

# Tests for Relevance and Redundancy of Moment Conditions\*

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## Abstract

In the paper, we consider generalized method of moments (GMM) and propose statistics to test relevance, conditional relevance and redundancy of moment conditions, as well as their modifications to check if moment conditions help with identification of a particular set of unknown parameters. The statistics have, under the null hypothesis and when the estimator of moments covariance matrix converges at the rate  $\sqrt{T}$ , a  $\chi^2$  limiting distribution. We show that, though the concepts of redundancy and conditional relevance are very close, necessary and sufficient conditions for them may differ, which happens in the case of partial redundancy and relevance.

## 1 Introduction

In GMM, the issue of identification is well examined. The consequences of weak (or lack of) identification are summarized in the review of Stock et al. (2002). A number of tests for weak (lack of) identification were proposed by Cragg and Donald (1993), Stock and Yogo (2005), Hahn and Hausman (2002), Wright (2003), Inoue and Rossi (2011). Various asymptotic approximations for the finite-sample behavior of estimators can be used (Staiger and Stock, 1997; Han and Phillips, 2006; Newey and Windmeijer, 2009). Inference techniques which are robust to the weak (lack of) identification are available (Wang and Zivot, 1998; Kleibergen, 2005; Moreira, 2009). Procedures to select relevant moment conditions in many-moments asymptotics

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are developed by, among others, Hall and Peixe (2003), Hall et al. (2007), Cheng and Liao (2015).

A one of the few white spots in this issue is the test for relevance of a particular set of moment conditions. In this paper, we color this white spot. We show that the necessary and sufficient condition for redundancy, developed by Breusch, Qian, Schmidt, and Wyhowski (1999), implies that the set of moment conditions is conditionally irrelevant in the sense that it won't help with identification — the curvature of the GMM population objective function at true values of parameters stays the same if we drop these moment conditions. We propose a test statistic based on this condition, which has, under the null hypothesis of moments redundancy (conditional irrelevance) and when the estimator of moments covariance matrix converges at the rate  $\sqrt{T}$ , a  $\chi^2$  limiting distribution. This statistic can also be used to test for the partial irrelevance, which means that moment conditions do not help with identification of a subset of unknown parameters. We show, however, that partial irrelevance does not imply partial redundancy, which requests additional restrictions to hold.

Note that condition of Breusch, Qian, Schmidt, and Wyhowski (1999) is for *conditional* relevance, which means that a set of moment conditions can help with identification *given other moment conditions*, which are used for estimation. We propose a statistic to test for *unconditional* relevance, which means that a set of moment conditions can, maybe potentially, help with identification *by itself*. This statistic is much simpler and its asymptotic behavior does not depend on the rate of convergence of moments covariance matrix estimator.

The statistics we propose can be useful in models with rational expectations, such as the consumption-based capital asset pricing model (C-CAPM). A simple version of C-CAPM, after a small transformation, implies that

$$\mathbb{E} \left( \left( \beta \left( \frac{C_{t+1}}{C_{t-1}} \right)^{-\gamma} R_{t,t+1} - \left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \right) z_t \right) = 0, \quad (1.1)$$

where  $C_t$  is consumption at time  $t$ ,  $R_{t,t+1}$  is the asset return between  $t$  and  $t + 1$ ,  $z_t$  is the vector of instrumental variables,  $\beta$  is the subjective discount factor, and  $\gamma$  is the coefficient of relative risk aversion. The condition for a subset  $z_{2t}$  of  $z_t$  to be partially relevant for identification of  $\beta$ ,

$$\mathbb{E} \left( \left( \frac{C_{t+1}}{C_{t-1}} \right)^{-\gamma} R_{t,t+1} z_{2t} \right) \neq 0, \quad (1.2)$$

implies that  $z_{2t}$  affects agents expectations of  $(C_{t+1}/C_{t-1})^{-\gamma} R_{t,t+1}$ , which determine current consumption level  $C_t$ . Thus, the test for partial relevance can be used to check if instruments  $z_{2t}$  affect agents expectations.

Another example, where these statistics can be used, is motivated by Cragg (1983), who shows that, in the linear regression model with heteroscedastic disturbances, the efficiency of an estimator can be improved by using squares of regressors and their cross-products as instruments in addition to original regressors. Thus, the rejection of redundancy of these additional instruments may indicate heteroscedasticity.

One important note is that, to derive limiting behavior of the statistics, we consider standard GMM asymptotics and assume that moment conditions, which are not under test, are enough for the strong identification of unknown parameters. We do not consider the problem of weak identification or the question of selecting relevant moment conditions under many-moments asymptotics.

The paper is organized as follows. In the second section, we make main assumptions. In the third and the forth sections, we discuss conditions for relevance, conditional relevance and redundancy, construct statistics and derive their limiting behavior. In the fifth section, we provide results for partial relevance and partial redundancy. In the sixth section, we discuss some special case. In the seventh, we provide results of Monte-Carlo simulations. The eighth section concludes.

We use the following notation throughout the paper:  $\mathbb{E}(X)$  and  $\text{var}(X)$  are respectively the expected value and the variance of the random variable  $X$ ,  $\text{cov}(X, Y)$  is the covariance of the random variables  $X$  and  $Y$ ,  $\text{rank}(A)$  is the rank of the matrix  $A$ ,  $\text{vec}(A)$  denotes column vectorization of the matrix  $A$ ,  $\text{vech}(A)$  denotes half-vectorization of the symmetric matrix  $A$ ,  $I_m$  is the  $m \times m$  identity matrix,  $0_{m \times n}$  is the  $m \times n$  zero matrix,  $D_m$  and  $L_m$  are, respectively, the duplication and elimination matrices, so that for the  $m \times m$  matrix  $A$  we have  $D_m \text{vech}(A) = \text{vec}(A)$  and  $L_m \text{vec}(A) = \text{vech}(A)$ ,  $\rightarrow_p$  indicates convergence in probability, and  $\rightarrow_d$  indicates convergence in distribution.

## 2 Assumptions

We consider the estimation of the  $k \times 1$  parameter vector  $\theta \in \Theta$ , for which the set of moment conditions

$$\mathbb{E}(f(Z_t, \theta_0)) = 0 \quad (2.1)$$

holds. Here  $f(Z_t, \theta)$  is the  $m \times 1$  vector of functions of  $\theta$ , which are finite, continuous, and twice continuous differentiable,  $\theta_0$  is the interior of  $\Theta$  and is the unique value of  $\theta$ , at which (2.1) holds,  $\Theta$  is a compact subset of  $\mathbb{R}^k$ , and  $\{Z_t : t = 1, \dots, T\}$  is the observed data set.

The optimal GMM estimator<sup>1</sup> of Hansen (1982) is given by

$$\hat{\theta} = \arg \min_{\theta \in \Theta} f_T(\theta)' \hat{\Omega}^{-1} f_T(\theta), \quad (2.2)$$

where  $f_T(\theta) = T^{-1} \sum_{t=1}^T f(Z_t, \theta)$ ,  $\hat{\Omega}$  is a consistent estimator of  $\Omega = \Omega(\theta_0)$ ,  $\Omega(\theta) = \lim_{T \rightarrow \infty} \text{var}(\sqrt{T} f_T(\theta))$  is the  $m \times m$  covariance matrix of  $f_T(\theta)$ .

