Final Project:

1)

$U_t - U_{xx} = f(x,t)$, $(x,t) \in (0,1) \times (0,1)$, $f(x,t) = (\pi^2 - 1)e^{-t}\sin(\pi x)$

w/ initial boundary conditions

$$U(x,0) = \sin(\pi x)$$
$$U(0,t) = U(1,t) = 0$$

$+$

$$U(x,t) = e^{-t}\sin(\pi x)$$

$$\frac{\delta u}{\delta t} = -e^{-t}\sin(\pi x)$$

$$\frac{\delta u}{\delta x} = \pi e^{-t}\cos(\pi x) \therefore \frac{\delta^2 u}{\delta t^2} = -\pi^2 e^{-t}\sin(\pi x)$$

$U_t - U_{xx} = f(x,t)$

$$\hookrightarrow \int_0^1 \frac{\delta u}{\delta t} v(x)\,dx - \int_0^1 \frac{\delta^2 u}{\delta x^2} v(x)\,dx = \int_0^1 f(x,t)\,v(x)\,dx$$

$$\hookrightarrow \int_0^1 \frac{\delta u}{\delta t} v(x)\,dx - \left[\frac{\delta u}{\delta x} v(x)\right]_0^1 + \int_0^1 \frac{\delta^2 u}{\delta x^2} v'(x)\,dx = \int_0 f(x,t)\,v(x)\,dx$$

$\hookrightarrow$ applying boundary conditions

$$\left[\cancel{\frac{\delta u}{\delta x} v(x)}\right]_0^1 + \int_0^1 \frac{\delta^2 u}{\delta x^2} v'(x)\,dx \quad + \text{ let } V(x) = \phi(x)$$

$$\hookrightarrow \int_0^1 \frac{\delta u}{\delta t} \cdot \phi_i(x)\,dx + \int_0^1 \frac{\delta^2 u}{\delta x^2} \frac{\delta \phi_i}{\delta x}\,dx = \int_0^1 f(x,t)\,\phi(x)\,dx$$

$$\hookrightarrow \int_0^1 \frac{\delta u}{\delta t} \cdot \phi_i(x)\,dx + \int_0^1 \frac{\delta^2 u}{\delta x^2} \frac{\delta \phi_i}{\delta x}\,dx - \int_0^1 f(x,t)\,\phi(x)\,dx = \emptyset$$