Integrated Programme/mainstream
Secondary Three Mathematics

***Name:* ___________________________________ ( ) *Class:* _________ *Date:* _____________**

Term 1 : Unit 2 ***Self Study***

At the end of the unit, students should be able to

1. present information in the form of a matrix of any order,
2. define equal, zero, identity matrices,
3. find unknowns in equal matrices,
4. perform addition and subtraction on matrices of same order,
5. perform scalar multiplication,
6. perform matrix multiplication on small order matrices,
7. find determinant of a 2 × 2 matrix,
8. understand singular and non-singular matrices,
9. find the inverse of a 2 × 2 non-singular matrix by formula,
10. express a pair of simultaneous linear equations in matrix form and solving the equations by inverse matrix method.

Introduction to Matrices *Worksheet 1*

A : Introduction

1. Matrix
A matrix is defined as a rectangular _______ of numbers arranged in the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

2. Elements
The numbers in the matrix (e.g. a_{12}, a_{1n} and a_{mn}) are called the elements.

3. Order
A matrix having *m* rows and *n* columns is called a $m\times n$ (*m* by *n*) matrix. A $m\times n$ matrix has order $m\times n$. E.g.
$\begin{pmatrix} 2 & 3 & 7 \\ 1 & 3 & 5 \end{pmatrix}$ has order _______.

4. Square Matrix
An $n\times n$ matrix is a square matrix of order *n* e.g. ________________.

5. Null or Zero Matrix
A null / zero matrix is a matrix with all elements zero e.g. ________________.

6. Identity Matrix
An identity matrix is a square matrix in which the elements on the main diagonal are 1 and the elements outside the main diagonal are all zero e.g. ________________.

7. Diagonal Matrix
A diagonal matrix is a square matrix in which the elements outside the main diagonal are all zero e.g. ________________.

B: Notation

We use capital letters (A, B, etc) to denote matrices.

For example, A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Note that a_{ij} is used to denote the element which appears at the ith row and jth column of the matrix.

The identity matrix and null matrix are represented as I and O respectively.

Skill Check 1:

1. In May, Suzanne bought 32 styrofoam balls and decorated them as toy figurines. In June, she sold 12 figurines. In May, Carrie bought 36 styrofoam balls to decorate and in June, she sold 22 figurines. Which matrix represents all of their May purchases and their June sales?

A. $\begin{array}{cc} & \begin{matrix} \text{May} & \text{June} \end{matrix} \\ \begin{matrix} \text{Suzanne} \\ \text{Carrie} \end{matrix} & \begin{bmatrix} 32 & 20 \\ 36 & 14 \end{bmatrix} \end{array}$

B. $\begin{array}{cc} & \begin{matrix} \text{May} & \text{June} \end{matrix} \\ \begin{matrix} \text{Suzanne} \\ \text{Carrie} \end{matrix} & \begin{bmatrix} 32 & 36 \\ -20 & -14 \end{bmatrix} \end{array}$

C. $\begin{array}{cc} & \begin{matrix} \text{May} & \text{June} \end{matrix} \\ \begin{matrix} \text{Suzanne} \\ \text{Carrie} \end{matrix} & \begin{bmatrix} 32 & -12 \\ 36 & -22 \end{bmatrix} \end{array}$

D. $\begin{array}{cc} & \begin{matrix} \text{May} & \text{June} \end{matrix} \\ \begin{matrix} \text{Suzanne} \\ \text{Carrie} \end{matrix} & \begin{bmatrix} 32 & 36 \\ 12 & 22 \end{bmatrix} \end{array}$

[ ]

2. What are the dimensions of $\begin{bmatrix} 3 \\ \sqrt{5} \end{bmatrix}$?

A. 1 × 2 B. 3 × 5

C. 2 × 1 D. 5 × 3 [ ]

3. Identify a_{31} in $\begin{bmatrix} -13 & -20 & -17 & 4 \\ -21 & 5 & -6 & 27 \\ 10 & 20 & 21 & 14 \end{bmatrix}$.

A. 31 B. -17

C. 10 D. -6 [ ]

4. What are the dimensions of $\begin{bmatrix} 19 & 16 & 13 & -11 & 20 \\ -9 & -5 & -10 & -18 & 15 \\ 14 & 7 & 2 & 1 & 17 \end{bmatrix}$?

