The stochastic spectator

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Abstract. We study the stochastic distribution of spectator fields predicted in different slow-roll inflation backgrounds. Spectator fields have a negligible energy density during inflation but may play an important dynamical role later, even giving rise to primordial density perturbations within our observational horizon today. During de Sitter expansion there is an equilibrium solution for the spectator field which is often used to estimate the stochastic distribution during slow-roll inflation. However slow roll only requires that the Hubble rate varies slowly compared to the Hubble time, while the time taken for the stochastic distribution to evolve to the de-Sitter equilibrium solution can be much longer than a Hubble time. We study both monomial (chaotic) and plateau inflaton potentials, with quadratic, quartic and axionic spectator fields. We give an adiabaticity condition for the spectator field distribution to relax to the de-Sitter equilibrium, and find that the de-Sitter approximation is never a reliable estimate for the typical distribution at the end of inflation for a quadratic spectator during monomial inflation. The existence of an adiabatic regime at early times can erase the dependence on initial conditions of the final distribution of field values. In these cases, spectator fields acquire sub-Planckian expectation values. Otherwise spectator fields may acquire much larger field displacements than suggested by the de Sitter equilibrium solution. We quantify the information about initial conditions that can be obtained from the final field distribution. Our results may have important consequences for the viability of spectator models for the origin of structure, such as the simplest curvaton models.

Keywords: physics of the early universe, inflation

Contents

1 Introduction

Inflation [1–6] is a phase of accelerated expansion at very high energy in the primordial Universe. During this epoch, vacuum quantum fluctuations of the gravitational and matter fields were amplified to large-scale cosmological perturbations [7–12], that later seeded the cosmic microwave background anisotropies and the large-scale structure of our Universe. At present, the full set of observations can be accounted for in a minimal setup, where inflation is driven by a single scalar inflaton field with canonical kinetic term, minimally coupled to gravity, and evolving in a flat potential in the slow-roll regime [13, 14]. From a theoretical point of view, however, inflation takes place in a regime that is far beyond the reach of accelerators, and the physical details of how the inflaton is connected with the standard model of particle physics and its extensions are still unclear. In particular, most physical setups that have been proposed to embed inflation contain extra scalar fields. This is notably the case in string theory models where many extra light moduli fields may be present [15–20].

Even if such fields are purely spectators during inflation (i.e. contribute a negligible amount to the total energy density of the Universe), they can still play an important dynamical role afterwards. The details of their post-inflationary contribution typically depend on the field displacement they acquire during inflation. In this context, if inflation provides initial conditions for cosmological perturbations, it should also be seen as a mechanism that generates a distribution of initial field displacements for light degrees of freedom. In this paper, we investigate what possibilities this second channel offers to probe the physics of inflation. In practice, we study how the field value acquired by light scalar spectator fields at the end of inflation depends on the inflaton field potential, on the spectator field potential and on the initial distribution of spectator field values.

As an illustration of post-inflationary physical processes for which the field value acquired by spectator fields during inflation plays an important role, we may consider the curvaton scenario [21–25]. In this model, the curvaton field σ is a light spectator field during inflation that can dominate the energy budget of the Universe afterwards. Its density perturbation is given by $\delta\rho_\sigma/\rho_\sigma \sim \delta\sigma/\sigma$, where ρ_σ denotes the energy density contained in σ, and the effect of this perturbation on the total density perturbation of the Universe is reduced by the relative energy density of the curvaton field to the total energy density. The curvaton field, like every light scalar field, is perturbed at Hubble radius exit by an amount $\delta\sigma \sim H_* \lesssim 10^{-6} M_{\rm Pl}$, where H_* is the Hubble parameter evaluated at the time of Hubble radius crossing during inflation and $M_{\rm Pl}$ is the reduced Planck mass. If the curvaton perturbations produce the entire observed primordial density perturbation with amplitude 10^{-5}, the average field value in our Hubble patch, σ, is of order $\sigma \sim 10^5 H_*$. An important question is therefore whether such a field value can naturally be given to the curvaton during inflation. In the limit of low energy scale inflation in particular, this implies that $\sigma \ll M_{\rm Pl}$. The requirement for a very sub-Planckian spectator field value in models where an initially isocurvature field perturbation is later converted into the observed adiabatic curvature perturbation is common but not completely generic, and may be intuitively understood by realising that if the spectator field fluctuations are negligible compared to the background value (i.e. $\delta\sigma < 10^{-5}\sigma$), then it is difficult to make the primordial density perturbation have a significant dependence on $\delta\sigma$ if the background value is not very sub-Planckian. This is discussed in the conclusions of Ref. [26], which shows that it typically also applies to scenarios such as modulated reheating [18, 27]. The dark energy model proposed in Ref. [28] also requires sub-Planckian spectator fields during inflation, and the new results we derive on the field value distribution of a spectator field with a quartic potential may have implications for the stability of the Higgs vacuum during inflation as well, see *e.g.* Refs. [29–31].

This naturally raises the question of whether having a sub-Planckian spectator field value represents a fine tuning of the initial conditions or not. Provided that inflation lasts long enough, we address this question here by calculating the stochastically generated distribution of spectator field values. We will show cases in which sub-Planckian field values are natural, and others in which super-Planckian field values are preferred.

If the spectator field value is driven to become significantly super-Planckian, it can drive a second period of inflation, which may have observable effects even if the inflaton

field perturbations dominate, because the observable scales exit the Hubble radius at a different time during the first period of inflation, when the inflaton is traversing a different part of the potential [32, 33]. In some cases, we will show that the spectator field value may naturally become so large that it drives more than 60 *e*-folds of inflation. In this case we would not observe the initial period of inflation at all, but its existence remains important for generating the initial conditions for the second, observable period of inflation.

If no isocurvature perturbations persist after reheating, the linear perturbations from the inflaton and spectator field are likely to be observationally degenerate. Non-linear perturbations, especially the coupling between primordial long- and short-wavelength perturbations, help to break this degeneracy. We will not study non Gaussianity in this paper, but highlight that the results calculated here help to motivate a prior distribution for the initial spectator field value, which is a crucial ingredient of model comparison between single- and multiple-field models of inflation [32–35].

The paper is organised as follows. In Sec. 1.1, the stochastic inflation formalism is introduced. This framework allows one to compute the field value acquired by a quantum scalar field on large scales during inflation, in the form of a classical probability distribution for the field values. In de-Sitter space-times, an equilibrium solution exists for this distribution that is often used as an estimate for more generic inflationary backgrounds as long as they are in the slow-roll regime, hence close to de Sitter. In Sec. 1.2, we explain why this "adiabatic" approximation is in fact not valid in general, and why the de-Sitter equilibrium cannot even be used as a proxy in most relevant cases (for a discussion on de-Sitter equilibrium for non-minimally coupled spectators, see Refs. [36–38]). This is why in Secs. 2, 3 and 4, we study the stochastic dynamics of spectator fields with quadratic, quartic and axionic (i.e. cosine) potentials respectively. Each case is investigated with two classes of inflationary potentials, namely plateau and monomial. In Sec. 5, the amount of information about the state of the spectator field at the onset of inflation that can be extracted from its field value at the end of inflation is quantified and discussed. Finally, in Sec. 6, we summarise our main results and draw a few conclusions. Various technical results are derived in Appendixes A and B.

1.1 Stochastic inflation

Let us now see how the field value acquired by quantum scalar fields during inflation can be calculated in practice. During inflation, scalar field perturbations are placed in squeezed states, which undergo quantum-to-classical transitions [39–44] in the sense that on super-Hubble scales, the non-commutative parts of the fields become small compared to their anti-commutative parts. This gives rise to the stochastic inflation formalism [10, 45–55], consisting of an effective theory for the long-wavelength parts of the quantum fields, which are "coarse grained" at a fixed physical scale larger than the Hubble radius during the whole inflationary period. In this framework, the short wavelength fluctuations behave as a classical noise acting on the dynamics of the super-Hubble scales as they cross the coarse-graining scale. The coarse-grained fields can thus be described

by a stochastic classical theory, following Langevin equations

$$\frac{\mathrm{d}\sigma}{\mathrm{d}N} = -\frac{V_{,\sigma}(\sigma)}{3H^2} + \frac{H}{2\pi}\xi \,. \tag{1.1}$$

In this expression, σ denotes a coarse-grained field with potential $V(\sigma)$ (a subscript ",σ" corresponds to partial derivation with respect to σ). The Hubble parameter H is defined as $H \equiv \dot{a}/a$, where a is the scale factor and a dot denotes differentiation with respect to cosmic time. The time variable $N \equiv \ln(a)$ has been used but the choice of the time variable is irrelevant for test fields [54–57]. Finally, ξ is a Gaussian white noise with vanishing mean and unit variance such that $\langle \xi(N) \rangle = 0$ and $\langle \xi(N_1)\xi(N_2) \rangle = \delta(N_1 - N_2)$, where $\langle \cdot \rangle$ denotes ensemble average. The Langevin equation (1.1) is valid at leading order in slow roll, hence for a light test field with $V_{,\sigma\sigma} \ll H$. In the Itô interpretation, it gives rise to a Fokker-Planck equation for the probability density $P(\sigma, N)$ of the coarse-grained field σ at time N [45, 58]

$$\frac{\partial P(\sigma, N)}{\partial N} = \frac{\partial}{\partial \sigma}\left[\frac{V_{,\sigma}(\sigma)}{3H^2}P(\sigma, N)\right] + \frac{H^2}{8\pi^2}\frac{\partial^2}{\partial \sigma^2}\left[P(\sigma, N)\right]. \tag{1.2}$$

This equation can be written as $\partial P/\partial N = -J_{,\sigma}$, where $J \equiv -V_{,\sigma}P/(3H^2) - H^2P_{,\sigma}/(8\pi^2)$ is the probability current.

When H is constant, a stationary solution P_{stat} to Eq. (1.2) can be found as follows. Since P_{stat} does not depend on time, the probability current does not depend on σ (or on time either). Therefore, if J vanishes at the boundaries of the field domain, it vanishes everywhere. This yields a first-order differential equation for $P_{\mathrm{stat}}(\sigma)$ that can be solved and one obtains

$$P_{\mathrm{stat}}(\sigma) \propto \exp\left[-\frac{8\pi^2 V(\sigma)}{3H^4}\right], \tag{1.3}$$

where the overall integration constant is fixed by requiring that the distribution is normalised, $\int P(\sigma)\mathrm{d}\sigma = 1$. In the following, the solution (1.3) will be referred to as the "de-Sitter equilibrium". For instance, if the spectator field has a quadratic potential $V(\sigma) = m^2\sigma^2/2$, the de-Sitter equilibrium is a Gaussian with standard deviation $\sqrt{\langle \sigma^2 \rangle} \sim H^2/m$. In this case, it will be shown in Sec. 2.1 that this equilibrium solution is in fact an attractor of Eq. (1.3), that is reached over a time scale $N_{\mathrm{relax}} \sim H^2/m^2$. Therefore, provided inflation lasts more than N_{relax} e-folds, the typical field displacement is of order H^2/m at the end of inflation in this case [59].

1.2 Limitations of the adiabatic approximation

It is commonly assumed that if one considers a more realistic inflationary background in the slow-roll regime, the de-Sitter results derived in Sec. 1.1 still apply. The reason is that H varies slowly during slow-roll inflation, which thus does not deviate much from de Sitter. This is why in practice, Eq. (1.3) is often used to estimate the field value acquired by spectator fields during inflation. However, one can already see why this "adiabatic" approximation, which assumes that one can simply replace H by $H(N)$ in Eq. (1.3) and track the local equilibrium at every time, is not always valid. Indeed, the

time scale over which H varies by a substantial amount in slow-roll inflation is given by $N_H = 1/\epsilon_1$, where $\epsilon_1 \equiv -\dot{H}/H^2$ is the first slow-roll parameter. During inflation, $\epsilon_1 \ll 1$, so that $N_H \gg 1$. However, in order to see whether a spectator field tracks the de-Sitter equilibrium, one should not compare N_H to 1, but to $N_{\rm relax}$, the number of e-folds required by the spectator field to relax towards the equilibrium. In other words, only if the adiabatic condition

$$N_{\rm relax} \ll N_H = \frac{1}{\epsilon_1} \tag{1.4}$$

holds can H be considered as a constant over the time required by the spectator field to relax to the equilibrium, and only in this case can the stationary distribution (1.3) be used.

If the inflaton potential is of the plateau type and asymptotes to a constant as the field value asymptotes to infinity, one typically has [60, 61] $\epsilon_1 \simeq \mathcal{O}(1)/(N_{\rm end} - N)^2$ in the limit where $N_{\rm end} - N \gg 1$, where $N_{\rm end}$ denotes the number of e-folds at the end of inflation where $\epsilon_1 \simeq 1$. This leads to

$$H \simeq H_{\rm plateau} \exp\left[\frac{\mathcal{O}(1)}{N - N_{\rm end} - 1}\right], \tag{1.5}$$

where $H_{\rm plateau}$ is the asymptotic value of H at large-field value, hence $N_H \simeq \mathcal{O}(1)\,(N_{\rm end} - N)^2$, meaning that H cannot change by more than a factor of order one throughout the entire inflationary phase. For instance, if one considers the Starobinsky potential [1] $V(\phi) \propto (1 - e^{-\sqrt{2/3}\phi/M_{\rm Pl}})^2$, one finds $\epsilon_1 \simeq 3/[4(N_{\rm end} - N)^2]$ and $H_{\rm end}/H_{\rm plateau} \simeq 0.53$. In this case, the de-Sitter equilibrium (1.3), $\langle V(\sigma)\rangle \sim H^4$, only changes by a relatively small fraction and therefore provides a useful estimate for the order of magnitude of spectator field displacements at the end of inflation [using either $H = H_{\rm plateau}$ or $H = H_{\rm end}$ in Eq. (1.3)]. Note that the same can be true for hilltop potentials where H also asymptotes a constant in the infinite past.

