We consider the following IBVP:

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \quad (x,t) \in (0,1) \times (0,T), \quad (T > 0), \tag{1}$$

with the initial condition,

$$u(0,x) = u_0(x), \quad 0 \le x \le 1, \tag{2}$$

where known functions $f \in C^1([0,1] \times [0,T]); \mathbb{R})$ and $u_0 \in C^1([0,1]; \mathbb{R})$, and satisfies the compatibility conditions $u_0(0) = u_0(1) = 0$, and the homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T. \tag{3}$$

1 Time Discretization

We fix $n \in \mathbb{N}$ with $n \ge 2$. We set $h = T/n$, for a give $u_0^n(x)$, $0 \le x \le 1$ and we consider the time discretized system of n BVPs,

$$h\frac{d^2 u_j^n}{dx^2}(x) - u_j^n(x) = -u_{j-1}^n(x) - hf_j^n(x), \quad u_j^n(0) = u_j^n(1) = 0, \quad j = 1, 2, \ldots, n; \tag{4}$$

where $f_j(x) = f(jh, x)$. The approximate solution $U^n(x,t)$ of IBVP (1)-(3) is given by

$$U^n(x,t) = u_{j-1}^n(x) + \frac{1}{h}[t - (j-1)h](u_j^n - u_{j-1}^n), \quad 0 \le x \le 1,\ (j-1)h < t \le jh;\ j = 1, 2, \ldots, n.$$

We now look closer at the system in (4). For $j = 1$ we have

$$h\frac{d^2 u_1^n}{dx^2}(x) - u_1^n(x) = -u_0(x) - hf_1^n(x), \quad u_1^n(0) = u_1^n(1) = 0. \tag{5}$$

The right hand side of (5) is known and hence can be solved for u_1^n. This u_1^n will be used as a known function in the equation for $j = 2$,

$$h\frac{d^2 u_2^n}{dx^2}(x) - u_2^n(x) = -u_1^n(x) - hf_2^n(x), \quad u_2^n(0) = u_2^n(1) = 0. \tag{6}$$

Continuing this way, we get u_{n-1}^n known and use it in

$$h\frac{d^2 u_n^n}{dx^2}(x) - u_n^n(x) = -u_{j-1}^n(x) - hf_n^n(x), \quad u_n^n(0) = u_n^n(1) = 0. \tag{7}$$

2 Neural Network

Residual Method:
We define

$$N(x,t) = \sum_{i=1}^{m} v_i \sigma(z_i(x,t)) \tag{8}$$

for an activation function σ where

$$z_i(x,t) = w_i x + k_i\, t + b_i,$$

and we approximate a solution using

$$\hat{u}_j^n(x) = N^n(x, jh), \tag{9}$$

by optimizing the hyperparameters v_i, w_i, k_i and b_i.

We have

$$\frac{d\hat{u}_j^n}{dx} = \frac{dN_j^n}{dx} = \sum_{i=1}^{m} v_i w_i \sigma'(z_i(x, jh))$$

and

$$\frac{d^2\hat{u}_j^n}{dx^2} = \sum_{i=1}^{m} v_i w_i^2 \sigma''(z_i(x, jh)) \tag{10}$$

Let

$$E_j^n(x) = \frac{\hat{u}_j^n(x) - \hat{u}_{j-1}^n(x)}{h} - \frac{d^2\hat{u}_j^n}{dx^2}(x) - f_j^n(x),$$

$$= \sum_{i=1}^{m} v_i \left(\frac{1}{h}(\sigma(z_i(x, jh)) - \sigma(z_i(x, (j-1)h)) - w_i^2 \sigma''(z_i(x, jh)) \right) - f(jh, x),$$

and

$$E^n(x) = \frac{1}{2n} \sum_{j=1}^{n} \left[\sum_{i=1}^{m} v_i \left(\frac{1}{h}(\sigma(z_i(x, jh)) - \sigma(z_i(x, (j-1)h)) - w_i^2 \sigma''(z_i(x, jh)) \right) - f(jh, x) \right]^2$$

$$+ \frac{\tau}{2} \left[\hat{u}_0^n(x) - u_0(x) \right]^2$$

We define

$$N(x, j) = \sum_{i=1}^{m} v_i \sigma(z_i(x, j)) \tag{8}$$

for an activation function σ where

$$z_i(x, j) = w_i x + k_i\, jh + b_i = w_i x + k_i\, \frac{jT}{n} + b_i,$$

and we approximate a solution using

$$\hat{u}_j^n(x) = jhx(x-1)N^n(x, j), \tag{9}$$

by optimizing the hyperparameters v_i, w_i, k_i and b_i.