Let  $G_T(\theta) = \partial f_T(\theta) / \partial \theta'$  be the  $m \times k$  matrix of the first derivatives of moment functions  $f_T(\theta)$ ,  $\Omega_T(\theta)$  be an estimator of  $\Omega(\theta)$ , which can be different from  $\hat{\Omega}$ ,  $g_T(\theta) = \text{vec}(G_T(\theta)')$  be the  $mk \times 1$  vector of elements of  $G_T(\theta)$ ,  $\omega_T(\theta) = \text{vech}(\Omega_T(\theta))$  be the  $(m+1)m/2$  vector of elements of  $G_T(\theta)$  and  $\Omega_T(\theta)$ , respectively,  $H_T(\theta) = \partial g_T(\theta) / \partial \theta'$  is the  $km \times k$  matrix of the second derivatives of  $f_T(\theta)$ , and  $Q_T(\theta) = \partial \omega_T(\theta) / \partial \theta'$  is the  $(m+1)m/2 \times k$  matrix of the first derivatives of  $\omega_T(\theta)$ . And let  $r_T(\theta)$  be the  $(m+mk+(m+1)m/2) \times 1$  vector, composed of the elements of moment functions  $f_T(\theta)$ , its first derivatives  $G_T(\theta)$ , and covariance matrix estimator  $\Omega_T(\theta)$ :

$$r_T(\theta) = \begin{pmatrix} f_T(\theta) \\ g_T(\theta) \\ v_T(\theta) \end{pmatrix}. \quad (2.3)$$

Below, we make some high-level assumptions about the estimator  $\hat{\theta}$  and the functions, defined above.

**Assumption 1.** *The estimator  $\hat{\theta}$  converges in probability to  $\theta_0$  and the limiting behavior of  $\hat{\theta}$  is such that*

$$\sqrt{T} (\hat{\theta} - \theta_0) \rightarrow_d N(0, \Sigma_{\hat{\theta}}), \quad (2.4)$$

where the asymptotic variance  $\Sigma_{\hat{\theta}}$  is given by

$$\Sigma_{\hat{\theta}} = (G' \Omega^{-1} G)^{-1}, \quad (2.5)$$

$G = \mathbb{E}(G_T(\theta_0))$  has full column rank, and  $\Omega$  is positive definite.

**Assumption 2.** *As  $T \rightarrow \infty$ , for  $\theta \in \Theta$ , the following conditions hold:*

- (i)  $G_T(\theta)$  converges uniformly in probability to  $G(\theta) = \mathbb{E}(G_T(\theta))$ ;
- (ii)  $H_T(\theta)$  converges uniformly in probability to  $H(\theta) = \mathbb{E}(H_T(\theta))$ ;
- (iii)  $\Omega_T(\theta)$  converges uniformly in probability to  $\Omega(\theta)$ ;
- (iv)  $Q_T(\theta)$  converges uniformly in probability to  $Q(\theta) = \partial \text{vech}(\Omega(\theta)) / \partial \theta$ .

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<sup>1</sup>Throughout the paper, for simplicity, we consider the optimal GMM estimator, where  $\Omega$  is estimated before getting  $\hat{\theta}$ . All the results, however, are valid for the continuously updating estimator (CUE) of Hansen, Heaton and Yaron (1996).

**Assumption 3.** As  $T \rightarrow \infty$ , the series  $r_T(\theta)$  at  $\theta = \theta_0$  accord with a central limit theorem

$$\sqrt{T}(r_T(\theta_0) - r) \rightarrow_d N(0, \Sigma_r), \quad (2.6)$$

where  $r = (\begin{array}{ccc} 0_{1 \times m} & \text{vec}(G')' & \text{vech}(\Omega)' \end{array})'$ , and  $\Sigma_r = \lim_{T \rightarrow \infty} \text{var}(\sqrt{T}r_T(\theta_0))$  is a positive semidefinite symmetric matrix.

Assumption 1 is a standard assumption about limiting behavior of the GMM estimator. Assumption 2 requests  $G_T(\theta)$ ,  $\Omega_T(\theta)$ , and the matrices, composed of their first derivatives, to be a consistent estimators of corresponding functions. This assumption is not restrictive at all as the necessary conditions for the uniform convergence in probability are quite weak (see, e.g., Newey and McFadden (1994), Lemma 2.4.). Assumption 3 requests a central limit theorem to hold for the series  $r_T(\theta_0)$ . This assumption requests the estimator of moments covariance matrix to converge at the rate  $\sqrt{T}$ , which may be violated, for example, for the heteroscedasticity autocorrelation consistent covariance (HACC) matrix estimators.

In order to give a definition for the relevance and the redundancy, we split  $f_T(\theta)$  into two subsets so that

$$f_T(\theta) = \begin{pmatrix} f_{1T}(\theta) \\ f_{2T}(\theta) \end{pmatrix}, \quad (2.7)$$

where  $f_1(Z_t, \theta)$  is of size  $m_1 \times 1$ ,  $m_1 \geq k$ , and  $f_2(Z_t, \theta)$  is of size  $m_2 \times 1$ . Below, we use subscripts 1 and 2 for functions and matrices to show that they are derived from  $f_1(Z_t, \theta)$  and  $f_2(Z_t, \theta)$ , respectively. For brevity, we use  $f_1$  and  $f_2$  to denote the first  $\mathbb{E}(f_{1T}(\theta_0)) = 0$  and the second  $\mathbb{E}(f_{2T}(\theta_0)) = 0$  sets of moment conditions, respectively.

The partition of the covariance matrix  $\Omega(\theta)$ , which corresponds to the partition of  $f(Z_t, \theta)$ , is

$$\Omega(\theta) = \begin{pmatrix} \Omega_{11}(\theta) & \Omega_{12}(\theta) \\ \Omega_{21}(\theta) & \Omega_{22}(\theta) \end{pmatrix}, \quad (2.8)$$

with  $\Omega_{11}(\theta)$  is the  $m_1 \times m_1$  covariance matrix of  $f_{1T}(\theta)$ ,  $\Omega_{22}(\theta)$  is the  $m_2 \times m_2$  covariance matrix of  $f_{2T}(\theta)$ , and  $\Omega_{21}(\theta) = \Omega'_{12}(\theta)$  is the  $m_2 \times m_1$  matrix of covariance between  $f_{2T}(\theta)$  and  $f_{1T}(\theta)'$ . Below, for brevity, we use  $\Omega_{11}$ ,  $\Omega_{22}$ , and  $\Omega_{21}$  to denote corresponding covariance matrices at  $\theta = \theta_0$ . We use the same notation for other functions of  $\theta$  as well.

We consider the case of the strong identification of  $\theta_0$ . To ensure that  $\theta_0$  is identified under the first set of moment conditions and that the matrix  $G'_1 \Omega_{11}^{-1} G_1$  is positive definite and nonsingular, we make an additional assumption on the matrix  $G$ :

**Assumption 4.** The matrix  $G_1$  has full column rank,  $\text{rank}(G_1) = k$ .

In order to give a definition for the *partial* relevance and the *partial* redundancy, we split  $\theta$  into two subsets so that

$$\theta = \begin{pmatrix} \theta_A \\ \theta_B \end{pmatrix}, \quad (2.9)$$

where  $\theta_A$  is of size  $k_A \times 1$ ,  $\theta_B$  is of size  $k_B \times 1$ , and  $k_A + k_B = k$ . We use subscripts  $A$  and  $B$  to denote blocks of vectors (matrices), which correspond to  $\theta_A$  and  $\theta_B$ , respectively.

