Name____ Date_______________________ Period______

Worksheet 7.5—Partial Fractions & Logistic Growth
Show all work. No calculator unless stated.

Multiple Choice

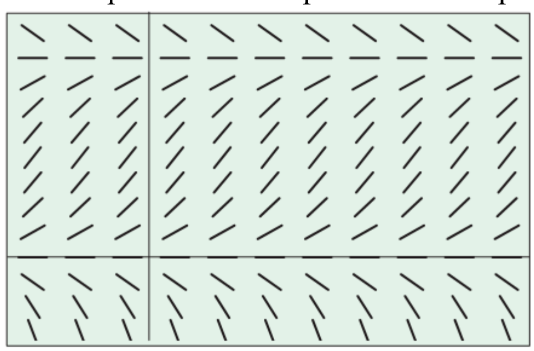
1. The spread of a disease through a community can be modeled with the logistic equation $y = \dfrac{600}{1+59e^{-0.1t}}$, where y is the number of people infected after t days. How many people are infected when the disease is spreading the fastest?
(A) 10 (B) 59 (C) 60 (D) 300 (E) 600

2. The spread of a disease through a community can be modeled with the logistic equation $y = \dfrac{0.9}{1+45e^{-0.15t}}$, where y is the proportion of people infected after t days. According to the model, what percentage of people in the community will not become infected?
(A) 2% (B) 10% (C) 15% (D) 45% (E) 90%

3. $\displaystyle\int_2^3 \frac{3}{(x-1)(x+2)}\,dx =$

(A) $-\frac{33}{20}$ (B) $-\frac{9}{20}$ (C) $\ln\left(\frac{5}{2}\right)$ (D) $\ln\left(\frac{8}{5}\right)$ (E) $\ln\left(\frac{2}{5}\right)$

4. Which of the following differential equations would produce the slope field shown below?



[–3, 8] by [–50, 150]

(A) $\frac{dy}{dx} = 0.01x(120-x)$ (B) $\frac{dy}{dx} = 0.01y(120-y)$ (C) $\frac{dy}{dx} = 0.01y(100-x)$

(D) $\frac{dy}{dx} = \frac{120}{1+60e^{-1.2x}}$ (E) $\frac{dy}{dx} = \frac{120}{1+60e^{-1.2y}}$

5. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is the time in years. What is $\lim_{t \to \infty} P(t)$?

(A) 2500 (B) 3000 (C) 4200 (D) 5000 (E) 10,000

6. Suppose a population of wolves grows according to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$, where P is the number of wolves at time t, in years. Which of the following statements are true?

I. $\lim_{t \to \infty} P(t) = 300$

II. The growth rate of the wolf population is greatest when $P = 150$.

III. If $P > 300$, the population of wolves is increasing.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

Short Answer/Free Response

Work the following on notebook paper.

7. Suppose the population of bears in a national park grows according to the logistic differential equation $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears at time t in years.

   (a) If $P(0) = 100$, then $\lim_{t\to\infty} P(t) =$____________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

   (b) If $P(0) = 1500$, $\lim_{t\to\infty} P(t) =$____________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

   (c) If $P(0) = 3000$, $\lim_{t\to\infty} P(t) =$____________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

   (d) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

8. (Calculator Permitted) A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P\left(100 - P\right)$, where t is measured in years.

   (a) If $P(0) = 20$, solve for P as a function of t.

   (b) Use your answer to (a) to find P when $t = 3$ years. Give exact and 3-decimal approximation.

   (c) Use your answer to (a) to find t when $P = 80$ animals. Give exact and 3-decimal approximation.

9. (Calculator Permitted) The rate at which a rumor spreads through a high school of 2000 students can be modeled by the differential equation $\frac{dP}{dt} = 0.003P(2000 - P)$, where P is the number of students who have heard the rumor t hours after 9AM.

   (a) How many students have heard the rumor when it is spreading the fastest?

   (b) If $P(0) = 5$, solve for P as a function of t.

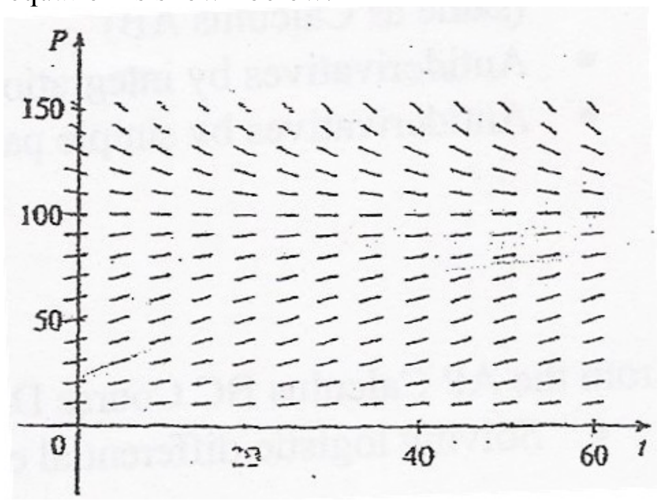
   (c) Use your answer to (b) to determine how many hours have passed when the rumor is spreading the fastest. Give exact and 3-decimal approximation.

