End Behavior of Rational Functions

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Objective:

Students will investigate the end behavior of rational functions. They will determine if the end behavior can be modeled with a horizontal line, with an oblique (slant) line, or as a polynomial. They will also determine whether the rational function intersects the function that models the end behavior.

Connections to Previous Learning:

Students should be familiar with solving systems of equations and checking solutions, limits at infinity, and long division of polynomials.

Connections to AP*:

AP Calculus Topic: Analysis of Functions

Materials:

Student Activity pages, graphing calculators

Teacher Notes:

The end behavior model of a rational function only describes the behavior of the function when $x \to \infty$ or $x \to -\infty$. For large values of *x*, the rational function can approach a horizontal line, an oblique (slant) line, or a polynomial. Knowing how the function is shaped for large values does not provide information about the function's behavior for smaller values of *x*; therefore, the statement that a function cannot cross an asymptote is not true for the horizontal or oblique asymptotes. Rational functions often intersect the lines or polynomials that describe their end behavior.

The end behavior of a rational function $y = \dfrac{f(x)}{g(x)}$ can have three forms:

- Horizontal Asymptote
  When the degree of $f(x)$ is less than or equal to the degree of $g(x)$, the rational function will have a horizontal asymptote.

- Oblique (Slant) Asymptote
  When the degree of $f(x)$ is one more than the degree of $g(x)$, the rational function will have an oblique asymptote.

- Asymptotic to a polynomial
  When the degree of $f(x)$ is at least two more than the degree of $g(x)$, the rational function will behave like a polynomial.

The equations of the oblique asymptotes and the end behavior polynomials are determined by dividing the polynomial $f(x)$ by the polynomial $g(x)$. The quotient, ignoring the remainder, is the equation for the end behavior model.

The process for judging whether the function touches or crosses its asymptote involves solving the system of equations that consists of the equation of the end behavior and the rational function. When the system results in a null solution, such as $12 = 5$, the graphs never touch. When the system has a solution, such as $x = 1.5$, the x-value provides the x-coordinate of the point of intersection.

End Behavior of Rational Functions

1. Determine the horizontal asymptote of $y = \dfrac{2x-4}{x+3}$ by evaluating the limit as x approaches $\pm$ infinity. Does the function intersect its horizontal asymptote? If so, at what point(s)?

2. Determine the horizontal asymptote of $y = \dfrac{2x-4}{x^2+3}$ by evaluating the limit as x approaches $\pm$ infinity. Does the function intersect its horizontal asymptote? If so, at what point(s)?

3. Determine the horizontal asymptote of $y = \dfrac{x^2+x-2}{x^2+3x-10}$ by evaluating the limit as x approaches $\pm$ infinity. Does the function intersect its horizontal asymptote? If so, at what point(s)?

4. The rational function, $y = \dfrac{x^3-3x+2}{x^2+3x-10}$, has an oblique asymptote. Use long division to divide the numerator by the denominator. The line, y = quotient, is the oblique asymptote. Does the rational function intersect its oblique asymptote? If so, at what point(s)?

5. Determine the oblique asymptote of $y = \dfrac{2x^3+5x^2-x-6}{x^2-x-20}$. Does the function intersect its oblique asymptote? If so, at what point(s)?

6. What is the rule that determines when a rational function has a horizontal asymptote and when it has an oblique asymptote?

7. When the degree of the numerator is two or more higher than the degree of the denominator, the end behavior of a rational function is modeled by a polynomial. Use long division to divide the numerator by the denominator. The polynomial is y = quotient. What is the polynomial that models the end behavior of the curve $y = \dfrac{x^3+2x^2-5x-6}{x-1}$ as x approaches infinity? Will the graph of the rational function intersect this polynomial? If so, at what point(s)?

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Answers:

1. $\lim\limits_{x\to\infty}\dfrac{2x-4}{x+3}=2$, so the horizontal asymptote is $y = 2$.
   To determine where the graph intersects $y = 2$, set these equations equal to each other and solve.
   $\dfrac{2x-4}{x+3}=2$,
   $2x - 4 = 2x + 6$
   No solution; therefore the graph does not intersect its horizontal asymptote.

2. $\lim\limits_{x\to\infty}\dfrac{2x-4}{x^2+3}=0$, so the horizontal asymptote is $y = 0$.
   To determine where the graph intersects $y = 0$, set these equations equal to each other and solve.
   $\dfrac{2x-4}{x^2+3}=0$, so $2x - 4 = 0$, and $x = 2$. The point of intersection is (2, 0) which is also the zero of the function.

3. To determine where the graph intersects $y = 1$, set these equations equal to each other and solve.
   $\dfrac{x^2+x-2}{x^2+3x-10}=1$
   $x^2+x-2=x^2+3x-10$
   $-2x = -8$
   $x = 4$.
   The point of intersection is (4, 1).

4. $(x^2+3x-10)\overline{\big)x^3-3x+2} = x-3+\dfrac{16x-28}{x^2+3x-10}$
   Ignoring the fractional part, the oblique asymptote is $y = x - 3$.
   To determine if the curve intersects this asymptote, set the curves equal and solve.

   $\dfrac{x^3-3x+2}{x^2+3x-10}=x-3$
   $x^3-3x+2=x^3-19x+30$
   $16x=28$
   $x=1.75 \quad and \quad y=-1.25$

   The point of intersection is (1.75, –1.25).

5. $(x^2 - x - 20)\overline{)2x^3 + 5x^2 - x - 6} = 2x + 7 + \dfrac{46x + 134}{x^2 - x - 20}$

Ignore the fractional part, the oblique asymptote is $y = 2x + 7$.
To determine if the curve intersects this asymptote, set the curves equal and solve.

$$\frac{2x^3 + 5x^2 - x - 6}{x^2 - x - 20} = 2x + 7$$

$$2x^3 + 5x^2 - x - 6 = 2x^3 + 5x^2 - 47x - 140$$

$$x = -2.913, \quad y = 1.174$$

The point of intersection is (–2.913, 1.174)

6. The following rules apply:
   - If the degree of the numerator and denominator are the same, then the horizontal asymptote is the ratio of the leading coefficients.
   - If the degree of the numerator is less than the degree of the denominator, then the horizontal asymptote is $y = 0$.
   - If the degree of the numerator is one degree higher than the degree of the denominator, then the asymptote is oblique.

7. $(x - 1)\overline{)x^3 + 2x^2 - 5x - 6} = x^2 + 3x - 2 + \dfrac{-8}{x - 1}$

Ignoring the fractional part, the end behavior polynomial is $y = x^2 + 3x - 2$.

Setting the polynomial equation equal to the function reveals that there is no solution; therefore the rational function is asymptotic to the polynomial, but the curves do not intersect.

Note: To determine if the rational function approaches the polynomial from above or below, substitute large positive values and large negative values in for x into the fraction, $\dfrac{-8}{x-1}$. If the value is positive the rational function will approach the polynomial from above. If value is negative, then the rational function approaches the polynomial from below.