In the context of single-field inflation however, plateau potentials are known to provide a good fit to the data only in the last ~ 50 e-folds of inflation. The shape of the inflaton potential is not constrained beyond this range and is typically expected to receive corrections when the field varies by more than the Planck scale. In multiple-field inflation, observations allow the inflaton potential to be of the large-field type all the way down to the end of inflation [33]. Therefore we also consider monomial inflaton potentials $V(\phi) \propto \phi^p$ with $p > 0$. In these models, one has

$$H(N) = H_{\rm end}\left[1 + \frac{4}{p}\left(N_{\rm end} - N\right)\right]^{\frac{p}{4}}. \tag{1.6}$$

If $p > 1$, this corresponds to convex inflaton potentials (meaning $V'' > 0$), while this describes concave inflaton potentials ($V'' < 0$) for $p < 1$, and the de-Sitter case is recovered in the limit $p \to 0$. From Eq. (1.6), one has $\epsilon_1 = (H_{\rm end}/H)^{4/p}$, so that $N_H = (H/H_{\rm end})^{4/p}$. If the spectator field has a quadratic potential for instance, as

mentioned above, it will be shown in Sec. 2.1 that $N_{\mathrm{relax}} \sim H^2/m^2$. In this case, the adiabatic condition (1.4) reads $(H/H_{\mathrm{end}})^{2/p-1} \gg H_{\mathrm{end}}/m$. If $p \geq 2$, one can see that this can never be realised since $H_{\mathrm{end}} > m$ and $H > H_{\mathrm{end}}$. If $p < 2$, the adiabatic condition is satisfied when H is sufficiently large, that is to say at early enough times when $N_{\mathrm{end}} - N > p[(H_{\mathrm{end}}/m)^{4/(2-p)} - 1]/4$. If $m/H_{\mathrm{end}} \sim 0.01$ for instance, this number of e-folds is larger than ~ 400 as soon as $p > 0.1$ (and larger than $\sim 10^7$ for $p > 1$), which means that even in this case, the adiabatic regime lies far away from the observable last 50 e-folds of inflation. One concludes that in most cases, the de-Sitter equilibrium solution does not provide a reliable estimate of the field value acquired by spectator fields during inflation. In the following, we therefore study the dynamics of such fields beyond the adiabatic approximation.

2 Quadratic spectator

In this section, we consider a quadratic spectator field, for which

$$V(\sigma) = \frac{m^2}{2}\sigma^2 \,. \tag{2.1}$$

In this case, the Langevin equation (1.1) is linear, which allows one to solve it analytically. In Appendix A, we explain how to calculate the first two statistical moments of the spectator field σ. The first moment is given by

$$\langle \sigma(N) \rangle = \langle \sigma(N_0) \rangle \exp\left[-\frac{m^2}{3} \int_{N_0}^{N} \frac{\mathrm{d}N'}{H^2(N')} \right] \,, \tag{2.2}$$

which corresponds to the classical solution of Eq. (1.1) in the absence of quantum diffusion, and where we have set $\langle \sigma \rangle = \langle \sigma(N_0) \rangle$ at the initial time N_0. For the second moment, one obtains

$$\begin{aligned}
\langle \sigma^2(N) \rangle =& \langle \sigma^2(N_0) \rangle \exp\left[-\frac{2m^2}{3} \int_{N_0}^{N} \frac{\mathrm{d}N'}{H^2(N')} \right] \\
&+ \int_{N_0}^{N} \mathrm{d}N' \frac{H^2(N')}{4\pi^2} \exp\left[\frac{2m^2}{3} \int_{N}^{N'} \frac{\mathrm{d}N''}{H^2(N'')} \right] \,.
\end{aligned} \tag{2.3}$$

In this expression, the structure of the first term in the right-hand side is similar to the first moment (2.2) while the second term is due to quantum diffusion, so that the variance of the distribution $\langle \sigma^2 \rangle - \langle \sigma \rangle^2$ is given by the same formula as the second moment [i.e. one can replace $\langle \sigma^2 \rangle$ by $\langle \sigma^2 \rangle - \langle \sigma \rangle^2$ in Eq. (2.3) and the formula is still valid].

One can also show that the Fokker-Planck equation (1.2) admits Gaussian solutions,

$$P(\sigma, N) = \frac{1}{\sqrt{2\pi \langle \sigma^2(N) \rangle}} \exp\left\{ -\frac{[\sigma - \langle \sigma(N) \rangle]^2}{2 \langle \sigma^2 \rangle} \right\} \,, \tag{2.4}$$

where $\langle \sigma(N) \rangle$ and $\langle \sigma^2(N) \rangle$ are given by Eqs. (2.2) and (2.3) respectively. However, let us stress that Eqs. (2.2) and (2.3) are valid for any (i.e. not only Gaussian) probability distributions.

2.1 Plateau inflation

As explained in Sec. 1.2, if the inflaton potential is of the plateau type, H can be approximated by a constant. In this case, the mean coarse-grained field (2.2) is given by

$$\langle \sigma(N) \rangle = \langle \sigma(N_0) \rangle \exp \left[-\frac{m^2}{3H^2} (N - N_0) \right] . \tag{2.5}$$

It follows the classical trajectory as already pointed out below Eq. (2.2), and becomes small when $N - N_0 \gg H^2/m^2$. For the second moment, Eq. (2.3) gives rise to

$$\langle \sigma^2(N) \rangle = \left[\langle \sigma^2(N_0) \rangle - \frac{3H^4}{8\pi^2 m^2} \right] \exp \left[-\frac{2m^2}{3H^2} (N - N_0) \right] + \frac{3H^4}{8\pi^2 m^2} . \tag{2.6}$$

When $N - N_0 \gg H^2/m^2$, it approaches the constant value $\langle \sigma^2 \rangle = 3H^4/(8\pi^2 m^2)$. One can check that this asymptotic value corresponds to the de-Sitter equilibrium in Eq. (1.3). Moreover, one can see that the typical relaxation time that is required to reach the attractor is given by

$$N_{\rm relax} = \frac{H^2}{m^2} , \tag{2.7}$$

which corresponds to the value reported in Sec. 1.1.

2.2 Monomial inflation

If the inflaton potential is monomial and of the form $V(\phi) \propto \phi^p$, the Hubble factor is given by Eq. (1.6). Substituting this expression for $H(N)$ into Eq. (2.2), one obtains (for $p \neq 2$)

$$\langle \sigma(H) \rangle = \langle \sigma(H_0) \rangle \exp \left\{ \frac{\mu}{2} \left[\left(\frac{H}{H_{\rm end}} \right)^{\frac{4}{p}-2} - \left(\frac{H_0}{H_{\rm end}} \right)^{\frac{4}{p}-2} \right] \right\} , \tag{2.8}$$

where H_0 is the value of H at an initial time N_0, and we have defined

$$\mu \equiv \frac{m^2}{3H_{\rm end}^2} \frac{p}{2-p} . \tag{2.9}$$

In Eq. (2.8), time is parametrised by H instead of N for convenience but the two are directly related through Eq. (1.6). For the second moment (or for the variance), by substituting Eq. (1.6) into Eq. (2.3), one obtains

$$\langle \sigma^2(H) \rangle = \langle \sigma^2(H_0) \rangle \exp \left\{ \mu \left[\left(\frac{H}{H_{\rm end}} \right)^{\frac{4}{p}-2} - \left(\frac{H_0}{H_{\rm end}} \right)^{\frac{4}{p}-2} \right] \right\}$$
$$+ \frac{p H_{\rm end}^2 \mu^{\frac{p+2}{p-2}}}{8\pi^2 (p-2)} e^{\mu \left(\frac{H}{H_{\rm end}} \right)^{\frac{4}{p}-2}} \left\{ \Gamma \left[\frac{2+p}{2-p}, \mu \left(\frac{H_0}{H_{\rm end}} \right)^{\frac{4}{p}-2} \right] - \Gamma \left[\frac{2+p}{2-p}, \mu \left(\frac{H}{H_{\rm end}} \right)^{\frac{4}{p}-2} \right] \right\}$$
$$\tag{2.10}$$

where Γ denotes the incomplete Gamma function. One can note that both Eqs. (2.8) and (2.10) can be expressed as functions $\mu(H/H_{\rm end})^{4/p-2}$ only, which is directly proportional to the ratio $N_H/N_{\rm relax}$. As noted in Sec. 1.2, for $p \geq 2$ this ratio is always small, while for $p < 2$, it is large unless H is sufficiently large. The two cases $p \geq 2$ and $p < 2$ must therefore be treated distinctly.

2.2.1 Case where $p \geq 2$

If $p > 2$, one has $N_H \ll N_{\rm relax}$ and the quantity $\mu(H/H_{\rm end})^{4/p-2}$ in Eqs. (2.8) and (2.10) is always much smaller than one. This implies that the argument of the exponential in Eq. (2.8) can be neglected, and $\langle \sigma(H) \rangle \simeq \langle \sigma(H_0) \rangle$ stays constant. Therefore, the distribution remains centred at the initial value. Note that the case $p = 2$ is singular and gives rise to

$$\langle \sigma(H) \rangle = \langle \sigma(H_0) \rangle \left(\frac{H}{H_0} \right)^{\frac{m^2}{3H_{\rm end}^2}} , \tag{2.11}$$

which also yields $\langle \sigma(H) \rangle \simeq \langle \sigma(H_0) \rangle$ unless $H_0/H_{\rm end} \gg \exp(3H_{\rm end}^2/m^2)$.

For the second moment, the second arguments of the incomplete Gamma functions in Eq. (2.10) are always much smaller than one and in this limit, one finds

$$\langle \sigma^2(H) \rangle \simeq \langle \sigma^2(H_0) \rangle + \frac{H_{\rm end}^2 p}{8\pi^2(p+2)} \left[\left(\frac{H_0}{H_{\rm end}} \right)^{2+\frac{4}{p}} - \left(\frac{H}{H_{\rm end}} \right)^{2+\frac{4}{p}} \right] . \tag{2.12}$$

In this expression, one can see that $\langle \sigma^2 \rangle$ can only increase as time proceeds, in a way that does not depend on the mass (as long as it is sub-Hubble). The result is therefore the same as if one set the mass to zero, and corresponds to a free diffusion process. This is consistent with the fact that $\langle \sigma \rangle$ stays constant in this case. If $p = 2$, Eq. (2.10) is singular and one has

$$\langle \sigma^2(H) \rangle = \langle \sigma^2(H_0) \rangle \left(\frac{H}{H_0} \right)^{\frac{2m^2}{3H_{\rm end}^2}} + \frac{H_{\rm end}^2}{8\pi^2 \left(2 - \frac{m^2}{3H_{\rm end}^2} \right)} \left(\frac{H}{H_{\rm end}} \right)^{\frac{2m^2}{3H_{\rm end}^2}}$$
$$\times \left[\left(\frac{H_0}{H_{\rm end}} \right)^{4-\frac{2m^2}{3H_{\rm end}^2}} - \left(\frac{H}{H_{\rm end}} \right)^{4-\frac{2m^2}{3H_{\rm end}^2}} \right] . \tag{2.13}$$

In this case, it was also shown in Sec. 1.2 that $N_H \ll N_{\rm relax}$ so there is no adiabatic regime either. Unless $H_0/H_{\rm end} \gg \exp(3H_{\rm end}^2/m^2)$, in the limit $m \ll H_{\rm end}$, Eq. (2.13) coincides with Eq. (2.12) evaluated at $p = 2$ so in practice the latter formula can be used for all values of $p \geq 2$.

An important feature of Eq. (2.12) is that it strongly depends on the initial conditions $\langle \sigma(H_0) \rangle$ and H_0. This is because there is no adiabatic regime in this case and hence no attractor that would erase initial conditions. As a consequence, the typical spectator field displacement at the end of inflation cannot be determined without specifying initial conditions.

One should also note that the present analysis relies on the assumption that the inflaton is not experiencing large stochastic diffusion, which allows us to use Eq. (1.6). This is in fact the case if $H \ll H_{\mathrm{eternal}}$, where

$$H_{\mathrm{eternal}} \equiv H_{\mathrm{end}} \left(\frac{M_{\mathrm{Pl}}}{H_{\mathrm{end}}} 2\pi\sqrt{2} \right)^{\frac{p}{2+p}} \tag{2.14}$$

is the scale above which a regime of so-called "eternal inflation" takes place.[1] For this reason, H_{eternal} is the largest value one can use for H_0 in order for the calculation to be valid. Setting $H_0 = H_{\mathrm{eternal}}$, and substituting Eq. (2.14) into Eq. (2.12), one obtains at the end of inflation

$$\left\langle \sigma_{\mathrm{end}}^2 \right\rangle \simeq \left\langle \sigma_{\mathrm{eternal}}^2 \right\rangle + \frac{p}{p+2} M_{\mathrm{Pl}}^2 . \tag{2.15}$$

This expression is displayed in the left panel of Fig. 1. It means that the field value of the spectator field is at least of the order of the Planck mass at the end of inflation. If one assumes the de-Sitter equilibrium distribution (1.3) at the end of eternal inflation for instance, $\langle \sigma_{\mathrm{eternal}}^2 \rangle = 3 H_{\mathrm{eternal}}^4 / (8\pi^2 m^2)$, even much larger field displacements are obtained at the end of inflation.