We have

$$\frac{d\hat{u}_j^n}{dx} = jhx(x-1)\frac{dN_j^n}{dx} + jh(2x-1)N_j^n$$

and

$$\frac{d^2\hat{u}_j^n}{dx^2} = jhx(x-1)\frac{d^2N_j^n}{dx^2} + 2jh(2x-1)\frac{dN_j^n}{dx} + 2jhN_j^n. \tag{10}$$

From (4), (8), and (10), we get

$$E_j^n(x) = \frac{\hat{u}_j^n(x) - \hat{u}_j^n(x)}{h} - \frac{d^2\hat{u}_j^n}{dx^2}(x) - f_j^n(x), \quad j = 1, 2, \ldots, n; \tag{11}$$

and we minimize our cost function

$$E^n(x) = \sum_{j=1}^{n} \left(E_j^n(x)\right)^2,$$

by iteratively optimizing the hyperparameters v_i, w_i, k_i and b_i by the gradient descent rule

$$\Delta\theta = -\frac{\lambda}{n}\frac{dE^n(x)}{d\theta} = -\sum_{j=1}^{n} \frac{2\lambda}{n} E_j^n(x) \frac{dE_j^n(x)}{d\theta}$$

where θ is an instance of our hyperparameters and λ is a chosen learning rate to tune step size at each iteration.

Now we can expand (11) as

$$E_j^n(x) = \sum_{i=1}^{m} \left[-jhx(x-1)v_i {w_i}^2 \sigma''(z_i(x,j)) - 2jh(2x-1)v_i w_i \sigma'(z_i(x,j)) - 2jhv_i\sigma(z_i(x,j))\right]$$

$$+x(x-1)\sum_{i=1}^{m} \left[jv_i\sigma(z_i(x,j)) - (j-1)v_i\sigma(z_i(x,j-1))\right] + f_j^n(x).$$

Finally, we can derive gradients for network parameters to get

$$\frac{dE_j^n(x)}{dv_i} = -jh\left[x(x-1){w_i}^2\sigma''(z_i(x,j)) + 2(2x-1)w_i\sigma'(z_i(x,j)) + 2\sigma(z_i(x,j))\right]$$

$$+x(x-1)\left[j\sigma(z_i(x,j)) - (j-1)\sigma(z_i(x,j-1))\right],$$

$$\frac{dE_j^n(x)}{dw_i} = -jhx(x-1)v_i\left[{w_i}^2\sigma'''(z_i(x,j))x + 2w_i\sigma''(z_i(x,j))\right] - 2jh(2x-1)v_i\left[w_i\sigma''(z_i(x,j))x + \sigma'(z_i(x,j))\right]$$

$$-2jhxv_i\sigma'(z_i(x,j)) - hx^2(x-1)v_i\left[j\sigma'(z_i(x,j)) - (j-1)\sigma'(z_i(x,j-1))\right],$$

$$\frac{dE_j^n(x)}{dk_i} = -j^2h^2x(x-1)v_i{w_i}^2\sigma'''(z_i(x,j)) - 2j^2h^2(2x-1)v_iw_i\sigma''(z_i(x,j)) - 2j^2h^2v_ij\sigma'(z_i(x,j))$$

$$+h^2x(x-1)\left[j^2v_i\sigma'(z_i(x,j)) - (j-1)^2v_i\sigma'(z_i(x,j-1))\right],$$

$$\frac{dE_j^n(x)}{db_i} = -jhx(x-1)v_i{w_i}^2\sigma'''(z_i(x,j)) - jh2(2x-1)v_iw_i\sigma''(z_i(x,j)) - 2jhv_i\sigma'(z_i(x,j))$$

$$+hx(x-1)\left[v_i\sigma'(jz_i(x,j)) - (j-1)v_i\sigma'(z_i(x,j-1))\right],$$

3 Examples

Consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \pi^2 \sin \pi x, \quad 0 < x < 1,\ 0 < t \leq 1,$$