   (d) Use your answer to (b) to determine the number of people who have heard the rumor after two hours. Give exact and 3-decimal approximation.

10. Suppose that a population develops according to the logistic equation $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks.

(a) What is the carrying capacity/limit to growth?

(b) A slope field for this equation is shown below.



I. Where are the slopes close to zero?

II. Where are they largest?

III. Which solutions are increasing?

IV. Which solutions are decreasing?

(c) Use the slope field to sketch solutions for initial populations of 20, 60, and 120.

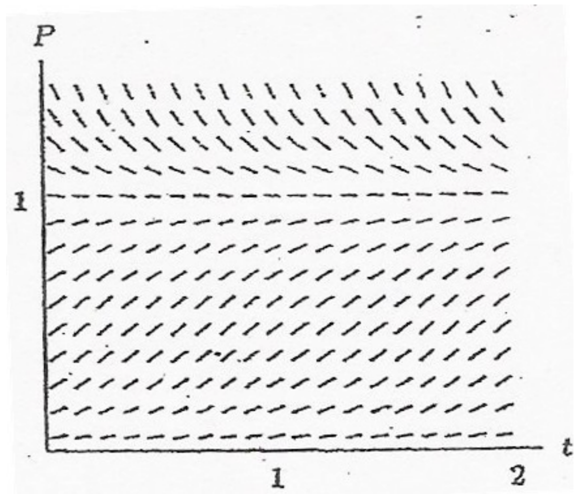
I. What do these solutions have in common?

II. How do they differ?

III. Which solutions have inflection points?

IV. At what population level do these inflection points occur?

11. The slope field show below gives general solutions for the differential equation given by $\frac{dP}{dt} = 3P - 3P^2$.



(a) On the graph above, sketch three solution curves showing three different types of behavior for the population P.

(b) Describe the meaning of the shape of the solution curves for the population.

I. Where is P increasing?

II. Where is P decreasing?

III. What happens in the long run (for large values of t)?

IV. Are there any inflection points? If so, where?

V. What do the inflection points mean for the population?

Multiple Choice II

12. $\int \frac{7x}{(2x-3)(x+2)}dx=$

(A) $\frac{3}{2}\ln|2x-3|+2\ln|x+2|+C$ (B) $3\ln|2x-3|+2\ln|x+2|+C$ (C) $3\ln|2x-3|-2\ln|x+2|+C$

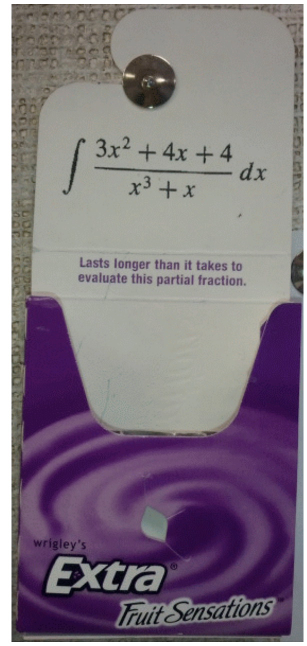
(D) $-\frac{6}{(2x-3)^2}-\frac{2}{(x+2)^2}+C$ (E) $-\frac{3}{(2x-3)^2}-\frac{2}{(x+2)^2}+C$

13. $\int \frac{2x}{x^2+3x+2}dx =$

(A) $\ln|x+2|+\ln|x+1|+C$ (B) $\ln|x+2|+\ln|x+1|-3x+C$ (C) $-4\ln|x+2|+2\ln|x+1|+C$

(D) $4\ln|x+2|-2\ln|x+1|+C$ (E) $2\ln|x|+\frac{2}{3}x+\frac{1}{2}x^2+C$

14. CHALLENGE:
On a pack of Extra Fruit Sensations gum package was written the following integral:



$$\int \frac{3x^2+4x+4}{x^3+x}dx$$

Beneath this, the gum maker claims that the gum's flavor "[l]asts longer than it takes to evaluate this partial fraction." Obviously this gum was not designed for an AP Calculus student, as a student of this caliber requires his gum to hold its flavor for much, much, much longer. Prove my point by evaluating this integral using partial fraction decomposition, but be careful, because of the quadratic factor in the denominator, the Heaviside Cover-Up Method does NOT work. Get chewing!!

