Numerical Methods: 24 Hours Take-home examination, August 2023

This examination consists of 5 exercises. Each question has been assigned a weight (points). The total number of points is 100.

For the exam, you will need to be able to read matrices and vectors from files having the following format for an $M \times N$ matrix:

$$\begin{array}{r} M \\ N \\ Row1 \\ \dots \\ RowM \end{array}$$

where vectors always have $N = 1$. An example with a 3×2 matrix could be

$$\begin{array}{ll} 3 & \\ 2 & \\ 3.04464 & 5.06464 \\ -0.6454 & 0.435435 \\ 4.05454 & -1222.933435 \end{array}$$

You must hand in a zip file containing a report and all your used code. Concerning the report, it needs to be CLEARLY readable (unreadable parts will be assessed as wrong). Whenever the word "state" is used in the questions, the answer MUST be present in the report. Wherever it says "Submit the used code", the used code MUST be handed in. Otherwise, you will get zero points for your answer. Clearly name your code files so that it is easy to see what exercise the code is used for.

Notice the following rules:

- It is allowed to use general purpose methods, text or code that are part of NR or elsewhere publicly available, including methods, text or code that has been uploaded by Kristine or me during the course. However, there must for each use be a CLEAR marking **stated in the report** including where the methods/text/used code was taken from. In all cases, you are solely responsible for the correctness of used methods, text and code. All methods, text and code that **explicitly handles the problems from the exam exercise** MUST be written by yourself.
- It is NOT allowed to share your answers (or parts of these), or to communicate with other people about the exercise. This includes other NM students taking this 24 hours take-home examination.

I wish you all the best for the forthcoming 24 hours!

Best regards, Henrik

Exercise 1 (20 points)

Consider a Linear Least Squares problem with a 40×6 design matrix $\mathbf{A}$ and the right hand side $\mathbf{b}$ that are given in *Ex1A.dat* and *Ex1b.dat* respectively.

i) (5 points) Compute the matrix $\mathbf{C} = \mathbf{A}^T\mathbf{A}$ and right hand side $\mathbf{c} = \mathbf{A}^T\mathbf{b}$ for the "Normal Equations". State $\mathbf{c}$ and the diagonal elements in $\mathbf{C}$. Submit the used code.

ii) (5 points) Find the LU-decomposition $\mathbf{C} = \mathbf{LU}$. State the diagonal elements in $\mathbf{U}$. State the sequence of the rows after the permutations. Submit the used code.

iii) (5 points) Use the LU-decomposition to compute the solution $\mathbf{x}$ to $\mathbf{Cx} = \mathbf{c}$. State the solution $\mathbf{x}$. Submit the used code.

iv) (5 points) Assume now that you would know the Singular Value Decomposition $\mathbf{A} = \mathbf{UWV}^T$. State a clear explanation on how you would then estimate the error on the solution $\mathbf{x}$ found in iii).

Exercise 2 (20 points)

Consider the equations

$$\begin{aligned} x_0 + 2\sin(x_1 - x_0) - \exp(-\sin(x_1 + x_0)) &\equiv 0 \\ x_0 \cos(x_1) + \sin(x_0) - 1 &\equiv 0 \end{aligned}$$

i) (3 points) With $x_0 = 1$ and $x_1 = 1$, state (with at least 6 digits) the values of the left hand sides of the two equations. (HINT: you should get approximately 0.597 and 0.382 respectively). Submit the used code.

ii) (4 points) State which methods from the course you can apply for this problem.

iii) (6 points) Perform 6 iterations with a method from the course using $x_0 = 1$ and $x_1 = 2$ as the initial guess and state the values of x_0 and x_1 after each of the 6 iterations. Submit the used code.

iv)(7 points) Provide an estimate of the error on the solution after 6 iterations. State the result and state a detailed explanation on how you arrived at the result

Exercise 3 (20 points)

Consider the following system

$$\begin{aligned} v_1'(t) &= \exp(-t)\cos(v_2(t)) + v_3(t)^2 - v_1(t); \qquad & v_1(0) = 1 \\ v_2'(t) &= \cos(v_3(t)^2) - v_2(t); & v_2(0) = 2 \\ v_3'(t) &= \cos(t)\exp(-v_1(t)^2) - v_3(t); & v_3(0) = 3 \end{aligned}$$

i) (5 points) State the result for $(v_1'(0), v_2'(0), v_3'(0))$ with at least 6 digits. HINT: It should be approximately $(7.58, -2.91, -2.63)$. Submit the used code.

ii) (10 points) Use the Trapezoidal method with $N = 50, 100, 200, 400, 800$ to generate solutions for $(v_1(5), v_2(5), v_3(5))$. State $(v_1(5), v_2(5), v_3(5))$ for each N. Submit the used code.

You may now assume that the global order is 2 as expected.

iii) (5 points) State the accuracy for $N = 800$. State a clear and detailed explanation of how you arrived at this accuracy.

Exercise 4 (15 points)

Consider the problem

$$\begin{cases} y''(x) = \cos(y(x)^2) + y'(x) - \exp(-x^2); & 0 < x < 2 \\ y(0) = -1; \quad y(2) = 3 \end{cases}$$

i) (8 points) Use the Finite Difference method to find an approximation to the solution curve $y(x)$. Use $N = 2, 4, 8, 16, 32, \dots, 32768$. State the numerical estimate of $y(1)$ for each N with at least 10 digits. Submit the used code.

ii) (7 points) Determine the smallest N at which you can obtain an estimated accuracy on $y(1)$ of less than 10^{-6}. State N, state your estimate of the accuracy, and state an explanation on how you found this estimate.

Exercise 5 (25 points)

Consider the integral

$$\int_a^b \frac{\cos(x)\exp(-x^3)}{(x-a)^{\frac{3}{4}}}dx$$

i) (5 points) For $N = 2$, state by hand computations an analytical expression for the approximation of the integral as obtained by the Extended Midpoint Method (see (4.1.19) in Numerical Recipes or the first slide in the presentation from Week 9).

We now consider the case $a = 1$, $b = 5$. We wish to approximate the integral using the Extended Midpoint Method method.

ii) (10 points) With $N - 1 = 2^k; \quad k = 1, \ldots, 20$ use the Extended Midpoint Method method to approximate the integral. State the results in a table similar to those used during the course. Submit the used code.

iii) (5 points) Use Richardson extrapolation to estimate the order at $N = 2^{20}$. State the result. If the estimated order is different than the expected order provide an explanation for the difference.

iv) (5 points) State the estimated accuracy on the result at $N = 2^{20}$ using the estimated order. State clearly how you compute the accuracy estimate.