Solving Systems of Linear Equations Using Matrices

What is a Matrix?

A matrix is a compact grid or array of numbers. It can be created from a system of equations and used to solve the system of equations. Matrices have many applications in science, engineering, and math courses. This handout will focus on how to solve a system of linear equations using matrices.

How to Solve a System of Equations Using Matrices

Matrices are useful for solving systems of equations. There are two main methods of solving systems of equations: Gaussian elimination and Gauss-Jordan elimination. Both processes begin the same way. To begin solving a system of equations with either method, the equations are first changed into a matrix. The *coefficient matrix* is a matrix comprised of the coefficients of the variables which is written such that each row represents one equation and each column contains the coefficients for the same variable in each equation. The *constant matrix* is the solution to each of the equations written in a single column and in the same order as the rows of the coefficient matrix. The *augmented matrix* is the coefficient matrix with the constant matrix as the last column.

> **Example**: Write the coefficient matrix, constant matrix, and augmented matrix for the following system of equations:
>
> $$-3x - 2y + 4z = 9$$
> $$3y - 2z = 5$$
> $$4x - 3y + 2z = 7$$
>
> **Solution**: The coefficient matrix is created by taking the coefficients of each variable and entering them into each row. The first equation will be the first row; the second equation will be the second row, and the third equation will be the third row. Also, the first column will represent the "x" variable; the second column will represent the "y" variable, and the third column will represent the "z" variable.
>
> $$\begin{bmatrix} -3 & -2 & 4 \\ 0 & 3 & -2 \\ 4 & -3 & 2 \end{bmatrix}$$

Because the second equation does not contain an "x" variable, a "0" has been entered into the "x" column in the second row.

The constant matrix is a single column matrix consisting of the solutions to the equations.

$$\begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

To create the augmented matrix, add the constant matrix as the last column of the coefficient matrix.

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 9 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right]$$

For the <u>Gaussian elimination</u> method, once the augmented matrix has been created, use elementary row operations to reduce the matrix to Row-Echelon form. There are three basic types of elementary row operations: (1) row swapping, (2) row multiplication, and (3) row addition. Row multiplication and row addition can be combined together.

(1) In *row swapping*, the rows exchange positions within the matrix. The matrix resulting from a row operation or sequence of row operations is called row equivalent to the original matrix.

    Example: Swap row one and row three

    Solution:

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 9 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 4 & -3 & 2 & 7 \\ 0 & 3 & -2 & 5 \\ -3 & -2 & 4 & 9 \end{array}\right]$$

(2) In *row multiplication*, every entry in a row is multiplied by the same constant.

    Example: Multiply row one by $-\frac{1}{3}$

    Solution:

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 9 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2/3 & -4/3 & -3 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right]$$

(3) In *row addition*, the column elements of row "A" are added to the column elements of row "B". The resulting sums replace the column elements of row "B" while row "A" remains unchanged.

Example: Add row one to row two

Solution:

$$\begin{bmatrix} -3 & -2 & 4 & | & 9 \\ 0 & 3 & -2 & | & 5 \\ 4 & -3 & 2 & | & 7 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} -3 & -2 & 4 & | & 9 \\ 0 + (-3) & 3 + (-2) & -2 + 4 & | & 5 + 9 \\ 4 & -3 & 2 & | & 7 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} -3 & -2 & 4 & | & 9 \\ -3 & 1 & 2 & | & 14 \\ 4 & -3 & 2 & | & 7 \end{bmatrix}$$

The previous examples all started from the original augmented matrix. In order to solve a system of equations, these row operations are performed back to back on the resulting matrix, instead of returning to the original matrix each time, until Row-Echelon form is achieved.

Row-Echelon form is characterized by having the furthest left non-zero entry in a row, the leading entry, with all zeros below it, and the leading entry of each row is in a column to the right of the leading entry in the row above it. For Pre-Calculus students, the leading entry in each row should be reduced to a 1; for Linear Algebra students, this leading entry could be any number unless otherwise specified in your assignment.

Example: Are the following matrices in Row-Echelon form?

a) $\begin{bmatrix} 1 & -9 & 2 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 7 & -6 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 6 & -8 & 11 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 5 & 12 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$

Solution a): No, this matrix is not in Row-Echelon form since the leading entry in row three is in a column to the left of the leading entry in row two. **Please note**: If we swapped row two and row three, then the matrix would be in Row-Echelon form.