2.2.2 Case where $p < 2$

If $p < 2$, whether the ratio N_H / N_{relax} is small or large depends on the value of H. More precisely, if $H \gg H_{\mathrm{adiab}}$, where

$$H_{\mathrm{adiab}} \equiv H_{\mathrm{end}} \left(\frac{H_{\mathrm{end}}}{m} \right)^{\frac{p}{2-p}} , \tag{2.16}$$

one is in the adiabatic regime and $N_H \gg N_{\mathrm{relax}}$. As soon as H drops below H_{adiab} however, one leaves the adiabatic regime. In order to set initial conditions during the adiabatic regime, it should apply after the eternal inflationary phase during which our calculation does not apply, which implies that $H_{\mathrm{adiab}} < H_{\mathrm{eternal}}$. Making use of Eqs. (2.14) and (2.16), this condition gives rise to

$$\frac{m}{H_{\mathrm{end}}} > \left(\frac{H_{\mathrm{end}}}{M_{\mathrm{Pl}}} \right)^{\frac{2-p}{2+p}} . \tag{2.17}$$

Let us distinguish the two cases where this relation is and is not satisfied.

Starting out in the adiabatic regime
If Eq. (2.17) is satisfied, one can set initial conditions for the spectator field σ in the adiabatic regime while being outside the eternal inflationary phase, that is to say one can take $H_{\mathrm{adiab}} < H_0 < H_{\mathrm{eternal}}$. From Eq. (2.8), this implies that $\langle \sigma_{\mathrm{end}} \rangle \ll \langle \sigma_0 \rangle$ and the

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[1]More precisely, H_{eternal} is defined as the scale above which, over the typical time scale of an e-fold, the mean quantum diffusion received by the inflaton field, $H/(2\pi)$, is larger than the classical drift, $\sqrt{2\epsilon_1} M_{\mathrm{Pl}}$. Since $\epsilon_1 = (H_{\mathrm{end}}/H)^{4/p}$ in monomial inflation (1.6), this condition gives rise to $H > H_{\mathrm{eternal}}$ where H_{eternal} is given by Eq. (2.14).
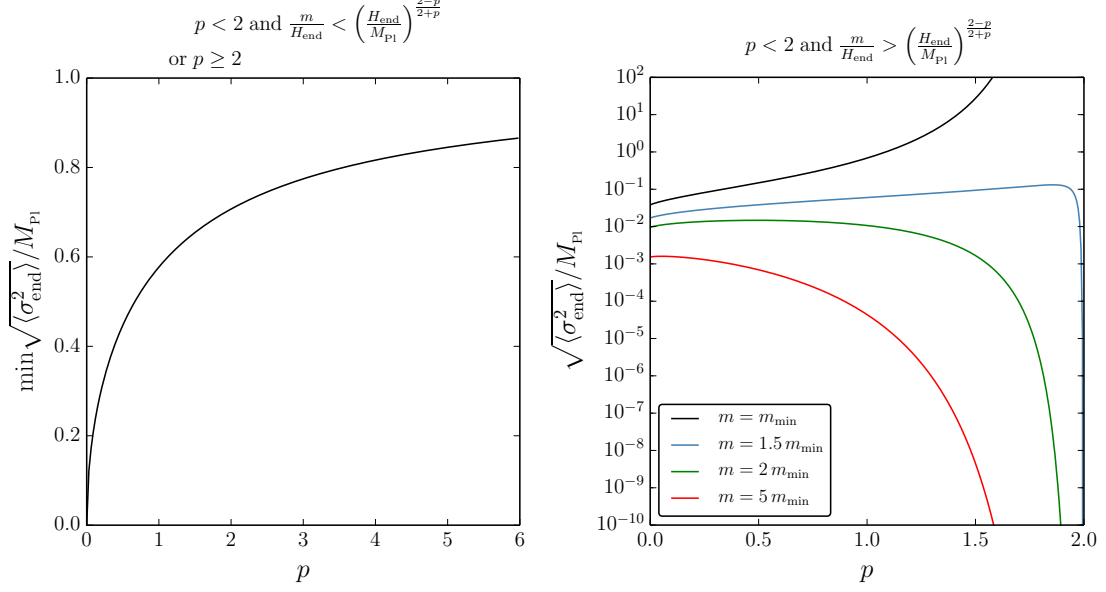
Figure 1. The typical field displacement $\sqrt{\langle\sigma_{\rm end}^2\rangle}$ acquired by a test field σ with quadratic potential $V(\sigma) = m^2\sigma^2/2$ at the end of an inflationary phase driven by an inflaton potential $V(\phi) \propto \phi^p$. In the left panel, the cases $p \geq 2$ and $p < 2$ with $m/H_{\rm end} < (H_{\rm end}/M_{\rm Pl})^{(2-p)/(2+p)}$ are displayed, where the minimum value of $\langle\sigma_{\rm end}^2\rangle$ is given if one initially sets $\langle\sigma^2\rangle = 0$ at the time $H = H_{\rm eternal}$ when stochastic corrections to the inflaton dynamics stop being large. This corresponds to Eq. (2.15) and shows that spectator fields are typically at least close to super-Planckian at the end of inflation in these cases. In the right panel, the case $p < 2$ with $m/H_{\rm end} > (H_{\rm end}/M_{\rm Pl})^{(2-p)/(2+p)}$ is displayed, where there is an early adiabatic regime that allows the dependence on initial condition to be erased. The typical field displacement is given by Eq. (2.19), which is expressed as a function of p and $m/m_{\rm min}$ in Eq. (2.20), where $m_{\rm min}$ is the lower bound on m associated to the condition $m/H_{\rm end} > (H_{\rm end}/M_{\rm Pl})^{(2-p)/(2+p)}$. One can check that, as soon as $m \gtrsim 1.5\,m_{\rm min}$, $\sqrt{\langle\sigma_{\rm end}^2\rangle}$ is always sub-Planckian in this case.

distribution becomes centred around smaller field values as time proceeds. Regarding the width of the distribution, two regimes of interest need to be considered.

At early time, i.e. when $H \gg H_{\rm adiab}$, the incomplete Gamma functions in Eq. (2.10) can be expanded in the large second argument limit and one obtains

$$\langle\sigma^2(H)\rangle \simeq \left[\langle\sigma^2(H_0)\rangle - \frac{3H_0^4}{8\pi^2 m^2}\right] \exp\left\{\mu\left[\left(\frac{H}{H_{\rm end}}\right)^{\frac{4}{p}-2} - \left(\frac{H_0}{H_{\rm end}}\right)^{\frac{4}{p}-2}\right]\right\} + \frac{3H^4}{8\pi^2 m^2}\,. \tag{2.18}$$

In this expression, one can see that as soon as H decreases from H_0, the first term is exponentially suppressed and one obtains $\langle\sigma^2\rangle \simeq 3H^4/(8\pi m^2)$, which corresponds to the de-Sitter equilibrium formula[2] and confirms that one is in the adiabatic regime. This

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[2]More precisely, in a de-Sitter universe where H is constant and equal to the instantaneous value

also shows that the de-Sitter equilibrium is an attractor of the stochastic dynamics in this case, and that it is reached within a number of e-folds $\sim H_0^2/m^2$, which exactly corresponds to N_{relax} given in Eq. (2.7) when $H = H_0$.

At later times, i.e. when $H \ll H_{\mathrm{adiab}}$, one leaves the adiabatic regime and while the first incomplete Gamma function in Eq. (2.10) can still be expanded in the large second argument limit, the second one must be expanded in the small second argument limit and this gives rise to

$$\left\langle \sigma^2(H) \right\rangle \simeq \frac{H_{\mathrm{end}}^2}{8\pi^2} \frac{p}{2-p} \Gamma\left(\frac{2+p}{2-p}\right) \left(\frac{3H_{\mathrm{end}}^2}{m^2} \frac{2-p}{p}\right)^{\frac{2+p}{2-p}} . \tag{2.19}$$

Interestingly, this expression does not depend on H, meaning that $\langle \sigma^2 \rangle$ stays constant as soon as one leaves the adiabatic regime (and obviously stops tracking the adiabatic solution). One can also check that in this expression, the limit $p \to 0$ gives rise to $\langle \sigma_{\mathrm{end}}^2 \rangle \simeq 3H_{\mathrm{end}}^4/(8\pi^2 m^2)$, that is to say the de-Sitter equilibrium formula.

An important consequence of this result is that in the case $p < 2$ and if $m > m_{\mathrm{min}}$, where m_{min} corresponds to the lower bound on m given by Eq. (2.17), even if the end of inflation lies far outside the adiabatic regime, the existence of an early adiabatic phase allows initial conditions to be erased. At the end of inflation, the field value of the spectator field only depends on m, H_{end} and p. This is in contrast with the case $p \geq 2$ where there is no adiabatic regime, even at early time, and initial conditions remain important even at the end of inflation. A second important consequence is that the typical field displacement is always sub-Planckian at the end of inflation in this case. Indeed, substituting the expression given for m_{min} by Eq. (2.17) into Eq. (2.19), one obtains

$$\frac{\left\langle \sigma_{\mathrm{end}}^2 \right\rangle}{M_{\mathrm{Pl}}^2} = \frac{1}{8\pi^2} \left(\frac{p}{2-p}\right)^{\frac{2p}{p-2}} \Gamma\left(\frac{2+p}{2-p}\right) \left(3\frac{m_{\mathrm{min}}^2}{m^2}\right)^{\frac{2+p}{2-p}} . \tag{2.20}$$

This expression is displayed in the right panel of Fig. 1 for a few values of m/m_{min}. One can see that as soon as $m \gtrsim 1.5\, m_{\mathrm{min}}$, the spectator field is always sub-Planckian at the end of inflation.

Starting out away from the adiabatic regime
If the condition (2.17) is not satisfied, the adiabatic regime cannot be used to erase initial conditions dependence. If both H_0 and H are much smaller than H_{adiab}, the incomplete Gamma functions in Eq. (2.10) can be expanded in the small second argument limit and one obtains Eq. (2.12) again. When H becomes small compared to H_0, $\langle \sigma^2 \rangle$ reaches a constant and the distribution remains frozen until the end of inflation. Letting $H_0 = H_{\mathrm{eternal}}$ as in Sec. 2.2.1, this gives rise to Eq. (2.15) and one concludes that, in this case, the spectator field acquires a super-Planckian field value at the end of inflation.

$H(N)$ for a given N in the case at hand, the asymptotic value reached by $\langle \sigma^2 \rangle$ at late time is the same as the instantaneous value $\langle \sigma^2(N) \rangle$ obtained from Eq. (2.18). In this sense, the time evolution of H can be neglected and this corresponds, by definition, to an adiabatic regime.
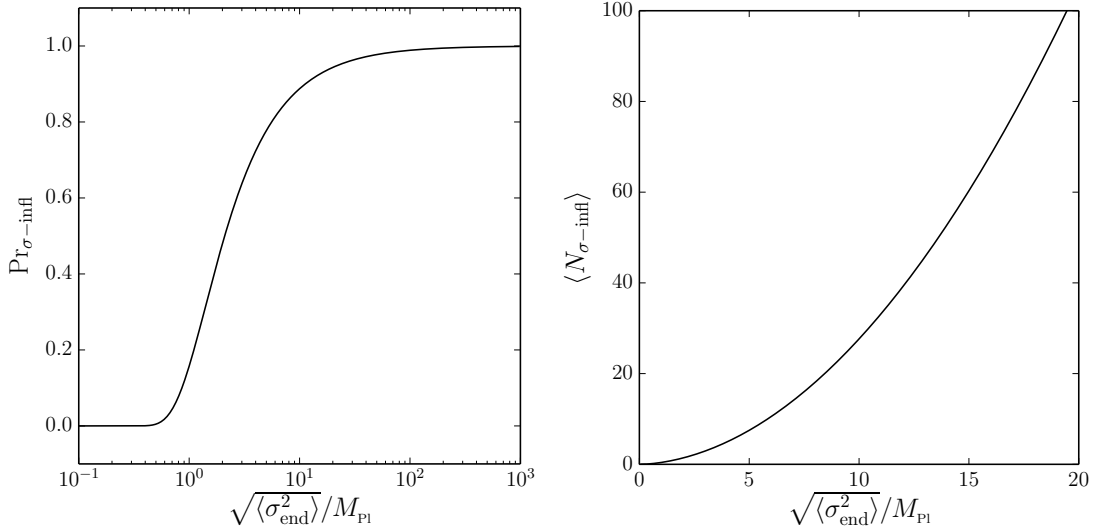
Figure 2. A quadratic spectator field σ can trigger a second phase of inflation if $|\sigma_{\rm end}| > \sqrt{2}M_{\rm Pl}$. Assuming a centred Gaussian distribution with variance $\langle\sigma^2_{\rm end}\rangle$, the left panel displays the probability for such a condition to be satisfied, while the mean number of e-folds realised in the second phase of inflation is given in the right panel.

