Numerical Methods: 24 Hours Take-home examination, June 2022

This examination consists of 5 exercises. Each question has been assigned a weight (points). The total number of points is 100.

For the exam, you will need to be able to read matrices and vectors from files having the following format for an $M \times N$ matrix:

$$\begin{array}{r} M \\ N \\ Row1 \\ \dots \\ RowM \end{array}$$

where vectors always have $N = 1$. An example with a 3×2 matrix could be

$$\begin{array}{ll} 3 & \\ 2 & \\ 3.04464 & 5.06464 \\ -0.6454 & 0.435435 \\ 4.05454 & -1222.933435 \end{array}$$

In addition you will need to be able to perform the following plots: Parametrized analytical expressions in $2D$ (curves) and $3D$ (curves or surfaces); lists of points in $2D$ or $3D$.

You must hand in a zip file containing a report and all your used code. Concerning the report, it needs to be CLEARLY readable (unreadable parts will be assessed as wrong). Whenever the word "state" is used in the questions, the answer must be present in the report. Wherever it says "Submit the used code", the used code MUST be handed in. Otherwise, you will get zero points for your answer. Clearly name your code files so that it is easy to see what exercise the code is used for.

Notice the following rules:

- It is allowed to use general purpose methods, text or code that are part of NR or elsewhere publicly available, including methods, text or code that has been uploaded by Ole or me during the course. However, there must for each use be a CLEAR marking **stated in the report** including where the methods/text/used code was taken from. In all cases, you are solely responsible for the correctness of used methods, text and code. All methods, text and code that **explicitly handles the problems from the exam exercise** MUST be written by yourself.
- It is NOT allowed to share your answers (or parts of these), or to communicate with other people about the exercise. This includes other NM students taking this 24 hours take-home examination.

I wish you all the best for the forthcoming 24 hours !. Best regards, Henrik

Exercise 1 (15 points)

Consider the linear equation $\mathbf{Ax} = \mathbf{b}$ where the 6×6 dimensional coefficient matrix A (which can be assumed to be symmetric and positive definite) and the right hand side $\mathbf{b}$ are given in *Ex1A.dat* and *Ex1b.dat*.

i) (5 points) Find the Cholesky Decomposition $\mathbf{A} = \mathbf{L}\mathbf{L}^T$. State the diagonal elements in $\mathbf{L}$. Submit the used code.

ii) (5 points) Use the Cholesky Decomposition to compute the solution $\mathbf{x}$ to $\mathbf{Ax} = \mathbf{b}$. State the solution $\mathbf{x}$. Submit the used code.

iii) (5 points) Consider an arbitrary symmetric $N \times N$ matrix $\mathbf{A}$ and assume that we have performed an SVD to obtain the matrices $\mathbf{U}$, $\mathbf{W}$ and $\mathbf{V}$ so that $\mathbf{A} = \mathbf{UWV}^T$. What can be said about the relation between $\mathbf{U}$ and $\mathbf{V}$? State your answer with a clear argument.

Exercise 2 (20 points)

Consider two parametrized curves in $\mathbb{R}^3$ with functions $\mathbf{r}_A(u_A)$ and $\mathbf{r}_B(u_B)$. We now want to find candidates for points on the two curves with minimal distance between them. That is, we want to minimize the distance

$$D(u_A, u_B) = \|\mathbf{r}_A(u_A) - \mathbf{r}_B(u_B)\|$$

A necessary condition for a point pair $\mathbf{r}_A(u_A)$ and $\mathbf{r}_B(u_B)$ to lead to a minimal distance is that the first order conditions are satisfied. These are

$$\begin{aligned} \frac{\partial D(u_A, u_B)}{\partial u_A} &= 0 \\ \frac{\partial D(u_A, u_B)}{\partial u_B} &= 0 \end{aligned}$$

which when inserting the parametrized curves and simplify can be written as

$$\begin{aligned} f_0(u_A, u_B) &\equiv \mathbf{r}'_A(u_A) \cdot (\mathbf{r}_A(u_A) - \mathbf{r}_B(u_B)) = 0 \\ f_1(u_A, u_B) &\equiv -\mathbf{r}'_B(u_B) \cdot (\mathbf{r}_A(u_A) - \mathbf{r}_B(u_B)) = 0 \end{aligned} \tag{1}$$

