Section 5.4 Properties of Rational Functions

Rational Function

- A rational function is a function of the form $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$and $Q(x)$ are polynomial functions and $Q(x) \neq 0$.
- Domain is the set of all real numbers x except the value(s) that make(s) $Q(x) = 0$.

Vertical Asymptotes

- **Vertical Asymptote** is the line $\boldsymbol{x = a}$ if $f(x)$ increases or decreases without bound as x approaches a.
- Location: At where the denominator of a rational function is equal to 0.

Horizontal Asymptotes

- **Horizontal Asymptote** is the line $\boldsymbol{y = b}$ if $f(x)$ approaches b as x increases or decreases without bound.
- Location: let n and m be the degree of the numerator and denominator, respectively.
  1. If $\boldsymbol{n < m}$ (the degree of denominator is greater than the degree of numerator), the **horizontal asymptote** of the graph is the ***x*-axis**, or $\boldsymbol{y = 0}$.
  2. If $\boldsymbol{n = m}$ (Same degree), the **horizontal asymptote** of the graph is
  $$y = \frac{\text{The leading coefficient of numerator}}{\text{The leading coefficient of Denominator}}$$
  3. If $\boldsymbol{n > m}$ (the degree of numerator is greater than the degree of denominator), the graph has **no horizontal asymptote**.

Oblique Asymptotes

- The graph of a rational function has an **oblique asymptote** if **the degree of the numerator is ONE MORE than the degree of the denominator.**
- The oblique asymptote(s) can be obtained by the long division.
- The quotient resulted from the long division is the oblique asymptote.

Exercises

1. Find the domain of the following rational function. $H(x) = \frac{-8x^2}{(x-2)(x+9)}$ Select the correct choice below and, if necessary, fill in the answer box to complete your choice. A. The domain of H(x) is $\{x \mid x \neq 2, -9\}$. (Use a comma to separate answers as needed.) B. The domain of H(x) has no restrictions.	(Solution 1) The rational function $R(x) = \frac{P(x)}{Q(x)}$, $Q(x) \neq 0$ Set Denominator = 0, then find zeros. $(x-2)(x+9) = 0$ $(x-2)$ and $(x+9)$ are factors of the equation, thus zeros are $x = 2$, $x = -9$. Thus, x cannot be 2 and -9.

2. Find the domain of the following rational function. $R(x) = \frac{3(x^2-2x-48)}{4(x^2-64)}$ Select the correct choice below and fill in any answer boxes within your choice. A. The domain of R(x) is $\{x \mid x \neq 8, -8\}$. (Use a comma to separate answers as needed.) B. There are no restrictions on the domain of R(x).	(Solution 2) Set Denominator = 0, then find zeros. $x^2 - 64 = 0$ $+64 \quad +64$ $x^2 = 64$ $\sqrt{x^2} = \pm\sqrt{64}$ $x = \pm 8$ Thus, x cannot be 8 and -8.

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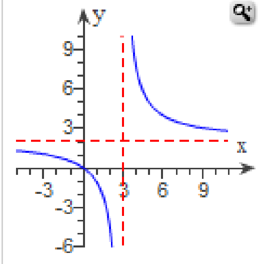
3. Use the graph shown to find the following.
   (a) The domain and range of the function
   (b) The intercepts, if any
   (c) Horizontal asymptotes, if any
   (d) Vertical asymptotes, if any
   (e) Oblique asymptotes, if any



(a) What is the domain? Select the correct choice below and fill in any answer boxes within your choice.

A. All real numbers x except x = 3 (Use a comma to separate answers as needed.)

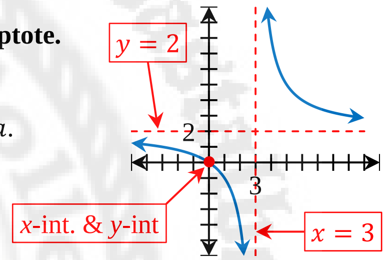
B. All real numbers x.

What is the range? Select the correct choice below and fill in any answer boxes within your choice.

A. All real numbers y except y = 2 (Use a comma to separate answers as needed.)

B. All real numbers y

(Solution 3.a) **A equation must be in simplest form for finding a vertical asymptote.**
If a function is not continuous at a point in its domain or range, the point must be removed from the domain or range of the function.
For the domain of a function, a discontinuous point is at a vertical asymptote $x = a$.
The vertical asymptote is $x = 3$, thus $x = 3$ is **not included** in the domain.
For the range of a function, a discontinuous point is at a vertical asymptote $y = b$.
The horizontal asymptote is $y = 2$, thus $y = 2$ is **not included** in the range.