OK, here's a hint, decompose the integrand by finding the values of A, B, and C in the decomposition form below.

$$\frac{3x^2+4x+4}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

(1) $y = \frac{600}{1+59e^{-0.1t}}$

600 = limit to growth = L

disease grows fastest when $\frac{L}{2} = \frac{600}{2} = \boxed{300 \text{ people}}$ are infected. $\boxed{D}$

(2) $y = \frac{0.9}{1+45e^{-0.15t}}$ is proportion of people infected.

0.9 (or 90%) is limit to growth.

So, eventually, 90% will become infected leaving only $\boxed{10\%}$ to remain uninfected. $\boxed{B}$

(3) $\int_2^3 \frac{3}{(x-1)(x+2)}dx$

$= \int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx$

$= \ln|x-1| - \ln|x+2| \Big|_2^3$

$= \ln\left|\frac{x-1}{x+2}\right| \Big|_2^3$

$= \ln\left(\frac{2}{5}\right) - \ln\left(\frac{1}{4}\right)$

$= \ln\left(\frac{2/5}{1/4}\right) = \boxed{\ln\left(\frac{8}{5}\right)}$

(4) Given slope field has zero slopes when $y=0$ and at $y=L$, the solutions appear to be logistic, so the differential equation is of form $\frac{dy}{dt} = ky(L-y)$.

Only choice $\boxed{B}$ fits this form.

(5) $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, $P(0) = 3000$

$\frac{dP}{dt} = \frac{1}{5000}P(10{,}000 - P)$

So $L = 10{,}000$ $\boxed{E}$

and $\boxed{\lim_{t\to\infty} P(t) = 10{,}000}$

(6) $\frac{dP}{dt} = 3P - 0.01P^2$

$\frac{dP}{dt} = \underset{K}{0.01}P(\underset{L}{300} - P)$

I. $\lim_{t\to\infty} P(t) = 300$ ✓ (true)

II. growth is fastest when $P = 150$ ✓ (true)

III. If $P > 300$, P is increasing. X (false)

* P is inc when $0 < P < 300$ and dec when $P > 300$

$\boxed{C}$ $\boxed{\text{I and II only}}$

(7) $\frac{dP}{dt} = 5P - 0.002P^2$

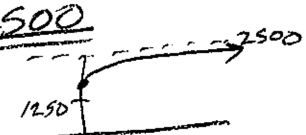
$\frac{dP}{dt} = \underset{K}{0.002}P(\underset{L}{2500} - P)$

(a) If $P(0) = 100$, $\lim_{t\to\infty} P(t) = \underline{\underline{2500}}$

P is inc $\forall t > 0$ since $P(0) < 2500$

(b) if $P(0) = 1500$, $\lim_{t\to\infty} P(t) = \underline{\underline{2500}}$

P is inc $\forall t > 0$ since $P(0) < 2500$



(c) if $P(0) = 3000$, $\lim_{t\to\infty} P(t) = \underline{\underline{2500}}$

P is dec $\forall t > 0$, since $P(0) > 2500$

(d) Pop is growing fastest when $\frac{2500}{2} = 1250$ bears are in the park. This is where P' is a maximum

(8) $\frac{dP}{dt} = \underset{K}{0.01}P(\underset{L}{100} - P)$

(a) $P(t) = \frac{L}{1+Ce^{-LKt}}$

$P(t) = \frac{100}{1+Ce^{-t}}$

at $(0, 20)$: $20 = \frac{100}{1+C}$, $\boxed{C = 4}$

so $\boxed{P(t) = \frac{100}{1+4e^{-t}}}$

(b) $\boxed{P(3) = \frac{100}{1+4e^{-3}} \approx 83.392 \text{ Animals}}$

(c) $\frac{100}{1+4e^{-t}} = 80$

$\frac{1}{4}\left(\frac{100}{80} - 1\right) = e^{-t}$

$t = -\ln\left(\frac{1}{16}\right)$

$\boxed{t = \ln 16 \text{ years} \approx 2.772 \text{ yrs}}$

(9) $\frac{dP}{dt} = 0.003P(2000-P)$

$\quad\quad k \uparrow \quad\quad \uparrow L$

(a) 1000 students $\left(\frac{2000}{2}\right)$ have heard the rumor when it's growing the fastest.

