Gauge Theory — Talk for the Imperial College Junior Geometry on 14 Dec 2018

Gauge theory is the study of principal bundle connections. In the talk I explained what these are, mentioned the existence of exotic $\mathbb{R}^4$'s as one remarkable application, and gave handwavy outlook at open problems.

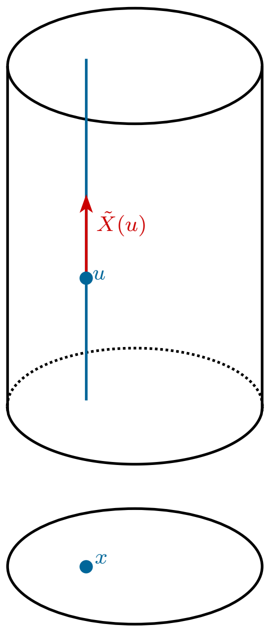
1 Definitions

References for this section are [Bau09, KN63].

Definition 1.1. Let $P \xrightarrow{\pi} M$ be a G-principal bundle. A one-form $A \in \Omega^1(P, \mathfrak{g})$ is called *connection* if

1. it is right-invariant, i.e. $r_g^* A = \mathrm{Ad}(g^{-1}) \circ A$ for all $g \in G$,
2. it reproduces fundamental vector fields, i.e. $A(\tilde{X}) = X$ for all $X \in \mathfrak{g}$.

Here $\tilde{X}(u) = \frac{d}{dt}(u \cdot \exp(tX))|_{t=0}$. Denote by $\mathcal{A}(P)$ the space of all connections on P.



Definition 1.2. Let $A \in \mathcal{A}(P)$. The map

$$\begin{aligned} D_A : \Omega^1(P, \mathfrak{g}) &\to \Omega^2(P, \mathfrak{g}) \\ \omega &\mapsto \mathrm{d}\omega + \frac{1}{2}[\omega \wedge A] \end{aligned} \tag{1}$$

is called the *absolute differential* with respect to A. $F_A := D_A A \in \Omega^2(P, \mathfrak{g})$ is called the *curvature of* A.

Proposition 1.3. *There is a 1:1-correspondence*

$$\Omega^k(P,\mathfrak{g})^{hor,G} \leftrightarrow \Omega^k(M,\mathrm{Ad}(P))$$
$$\begin{Bmatrix} \textit{horizontal, right-invariant} \\ \textit{forms with values in } \mathfrak{g} \textit{ on } P \end{Bmatrix} \overset{1:1}{\leftrightarrow} \begin{Bmatrix} \textit{forms with values in} \\ \mathrm{Ad}(P) = P \times_{\mathrm{Ad},G} \mathfrak{g} \textit{ on } M \end{Bmatrix}. \tag{2}$$

For $\overline{\omega} \in \Omega^k(P,\mathfrak{g})^{hor,G}$ and $\omega \in \Omega^k(M,\mathrm{Ad}(P))$ the correspondence is given by:

$$\omega(\mathrm{d}\pi(X_1),\ldots,\mathrm{d}\pi(X_k)) = [u,\overline{\omega}(X_1,\ldots,X_k)], \tag{3}$$

where $u \in P$, and X_1, dots, $X_k \in T_uP$.

Together with line 2, the map D_A from line 1 induces a differential $d_a : \Omega^1(M,\mathrm{Ad}\,P) \to \Omega^2(M,\mathrm{Ad}\,P)$.

Proposition 1.4. *Let A_1, $A_2 \in \mathcal{A}(P)$. Then $A_1 - A_2$ is a horizontal and right-invariant form on P. Thus, $\mathcal{A}(P)$ is an affine space with vector space $\Omega^1(M,\mathrm{Ad}\,P)$.*

Proposition 1.5. *For $A \in \mathcal{A}(P)$ its curvature F_A is horizontal and right-invariant.*

Thus we can consider F_A as an element in $\Omega^2(M,\mathrm{Ad}(P))$, which is the viewpoint we will adopt for the rest of the document.

2 ASD-Instantons

References for this section are [Bau09, Siv18].

Let M and G be compact.

