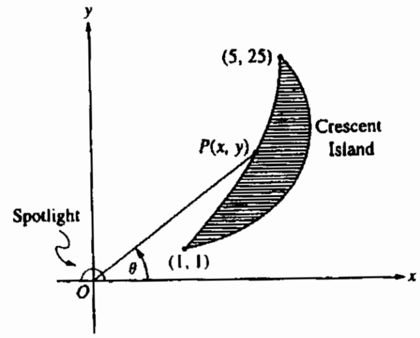
AP Calculus BC - Polar Free Response Practice

1996 BC6

CALCULATOR PROBLEM



<u>Note:</u> **Figure not drawn to scale.**

The figure above shows a spotlight shining on point $P(x, y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point $(1,1)$ to the point $(5,25)$. Let θ be the angle between the beam of light and the positive x-axis.

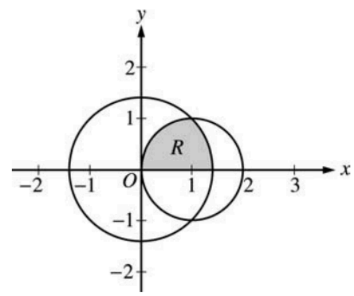
(a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?

(b) Find the x- and y-coordinates of point P in terms of $\tan\theta$.

(c) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point $(3,9)$?

CALCULATOR PROBLEM

Question 2

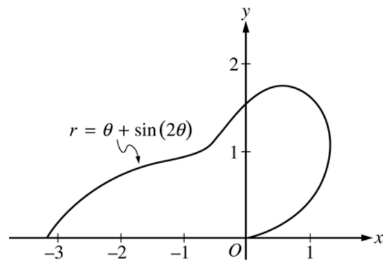


The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1,1)$ and $(1,-1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.

(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.



CALCULATOR PROBLEM

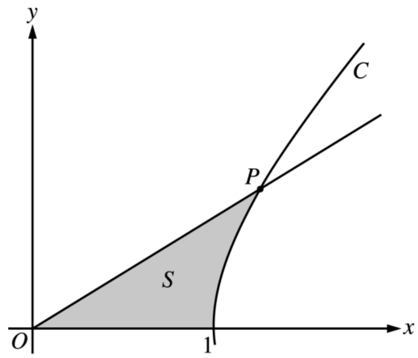
2. The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.



NO CALCULATOR

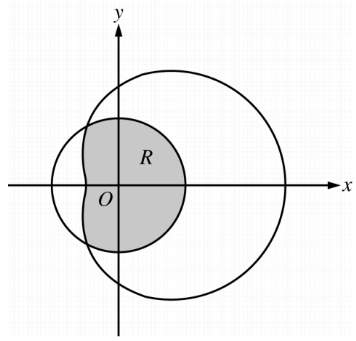
3. The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.



NO CALCULATOR

3. The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

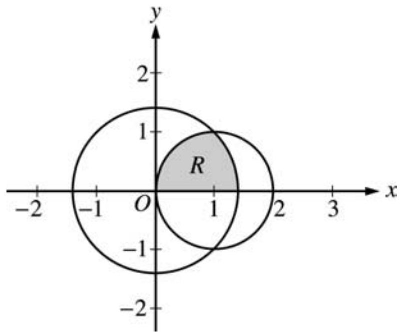
(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

Question 2

The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1,1)$ and $(1,-1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.



