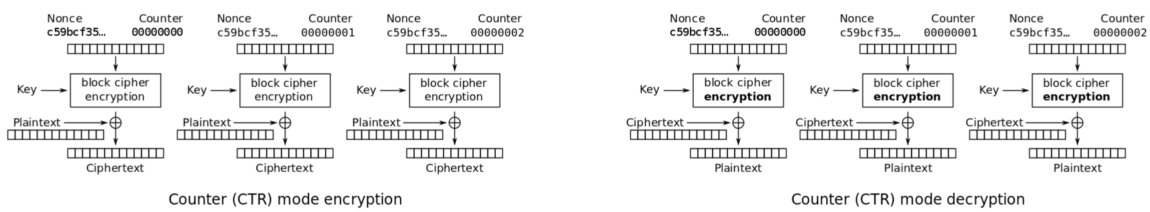


Figure 13.5: CTR Mode

We can similarly alter the equation as OFB so that it can be used as a stream cipher.

Characteristics

Table 13.1 shows the characteristics for each modes of operation.

- Block Pattern: Whether if the overall pattern is kept after encryption
- Preprocessing: Whether if preprocessing is possible on encryption and decryption
- Parallel Processing: Whether if parallel processing is possible on encryption or decryption
- Error Propagation: If there is an error in the encryption/decryption process, whether if the error spreads through other blocks
- Encryption Unit: The minimum requirement byte for encryption

		ECB	CBC	CFB	OFB	CTR
Block Pattern		O	X	X	X	X
Preprocessing		X	X	X	O	O
Parallel Processing	Encryption	O	X	X	O	O
	Decryption	O	O	O	O	O
Error Propagation		X	(P_i, P_{i+1})	$\lceil \frac{n}{r} \rceil$ blocks	X	X
Encryption Unit		n	n	$r(\leq n)$	$r(\leq n)$	$r(\leq n)$

Table 13.1: Characteristics for Each Modes of Operation

13.3 Types of Attack

13.3.1 Attacking Classical Cryptosystems

Classical Cryptosystems are typically a substitution cipher and/or a transposition cipher. Since most, if not all, the classical cryptosystems are broken, the two valid ways to attack any classical cryptosystems is given here.

- Brute Force Attack

  When the attacker gains the ciphertext c, the attacker uses every key possibility to try to gain m. This is otherwise known as the Exhaustive Key Search Attack. Theoretically this can be done to any symmetric-key cipher; but this is inapplicable to most modern cryptosystems as they have an extremely large key space.

- Frequency Analysis

  The plaintext having some pattern, such as the alphabet 'e' appearing with the most frequency, will help the attacker gain knowledge on the plaintext just by seeing the ciphertext.

13.4 Cryptographic Hash Functions

A general hash function has the following properties:

- They take an arbitrary size of data as input, and;
- They produce a constant and fixed length data as output.

A <u>cryptographic hash function</u>, in addition to the properties above, must have the following properties:

- Preimage Resistance

  If the hash value y is given, it must be hard to find an x such that $h(x) = y$, that is, the hash function must have one-wayness.

- Second Preimage Resistance

  If the message x is given, it must be hard to find an $x' \neq x$ such that $h(x) = h(x')$.

- Collision Resistance

  It must be hard to find $x \neq x'$ such that $h(x) = h(x')$. The pair (x, x') is called the collision pair.

13.5 Attacking the Cyrptosystems

Attacks on cryptosystems are classified into passive and active attack. Passive attacks simply eavesdrops the transmission, and gains what the attacker wants without modification of the message. This type of attacking includes eavesdropping, of which the attacker intercepts the message in the middle to check the plaintext. This type of attacker is often referred to as ”Eve”(as in eavesdropping) in theories. Active attackers will modify the message, which includes Modification, Deletion, Impersonation, and Replay. This type of attacker can also be referred to as ”Eve”, but sometimes is referred to as ”Mallory”, for malicious user.

- Modification: Changes the order of the message or changes a part of it to alter the meaning.
- Deletion: Intercepts the message and does not send it, interrupting the communication.
- Impersonation: Fakes their own identity to be identified as a correct user.
- Replay: Send a message again after eavesdropping, expecting some kind of result.

The four methods above are just the general ways to attack. We need to attack the system itself to know how to attack it. There are four methods of attack on system:

- **Ciphertext Only Attack**

  The attacker knows only the ciphertext.

- **Known Plaintext Attack**

  The attacker knows a list of (message, ciphertext) pair, and attempts to crack a ciphertext not in the list.