### 3 Relevance

To define relevance, we start from the notion of local identifiability, used by Fisher (1959), and apply it to a set of moment conditions. Moment conditions  $f_2$  can (potentially) help with local identification of  $\theta_0$ , if there exists some neighborhood  $\Theta_0$  of  $\theta_0$ , such that  $\theta_0$  is the interior point of  $\Theta_0$  and

$$\mathbb{E}(f_{2T}(\theta)) \neq 0 \text{ for any } \theta \neq \theta_0 \text{ and } \theta \in \Theta_0. \quad (3.1)$$

Since  $f_{2T}(\theta)$  is twice continuous differentiable, condition (3.1) holds if the matrix of the first derivatives of  $\mathbb{E}(f_{2T}(\theta))$  at  $\theta = \theta_0$  is not zero. We use this as a condition for relevance.

**Definition 1.** *We say that moment conditions  $f_2$  are relevant if and only if*

$$G_2 \neq 0. \quad (3.2)$$

As noted by Sargan (1983),  $G_2 = 0$  does not necessarily mean that moment conditions do not help with local identification of  $\theta_0$ . Indeed, even if  $G_2 = 0$ , equation (3.1) may hold, when higher order derivatives of  $\mathbb{E}(f_{2T}(\theta))$  are not zero at  $\theta = \theta_0$ . However, we require condition (3.2) to hold, because, otherwise, moment conditions may potentially change the limiting behavior of the GMM estimator. Just to illustrate this statement, consider the case, when Assumption 4 does not hold and the rank of  $G_1$  is  $k - 1$ . Then, if columns of  $G_2$  are not linear combinations of columns of  $G_1$ , condition (3.2) is necessary for the asymptotic variance  $\Sigma_{\hat{\theta}}$  to exist and, hence, for the asymptotic results in Assumption 1 to be valid.

To derive a statistic for testing relevance, we first derive asymptotic behavior of an estimator  $g_{2T}(\hat{\theta})$  of  $g_2 = \text{vec}(G'_2)$ .

**Theorem 1.** *Under Assumptions 1-4 and  $H_0 : G_2 = 0$ , the limiting behavior of  $g_{2T}(\hat{\theta})$  reads*

$$\sqrt{T}g_{2T}(\hat{\theta}) \rightarrow_d N(0, \Sigma_{\hat{g}_2}), \quad (3.3)$$

where  $\Sigma_{\hat{g}_2} = B\Sigma_r B'$ , with  $B$  given in equation A.20.

*Proof.* See the Appendix. □

Let  $\widehat{\Sigma}_{\hat{g}_2}$  be a consistent estimator of  $\Sigma_{\hat{g}_2}$ . Then, the statistic to test the null hypothesis of moment conditions irrelevance follows from Theorem 1.

**Definition 2.** *The statistic for testing  $H_0 : G_2 = 0$  reads*

$$W = T g_{2T}(\hat{\theta})' \widehat{\Sigma}_{\hat{g}_2}^{-1} g_{2T}(\hat{\theta}) \quad (3.4)$$

and has, under  $H_0$  and Assumptions 1-4, a  $\chi^2(m_2 k)$  limiting distribution.

Note that some of the assumptions in Theorem 1 and Definition 2 are excessive. Statistic (3.4) has a  $\chi^2(m_2 k)$  limiting distribution, even if Assumptions 4(iii)-(iv) does not hold and/or  $v_T(\theta_0)$  does not have normal limiting distribution. What is important is that it allows the estimator of  $\Omega$  to converge at the speed lower than  $\sqrt{T}$  and, hence, HACC estimator can be used.

## 4 Conditional relevance and redundancy

With  $T \rightarrow \infty$ , the optimization problem (2.2) can be written in the form, convenient for the analysis of identification

$$\theta_0 = \arg \min_{\theta \in \Theta} S(\theta), \quad (4.1)$$

where

$$S(\theta) = \mathbb{E}(f_T(\theta))' \Omega^{-1} \mathbb{E}(f_T(\theta)). \quad (4.2)$$

We use the idea that the strength of identification can be measured by the curvature of population objective function  $S(\theta)$  at  $\theta = \theta_0$ . We say that moment conditions  $f_2$  are conditionally relevant, if its exclusion deteriorate identification by decreasing the hessian of  $S(\theta)$ , which describes the curvature of  $S(\theta)$ .

To extract the impact of excluding  $f_2$  from the set of moment conditions, we can write equation (4.2) as

$$S(\theta) = \mathbb{E}(f_{1T}(\theta))' \Omega_{11}^{-1} \mathbb{E}(f_{1T}(\theta)) + \mathbb{E}(f_{\Delta T}(\theta))' \Omega_{\Delta}^{-1} \mathbb{E}(f_{\Delta T}(\theta)), \quad (4.3)$$

where  $f_{\Delta T}(\theta) = f_{2T}(\theta) - \Omega_{21} \Omega_{11}^{-1} f_{1T}(\theta)$ , and  $\Omega_{\Delta} = \Omega_{22} - \Omega_{21} \Omega_{11}^{-1} \Omega_{12}$ . From equation (4.3), the hessian of  $S_T(\theta)$  at  $\theta = \theta_0$  reads

$$\left. \frac{\partial^2 S(\theta)}{\partial \theta \partial \theta'} \right|_{\theta=\theta_0} = G_1' \Omega_{11}^{-1} G_1 + G_{\Delta}' \Omega_{\Delta}^{-1} G_{\Delta}, \quad (4.4)$$

where  $G_{\Delta} = \mathbb{E}(G_{\Delta T}(\theta_0))$  and  $G_{\Delta T} = \partial f_{\Delta T}(\theta)/\partial \theta'$ .

From the Schur decomposition of  $\Omega$ , it follows that  $\Omega_{\Delta}^{-1}$  is the lower-right block of  $\Omega^{-1}$ . Hence,  $\Omega_{\Delta}^{-1}$  is positive definite and  $G_{\Delta}' \Omega_{\Delta}^{-1} G_{\Delta}$  is positive semidefinite. It follows that the exclusion of  $f_2$  does not affect the curvature of  $S(\theta)$  at  $\theta = \theta_0$ , if and only if  $G_{\Delta} = 0$ . Otherwise, the curvature of  $S(\theta)$  will become lower. This gives us a definition of conditional relevance.

**Definition 3.** We say that moment conditions  $f_2$  are conditionally relevant given  $f_1$  if and only if

$$G_\Delta \neq 0. \quad (4.5)$$

The intuition behind condition (4.5) is that  $f_{\Delta T}(\theta_0)$  is the residual from projection of  $f_{2T}(\theta)$  on  $f_{1T}(\theta)$  at  $\theta = \theta_0$  and, hence, it states for the information contained only in moment conditions  $f_2$ . Thus, for the conditional relevance, a set of moment conditions must remain relevant by itself after subtracting the information contained in other moment conditions.

Breusch, Qian, Schmidt, and Wyhowski (1999) show that condition (4.5) is also a necessary and sufficient condition for  $f_2$  not to be redundant given  $f_1$ . They define moment conditions  $f_2$  to be redundant given  $f_1$  if exclusion of  $f_2$  does not raise asymptotic variance of  $\hat{\theta}$ . Thus, conditional relevance implies non-redundancy and vice versa.