(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

(a) $\text{Area} = \int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

OR

$\text{Area} = \frac{1}{4}\left(\pi \cdot 1^2\right) + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

$$3: \begin{cases} 1: \text{integrand for larger circle} \\ 1: \text{integrand or geometric area} \\ \quad \text{for smaller circle} \\ 1: \text{limits on integral(s)} \end{cases}$$

Note: < –1 > if no addition of terms

(b) $\text{Area} = \int_0^1 \left(\sqrt{2-y^2} - \left(1 - \sqrt{1-y^2}\right)\right) dy$

$$3: \begin{cases} 1: \text{limits} \\ 2: \text{integrand} \\ \quad < -1 > \text{ reversal} \\ \quad < -1 > \text{ algebra error} \\ \quad\quad \text{in solving for } x \\ \quad < -1 > \text{ add rather} \\ \quad\quad \text{than subtract} \\ \quad < -2 > \text{ other errors} \end{cases}$$

(c) $\text{Area} = \int_0^{\pi/4} \frac{1}{2}\left(\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

OR

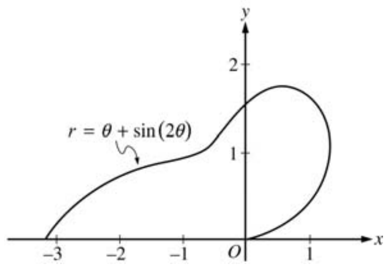
$\text{Area} = \frac{1}{8}\pi\left(\sqrt{2}\right)^2 + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

$$3: \begin{cases} 1: \text{integrand or geometric area} \\ \quad \text{for larger circle} \\ 1: \text{integrand for smaller circle} \\ 1: \text{limits on integral(s)} \end{cases}$$

Note: < –1 > if no addition of terms

Question 2

The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.



(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) Area $= \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta = 4.382$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$
$\theta = 2.786$

$2 : \begin{cases} 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

$2 : \begin{cases} 1 : \text{information about } r \\ 1 : \text{information about the curve} \end{cases}$

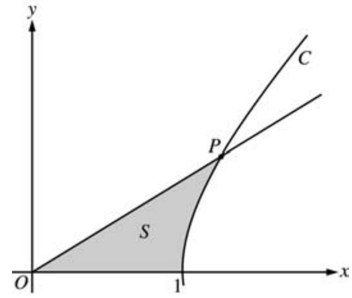
(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs when $\theta = \frac{\pi}{3}$.

$2 : \begin{cases} 1 : \theta = \frac{\pi}{3} \text{ or } 1.047 \\ 1 : \text{answer with justification} \end{cases}$

Question 3



The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

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(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a) At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$. Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$. $\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}$. At P, $\frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}$.	$2 : \begin{cases} 1 : \text{coordinates of } P \\ 1 : \frac{dx}{dy} \text{ at } P \end{cases}$
(b) Area $= \int_0^{3/4} \left(\sqrt{1+y^2} - \frac{5}{3}y\right) dy$ $= 0.346$ or 0.347	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) $x = r\cos\theta\,;\ y = r\sin\theta$ $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$ $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$	$2 : \begin{cases} 1 : \text{substitutes } x = r\cos\theta \text{ and} \\ \quad y = r\sin\theta \text{ into } x^2 - y^2 = 1 \\ 1 : \text{isolates } r^2 \end{cases}$
(d) Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$. Area $= \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\, d\theta$ $= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\, d\theta$	$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand and constant} \end{cases}$

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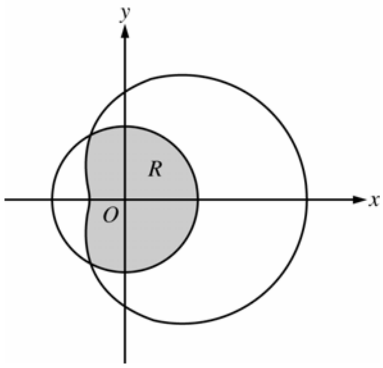
Question 3



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(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\,d\theta$
$= 10.370$

$$4 : \begin{cases} 1 : \text{area of circular sector} \\ 2 : \text{integral for section of limaçon} \\ \quad 1 : \text{integrand} \\ \quad 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$$

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

$$2 : \begin{cases} 1 : \left.\frac{dr}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

$$3 : \begin{cases} 1 : \text{expression for } y \text{ in terms of } \theta \\ 1 : \left.\frac{dy}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$