Solution b): Yes, this matrix is in Row-Echelon form as the leading entry in each row has 0's below, and the leading entry in each row is to the right of the leading entry in the row above. Notice the leading entry for row three is in column 4 not column 3. The leading

entry **is** allowed to skip columns, but it **cannot** be to the left of the leading entry in any row above it.

Solution c): Yes, this matrix is in Row-Echelon form. Each leading entry in each row is to the right of the leading entry in the row above it, and each leading entry contains only 0's below it.

The following example will demonstrate how to use the elementary row operations to reduce the augmented matrix from a system of equations to Row-Echelon form. After Row-Echelon form is achieved, back substitution can be used to find the solution to the system of equations.

Example: Solve the following system of equations using Gaussian Elimination:

$$-3x - 2y + 4z = 9$$
$$3y - 2z = 5$$
$$4x - 3y + 2z = 7$$

Solution: First, create the augmented matrix for the system.

$$\begin{bmatrix} -3 & -2 & 4 & | & 9 \\ 0 & 3 & -2 & | & 5 \\ 4 & -3 & 2 & | & 7 \end{bmatrix}$$

Next use the elementary row operations to reduce the matrix to Row-Echelon form.

$$\begin{bmatrix} -3 & -2 & 4 & | & 9 \\ 0 & 3 & -2 & | & 5 \\ 4 & -3 & 2 & | & 7 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{4}{3} & | & -3 \\ 0 & 3 & -2 & | & 5 \\ 4 & -3 & 2 & | & 7 \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{4}{3} & | & -3 \\ 0 & 3 & -2 & | & 5 \\ 0 & -\frac{17}{3} & \frac{22}{3} & | & 19 \end{bmatrix} \xrightarrow{-\frac{3}{17}R_3}$$

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{4}{3} & | & -3 \\ 0 & 3 & -2 & | & 5 \\ 0 & 1 & -\frac{22}{17} & | & -\frac{57}{17} \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{4}{3} & | & -3 \\ 0 & 1 & -\frac{22}{17} & | & -\frac{57}{17} \\ 0 & 3 & -2 & | & 5 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{4}{3} & | & -3 \\ 0 & 1 & -\frac{22}{17} & | & -\frac{57}{17} \\ 0 & 0 & \frac{32}{17} & | & \frac{256}{17} \end{bmatrix} \xrightarrow{\frac{17}{32}R_3}$$

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{4}{3} & \Big| & -3 \\ 0 & 1 & -\frac{22}{17} & \Big| & -\frac{57}{17} \\ 0 & 0 & 1 & \Big| & 8 \end{bmatrix}$$

Finally, rewrite the matrix as a system of reduced equations and back substitute to find the solution.

$$1x + \frac{2}{3}y - \frac{4}{3}z = -3$$
$$1y - \frac{22}{17}z = -\frac{57}{17}$$
$$1z = 8$$

The reduced equations show that $z = 8$. Substitute 8 for z and solve for y in the second equation.

$$1y - \frac{22}{17}(8) = -\frac{57}{17}$$
$$y - \frac{176}{17} = -\frac{57}{17}$$
$$y = 7$$

Substitute 8 for z and 7 for y in the first equation and solve for x.

$$x + \frac{2}{3}(7) - \frac{4}{3}(8) = -3$$
$$x + \frac{14}{3} - \frac{32}{3} = -3$$
$$x - 6 = -3$$
$$x = 3$$

The solution to the system of equations is $(3,7,8)$.

An alternative method, the Gauss-Jordan elimination method, can be used to solve the system of equations. This involves reducing the augmented matrix to Reduced Row-Echelon form. The Reduced Row-Echelon form is similar to the Row-Echelon form except that the leading entry in each row must be a 1 and all other entries in the same column as a leading entry must be 0. Unlike the Row-Echelon form, there is one and only one Reduced Row-Echelon form for a system of equations.

Example: Solve the following system of equations using Gauss-Jordan Elimination:

$$-3x - 2y + 4z = 9$$

$$3y - 2z = 5$$

$$4x - 3y + 2z = 7$$

Solution: First, create the augmented matrix for the system.