$$u(x,0) = 0, \quad 0 \leq t \leq 1,$$

$$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq 1.$$

The exact solution is given by $u(x,t) = (1 - e^{\pi^2 t}) \sin \pi x$, $(x,t) \in [0,1] \times [0,1]$. We take the NN solution as

$$\hat{u}(x,t) = tx(1-x)N(x,t)$$

where $N(x,t) = \sum_{i=1}^{m} v_i \sigma(z_i)$, $z_i = w_i x + k_i t + b_i$, $\Theta = \{\theta_i = (v_i, w_i, b_i)\}_{i=1}^{m}$ is the parameter set and the activation sigmoid function is $\sigma(x) = 1/(1 + e^{-x})$. We set

$$N(x_p,t_q)=\sum_{i=1}^{m} v_i\sigma(z_i),\quad z_i(x,t)=w_ix_p+k_it_q+b_i,\quad 0\le x\le 1,\ 0\le t\le 1,\ p=0,1,\dots,n_x;\ q=0,1,\dots,n_t.$$

Let

$$F=\frac{1}{(n_x-1)n_t}\sum_{p=1}^{n_x-1}\sum_{q=1}^{n_t}\sum_{i=1}^{m}\left[v_ik_i\sigma'(z_i(x_p,t_q))-v_iw_i^2\sigma''(z_i(x_p,t_q))\right]-\frac{\pi^2 m}{n_x-1}\sum_{p=1}^{n_x-1}\sin\pi x_p,$$

$$G=\frac{1}{n_x-1}\sum_{p=1}^{n_x-1}\sum_{i=1}^{m}v_i\sigma(w_ix_p+b_i)+\frac{1}{n_t+1}\sum_{q=0}^{n_t}\sum_{i=1}^{m}v_i\sigma(k_it_q+b_i)+\frac{1}{n_t+1}\sum_{q=0}^{n_t}\sum_{i=1}^{m}v_i\sigma(w_i+k_it_q+b_i).$$

$$E=\frac{1}{2}F^2+\frac{\tau}{2}G^2,$$

We have

$$\frac{\partial E}{\partial\theta}=E\frac{\partial F}{\partial\theta}+\tau G\frac{\partial F}{\partial\theta},\quad \theta\in\{v_i,w_i,k_i,b_i\}.$$

now,

$$\frac{\partial F}{\partial v_i}=\frac{1}{(n_x-1)n_t}\sum_{p=1}^{n_x-1}\sum_{q=1}^{n_t}\left[k_i\sigma'(z_i(x_p,t_q))-w_i^2\sigma''(z_i(x_p,t_q))\right]$$

$$\frac{\partial G}{\partial v_i}=\frac{\tau}{n_x-1}\sum_{p=1}^{n_x-1}\sigma(w_ix_p+b_i)+\frac{\tau}{n_t+1}\sum_{q=0}^{n_t}\sigma(k_it_q+b_i)+\frac{\tau}{n_t+1}\sum_{q=0}^{n_t}\sigma(w_i+k_it_q+b_i).$$

$$\frac{\partial F}{\partial w_i}=\frac{1}{(n_x-1)n_t}\sum_{p=1}^{n_x-1}\sum_{q=1}^{n_t}\left[v_i(k_ix_p-2w_i)\sigma''(z_i(x_p,t_q))-v_iw_i^2x_p\sigma'''(z_i(x_p,t_q))\right]$$

$$\frac{\partial G}{\partial w_i}=\frac{\tau}{n_x-1}\sum_{p=1}^{n_x-1}v_ix_p\sigma'(w_ix_p+b_i)+\frac{\tau}{n_t+1}\sum_{q=0}^{n_t}v_i\sigma'(w_i+k_it_q+b_i).$$

$$\frac{\partial F}{\partial k_i}=\frac{1}{(n_x-1)n_t}\sum_{p=1}^{n_x-1}\sum_{q=1}^{n_t}\left[v_i\sigma'(z_i(x_p,t_q))+v_ik_it_q\sigma''(z_i(x_p,t_q))-v_iw_i^2t_q\sigma'''(z_i(x_p,t_q))\right]$$