Below, you may use the finite difference approximations

$$\begin{aligned} \mathbf{r}'_A(u_A) &\simeq (\mathbf{r}_A(u_A + \epsilon) - \mathbf{r}_A(u_A))/\epsilon \\ \mathbf{r}'_B(u_B) &\simeq (\mathbf{r}_B(u_B + \epsilon) - \mathbf{r}_B(u_B))/\epsilon \end{aligned}$$

using $\epsilon = 10^{-8}$.

We now consider the example

$$\begin{aligned} \mathbf{r}_A(u_A) &= \left(a_1 \cos(1 + u_A)^3, a_2 u_A^2, a_3 u_A \sin(u_A)\right) \\ \mathbf{r}_B(u_B) &= \left(b_1(u_B + \exp[-u_B^2]), b_2 u_B^3, b_3 \cos(u_B)\right) \end{aligned}$$

where $a_1 = 1.1$, $a_2 = 2.1$, $a_3 = 0.8$, $b_1 = 0.4$, $b_2 = 1.3$, $b_3 = 0.5$

i) (5 points) State (with at least 6 digits) the values $f_0(u_A, u_B)$ and $f_1(u_A, u_B)$ for $u_A = 1$ and $u_B = -1$ using your code (HINT: You should obtain $f_0 \simeq 14.6$ and $f_1 \simeq -13.6$). Submit the used code.

ii) (8 points) With the initial guess $u_A = 1$ and $u_B = -1$, use Newtons method to find the solution to the equations in Eq.(1) after 10 iterations. State the solution. Submit the used code.

iii) (7 points) In the method used in [ii)], what was the smallest number of iterations needed to obtain a proven accuracy of around 10^{-4}. State how you arrive at your result including how you estimate the accuracy and (if relevant) which convergence constant you used.

Exercise 3 (25 points)

Dynamic Movement (or Motion) Primitives (abbreviated DMP's) deploy differential equations to guide a robot towards a goal using a taught trajectory. Here, we will study this method in its most simplistic formulation.

$$\begin{aligned}
y_1''(t) &= a_1\{b_1[g_1 - y_1(t)] - y_1'(t)\} + f_1(x(t)); \quad y_1(0) = Y_1; \quad y_1'(0) = 0 \\
y_2''(t) &= a_2\{b_2[g_2 - y_2(t)] - y_2'(t)\} + f_2(x(t)); \quad y_2(0) = Y_2; \quad y_2'(0) = 0 \\
x'(t) &= -x(t); \quad x(0) = 1
\end{aligned} \tag{2}$$

where we require a_1, a_2, b_1, b_2 are all positive. If we set $f_1(x(t)) = f_2(x(t)) = 0$, the y_1 and y_2 equations are independent damped mass-spring equations with equilibrium at g_1 and g_2 respectively. In robotics, (g_1, g_2) can be viewed as the goal for a $2D$ point robot with position $(y_1(t), y_2(t))$ where t is time. Hence the robot will reach (g_1, g_2) but there is no control over which 2D trajectory it takes.Therefore, the forcing terms $f_1(x)$ and $f_2(x)$ are introduced to force the robot to move in a particular way towards the goal. The coordination of the forcing is handled via the phase $x(t)$. As can be seen, the phase $x(t)$ goes from one to zero as time t goes from zero to infinity.

i) (8 points) Reformulate the differential equations in (2) to the initial value problem in the usual format of a set of first order ODE's. State each of the resulting differential equations including the initial value.