Definition 2.1. $A \in \mathcal{A}(M)$ is called Yang-Mills connection if $\delta_A F_A = 0$. Here $\delta_A = *d_A* : \Omega^2(M,\mathrm{Ad}\,P) \to \Omega^1(M,\mathrm{Ad}\,P)$ denotes the co-differential of d_A.

Proposition 2.2. *Let $A \in \mathcal{A}(M)$. Then the following are equivalent:*

1. *A is a Yang-Mills connections.*
2. *A is a critical point of the Yang-Mills functional*

$$\begin{aligned} L : \Omega^1(M,\mathrm{Ad}\,P) &\to \mathbb{R} \\ A &\mapsto \langle F_A, F_A\rangle_{L^2}. \end{aligned} \tag{4}$$

On a four-dimensional manifold we have $* : \Omega^2(M) \to \Omega^2(M)$, $*^2 = \mathrm{Id}$, and therefor

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M) = \{\omega : *\omega = \omega\} \oplus \{\omega : *\omega = -\omega\}. \tag{5}$$

This allows the following definition:

Definition 2.3. $A \in \mathcal{A}(P)$ is called anti-self-dual (ASD) isntantons if $F_A \in \Omega^2_-(M, \mathrm{Ad}\, P)$.

Proposition 2.4. *If $A \in \mathcal{A}(M)$ ASD instanton, then A is a Yang-Mills connection.*

Proof.

$$\delta_A F_A = * \mathrm{d}_A * F_A = - * \underbrace{\mathrm{d}_A F_A} = 0,$$

where the last step followed from the Bianchi identity $\mathrm{d}_A F_A = 0$, which holds for all connections. □

The Yang-Mills connection is a second order PDE, the ASD condition is a first order PDE. In general it is not true that all Yang-Mills connections are ASD instantons, however ASD instantons are the same as minimisers of the Yang-Mills functional, as stated in the following proposition:

Proposition 2.5. *We have $p_1(\mathrm{Ad}\, P) = \frac{1}{8\pi^2}\langle F_A, *F_A\rangle_{L^2}$ for any connection $A \in \mathcal{A}(P)$ and therefore:*

1. *$p_1(\mathrm{Ad}\, P) > 0$ implies that there exist not ASD instantons on P.*
2. *A is a minimum of the Yang-Mills function L if and only if A is and ASD instanton.*

Proof. The identity $p_1(\mathrm{Ad}\, P) = \frac{1}{8\pi^2}\langle F_A, *F_A\rangle_{L^2}$ follows from Chern-Weil theory and we omit the proof here.

1. If $A \in \mathcal{A}(P)$ is an ASD instantons, then $\frac{1}{8\pi^2}\langle F_A, *F_A\rangle_{L^2} \leq 0$, thus there can be no ASD instantons under the assumption $p_1(\mathrm{Ad}\, P) > 0$.
2. We have

$$\begin{aligned} L(A) &= \langle F_A, F_A\rangle \\ &= \frac{1}{2}(\langle F_A, F_A\rangle + \langle *F_A, *F_A\rangle) \\ &= \frac{1}{2}(\langle F_A, F_A\rangle + \langle *F_A, *F_A\rangle + 2\langle F_A, *F_A\rangle) - \langle F_A, *F_A\rangle \\ &= \frac{1}{2}\langle F_A + *F_A, F_A + *F_A\rangle - \langle F_A, *F_A\rangle \\ &\geq -\langle F_A, *F_A\rangle \end{aligned}$$

with equality if and only if A is and ASD instanton.

□

3 Deformation Theory

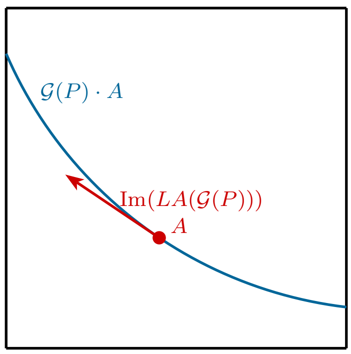
Write $\hat{\mathcal{M}} = \{A \in \mathcal{A}(P) : A \text{ ASD instanton on } P\}$. Our goal is to describe $\hat{\mathcal{M}}$. There is a problem with this, as described below.

