EE2160 CAD Laboratory - Python Exercise 4

Finite Differences Method

1 Theory

1.1 1-D Functions

Consider the second-order differential equation

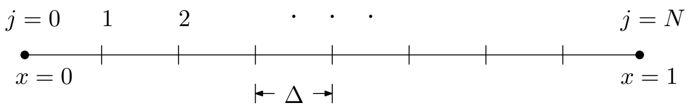
$$\frac{d^2\Phi(x)}{dx^2} = f(x), \text{ for } x \in [0, 1],$$

with the boundary conditions

$$\Phi(0) = g, \ \left.\frac{d\Phi(x)}{dx}\right|_{x=1} = h.$$

Steps:

1. Discretize the domain into N segments each of length Δ



2. Use finite differences approximation of the derivative for the interior points, $j = 1, \ldots, N-1$,

$$\left.\frac{d^2\Phi(x)}{dx^2}\right|_{x=x_j} = \frac{\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}}{\Delta^2} = f(x_j).$$

3. Incorporating the Dirichlet boundary condition, $\Phi(0) = g$:

$$\Phi_0 = g.$$

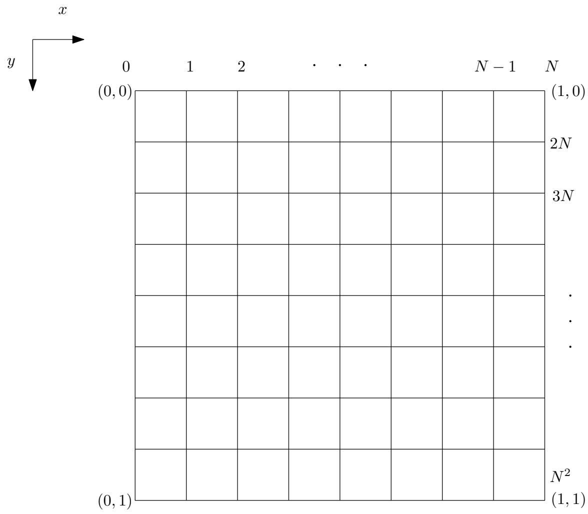
4. Incorporating the Neumann boundary condition, $\left.\frac{d\Phi(x)}{dx}\right|_{x=1} = h$:

$$\left.\frac{d\Phi(x)}{dx}\right|_{x=x_N} = \frac{\Phi_N - \Phi_{N-1}}{\Delta} = h.$$

5. Solve the system of linear equations for $\Phi_0, \ldots, \Phi_N$

1.2 2-D Functions

1. Discretize the domain into N^2 segments each of length Δ in $x-$ and y-directions



2. Approximate the Laplace operator at (m, n) as

$$\nabla^2 \Phi|_{(m,n)} = \frac{\Phi_{m+1,n} + \Phi_{m-1,n} + \Phi_{m,n+1} + \Phi_{m,n-1} - 4\Phi_{m,n}}{\Delta^2}$$

3. Incorporate boundary conditions in a manner similar to before

4. Generate the linear system of equations and then solve it

2 Exercises

1-D Functions

1. Solve the second order differential equation in Section 1.1 analytically for $f(x) = -1$, $g = 0$, and $h = 0$.

2. Write a Python program to solve the differential equation via the finite differences method.

3. Define error $e(N)$ as a function of discretization points as

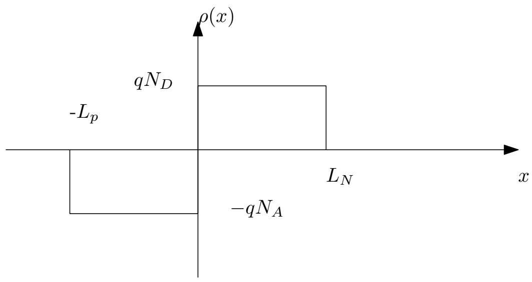
$$e(N) = \frac{1}{N+1} \sum_{j=0}^{N} \frac{|\Phi_{\text{numerical}}(x) - \Phi_{\text{analytical}(x)}|}{|\Phi_{\text{analytical}(x)}|}.$$

Plot error $e(N)$ versus N.

4. Consider the following differential equation for a P-N diode with $N_A = N_D = 10^{15}\,\text{cm}^{-3}$, $L_N = L_P = 1\mu m$, and $\epsilon_s = 12\epsilon_0$:

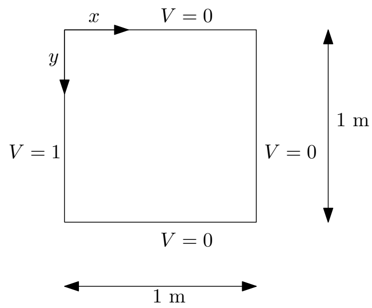
$$\frac{d^2\Phi}{dx^2} = -\frac{\rho(x)}{\epsilon_s}.$$

Solve for $\Phi(x)$ assuming $\Phi(-3L_P) = 0$ and $\Phi'(3L_N) = 0$.



2-D Laplace Equation

Write a Python program to solve $\nabla^2 V = 0$ with the boundary conditions $V(0, y) = 1$ V, $V(1, y) = 0$ V, $V(x, 0) = 0$ V, and $V(x, 1) = 0$ V.



Compute the error with the analytic solution:

$$V(x, y) = \sum_{n=1,3,5,\ldots} \left(\frac{4}{n\pi \sinh(n\pi)} \right) \sinh(m\pi(1 - x)) \sin(m\pi y).$$