Riemann Sums

2003 SCORING GUIDELINES (Form B)

Question 3

CALCULATOR

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\frac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

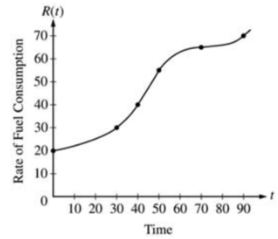
(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

2003 SCORING GUIDELINES

Question 3

CALCULATOR

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

2004 SCORING GUIDELINES (Form B)

Question 3

CALCULATOR

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

CALCULATOR

2005 SCORING GUIDELINES

Question 3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

2006 SCORING GUIDELINES

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

NO CALCULATOR

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\, dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\, dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\frac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360} \int_0^{360} \frac{B(x)}{2}\, dx$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $\frac{1}{360}\left[120\left(\frac{B(60)}{2} + \frac{B(180)}{2} + \frac{B(300)}{2} \right) \right] =$

$\frac{1}{360}[60(30 + 30 + 24)] = 14$

$2 : \begin{cases} 1 : B(60) + B(180) + B(300) \\ 1 : \text{answer} \end{cases}$

(c) $\frac{B(x)}{2}$ is the radius, so $\pi \left(\frac{B(x)}{2} \right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

$2 : \begin{cases} 1 : \text{volume in mm}^3 \\ 1 : \text{between } x = 125 \text{ and } x = 275 \end{cases}$

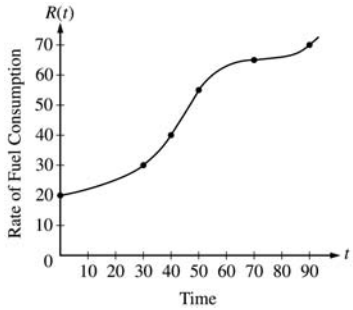
(d) By the MVT, $B'(c_1) = 0$ for some c_1 in $(60, 180)$ and $B'(c_2) = 0$ for some c_2 in $(240, 360)$. The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2).

$3 : \begin{cases} 2 : \text{explains why there are two values of } x \text{ where } B'(x) \text{ has the same value} \\ 1 : \text{explains why that means } B''(x) = 0 \text{ for } 0 < x < 360 \end{cases}$

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in $(0, 360)$.

Question 3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

(a) $R'(45) \approx \dfrac{R(50) - R(40)}{50 - 40} = \dfrac{55 - 40}{10}$
$= 1.5 \text{ gal/min}^2$

$2 : \begin{cases} 1 : \text{a difference quotient using numbers from table and interval that contains 45} \\ 1 : 1.5 \text{ gal/min}^2 \end{cases}$

(b) $R''(45) = 0$ since $R'(t)$ has a maximum at $t = 45$.

$2 : \begin{cases} 1 : R''(45) = 0 \\ 1 : \text{reason} \end{cases}$

(c) $\int_0^{90} R(t)\,dt \approx (30)(20) + (10)(30) + (10)(40) + (20)(55) + (20)(65) = 3700$

Yes, this approximation is less because the graph of R is increasing on the interval.

$2 : \begin{cases} 1 : \text{value of left Riemann sum} \\ 1 : \text{"less" with reason} \end{cases}$

(d) $\int_0^{b} R(t)\,dt$ is the total amount of fuel in gallons consumed for the first b minutes.

$\frac{1}{b}\int_0^{b} R(t)\,dt$ is the average value of the rate of fuel consumption in gallons/min during the first b minutes.

$3 : \begin{cases} 2 : \text{meanings} \\ \quad 1 : \text{meaning of } \int_0^b R(t)\,dt \\ \quad 1 : \text{meaning of } \frac{1}{b}\int_0^b R(t)\,dt \\ \quad < -1 > \text{ if no reference to time } b \\ 1 : \text{units in both answers} \end{cases}$

Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

(a) Midpoint Riemann sum is $10\cdot[v(5) + v(15) + v(25) + v(35)]$ $= 10\cdot[9.2 + 7.0 + 2.4 + 4.3] = 229$. The integral gives the total distance in miles that the plane flies during the 40 minutes.	$3: \begin{cases} 1: v(5)+v(15)+v(25)+v(35) \\ 1: \text{answer} \\ 1: \text{meaning with units} \end{cases}$
(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.	$2: \begin{cases} 1: \text{two instances} \\ 1: \text{justification} \end{cases}$
(c) $f'(23) = -0.407$ or -0.408 miles per minute^2	1 : answer with units
(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$ $= 5.916$ miles per minute	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

Question 3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8) - T(6)}{8 - 6} = \dfrac{55 - 62}{2} = -\dfrac{7}{2}$ °C/cm

1 : answer

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100 + 93}{2}\cdot 1 + \frac{93 + 70}{2}\cdot 4 + \frac{70 + 62}{2}\cdot 1 + \frac{62 + 55}{2}\cdot 2$$

Average temperature $\approx \dfrac{1}{8}A = 75.6875$°C

$3 : \begin{cases} 1 : \dfrac{1}{8}\int_0^8 T(x)\,dx \\ 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45$°C

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning} \end{cases}$

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70 - 93}{5 - 1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62 - 70}{6 - 5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

$2 : \begin{cases} 1 : \text{two slopes of secant lines} \\ 1 : \text{answer with explanation} \end{cases}$

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20} \text{ ft/sec}^2$$

1 : answer

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is

$$20[v(20) + v(40) + v(60)]$$
$$= 20[22 + 35 + 44] = 2020 \text{ ft}$$

$$3 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{uses } v(20),\ v(40),\ v(60) \\ 1 : \text{value} \end{cases}$$

(c) Let $v_B(t)$ be the velocity of rocket B at time t.

$$v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$$
$$2 = v_B(0) = 6 + C$$
$$v_B(t) = 6\sqrt{t+1} - 4$$
$$v_B(80) = 50 > 49 = v(80)$$

Rocket B is traveling faster at time $t = 80$ seconds.

$$4 : \begin{cases} 1 : 6\sqrt{t+1} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{finds } v_B(80) \text{, compares to } v(80)\text{,} \\ \quad \text{and draws a conclusion} \end{cases}$$

Units of ft/sec^2 in (a) and ft in (b)

1 : units in (a) and (b)