Selected Topics in Probability Theory

Lecture Notes 2026

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Vorwort

Diese Notizen begleiten die Mastervorlesung „Selected Topics in Probability Theory“. Sie sind auf Englisch abgefasst, also in der Sprache der Vorlesung. Das Ziel ist es diese Notizen regelmässig zu aktualisieren und hochzuladen. Die Literatur zur Vorlesung finden Sie im Vorlesungsverzeichnis.

> **Achtung: Diese Mitschrift wird sicherlich viele Typos enthalten. Ich bin dankbar für jegliche Korrekturhinweise!**

[todo: Quellenverzeichnis]

Organization

- No exam at the end, just hand in exercises.
- (official) notes are on ADAM.

Contents

0 Introduction

[todo: Muss noch abgetippt werden]

1 Bond Percolation

We start with a few reminders from graph theory.

Definition 1.1. A (unoriented) graph is a tuple $G = (V, E)$ where V is the set of vertices/sites and E is the set of edges/bonds, i.e.,

$$E \subseteq \{\{x, y\} : x, y \in V\}.$$

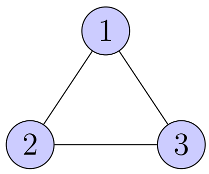
If $e = \{x, y\} \in E$ is an edge we call x, y endpoints. We also say that x, y are neighbours in the graph G.

We will mostly denote edges by e, f and vertices by x, y, z.

Remark. Graphs are almost always represented in pictorial way. For example the graph $G = (V, E)$ with

$$V = \{1, 2, 3\}, \quad E = \Big\{\{1, 2\}, \{2, 3\}, \{3, 1\}\Big\}$$

may be represented by



We will mostly use a certain type of graph called hypercatic lattice: $G = (\mathbb{Z}^d, E_d)$, where

$$E_d = \{\{x, y\} \mid \|x - y\|_1 = 1\}.$$

[todo: Image in 2-D].

Definition 1.2. Let $G = (V, E)$ be a graph. A path from $x \in V$ to $y \in V$ of length ℓ is a sequence $\gamma = (\gamma_0, \ldots, \gamma_\ell)$ such that $\gamma_0 = x$, $\gamma_\ell = y$ with $\{\gamma_i, \gamma_{i+1}\} \in E$ with γ_i distinct. We define the distance between two points $x, y \in V$ as

$$d(x, y) = \inf\{\text{length}(\gamma) : \gamma \text{ is a path from } x \text{ to } y\}.$$

A typical distance on $\mathbb{Z}^d$ would be $d(x, y) = \|x - y\|_1$.

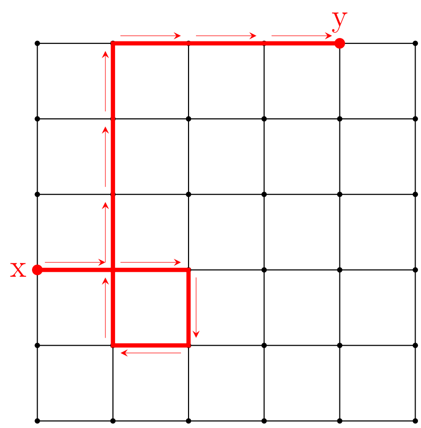


Figure 1.1: This is *not* a path, because we have a loop

Definition 1.3. Let $G = (V, E)$ be a graph and let $A \subseteq V$. We define

$$\partial A := \left\{x \in A : \exists y \in A^{\complement} \text{ s.t. } x \sim y\right\}$$

as the internal boundary and

$$\partial_e A := \partial A := \left\{x \in A^{\complement} : \exists y \in A \text{ s.t. } x \sim y\right\}$$

as the external boundary and

$$\Delta A = \{e \in E : e = \{x, y\}, x \in A, y \in A^{\complement}\}$$

and on $\mathbb{Z}^d$

$$\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d.$$

Let $G = (V, E)$ be a graph, $\Omega = \{0, 1\}^E$ the *percolation configuration.* If $\omega(e) = 1$ we say that the edge e is *open* otherwise, if $\omega(e) = 0$ we say that e is *closed*. Let $\mathcal{A}$ be the σ-algebra on Ω, generated by events depending on the state of only finitely many edges (cylinder σ-algebra). Moreover,

$$\mathbb{P}_p = \mu^{\otimes E}, \quad \text{where} \quad \mu(\{1\}) = 1 - \mu(\{0\}) = p,$$

the *percolation measure*. This measure is characterized as follows: let $e_1, \ldots, e_n \in E$ and $\omega_1, \ldots, \omega_n \in \{0, 1\}$, then

$$\mathbb{P}_p\left(\bigcap_{i=1}^{n}\{\omega(e_i) = \omega_i\}\right) = p^{\sum_{i=1}^{n}\omega_i}(1-p)^{\sum_{i=1}^{n}(1-\omega_i)}.$$