$$\frac{\partial G}{\partial k_i}=\frac{\tau}{n_t+1}\sum_{q=0}^{n_t}v_it_q[\sigma'(k_it_q+b_i)+\sigma'(w_i+k_it_q+b_i)].$$

$$\frac{\partial F}{\partial b_i}=\frac{1}{(n_x-1)n_t}\sum_{p=1}^{n_x-1}\sum_{q=1}^{n_t}\left[v_ik_i\sigma''(z_i(x_p,t_q))-v_iw_i^2\sigma'''(z_i(x_p,t_q))\right]$$

$$\frac{\partial G}{\partial b_i}=\frac{\tau}{n_t+1}\sum_{q=0}^{n_t}v_i[\sigma'(k_it_q+b_i)+\sigma'(w_i+k_it_q+b_i)].$$

Recurrent NN: We take $u(x_p, 0) = U_0(x_p)$ and

$$u(x_p, t_q) = U_0(x_p) + (1 - e^{-\alpha^2 t_q}) x_p(x_p - 1) \sum_{i=1}^{m} v_i \sigma(z(i,p,q)),$$

where $x_p = p/nx$, $p = 0, 1, \ldots, nx$; are $nx + 1$ points on the inerval $[0, 1]$, $t_q = qT/nt$, $q = 0, 1, 2, \ldots, nt$; are the $nt + 1$ points on the interval $[0, T]$, and

$$z(i,p,q) = w_i x_p + k_i t_q + l_i u(x_p, t_{q-1}) + b_i,$$

$$i = 1, 2, \ldots, m;\ p = 0, 1, 2, \ldots, nx;\ q = 0, 1, 2, \ldots, nt.$$

We have $\partial u(x_p, 0)/\partial x = \partial U_0(x_p)/\partial x$ and $\partial^2 u(x_p, 0)/\partial x^2 = \partial^2 U_0(x_p)/\partial x^2$. To simplify notations, we write

$$s_i = \sigma(z(i,p,q),\ a = 1 - e^{-\alpha^2 t_q},\ a_1 = \alpha^2 e^{-\alpha^2 t_q},\ b = x_p(x_p - 1),\ b_1 = 2x_p - 1.$$

$$u = u(x_p, t_q),\ u_- = u(x_p, t_{q-1}),\ u_{-,\xi} = (\partial u_-/\partial \xi),\ u_{-,\xi\eta} = (\partial^2 u_-/\partial \xi \partial \eta),\ \text{etc.}$$

we calculate

$$(\partial u/\partial t) = a_1 b \sum_{i=1}^{m} v_i s_i + ab \sum_{i=1}^{m} v_i (k_i + l_i(u_{-,t}) s_i'.$$

Similarly,

$$\partial u/\partial x = U_0'(x_p) + ab_1 \sum_{i=1}^{m} v_i s_i + ab \sum_{i=1}^{m} v_i (w_i + l_i(u_{-,x})) s_i',$$

and

$$\partial^2 u/\partial x^2 = U_0''(x_p) + 2a \sum_{i=1}^{m} v_i s_i + 2ab_1 \sum_{i=1}^{m} v_i (w_i + l_i(u_{-,x})) s_i'$$

$$+ab \sum_{i=1}^{m} v_i l_i (u_{-,xx}) s_i' + ab \sum_{i=1}^{m} v_i (w_i + l_i(u_{-,x}))^2 s_i''.$$

The cost function is given by

$$E = \frac{1}{2(nt-1)(nx-2)} \sum_{p=1}^{nx-1} \sum_{q=1}^{nt} \left((\partial u/\partial t) - (\partial^2 u/\partial x^2) - f(x_p, t_q) \right)^2.$$

We express $(\partial^2 u/\partial x \partial \theta)$, $(\partial^2 u/\partial t \partial \theta)$, $(\partial^3 u/\partial x^2 \partial \theta)$ in terms of $(\partial u_-/\partial x)$, $(\partial u_-/\partial t)$, $(\partial^2 u_-/\partial x^2)$, $(\partial^2 u_-/\partial x \partial \theta)$, $(\partial^2 u_-/\partial t \partial \theta)$, $(\partial^3 u_-/\partial x^2 \partial \theta)$.