(b) Find the x-intercepts, if there are any. Select the correct choice below and fill in any answer boxes within your choice.

A. x = 0 (Use a comma to separate answers as needed.) B. There are no x-intercepts.

Find the y-intercepts, if there are any. Select the correct choice below and fill in any answer boxes within your choic

A. y = 0 (Use a comma to separate answers as needed.) B. There are no y-intercepts.

(c) Find the horizontal asymptotes, if there are any. Select the correct choice below and fill in any answer boxes within your choice.

A. y = 2 (Use a comma to separate answers as needed.)

(d) Find the vertical asymptotes, if there are any. Select the correct choice below and fill in any answer boxes within your choice.

A. x = 3 (Use a comma to separate answers as needed.)

B. There are no vertical asymptotes.

(e) Find the oblique asymptotes, if there are any. Select the correct choice below and fill in any answer boxes within your choice.

A. y = (Simplify your answer. Use a comma to separate answers as needed.)

B. There are no oblique asymptotes.

(Solution 3.e)
The graph has a horizontal asymptote if the degree of its denominator is greater than that of the numerator.
The graph has an oblique asymptote if the degree of its numerator is one more than that of the denominator.
Thus, there is no function that has both the horizontal and oblique asymptote.
Because the given function has a horizontal asymptote, no oblique asymptote for the given function.

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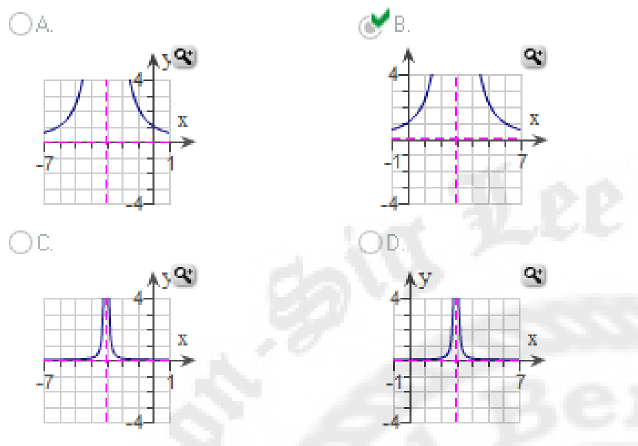
4. Graph the following rational function using transformations.

$$R(x) = \frac{9}{(x-3)^2}$$

Select the correct graph.



(Solution 4)

A simplest equation must be used for finding a vertical asymptote.

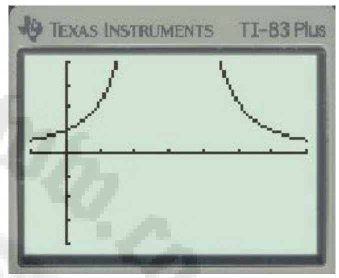
Vertical Asymptote at $Denominator = 0$

$(x-3)^2 = 0 \Rightarrow x = 3$

Thus, the vertical asymptote is $\boldsymbol{x = 3}$

The horizontal asymptote is $\boldsymbol{y = 0}$ because the degree of the denominator is greater than that of the numerator.

By TI-83:

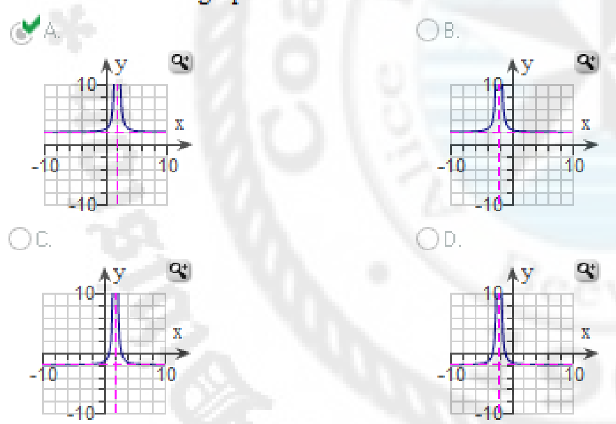


$[-1, 7, 1]$by$[-4, 4, 1]$

5. Graph the following rational function using transformations.

$$G(x) = 2 + \frac{2}{(x-2)^2}$$

Select the correct graph.



(Solution 5)

$$G(x) = 2 + \frac{2}{(x-2)^2} = \frac{\mathbf{2(x-2)^2+2}}{\mathbf{(x-2)^2}}$$

$\mathbf{2(x-2)^2+2} \Rightarrow$ Degree is 2.

$\mathbf{(x-2)^2} \Rightarrow$ Degree is 2.

