AK High-Dimensional Projection Structural Theory: Formal Proofs of Lemmas

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Overview

This paper provides formal mathematical proofs of the lemmas introduced in the AK High-Dimensional Projection Structural Theory, reinforcing the logical rigor and structural reliability of the framework.

1 Formal Proofs of Lemmas

Lemma 1.1 (Structural Group Connectivity Lemma). *Given Axiom A2, if each group* $\{G_i\}$ *is a connected component and satisfies the adjacency condition* $\overline{G_i} \cap \overline{G_{i+1}} \neq \emptyset$*, then the projected space* $\mathcal{P} = \bigcup_i G_i$ *is connected.*

Proof. Each G_i is assumed to be a connected closed set. The adjacency ensures that closures of adjacent groups intersect nontrivially. As the union of a finite chain of connected sets with non-empty intersections remains connected, $\mathcal{P} = \bigcup_i G_i$ is connected. □

Lemma 1.2 (Local Smoothness Lemma). *Given Axiom A3, for any group* G_i*, if the structural stability function* $S_i(t)$ *exists and* $\lim_{t\to\infty} S_i(t) = 0$*, then the inverse map* $\Phi^{-1}|_{G_i}$ *is smooth.*

Proof. Assume $S_i(t)$ quantifies structural features such as topological persistence (e.g., PH distance) or analytical features (e.g., energy gradient). Convergence of $S_i(t)$ to zero implies the disappearance of structural perturbations over time, yielding a steady geometric/topological regime. Consequently, the inverse $\Phi^{-1}|_{G_i}$ preserves differentiability and continuity, hence is smooth. □

Lemma 1.3 (Inverse Projection Continuity Lemma). *Given Axiom A4 and the above lemmas, the inverse projection* Φ^{-1} *is continuous on each* G_i*, and hence on the whole space* $\mathcal{P}$*.*

Proof. Since Φ is a structure-preserving continuous map and each G_i satisfies structural stability, the restriction $\Phi^{-1}|_{G_i}$ is continuous. As the decomposition $\{G_i\}$ is MECE, the total map Φ^{-1} is a union of local continuous maps and hence globally continuous on $\mathcal{P}$. □

2 Main Theorem and Corollary

Theorem 2.1 (Global Smoothness Theorem (Complete Form)). *Given Axioms A1 through A4, and if each structural stability function* $S_i(t)$ *is monotonically decreasing and convergent, then the original space* $\mathcal{X}(t)$ *evolves smoothly over time.*

Proof. Lemma 1 shows that the projected space $\mathcal{P}$ is connected. Lemmas 2 and 3 establish smoothness on each group and continuity of the inverse map. Together, they imply global smoothness on $\mathcal{X}(t)$. $\square$

Corollary 2.2 (Non-Singularity Corollary)**.** *Under the above conditions, no singularities (blow-up or non-differentiable points) emerge in the evolution of the original space* $\mathcal{X}(t)$*.*

Proof. The preservation of smoothness implies absence of singularities. Therefore, continuity and differentiability hold globally for all t. $\square$