Ch1.

1、理想弦的振动方程:

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+f(t,x).$$

$$a=\sqrt{\frac{T}{\rho}},\ f(t,x)=\frac{g(t,x)}{\rho}.$$

2、热传导方程:

$$\frac{\partial u}{\partial t}=a^2\Delta u+\frac{1}{c\rho}f(t,x,y,z).$$

$$a=\sqrt{\frac{k}{c\rho}}.$$

3、扩散方程: $\frac{\partial u}{\partial t}=a^2\Delta_3 u$. $a=\sqrt{D}$, $D>0$ 是扩散系数.

4、场势方程: $\Delta\varphi=-\frac{\rho}{\varepsilon}$.

5. $$\begin{cases} u_{tt}=a^2u_{xx} & (-\infty<x<+\infty,\ t>0) \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x) & (-\infty<x<+\infty) \end{cases}$$

$$\Rightarrow u(t,x)=\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi.$$

6、齐次化原理1: 设 $w(t,M;\tau)$ 满足

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}=Lw & (M\in R^3,\ t>\tau) \\ w|_{t=\tau}=0,\ \frac{\partial w}{\partial t}\Big|_{t=\tau}=f(\tau,M) \end{cases}$$

则 $$\begin{cases} \frac{\partial^2 u}{\partial t^2}=Lu+f(t,M) \\ u|_{t=0}=0,\ \frac{\partial u}{\partial t}\Big|_{t=0}=0 \end{cases}$$

的解为 $u=\int_0^t w(t,M;\tau)d\tau$.

齐次化原理2: 设 $w(t,M;\tau)$ 满足

$$\begin{cases} \frac{\partial w}{\partial t}=Lw & (M\in R^3,\ t>\tau), \\ w|_{t=\tau}=f(\tau,M), \end{cases}$$

则柯西问题

$$\begin{cases} \frac{\partial u}{\partial t}=Lu+f(t,M) & (M\in R^3,\ t>0), \\ u|_{t=0}=0 \end{cases}$$

的解为 $u=\int_0^t w(t,M;\tau)d\tau$.

Ch 2.

1. $\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

2. $\Delta_2 u = 0$ 的分离变量解为

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{k=1}^{+\infty} (A_k r^k + B_k r^{-k})(C_k \cos k\theta + D_k \sin k\theta)$$

3. 考虑方程 $b_0(x)y''(x) + b_1(x)y'(x) + b_2(x)y(x) + \lambda y(x) = 0$.

令 $\rho(x) = \frac{1}{b_0(x)} e^{\int \frac{b_1(x)}{b_0(x)} dx}$, $k(x) = \rho(x) b_0(x)$, $-q(x) = \rho(x) b_2(x)$.

则方程可化为 $\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - q(x)y + \lambda \rho(x) y = 0$.

两个假定:

1° $k(x), k'(x), \rho(x)$ 在 $[a,b]$ 连续; $x \in (a,b)$ 时 $k(x) > 0$, $\rho(x) > 0$, $q(x) \geq 0$. a, b 至多是 $k(x)$ 和 $\rho(x)$ 的1级零点

2° q 在 (a,b) 连续, 在端点处至多1级极点

当 $k(a) > 0$ 且 q 在 a 点连续: 一~三类边界条件.

当 $k(a) = k(b) > 0$: 周期性边界条件;

$k(a)k(b) = 0$: 有界性条件.

1° $\lambda_n = 0 \Leftrightarrow q(x) \equiv 0$ 且 a、b 两端都不取第一、三类边界条件.

2° $y_m(x)$ 和 $y_n(x)$ 在 $[a,b]$ 带权 $\rho(x)$ 正交, 即 $\int_a^b \rho(x) y_m(x) y_n(x) = 0$.