We denote

$$\{x \leftrightarrow y\} := \{\omega \in \Omega : \text{there is an open path from } x \text{ to } y \text{ in } \omega\}.$$

An open path is a path that only contains open edges. For $A, B \subseteq V$ we define

$$\{A \leftrightarrow B\} := \bigcup_{x \in A} \bigcup_{y \in B} \{x \leftrightarrow y\}$$

and

$$\{x \leftrightarrow \infty\} = \bigcap_{n \in \mathbb{N}} \{\exists y \in V : d(x, y) = n, x \leftrightarrow y\}.$$

Exercise 1.4. Show that these sets are events.

Definition 1.5. A *cluster* is a connected component of $(V, \text{“open edges”})$. We write $\mathcal{C}_x$ for a cluster containing $x \in V$.

Monotonicity Let $x, y \in V$. Is the map

$$[0, 1] \ni p \mapsto \mathbb{P}_p(\{x \leftrightarrow y\})$$

increasing? To answer this question rigorously we have to talk about orderings of Ω. Let $\omega, \eta \in \Omega$. We say that

$$\omega < \eta \quad \text{if } \omega(e) \leq \eta(e) \text{ for all } e \in E.$$

(this is obviously not a total order). A function $f : \Omega \to \mathbb{R}$ is said to be increasing if

$$\omega < \eta \Rightarrow f(\omega) \leq f(\eta).$$

We say that $A \in \mathcal{A}$ is increasing if $\omega \mapsto \mathbf{1}_A(\omega)$ is an increasing function. Which is equivalent to

$$\omega \in A, \eta > \omega \Rightarrow \eta \in A.$$

Example 1.6. The event $\{x \leftrightarrow y\}$ is increasing, as well as $\{|\mathcal{C}_x| \geq 17\}$. The event $|\mathcal{C}_x| = 10$ is neither increasing nor decreasing.

Proposition 1.7. With the above notation, the following holds:

1. Let A be an increasing event, then $[0, 1] \ni p \mapsto \mathbb{P}_p(A)$is non-decreasing.
2. If $f : \Omega \to \mathbb{R}$ is increasing, then $[0, 1] \ni p \mapsto \mathbb{E}_p[f]$ is non-decreasing where $\mathbb{E}_p[f] := \int_\Omega f \mathrm{d}\mathbb{P}_p$.

Proof. Note that the second point implies the first. We introduce monotonic coupling: Let

$$\Omega_1 = [0, 1]^E, \quad (U_\ell)_{\ell \in E} \text{ i.i.d. random variables, uniformly on } [0, 1].$$

Then we define $X_p(e) = \mathbf{1}_{U_e \leq p}$, $e \in E$, which is, for p fixed a collection of i.i.d. random variables with $\mathbb{P}(X_p(e) = 1) = p$. This means $(X_p(e))_{e\in E}$ has distribution $\mathbb{P}_p$.

If $p < p'$ we have $X_p(e) \leq X_{p'}(e)$ and thus $X_p < X_{p'}$ as elements of Ω. Now, using that f is increasing,

$$\mathbb{E}_p[f] = \mathbb{E}[f(X_p)] \leq \mathbb{E}[f(X_{p'})] = \mathbb{E}_{p'}[f]. \qquad \square$$

Phase transition in bond percolation Define the percolation probability $\theta(p)$ by

$$\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty) = \mathbb{P}_p(|\mathcal{C}_x| = \infty) = \lim_{n\to+\infty} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)$$

Note that $0 \leftrightarrow \partial\Lambda_n$ is an event as it depends only on finitely many edges meaning that the other guys are also events.