The situation is summarised in the first line of table 1 in Sec. 6. If $p < 2$ and $m/H_{\rm end} > (H_{\rm end}/M_{\rm Pl})^{(2-p)/(2+p)}$, quadratic spectator fields acquire sub-Planckian field values at the end of inflation, while if $p \geq 2$ or if $p < 2$ with $m/H_{\rm end} < (H_{\rm end}/M_{\rm Pl})^{(2-p)/(2+p)}$, they are typically super-Planckian.

2.3 Can a spectator field drive a second phase of inflation?

If inflation is driven by a monomial potential $V \propto \phi^p$ with $p \geq 2$, in Sec. 2.2 it was shown that quadratic spectator fields typically acquire super-Planckian field values at the end of inflation. This can have important consequences as discussed in Sec. 1, amongst which is the ability for the spectator field to drive a second phase of inflation. This can happen if $|\sigma_{\rm end}| > \sqrt{2}M_{\rm Pl}$, and the probability associated to this condition is given by

$$\mathrm{Pr}_{\sigma\text{-infl}} = \int_{|\sigma|>\sqrt{2}M_{\rm Pl}} P(\sigma, N_{\rm end})\mathrm{d}\sigma = \mathrm{erfc}\left(\frac{M_{\rm Pl}}{\sqrt{\langle\sigma^2_{\rm end}\rangle}}\right). \tag{2.21}$$

In the second expression, we have assumed that the probability distribution of the spectator field value at the end of inflation is a Gaussian with vanishing mean and variance $\langle\sigma^2_{\rm end}\rangle$, and erfc denotes the complementary error function. This probability is displayed in the left panel of Fig. 2. If a second phase of inflation starts driven by the quadratic potential with initial field value $\sigma_{\rm end}$, then the number of e-folds realised is given by $\sigma^2_{\rm end}/(4M^2_{\rm Pl}) - 1/2$. The mean duration of this additional inflationary period can thus

be calculated according to

$$
\begin{aligned}
\langle N_{\sigma\text{-infl}}\rangle &= \frac{1}{\Pr\left(\sigma\text{-infl}\right)} \int_{|\sigma|>\sqrt{2}M_{\mathrm{Pl}}} \left(\frac{\sigma^2}{4M_{\mathrm{Pl}}^2} - \frac{1}{2}\right) P(\sigma, N_{\mathrm{end}})\mathrm{d}\sigma \\
&= \left(\frac{\langle\sigma_{\mathrm{end}}^2\rangle}{4M_{\mathrm{Pl}}^2} - \frac{1}{2}\right) + \frac{\sqrt{\langle\sigma_{\mathrm{end}}^2\rangle}}{2\sqrt{\pi}M_{\mathrm{Pl}}} \frac{\exp\left(-\frac{M_{\mathrm{Pl}}^2}{\langle\sigma_{\mathrm{end}}^2\rangle}\right)}{\mathrm{erfc}\left(\frac{M_{\mathrm{Pl}}}{\sqrt{\langle\sigma_{\mathrm{end}}^2\rangle}}\right)},
\end{aligned}
\tag{2.22}
$$

where in the second expression, again, we have assumed that the probability distribution of the spectator field value at the end of inflation is a centred Gaussian. This mean number of e-folds is shown in the right panel of Fig. 2. When $\langle\sigma_{\mathrm{end}}^2\rangle$ is super-Planckian, one has a non-negligible probability of a second phase of inflation. For instance, with $\sqrt{\langle\sigma_{\mathrm{end}}^2\rangle} = 5M_{\mathrm{Pl}}$, one finds $\Pr\left(\sigma\text{-infl}\right) \simeq 0.77$ and $\langle N_{\sigma\text{-infl}}\rangle = 7.5$.

3 Quartic spectator

In Sec. 2, it was shown that quadratic spectator fields with potential $V(\sigma) = m^2\sigma^2/2$ typically acquire super-Planckian field displacements at the end of inflation if the inflaton potential is of the form $V(\phi) \propto \phi^p$ with $p \geq 2$ at large-field values or with $p < 2$ and $m/H_{\mathrm{end}} < (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{(2-p)/(2+p)}$. In this section, we investigate whether these super-Planckian field values can be tamed by making the spectator field potential steeper at large-field values. In practice, we consider a quartic spectator field,

$$
V(\sigma) = \lambda\sigma^4,
\tag{3.1}
$$

where λ is a dimensionless constant. Contrary to the quadratic case in Sec. 2, the Langevin equation (1.1) is not linear for quartic spectators and cannot be solved analytically. Numerical solutions are therefore presented in this section, where a large number (typically 10^5 or 10^6) of realisations of Eq. (1.1) are generated with a fourth order Runge-Kutta method, over which moments of the spectator field value are calculated at fixed times. These results have been checked with independent numerical solutions of the Fokker-Planck equation (1.2).

3.1 Plateau inflation

As explained in Sec. 1.2, if the inflaton potential is of the plateau type, H can be approximated by a constant and the spectator field value reaches the de-Sitter equilibrium (1.3) where the typical field displacement, for the quartic spectator potential (3.1), is given by

$$
\langle\sigma^2\rangle = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{\frac{3}{2\lambda}} \frac{H^2}{2\pi}.
\tag{3.2}
$$

The relaxation time required to reach this asymptotic value can be assessed as follows. Since the equilibrium (1.3) is of the form $P(\sigma) \propto e^{-\alpha\sigma^4}$, with $\alpha = 8\pi^2\lambda/(3H^4)$, let us

assume that the time evolving distribution for σ is more generally given by

$$P(\sigma, N) = \frac{2\alpha^{1/4}(N)}{\Gamma\left(\frac{1}{4}\right)} \exp\left[-\alpha(N)\sigma^4\right] , \tag{3.3}$$

where $\alpha(N)$ is a free function of time and the prefactor is set so that the distribution remains normalised, and track the stochastic dynamics with this ansatz. By substituting Eq. (3.3) into Eq. (1.2), an ordinary differential equation for $\alpha(N)$ is derived in Appendix B, that reads

$$\frac{\mathrm{d}\alpha}{\mathrm{d}N} = \frac{\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)} \left(\frac{\lambda}{H^2}\alpha^{1/2} - \frac{3H^2}{8\pi^2}\alpha^{3/2}\right) . \tag{3.4}$$

If H is a constant, this equation can be solved analytically and the solution is given by Eq. (B.6). Since Eq. (3.3) gives rise to $\langle\sigma^2\rangle = \alpha^{-1/2}\Gamma(3/4)/\Gamma(1/4)$, one obtains for the second moment

$$\langle\sigma^2(N)\rangle = \frac{\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{\frac{3H^4}{8\pi^2\lambda}}}{\tanh\left\{\sqrt{\frac{3\lambda}{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{8\pi\Gamma\left(\frac{3}{4}\right)}(N-N_0) + \mathrm{atanh}\left[\sqrt{\frac{3H^4}{8\pi^2\lambda}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\langle\sigma^2(N_0)\rangle}\right]\right\}} . \tag{3.5}$$

In the late time limit, one recovers the de-Sitter equilibrium value (3.2). Let us stress however that Eq. (3.5) is not an exact solution to Eq. (1.2) but only provides an approximation under the ansatz (3.3). This approximation will be shown to be reasonably accurate in Sec. 3.2, but for now, expanding $\tanh(x) \simeq 1 - 2e^{-2x}$ when $x \gg 1$ at late time, it provides an estimate of the relaxation time as

$$N_{\mathrm{relax}} = \frac{1}{\sqrt{\lambda}} . \tag{3.6}$$

It is interesting to notice that this expression is consistent with the numerical exploration of Ref. [59], see Eq. (2.12) of this reference.

3.2 Monomial inflation

If the inflaton potential is monomial and of the form $V(\phi) \propto \phi^p$, the Hubble factor is given by Eq. (1.6) and varies over time scales of order $N_H = (H/H_{\mathrm{end}})^{4/p}$ as explained in Sec. 1.2. Making use of Eq. (3.6), the adiabatic condition $N_H \gg N_{\mathrm{relax}}$ then requires $H \gg H_{\mathrm{adiab}}$, where

$$H_{\mathrm{adiab}} \equiv \lambda^{-p/8} H_{\mathrm{end}} . \tag{3.7}$$

A fundamental difference with the quadratic spectator is that in the quartic case, for all values of p, there always exists an adiabatic regime at early times. However, it is not guaranteed that this regime is consistent with the classical inflaton solution (1.6),
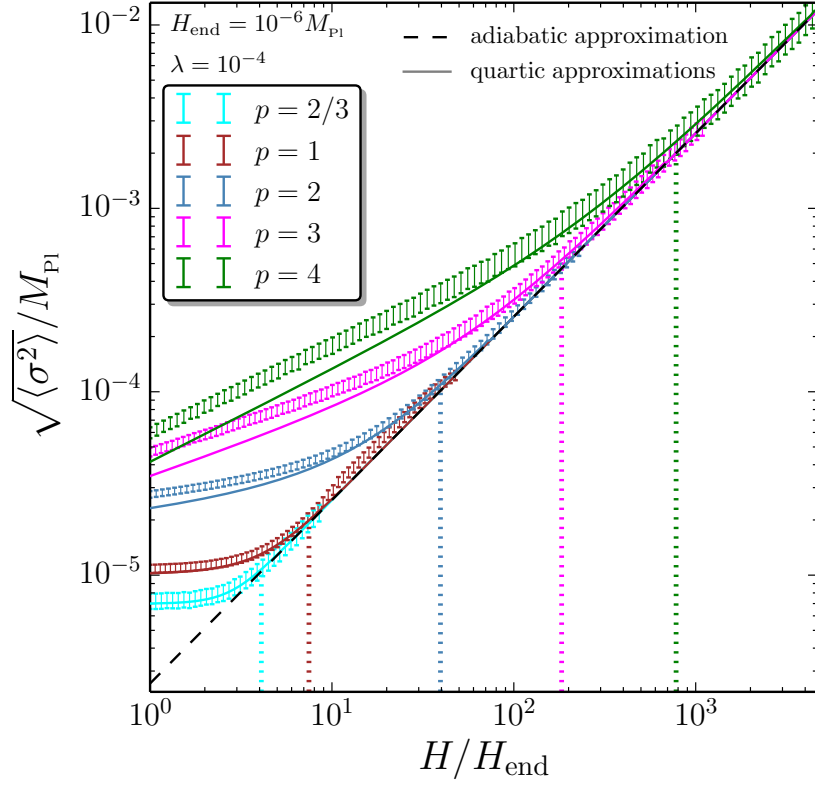
Figure 3. Standard deviation $\sqrt{\langle\sigma^2\rangle}$ of the distribution of a quartic spectator field with potential $V(\sigma) = \lambda\sigma^4$ with $\lambda = 10^{-4}$, as a function of time parametrised by the Hubble scale H (time flows from the right to the left). The inflaton potential is of the monomial type $V(\phi) \propto \phi^p$, with $H_{\mathrm{end}} = 10^{-6}M_{\mathrm{Pl}}$. The coloured I symbols correspond to numerical solutions of the Langevin equation where 10^5 realisations of Eq. (1.1) have been produced for the values of p displayed in the legend. The centres of the vertical bars correspond to ensemble averages of σ^2 while the heights of the bars are statistical noise estimates (due to having a finite number of realisations only) obtained from the jackknife resampling technique. The realisations are initially drawn according to the adiabatic distribution (1.3). The black dashed line corresponds to this adiabatic value (3.2) for $\langle\sigma^2\rangle$. The coloured dashed vertical lines denote the value of H such that the argument of the Bessel functions in Eq. (3.9) equals one, which corresponds to H_{adiab} given by Eq. (3.7) up to an order one prefactor. One can see that when H drops below H_{adiab}, the numerical solutions depart from the de-Sitter equilibrium, denoting the end of the adiabatic regime. Finally, the coloured solid lines correspond to the quartic approximation (3.9).

i.e. extends beyond the eternal inflationary phase. This is the case only if $H_{\mathrm{adiab}} < H_{\mathrm{eternal}}$, where H_{eternal} is given in Eq. (2.14), that is to say if λ is large enough,

$$\lambda > \left(\frac{H_{\mathrm{end}}}{M_{\mathrm{Pl}}}\right)^{\frac{8}{p+2}} . \tag{3.8}$$

Let us distinguish the case where this condition is satisfied and one can use the stationary solution (1.3) to describe the distribution in the adiabatic regime independently of initial conditions, and the case where this is not possible.

3.2.1 Starting out in the adiabatic regime

If the condition (3.8) is satisfied, one can set initial conditions for the spectator field σ in the adiabatic regime after the eternal inflationary phase. In Fig. 3, we present the results of a numerical integration of the Langevin equation (1.1) in this case [with the values used for H_{end} and λ, one can check that Eq. (3.8) is satisfied up to $p = 10$]. The values of H_{adiab} given by Eq. (3.7) are denoted by the vertical coloured dashed lines. When $H \gg H_{\mathrm{adiab}}$, the numerical results follow the de-Sitter stationary solution (3.2) represented by the black dashed line. When H drops below H_{adiab}, this is not the case anymore, and the distributions are wider at the end of inflation than the adiabatic approximation would naively suggest.

