Lagrange Error Bound Worksheet

1) Let *f* be a function with 5 derivatives on the interval [2, 3]. Assume $\left|f^{(5)}(x)\right| < 0.2$ for all *x* in the interval [2, 3] and that a fourth-degree Taylor polynomial for *f* at *c* = 2 is used to estimate $f(3)$.
   a. How accurate is this approximation? Round your answer to five decimal places.
   b. Suppose that $P_4(3) = 1.763$. Use your answer from part (a) to find an interval in which $f(3)$ must reside.
   c. Could $f(3) = 1.778$? Explain your reasoning.
   d. Could $f(3) = 1.764$. Explain your reasoning.

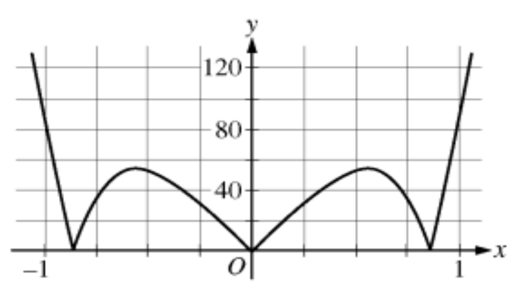
2) $f(x) = \sin x$
   a. Find the fifth-degree Maclaurin polynomial for $f(x) = \sin x$.
   b. Use the polynomial found in part (a) to approximate $\sin 1$.
   c. Use Taylor's Theorem to find the maximum error for your approximation.

3) $f(x) = e^x$.
   a. Write the fourth-degree Maclaurin polynomial for $f(x) = e^x$.
   b. Using your answer from part (a), approximate the value of *e*.
   c. Find a Lagrange error bound for the maximum error involved in the approximation found in part (b).

4) The function has derivatives of all orders for all real number *x*. Assume that $f(2) = 6$, $f'(2) = 4$, $f''(2) = -7$ and $f'''(2) = 8$.
   a. Write the third-degree Taylor polynomial for *f* about *x*=2, and use it to approximate $f(2.3)$.
   b. The fourth derivative of *f* satisfies the inequality $\left|f^{(4)}(x)\right| \le 9$ for all *x*. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval [a, b] such that $a \le f(2.3) \le b$.
   c. Based on the information above, could $f(2.3) = 6.992$? Explain your reasoning.

5)

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

6) Let $f(x)=e^{x/2}$. If the second-degree Maclaurin polynomial for f is used to approximate f on the interval [0, 2], what is the Lagrange error bound for the maximum error on the interval [0, 2]?

a. 0.028
b. 0.113
c. 0.453
d. 0.499
e. 0.517

7) Let f be a function having 5 derivatives on the interval [2, 2.9] and assume that $\left|f^{(5)}(x)\right| \le 0.8$ for all x in the interval [2, 2.9]. If the fourth-degree Taylor polynomial for f about $x=2$ is used to approximate f on the interval [2, 2.9], what is the Lagrange error bound for the maximum error on the interval [2, 2.9]?

a. 0.004
b. 0.011
c. 0.022
d. 0.033
e. 0.044

ANSWERS

1)
a) Max Error= 1/600
b) $1.761 \le f(3) \le 1.765$
c) No, since 1.778 does not fall in the interval found in part (b), the IVT does not guarantee 1.778 to be a possible value of $f(3)$.
d) Yes, since 1.764 does fall in the interval found in part (b), the IVT does guarantee 1.764 to be a possible value of $f(3)$.

2)
a) $P_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}$
b) 101/120
c) Max Error = 1/5,040

3)
a) $P_4(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$
b) 65/24 ≈ 2.708
c) e/120

4)
a)
$P_3(x)=6+4(x-2)-\frac{7(x-2)^2}{2!}+\frac{8(x-2)^3}{3!}$
$f(2.3)\approx P_3(2.3)=6.0216$
b) $6.020 \le f(2.3) \le 6.023$
c) No, since 6.992 does not fall in the interval found in part (b), the IVT does not guarantee 6.992 to be a possible value of $f(2.3)$.

5)
a) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$
$\sin x^2 = x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots$
b) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

$f(x)=1+\frac{x^2}{2}+\frac{x^4}{24}-\frac{121x^6}{720}+\ldots$
c) $f^6(0)=-121$
d) Max Error= $\frac{1}{3072}$

6) C
7) A