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 9 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right]$$

Next, use the elementary row operations to reduce the matrix to Reduced Row-Echelon form.

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 9 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 3 & -2 & 5 \\ 4 & -3 & 2 & 7 \end{array}\right] \xrightarrow{-4R_1+R_3} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 3 & -2 & 5 \\ 0 & -\frac{17}{3} & \frac{22}{3} & 19 \end{array}\right] \xrightarrow{-\frac{3}{17}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 3 & -2 & 5 \\ 0 & 1 & -\frac{22}{17} & -\frac{57}{17} \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 1 & -\frac{22}{17} & -\frac{57}{17} \\ 0 & 3 & -2 & 5 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 1 & -\frac{22}{17} & -\frac{57}{17} \\ 0 & 0 & \frac{32}{17} & \frac{256}{17} \end{array}\right] \xrightarrow{\frac{17}{32}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 1 & -\frac{22}{17} & -\frac{57}{17} \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow{\frac{22}{17}R_3+R_2} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{4}{3} & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow{\frac{4}{3}R_3+R_1} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & 0 & \frac{23}{3} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow{-\frac{2}{3}R_2+R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

The solution to the system can be written directly from the Reduced Row-Echelon form by converting the matrix back to equation form.

$$x = 3$$

$$y = 7$$

$$z = 8$$

Thus, the solution to the system of equations is $(3,7,8)$. This is the same solution obtained by using the Gaussian elimination method in the previous example.

The system of equations above is an example of a consistent system of equations. A *consistent system of equations* is characterized by having a leading coefficient in each column of the coefficient matrix

when it is row reduced to either Row-Echelon form or Reduced Row-Echelon form. In other words, each variable represented by a column can be solved for a specific number. With an *inconsistent system of equations*, the leading coefficient in one of the rows will be in the last column of the augmented matrix.

Example: Determine if the following system of equations is consistent or inconsistent and state the solution.

$$2x - 4y + z = 3$$
$$x - 3y + z = 5$$
$$3x - 7y + 2z = 12$$

Solution: First, create the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 3 \\ 1 & -3 & 1 & 5 \\ 3 & -7 & 2 & 12 \end{array}\right]$$

Use the elementary row operations to obtain a Row-Echelon form.

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 3 \\ 1 & -3 & 1 & 5 \\ 3 & -7 & 2 & 12 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 2 & -4 & 1 & 3 \\ 3 & -7 & 2 & 12 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -7 \\ 3 & -7 & 2 & 12 \end{array}\right] \xrightarrow{-3R_1+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -7 \\ 0 & 2 & -1 & -3 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 0 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 0 & 4 \end{array}\right]$$

The last row indicates the system is inconsistent. This can most easily be seen if the last row is converted back to an equation.

$$0x + 0y + 0z = 4$$

According to this equation, there are not any values of x, y, or z that will make the above equation true. Therefore, the system has no solution; this is represented by the symbol for the null set, $\emptyset$. Any augmented system of equations is inconsistent if the Row-Echelon form

contains a row with the coefficient portion of the row containing all 0's and the augmented column containing any number except 0.

A system of equations can also be dependent. In the case of a *dependent system*, one of the columns of the coefficient portion of the augmented matrix will lack a leading coefficient. In some cases, the row corresponding to the missing leading coefficient will contain only 0's. In other cases, there will be fewer rows than columns. Be careful because the presence of a row of 0's does not automatically

indicate a dependent system. If a system of three equations contains only two variables, then a row of 0's does not indicate a dependent system. However, a system of three equations with three variables that contains a row of 0s indicates a dependent system.

Example 1: Determine if the following system of equations is consistent, inconsistent, or dependent, and state the solution.

$$2x - 3y = -21$$
$$3x + 2y = 1$$
$$8x - 5y = -49$$

Solution: Create the augmented matrix and use elementary row operations to reduce the ` matrix to Row-Echelon form.