In this regime, the behaviour of $\langle \sigma^2 \rangle$ can in fact still be tracked analytically by making use of the quartic ansatz (3.3) introduced in Sec. 3.1. Indeed, in the case where H is given by Eq. (1.6), one can cast Eq. (3.4) into a Ricatti equation and in Appendix B it is shown that its solution reads

$$\langle \sigma^2(H) \rangle = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{\frac{3}{2\lambda}} \frac{H^2}{2\pi} \frac{K_{\frac{p}{4}+\frac{1}{2}}\left[\frac{p}{4\pi}\sqrt{\frac{\lambda}{6}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(\frac{H}{H_{\mathrm{end}}}\right)^{4/p}\right]}{K_{\frac{p}{4}-\frac{1}{2}}\left[\frac{p}{4\pi}\sqrt{\frac{\lambda}{6}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(\frac{H}{H_{\mathrm{end}}}\right)^{4/p}\right]}. \tag{3.9}$$

In this expression, K is a modified Bessel function of the second kind. One can note that the argument of the Bessel functions is directly proportional to N_H/N_{relax}, confirming that this ratio controls the departure from the adiabatic solution (3.2). At early times when $N_H \gg N_{\mathrm{relax}}$, or equivalently $H \gg H_{\mathrm{adiab}}$, one can expand the Bessel functions in the large argument limit, $K_\alpha(x) \simeq \sqrt{\pi/(2x)}e^{-x}$, and one recovers the adiabatic approximation (3.2). The formula (3.9) is displayed in Fig. 3 with the solid coloured lines. One can see that even when $H < H_{\mathrm{adiab}}$, it still provides a reasonable approximation to the numerical solutions. One can also notice that the lower p is, the better this quartic approximation. At the end of inflation, $N_H/N_{\mathrm{relax}} = \sqrt{\lambda} \ll 1$, so the Bessel functions can be expanded in the small argument limit, which depends on the sign of the index of the Bessel function.[3] Because the index of the Bessel function in the denominator of Eq. (3.9) is proportional to $p - 2$, this leads to different results whether p is smaller or

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[3]In the limit $x \ll 1$, if $\alpha < 0$, $K_\alpha(x) \simeq \Gamma(-\alpha)2^{-1-\alpha}x^\alpha$, if $\alpha > 0$, $K_\alpha(x) \simeq \Gamma(\alpha)2^{\alpha-1}x^{-\alpha}$ and if $\alpha = 0$, $K_\alpha(x) \simeq \ln(2/x) - \gamma$, where $\gamma \simeq 0.577$ is the Euler constant [62].

larger than 2, namely

$$\langle \sigma_{\mathrm{end}}^2 \rangle \simeq \begin{cases} \dfrac{\Gamma\left(\frac{1}{2}+\frac{p}{4}\right)}{\Gamma\left(\frac{1}{2}-\frac{p}{4}\right)} \left[\sqrt{\dfrac{3}{2}}\dfrac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]^{1+\frac{p}{2}} \dfrac{\left(\frac{16\pi}{p}\right)^{\frac{p}{2}}}{2\pi} \dfrac{H_{\mathrm{end}}^2}{\lambda^{\frac{1}{2}+\frac{p}{4}}} & \text{if } p < 2 \\[3ex] \dfrac{6}{\bar{\gamma}-\ln(\lambda)} \dfrac{\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \dfrac{H_{\mathrm{end}}^2}{\lambda} & \text{if } p = 2 \\[3ex] \left(3-\dfrac{6}{p}\right) \dfrac{\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \dfrac{H_{\mathrm{end}}^2}{\lambda} & \text{if } p > 2 \end{cases} , \qquad (3.10)$$

where we have defined $\bar{\gamma} \equiv 2\ln[4\pi\sqrt{6}\Gamma(3/4)/\Gamma(1/4)] - 2\gamma \simeq 3.53$, where γ is the Euler constant. Ignoring the overall constants of order one, if $p \geq 2$, one finds $\langle \sigma_{\mathrm{end}}^2 \rangle \sim H_{\mathrm{end}}^2/\lambda$, and if $p < 2$, $\langle \sigma_{\mathrm{end}}^2 \rangle \sim H_{\mathrm{end}}^2/\lambda^{1/2+p/4}$. This needs to be compared to the de-Sitter case (3.2) where $\langle \sigma_{\mathrm{end}}^2 \rangle \sim H_{\mathrm{end}}^2/\sqrt{\lambda}$. In monomial inflation, $\langle \sigma_{\mathrm{end}}^2 \rangle$ is therefore larger than in plateau inflation for the same value of H_{end}, by a factor $\lambda^{-p/4}$ if $p < 2$ and $\lambda^{-1/2}$ if $p \geq 2$. One should also note that the condition (3.8) for the adiabatic regime to extend beyond the eternal inflationary phase can be substituted into Eq. (3.10) and gives rise to $\sqrt{\langle \sigma_{\mathrm{end}}^2 \rangle}/M_{\mathrm{Pl}} \ll (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{(p-2)/(p+2)}$ if $p \geq 2$ and $\sqrt{\langle \sigma_{\mathrm{end}}^2 \rangle}/M_{\mathrm{Pl}} \ll 1$ if $p < 2$. In both cases, the spectator field displacement at the end of inflation is therefore sub-Planckian.

3.2.2 Starting out away from the adiabatic regime

If the condition (3.8) is not satisfied, the adiabatic regime lies entirely within the eternal inflationary phase and cannot be used to erase initial conditions. In this case, the spectator field displacement at the end of inflation is thus strongly dependent on initial conditions at the start of the classical inflaton evolution. In this section, we derive a lower bound on $\langle \sigma_{\mathrm{end}}^2 \rangle$, assuming that it vanishes when $H = H_{\mathrm{eternal}}$ and solving the subsequent stochastic dynamics numerically. The result is presented in Fig. 4 where $\langle \sigma_{\mathrm{end}}^2 \rangle$ is displayed as a function of $\lambda M_{\mathrm{Pl}}^2/H_{\mathrm{end}}^2$ for $H_{\mathrm{end}} = 10^{-6}M_{\mathrm{Pl}}$ (left panel) and for $H_{\mathrm{end}} = 10^{-10}M_{\mathrm{Pl}}$ (left panel). The two cases $p \geq 2$ and $p < 2$ must be treated separately.

Case where $p \geq 2$
If $p \geq 2$, it was shown in Sec. 2.2.1 that a light quadratic spectator field always acquires a super-Planckian field value at the end of inflation. The mean effective mass of the quartic spectator field is given by

$$m_{\mathrm{eff}}^2 = 12\lambda \left\langle \sigma^2 \right\rangle , \qquad (3.11)$$

and is smaller than H_{end} for $\sqrt{\langle \sigma_{\mathrm{end}}^2 \rangle} \sim M_{\mathrm{Pl}}$ if $\lambda < H_{\mathrm{end}}^2/M_{\mathrm{Pl}}^2$. This explains why, in Fig. 4, in the regime $\lambda < H_{\mathrm{end}}^2/M_{\mathrm{Pl}}^2$, one recovers Eq. (2.15) that is displayed with the horizontal coloured lines, and which shows that the spectator field acquires a super-Planckian field value in this case. Otherwise, if $H_{\mathrm{end}}^2/M_{\mathrm{Pl}}^2 < \lambda < (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{8/(p+2)}$ [the
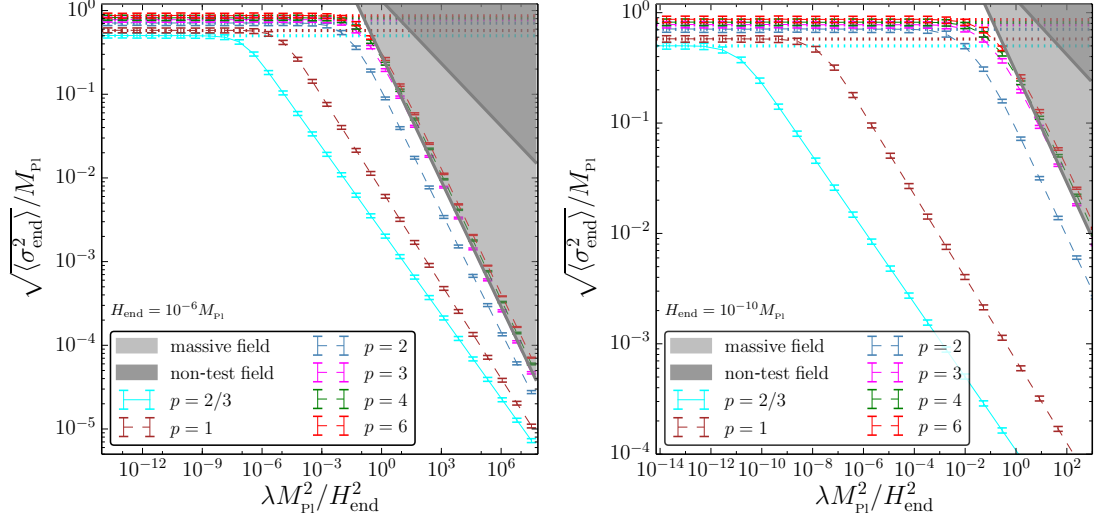
Figure 4. Field displacement $\sqrt{\langle \sigma^2 \rangle}$ acquired by a quartic spectator field with potential $V(\sigma) = \lambda \sigma^4$ at the end of inflation, as a function of $\lambda M_{\mathrm{Pl}}^2 / H_{\mathrm{end}}^2$, for $H_{\mathrm{end}} = 10^{-6} M_{\mathrm{Pl}}$ (left panel) and $H_{\mathrm{end}} = 10^{-10} M_{\mathrm{Pl}}$ (right panel). The inflaton potential is of the monomial type $V(\phi) \propto \phi^p$. The coloured I symbols correspond to numerical solutions of the Langevin equation where 10^5 realisations of Eq. (1.1) have been produced for the values of p displayed in the legend. The centres of the vertical bars correspond to ensemble averages of σ^2 while the heights of the bars are statistical noise estimates (due to having a finite number of realisations only) obtained from the jackknife resampling technique. All realisations are initiated with $\sigma = 0$ at $H = H_{\mathrm{eternal}}$. The horizontal dashed lines correspond to Eq. (2.15) to which the numerical results asymptote in the limit $\lambda \to 0$. The pale grey region corresponds to $m_{\mathrm{eff}} > H$ where the spectator field is not light and our calculation does not apply, and the dark region stands for $\lambda \sigma_{\mathrm{end}}^4 > 3 M_{\mathrm{Pl}}^2 H_{\mathrm{end}}^2$ where σ cannot be considered as a spectator field anymore.

upper bound coming from breaking the inequality (3.8)], one can see in Fig. 4 that the field displacement can be made sub-Planckian, but that its effective mass becomes of order H.[4] In this regime, the spectator field cannot be considered as light anymore.

Case where $p < 2$

If $p < 2$, it was shown in Sec. 2.2.2 that a quadratic spectator field acquires a super-Planckian field value at the end of inflation if its mass is smaller than $H_{\mathrm{end}}(H_{\mathrm{end}}/M_{\mathrm{Pl}})^{(2-p)/(2+p)}$, see Eq. (2.17). When evaluated at the Planck scale, the effective mass (3.11) of the quartic spectator field is smaller than this threshold when $\lambda < (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{8/(2+p)}$, which exactly corresponds to breaking the inequality (3.8). One

---

[4]Strictly speaking, the present calculation does not apply when the effective mass of the spectator field is of order H or larger. However, if the effects of the mass were taken into account, the amplitude of the noise term in Eq. (1.1) would not be $H/(2\pi)$ but would become smaller as m_{eff} approaches H. This would result in a smaller value for $\langle \sigma^2 \rangle$, hence for m_{eff}, and therefore a larger noise amplitude. One can expect the two effects to compensate for a value of m_{eff} around H.

can check in Fig. 4 that when $\lambda < (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{8/(2+p)}$, one does indeed recover Eq. (2.15) which is displayed with the horizontal dashed coloured lines. One concludes that in this case, the spectator field always acquires a field value at least of order the Planck mass at the end of inflation.

The situation is summarised in the second line of table 1 in Sec. 6. If $\lambda > (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{8/(p+2)}$, the spectator field is sub-Planckian at the end of inflation. Otherwise, if $p \geq 2$, either the spectator field is super-Planckian or not light at the end of inflation, and if $p < 2$, it is always super-Planckian. Considering the quadratic spectator discussed in Sec. 2 where it was shown that super-Planckian field displacements are usually generated at the end of inflation, one thus concludes that an additional self-interacting term $\lambda\sigma^4$ in the potential can render the field value sub-Planckian if λ is large enough, namely if $\lambda > (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{8/(p+2)}$. One can check that for such a value of λ, if $V(\sigma) = m^2\sigma^2/2 + \lambda\sigma^4$ with $m < H_{\mathrm{end}}$, the quartic term always dominates over the quadratic one when $\sigma \sim M_{\mathrm{Pl}}$, which is consistent.