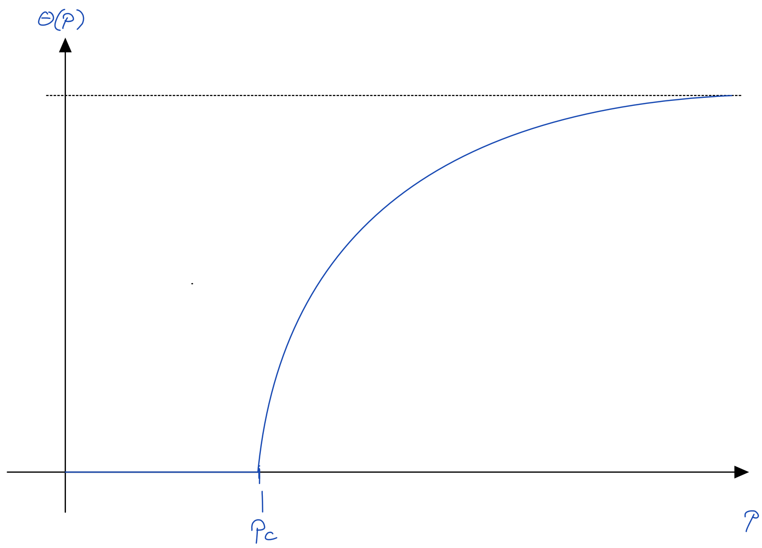


Figure 1.2: possible graph of p_c. But actually no one knows what it looks like.

Define

$$p_c := \inf\{p \in [0,1] : \theta(p) > 0\}.$$

There arise some questions:

Question 1.8. Does $0 < p_c < 1$ occur?

Exercise 1.9. Prove that in $d = 1$,

$$\theta_1(p) = 0 \text{ if } p < 1 \text{ and } 1 \text{ else}$$

Define

$$\begin{aligned} I &= \{w \in \Omega : \omega \text{ contains an infinite cluster}\} \\ &= \bigcup_{x \in \mathbb{Z}^d} \{|\mathcal{C}_x| = +\infty\} \end{aligned}$$

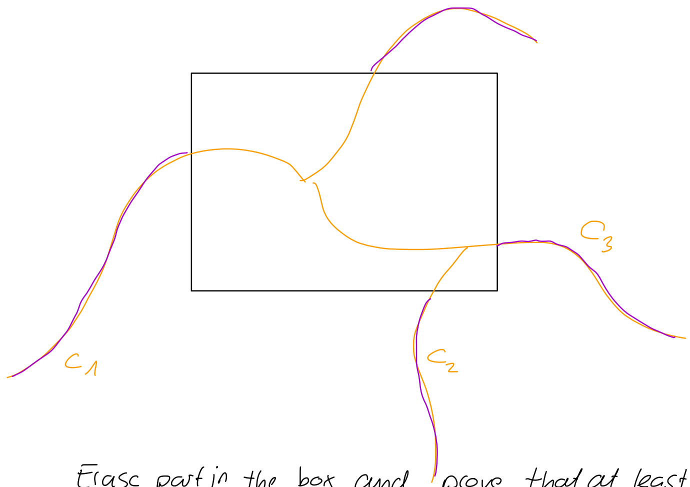
Proposition 1.10. The following statements hold:

1. It holds that $\mathbb{P}_p(I) \in \{0, 1\}$.
2. $\mathbb{P}_p(I) = 0$ if and only if $\theta(p) = 0$.

Proof. 1 Enumerate edges of E_d by increasing distance from the origin and write $E_d = \{\omega_1, \omega_2, \ldots\}$ Define

$$I_n := \left\{\omega \in \Omega : \text{ there is infinite cluster in } \mathbb{Z}^d \setminus \Lambda_n\right\}.$$

We now claim that $I = I_n$. Indeed, if we show this, the claim follows by Kolmogorov's 0-1-law. The implication $I \supseteq I_n$ is obvious. To prove the opposite inclusion assume that $\omega \in I$.



Erase part in the box and prove that at least one of the remaining clusters is infinite and that the number of clusters that are generated by erasing is finite.

We claim that any of the C_i's must intersect $\partial\Lambda_m$ but $|\partial\Lambda_m|$ is finite so the claim follows.

2 We estimate.

$$\theta(p) = \mathbb{P}_p(|\mathcal{C}_0| = \infty) \le \mathbb{P}_p(I) = \mathbb{P}\left(\bigcup_{x\in\mathbb{Z}^d} \{|\mathcal{C}_x| = \infty\}\right)$$

$$\overset{!}{\le} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(|\mathcal{C}_x| = \infty) = \sum_{x\in\mathbb{Z}^d} \theta(p),$$

where ! follows from the tranlation invariance of $\mathbb{P}_p(|\mathcal{C}_x| = \infty)$. □

Theorem 1.11. For $d \ge 2$, $p_c(d) \in (0,1)$.