3° 设 $f(x) = \sum_{n=1}^{\infty} f_n y_n(x)$, 则 $f_k = \frac{1}{\|y_k(x)\|^2} \int_a^b \rho(x) f(x) y_k(x) dx$.

$$\|y_k(x)\|^2 = \int_a^b \rho(x) y_k^2(x) dx.$$

Ch 3.

1. $\cancel{x^2y''}$ $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$ $(\nu \geq 0)$ 的解:

1° $\nu \notin \mathbb{Z}$ 时, $y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$.

$\nu \in \mathbb{N}$ 时, $y = C_1 J_\nu(x) + C_2 N_\nu(x)$.

$$J_\nu(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt \quad (x > 0).$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

2. $\cancel{J_n(x+y)}$

$$\exp\left\{\frac{x}{2}(\zeta - \zeta^{-1})\right\} = \sum_{n=-\infty}^{\infty} J_n(x) \zeta^n.$$

$J_n(x)=\frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta-n\theta)d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{i(x\sin\theta-n\theta)\}d\theta.$

3. $J_n(x+y)=\sum_{k=-\infty}^{\infty}J_k(x)J_{n-k}(y).$

$\lim_{x\to 0}N_\nu(x)=\infty$

4. $\frac{d}{dx}(x^\nu J_\nu(x))=x^\nu J_{\nu-1}(x)$

$\frac{d}{dx}\left(\frac{J_\nu(x)}{x^\nu}\right)=-\frac{J_{\nu+1}(x)}{x^\nu}$

$$\begin{cases} J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2\nu}{x}J_\nu(x) \\ J_{\nu-1}(x)-J_{\nu+1}(x)=2J_\nu'(x). \end{cases}$$

5. $$\begin{cases} (1-x^2)y''-2xy'+\lambda y=0 \quad (-1<x<1) \\ |y(\pm1)|<+\infty \end{cases}$$

则解：$\lambda_n=n(n+1)$，$y_n(x)=\frac{d^n}{dx^n}(x^2-1)^n \ (n\in N)$

勒让德多项式：$P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$

6. $\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n.$

7. $$\begin{cases} (n+1)P_{n+1}(x)-x(2n+1)P_n(x)+nP_{n-1}(x)=0. \\ nP_n(x)-xP_n'(x)+P_{n-1}'(x)=0 \\ nP_{n-1}(x)-P_n'(x)+xP_{n-1}'(x)=0 \\ P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x). \end{cases}$$

8. 记 $f(m,n)=\int_0^1 x^m P_n(x)dx$，则

$f(m,n)=\frac{m}{m+n+1}f(m-1,n-1).$

$2|n$时，$\int_0^1 P_n(x)dx=0.$

9. $\|P_n(x)\|^2=\frac{2}{2n+1}.$

10. 设 $f(x)=\sum_{n=0}^{\infty}C_nP_n(x)$

则 $C_n=\frac{2n+1}{2}\int_0^1 f(x)P_n(x)dx.$

ch 4.

1. $F[f] = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{F[\lambda]} F(\lambda) e^{-i\lambda x} d\lambda.$$

2. $F[f(x)e^{i\lambda_0 x}] = F(\lambda+\lambda_0)$

$F[f(x+a)] = F(\lambda)e^{-i\lambda a}$

$F[f(ax)] = \frac{1}{a}F(\frac{\lambda}{a}).$

$F[f^{(n)}(x)] = (-i\lambda)^n F(\lambda)$

3. $F[e^{-ax^2}] = \sqrt{\frac{\pi}{a}} e^{-\frac{\lambda^2}{4a}}$

$F^{-1}[e^{-a\lambda^2}] = \frac{1}{2\sqrt{\pi a}} e^{-\frac{x^2}{4a}}$

4. $F(\lambda,\mu,\nu) = \iiint f(x,y,z) e^{i(\lambda x+\mu y+\nu z)} dx dy dz.$

$f(x,y,z) = \frac{1}{(2\pi)^3}\iiint F(\lambda,\mu,\nu) e^{-i(\lambda x+\mu y+\nu z)} d\lambda d\mu d\nu.$