4 Axionic spectator

In Sec. 2, it was shown that quadratic spectator fields with potential $V(\sigma) = m^2\sigma^2/2$ typically acquire super-Planckian field displacements at the end of inflation if the inflaton potential is of the form $V(\phi) \propto \phi^p$ with $p \geq 2$ at large-field value or with $p < 2$ and $m/H_{\mathrm{end}} < (H_{\mathrm{end}}/M_{\mathrm{Pl}})^{(2-p)/(2+p)}$. In Sec. 3, we discussed how adding a quartic self-interaction term in the potential could help to tame these super-Planckian values. In this section, we investigate another possibility, which consists in making the field space compact and of sub-Planckian extent. This is typically the case for axionic fields, with periodic potentials of the type

$$V(\sigma) = \Lambda^4 \left[1 - \cos\left(\frac{\sigma}{f}\right) \right] . \tag{4.1}$$

In this expression, Λ and f are two mass scales that must satisfy $\Lambda^2 < fH_{\mathrm{end}}$ in order for the curvature of the potential to remain smaller than the Hubble scale throughout inflation, i.e. for the axionic field to remain light, which we will assume in the following.

4.1 Plateau inflation

As explained in Sec. 1.2, if the inflaton potential is of the plateau type, H can be approximated by a constant and the spectator field value reaches the de-Sitter equilibrium (1.3). If $H \gg \Lambda$, such a distribution is approximately flat, in which case $\langle\sigma^2\rangle \simeq \pi^2 f^2/3$ if σ is restricted to one period of the potential (4.1). In this regime, the classical drift due to the potential gradient in Eq. (1.1) can be neglected and the spectator field experiences a free diffusion process. The relaxation time is therefore the time it takes to randomise σ over the period of the potential and is given by $N_{\mathrm{relax}} \simeq (\pi^2 f/H)^2$. In the opposite limit when $H \ll \Lambda$, the distribution is localised close to the minimum of the potential where it can be approximated by a quadratic function $V(\sigma) \simeq m^2\sigma^2/2$ with mass $m^2 = \Lambda^4/f^2$.
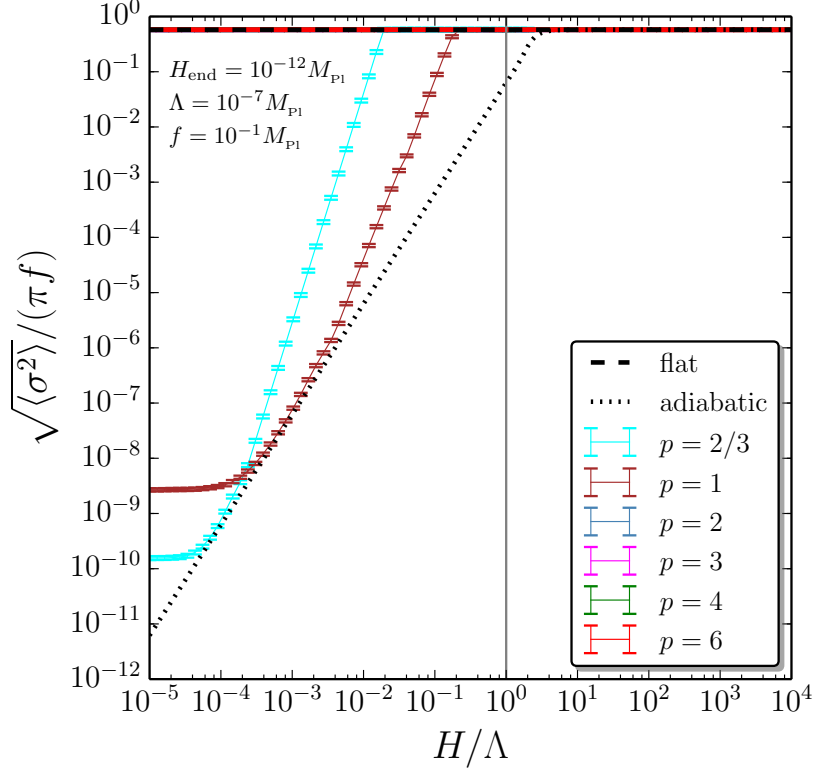
Figure 5. Standard deviation $\sqrt{\langle\sigma^2\rangle}$ of the distribution of an axionic spectator field with potential $V(\sigma) = \Lambda^4[1 - \cos(\sigma/f)]$ with $\Lambda = 10^{-7}M_{\mathrm{Pl}}$ and $f = 10^{-1}M_{\mathrm{Pl}}$, as a function of time parametrised by the Hubble scale H (time flows from the right to the left). The inflaton potential is of the monomial type $V(\phi) \propto \phi^p$, with $H_{\mathrm{end}} = 10^{-12}M_{\mathrm{Pl}}$. The coloured I symbols correspond to numerical solutions of the Langevin equation where 10^6 realisations of Eq. (1.1) have been produced for the values of p displayed in the legend. The centres of the vertical bars correspond to ensemble averages of σ^2 while the heights of the bars are statistical noise estimates (due to having a finite number of realisations only) obtained from the jackknife resampling technique. The realisations are initially drawn according to a flat distribution when $H/\Lambda = 10^4$. The black dashed line corresponds to the standard deviation of a distribution that is flat over one period of the potential, $\langle\sigma^2\rangle = \pi^2 f^2/3$. The black dotted line corresponds to the adiabatic solution (1.3), which remains flat when H is larger than Λ, represented by the grey vertical line. When $p \geq 2$ the distributions remain flat until the end of inflation (and one cannot distinguish the different values of p that are superimposed). When $p < 2$, the distributions narrow down once $H \ll \Lambda$ since the parameters have been chosen to satisfy Eq. (4.2).

In this case, according to Sec. 2.1, one has $\langle\sigma^2\rangle = 3H^4 f^2/(8\pi^2\Lambda^4)$, and the relaxation time is of order $N_{\mathrm{relax}} = H^2/m^2 \simeq H^2 f^2/\Lambda^4$.

4.2 Monomial inflation

If inflation is realised by a monomial potential $V(\phi) \propto \phi^p$, there is always an epoch when $H > \Lambda$ in the past and during which the spectator field distribution is made flat within a number of e-folds of order $N_{\text{relax}} \simeq (\pi^2 f/H)^2$. Therefore, contrary to the quadratic and to the quartic spectators, the field displacement of an axionic spectator at the end of inflation is always independent of initial conditions, provided that inflation lasts long enough. If $\Lambda < H_{\text{end}}$, the distribution remains flat until the end of inflation. In the opposite case, when H drops below Λ, the subsequent dynamics of σ depends on whether $p \geq 2$ or $p < 2$.

4.2.1 Case where $p \geq 2$

If $p \geq 2$, in Sec. 2 it was shown that the evolution of a quadratic field with mass $m < H_{\text{end}}$ is effectively described by a free-diffusion process where the potential drift can be neglected. For an axionic spectator, the potential is always flatter than its quadratic expansion around its minimum and can therefore also be neglected. As a consequence, the distribution remains flat until the end of inflation and one finds $\langle \sigma_{\text{end}}^2 \rangle \simeq \pi^2 f^2/3$.

4.2.2 Case where $p < 2$

If $p < 2$, in Sec. 2 it was shown that the distribution of a quadratic field with mass $m < H_{\text{end}}$ tracks the adiabatic equilibrium until $H = H_{\text{adiab}}$, where H_{adiab} is given by Eq. (2.16), and remains frozen afterwards. This implies that an axionic spectator distribution narrows down from a flat profile if $H_{\text{adiab}} < \Lambda$, which gives rise to

$$\frac{\Lambda}{H_{\text{end}}} > \left(\frac{f}{H_{\text{end}}} \right)^{\frac{p}{p+2}} . \tag{4.2}$$

Notice that for this condition to be compatible with the light-field prescription given below Eq. (4.1), one must have $H_{\text{end}} < f$ for $p < 2$ (which makes sense, otherwise the distribution would be randomised over one e-fold even towards the end of inflation). In this case, $\langle \sigma^2 \rangle$ settles down to $3H_{\text{adiab}}^4/(8\pi^2 m^2)$, which gives rise to

$$\sqrt{\langle \sigma_{\text{end}}^2 \rangle} \simeq \sqrt{\frac{3}{2}} \frac{H_{\text{end}}}{2\pi} \left(\frac{H_{\text{end}} f}{\Lambda^2} \right)^{\frac{2+p}{2-p}} . \tag{4.3}$$

If Eq. (4.2) is not satisfied however, the field distribution remains flat until the end of inflation and one has $\langle \sigma_{\text{end}}^2 \rangle \simeq \pi^2 f^2/3$.

In order to check the validity of these considerations, in Fig. 5 we present numerical solutions of the Langevin equation (1.1). When $p \geq 2$, one can check that the distributions remain flat until the end of inflation. The values of the parameters Λ, f and H_{end} have been chosen to satisfy Eq. (4.2), which explains why for $p < 2$, the distributions narrow down once H drops below Λ (otherwise, we have checked that even when $p < 2$, the distributions remain flat). However, one can see that when the distributions start moving away from the flat configuration, they do not exactly follow the adiabatic

solution displayed with the black dotted line, even though $H > H_{\mathrm{adiab}}$. This is because in the above discussion, we have approximated the axionic potential with its quadratic expansion around its minimum, which is not strictly valid at the stage where the distribution is still flat and sensitive to the full potential shape. Nonetheless, the distributions converge towards the adiabatic profile at later time and the final value of $\langle \sigma^2 \rangle$ is well described by Eq. (4.3).

The situation is summarised in the third line of table 1 in Sec. 6. If $H_{\mathrm{end}} > \Lambda$, $p \geq 2$, or $p < 2$ with $\Lambda < H_{\mathrm{end}}(f/H_{\mathrm{end}})^{p/(p+2)}$, the distribution of the axionic spectator remains flat until the end of inflation and $\sqrt{\langle \sigma^2_{\mathrm{end}} \rangle} = \pi f/\sqrt{3}$. Only if $p < 2$ with $\Lambda > H_{\mathrm{end}}(f/H_{\mathrm{end}})^{p/(p+2)}$ does the distribution narrow down and $\sqrt{\langle \sigma^2_{\mathrm{end}} \rangle} \simeq H_{\mathrm{end}}(H_{\mathrm{end}}f/\Lambda^2)^{(2+p)/(2-p)}$. In all cases, if f is sub-Planckian, the typical field displacement obviously remains sub-Planckian as well.

5   Information retention from initial conditions

When calculating the field value acquired by spectator fields at the end of inflation, we have found situations in which initial conditions are erased by the existence of an adiabatic regime at early times, and situations in which this is not the case. In this section, we propose to quantify this memory effect using information theory in order to better describe the amount of information about early time physics (potentially pre-inflationary) available in the final field displacements of spectator fields. The relative information between two distributions $P_1(\sigma)$ and $P_2(\sigma)$ can be measured using the Kullback-Leibler divergence [63] D_{KL},

$$D_{\mathrm{KL}}(P_1||P_2) \equiv \int_{-\infty}^{\infty} P_1(\sigma) \log_2\left[\frac{P_1(\sigma)}{P_2(\sigma)}\right] \mathrm{d}\sigma \,. \tag{5.1}$$

It is invariant under any reparametrisation $\sigma' = f(\sigma)$, and since it uses a logarithmic score function as in the Shannon's entropy, it is a well-behaved measure of information [64]. Considering two initial distributions separated by an amount of information δD_{KL}^0, giving rise to two final distributions separated by $\delta D_{\mathrm{KL}}^{\mathrm{end}}$, we define the information retention criterion by

$$\mathcal{I} \equiv \frac{\delta D_{\mathrm{KL}}^{\mathrm{end}}}{\delta D_{\mathrm{KL}}^0} \,. \tag{5.2}$$

