1.2.P1 If $\det A=0$, then $\cancel{S_n=\lambda_1\lambda_2\cdots\lambda}$ $E_n(A)=\det A=0$.

$\therefore S_n(A)=E_n(A)=0$.

Since $S_n(A)=\lambda_1\lambda_2\cdots\lambda_n$, $\exists i$, s.t. $\lambda_i=0$. $\therefore 0\in\sigma(A)$.

If $0\in\sigma(A)$, then $S_n(A)=\lambda_1\lambda_2\cdots\lambda_n=0=E(A)=\det A$.

$\therefore \det A=0 \Leftrightarrow 0\in\sigma(A)$.

1.2.P2 1> $[AB]_{ii}=\sum_{j=1}^{n}A_{ji}B_{ij}$

$\therefore tr(AB)=\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ji}B_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{n}B_{ij}A_{ji}=tr(BA)$.

2> $\because tr(AB)=tr(BA)$

$\therefore tr(S^{-1}AS)=tr(S^{-1}A\cdot S)=tr(SS^{-1}A)=trA$.

3> $\det(S^{-1}AS)=\det(S^{-1})\det(A)\det(S)=\det(S^{-1})\det(S)\det(A)$
$=\det(S^{-1}S)\det(A)=\det(I)\det(A)=\det(A)$.

1.2.P3 Suppose $D=\begin{bmatrix} d_{11} & & 0\\ & d_{22} & \\ 0 & & \ddots\ d_{nn}\end{bmatrix}$. Then $\lambda I-D=\begin{bmatrix}\lambda-d_{11} & & 0\\ & \ddots & \\ 0 & & \lambda-d_{nn}\end{bmatrix}$

$\therefore p_D(t)=|tI-D|=(t-d_{11})(t-d_{22})\cdots(t-d_{nn})$.

$p_D(D)=(D-d_{11}I)(D-d_{22}I)\cdots(D-d_{nn}I)$.

Easy to show that $\prod_{i=1}^{k}(D-d_{ii}I)=\begin{bmatrix}0_k & 0\\ 0 & D'\end{bmatrix}$, where D' is a diagonal matrix.

$\therefore p_D(D)=0_n=0$.

1.2.P4. $A^2=A\Rightarrow\lambda_1,\ldots,\lambda_n$ are integers $\Rightarrow\forall k=1,\ldots,n$, $S_k(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}$.

So the coefficients of $p_A(t)$ (i.e. $(-1)^k\underbrace{S_k(A)}_{(k=1,2,\ldots,n)}$ and 1) are integers.

1.2.P5 If A is a nilpotent matrix, then (1.1.P6) shows that $\lambda_1=\lambda_2=\cdots=\lambda_n=0$.

$\therefore tr(A)=\lambda_1+\cdots+\lambda_n=0$.

And for any $k=1,2,\ldots,n$, $S_k(\lambda_1,\ldots,\lambda_n)=0$.

$\therefore p_A(t)=t^n$.

扫描全能王 创建

1.2.P6 $\text{rank}(A-\lambda I)=n-1 \Rightarrow |A-\lambda I|=0 \Rightarrow \lambda\in\sigma(A)$.

If $\text{rank}(A-\lambda I)=n-1$, the multiplicity of λ is not certainly 1.

eg. $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\lambda=1$.

In fact, all $A=\begin{pmatrix}\lambda&1&&\\&\lambda&\ddots&\\&&\ddots&1\\&&&\lambda\end{pmatrix}$ are contradictory cases.

1.2.P7 $E_1(A)=5$, $E_2(A)=3+2+1=6$ (i、j不同且不相邻时，$|A[\{i,j\}]|=1$).

$E_3(A)=\begin{vmatrix}1&1&0\\1&1&1\\0&1&1\end{vmatrix}+\begin{vmatrix}1&1&0\\1&1&0\\0&0&1\end{vmatrix}+\begin{vmatrix}1&1&0\\1&1&0\\0&0&1\end{vmatrix}+\begin{vmatrix}1&0&0\\0&1&1\\0&1&1\end{vmatrix}+\dots$

Note that $\det A[\{i,j,k\}]\neq 0 \Leftrightarrow i,j,k$ are continuous integers.