Because 'Numerator Degree = Denominator Degree, the horizontal asymptote is

$$y = \frac{\text{Leading Coefficient of Numerator}}{\text{Leading Coefficient of Denominator}} = \frac{2}{1} = 2$$

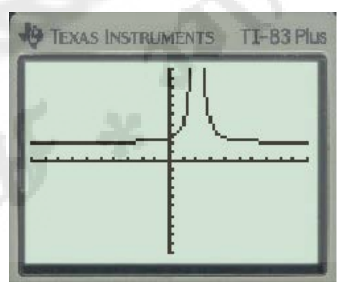
The horizontal asymptote is $\boldsymbol{y = 2}$

A simplest equation must be used for finding a vertical asymptote.

Vertical Asymptote at $Denominator = 0$

$(x-2)^2 = 0 \Rightarrow \boldsymbol{x = 2}$

TI-83:



6. Find the vertical, horizontal, and oblique asymptotes, if any, for the following rational function.

$$R(x) = \frac{15x}{x+10}$$

Select the correct choice below and fill in any answer boxes within your choice.

A. The vertical asymptote(s) is/are x = -10 . (Use a comma to separate answers as needed.)

B. There is no vertical asymptote.

Select the correct choice below and fill in any answer boxes within your choice.

A. The horizontal asymptote(s) is/are y = 15 . (Use a comma to separate answers as needed.)

B. There is no horizontal asymptote.

Select the correct choice below and fill in any answer boxes within your choice.

A. The oblique asymptote(s) is/are y = . (Use a comma to separate answers as needed.)

B. There is no oblique asymptote.

(Solution 6)

A simplest equation must be used for finding a vertical asymptote. Vertical asymptote at Denominator = 0
$x + 10 = 0 \Rightarrow x = -10$. Therefore, the vertical asymptote $\boldsymbol{x = -10}$.

Because 'Numerator Degree = Denominator Degree, the horizontal asymptote is $\boldsymbol{y = 15}$

$$y = \frac{\text{Leading Coefficient of Numerator}}{\text{Leading Coefficient of Denominator}} = \frac{15}{1} = 15$$

A function has the oblique asymptote(s) if the degree of numerator is **ONE** more than that of denominator.
Because 'Numerator Degree = Denominator Degree, there is no oblique asymptote.

7. Find the vertical, horizontal, and oblique asymptotes, if any, for the given rational function.

$$R(x) = \frac{6x^2 + 7x - 20}{2x + 5}$$

Select the correct choice below and fill in any answer boxes within your choice.

- A. The vertical asymptote(s) is/are x = ___.
  (Use a comma to separate answers as needed. Use integers or fractions for any numbers in the expression.)
- B. There is no vertical asymptote. ✔

Select the correct choice below and fill in any answer boxes within your choice.

- A. The horizontal asymptote(s) is/are y = ___.
  (Use a comma to separate answers as needed. Use integers or fractions for any numbers in the expression.)
- B. There is no horizontal asymptote. ✔

Select the correct choice below and fill in any answer boxes within your choice.

- A. The oblique asymptote(s) is/are y = 3x − 4. ✔
  (Use a comma to separate answers as needed. Use integers or fractions for any numbers in the expression.)
- B. There is no oblique asymptote.

(Solution 7)
An equation must be in the simplest form for finding a vertical asymptote.

$$R(x) = \frac{6x^2 + 7x - 20}{2x + 5} = \frac{(2x + 5)(3x - 4)}{2x + 5} = 3x - 4$$

Because the function $R(x)$ has no variable in its denominator, the function is continuous for all real numbers. Therefore, there is not vertical asymptote.

No vertical asymptote because the degree of the numerator is greater than the degree of the denominator,

The function has the oblique asymptote(s) because the degree of the numerator is **ONE** more than the degree of the denominator. Apply the long division to find the oblique asymptote.
If the given function is reducible, apply the long division after simplifying the function. **The oblique asymptote is the quotient resulted from the long division.**
Therefore, the oblique asymptote is $\boldsymbol{y = 3x - 4}$

① $\frac{6x^2}{2x}$ ③ $\frac{-8x}{2x}$

$$\begin{array}{r|l} & 3x - 4 \\ \hline 2x + 5 & 6x^2 + 7x - 20 \\ & \underline{-6x^2 - 15x} \quad \xleftarrow{\times(-1)} ②\ 3x(2x+5) = 6x^2 + 15x \\ & -8x - 20 \\ & \underline{8x + 20} \quad \xleftarrow{\times(-1)} ④\ -4(2x+5) = -8x - 20 \\ & 0 \end{array}$$