Selected Topics in Probability Theory

Lecture Notes 2026

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Vorwort

Diese Notizen begleiten die Mastervorlesung „Selected Topics in Probability Theory“. Sie sind auf Englisch abgefasst, also in der Sprache der Vorlesung. Das Ziel ist es diese Notizen regelmässig zu aktualisieren und hochzuladen. Die Literatur zur Vorlesung finden Sie im Vorlesungsverzeichnis.

> **Achtung: Diese Mitschrift wird sicherlich viele Typos enthalten. Ich bin dankbar für jegliche Korrekturhinweise!**

[todo: Quellenverzeichnis]

Organization

- No exam at the end, just hand in exercises.
- (official) notes are on ADAM.

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0 Introduction

[todo: Muss noch abgetippt werden]

1 Bond Percolation

We start with a few reminders from graph theory.

Definition 1.1. A (unoriented) graph is a tuple $G = (V, E)$ where V is the set of vertices/sites and E is the set of edges/bonds, i.e.,

$$E \subseteq \{\{x, y\} : x, y \in V\}.$$

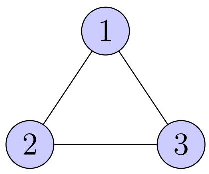
If $e = \{x, y\} \in E$ is an edge we call x, y endpoints. We also say that x, y are neighbours in the graph G.

We will mostly denote edges by e, f and vertices by x, y, z.

Remark. Graphs are almost always represented in pictorial way. For example the graph $G = (V, E)$ with

$$V = \{1, 2, 3\}, \quad E = \Big\{\{1, 2\}, \{2, 3\}, \{3, 1\}\Big\}$$

may be represented by



We will mostly use a certain type of graph called hypercatic lattice: $G = (\mathbb{Z}^d, E_d)$, where

$$E_d = \{\{x, y\} \mid \|x - y\|_1 = 1\}.$$

[todo: Image in 2-D].

Definition 1.2. Let $G = (V, E)$ be a graph. A path from $x \in V$ to $y \in V$ of length ℓ is a sequence $\gamma = (\gamma_0, \ldots, \gamma_\ell)$ such that $\gamma_0 = x$, $\gamma_\ell = y$ with $\{\gamma_i, \gamma_{i+1}\} \in E$ with γ_i distinct. We define the distance between two points $x, y \in V$ as

$$d(x, y) = \inf \{\text{length}(\gamma) : \gamma \text{ is a path from } x \text{ to } y\}.$$

A typical distance on $\mathbb{Z}^d$ would be $d(x, y) = \|x - y\|_1$.

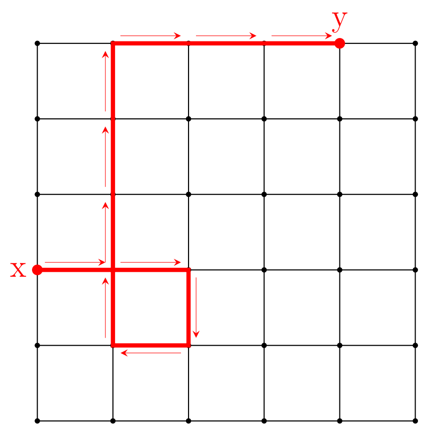


Figure 1.1: This is *not* a path, because we have a loop

Definition 1.3. Let $G = (V, E)$ be a graph and let $A \subseteq V$. We define

$$\partial A := \left\{x \in A : \exists y \in A^\complement \text{ s.t. } x \sim y\right\}$$

as the internal boundary and

$$\partial_e A := \partial A := \left\{x \in A^\complement : \exists y \in A \text{ s.t. } x \sim y\right\}$$

as the external boundary and

$$\Delta A = \{e \in E : e = \{x, y\}, x \in A, y \in A^\complement\}$$

and on $\mathbb{Z}^d$

$$\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d.$$

Let $G = (V, E)$ be a graph, $\Omega = \{0, 1\}^E$ the *percolation configuration.* If $\omega(e) = 1$ we say that the edge e is *open* otherwise, if $\omega(e) = 0$ we say that e is *closed*. Let $\mathcal{A}$ be the σ-algebra on Ω, generated by events depending on the state of only finitely many edges (cylinder σ-algebra). Moreover,

$$\mathbb{P}_p = \mu^{\otimes E}, \quad \text{where} \quad \mu(\{1\}) = 1 - \mu(\{0\}) = p,$$

the *percolation measure*. This measure is characterized as follows: let $e_1, \ldots, e_n \in E$ and $\omega_1, \ldots, \omega_n \in \{0, 1\}$, then

$$\mathbb{P}_p\left(\bigcap_{i=1}^n \{\omega(e_i) = \omega_i\}\right) = p^{\sum_{i=1}^n \omega_i}(1-p)^{\sum_{i=1}^n (1-\omega_i)}.$$

We denote

$$\{x \leftrightarrow y\} := \{\omega \in \Omega : \text{there is an open path from } x \text{ to } y \text{ in } \omega\}.$$

An open path is a path that only contains open edges. For $A, B \subseteq V$ we define

$$\{A \leftrightarrow B\} := \bigcup_{x \in A} \bigcup_{y \in B} \{x \leftrightarrow y\}$$

and

$$\{x \leftrightarrow \infty\} = \bigcap_{n \in \mathbb{N}} \{\exists y \in V : d(x, y) = n, x \leftrightarrow y\}.$$

Exercise 1.4. Show that these sets are events.