$$\begin{bmatrix} 2 & -3 & | & -21 \\ 3 & 2 & | & 1 \\ 8 & -5 & | & -49 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -\frac{3}{2} & | & -\frac{21}{2} \\ 3 & 2 & | & 1 \\ 8 & -5 & | & -49 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & -\frac{3}{2} & | & -\frac{21}{2} \\ 0 & \frac{13}{2} & | & \frac{65}{2} \\ 8 & -5 & | & -49 \end{bmatrix} \xrightarrow{-8R_1+R_3}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & | & -\frac{21}{2} \\ 0 & \frac{13}{2} & | & \frac{65}{2} \\ 0 & 7 & | & 35 \end{bmatrix} \xrightarrow{\frac{2}{13}R_2} \begin{bmatrix} 1 & -\frac{3}{2} & | & -\frac{21}{2} \\ 0 & 1 & | & 5 \\ 0 & 7 & | & 35 \end{bmatrix} \xrightarrow{-7R_2+R_3} \begin{bmatrix} 1 & -\frac{3}{2} & | & -\frac{21}{2} \\ 0 & 1 & | & 5 \\ 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{\frac{3}{2}R_2+R_1} \begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 5 \\ 0 & 0 & | & 0 \end{bmatrix}$$

This system of equations is consistent even though there is a row of 0's at the bottom. The original system of equations had only two variables, x and y, thus the solution to the system only contains two numbers. The solution to the system of equations is $(-3, 5)$.

Example 2: Determine if the following system of equations is consistent, inconsistent, or dependent, and state the solution.

$$8x + 5y + 11z = 30$$
$$-x - 4y + 2z = 3$$
$$2x - y + 5z = 12$$

Solution 2: Create the augmented matrix and use elementary row operations to reduce the matrix to Row-Echelon form.

$$\left[\begin{array}{ccc|c} 8 & 5 & 11 & 30 \\ -1 & -4 & 2 & 3 \\ 2 & -1 & 5 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & -4 & 2 & 3 \\ 8 & 5 & 11 & 30 \\ 2 & -1 & 5 & 12 \end{array}\right] \xrightarrow{-1R_1} \left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 8 & 5 & 11 & 30 \\ 2 & -1 & 5 & 12 \end{array}\right] \xrightarrow{-8R_1+R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 0 & -27 & 27 & 54 \\ 2 & -1 & 5 & 12 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 0 & -27 & 27 & 54 \\ 0 & -9 & 9 & 18 \end{array}\right] \xrightarrow{-\frac{1}{27}R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & -9 & 9 & 18 \end{array}\right] \xrightarrow{9R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is dependent because the "z" column does not have a leading coefficient, and the last row of the matrix contains only 0s. Next, solve for the dependent solution to the system of equations.

To begin, write the resulting system of equations.

$$x + 4y - 2z = -3$$
$$y - z = -2$$
$$0 = 0$$

Then, solve the second equation for y in terms of z.

$$y - z = -2$$
$$y = z - 2$$

Substitute the result for y into the first equation, and solve for x in terms of z.

$$x + 4y - 2z = -3$$

$$x + 4(z - 2) - 2z = -3$$
$$x + 4z - 8 - 2z = -3$$
$$x + 2z - 8 = -3$$
$$x - 8 = -2z - 3$$
$$x = -2z + 5$$

The solution to the dependent system of equations is $(-2z + 5, z - 2, z)$.

Practice Problems

Solve the following systems of equations by:

Gaussian Elimination

1.
$$2x - 3y + 2z = 13$$
$$3x + y - z = 2$$
$$3x - 4y - 3z = 1$$

2.
$$x + 3y + 4z = 11$$
$$2x + 3y + 2z = 7$$
$$4x + 9y + 10z = 20$$
$$3x - 2y + z = 1$$

3.
$$t - u + 2v - 3w = 9$$
$$4t + 11v - 10w = 46$$
$$3t - u + 8v - 6w = 27$$

Gauss-Jordan Elimination:

4.
$$x - 3y + z = 8$$
$$2x - 5y - 3z = 2$$
$$x + 4y + z = 2$$

5.
$$x - 3y + 2z = 0$$
$$2x - 5y - 2z = 0$$
$$4x - 11y + 2z = 0$$

6.
$$2x + 5y + 2z = -1$$
$$x + 2y - 3z = 5$$
$$5x + 12y + z = 10$$

Solutions

1. $(2, -1, 3)$

2. No Solution

3. $\left(\frac{27}{2}c + 39, \frac{5}{2}c + 10, -4c - 10, c\right)$

4. $\left(\frac{12}{5}, -1, \frac{13}{5}\right)$

5. $(16c, 6c, c)$

6. No Solution