(b) $P(t) = \frac{L}{1+Ce^{-Lkt}}$

$P(t) = \frac{2000}{1+Ce^{-6t}}$

at $(0,5)$: $5 = \frac{2000}{1+C}$, $\boxed{C = 399}$

so $\boxed{P(t) = \frac{2000}{1+399e^{-6t}}}$

(c) $\frac{2000}{1+399e^{-6t}} = 1000$

$\ln\left(\frac{1}{399}\right) = -6t$

$\boxed{t = \frac{1}{6}\ln(399) \text{ hrs after 9AM}}$

$t \approx 0.998$ hrs

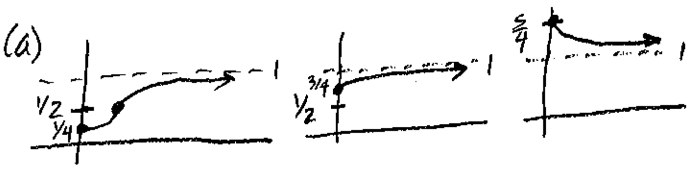
(d) $\boxed{P(2) = \frac{2000}{1+399e^{-12}} \text{ students}}$

$P(2) \approx 1995.108$ students

(11) $\frac{dP}{dt} = 3P - 3P^2 = 3P(1-P)$

$\quad\quad k \uparrow \quad \uparrow L$

(a)



(b) I. P is inc for all $0<P<1$

II. P is dec for all $P>1$

III. In the long run, as $t\to\infty$, $P(t)\to 1$

IV. P only has inflection pts for $0<P(0)<\frac{1}{2}$. These occur where $P(t) = \frac{1}{2}$

V. Inflection pts indicate the place where the population is growing at the fastest rate.

(10) $\frac{dP}{dt} = 0.05P - 0.0005P^2$

$\frac{dP}{dt} = 0.0005P(100-P)$

$\quad\quad k \uparrow \quad\quad \uparrow L$

(a) Carrying Capacity $= L = 100$

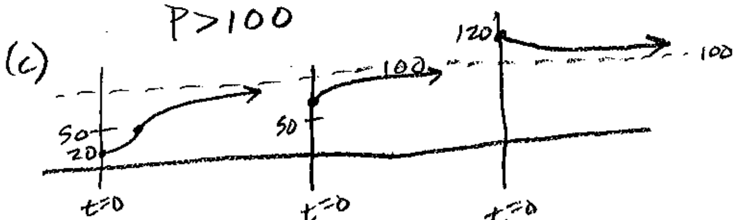
(b) I. Slopes approach zero near $P=0$ and $P=100$ (the two horizontal asymptotes).

II. Slopes are largest when $P = \frac{100}{2} = 50$

III. Solutions are increasing for $0<P<100$

IV. solutions are decreasing for $P>100$

(c)



I. For all these solutions, $\lim_{t\to\infty} P(t) = 100$

II. For $P(0)<100$, solutions are increasing

For $P(0)>100$, solution is decreasing

For $0<P(0)<50$, solution has inflection pt.

For $P(0)>50$, solutions have no inflection pt.

III. only for $P(0)<50=\frac{100}{2}$ does the solution have an inflection point.

IV. These inflection pts occur when pop. grows fastest, which is when $P = \frac{L}{2} = \frac{100}{2} = 50$.

(12) $\int \frac{7x}{(2x-3)(x+2)}dx = \int\left[\frac{\frac{7(3/2)}{7/2}}{2x-3} + \frac{\frac{-14}{-7}}{x+2}\right]dx$

$= \int\left[\frac{3}{2x-3} + \frac{2}{x+2}\right]dx$

$= (3)\left(\frac{1}{2}\right)\ln|2x-3| + 2\ln|x+2| + C$

rider corr

$\boxed{\frac{3}{2}\ln|2x-3| + 2\ln|x+2| + C}$

$\boxed{A}$

(13) $\int \frac{2x}{x^2+3x+2}dx = \int \frac{2x}{(x+1)(x+2)}dx = \int\left[\frac{\frac{-2}{1}}{x+1} + \frac{\frac{-4}{-1}}{x+2}\right]dx = \int\left[\frac{4}{x+2} - \frac{2}{x+1}\right]dx$

$= \boxed{4\ln|x+2| - 2\ln|x+1| + C}$ $\boxed{D}$

(14) $\int \frac{3x^2+4x+4}{x^3+x}dx$ *working first with the integrand

$\frac{3x^2+4x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$, multiplying both sides by $x(x^2+1)$

$3x^2+4x+4 = A(x^2+1) + (Bx+C)x$, distributing right side

$3x^2+4x+4 = Ax^2 + A + Bx^2 + Cx$, reorganizing right side

$\underline{3}x^2 + \underline{4}x + \underline{4} = \underline{(A+B)}x^2 + \underline{C}x + \underline{A}$

so $\boxed{A=4}$, $\boxed{C=4}$, and $A+B=3$, so $B = 3-A = 3-4$, $\boxed{B=-1}$

So $\int\left[\frac{4}{x} + \frac{-x+4}{x^2+1}\right]dx$

$= \int\left[\frac{4}{x} - \frac{x}{x^2+1} + \frac{4}{x^2+1}\right]dx$

$= \boxed{4\ln|x| - \frac{1}{2}\ln|x^2+1| + 4\arctan x + C}$