We get

$$(\partial u/\partial v_i) = abs_i + abv_i l_i (u_{-,v_i}) s_i',$$

$$(\partial u/\partial w_i) = abv_i (x_p + l_i(u_{-,w_i})) s_i',$$

$$(\partial u/\partial k_i) = abv_i (t_q + l_i(u_{-,k_i})) s_i',$$

$$(\partial u/\partial b_i) = abv_i (1 + l_i(u_{-,b_i})) s_i',$$

$$(\partial u/\partial l_i) = abv_i (u_- + l_i(u_{-,l_i})) s_i',$$

$$(\partial u/\partial \alpha) = 2\alpha t_q(1-a)b\sum_{i=1}^{m} v_i s_i + ab\sum_{i=1}^{m} v_i l_i(u_{-,\alpha})s_i'.$$

Furthermore, we have

$$(\partial^2 u/\partial t\partial v_i) = a_1 b s_i + a_1 b v_i l_i(u_{-,v_i})s_i' + ab(k_i + l_i(u_{-,t}))s_i'$$
$$+abv_i l_i(u_{-,tv_i})s_i' + abv_i l_i(k_i + l_i(u_{-,t}))(u_{-,v_i})s_i''.$$

$$(\partial^2 u/\partial x\partial v_i) = ab_1 s_i + ab_1 v_i(u_{-,v_i})s_i' + ab(w_i + l_i(u_{-,x}))s_i',$$
$$+abv_i l_i(u_{-,xv_i})s_i' + abv_i l_i(w_i + l_i(u_{-,x}))(u_{-,v_i})s_i'',$$

$$(\partial^3 u/\partial x^2\partial v_i) = 2as_i + 2av_i l_i(u_{-,v_i}))s_i' + 2ab_1(w_i + l_i(u_{-,x}))s_i'$$
$$+2ab_1 v_i l_i(u_{-,xv_i}))s_i' + 2ab_1 v_i l_i(w_i + l_i(u_{-,x}))(u_{-,v_i})s_i''$$
$$+abl_i(u_{-,xx}))s_i' + abv_i l_i(u_{-,xxv_i})s_i' + abv_i l_i^2(u_{-xx})(u_{-,v_i})s_i''$$
$$+ab(w_i + l_i(u_{-,x}))^2 s_i'' + 2abv_i l_i(w_i + l_i((u_{-,x}))(u_{-,v_i})s_i''$$
$$+abv_i l_i(w_i + l_i(u_{-,x}))^2(u_{-,v_i})s_i'''.$$

Similarly,

$$(\partial^2 u/\partial t\partial w_i) = a_1 bv_i(x_p + l_i(u_{-,w_i}))s_i' + abv_i l_i(u_{-,tw_i})s_i' + abv_i(k_i + l_i(u_{-,t}))(x_p + l_i(u_{-,w_i}))s_i''.$$

$$(\partial^2 u/\partial x\partial w_i) = ab_1 v_i(x_p + l_i(u_{-,w_i}))s_i' + abv_i(1 + l_i(u_{-,xw_i}))s_i' + abv_i(w_i + l_i(u_{-,x}))(x_p + l_i(u_{-,w_i}))s_i''$$

$$(\partial^3 u/\partial x^2\partial w_i) = 2av_i(x_p + l_i(u_{-,w_i}))s_i' + 2ab_1 v_i(1 + l_i(u_{-,xw_i}))s_i'$$
$$+2ab_1 v_i(w_i + l_i(u_{-,x}))(x_p + l_i(u_{-,w_i}))s_i'' + abv_i l_i(u_{-,xxw_i})s_i'$$
$$+abv_i l_i(u_{-,xx})(x_p + l_i(u_{-,w_i}))s_i'' + 2abv_i(w_i + l_i(u_{-,x}))(1 + l_i(u_{-,xw_i}))s_i''$$
$$+abv_i(w_i + l_i(u_{-,x}))^2(x_p + l_i(u_{-,w_i}))s_i'''.$$

$$(\partial^2 u/\partial t\partial k_i) = a_1 bv_i(t_q + l_i(u_{-,k_i}))s_i' + abv_i(1 + l_i(u_{-,tk_i}))s_i'$$
$$+abv_i(k_i + l_i(u_{-,t}))(t_q + l_i(u_{-,k_i}))s_i'',$$