Proof. Step 1. We show that $\theta_d\left(\frac{1}{2d}\right) = 0$ which implies that $p_c(d) \ge \frac{1}{2d}$. Let

$$\mathcal{P}_n := \{\gamma : \gamma \text{ is a path starting in } 0 \text{ of length } n\}$$

and observe that

$$\{|\mathcal{C}_0| = \infty\} \subseteq \{\exists \gamma \in \mathcal{P}_n \text{ where all edges of } \gamma \text{ are open}\}$$

so

$$\begin{aligned}\theta(p) =& \mathbb{P}_p(|\mathcal{C}_0| = \infty) \leq \mathbb{P}_p(\exists \gamma \in \mathcal{P}_n,\ \gamma \text{ is open })\\ &\leq \sum_{\gamma \in \mathcal{P}_n} \underbrace{\mathbb{P}_p(\gamma \text{ is open})}_{=p^n} = p^n |\mathcal{P}_n|.\end{aligned}$$

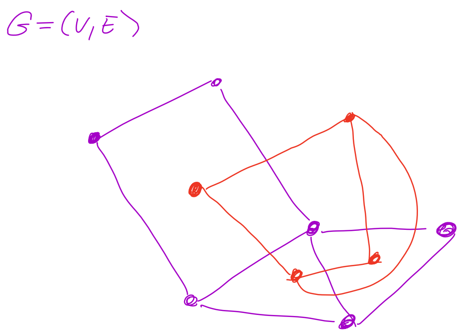
By simple combinatorics one can show that

$$|\mathcal{P}_n| \leq (2d)(2d-1)^{n-1}.$$

Plugging this into the above line and letting $n \to +\infty$ with $p \leq \frac{1}{2d}$, we obtain that $\theta(p) = 0$.

Step 2. Let $d = 2$. We claim that for any $p > \frac{3}{4}$, $\theta_2(p) > 0$ (which again implies that $p_c(2) \leq \frac{3}{4}$). We will use something called *Peierls argument*.

Dual graphs Let G be a *planar* graph (that is a graph that can be drawn without intersecting edges that aren't vertices).



Dual: $G^* = (V^*, E^*)$

$E^* = \{ \{x^*, y^*\} : x^*, y^* \text{ are neighbouring faces} \}$

Claim. There is a bijection between E and $E^\star$ via $e^\star \mapsto e$.

In $\mathbb{Z}^2$, the dual of $\mathbb{Z}^2$ is again $\mathbb{Z}^2$ which we call *self dual.* We now define a *dual configuration* living on the dual graph. Let $\omega \in \Omega$. Define

$$\omega^\star(e^\star) = \begin{cases} 1 & \text{if } \omega(e) = 0 \\ 0 & \text{if } \omega(e) = 1 \end{cases}$$

If $\omega \sim \mathbb{P}_p$, then $\omega^\star \sim \mathbb{P}_{1-p}$ and if $\mathcal{C}_0$ is finite, then there is a cycle in $\omega^\star$ surrounding 0 (we are not going to show this, exercise). Now,

$$\begin{aligned} 1 - \theta(p) =& \mathbb{P}_p(|\mathcal{C}_0| < \infty) \\ =& \mathbb{P}_p\left(\bigcup_{n\in\mathbb{N}_0} \{\omega^\star \text{ contains an open cycle of length } n \text{ surrounding } 0\}\right) \\ \leq& \sum_{n\in\mathbb{N}_0} \underbrace{\mathbb{P}_p(\ldots)}_{(1-p)^n} \\ \leq& \sum_{n\in\mathbb{N}} (1-p)^n \cdot \#\{\text{cycles around 0 of length } n\}. \end{aligned}$$

If n is odd or $n = 2$ there are no cycles. A cycle of length n intersects x^+-axis at a point at distance at most $\frac{n}{2}$. From this,

$$\#\{\text{cycles around 0 of length } n\} \leq \frac{n}{2} \cdot 3^{n-1}.$$

Plugging this the claim is proved.

Step 3. We claim

$$p_c(d) \leq p_c(2) \quad \text{for all } d \geq 2. \tag{1.1}$$

Using that $\mathbb{Z}^2$ can be embedded in $\mathbb{Z}^d$ for $d \geq 2$ we have that

$$\mathbb{P}_p(|\mathcal{C}_0| = \infty) \geq \mathbb{P}_p(|\mathcal{C}_0 \cap \mathbb{Z}^2| = \infty). \qquad \square$$

We have seen some tools in our "percolation toolbox" namely, monotone coupling and duality (on $\mathbb{Z}^2$).

The goal now is to introduce a new tool: Harris-FKG-inequality. Motivation: If A, B are events and $\mathbb{P}(A), \mathbb{P}(B)$ known, can we estimate $\mathbb{P}(A \cap B)$? This is trivially easy if A and B are independent but not otherwise.