Definition 3.1. Denote by

$$\mathcal{G}(P) = \{f : P \to P : \pi f = f, f(pg) = f(p)g \text{ for all } p \in P, g \in G\} \tag{6}$$

the gauge group of P.

Let $A \in \hat{\mathcal{M}}$, and $f \in \mathcal{G}(P)$, then $f^*A \in \hat{M}$. In other words: if we fix just one ASD instanton, we already get a whole infinite-dimensional family of them as the orbit under the action of the gauge group.



So, what we really want to study is

$$\mathcal{M} = \hat{\mathcal{M}}/\mathcal{G}(P). \tag{7}$$

As a first attempt, let us determine the dimension of $\mathcal{M}$, wherever it is a smooth manifold.

- Describe $T_A\hat{\mathcal{M}}$:

  Assume we have a deformation of $A \in \hat{\mathcal{M}}$, say $\omega \in \Omega^1(M, \operatorname{Ad} P)$ such that $A + t\omega \in \hat{\mathcal{M}}$ for $t \in (-\epsilon, \epsilon)$. Then

  $$0 = F^+_{A+t\omega} = (F_A + t\,\mathrm{d}_A\omega + \frac{1}{2}[\omega \wedge \omega])^+$$

  and differentiating gives $0 = \mathrm{d}^+_A\omega$, i.e. $T_A\hat{\mathcal{M}} = \operatorname{Ker} \mathrm{d}^+_A$.

- Quotient out by the infinitesimal action of the gauge group $\mathcal{G}(P)$:

  We have an action of $\mathcal{G}(P)$ on $\hat{\mathcal{M}}$, and therefor an infinitesimal action of the Lie algebra $LA(\mathcal{G}(P))$ on $\hat{\mathcal{M}}$, i.e. $\theta \in LA(\mathcal{G}(P))$ defines a vector field X_θ on $\hat{\mathcal{M}}$.

  We now compute $X_\theta(A)$ for $\theta \in LA(\mathcal{G}(P))$:

$$\begin{aligned} X_\theta(A) &= \frac{d}{dt}\exp(t\theta)^*A|_{t=0} \\ &= \frac{d}{dt}\exp(t\theta)\cdot A\cdot\exp(-t\theta) - (\mathrm{d}\exp(t\theta))\cdot\exp(-t\theta)|_{t=0} \\ &= \theta A - A\theta - \mathrm{d}\theta \\ &= [\theta, A] - \mathrm{d}\theta \\ &= -\,\mathrm{d}_A\theta, \end{aligned}$$

  where we assumed that G is a matrix Lie group and used $\cdot$ to denote matrix multiplication. The first equality is [DK90, line 2.1.9]. So, $\{X_\theta : \theta \in LA(\mathcal{G}(P))\} = \operatorname{Im}\mathrm{d}_A$.

Combine the two points in a complex, where we omit the $(M, \operatorname{Ad}P)$ from the notation:

$$0 \to \Omega^0 \xrightarrow{\mathrm{d}_A} \Omega^1 \xrightarrow{\mathrm{d}_A^+} \Omega^2_+ \to 0, \tag{(*)}$$

This is called the *deformation complex*. Then

$$T_{[A]}\mathcal{M} = \frac{\operatorname{Ker}\mathrm{d}_A^+}{\operatorname{Im}\mathrm{d}_A} = H_1(*), \tag{8}$$

and $\operatorname{ind}(*) = \operatorname{ind}(\mathrm{d}_A \oplus \mathrm{d}_A^+) = \dim H_0(*) - \dim H_1(*) + \dim H_2(*)$, thus

$$\dim\mathcal{M} = -\operatorname{ind}(*) + \dim H_0(*) + \dim H_2(*). \tag{9}$$

For special choices of M and G we have $H_0(*) = H_2(*) = 0$. In these cases $\dim\mathcal{M}$ can be computed through an Atiyah-Singer index formula.

4 Generalized Instantons

A reference for this section is [LM17].

