Production FUnctions

Massimiliano

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1 Example of CES function

$$F(K,L) = \left[\gamma K^{\frac{(\sigma-1)}{\sigma}} + (1-\gamma)L^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

2 Example of Cobb-Douglas function

$$F(K,L) = K^{\alpha}L^{\beta}$$

3 Example of Leontief function (perfect complements)

$$F(K,L) = \min\{\alpha K, \beta L\}$$

4 CES Function

4.1 Form

Following Chen (2011), we provide a proof of the form of CES function. For more details on CES function, read Arrow et. al (1961)

The production function $V = F(K,L)$ in capital version can be written as

$$\frac{V}{L} = F\left(\frac{K}{L}, 1\right)$$

Let us start from $\ln \frac{V}{L} = \ln a + b \ln w$

where,

$$\begin{aligned}
&\frac{V}{L} = y \\
&\frac{K}{L} = k \\
&MPK = f'(k) \\
&MPL = f(k) - kf'(k)
\end{aligned}$$

We can write

$$\ln y = \ln a + b \ln[f(k) - kf'(k)]$$

Solving for $f'(k)$

$$
\begin{aligned}
&\ln y - \ln a = b \ln[y - kf'(k)] \\
&\ln \frac{y}{a} = b \ln[y - kf'(k)] \\
&\ln \left(\frac{y}{a}\right)^{\frac{1}{b}} = \ln[y - kf'(k)] \\
&kf'(k) = y - \left(\frac{y}{a}\right)^{\frac{1}{b}} \\
&kf'(k) = y - \frac{y^{\frac{1}{b}}}{a^{\frac{1}{b}}} \\
&f'(k) = \frac{a^{\frac{1}{b}} y - y^{\frac{1}{b}}}{k a^{\frac{1}{b}}}
\end{aligned}
$$

where

$$
\frac{a^{\frac{1}{b}} y - y^{\frac{1}{b}}}{k a^{\frac{1}{b}}} = \frac{a^{\frac{-1}{b}} (a^{\frac{1}{b}} y - y^{\frac{1}{b}})}{k} = \frac{y - a^{\frac{-1}{b}} y^{\frac{1}{b}}}{k}
$$

Setting $\alpha = a^{\frac{-1}{b}}$ and $\rho = \frac{1}{b} - 1$, we can write

$$
\frac{dy}{dk} = \frac{y - \alpha y^{\frac{1}{b}}}{k} = \frac{y(1 - \alpha y^{\frac{1}{b}-1})}{k} = \frac{y(1 - \alpha y^{\rho})}{k}
$$

Becuase it is a separable differential equation, first let us separate y and k, perform partial fraction decomposition, and take integral on both sides:

$$
\begin{aligned}
&\frac{dk}{k} = \frac{dy}{y(1 - \alpha y^{\rho})} \\
&\frac{dk}{k} = \frac{dy}{y} + \frac{\alpha y^{\rho-1} dy}{1 - \alpha y^{\rho}} \\
&\ln k = \ln y - \frac{1}{\rho} \ln(1 - \alpha y^{\rho}) + \frac{1}{\rho} \ln \beta
\end{aligned}
$$

where $\ln \beta$ is a constat term after indifinite integration. Take antilog on both sides and raise the power of ρ to get

$$
k^{\rho} = \frac{\beta y^{\rho}}{1 - \alpha y^{\rho}}
$$

and solve for y

$$
\begin{aligned}
&k^{\rho}(1 - \alpha y^{\rho}) = \beta y^{\rho} \\
&k^{\rho} - \alpha k^{\rho} y^{\rho} = \beta y^{\rho} \\
&k^{\rho} = \beta y^{\rho} + \alpha k^{\rho} y^{\rho} \\
&k^{\rho} = (\beta + \alpha k^{\rho}) y^{\rho} \\
&y^{\rho} = \frac{k^{\rho}}{\beta + \alpha k^{\rho}} \\
&y = \left(\frac{k^{\rho}}{\beta + \alpha k^{\rho}}\right)^{\frac{1}{\rho}} \\
&y = k(\beta + \alpha k^{\rho})^{\frac{-1}{\rho}} \\
&y = \left[k^{-\rho}(\beta + \alpha k^{\rho})\right]^{\frac{-1}{\rho}} \\
&y = (\beta k^{-\rho} + \alpha)^{\frac{-1}{\rho}}
\end{aligned}
$$

It can be written in the aggregate form as

$$F(K, L) = (\beta K^{-\rho} + \alpha L)^{\frac{-1}{\rho}}$$

5 References

K. J. Arrow, H. B. Chenery, B. S. Minhas, R. M. Solow (1961). Capital-Labor Substitution and Economic Efficiency, *The Review of Economics and Statistics*, Vol. 43, No. 3, pp. 225-250.

Weijie Chen (2011). CES functions and Dixit-Stiglitz Formulation, Department of Political and Economic Studies, University of Helsinki.