To derive a statistic for testing  $H_0 : G_\Delta = 0$ , we first derive asymptotic behavior of an estimator  $\hat{g}_{\Delta T}(\hat{\theta})$  of  $g_\Delta = \text{vec}(G'_\Delta)$ ,

$$\hat{g}_{\Delta T}(\hat{\theta}) = \text{vec} \left( G_{2T}(\hat{\theta}) - \Omega_{21T}(\hat{\theta}) \Omega_{11T}^{-1}(\hat{\theta}) G_{1T}(\hat{\theta}) \right). \quad (4.6)$$

**Theorem 2.** Under Assumptions 1-4 and  $H_0 : G_\Delta = 0$ , the limiting behavior of  $\hat{g}_{\Delta T}(\hat{\theta})$  reads

$$\sqrt{T} \hat{g}_{\Delta T}(\hat{\theta}) \rightarrow_d N(0, \Sigma_{\hat{g}_\Delta}), \quad (4.7)$$

where  $\Sigma_{\hat{g}_\Delta} = C \Sigma_r C'$ , and  $C$  is given in equation (A.31).

*Proof.* See the Appendix. □

Denote  $\hat{\Sigma}_{\hat{g}}$  to be a consistent estimator of  $\Sigma_{\hat{g}}$ . Then, the statistic for testing the null hypothesis of conditional irrelevance (or redundancy) follows from Theorem 1.

**Definition 4.** The statistic for testing  $H_0 : G_\Delta = 0$  reads

$$W = T \hat{g}_{\Delta T}(\hat{\theta})' \hat{\Sigma}_{\hat{g}}^{-1} \hat{g}_{\Delta T}(\hat{\theta}) \quad (4.8)$$

and has, under  $H_0$  and Assumptions 1-4, a  $\chi^2(m_2 k)$  limiting distribution.

## 5 Partial relevance and partial redundancy

In some empirical applications of GMM, only a particular subset of parameters  $\theta_0$  is of interest. In this case the notions of partial relevance, partial conditional relevance and partial redundancy become useful. Suppose that we want to test if moment conditions  $f_2$  can help with estimation of  $\theta_{0A}$ , which is the first  $k_A$  elements of  $\theta_0$ .

To define partial relevance, we use the condition for local identification and apply it to  $\theta_{0A}$ . The reasoning for Definition 5 is analogous to those, proposed for the (unconditional) relevance.

**Definition 5.** We say that moment conditions  $f_2$  are (at least) partially relevant for the estimation of  $\theta_{0A}$  if and only if

$$G_{2A} \neq 0, \quad (5.1)$$

where  $G_{2A} = \partial \mathbb{E}(f_{2T}(\theta)) / \partial \theta'_A$  at  $\theta = \theta_0$ .

Let  $R_A = (I_{k_A} \ 0_{k_A \times k_B})$ , so that  $G_{2A} = G_2 R'_A$ . Then, the estimator of  $\text{vec}(G'_{2A})$  reads  $g_{2TA}(\hat{\theta}) = (I_{m_2} \otimes R_A)g_{2T}(\hat{\theta})$ , and the estimator of its covariance matrix is given by  $\widehat{\Sigma}_{\hat{g}_{2A}} = (I_{m_2} \otimes R_A)\widehat{\Sigma}_{\hat{g}_2}(I_{m_2} \otimes R'_A)$ . The statistic for testing the null hypothesis of partial irrelevance follows from Theorem 1.

**Definition 6.** The statistic for testing  $H_0 : G_{2A} = 0$  reads

$$W = T g_{2TA}(\hat{\theta})' \widehat{\Sigma}_{\hat{g}_{2A}}^{-1} g_{2TA}(\hat{\theta}) \quad (5.2)$$

and has, under  $H_0$  and Assumptions 1-4, a  $\chi^2(m_2 k_A)$  limiting distribution.

To define partial conditional relevance, we consider the curvature of  $S(\theta)$  in  $\theta_A$  coordinates. From expression (4.4), it follows that, at  $\theta = \theta_0$ , the impact of moment conditions  $f_2$  onto the hessian of  $S(\theta)$  with respect to  $\theta_A$  reads  $G'_{\Delta A} \Omega_{\Delta}^{-1} G_{\Delta A}$ , where  $G_{\Delta A} = \partial \mathbb{E}(f_{\Delta}(\theta)) / \partial \theta_A$  at  $\theta = \theta_0$ . Hence, the curvature of  $S(\theta)$  in  $\theta_A$  coordinates will not change after excluding  $f_2(\theta)$  if and only if  $G_{\Delta A} = 0$ . Otherwise, the curvature will decrease.

**Definition 7.** We say that moment conditions  $f_2$  is (at least) partially conditionally relevant for the estimation of  $\theta_{0A}$  given  $f_1$  if and only if

$$G_{\Delta A} \neq 0. \quad (5.3)$$

The estimator of  $\text{vec}(G'_{\Delta A})$  reads  $g_{\Delta TA}(\hat{\theta}) = (I_{m_2} \otimes R_A)g_{\Delta T}(\hat{\theta})$ , and the estimator of its covariance matrix is given by  $\widehat{\Sigma}_{\hat{g}_{\Delta A}} = (I_{m_2} \otimes R_A)\widehat{\Sigma}_{\hat{g}_{\Delta}}(I_{m_2} \otimes R'_A)$ . The statistic for testing the null hypothesis of partial irrelevance follows from Theorem 2.

**Definition 8.** The statistic for testing  $H_0 : G_{\Delta A} = 0$  reads

$$W = T g_{\Delta TA}(\hat{\theta})' \widehat{\Sigma}_{\hat{g}_{\Delta A}}^{-1} g_{\Delta TA}(\hat{\theta}) \quad (5.4)$$

and has, under  $H_0$  and Assumptions 1-4, a  $\chi^2(m_2 k_A)$  limiting distribution.

Note that  $G_{2A} = 0$  does not imply that  $f_2$  is partially redundant. A condition for partial redundancy is given by Breusch, Qian, Schmidt, and Wyhowski (1999). To analyze the relation between partial redundancy and partial conditional relevance, we reformulate their condition in a more convenient form.

**Theorem 3.** Under Assumptions 1 and 4, moment conditions  $f_2$  are partially redundant for the estimation of  $\theta_A$  given  $f_1$  if and only if

$$G_{\Delta A}\Sigma_{1AA} + G_{\Delta B}\Sigma_{1BA} = 0, \quad (5.5)$$

where  $G_{\Delta B} = \partial \mathbb{E}(f_{\Delta}(\theta)) / \partial \theta_B$  at  $\theta = \theta_0$ ,  $\Sigma_{1AA}$  is the  $k_A \times k_A$  upper-left block and  $\Sigma_{1BA}$  is the  $k_B \times k_B$  lower-left block of matrix  $\Sigma_1$ ,  $\Sigma_1 = (G'_1 \Omega_{11}^{-1} G_1)^{-1}$  is the asymptotic covariance matrix of  $\hat{\theta}$ , obtained under the first set of moment conditions only.

*Proof.* See the Appendix. □

Theorem 3 implies that for the partial redundancy one of the following conditions is enough: (i)  $f_2$  is conditionally irrelevant for the estimation of both  $\theta_{0A}$  and  $\theta_{0B}$  or (ii)  $f_2$  is conditionally irrelevant for the estimation of  $\theta_{0A}$  and estimators of  $\theta_{0A}$  and  $\theta_{0B}$  are uncorrelated<sup>2</sup>. The intuition behind this result is that the variance of  $\hat{\theta}$  is affected by two factors: the variance of objective function  $S(\theta)$  estimator and the curvature of  $S(\theta)$ . Suppose  $f_2$  is partially conditionally irrelevant for the estimation of  $\theta_A$ , but it is partially conditionally relevant for the estimation of  $\theta_B$ . It implies, that  $f_2$  the curvature of the objective function  $S(\theta)$  in  $\theta_B$  and, hence, improves the efficiency of  $\hat{\theta}_B$ . Since  $\hat{\theta}_B$  becomes more precise, it improves the efficiency of  $S(\theta)$  estimator, which we can consider as a function of  $\theta_A$ , and, hence, it may improve the efficiency of  $\hat{\theta}_A$ .