$F[\frac{\partial f}{\partial x}] = -i\lambda F[f]$ $\quad (F[\frac{\partial^2 f}{\partial x^2}] = -\lambda^2 F[f].$

$\therefore F[\Delta_3 f] = -(\lambda^2+\mu^2+\nu^2) F[f]$

5. ~~则~~ $f_s(\lambda) = \int_0^{\infty} f(x) \sin\lambda x \, dx$

$f_c(\lambda) = \int_0^{\infty} f(x) \cos\lambda x \, dx.$

$f(x) = \frac{2}{\pi}\int_0^{\infty} f_s(x) \sin\lambda x \, d\lambda = \frac{2}{\pi}\int_0^{\infty} f_c(x) \cos\lambda x \, d\lambda$

6. $(f')_s = -\lambda f_c \quad (f')_c = \lambda f_s - f(0)$

二阶微分方程通解？

7. $L[f] = \int_0^{\infty} f(t) e^{-pt} dt$

$f(t) = \frac{1}{2\pi}\int_{\sigma-\infty}^{\sigma+\infty} L(p) e^{pt} dp. = \sum Res(L(p)e^{pt}, p_i).$

8. $L[f(t)e^{\lambda t}] = L(p-\lambda).$

$L[f(t-\tau)] = e^{-p\tau} L(p).$

$L[f(at)] = \frac{1}{a}F(\frac{p}{a})$

$L[f^{(n)}(t)] = p^n L(p) - p^{n-1} f(0+) - p^{n-2} f'(0+) - \ldots$

$L[f(t)]^{(n)} = L[(-t)^n f]$.

$L[\int_0^t f(s)ds] = \frac{L[f]}{p}$.

9、 $L[e^{\lambda t}] = \frac{1}{p-\lambda}$

$L[\frac{t^n}{n!}] = \frac{1}{p^{n+1}}$

$L[\sin\omega t] = \frac{\omega}{p^2+\omega^2}$

$L[\cos\omega t] = \frac{p}{p^2+\omega^2}$.

Ch 5

1. $F[\delta(x)] = 1$.

$\delta^{(n)}(x): f \to (-1)^n f^{(n)}(0)$.

$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\lambda x \, d\lambda$.

$F^{-1}[e^{i\lambda\xi}] = \delta(x-\xi)$.

2. $G(M;M_0)$满足:

$$\begin{cases} \Delta u = -\delta(M-M_0), M \in V \\ u|_{\partial V} = 0. \end{cases}$$

3. $$\begin{cases} \Delta u = -f(x,y,z), M \in V \\ u|_{\partial V} = \varphi(x,y,z) \end{cases}$$

的解为$u(M) = -\int_S \varphi(M_0)\frac{\partial_{M_0} G}{\partial n} dS + \int_V G(M;M_0) f(M_0) dM_0$.

4、球形域上的格林函数:

设球内$M_0(\xi,\eta,\zeta)$处有ε_0的电荷, $\rho_0 = r(O,M_0)$, 在$M_0' = \frac{R^2}{\rho_0^2}(\xi,\eta,\zeta)$摆放电荷, 电量为$-\frac{R}{\rho_0}\varepsilon_0$.

$$G(M;M_0) = \frac{1}{4\pi}\left(\frac{1}{r(M,M_0)} - \frac{R}{\rho_0 r(M,M_0')}\right).$$

$$= \frac{1}{4\pi}\left(\frac{1}{\sqrt{r^2+\rho_0^2-2r\rho_0\cos\psi}} - \frac{R}{\sqrt{r^2\rho_0^2+R^4-2R^2 r\rho_0\cos\psi}}\right).$$

$r = r(O,M)$, ψ为OM_0与OM间夹角.