So $E_3(A)=|\{(1,2,3),(2,3,4),(3,4,5)\}|=-3$ $\left(\begin{vmatrix}1&1&0\\1&1&1\\0&1&1\end{vmatrix}=-1\right)$.

$E_4(A)=\begin{vmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{vmatrix}+\begin{vmatrix}1&0&0&0\\0&1&1&0\\0&1&1&1\\0&0&1&1\end{vmatrix}+\begin{vmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{vmatrix}+\begin{vmatrix}1&1&0&0\\1&1&1&0\\0&1&1&0\\0&0&0&1\end{vmatrix}+$

$\begin{vmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{vmatrix}=-1-1+0-1-1=-4$.

$E_5(A)=\det A=\begin{vmatrix}0&1&0&0&0\\0&1&1&0&0\\1&1&1&1&0\\0&0&1&1&1\\0&0&0&1&1\end{vmatrix}=-\begin{vmatrix}0&0&0&0\\1&1&0&0\\0&1&1&1\\0&0&1&1\end{vmatrix}=0$.

$\therefore p_A(t)=t^5-5t^4+6t^3+3t^2-4t$.

1.2.P8. $p_{A+\lambda I}(t)=|tI-A-\lambda I|=|(t-\lambda)I-A|=p_A(t-\lambda)$.

Let $t=\lambda+\lambda_1, \lambda+\lambda_2, \dots, \lambda+\lambda_n$, then $p_A(t-\lambda)=0=p_{A+\lambda I}(t)$

$\therefore$ The eigenvalues of $A+\lambda I$ are $\lambda_1+\lambda, \dots, \lambda_n+\lambda$.

1.2.P9. $S_2(\lambda_1,\dots,\lambda_6)=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\dots+\lambda_1\lambda_6+\lambda_2\lambda_3+\dots+\lambda_2\lambda_6+\lambda_3\lambda_4+\dots+\lambda_5\lambda_6$.

$S_3(\lambda_1,\dots,\lambda_6)=\lambda_1\lambda_2\lambda_3+\dots+\lambda_1\lambda_2\lambda_6+\lambda_1\lambda_3\lambda_4+\dots+\lambda_1\lambda_5\lambda_6+\dots+\lambda_4\lambda_5\lambda_6$.

$S_4(\lambda_1,\dots,\lambda_6)=\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_6+\lambda_1\lambda_2\lambda_4\lambda_5+\dots+\lambda_1\lambda_4\lambda_5\lambda_6$
$+\dots+\lambda_3\lambda_4\lambda_5\lambda_6$.

$S_5(\lambda_1,\dots,\lambda_6)=\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6+\lambda_1\lambda_2\lambda_3\lambda_5\lambda_6+\lambda_1\lambda_2\lambda_4\lambda_5\lambda_6+\lambda_1\lambda_3\lambda_4\lambda_5\lambda_6+\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6$.

$S_6(\lambda_1,\dots,\lambda_6)=\lambda_1\lambda_2\dots\lambda_6$.

1.2.P10. The degree of $p_A(t)$ is odd and use the fact that

$\lambda \in \sigma(A) \Rightarrow \bar{\lambda} \in \sigma(A)$.

1.2.P11. The first conclusion can be drawn since $\deg p_A(t) \geq 1$.

1.1.P10 is the contradictory case of infinite dimensional.

If $F \neq \mathbb{C}$, suppose $F=\mathbb{R}$, $V=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, then $p_V(t)=t^2+1$. No zero in $\mathbb{R}$.

1.2.P12. (a) Let $\tilde{A} = \begin{bmatrix} t & & \\ & \ddots & \\ & & t \end{bmatrix}$ be a diagonal matrix.

Cauchy's expansion gives that

~~$\det A = a\det\tilde{A} - y^*(\operatorname{adj}\tilde{A})x$~~.

~~det~~ $\det(tI-A) = (t-a)\det\tilde{A} - y^*(\operatorname{adj}\tilde{A})x$.

$\det\tilde{A} = t^n$, $\operatorname{adj}\tilde{A} = \begin{bmatrix} t^{n-1} & & \\ & \ddots & \\ & & t^{n-1} \end{bmatrix} \Rightarrow y^*(\operatorname{adj}\tilde{A}) = t^{n-1}y^*$.