From the expression for  $\Sigma_1$  as the inverse of partitioned matrix  $G'_1 \Omega_{11}^{-1} G_1$ , it follows that the necessary and sufficient condition for  $\Sigma_{1BA} = 0$  reads

$$G'_{1B} \Omega_{11}^{-1} G_{1A} = 0. \quad (5.6)$$

It implies that, when  $\theta_{0A}$  and  $\theta_{0B}$  are in separate moment conditions, their estimators are uncorrelated under  $f_1$ , and condition (5.1) is enough to test partial redundancy.

In the general case, the asymptotic behavior of  $G_{\Delta A}\Sigma_{1AA} + G_{\Delta B}\Sigma_{1BA}$  estimator is hard to derive. One may use sufficient conditions for the partial redundancy, instead, and check it by testing  $G_{\Delta 1} = 0$ ,  $G_{\Delta 2} = 0$ , and  $\Sigma_{\hat{\theta}BA} = 0$ .

## 6 Some Special Cases

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<sup>2</sup>Of course, partial redundancy may hold under other conditions, which, among others, include various mixes of (i) and (ii).

## 7 Monte Carlo Simulations

We carry out Monte Carlo simulations for the C-CAPM

$$\mathbb{E} \left( \left( \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t,t+1} - 1 \right) z_t \right) = 0, \quad (7.1)$$

where  $\beta = 0.97$  and  $\gamma = 5$ .

To generate artificial data we use the design, proposed by [Tauchen \(1986\)](#). To ensure strong identification, we follow [Kleibergen \(2005\)](#) and calibrate  $10^2$ -dimensional Markov chain to approximate first-order vector autoregression (VAR) for consumption and dividend growth:

$$\begin{pmatrix} c_t \\ d_t \end{pmatrix} = 2 \begin{pmatrix} 0.021 \\ 0.004 \end{pmatrix} + 2 \begin{pmatrix} -0.161 & 0.017 \\ 0.004 & 0.117 \end{pmatrix} \begin{pmatrix} c_{t-1} \\ d_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{c,t} \\ \varepsilon_{d,t} \end{pmatrix}, \quad (7.2)$$

where  $c_t$  is the logarithm of consumption growth rate,  $d_t$  is the logarithm of real dividends growth,  $(\varepsilon_{c,t} \ \varepsilon_{d,t})$  are independently normally distributed disturbances with zero mean and  $\text{var}(\varepsilon_{c,t}) = 0.014$ ,  $\text{var}(\varepsilon_{d,t}) = 0.0012$ ,  $\text{corr}(\varepsilon_{c,t}, \varepsilon_{d,t}) = 0.43$ .

For the estimation and testing, we use  $z_t = (\text{Const } C_t/C_{t-1} \ R_{t-1} \ Z_{1t} \ Z_{2t})$ , where  $\text{Const} = 1$ . Artificial instrumental variables  $Z_{1t}$  and  $Z_{2t}$  are generated to illustrate various definitions of relevance,

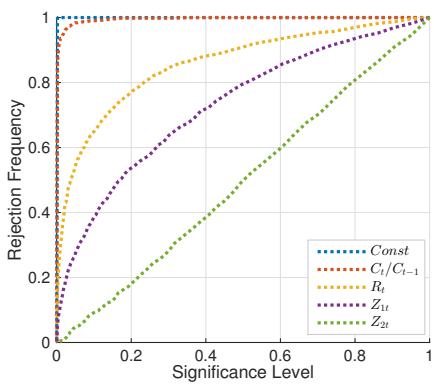
$$Z_{1t} = \exp(0.5c_t + 0.5\varepsilon_{1,t}), \quad (7.3)$$

$$Z_{2t} = \exp(\varepsilon_{2,t}), \quad (7.4)$$

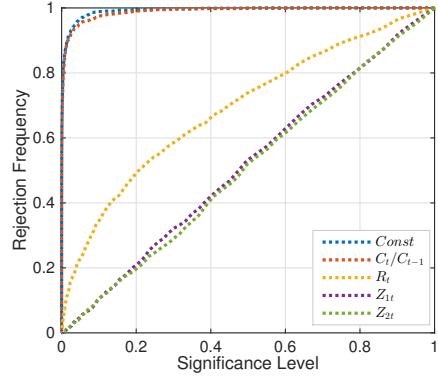
where  $(\varepsilon_{1,t} \ \varepsilon_{2,t})$  are independently normally distributed disturbances with mean zero and  $\text{var}(\varepsilon_{c,t}) = 0.014$ ,  $\text{var}(\varepsilon_{d,t}) = 0.0012$ ,  $\text{corr}(\varepsilon_{c,t}, \varepsilon_{d,t}) = 0.0$ . The number of observations is 250.

Figure 1 shows rejection frequencies of the null hypothesis of instrument irrelevance. Panel A shows rejection frequencies for the test for relevance, Panel B — for conditional relevance (redundancy), Panels C and D — for partial relevance, Panel E and F — for conditional partial relevance. Since the constant is relevant in the setting we consider, any random variable with non-zero mean can be relevant, even if it has nothing to do with the model and its parameters. That is why, for all the instruments, except the constant, we do not test for (partial) relevance itself. We test for (partial) relevance conditional on a constant (panels A, C, and D).

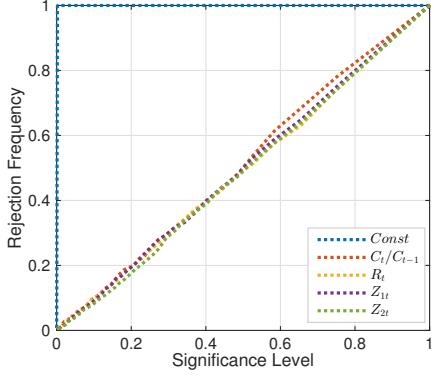
For irrelevant instruments, we expect rejection frequencies to be close to significance levels so that the rejection frequency curve lies on a  $45^\circ$  line. The results for the artificial instruments  $Z_{1t}$  and  $Z_{2t}$  are as expected. For  $Z_{1t}$ , which is not correlated with any variable of the model, rejection frequencies are close to significance levels for each test. For  $Z_{2t}$ , which is correlated with consumption growth, the hypothesis of (partial) irrelevance is rejected



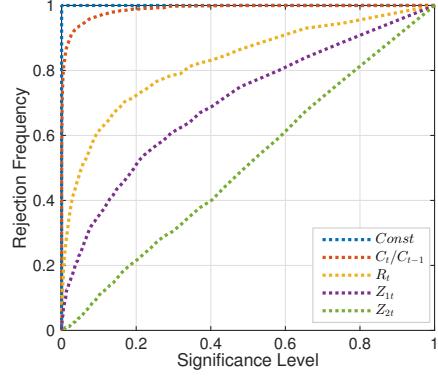
Panel A: Irrelevance



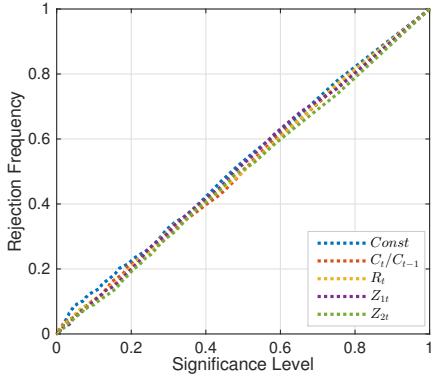
Panel B: Conditional irrelevance (redundancy)