Intuition. $x, y, u, v \in \mathbb{Z}^d$ and let e be an edge. One would expect

$$\mathbb{P}_p(x \leftrightarrow y) \leq \mathbb{P}_p\left(x \leftrightarrow y \mid e \text{ open}\right).$$

Similarly,

$$\mathbb{P}_p(x \to y) \leq \mathbb{P}_p(x \leftrightarrow y \mid u \leftrightarrow v).$$

Theorem 1.12 (Harris-FKG-inequality).

1. If A, B are increasing events
$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$
2. If $f, g \in L^2(\mathbb{P}_p)$ are increasing
$$\mathbb{E}_p[fg] \geq \mathbb{E}_p[f]\mathbb{E}_p[g].$$

Remark.

- If we replace by f by $-f$ and g by $-g$ we can also put decreasing and the equality still holds.
- The proof does not use the geometry of $\mathbb{Z}^d$, it only uses the properties of the product measure.
- FKG holds even for some non-product measure.

of Theorem 1.12. Step 1: Assume that f, g only depend on finitely many edges. We again enumerate the edges $E_d = \{e_1, e_2, \dots\}$ and write $\omega_i := \omega(e_i)$. If $f, g : \{0,1\}^n \to \mathbb{R}$, we claim

$$\mathbb{E}_p[f(\omega_1, \dots, \omega_n)g(\omega_1, \dots, \omega_n)] \geq \mathbb{E}_p[f(\omega_1, \dots, \omega_n)]\mathbb{E}_p[g(\omega_1, \dots, \omega_n)]. \tag{1.2}$$

We prove this claim by induction: For $n = 1$, $a, b \in \{0,1\}$we have

$$(f(a) - f(b))(g(a) - g(b)) \geq 0$$

so multiplying by $\mathbb{P}_p(\omega_1 = a)\mathbb{P}(\omega_1 = b) \geq 0$ and summing over all a, b gives us

$$2(\mathbb{E}_p[fg] - \mathbb{E}_p[f]\mathbb{E}_p[g]) \geq 0.$$

Assume that (1.2) holds for $n \leq k$. Then

$$\begin{aligned}
\mathbb{E}_p[(f \cdot g)(\omega_1, \dots, \omega_{k+1})] =& p\mathbb{E}_p[(f \cdot g)(\omega_1, \dots, \omega_k, 1)] \\
&+ (1-p)\mathbb{E}_p[(f \cdot g)(\omega_1, \dots, \omega_k, 0)] \\
\overset{\text{Ind}}{\geq}& p \underbrace{\mathbb{E}_p[f(\omega_1, \dots, \omega_k, 1)]}_{\tilde{f}(1)} \underbrace{\mathbb{E}_p[g(\omega_1, \dots, \omega_k, 1)]}_{\tilde{g}(1)} \\
&+ (1-p) \underbrace{\mathbb{E}_p[f(\omega_1, \dots, \omega_k, 0)]}_{\tilde{f}(0)} \underbrace{\mathbb{E}_p[g(\omega_1, \dots, \omega_k, 0)]}_{\tilde{g}(0)} \\
=& \mathbb{E}_p[\tilde{f}\tilde{g}] \\
\overset{\text{Ind}}{\geq}& \mathbb{E}_p[\tilde{f}]\mathbb{E}_p[\tilde{g}] \\
=& \mathbb{E}_p[f]\mathbb{E}_p[g],
\end{aligned}$$

where we used that $\tilde{f}$ and $\tilde{g}$ are increasing.

Step 2: (infinite volume). Take f, g as in the theorem and define

$$f_n := \mathbb{E}_p[f \mid \sigma(\omega_1, \dots, \omega_n)], \quad g_n := \mathbb{E}_p[g \mid \sigma(\omega_1, \dots, \omega_n)].$$

Then f_n, g_n only depend on $\omega_1, \dots, \omega_n$ and are increasing. Then f_n, g_n, $n \in \mathbb{N}$ are martingales bounded in $L^2(\mathbb{P}_p)$. So by martingale convergence theorem we have

$$f_n \xrightarrow{L^2} f, \quad g_n \xrightarrow{L^2} g$$

but convergence in L^2 implies convergence in L^1 and thus we have convergence of the products $f_n g_n$. So by this reasoning and the previous step,

$$\begin{aligned}\mathbb{E}_p[fg] &= \lim_{n \to +\infty} \mathbb{E}_p[f_n g_n] \geq \lim_{n \to +\infty} \mathbb{E}_p[f_n]\mathbb{E}[g_n] \\ &= \mathbb{E}_p[f]\mathbb{E}_p[g].\end{aligned}$$

□