A. 5 x 3 B. 3 x 5

C. 4 x 5 D. 5 x 4 [ ]

Matrix Operations ***Worksheet 2***

A: Equality of Matrices

Two matrices are equal only if they fulfil both of the following conditions:

a. they have the same order;

b. all corresponding elements are equal.

For example, $\begin{pmatrix} 1 & 1\frac{1}{2} \\ 3.5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ \frac{7}{2} & 4 \end{pmatrix}$ but $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \end{pmatrix}$.

Skill Check 2 :

1. Evaluate the unknowns:

(a) $\begin{pmatrix} 2 & a & 5 \end{pmatrix} = \begin{pmatrix} b^2 & 7 & \sqrt{c} \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 4 \\ 2x^3 & z-2 \end{pmatrix} = \begin{pmatrix} y\sqrt{3} & 4 \\ 54 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 3x-y & x+2y \\ 2xy & 4 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ -6 & 4 \end{pmatrix}$

B: Matric Operations

i) Addition of Matrices

Two matrices can be added together only if they have the same order.

Example

The following table shows the stock of Waris Dirie's Desert Flower in three bookstores.

	Hardcover	Softcover
Bookstore 1	23	17
Bookstore 2	15	7
Bookstore 3	26	34

Represent the given information as a matrix A.

$A =$

The next table shows the replenishments which has just arrived at the bookstores.

	Hardcover	Softcover
Bookstore 1	2	10
Bookstore 2	3	4
Bookstore 3	5	11

Represent this information as a matrix B.

$B =$

Adding these two matrices is easy; simply add up the corresponding elements of A and B.

$\therefore A + B =$

What does the answer to $A + B$ represent?

ii) Subtraction of Matrices

Subtraction of matrices is similar to addition. Two matrices can be subtracted only if they have the same order. Instead of adding up the corresponding elements of the matrices, we subtract them.

E.g. $\begin{pmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{-2} \end{pmatrix} - \begin{pmatrix} \mathbf{-1} & \mathbf{4} \\ \mathbf{0.5} & \mathbf{8} \end{pmatrix} =$

iii) Scalar Multiplication of Matrices

Example

John wants to organise a barbeque for his classmates. He decided to buy sausages, chicken wings and satay. The following table shows 2 proposals:

	Sausages	Chicken wings	Satay
Proposal 1	24	32	48
Proposal 2	28	26	56

Represent the above information as a matrix A.

$$A =$$

However there is a change of plan and everyone decides to bring along a partner. Therefore the amount of food has to be doubled.

$2A =$

The answer to $2A$ can easily be obtained by multiplying every element by 2.
What does the answer to $2A$ represent?

Matrix Multiplication and Word Problems *Worksheet 3*

A: Matrix Multiplication

In summary,

1. Two matrices, A and B can be multiplied together iff the number of __________ in A is equal to the number of __________ in B.
2. The multiplication of two matrices, A and B, will give rise to a matrix AB with the number of __________ of A and the number of __________of B, i.e. $A_{m \times n} \times B_{n \times p} = AB_{m \times p}$.
3. The product of two matrices, A and B, will give rise to a matrix AB, whose element in the ith row and jth column is the sum of the products formed by multiplying each element in the ith row of A by the corresponding element in the jth column of B.

Skill Check 3 :

1. Compute the products AB and BA for the following, if possible:

(i) $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 4 \\ 2 & 0.5 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0.5 & -1 \\ 1 & 3 & 4 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 4 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 4 \end{pmatrix}$

<u>B: Laws of Matrix Multiplication</u>

1. Given that $A = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, find **(i)** $A(BC)$**,** **(ii)** $(AB)C$**.**

Note: Q $A(BC) = (AB)C$, this question illustrates that matrix multiplication is **associative**.

2. Given that $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, find

(i) $A(B+C)$ **(ii)** $AB + AC$ **(iii)** $(B+C)A$ **(iv)** $BA + CA$

Note: Q $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$, this illustrates that matrix multiplication is **distributive** over addition.