When $\mathcal{I} < 1$, the initial information is contracted by the dynamics of the distributions. This is typically the case when there is an attractor, or an adiabatic regime, which tends to erase the initial conditions dependence of final states. When $\mathcal{I} > 1$, the initial information is amplified and the final state is sensitive to initial conditions. Values of $\mathcal{I} \gg 1$ might signal the presence of chaotic dynamics in which case initial conditions are difficult to infer. For this reason, $\mathcal{I} = \mathcal{O}(1)$ represents an optimal situation in terms of initial conditions reconstruction. In practice, $\mathcal{I}$ depends both on the initial (or final) state around which the infinitesimal variation is performed, and on the direction in the space of distributions along which it is performed.
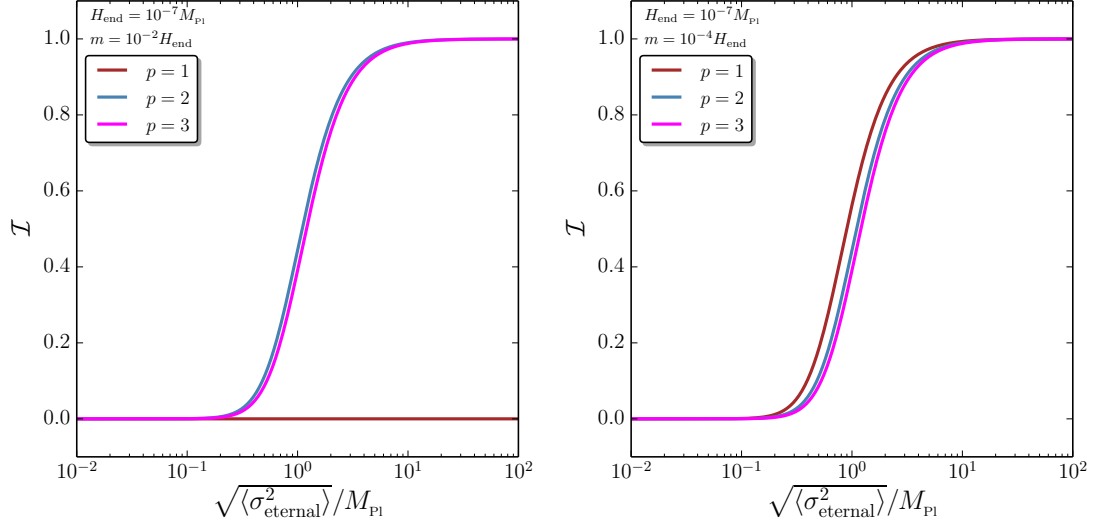
Figure 6. Information retention criterion (5.2) as a function of the initial standard deviation $\sqrt{\langle \sigma^2_{\mathrm{eternal}} \rangle}$ for a quadratic spectator field with potential $V(\sigma) = m^2\sigma^2/2$, if inflation is driven by a monomial potential $V(\phi) \propto \phi^p$. Initial conditions are set at $H_0 = H_{\mathrm{eternal}}$ where the inflaton exits the eternal inflationary regime. In both panels, $H_{\mathrm{end}} = 10^{-7} M_{\mathrm{Pl}}$, and $m = 10^{-2} H_{\mathrm{end}}$ in the left panel and $m = 10^{-4} H_{\mathrm{end}}$ in the right panel. Different colours represent different values of p. If $p \geq 2$, initial conditions are not erased but provide a subdominant contribution to the final distribution if the field displacement is initially sub-Planckian. This is why, if $\sqrt{\langle \sigma^2_{\mathrm{eternal}} \rangle} \ll M_{\mathrm{Pl}}$, $\mathcal{I} \simeq 0$, while if $\sqrt{\langle \sigma^2_{\mathrm{eternal}} \rangle} \gg M_{\mathrm{Pl}}$, $\mathcal{I} \simeq 1$. In the left panel, the condition (2.17) is satisfied for $p = 1$, so initial conditions are erased ($\mathcal{I} \simeq 0$), while in the right panel, the condition (2.17) is not satisfied for $p = 1$ which therefore behaves as the cases $p \geq 2$.

For concreteness, let us restrict the analysis to the space of symmetric Gaussian distributions, fully characterised by a single parameter, $\langle \sigma^2 \rangle$. In this case, Eq. (5.1) gives rise to

$$D_{\mathrm{KL}} \left(P_1 || P_2 \right) = \frac{1}{2 \ln 2} \left[\frac{\langle \sigma_2^2 \rangle}{\langle \sigma_1^2 \rangle} - \ln \left(\frac{\langle \sigma_2^2 \rangle}{\langle \sigma_1^2 \rangle} \right) - 1 \right] , \tag{5.3}$$

where $\langle \sigma_1^2 \rangle$ (respectively $\langle \sigma_2^2 \rangle$) is the variance of P_1 (respectively P_2). One then has

$\delta D_{\mathrm{KL}} = (\delta\langle\sigma^2\rangle/\langle\sigma^2\rangle)^2/(4\ln 2)$, which gives rise to[5]

$$\mathcal{I} = \left(\frac{\partial \ln\langle\sigma_{\mathrm{end}}^2\rangle}{\partial \ln\langle\sigma_0^2\rangle} \right)^2 . \tag{5.5}$$

In practice, the functional relationship between $\langle\sigma_0^2\rangle$ and $\langle\sigma_{\mathrm{end}}^2\rangle$ depends on the details of the stochastic dynamics followed by σ. When $\langle\sigma_{\mathrm{end}}^2\rangle$ is independent of $\langle\sigma_0^2\rangle$ for instance, initial conditions are irrelevant to determine the final state and $\mathcal{I} = 0$.

For quadratic spectator fields, in Sec. 2 it was shown that the distributions remain Gaussian if they were so initially, and the relationship (2.3) between $\langle\sigma_0^2\rangle$ and $\langle\sigma_{\mathrm{end}}^2\rangle$ was derived. The formula (5.5) can therefore directly be evaluated, and it is displayed in Fig. 6 in the case where inflation is driven by a monomial potential $V \propto \phi^p$ and initial conditions are taken at the time when the inflaton exits the eternal inflationary epoch. When $p \geq 2$, there is no adiabatic regime and therefore no erasure of initial conditions. Since quantum diffusion contributes a field displacement of order the Planck mass, if the initial field value is much smaller than the Planck mass, it provides a negligible contribution to the final field value and one has $\mathcal{I} \simeq 0$. If it is much larger than the Planck mass it provides the dominant contribution to the final field value and $\mathcal{I} \simeq 1$. In the left panel, the value of m has been chosen so that the condition (2.17) is satisfied for $p = 1$. In this case, initial conditions are erased during the adiabatic regime and one has $\mathcal{I} \simeq 0$. In the right panel, the value chosen for m is such that Eq. (2.17) is not satisfied and the situation for $p = 1$ is similar to the cases $p \geq 2$.

For quartic spectator fields, in Sec. 3 it was shown that either the condition (3.8) is satisfied and initial conditions are erased during an early adiabatic phase, leading to $\mathcal{I} \simeq 0$; or if the condition (3.8) is not satisfied, the dynamics of the spectator field is described by a free diffusion process and the situation is the same as in the right panel of Fig. 6.

For axionic spectator fields finally, in Sec. 4, initial conditions were shown to always be erased at early times, yielding $\mathcal{I} \simeq 0$.

The amount of information one can recover about the initial state from the final one therefore depends both on the potential of the spectator field and on the inflationary background. Let us stress that in some situations, initial conditions are not erased ($\mathcal{I} \simeq 1$). This suggests that, if observations yield non-trivial constraints on spectator

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[5]The same expression is obtained if one uses the Jensen-Shannon divergence as a measure of the relative information between two distributions,

$$D_{\mathrm{JS}}\left(P_1||P_2\right) = \frac{1}{2}D_{\mathrm{KL}}\left(P_1 \left|\left| \frac{P_1 + P_2}{2} \right.\right.\right) + \frac{1}{2}D_{\mathrm{KL}}\left(P_2 \left|\left| \frac{P_1 + P_2}{2} \right.\right.\right), \tag{5.4}$$

which is a symmetrised and smoothed version of the Kullback-Leibler divergence. The Jensen-Shannon divergence between two Gaussian distributions cannot be expressed in a closed form comparable to Eq. (5.3). However, in the limit where the two Gaussian distributions have variances $\langle\sigma^2\rangle$ and $\langle\sigma^2\rangle + \delta\langle\sigma^2\rangle$ infinitesimally close one to the other, one can expand the integrands of Eq. (5.4) at quadratic order in $\delta\langle\sigma^2\rangle$ and obtain $\delta D_{\mathrm{JS}} = (\delta\langle\sigma^2\rangle/\langle\sigma^2\rangle)^2/(16\ln 2) = \delta D_{\mathrm{KL}}/4$. As a consequence, $\delta D_{\mathrm{JS}}^{\mathrm{end}}/\delta D_{\mathrm{JS}}^0 = \delta D_{\mathrm{KL}}^{\mathrm{end}}/\delta D_{\mathrm{KL}}^0$ and the same information retention criterion is obtained.

field values at the end of inflation in our local patch, one may be able to infer a non-trivial probability distribution on its field value at much earlier time, for instance when one leaves the regime of eternal inflation. This might be relevant to the question [65] of whether observations can give access to scales beyond the observational horizon.

6 Conclusion

The typical field value acquired by spectator fields during inflation is an important parameter of many post-inflationary physical processes. Often, in slow-roll inflationary backgrounds, it is estimated using the stochastic equilibrium solution in de-Sitter space-times (1.3), since slow roll is parametrically close to de Sitter. However, slow roll only implies that the Hubble scale H varies over time scales larger than one e-fold. Since the relaxation time of a spectator field distribution towards the de-Sitter equilibrium is typically much larger than an e-fold, this does not guarantee that the spectator distribution adiabatically tracks the de-Sitter solution. In practice, we have found that when the inflaton potential is monomial at large-field values, the de-Sitter approximation is never a reliable estimate of the spectator typical field value at the end of inflation. Instead, spectator fields acquire field displacements that depend on the details of both the spectator potential and the inflationary background. These results are summarised in table 1.

In some cases, the existence of an adiabatic regime at early times leads to an erasure of initial conditions and the spectator field distribution is fully determined by the microphysical parameters of the model. When this is the case, we have showed that spectator fields always acquire sub-Planckian field values at the end of inflation. However, it can also happen that adiabatic regimes either do not exist or take place at a stage where quantum corrections to the inflaton dynamics are large and our calculation does not apply. In such cases, a dependence on the initial conditions is unavoidable, which we have quantified in the context of information theory. This suggests that observations might have the potential to give access to scales beyond the observable horizon, through processes that are integrated over the whole inflationary period, such as spectator field displacements.

In general, we have found that light spectator fields acquire much larger field displacements during inflation than the de-Sitter approximation suggests, which has important consequences. As an illustration, let us mention one of the curvaton models which is favoured by observations, where inflation is driven by a quartic potential in the presence of a quadratic spectator field, the curvaton, that later dominates the energy budget of the Universe and provides the main source of cosmological perturbations. In order for this model to provide a good fit to the data, the field value of the curvaton at the end of inflation should lie in the range [32, 33] $\Gamma_\sigma/\Gamma_\phi \ll \sigma_{\rm end}/M_{\rm Pl} \ll 1$, where Γ_ϕ and Γ_σ are the decay rates of the inflaton and of the curvaton, respectively. In this case however, we have found that if inflation starts from the eternal inflation regime, then the curvaton typically acquires a super-Planckian field value at the end of inflation, which challenges this model, at least in its simplest form. As shown in this work, a possible

solution could be to add a quartic coupling term to the curvaton potential or to consider axionic curvaton potentials. Whether the model is still in agreement with the data in this case is an important question that we plan to study in a future work.

Background $V(\phi)\propto\phi^p$ / Spectator	$p=0$ (plateau)	$0<p<2$	$p\geq 2$
Quadratic $V(\sigma)=\dfrac{m^2}{2}\sigma^2$	$\langle\sigma_{\rm end}^2\rangle\sim\dfrac{H^4}{m^2}$ $N_{\rm relax}\sim\dfrac{H^2}{m^2}$	$\langle\sigma_{\rm end}^2\rangle\gtrsim M_{\rm Pl}^2$ $\quad$ $\langle\sigma_{\rm end}^2\rangle\sim H_{\rm end}^2\left(\dfrac{H_{\rm end}^2}{m^2}\right)^{\frac{2+p}{2-p}}$ $\xrightarrow{\hspace{2cm}}$ init. cond. dep. $\left(\dfrac{H_{\rm end}}{M_{\rm Pl}}\right)^{\frac{2-p}{2+p}}\qquad\dfrac{m}{H_{\rm end}}$	$\langle\sigma_{\rm end}^2\rangle\gtrsim M_{\rm Pl}^2$ init. cond. dep.
Quartic $V(\sigma)=\lambda\sigma^4$	$\langle\sigma_{\rm end}^2\rangle\sim\dfrac{H^2}{\sqrt{\lambda}}$ $N_{\rm relax}\sim\dfrac{1}{\sqrt{\lambda}}$	$\langle\sigma_{\rm end}^2\rangle\gtrsim M_{\rm Pl}^2$ $\quad$ $\langle\sigma_{\rm end}^2\rangle\sim\dfrac{H_{\rm end}^2}{\lambda^{\frac{1}{2}+\frac{p}{4}}}$ $\xrightarrow{\hspace{2cm}}$ init. cond. dep. $\left(\dfrac{H_{\rm end}}{M_{\rm Pl}}\right)^{\frac{8}{2+p}}\qquad\lambda$	$\langle\sigma_{\rm end}^2\rangle\gtrsim M_{\rm Pl}^2$ $\;$ $m_{\rm eff}\gtrsim H$ $\;$ $\langle\sigma_{\rm end}^2\rangle\sim\dfrac{H_{\rm end}^2}{\lambda}$ $\xrightarrow{\hspace{1.5cm}}$ init. cond. dep. $\left(\dfrac{H_{\rm end}}{M_{\rm Pl}}\right)^2\quad\left(\dfrac{H_{\rm end}}{M_{\rm Pl}}\right)^{\frac{8}{2+p}}\quad\lambda$
Axionic $V(\sigma)=M^4\left[1-\cos\left(\dfrac{\sigma}{f}\right)\right]$ $|\sigma|<\pi f$	$\langle\sigma_{\rm end}^2\rangle\sim f^2$ $\quad$ $\langle\sigma_{\rm end}^2\rangle\sim\dfrac{H^4 f^2}{\Lambda^4}$ $N_{\rm relax}\sim\dfrac{f^2}{H^2}$ $\quad$ $N_{\rm relax}\sim\dfrac{H^2 f^2}{\Lambda^4}$ $\xrightarrow{\hspace{2cm}}$ $H_{\rm end}\qquad\Lambda$	$\langle\sigma_{\rm end}^2\rangle\sim f^2$ $\quad$ $\langle\sigma_{\rm end}^2\rangle\sim H_{\rm end}^2\left(\dfrac{H_{\rm end}^2 f^2}{\Lambda^4}\right)^{\frac{2+p}{2-p}}$ $\xrightarrow{\hspace{2cm}}$ $H_{\rm end}\left(\dfrac{f}{H_{\rm end}}\right)^{\frac{p}{p+2}}\qquad\Lambda$	$\langle\sigma_{\rm end}^2\rangle\sim f^2$

Table 1. Summary of the results obtained in this paper. The stochastic dynamics of spectator scalar fields with quadratic, quartic and axionic potentials have been studied in inflationary backgrounds driven by plateau and monomial potentials. In each case, the typical field displacement $\langle\sigma_{\rm end}^2\rangle$ acquired by spectator fields at the end of inflation is given in this table. When inflation is realised with a plateau potential, the de-Sitter equilibrium is reached within a number of e-folds $N_{\rm relax}$ also given in the table. If the inflaton potential receives monomial corrections at large-field values, the de-Sitter approximation is never a reliable estimate of the spectator typical field value, and the result depends on the details of both the spectator potential and the inflationary background. In some cases, the lack of adiabatic attractors at early time also introduces initial conditions dependence (denoted by "init. cond. dep".).