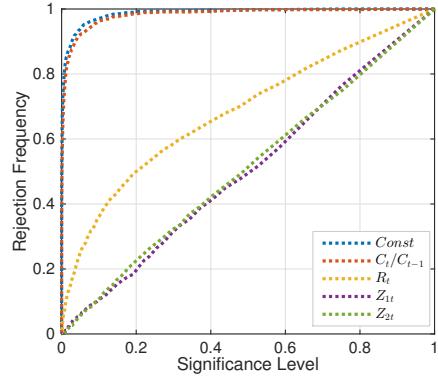
Panel C: Partial irrelevance for the identification of  $\beta$



Panel D: Partial irrelevance for the identification of  $\gamma$



Panel E: Conditional partial irrelevance for the identification of  $\beta$



Panel F: Conditional partial irrelevance for the identification of  $\gamma$

FIGURE 1.—Rejection frequencies of the null hypothesis of instrument irrelevance. On panels A, C, and D, for each instrument, except the constant, irrelevance conditional on a constant is tested. On panels B, D, and E, irrelevance conditional on all the other instruments is tested.

frequently. Since  $Z_{2t}$  does not have any additional information, rejection frequencies of the hypothesis of conditional (partial) irrelevance are close to significance levels.

An interesting result is that only the constant helps with identification of the subjective discount factor  $\beta$ . Consumption growth  $C_t/C_{t-1}$  helps only with identification of the coefficient of relative risk aversion  $\gamma$ . Asset return  $R_t$  helps with identification of  $\gamma$  as well, but rejection frequencies are low, which may potentially indicate that  $R_t$  is a weak instrument.

## 8 Conclusion

We propose statistics for testing relevance in GMM and derive their asymptotic behavior. The statistics can be used to test for relevance and conditional relevance, as well as to check if moment conditions can help with identification of particular parameters of interest. We show that the statistic for conditional relevance cannot be used when moments covariance matrix estimator converges at the rate slower than  $\sqrt{T}$  (when HAC estimator is used, for example). This statistic can be used for testing redundancy of moment conditions. For partial redundancy, however, additional conditions should be checked.

## Appendix

**Proof of Theorem 1.** By a mean-value expansion,

$$g_{2T}(\hat{\theta}) = g_{2T}(\theta_0) + \sum_{j=1}^{m_2 k} \lambda_j H_{2T}(\tilde{\theta}_j)(\hat{\theta} - \theta_0), \quad (\text{A.1})$$

where  $\sum_{j=1}^{m_2 k} \lambda_j = 1$ ,  $\lambda_j \geq 0$ ,  $\tilde{\theta}_j$  lies on the line joining  $\hat{\theta}$  and  $\theta_0$ , and, hence,  $\tilde{\theta}_j \rightarrow_p \theta_0$  as  $\hat{\theta} \rightarrow_p \theta_0$  for  $j = 1, \dots, m_2 k$ .

First, let us express  $H_{2T}(\tilde{\theta}_j)(\hat{\theta} - \theta_0)$  in terms of  $f_T(\theta_0)$ . The first-order condition for the GMM maximization problem (2.2) reads

$$G_T(\hat{\theta})' \widehat{\Omega} f_T(\hat{\theta}) = 0. \quad (\text{A.2})$$

Using mean-value expansion, we obtain the expression for  $\hat{\theta} - \theta_0$

$$\hat{\theta} - \theta_0 = \left( G_T(\hat{\theta})' \widehat{\Omega}^{-1} G_T(\bar{\theta}) \right)^{-1} G_T(\hat{\theta})' \widehat{\Omega}^{-1} f_T(\theta_0), \quad (\text{A.3})$$

where  $\bar{\theta}$  lies on the line joining  $\hat{\theta}$  and  $\theta_0$ , and, hence,  $\bar{\theta} \rightarrow_p \theta_0$  as  $\hat{\theta} \rightarrow_p \theta_0$ . Hence,

$$\sum_{j=1}^{m_2 k} \lambda_j H_{2T}(\tilde{\theta}_j)(\hat{\theta} - \theta_0) = \tilde{B}_f f_T(\theta_0), \quad (\text{A.4})$$

where

$$\tilde{B}_f = \sum_{j=1}^{m_2 k} \lambda_j H_{2T}(\tilde{\theta}_j) \left( G_T(\hat{\theta})' \widehat{\Omega}^{-1} G_T(\bar{\theta}) \right)^{-1} G_T(\hat{\theta})' \widehat{\Omega}^{-1}. \quad (\text{A.5})$$

Second, let us express  $g_{2T}(\hat{\theta})$  in terms of  $r_T(\theta_0)$  and  $r$ . Substitute (A.14) into (A.1) to obtain

$$g_{2T}(\hat{\theta}) = g_{2T}(\theta_0) + \tilde{B}_f f_T(\theta_0). \quad (\text{A.6})$$

Note that, under the null hypothesis,  $g_2 = 0$ , and the expression for  $g_{2T}(\hat{\theta})$  reads

$$g_{2T}(\hat{\theta}) = \tilde{B} (r_T(\theta_0) - r), \quad (\text{A.7})$$

where  $\tilde{B} = \begin{pmatrix} \tilde{B}_f & 0_{m_2 k \times m_1 k} & I_{m_2 k} & 0_{m_2 k \times (m+1)m/2} \end{pmatrix}$ .

Third, let us show that  $\tilde{B} \rightarrow_p B$ , where

$$B = \begin{pmatrix} B_f & 0_{m_2 k \times m_1 k} & I_{m_2 k} & 0_{m_2 k \times (m+1)m/2} \end{pmatrix}, \quad (\text{A.8})$$

and  $B_f = H_2 (G' \Omega^{-1} G)^{-1} G' \Omega^{-1}$ . Recall that  $\tilde{\theta}_j \rightarrow_p \theta_0$ . Hence, by Assumption 2, Lemma 4 of Amemiya (1973) and Slutsky's theorem,  $H_{2T}(\tilde{\theta}_j) \rightarrow_p H_2$

for  $j = 1, \dots, m_2 k$ . Hence, from  $\sum_{j=1}^{m_2 k} \lambda_j = 1$  it follows that  $\sum_{j=1}^{m_2 k} \lambda_j H_{2T}(\tilde{\theta}_j) \rightarrow_p H$ . Recall that  $\widehat{\Omega} \rightarrow_p \Omega$ ,  $\widehat{\theta} \rightarrow_p \theta$ , and  $\bar{\theta} \rightarrow_p \theta_0$ . Hence, by Assumption 2, Lemma 4 of Amemiya (1973) and Slutsky's theorem,  $\widetilde{B}_f \rightarrow_p B_f$ . Hence,  $\widetilde{B} \rightarrow_p B$ .

Finally, by Assumption 3 and Slutsky's theorem,

$$\widetilde{B}(r_T(\theta_0) - r) \rightarrow_p N(0, \Sigma_{\hat{g}}), \quad (\text{A.9})$$

with  $\Sigma_{\hat{g}} = B \Sigma_r B'$ . Theorem 1 follows straightforwardly from (A.19) and (A.23).