- On $\mathbb{R}^4$:

  $\Lambda^2\mathbb{R}^4 \simeq \mathfrak{so}(4)$ and $\mathfrak{su}(2) \subset \mathfrak{so}(4)$ by forgetting the complex structure. Under this inclusion $\mathfrak{su}(2) = \Lambda^2_-\mathbb{R}^4$ (check!). Thus:

$$*F_A = -F_A \text{ in a point } \leftrightarrow F_A \in \mathfrak{su}(2)\otimes\operatorname{Ad}(P) \subset \Lambda^2\mathbb{R}^4\otimes\operatorname{Ad}(P). \tag{10}$$

- On M^4:

  We have $\Omega^2(M) = Fr \times_{\mathrm{SO}(4)} \Lambda^2\mathbb{R}^4 \simeq Fr \times_{\mathrm{SO}(4)} \mathfrak{so}(4)$. Now assume the manifold has an SU(2)-structure, then

  $$\Omega^2_-(M) = Fr \times_{\mathrm{SU}(2)} \mathfrak{su}(2) \subset Fr \times_{\mathrm{SU}(2)} \Lambda^2\mathbb{R}^4, \tag{11}$$

  thus we can characterise ASD instantons as connections which satisfy

  $$F_A \in (Fr \times_{\mathrm{SU}(2)} \mathfrak{su}(2)) \otimes \mathrm{Ad}\, P. \tag{12}$$

This allows us to make the following definition:

Definition 4.1. Let M be a manifold with H-structure and P a G-bundle over M. We say that $A \in \mathcal{A}(P)$ is an H-instanton if

$$F_A \in (Fr \times_H \mathfrak{h}) \otimes \mathrm{Ad}\, P. \tag{13}$$

Example 4.2. Assume M to have holonomy contained in H. This gives rise to an H-structure and therefor to the definition of H-instantons on M.

Then the Levi-Civita connection A on the frame bundle of M is an H-instanton. This is because in a local trivialisation the Riemann curvature tensor R can be regarded as a two-form on M with values in $\mathfrak{h}$, i.e. $R \in (M \times \mathfrak{so}(n)) \otimes \mathfrak{h}$. And because of its symmetries it even belongs to the subbundle $M \times (\mathfrak{h} \otimes \mathfrak{h})$. (cf. [Car98, p. 2])

Thus, the Levi-Civita connection on a K3-surface is an ASD-instanton.

So, why all of this?

M. Freedman showed that the homeomorphism type of a 4-manifold depends only on its intersection form and an additional invariant in its top cohomology group. S. Donaldson assigned polynomial invariants to the moduli space of ASD-instantons which only depend on the smooth structure, and not on the metric structure. There then exist examples of manifolds which are homeomorphic by Freedman's results, but cannot be diffeomorphic because they have different polynomial invariants. This led to the discovery of exotic $\mathbb{R}^4$'s. (cf. [DK90])

It is hoped that one can create a similar theory for higher dimensional manifolds. For example, there is a hope that G_2-instantons can be used to find non-homotopic G_2-structures on a manifold. All of the results presented in this talk carry over from the ASD case to the G_2 case, but the construction of a suitable compactification of the moduli space of instantons turned out to be more difficult in the G_2 case and remains an open problem as of now.

References

[Bau09] H. Baum. *Eichfeldtheorie*. Springer-Lehrbuch Masterclass Series. Springer Berlin Heidelberg, 2009. 1, 2

[Car98] Ramón Reyes Carrión. A generalization of the notion of instanton. *Differential Geometry and its Applications*, 8(1):1 – 20, 1998. 6

[DK90] S.K. Donaldson and P.B. Kronheimer. *The Geometry of Four-manifolds*. Oxford mathematical monographs. Clarendon Press, 1990. 5, 6

[KN63] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Number v. 2 in Interscience tracts in pure and applied mathematics, no. 15 v. 1-2. Interscience Publishers, 1963. 1

[LM17] Jason D Lotay and Thomas Bruun Madsen. Instantons and special geometry. In *Special Metrics and Group Actions in Geometry*, pages 241–267. Springer, 2017. 5

[Siv18] Stiven Sivek. Lectures on instantons and contact structures. `http://wwwf.imperial.ac.uk/~ssivek/papers/regensburg-notes.pdf`, 2018. 2