Hints for 2023-09-18

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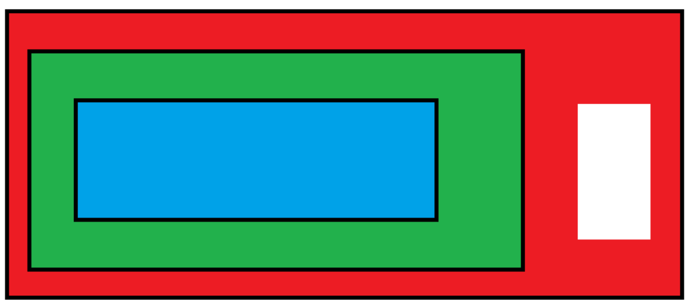
If you are stuck on any problems / hints for too long, look up "CLMC getting ready", then scroll down to "Try the Challenging Practice Problems". The complete solution should be there.

1 Hints for Problem 1

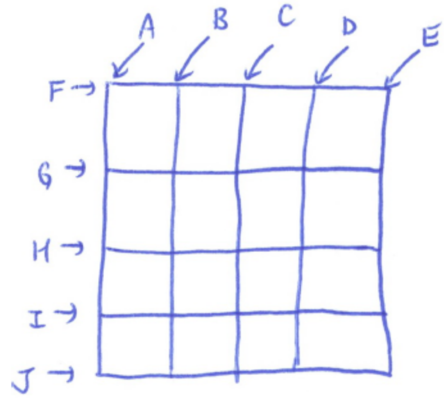
Clarification: the "True" in the problem means "ALWAYS True". Consider the following diagram:



1. The problems said, "All yoms are yems". Should the yom rectangle completely include the yem rectangle, or the other way round? Or in other words, should the yom rectangle be smaller or larger than the yem rectangle?
2. Match Red, Green and Blue to yoms, yums and yems.
3. Can the white rectangle in the figure be blue? What color(s) could it be otherwise?
4. Can some rectangles(red and green, green and blue, red and green and blue) have the same size?
5. (Extra thoughts: what if I changed the question to "How many of the statements COULD BE TRUE?)

2 Hints for Problem 2

There are a few alternatives that you can do with this problems:



1. How many 1*1 squares/rectangles are there? What about 1*2? (rectangle going downwards?) 2*1? (rectangle going horizontally) 3*2 (horizontal 3, vertical 2)? (There should be 6) Repeat this process with all possible dimensions and add them all up.
2. How many squares/rectangles have THE UPPER-RIGHT CORNER of BI? What about CI, BH and CH? (Alternatively, you are counting how many possible lower-left corner are there for a fixed upper-right corner) Repeat this with all the points in the grid and add them all up.
3. (Advanced but quick) First, try to analyze where could your rectangle/square's vertical sides lie on. Forget the length of the sides, just which column does your vertical sides lie on. One possibility is: the left vertical side lies on B, and the right vertical side lies on E.
   (a) List the rest of the possibilities for the vertical sides. How many possibilities are there in total (including the given example)?
   (b) List all of the possibilities for the horizontal sides. How many are there?
   (c) Convince yourself that the final answer should be the product of the above two answers. (Hint: side B, E, G, H gives a possible rectangle, and sides C, D, F, J gives another)
   (d) (Further thinking) If you don't list all the possibilities, can you still count how many possibilities for 1 and 2? If you know what C_r^n is, try to use that. Or else, prove that the answer should be $\frac{(5)(4)}{2}$
   (e) (Further thinking) Why are the answers of 1 and 2 the same?

3 Hints for Problem 3 (Very Challenging!)

1. $\frac{4}{4+1}+\frac{4}{4+2}+\frac{4}{4+4}+\frac{4}{4+8}+\frac{4}{4+16}=\frac{4}{5}+\frac{2}{3}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}$. Do we have to find the lowest common denominator (or the least common multiple) of 2, 3 and 5, or is there a more clever way to add these numbers? Check to see if the answer is $\frac{5}{2}$.

2. Calculate $f(9)$.

3. Prove that your "little trick" of adding works works for any n. (Hint: $1*16=2*8=4*4=16=4^2$, try this for other ns to see the trick. Then, pair up the first fraction with the last, then second fraction with the second last, up until the middle fraction. Ignore the middle fraction for now.)

4. prove that the middle fraction is always $\frac{1}{2}$.

5. prove that if n^2 has m factors, then $f(n)=\frac{m}{2}$.

6. Find the prime factorization of 2023.

7. Find how many positive factors does 2023^2 have, using the following theorem (Source: look up "cemc contest properties of numbers", it is a pdf)

- If $n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is the unique representation of a positive integer $n\geq 2$ as a product of distinct primes $p_1,p_2,\ldots,p_k$, then n has $(a_1+1)(a_2+1)\cdots(a_k+1)$ positive divisors.

8. Prove that from the given, $(2023-2x)^2$ has 15 positive factors

9. Prove that the prime factorization of $2023-2x$ is of the form p^7 or qr^2, where p, q and r are primes.

10. Prove that p^7 is impossible. (Hint: notice that $2023-2x$ is odd. Can $p=2$? what about $p=3$? From these two cases, you should see why this case is impossible.

11. We now know $2023-2x=qr^2$, and now we have two choices here: fixing prime q to find the largest perfect square of a prime r, or fixing prime r and find the largest q. Even though both options will lead you to a correct answer, it turns out that the latter is much more feasible in contest, without a calculator. After finishing the quicker solution, try to come back here later and do the slower way.

12. If $r=3$, $r=5$ and $r=7$, what is the largest q? What is the corresponding x?(Also, why r cannot be 2?) (The x for $r=7$ should be quite small!)

13. (FINALLY!!!) Show that no smaller x works.

4 Complete Solutions

Look up "CLMC getting ready", then scroll down to "Try the Challenging Practice Problems".