Definition 1.5. A *cluster* is a connected component of $(V, \text{“open edges”})$. We write $\mathcal{C}_x$ for a cluster containing $x \in V$.

Monotonicity Let $x, y \in V$. Is the map

$$[0, 1] \ni p \mapsto \mathbb{P}_p(\{x \leftrightarrow y\})$$

increasing? To answer this question rigorously we have to talk about orderings of Ω. Let $\omega, \eta \in \Omega$. We say that

$$\omega < \eta \quad \text{if } \omega(e) \leq \eta(e) \text{ for all } e \in E.$$

(this is obviously not a total order). A function $f : \Omega \to \mathbb{R}$ is said to be increasing if

$$\omega < \eta \Rightarrow f(\omega) \leq f(\eta).$$

We say that $A \in \mathcal{A}$ is increasing if $\omega \mapsto \mathbf{1}_A(\omega)$ is an increasing function. Which is equivalent to

$$\omega \in A, \eta > \omega \Rightarrow \eta \in A.$$

Example 1.6. The event $\{x \leftrightarrow y\}$ is increasing, as well as $\{|\mathcal{C}_x| \geq 17\}$. The event $|\mathcal{C}_x| = 10$ is neither increasing nor decreasing.

Proposition 1.7. With the above notation, the following holds:

1. Let A be an increasing event, then $[0, 1] \ni p \mapsto \mathbb{P}_p(A)$is non-decreasing.
2. If $f : \Omega \to \mathbb{R}$ is increasing, then $[0, 1] \ni p \mapsto \mathbb{E}_p[f]$ is non-decreasing where $\mathbb{E}_p[f] := \int_\Omega f \mathrm{d}\mathbb{P}_p$.

Proof. Note that the second point implies the first. We introduce monotonic coupling: Let

$$\Omega_1 = [0, 1]^E, \quad (U_\ell)_{\ell \in E} \text{ i.i.d. random variables, uniformly on } [0, 1].$$

Then we define $X_p(e) = \mathbf{1}_{U_e \leq p}$, $e \in E$, which is, for p fixed a collection of i.i.d. random variables with $\mathbb{P}(X_p(e) = 1) = p$. This means $(X_p(e))_{e \in E}$ has distribution $\mathbb{P}_p$.

If $p < p'$ we have $X_p(e) \leq X_{p'}(e)$ and thus $X_p < X_{p'}$ as elements of Ω. Now, using that f is increasing,

$$\mathbb{E}_p[f] = \mathbb{E}[f(X_p)] \leq \mathbb{E}[f(X_{p'})] = \mathbb{E}_{p'}[f]. \qquad \square$$

Phase transition in bond percolation Define the percolation probability $\theta(p)$ by

$$\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty) = \mathbb{P}_p(|\mathcal{C}_x| = \infty) = \lim_{n \to +\infty} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)$$

Note that $0 \leftrightarrow \partial\Lambda_n$ is an event as it depends only on finitely many edges meaning that the other guys are also events.

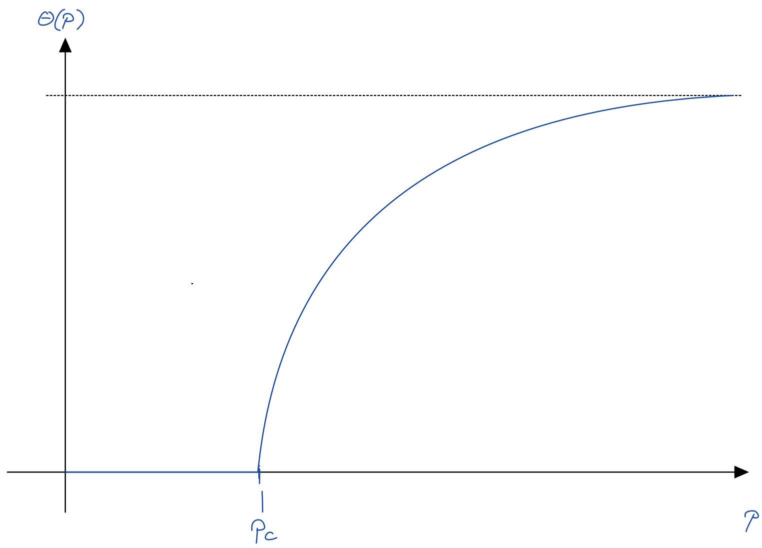


Figure 1.2: possible graph of p_c. But actually no one knows what it looks like.

Define

$$p_c := \inf\{p \in [0,1] : \theta(p) > 0\}.$$