Bayes' Theorem

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Outline

1. Introduction
2. Activities (Think-Pair-Share)

Formula

The general version of Bayes' theorem is given by:

$$P(A_i \mid B) = \frac{P(B \mid A_i) \cdot P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j) \cdot P(A_j)}$$

$A_1, A_2, \ldots, A_n$: A set of mutually exclusive and collectively exhaustive events.

$P(A_i \mid B)$: The posterior probability of A_i given B.

$P(B \mid A_i)$: The likelihood of B given A_i.

$P(A_i)$: The prior probability of A_i.

$\sum_{j=1}^{n} P(B \mid A_j) \cdot P(A_j)$: The total probability of B.

☞Activity 1

Problem: Drawing a Red Ball from a Bag

You have two bags:

Bag 1 contains 3 red balls and 7 blue balls.

Bag 2 contains 5 red balls and 5 blue balls.

You randomly select one of the two bags with equal probability. Then, you randomly draw one ball from the chosen bag. If the ball is red, what is the probability it came from Bag 2?

☞Activity 1

Solution:

☞Activity 1

Solution:

Let:

A_1: The event that Bag 1 is chosen.

A_2: The event that Bag 2 is chosen.

B: The event of drawing a red ball.

We want to calculate $P(A_2 \mid B)$, the probability that the red ball came from Bag 2.

☞Activity 1

Solution:

Using Bayes' theorem:

$$P(A_2 \mid B) = \frac{P(B \mid A_2) \cdot P(A_2)}{P(B)}$$

The bags are chosen with equal probability:

$$P(A_1) = P(A_2) = 0.5.$$

The probability of drawing a red ball from each bag is:

$$P(B \mid A_1) = \frac{3}{10}, \quad P(B \mid A_2) = \frac{5}{10}.$$

☞Activity 1

Solution:

Using the law of total probability:

$$P(B) = P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2)$$

$$P(B) = \left(\frac{3}{10} \cdot 0.5\right) + \left(\frac{5}{10} \cdot 0.5\right)$$

$$P(B) = \frac{3}{20} + \frac{5}{20} = \frac{8}{20} = 0.4.$$

Substitute into Bayes' theorem: $P(A_2 \mid B) = \frac{P(B|A_2) \cdot P(A_2)}{P(B)}$

$P(A_2 \mid B) = \frac{\frac{5}{10} \cdot 0.5}{0.4}$ $P(A_2 \mid B) = \frac{0.25}{0.4} = 0.625.$

☞ Activity 2

Problem: Coin Toss with a Biased Coin

You have two coins:

Coin 1 is a fair coin ($P(\text{Heads}) = 0.5$).

Coin 2 is a biased coin ($P(\text{Heads}) = 0.75$).

You randomly pick one of the two coins with equal probability and toss it. The result is Heads. What is the probability that you picked Coin 2?

☞ Activity 2

Coin Toss with a Biased Coin

Solution:

☞ Activity 2

Coin Toss with a Biased Coin

Solution:

Let:

A_1: The event of choosing Coin 1.

A_2: The event of choosing Coin 2.

B: The event of getting Heads.

We want to calculate $P(A_2 \mid B)$, the probability that Coin 2 was chosen given that the toss resulted in Heads.

Using Bayes' theorem:

$$P(A_2 \mid B) = \frac{P(B \mid A_2) \cdot P(A_2)}{P(B)}$$

☞ Activity 2

Coin Toss with a Biased Coin

Solution: The coins are chosen with equal probability:

$$P(A_1) = P(A_2) = 0.5.$$

The probabilities of getting Heads with each coin are:

$$P(B \mid A_1) = 0.5, \quad P(B \mid A_2) = 0.75.$$

☞ Activity 2

Coin Toss with a Biased Coin

Using the law of total probability:

$$P(B) = P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2)$$

$$P(B) = (0.5 \cdot 0.5) + (0.75 \cdot 0.5)$$

$$P(B) = 0.25 + 0.375 = 0.625.$$

Substitute into Bayes' theorem:

$$P(A_2 \mid B) = \frac{P(B \mid A_2) \cdot P(A_2)}{P(B)}$$

$$P(A_2 \mid B) = \frac{0.75 \cdot 0.5}{0.625}$$

$P(A_2 \mid B) = \frac{0.375}{0.625} = 0.6.$


THANK YOU