$$(\partial^2 u/\partial x\partial k_i) = ab_1 v_i(t_q + l_i(u_{-,k_i}))s_i' + abv_i l_i(u_{-,xk_i})s_i'$$
$$+abv_i(w_i + l_i(u_{-,x}))(t_q + l_i(u_{-,k_i}))s_i''$$

$$(\partial^3 u/\partial x^2\partial k_i) = 2av_i(t_q + l_i(u_{-,k_i}))s_i' + 2ab_1 v_i l_i(u_{-,xk_i})s_i' + 2ab_1 v_i(w_i + l_i(u_{-,x}))(t_q + l_i(u_{-,k_i}))s_i''$$
$$+abv_i l_i(u_{-,xxk_i})s_i' + abv_i l_i(u_{,xx})(t_q + l_i(u_{-,k_i}))s_i''$$

$$+2abv_i(w_i+l_i(u_{-,x}))l_i(u_{-,xk_i})s_i''+abv_i(w_i+l_i(u_{-,x}))^2(t_q+l_i(u_{-,k_i}))s_i'''.$$

$$(\partial^2 u/\partial t\partial b_i)=a_1bv_i(1+l_i(u_{-,b_i}))s_i'+abv_il_i(u_{-,tb_i})s_i'+abv_i(k_i+l_i(u_{-,t}))(1+l_i(u_{-,b_i}))s_i''.$$

$$(\partial^2 u/\partial x\partial b_i)=ab_1v_i(1+l_i(u_{-,b_i}))s_i'+abv_il_i(u_{-,xb_i})s_i'+abv_i(w_i+l_i(u_{-,x}))(1+l_i(u_{-,b_i}))s_i''.$$

$$(\partial^3 u/\partial x^2\partial b_i)=2av_i(1+l_i(u_{-,b_i}))s_i'+2ab_1v_il_i(u_{-,xb_i})s_i'+2ab_1v_i(w_i+l_i(u_{-,x}))(1+l_i(u_{-,b_i}))s_i''$$
$$+abv_il_i(u_{-,xxb_i})s_i'+abv_il_i(u_{-,xx})(1+l_i(u_{-,b_i}))s_i''+2abv_il_i(w_i+l_i(u_{-,x}))(u_{-,xb_i})s_i''$$
$$+abv_i(w_i+l_i(u_{-,x}))^2(1+l_i(u_{-,b_i}))s_i'''.$$

$$(\partial^2 u/\partial t\partial l_i)=a_1bv_i(u_-+l_i(u_{-,l_i}))s_i'+abv_i((u_{-,t})+l_i(u_{-,tl_i}))s_i'+abv_i(k_i+l_i(u_{-,t}))(u_-+l_i(u_{-,l_i}))s_i'',$$

$$(\partial^2 u/\partial x\partial l_i)=ab_1v_i(u_-+l_i(u_{-,l_i}))s_i'++abv_i((u_{-,x})+l_i(u_{-,xl_i}))s_i'+abv_i(w_i+l_i(u_{-,x}))(u_-+l_i(u_{-,l_i}))s_i'',$$

$$(\partial^3 u/\partial x^2\partial l_i)=2av_i(u_-+l_i(u_{-,l_i}))s_i'+2ab_1v_i((u_{-,x})+l_i(u_{-,xl_i}))s_i'+2ab_1v_i(w_i+l_i(u_{-,x}))(u_-+l_i(u_{-,l_i}))s_i''$$
$$+abv_i((u_{-,xx})+l_i(u_{-,xxl_i}))s_i'+abv_il_i(u_{-,xx})(u_-+l_i(u_{-,l_i}))s_i''$$
$$+2abv_i(w_i+l_iu_{-,x})((u_{-,x})+l_i(u_{-,xl_i}))s_i''+abv_i(w_i+l_i(u_{-,x}))^2(u_-+l_i(u_{-,l_i}))s_i'''.$$