$\therefore p_A(t) = \det(tI-A) = (t-a)t^n - t^{n-1}y^*x = t^{n-1}(t^2-at-y^*x)$

(b) Suppose there's a submatrix of A (denoted by $\tilde{A}$) such that $\tilde{A} \in M_3$, $\det\tilde{A} \neq 0$. Then $\tilde{A}$ must contain a (or $\det\tilde{A}=0$).

Suppose $\tilde{A} = \begin{bmatrix} 0 & 0 & x_1\\ 0 & 0 & x_2\\ y_1 & y_2 & a \end{bmatrix}$. Then $\det\tilde{A} = x_1\begin{vmatrix} 0 & 0\\ y_1 & y_2 \end{vmatrix} = 0$.

$\therefore \operatorname{rank} A \leq 2$.

$\therefore \forall k \geq 3, E_k(A) = 0$.

$E_1(A) = a$, $E_2(A) = -y^*x$.

$\therefore p_A(t) = t^{n+1} - at^n - y^*xt^{n-1} = t^{n-1}(t^2-at-y^*x)$.

$p_A(t)=0 \Rightarrow t=0$ or $\frac{a\pm\sqrt{a^2+4y^*x}}{2}$, and the multiplicity of 0 is $n-1$.

$\therefore$ The eigenvalues of A are $(a\pm\sqrt{a^2+4y^*x})/2$ together with $n-1$ eigenvalues.

1.2.P13 (a) $\lambda I - A = \begin{bmatrix} \lambda I - B & x \\ y^* & \lambda - a \end{bmatrix}$.

$\therefore det(tI-A) = (t-a) det(tI-B) - y^* adj((tI-B))x$

$\Rightarrow p_A(t) = (t-a)p_B(t) - y^*(adj(tI-B))x$.

(b) If $B = \lambda I_n$, then $det(tI-B) = det((t-\lambda)I_n) = (t-\lambda)^n$.

$y^* adj((tI-B)) = (t-\lambda)^{n-1} y^*$

$\therefore p_A(t) = (t-a)(t-\lambda)^n - (t-\lambda)^{n-1} y^* x$

$= (t-\lambda)^{n-1}(t^2 - (a+\lambda)t + a\lambda - y^*x)$.

The zeros of $p_A(t)$ are 0, and $\frac{(a+\lambda) \pm \sqrt{(a-\lambda)^2 + 4y^*x}}{2}$,

and the multiplicity of $t=0$ is $n-1$.

1.2.P14 $tI - A = \begin{bmatrix} t-\lambda & \star & \star \\ 0 & t-\mu & 0 \\ 0 & \star & B \end{bmatrix}$

$\therefore det(tI-A) = (t-\lambda) det(\begin{bmatrix} t-\mu & 0 \\ \star & tI-B \end{bmatrix})$

$= (t-\lambda)(t-\mu)\cancel{det}(tI-B)$.

$\therefore p_A(t) = (t-\lambda)(t-\mu)p_B(t)$.

1.2.P15. If $x_1(t_0), \ldots, x_n(t_0)$ are linearly independent, then $W(t_0) \neq 0$. $\Rightarrow W(t) = W(t_0) e^{\int_{t_0}^{t} tr A(s) ds} \neq 0$

$\therefore x_1(t), \ldots, x_n(t)$ are linearly dependent for all t.

$tr BC = tr CB$ is used in the identity $tr((adj X(t))A(t)X(t))$ $= W(t) tr A(t)$.

1.2.P16. $det(A + txy^T) = det A + ty^T(adj A)x$

$\therefore \beta = y^T(adj A)x$.

$\begin{cases} f(t_1) = det A + \beta t_1 \\ f(t_2) = det A + \beta t_2 \end{cases} \Rightarrow det A = \frac{t_2 f(t_1) - t_1 f(t_2)}{t_2 - t_1}$.

令 $f(t) = det(A \cancel{+} txy^T)$.

则 $q(c) = f(c)$, $\cancel{q(d) = f(d)}$ $q(b) = f(b)$

$\therefore det A = (bf(c) - cf(b))/(b-c) = (bq(c) - cq(b))/b-c$ if $b \neq c$.