- **Chosen Plaintext Attack**

  The attacker has access to an oracle that can encrypt the message, and attempts to crack a ciphertext.

- **Chosen Ciphertext Attack**

  The attacker has access to an oracle that can decrypt a ciphertext, except for the target ciphertext.

There are three important properties to encryption schemes:

- Semantic Security

  A semantically secure encryption scheme is infeasible for any computationally bounded adversary to derive a significant information about the original plaintext when given only its ciphertext and the corresponding public key if any. This can be represented as a game between the oracle and the adversary, as below:

  1. The oracle generates a key for the challenge.
  2. The adversary is given the encryption oracle(or the public key, in the case of public key cryptosystem).
  3. The adversary can perform any number of polynomially bounded number of encryptions or operations.

4. The adversary generates two equal-length messages m_0 and m_1, and transmits it to the oracle.
5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.
6. The adversary, upon receiving C, guesses b.

If the adversary cannot guess b correctly with significantly greater than 50% probability, then the scheme is said to be semantically secure under CPA.

- Indistinguishability

  If a cryptosystem is indistinguishable, then an adversary would not be able to distinguish pairs of ciphertexts based on the message they encrypt. There are three types: IND-CPA, IND-CCA1, and IND-CCA2. They can be represented as a game between the oracle and the adversary. In both cases, they are said to be secure if the adversary does not have a clear advantage.

  - **IND-CPA**
    1. The oracle generates a key for the challenge.
    2. The adversary is given the encryption oracle(or the public key, in the case of public key cryptosystem).
    3. The adversary can perform any number of polynomially bounded number of encryptions or operations.
    4. The adversary generates two distinct equal-length messages m_0 and m_1, and transmits it to the oracle.
    5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.
    6. The adversary, upon receiving C, performs polynomially bounded encryptions or operations, and guesses b.
  - **IND-CCA**
    1. The oracle generates a key for the challenge.
    2. The adversary is given the decryption oracle and the public key, in the case of public key cryptosystem.

       Note that in the case of the public key cryptosystem, the encryption oracle is also given.
    3. The adversary can perform any number of polynomially bounded number of decryptions or operations.
    4. The adversary generates two distinct equal-length messages m_0 and m_1, and tramsmits it to the oracle.
    5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.
    6. The adversary, upon receiving C, performs polynomially bounded operations.

       In the case of IND-CCA1, the adversary may not make further calls to the decryption oracle.

       In the case of IND-CCA2, the adversary may make further calls to the decryption oracle, but may not submit C.
    7. The adversary guesses b.

  This can be said with a random oracle. In that case, the adversary submits only one message and the oracle returns the encryption of the message or the random string equal to the length of the encryption with a fair chance. The adversary then guesses whether if the message is randomly generated or encrypted.

- Non-malleability

  Cryptosystems are called "malleable" if it is possible to transform a ciphertext into another ciphertext which decrypts to a related plaintext. Cryptosystems that are not malleable are called non-malleable. These, similar to indistinguishability, can be represented as a game between the oracle and the adversary, and are called NM-CPA, NM-CCA1, NM-CCA2.

Theorem 104
The following relations for each security properties hold:

- IND-CPA $\Leftrightarrow$ Semantic security under CPA
- NM-CPA $\Rightarrow$ IND-CPA
- NM-CCA2 $\Leftrightarrow$ IND-CCA2
- NM-CPA does not necessarily imply IND-CCA2.

13.6 Digital Signatures

Digital signatures are used in pair with the public key cryptosystems to verify the sender of the messages. When attacking, there are three major methods:

- **Key-Only Attack**

  The attacker only has access to the digital signature algorithm and the public key of the signer, pk_A. This is similar to the Ciphertext Only attack.

- **Known Message Attack**

  The attacker has access to the digital signature algorithm, the public key of the signer, and a list of (message, signature) pairs. This is similar to the Known Plaintext attack.

- **Chosen Message Attack**

  The attacker has access to the digital signature algorithm, the public key of the signer, and an oracle that takes a message as an input and returns signature as an output.

The attacker can have three different purposes:

- **Total Break**

  The attacker wants to gain the private key of the signer.

- **Selective Forgery**

  The attacker wants to generate a valid signature for a message the attacker wants(i.e. any message for that matter).

- **Existential Forgery**

  The attacker wants to generate a valid (message, signature) pair for any message.

It is said that an attack is valid if the attack succeeds with a non-negligible probability.

13.7 Zero-Knowledge Authentication

13.8 RSA Cryptosystem

13.9 Rabin Cryptosystem

13.10 ElGamal Cryptosystem

13.11 NTRU Cryptosystem