We now study an example where the robot starts at $(Y_1, Y_2) = (0, 0)$ with a goal $(g_1, g_2) = (2, 1)$. The forcing terms are $f_1(x) = x$ and $f_2(x) = 3x(1 - x^2)$. We use the parameters $a_1 = b_1 = 1$ and $a_2 = b_2 = 0.25$.

ii) (8 points) Use the Midpoint method with $h = 0.001$ to compute the robot trajectory $(y_1(t), y_2(t))$ for $0 \le t \le 20$. State a plot of the trajectory . Submit the used code.

iii) (9 points) Use the Midpoint method with $N = 5, 10, 20, 40, \ldots; h = 5.0/N$ to generate solutions for $y_2(5)$. You may assume that the global order is 2 as expected. Use as few number of subdivisions as possible to reach a proven accuracy of better than 10^{-4}.

Exercise 4 (20 points)

Consider a surface S in three dimensional space and consider any two points $\mathbf{r}_A$ and $\mathbf{r}_B$ on the surface. The geodesic curve between the two points is then the curve residing on the surface, which has the shortest length. Hence if the surface S is a plane, the geodesic curves will always be the straight line segment between $\mathbf{r}_A$ and $\mathbf{r}_B$. Another well known example is a sphere, where the geodesic curves are segments of grand circles between $\mathbf{r}_A$ and $\mathbf{r}_B$. Deriving geodesic curves for more arbitrary surfaces is relevant for many engineering applications such as in composite engineering.

In the remainder of this exercise, we consider a case, where the surface S can be written as a parametrization

$$\mathbf{r}(u,v) \equiv (u, v, v^4 - 2u^4); \quad -1 \le u, v \le 1$$

where we now wish to numerically estimate the geodesic curve between $\mathbf{r}_A = \mathbf{r}(u_A, v_A)$ and $\mathbf{r}_B = \mathbf{r}(u_B, v_B)$. Assuming that $u_A < u_B$ and that the geodesic curve runs purely on the interior of the surface, it can by calculus of variations be shown that the geodesic curve is given as $\mathbf{r}(u, V(u))$ where $V(u)$ is the solution to the boundary value problem

$$\begin{cases} V''(u) = \frac{48[V(u)^3 + 2u^3 V'(u)][2u^2 - V(u)^2 V'(u)^2]}{1 + 64u^6 + 16V(u)^6} & u_A < u < u_B \\ V(u_A) = v_A; \quad V(u_B) = v_B & \end{cases}$$

We select the end point parameters $u_A = -0.9$, $v_A = -0.85$, $u_B = 0.8$, $v_B = -0.9$.

i) (15 points) Use the Finite Difference method to find an approximation to the solution curve $V(u)$. Determine $V(\frac{u_A+u_B}{2})$ with a proven accuracy ϵ satisfying that $\epsilon \le 10^{-6}$. State your approximations for $V(\frac{u_A+u_B}{2})$ (preferably in a table similar to those used in the course. Use $N = 2, 4, 8, 16, 32, \ldots$). Also state the value of N where your accuracy was fulfilled.

ii) (5 points) State in the report: A plot of the surface S together with the points on your solution curve $\mathbf{r}(u_i, v_i)$; $\quad i = 0, \ldots, N$ where the u_i, v_i's denote the points from the largest N used. Make a statement whether you believe that your result is correct (based on a qualitative assessment of the solution curve).

Exercise 5 (20 points)

Consider the integral

$$\int_a^b \frac{\cos(x^3)\exp(-x)}{\sqrt{x}}dx$$

i) (5 points) For $N = 2$ (only one midpoint), state by hand computations an analytical expression for the approximation of the integral as obtained by the Extended Midpoint method.

We now consider the case $a = 0$, $b = 4$. We wish to approximate the integral using the Extended Midpoint method. For this, we as usual split the integration interval into N equidistant subintervals.

ii) (10 points) With $N = 2, 3, 5, 9, ...$ (corresponding to $N - 1 = 1, 2, 4, 8, ...$) use the Extended Midpoint method to approximate the integral. Terminate the subdivisions when you reach a proven accuracy of better than 10^{-3}. State your results in a table similar to those used during the course and state how you computed the accuracy. State also how many f-computations (computations of the integrand for a given x) were needed. Submit the used code.

iii) (5 points) Use DErule to approximate the integral. State your result and how many f-computations were applied. Submit the used code.