If $b=c$, then $det A = det(diag\{d_1-b, \ldots, d_n-b\} + bxy^T) = q(b) + by^T(adj(diag\{d_1-b, \ldots, d_n-b\}))x$ $= q(b) - bq'(b)$.

扫描全能王 创建

Http://www.zjut.edu.cn
浙江工業大學
Zhejiang University of Technology
地址：中国浙江省杭州市潮王路18号，310014（朝晖校区） 中国浙江省杭州市留和路288号，310023（屏峰校区）
18 Chaowang Road, Hangzhou, Zhejiang 310014, China (Zhaohui Campus) · 288 Liuhe Road, Hangzhou, Zhejiang 310023, China(Pingfeng Campus)

~~$p_A(t) =$~~

Let $\tilde{A} = \begin{bmatrix} t & -b & \cdots & -b \\ -c & t & & \vdots \\ \vdots & & \ddots & -b \\ -c & \cdots & -c & t \end{bmatrix}$, then $\det \tilde{A} = (bq(-c) - cq(-b))/(b-c) = (b(t+c)^n - c(t+b)^n)/(b-c)$

$\tilde{A} = tI - A$. $\therefore p_A(t) = \det \tilde{A} = ((b(t+c)^n - c(t+b)^n)/(b-c)$, if $b \neq c$.

If $b = c$, then $\det \tilde{A} = q(-b) + bq'(-b) = (t+b)^n - nb(t+b)^{n-1} = (t+b)^{n-1}(t-(n-1)b)$.

$\therefore p_A(t) = (t+b)^{n-1}(t-(n-1)b)$ if $b = c$.

1.2.P17 ~~$p_C(t) = \det(tI-C) = \det((tI-C)^T) = \det(tI-C^T) = \det(\begin{bmatrix} tI-B^T & 0_n \\ 0_n & \end{bmatrix}$~~

$p_C(t) = \det(tI - C) = \det(\begin{bmatrix} tI_n & -A \\ -B & tI_n \end{bmatrix}) = \det(t^2 I_n - AB) = \det(t^2 I_n - BA)$.

0.8.5.13 0.8.5.14

$\therefore p_C(t) = p_{AB}(t^2) = p_{BA}(t^2)$.

$p_{AB}(t^2) = p_{BA}(t^2) \Rightarrow$ The coefficients of p_{AB} and p_{BA} are the same.

$\therefore p_{AB}$ and p_{BA} have the same zeros $\Rightarrow$ AB and BA have the same eigenvalues. $\therefore \operatorname{tr} AB = \lambda_1 + \dots + \lambda_n = \mu_1 + \dots + \mu_n = \operatorname{tr} BA$,

$\sigma(AB)$ $\sigma(BA)$

$\det AB = \lambda_1 \dots \lambda_n = \mu_1 \dots \mu_n = \det BA$.

1.2.P20 shows that $\det(I+AB) = \det(I+BA)$.

1.2.P18. $E_1(A) = \operatorname{tr}(A)$, $E_2(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = A_{11} + A_{22} + A_{33} =$ $\operatorname{tr}(\operatorname{adj} A)$. $E_3(A) = \det A$

$\therefore p_A(t) = t^3 - E_1(A)t^2 + E_2(A)t - E_3(A) = t^3 - (\operatorname{tr} A)t^2 + (\operatorname{tr} \operatorname{adj} A)t - \det A$.

1.2.P19 $\lambda_1, \dots, \lambda_n > 0 \Rightarrow \det A = \lambda_1 \dots \lambda_n > 0$.

$a_{ij} \in N$ $(\forall i, j = 1 \sim n) \Rightarrow \det A \in N$

$\therefore \det A \in N_+$

The inequality shows that $\det A = 1$, $\lambda_1 = \dots = \lambda_n$, $\operatorname{tr} A = n$

$\therefore \lambda_1 = \dots = \lambda_n = 1$, $a_{11} = \dots = a_{nn} = 1$.

1.2.P20. Suppose the eigenvalues of A are $\lambda_1, \dots, \lambda_n$.