4. ~~三维~~ $\Delta_3 u = -\delta(x,y,z) \Rightarrow u = \frac{1}{4\pi r}$

$\Delta_2 u = -\delta(x,y) \Rightarrow u = \frac{-\ln r}{2\pi}$. (注意符号).

dx

5. 圆周上的Green函数：

$$G(M;M_0)=\frac{1}{2\pi}\ln\frac{\rho_0 r(M,M_0')}{R\,r(M,M_0)}$$

$$\begin{cases}\Delta_2 u=0,\ 0\le r\le R\\ u|_{r=R}=\varphi(\theta)\end{cases}$$

解为 $u(M)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\varphi(\theta)(R^2-r^2)}{r^2+R^2-2rR\cos(\theta-\varphi)}d\theta$.

6、设 II_1: $\begin{cases}u_t=Lu,\ t>0,-\infty<x,y,z<\infty\\ u(0,x,y,z)=\delta(x,y,z)\end{cases}$

为 II_2: $\begin{cases}u_t=Lu+f(t,x,y,z),\ t>0,-\infty<x,y,z<\infty\\ u(0,x,y,z)=\varphi(x,y,z)\end{cases}$

的基本解.

设 II_1 解为 $U(t,M)$,

则 II_2 解为

$$U(t,M)*\varphi(M)+\int_0^t U(t-\tau,M)*f(\tau,M)d\tau.$$

$$\begin{cases}u_t=a^2\Delta_3 u,\ t>0,-\infty<x,y,z<\infty\\ u(0,x,y,z)=\delta(x,y,z)\end{cases}$$

$$\Rightarrow U(t,x,y,z)=\left(\frac{1}{2a\sqrt{\pi t}}\right)^3 e^{-\frac{x^2+y^2+z^2}{4a^2t}}$$

二维：$U(t,x,y)=\left(\frac{1}{2a\sqrt{\pi t}}\right)^2 e^{-\frac{x^2+y^2}{4a^2t}}$

一维：$U(t,x)=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$

II_1: $\begin{cases}u_{tt}=Lu,\ t>0,-\infty<x,y,z<\infty\\ u(0,x,y,z)=0,\ u_t(0,x,y,z)=\delta(x,y,z)\end{cases}$

为 II_2: $\begin{cases}u_{tt}=Lu+f(t,x,y,z),\ t>0,-\infty<x,y,z<\infty\\ u(0,x,y,z)=\varphi(x,y,z),\ u_t(0,x,y,z)=\psi(x,y,z)\end{cases}$

的基本解.

$$u(t,M)=\frac{\partial}{\partial t}[U(t,M)*\varphi(M)]+U(t,M)*\psi(M)+\int_0^t U(t-\tau,M)*f(\tau,M)d$$

$\cancel{U(t,M)=\frac{\quad}{4\pi ar}}$

$$\begin{cases}u_{tt}=a^2\Delta_3 u,\ t>0,-\infty<x,y,z<\infty\\ u(0,x,y,z)=0,\ u_t(0,x,y,z)=\delta(x,y,z).\end{cases}$$

的解为 $U(t,r)=\frac{\delta(r-at)}{4\pi ar}$.

一维波动方程

$$\begin{cases} u_{tt}=a^2u_{xx},\ t>0,\ -\infty<x<\infty \\ u(0,x)=0,\ u_t(0,x)=\delta(x)\end{cases}$$

的解为

$$u(t,x)=\begin{cases}\frac{1}{2a},\ x\in[-at,at]\\ 0,\ 其他.\end{cases}$$

欧拉方程 $x^2y''+axy'+by=0$ 解法:

设 $x=e^t$, 则

$$\frac{dy}{dx}=\frac{dy}{de^t}=e^{-t}\frac{dy}{dt}.$$

$$\frac{d}{dx}\frac{dy}{dx}=\frac{d}{de^t}\left(e^{-t}\frac{dy}{dt}\right)=e^{-t}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)=e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2y}{dt^2}+(a-1)\frac{dy}{dt}+by=0.$$