**Proof of Theorem 2.** By a mean-value expansion,

$$\hat{g}_{\Delta T}(\hat{\theta}) = \hat{g}_{\Delta T}(\theta_0) + \sum_{j=1}^{m_2 k} \lambda_j \widehat{H}_{\Delta T}(\tilde{\theta}_j)(\hat{\theta} - \theta_0), \quad (\text{A.10})$$

where  $\widehat{H}_{\Delta T}(\theta) = \partial \hat{g}_{\Delta T}(\theta) / \partial \theta$ ,  $\sum_{j=1}^{m_2 k} \lambda_j = 1$ ,  $\lambda_j \geq 0$ ,  $\tilde{\theta}_j$  lies on the line joining  $\hat{\theta}$  and  $\theta_0$ , and, hence,  $\tilde{\theta}_j \rightarrow_p \theta_0$  as  $\hat{\theta} \rightarrow_p \theta_0$  for  $j = 1, \dots, m_2 k$ .

First, let us express  $\hat{g}_{\Delta T}(\theta_0)$  in terms of  $g_{1T}(\theta_0)$ ,  $g_{2T}(\theta_0)$  and  $v_T(\theta_0)$ . Recall that

$$\hat{g}_{\Delta T}(\theta_0) = \text{vec} \left( (G_{2T}(\theta_0) - \Omega_{21T}(\theta_0) \Omega_{11T}^{-1}(\theta_0) G_{1T}(\theta_0))' \right). \quad (\text{A.11})$$

The expression for  $\Omega_{21T}(\theta_0) \Omega_{11T}^{-1}(\theta_0)$  reads

$$\begin{aligned} \Omega_{21T}(\theta_0) \Omega_{11T}^{-1}(\theta_0) &= \Omega_{21} \Omega_{11}^{-1} + \Omega_{21T}(\theta_0) (\Omega_{11T}^{-1}(\theta_0) - \Omega_{11}^{-1}) \\ &\quad + (\Omega_{21T}(\theta_0) - \Omega_{21}) \Omega_{11}^{-1}. \end{aligned} \quad (\text{A.12})$$

Substituting (A.12) into (A.11), we obtain

$$\begin{aligned} \hat{g}_{\Delta T}(\theta_0) &= -(\Omega_{21} \Omega_{11}^{-1} \otimes I_k) g_{1T}(\theta_0) + g_{2T}(\theta_0) \\ &\quad - (\Omega_{21T}(\theta_0) \otimes G'_{1T}(\theta_0)) \text{vec} (\Omega_{11T}^{-1}(\theta_0) - \Omega_{11}^{-1}) \\ &\quad - ((I_{m_2} \otimes G'_{1T}(\theta_0) \Omega_{11}^{-1})) \text{vec} (\Omega'_{21T}(\theta_0) - \Omega'_{21}). \end{aligned} \quad (\text{A.13})$$

By a mean-value expansion,

$$\text{vec} (\Omega_{11T}^{-1}(\theta_0)) = \text{vec} (\Omega_{11}^{-1}) + \frac{\partial \text{vec} (\widetilde{\Omega}_{11T}^{-1})}{\partial \text{vech} (\widetilde{\Omega}_{11T})} \text{vech} (\Omega_{11T}(\theta_0) - \Omega_{11}), \quad (\text{A.14})$$

where  $\widetilde{\Omega}_{11T}$  is symmetric and  $\text{vec} (\widetilde{\Omega}_{11T})$  lies between  $\text{vec} (\Omega_{11T}(\theta_0))$  and  $\text{vec} (\Omega_{11})$  and, hence,  $\widetilde{\Omega}_{11T} \rightarrow_p \Omega_{11}$  as  $\Omega_{11T}(\theta_0) \rightarrow_p \Omega_{11}$ . The partial derivative is given by

$$\frac{\partial \text{vec} (\widetilde{\Omega}_{11T}^{-1})}{\partial \text{vech} (\widetilde{\Omega}_{11T})} = - \left( \widetilde{\Omega}_{11T}^{-1} \otimes \widetilde{\Omega}_{11T}^{-1} \right) D_{m_1}. \quad (\text{A.15})$$

Let

$$M_{11} = \begin{pmatrix} I_{m_1} & 0_{m_2 \times m_1} \end{pmatrix}, \quad (\text{A.16})$$

$$M_{21L} = \begin{pmatrix} 0_{m_2 \times m_1} & I_{m_2} \end{pmatrix}, \quad (\text{A.17})$$

$$M_{21R} = \begin{pmatrix} I_{m_1} & 0_{m_1 \times m_2} \end{pmatrix}, \quad (\text{A.18})$$

so that  $\Omega_{11T}(\theta) = M_{11}\Omega_T(\theta)M'_{11}$ , and  $\Omega_{21T}(\theta) = M_{21L}\Omega_T(\theta)M'_{21R}$ . Then

$$\text{vech}(\Omega_{11T}(\theta_0) - \Omega_{11}) = L_{m_1}(M_{11} \otimes M_{11})D_m \text{vech}(\Omega_T(\theta_0) - \Omega), \quad (\text{A.19})$$

and

$$\text{vec}(\Omega_{21T}(\theta_0) - \Omega_{21}) = (M_{21R} \otimes M_{21L})D_m \text{vech}(\Omega_T(\theta_0) - \Omega). \quad (\text{A.20})$$

Let

$$C_{g1} = -(\Omega_{21}\Omega_{11}^{-1} \otimes I_k), \quad (\text{A.21})$$

$$C_{g2} = I_{m_2 k}. \quad (\text{A.22})$$

Combining (A.13)–(A.22), we obtain

$$\hat{g}_{\Delta T}(\theta_0) = C_{g1}g_{1T}(\theta_0) + C_{g2}g_{2T}(\theta_0) + \tilde{C}_\omega \text{vech}(\Omega_T(\theta_0) - \Omega), \quad (\text{A.23})$$

where

$$\begin{aligned} \tilde{C}_\omega = & \left( \Omega_{21T}(\theta_0)\tilde{\Omega}_{11T}^{-1}M_{11} \otimes G'_{1T}(\theta_0)\tilde{\Omega}_{11T}^{-1}M_{11} \right) D_m \\ & - (I_{m_2}M_{21L} \otimes G'_{1T}(\theta_0)\Omega_{11}^{-1}M_{21R}) D_m. \end{aligned} \quad (\text{A.24})$$

Second, let us express  $\hat{H}_{\Delta T}(\tilde{\theta}_1, \dots, \tilde{\theta}_{m_2 k})(\hat{\theta} - \theta_0)$  in terms of  $f_T(\theta_0)$ . The first-order condition for the GMM maximization problem (2.2) reads

$$G_T(\hat{\theta})'\hat{\Omega}f_T(\hat{\theta}) = 0. \quad (\text{A.25})$$

Using mean-value expansion, we obtain the expression for  $\hat{\theta} - \theta_0$

$$\hat{\theta} - \theta_0 = \left( G_T(\hat{\theta})'\hat{\Omega}^{-1}G_T(\bar{\theta}) \right)^{-1} G_T(\hat{\theta})'\hat{\Omega}^{-1}f_T(\theta_0), \quad (\text{A.26})$$

where  $\bar{\theta}$  lies on the line joining  $\hat{\theta}$  and  $\theta_0$ , and, hence,  $\bar{\theta} \rightarrow_p \theta_0$  as  $\hat{\theta} \rightarrow_p \theta_0$ . Hence,

$$\sum_{j=1}^{m_2 k} \lambda_j \hat{H}_{\Delta T}(\tilde{\theta}_j)(\hat{\theta} - \theta_0) = \tilde{C}_f f_T(\theta_0), \quad (\text{A.27})$$

where

$$\tilde{C}_f = \sum_{j=1}^{m_2 k} \lambda_j \hat{H}_{\Delta T}(\tilde{\theta}_j) \left( G_T(\hat{\theta})'\hat{\Omega}^{-1}G_T(\bar{\theta}) \right)^{-1} G_T(\hat{\theta})'\hat{\Omega}^{-1}. \quad (\text{A.28})$$