Acknowledgements

A Statistical moments of quadratic spectators

In this section, we derive the first two statistical moments of quadratic spectator fields, for which $V(\sigma) = m^2 \sigma^2/2$. If the initial distribution is Gaussian, it remains so throughout the entire evolution so these two moments fully characterise the distribution at any time. Otherwise, higher-order moments can be derived along the same lines.

The first moment can be obtained by taking the stochastic average of Eq. (1.1), which gives rise to

$$\frac{\mathrm{d}\langle\sigma\rangle}{\mathrm{d}N} = -\frac{m^2}{3H^2}\langle\sigma\rangle\,. \tag{A.1}$$

In this expression, the fact that σ is a test field plays an important role since it implies that H does not depend on σ and is thus a classical (i.e. non-stochastic) quantity. Interestingly, Eq. (A.1) is the same as Eq. (1.1) in the absence of quantum diffusion, which is why $\langle\sigma\rangle$ follows the classical dynamics

$$\langle\sigma\left(N\right)\rangle = \langle\sigma\left(N_0\right)\rangle \exp\left[-\frac{m^2}{3}\int_{N_0}^{N}\frac{\mathrm{d}N'}{H^2(N')}\right]\,, \tag{A.2}$$

where $\langle\sigma\left(N_0\right)\rangle$ is the value of $\langle\sigma\rangle$ at the initial time N_0.

The second moment can be obtained by multiplying Eq. (1.1) by σ and taking the stochastic average, which leads to

$$\frac{1}{2}\frac{\mathrm{d}\langle\sigma^2\rangle}{\mathrm{d}N} = -\frac{m^2}{3H^2}\langle\sigma^2\rangle + \frac{H}{2\pi}\langle\sigma\xi\rangle, \tag{A.3}$$

where $\langle\sigma\xi\rangle$ needs to be calculated separately. This can be done by noticing that a formal solution to Eq. (1.1) is given by

$$\sigma = \int_{A}^{N}\mathrm{d}N'\frac{H(N')}{2\pi}\xi(N')\exp\left[\int_{N}^{N'}\frac{m^2}{3H^2(N'')}\mathrm{d}N''\right]\,, \tag{A.4}$$

where A is an integration constant. This gives rise to

$$\langle \sigma(N)\xi(N) \rangle = \int_A^N \mathrm{d}N' \frac{H(N')}{2\pi} \langle \xi(N)\xi(N') \rangle \exp\left[\int_N^{N'} \frac{m^2}{3H^2(N'')}\mathrm{d}N''\right] \tag{A.5}$$

$$= \int_A^N \mathrm{d}N' \frac{H(N')}{2\pi} \delta(N - N') \exp\left[\int_N^{N'} \frac{m^2}{3H^2(N'')}\mathrm{d}N''\right] \tag{A.6}$$

$$= \frac{1}{2}\frac{H(N)}{2\pi}, \tag{A.7}$$

where the factor $1/2$ comes from the fact that the delta function is centred at one of the boundaries of the integral [recall that $\int_{x_0}^{x_1} f(x)\delta(x-x_0) = f(x_0)/2$]. One can then write Eq. (A.3) as

$$\frac{1}{2}\frac{\mathrm{d}\langle\sigma^2\rangle}{\mathrm{d}N} = -\frac{m^2}{3H^2}\langle\sigma^2\rangle + \frac{H^2}{8\pi^2}. \tag{A.8}$$

This equation can be solved and one obtains

$$\langle\sigma^2(N)\rangle = \int_B^N \frac{\mathrm{d}N'}{4\pi^2} H^2(N') \exp\left[\frac{2m^2}{3}\int_N^{N'} \frac{\mathrm{d}N''}{H^2(N'')}\right]. \tag{A.9}$$

In this expression, B is an integration constant that can be solved requiring that $\langle\sigma^2\rangle = \langle\sigma^2(N_0)\rangle$ at the initial time N_0. This gives rise to

$$\begin{aligned}
\langle\sigma^2(N)\rangle = {} & \langle\sigma^2(N_0)\rangle \exp\left[-\frac{2m^2}{3}\int_{N_0}^N \frac{\mathrm{d}N'}{H^2(N')}\right] \\
& + \int_{N_0}^N \mathrm{d}N' \frac{H^2(N')}{4\pi^2} \exp\left[\frac{2m^2}{3}\int_N^{N'} \frac{\mathrm{d}N''}{H^2(N'')}\right].
\end{aligned} \tag{A.10}$$

In this expression, the structure of the first term is similar to the first moment (A.2), so that the variance of the distribution $\langle\sigma^2\rangle - \langle\sigma\rangle^2$ evolves according to the same formula as the second moment [i.e. one can replace $\langle\sigma^2\rangle$ by $\langle\sigma^2\rangle - \langle\sigma\rangle^2$ in Eq. (A.10) and the formula is still valid].

B   Adiabatic solution for quartic spectators

For quartic spectator fields, the Langevin equation is not linear anymore and cannot be solved analytically. In this section we provide a solution using the ansatz

$$P(\sigma, N) = \frac{2\alpha^{1/4}(N)}{\Gamma\left(\frac{1}{4}\right)} \exp\left[-\alpha(N)\sigma^4\right]. \tag{B.1}$$

This ansatz is satisfied by the de-Sitter equilibrium (1.3), so we expect the solution to be valid at least in the adiabatic regime and potentially beyond. By plugging Eq. (B.1)

into Eq. (1.2), one obtains

$$\left(\frac{1}{4\alpha} - \sigma^4 \right) \frac{\mathrm{d}\alpha}{\mathrm{d}N} P(\sigma, N) = \left(\frac{4\lambda}{H^2} - \frac{3H^2\alpha}{2\pi^2} \right) \sigma^2 P(\sigma, N) + \left(\frac{2H^2\alpha^2}{\pi^2} - \frac{16\lambda\alpha}{3H^2} \right) \sigma^6 P(\sigma, N) .$$
(B.2)

Multiplying this equation by σ^2 and integrating over σ, this gives rise to

$$\left(\frac{\langle \sigma^2 \rangle}{4\alpha} - \langle \sigma^6 \rangle \right) \frac{\mathrm{d}\alpha}{\mathrm{d}N} = \left(\frac{4\lambda}{H^2} - \frac{3H^2}{2\pi^2}\alpha \right) \langle \sigma^4 \rangle + \left(\frac{2H^2}{\pi^2}\alpha^2 - \frac{16\lambda}{3H^2}\alpha \right) \langle \sigma^8 \rangle P(\sigma, N) .$$
(B.3)

From the ansatz (B.1), the moments $\langle \sigma^2 \rangle$, $\langle \sigma^4 \rangle$, $\langle \sigma^6 \rangle$ and $\langle \sigma^8 \rangle$ are directly related to α, through

$$\langle \sigma^2 \rangle = \frac{\Gamma\left(\frac{3}{4}\right)}{\alpha^{1/2}\Gamma\left(\frac{1}{4}\right)} , \qquad \langle \sigma^4 \rangle = \frac{1}{4\alpha} , \qquad \langle \sigma^6 \rangle = \frac{3\Gamma\left(\frac{3}{4}\right)}{4\alpha^{3/2}\Gamma\left(\frac{1}{4}\right)} , \qquad \langle \sigma^8 \rangle = \frac{5}{16\alpha^2} . \quad (B.4)$$

By substituting these expressions into Eq. (B.3), one obtains

$$\frac{\mathrm{d}\alpha}{\mathrm{d}N} = \frac{\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)} \left(\frac{\lambda}{H^2}\alpha^{1/2} - \frac{3H^2}{8\pi^2}\alpha^{3/2} \right) .$$
(B.5)

Notice that if one had directly integrated Eq. (B.2) over σ and substituted Eq. (B.4), one would have obtained a trivial relationship, which is why we first multiplied Eq. (B.2) by σ^2 before integrating over σ.

If the inflaton potential is of the plateau type and H can be approximated by a constant, this equation can be solved and one finds

$$\alpha(N) = \frac{8\pi^2\lambda}{3H^4} \tanh^2 \left\{ \sqrt{\frac{3\lambda}{2}} \frac{\Gamma\left(\frac{1}{4}\right)}{8\pi\Gamma\left(\frac{3}{4}\right)} (N - N_0) + \mathrm{arctanh}\left[\sqrt{\frac{3H^4\alpha(N_0)}{8\pi^2\lambda}} \right] \right\} , \quad (B.6)$$

which gives rise to Eq. (3.5) for the second moment $\langle \sigma^2 \rangle$.

If the inflaton potential is monomial, the function $H(N)$ is given by Eq. (1.6) and although an analytical solution still exists, it is less straightforward to derive. The first step consists of writing Eq. (B.5) in terms of an equation for $\langle \sigma^2 \rangle$ using Eq. (B.4),

$$\frac{\mathrm{d}\langle \sigma^2 \rangle}{\mathrm{d}N} = -\frac{2}{3} \left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \right]^2 \frac{\lambda\langle \sigma^2 \rangle^2}{H^2} + \frac{H^2}{4\pi^2} .$$
(B.7)

The next step is to use $x \equiv H/H_{\mathrm{end}}$ as a time variable, which gives rise to

$$\frac{\mathrm{d}\langle \sigma^2 \rangle}{\mathrm{d}x} = \frac{2}{3} \left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \right]^2 \frac{\lambda x^{4/p-3}}{H_{\mathrm{end}}^2}\langle \sigma^2 \rangle^2 - \frac{H_{\mathrm{end}}^2 x^{4/p+1}}{4\pi^2} .$$
(B.8)

This equation is of the Ricatti type and can be transformed into a second-order linear differential equation making use of the change of variables

$$\langle \sigma^2 \rangle = -\frac{3}{2} \frac{H_{\mathrm{end}}^2}{\lambda} \left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right]^2 \frac{1}{f(x)} \frac{\mathrm{d}f}{\mathrm{d}x} .$$
(B.9)

By plugging Eq. (B.9) into Eq. (B.8), one obtains

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + \left(3 - \frac{4}{p}\right) \frac{1}{x} \frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\lambda}{6\pi^2} \left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]^2 x^{8/p-2} f = 0\,. \tag{B.10}$$

This equation can be solved in terms of modified Bessel functions of the first kind I. Making use of Eq. (B.9), the solution one obtains gives rise to

$$\begin{aligned}
\langle \sigma^2(x) \rangle = &\frac{x^{2-4/p}}{2A} \left(2 - \frac{4}{p}\right) \\
&- \sqrt{\frac{B}{A}} \frac{x^2}{2} \left\{ \frac{I_{-\frac{p}{4}-\frac{1}{2}}(W) + I_{-\frac{p}{4}+\frac{3}{2}}(W) + C\left[I_{\frac{p}{4}+\frac{1}{2}}(W) + I_{\frac{p}{4}-\frac{3}{2}}(W)\right]}{I_{\frac{p}{4}-\frac{1}{2}}(W) + CI_{-\frac{p}{4}+\frac{1}{2}}(W)} \right\}\,,
\end{aligned} \tag{B.11}$$

where we have defined

$$A = \frac{2}{3} \left[\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]^2 \frac{\lambda}{H_{\mathrm{end}}^2}\,, \qquad B = \frac{H_{\mathrm{end}}^2}{4\pi^2}\,, \qquad W = \frac{p}{4}\sqrt{AB}\,x^{4/p}\,, \tag{B.12}$$

where C is an integration constant that can be set as follows: In the asymptotic past, $W \gg 1$ and the Bessel functions can be expanded in this limit, $I_\alpha(W) \simeq e^W/\sqrt{2\pi W}$. Unless $C = -1$, the term inside square brackets in Eq. (B.11) goes to 1 and one finds $\langle \sigma^2 \rangle \simeq -\sqrt{B/A}\,x^2/2 < 0$ which would not be consistent. As a consequence, $C = -1$ is the only choice that allows the solution (B.11) to be defined over the entire inflationary period. Setting $C = -1$, Eq. (B.11) can be simplified and one obtains

$$\langle \sigma^2(N) \rangle = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{\frac{3}{2\lambda}} \frac{H^2}{2\pi} \frac{K_{\frac{p}{4}+\frac{1}{2}}(W)}{K_{\frac{p}{4}-\frac{1}{2}}(W)}\,, \tag{B.13}$$

where K is the modified Bessel function of the second kind.

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