Finally,

$$(\partial^2 u/\partial t\partial\alpha)=2\alpha(1-a)(1-\alpha^2t_q)b\sum_{i=1}^m v_is_i+a_1b\sum_{i=1}^m v_il_i(u_{-,\alpha})s_i'+2\alpha t_q(1-a)b\sum_{i=1}^m v_i(k_i+l_i(u_{-,t}))s_i'$$
$$+ab\sum_{i=1}^m v_il_i(u_{-,t\alpha})s_i'+ab\sum_{i=1}^m v_il_i(k_i+l_i(u_{-,t}))(u_{-,\alpha})s_i'',$$
$$(\partial^2 u/\partial x\partial\alpha)=2\alpha t_q(1-a)b_1\sum_{i=1}^m v_is_i+ab_1\sum_{i=1}^m v_il_i(u_{-,\alpha})s_i'+2\alpha t_q(1-a)b\sum_{i=1}^m v_i(w_i+l_i(u_{-,x}))s_i'$$
$$+ab\sum_{i=1}^m v_il_i(u_{-,x\alpha})s_i'+ab\sum_{i=1}^m v_il_i(w_i+l_i(u_{-,x}))(u_{-,\alpha})s_i'',$$
$$(\partial^3 u/\partial x^2\partial\alpha)=4\alpha t_q(1-a)\sum_{i=1}^m v_is_i+2a\sum_{i=1}^m v_il_i(u_{-,\alpha})s_i'+4\alpha t_q(1-a)b_1\sum_{i=1}^m v_i(w_i+l_i(u_{-,x}))s_i'$$
$$+2ab_1\sum_{i=1}^m v_il_i(u_{-,x\alpha})s_i'+2ab_1\sum_{i=1}^m v_il_i(w_i+l_i(u_{-,x}))(u_{-,\alpha})s_i''$$

$$+2\alpha t_q(1-a)b\sum_{i=1}^{m} v_i l_i(u_{-,xx})s_i' + ab\sum_{i=1}^{m} v_i l_i(u_{-,xx\alpha})s_i' + ab\sum_{i=1}^{m} v_i l_i^2(u_{-,xx})(u_{-,\alpha})s_i''$$

$$2\alpha t_q(1-a)b\sum_{i=1}^{m} v_i(w_i + l_i(u_{-,x}))^2 s_i'' + 2ab\sum_{i=1}^{m} v_i l_i(w_i + l_i(u_{-,x}))(u_{-,x\alpha})s_i''$$

$$ab\sum_{i=1}^{m} v_i l_i(w_i + l_i(u_{-,x}))^2(u_{-,\alpha})s_i'''.$$

References

[1] George A. Anastassiou, Intelligent Systems II: Complete Approximation by Neural Network Operators, Studies in Computational Intelligence Vol. 608, Springer International Publishing, Switzerland, 2016.

[2] A. M. Bruaset and A. Tveito, Numerical Solution of Partial Differential Equations on Parallel Computers, Springer-Verlag, 2006.

[3] J. Fang, C. Liu , T.E. Simos and I.T. Famelis, Neural Network Solution of Single-Delay Differential Equations, Mediterr. J. Math. (2020) 17:30, https://doi.org/10.1007/s00009-019-1452-51660-5446/20/010001-15

[4] H. Allia, A. Ucar and Y. Demir, The solutions of vibration control problems using artificial neural networks, Journal of the Franklin Institute 340 (2003) 307–325.

[5] Z. Liu, Y. Yang, Q. Cai, Neural network as a function approximator and its application in solving differential equations, Appl. Math. Mech. -Engl. Ed., 40(2), 237–248 (2019), https://doi.org/10.1007/s10483-019-2429-8

[6] E. Shi and C. Xu, A comparative investigation of neural networks in solving differential equations, Journal of Algorithms & Computational Technology, Vol. 15: 1–15.

[7] R. S. Beidokhti and A. Malek, Solving initial-boundary value problems for systems of partial differential equations using neural networks and optimization techniques, Journal of the Franklin Institute 346 (2009) 898–913.

[8] L. Ruthotto1 and E. Haber, Deep Neural Networks Motivated by Partial Differential Equations, Journal of Mathematical Imaging and Vision (2020) 62:352–364 https://doi.org/10.1007/s10851-019-00903-1

[9] S. Chakraverty and S. Mall, Artificial Neural Networks for Engineers and Scientists Solving Ordinary Differential Equations, CRC Press, Taylor & Francis Group, Boca Raton, 2017.

[10] E. Kharazmi, Z. Zhang G.E. Karniadakis, VPINNs: Variational Physics-informed Neural Networks for solving Partial Differential Equations, arXiv:1912.00873 [cs.NE]