Third, let us express  $\hat{g}_{\Delta T}(\hat{\theta})$  in terms of  $r_T(\theta_0)$  and  $r$ . Substituting (A.27) and (A.23) into (A.10), we obtain

$$\begin{aligned}\hat{g}_{\Delta T}(\hat{\theta}) &= \tilde{C}_f f_T(\theta_0) + C_{g_1} g_{1T}(\theta_0) + C_{g_2} g_{2T}(\theta_0) \\ &\quad + \tilde{C}_\omega \text{vech}(\Omega_T(\theta_0) - \Omega).\end{aligned}\tag{A.29}$$

Note that  $\text{vec}(G_\Delta) = C_{g_1} g_1 + C_{g_2} g_2$ , so that, under the null hypothesis,  $C_{g_1} g_1 + C_{g_2} g_2 = 0$  and the expression for  $\hat{g}_{\Delta T}(\hat{\theta})$  reads

$$\hat{g}_{\Delta T}(\hat{\theta}) = \tilde{C}(r_T(\theta_0) - r),\tag{A.30}$$

where  $\tilde{C} = \begin{pmatrix} \tilde{C}_f & C_{g_1} & C_{g_2} & \tilde{C}_\omega \end{pmatrix}$ .

Forth, let us show, that  $\tilde{C} \rightarrow_p C$ , where

$$C = (C_f \ C_{g_1} \ C_{g_2} \ C_\omega),\tag{A.31}$$

$$C_f = (C_h H + C_v Q)(G' \Omega^{-1} G)^{-1} G' \Omega^{-1},\tag{A.32}$$

$$C_h = (M_{21L} \otimes I_k - \Omega_{21} \Omega_{11T}^{-1} M_{11} \otimes I_k),\tag{A.33}$$

$$\begin{aligned}C_\omega &= (\Omega_{21} \Omega_{11}^{-1} M_{11} \otimes G'_1 \Omega_{11}^{-1} M_{11}) D_m \\ &\quad - (I_{m_2} M_{21L} \otimes G'_1 \Omega_{11}^{-1} M_{21R}) D_m\end{aligned}\tag{A.34}$$

Recall that

$$\hat{g}_{\Delta T}(\theta) = \text{vec}((G_{2T}(\theta)) - \text{vec}(\Omega_{21T}(\theta) \Omega_{11T}^{-1}(\theta) G_{1T}(\theta)').)\tag{A.35}$$

Hence,

$$\begin{aligned}\widehat{H}_{\Delta T}(\theta) &= \frac{\partial \text{vec}(G_{2T}(\theta)')}{\partial \theta} - (\Omega_{21}(\theta) \Omega_{11T}^{-1}(\theta) \otimes I_k) \frac{\partial \text{vec}(G_{1T}(\theta)')}{\partial \theta} \\ &\quad - (\Omega_{21T}(\theta) \Omega_{11T}^{-1}(\theta) \otimes G'_{1T}(\theta) \Omega_{11T}^{-1}(\theta)) \frac{\partial \text{vec}(\Omega_{11}(\theta))}{\partial \theta} \\ &\quad - (I_{m_2} \otimes G'_{1T}(\theta) \Omega_{11T}^{-1}(\theta)) \frac{\partial \text{vec}(\Omega_{21T}(\theta)')}{\partial \theta} \\ &= \tilde{C}_h(\theta) H_T(\theta) + \tilde{C}_\omega^*(\theta) Q_T(\theta),\end{aligned}\tag{A.36}$$

where

$$\tilde{C}_h(\theta) = (M_{21L} \otimes I_k - \Omega_{21T}(\theta) \Omega_{11T}^{-1}(\theta) M_{11} \otimes I_k),\tag{A.37}$$

and

$$\begin{aligned}\tilde{C}_\omega^*(\theta) &= (\Omega_{21T}(\theta) \Omega_{11T}^{-1}(\theta) M_{11} \otimes G'_{1T}(\theta) \Omega_{11T}^{-1}(\theta) M_{11}) D_m \\ &\quad - (I_{m_2} M_{21L} \otimes G'_{1T}(\theta) \Omega_{11T}^{-1}(\theta) M_{21R}) D_m.\end{aligned}\tag{A.38}$$

Recall that  $\tilde{\theta}_j \rightarrow_p \theta_0$ . Hence, by Assumption 2, Lemma 4 of Amemiya (1973) and Slutsky's theorem,  $\widehat{H}_{\Delta T}(\tilde{\theta}_j) \rightarrow_p C_h H + C_\omega Q$ .

Hence, from  $\sum_{j=1}^{m_2k} \lambda_j = 1$  it follows that  $\sum_{j=1}^{m_2k} \lambda_j \widehat{H}_{\Delta T}(\tilde{\theta}_j) \rightarrow_p C_h H + C_v Q$ . Recall that  $\widehat{\Omega} \rightarrow_p \Omega$ ,  $\widehat{\theta} \rightarrow_p \theta$ , and  $\bar{\theta} \rightarrow_p \theta_0$ . Hence, by Assumption 2, Lemma 4 of Amemiya (1973) and Slutsky's theorem,  $\widetilde{C}_f \rightarrow_p C_f$ . Recall also that  $\widetilde{V}_{11T} \rightarrow_p \Omega_{11}$ . Hence, by Assumption 2, Lemma 4 of Amemiya (1973) and Slutsky's theorem,  $\widetilde{C}_\omega \rightarrow_p C_v$  and, hence,  $\widetilde{C} \rightarrow_p C$ .

Finally, by Assumption 3 and Slutsky's theorem,

$$\widetilde{C}(r_T(\theta_0) - r) \rightarrow_p N(0, \Sigma_{\hat{g}}), \quad (\text{A.39})$$

with  $\Sigma_{\hat{g}} = C\Sigma_r C'$ . Theorem 2 follows straightforwardly from (A.30) and (A.39).

**Proof of Theorem 3.** Multiplying both sides of equation 5.5 from the right with  $\Sigma_{1AA}^{-1}$ , we obtain

$$G_{\Delta A} + G_{\Delta B}\Sigma_{1BA}\Sigma_{1AA}^{-1} = 0. \quad (\text{A.40})$$

Inverse of the partitioned matrix  $\Sigma_1$  implies that

$$\Sigma_{1BA}\Sigma_{1AA}^{-1} = -(G'_{1B}\Omega^{-1}G_{1B})^{-1}(G'_{1B}\Omega^{-1}G_{1A}). \quad (\text{A.41})$$

Substituting (A.41) into (A.41), we obtain

$$G_{\Delta A} = G_{\Delta B}(G'_{1B}\Omega_{11}^{-1}G_{1B})^{-1}(G'_{1B}\Omega_{11}^{-1}G_{1A}), \quad (\text{A.42})$$

which is, up to notations, the condition of Breusch, Qian, Schmidt, and Wyhowski (1999).

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