3. Given that $A = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & -1 \\ 1 & 3 \end{pmatrix}$, find

(i) AB **(ii)** BA

Note: Q $AB \neq BA$, this question illustrates that matrix multiplication is **not commutative.**

Summary:

1. Matrix multiplication is **associative.**

2. Matrix multiplication is **distributive** over addition and subtraction.

3. Matrix multiplication is **not commutative.**

C: Word Problems

1. An ice-cream stall sells both green tea and mocha ice cream. A small portion of either costs \$0.75 and a large portion costs \$1.25. During a short period of time, the number of ice creams sold is own in the table below.

	small	large
Green Tea	3	4
Mocha	6	3

(i) Write down a column matrix N, representing the cost of each portion of ice cream.

(ii) Given that $M = \begin{pmatrix} 3 & 4 \\ 6 & 3 \end{pmatrix}$, evaluate MN.

(iii) Explain what the numbers given in your answer in (ii) signify.

Answers:

(i) $N = \begin{pmatrix} 0.75 \\ 1.25 \end{pmatrix}$

(ii) $$MN = \begin{pmatrix} 3 & 4 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 0.75 \\ 1.25 \end{pmatrix}$$
$$= \begin{pmatrix} 7.25 \\ 8.25 \end{pmatrix}$$

(iii) \$7.25 is the amount received from the sales of green tea ice cream while \$8.25 is the amount received from the sale of mocha ice cream.

As we can see from the example above, matrices can be used in solving word problems.

Skill Check 4:

The matrix below shows the results of John and Paul in a multiple choice test.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Correct} & \text{No attempt} & \text{Incorrect} \end{array} \\ P = & \begin{pmatrix} 18 & 7 & 5 \\ 20 & 0 & 10 \end{pmatrix} \begin{array}{l} \textit{John} \\ \textit{Paul} \end{array} \end{array}$$

The method to award marks is given by:

$$\begin{array}{cc} & \text{Marks} \\ Q = & \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \begin{array}{l} \textit{Correct} \\ \textit{No attempt} \\ \textit{Incorrect} \end{array} \end{array}$$

(i) **Find the matrix product *PQ*.**

(ii) **Explain clearly what your answer to (a) represents.**

A: Determinant

The determinant of a 2 x 2 matrix is defined as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det A = ad - bc$$

Skill Check 5:

No.	M	det M= ad - bc
1.	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
2.	$\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$	
3.	$\begin{pmatrix} -1 & 4 \\ -2 & 7 \end{pmatrix}$	
4.	$\begin{pmatrix} 6 & 5 \\ -2 & -1 \end{pmatrix}$	
5.	$\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$	

B: Inverse Matrix

The **inverse** of a square matrix M, denoted by $\mathbf{M^{-1}}$, is defined in such a way that $\mathbf{MM^{-1} = M^{-1}M = I}$, where I is the identity matrix.

For example,

$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $N = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

$$MN = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$
$$= -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Hence, $M^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ since $MM^{-1} = I$.

Summary :

Given M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, det M = ad – bc.

Hence, $M^{-1} = \dfrac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

A matrix whose **determinant is zero** is called a **singular matrix** because it does not have an inverse (no 'partner'). A non-singular matrix has a non-zero determinant and thus has an inverse.

Skill Check 6:

No.	N	det N	N^{-1}
1.	$\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$		
2.	$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$		
3.	$\begin{pmatrix} 7 & -4 \\ 2 & -1 \end{pmatrix}$		
4.	$\begin{pmatrix} -1 & -5 \\ 2 & 6 \end{pmatrix}$		

C: Using matrix method to solve simultaneous equations

Given the pair of simultaneous equations : $x - 2y = 1$ and $x + 4y = 8$. Solve for x and y using matrix method.

Answer:

$x - 2y = 1$
$x + 4y = 8$

Step 1: Transform the equations into matrix form

$$\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

Step 2: Define the matrices.

Let A = $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, so A $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$

Step 3: As A A^{-1} = I, find A^{-1}

det A = 1(4) –(-2)(1)

$= 6$

$A^{-1} = \frac{1}{6}\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$

Step 4: Using matrix multiplication , solve for x and y.

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ \frac{7}{6} \end{pmatrix}$$

So, x = $3\frac{1}{3}$, y = $\frac{7}{6}$.

Skill Check 7:
Using matrix method , find the value of x and y.

$x + 5y = 10$

$7x - 4y = 10$