Then those of $I+A$ are $1+\lambda_1, \dots, 1+\lambda_n$.

$$\begin{aligned} \therefore \det(I+A) &= \prod_{i=1}^{n}(1+\lambda_i) = 1 + S_1(\lambda_1, \dots, \lambda_n) + \dots + S_n(\lambda_1, \dots, \lambda_n) \\ &= 1 + E_1(\lambda_1, \dots, \lambda_n) + \dots + E_n(\lambda_1, \dots, \lambda_n) \\ &= 1 + E_1(A) + \dots + E_n(A). \end{aligned}$$

1.2.P21. $\det(tI - A(c)) = \det((tI - cA) + (c-1)\lambda x v^*)$

$= \det(tI - cA) + (c-1)\lambda v^*(\operatorname{adj}(tI - cA))x$

1.2.8a tells that $(t - c\lambda)\operatorname{adj}(tI - cA)x = p_A(t)x$.

$\therefore (t - c\lambda)p_{A(c)}(t) = (t - c\lambda)p_A(t) + (c-1)p_A(t)v^*x$

$= (t - c\lambda)p_A(t) + (c-1)\lambda p_A(t)$

$= (t - \lambda)p_A(t)$.

~~The eigenvalues of $p_{CA}(t)$ are $c\lambda_2, \dots, c\lambda_n$.~~

If ~~$t \neq c\lambda$, then the zeros of $p_{A(c)}(t)$ are~~

The zeros of $\underbrace{p_{A(c)}(t)}_{(t-c\lambda)}$ are $c\lambda$, $\cancel{c\lambda_1, \dots, c\lambda_n, c\lambda}$ zeros of $p_{A(c)}(t)$.

Those of $(t-\lambda)p_{cA}(t)$ are $\lambda, c\lambda_1, \dots, c\lambda_n, c\lambda$.

$\therefore$ Zeros of $p_{A(c)}(t)$ are $\lambda, c\lambda_1, \dots, c\lambda_n$.

$\therefore$ Eigenvalues of $A(c)$ are $\lambda, c\lambda_1, \dots, c\lambda_n$.

1.2.P22 $tI - C_n(\varepsilon) = \begin{bmatrix} t & -1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & & & -1 \\ -\varepsilon & 0 & \cdots\ 0 & t \end{bmatrix}$

$\therefore |tI - C_n(\varepsilon)| = t^n + \cancel{\begin{vmatrix} 0 & -1 & & \\ 0 & 0 & -1 & \\ \vdots & & t & \\ -\varepsilon & & & \ddots \\ -1 & & & t \end{vmatrix}} \begin{vmatrix} 0 & -1 & & 0 \\ 0 & t & \ddots & \\ \vdots & & & -1 \\ -\varepsilon & & & t \end{vmatrix}$

$= t^n + (-1)^{n-1}\varepsilon \begin{vmatrix} -1 & & 0 \\ & \ddots & \\ * & & -1 \end{vmatrix} = t^n - \varepsilon$.

$\therefore p_{C_n(\varepsilon)}(t) = t^n - \varepsilon$.

$\Rightarrow \sigma(C_n(\varepsilon)) = \{\varepsilon^{1/n} e^{2\pi i k/n} \mid k = 0, 1, \dots, n-1\}$.

$\sigma(I + C_n(\varepsilon)) = 1 + \sigma(C_n(\varepsilon))$.

$|1 + \varepsilon^{1/n} e^{2\pi i k/n}| \leq 1 + \varepsilon^{1/n}$. The equality qualifies if and only if $\cancel{n =} k = 0$. $\therefore \rho(I + C_n(\varepsilon)) = 1 + \varepsilon^{\frac{1}{n}}$.

1.2.P23. $\det A = 0 \Rightarrow 0 \in \sigma(A)$. Suppose the eigenvalues of A are $0, \lambda_2, \dots, \lambda_n$. Since they are distinct, $\lambda_2, \dots, \lambda_n \neq 0$.

$\therefore S_{n-1}(A) = \lambda_2 \cdots \lambda_n \neq 0 \Rightarrow E_{n-2}(A) = S_{n-2}(A) \neq 0$.

$\therefore$ A has a nonsingular principal minor of size n-1.