1.1.P1 $Ax=\lambda x \Rightarrow A^{-1}Ax=\lambda A^{-1}x \Rightarrow \lambda^{-1}x=A^{-1}x$.

1.1.P2 (a) If $1, e$ is an eigenpair of A, then

$$\cancel{\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}} \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Rightarrow A_1 + \dots + A_n = \begin{bmatrix} \lambda \\ \vdots \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

So the sum of each row is 1.

If the sum of each row is 1, we can solve that $\lambda=1$.

So $1, e$ is an eigenpair of A.

(b) ~~Accord~~ If $1, e$ is an eigenpair of A, according to P1, $1, e$ is also an eigenpair of A^{-1}. And apply (a) ~~to~~ to prove.

e is an ~~eigenpair~~ eigenvector of $p(A)$. So the sums of the entries in each row $p(A)$ are equal. Equal to $\lambda = p(1)$.

1.1.P3. $A(u+iv)=\lambda(u+iv)$. ~~Since~~ Since $A \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we have $Au=\lambda u$, $Av=\lambda v$.

If $u=0$ and $v=0$, then $x=0$, which is contradictory to $x \neq 0$.

The nonzero item ~~in~~ in u and v is the real eigenvector associated with λ.

u and v haven't to both be eigenvectors. Since A is real, if its eigenvector is ~~not~~ real, the eigenvalue can't be imaginary.

1.1.P4. (a) $\lambda \in \sigma(A) \Rightarrow |A-\lambda I|=0$. $A-\lambda I = \begin{bmatrix} A_{11}-\lambda I & 0 \\ 0 & A_{22}-\lambda I \end{bmatrix}$

$\therefore |A-\lambda I| = |A_{11}-\lambda I||A_{22}-\lambda I| = 0$.

So λ is an eigenvalue of either A_{11} or A_{22}.

(b) if $\lambda \in \sigma(A_{11})$, then $|A_{11}-\lambda I|=0 \Rightarrow |A-\lambda I|=0$

$\therefore \lambda \in \sigma(A)$.

(~~b~~ c). Similar to (b).

So $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$.

1.1.P5. Suppose $Ax=\lambda x$, $x \neq 0$.

Then $A^2x = \lambda Ax \Rightarrow Ax = \lambda Ax$.

$\Rightarrow \lambda x = \lambda^2 x$

$\therefore \lambda^2 = \lambda$, $\lambda = 0$ or 1.

$A^2 = A \Rightarrow A(A-I)=0$.

If A is nonsingular, then $A-I = 0 \cdot A^{-1} = 0$. $\Rightarrow A=I$.

1.1.P6 Suppose $Ax=\lambda x$ and $x \neq 0$. k is the index of A.

~~We have~~ We have $A^k x = \lambda A^{k-1} x = \lambda^k x$

Since $A^k = 0$, $\lambda^k x = 0 \Rightarrow \lambda = 0$.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a nonzero nipotent matrix.

If $A^k = 0$ and $A^2 = A$, then $A^2 \cdot A^{k-2} = 0 \Rightarrow A^{k-1} = 0 \Rightarrow \dots \Rightarrow A = 0$.

So 0 is the only nonsingular idempotent matrix.

1.1.P7. Suppose $Ax = \lambda x$ and $x \neq 0$.

Then $x^* A x = \lambda x^* x$.

Where $x^* A x = (A^* x)^* x = (Ax)^* x = \bar{\lambda}\, x^* x$.

$x^* x > 0 \Rightarrow \lambda = \bar{\lambda}$. So λ is real.

1.1.P8. Omitted. (Wrong in the problem).

1.1.P9. $|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda = \pm i$.

There's no real eigenvalue.

1.1.P10. Suppose $x = (a_1, a_2, \dots)$, then $Sx = (0, a_1, a_2, \dots) = \lambda x = (\lambda a_1, \lambda a_2, \dots)$

So $\lambda a_1 = 0 \Rightarrow \lambda = 0$ or $a_1 = 0$.

If $\lambda = 0$, then $Sx = 0 \Rightarrow \cancel{x=0}\ a_1 = a_2 = \dots = 0 \Rightarrow x = 0$.

If $\lambda \neq 0$ and $a_1 = 0$, then $\lambda a_2 = 0 \Rightarrow a_2 = 0$.

Similarly, $a_3 = a_4 = \dots = 0$. $\therefore x = 0$.

In either situation, we have $x = 0$.

1.1.P11 If $\operatorname{rank}(A - \lambda I) = n-1$, then

$\operatorname{rank}(A-\lambda I) + \operatorname{rank}(\operatorname{adj}(A-\lambda I)) - n \leq \operatorname{rank}((A-\lambda I)\operatorname{adj}(A-\lambda I)) = 0$.

$\therefore \operatorname{rank}(\operatorname{adj}(A-\lambda I)) \leq n - (n-1) = 1$.

If $\operatorname{rank}(\operatorname{adj}(A-\lambda I)) = 0$, then $y=0$, or according ~~th~~ to the full rank factorization theorem, there exists x and y ($\in \mathbb{C}^n$), such that $\operatorname{adj}(A - \lambda I) = xy^*$.

The second conclusion equivalents to that every nonzero column of $\operatorname{adj}(A-\lambda I)$ is an eigenvector of $A - \lambda I$ associated with the eigenvalue 0. This can be drawn from the fact that $(A-\lambda I)\operatorname{adj}(A-\lambda I) = 0$

1.1. P12. $\operatorname{adj}(A-\lambda I) = \begin{bmatrix} d-\lambda & -b \\ -c & a-\lambda \end{bmatrix}$. The first conclusion is easy to draw.

One of the columns must be a scalar multiple of the other is because ~~they are the eigenvector (or 0) of the same~~ eigenvalue. $\begin{bmatrix} -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$\operatorname{rank}(A-\lambda I) = 0$ or 1.

1-1. P13. $Ax = \lambda x \Rightarrow (\operatorname{adj} A) Ax = \lambda (\operatorname{adj} A) x$.

$\therefore$ ~~adj~~ If $\lambda \neq 0$, then $(\operatorname{adj} A) x = \lambda^{-1} (\det A) x$.

If $\lambda = 0$, then $\operatorname{adj} A = 0$ or $\operatorname{adj} A = xy^T$. $\Rightarrow (\operatorname{adj} A) x = (y^T x) x$.

In Either case x is an eigenvector of $\operatorname{adj} A$.