An Analysis of Games Modelling the Competition Between Uber Drivers

Koen Dubrow

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1 Abstract

The sudden prolific use of ride-hailing platforms like Uber has introduced a new type of agent to the modern economy, whose utility is dependent on very short "contracts" that may only be fulfilled sequentially. There are many aspects in which such an agent's utility deviates from typical economies in significant ways. We wish to study the nature of competition between these agents as they seek to minimize the cost of reaching a client and maximizing their chances of securing clients. The key results found are that the variance governing the appearance of clients has at most a very obtuse effect on agent behavior, as opposed to the intuition that given high variance agents would choose more conservative actions. In some contrived situations we also observe that the incentivised positioning of Uber drivers is not conducive to low client waiting times. The work here seeks to expand on the work of Castillo and Knoepfle (2017) which examines the broader behavior of the ride-sharing economy in terms of pricing, instead we examine a small portion of the economy with an emphasis on position.

2 Introduction

Castillo and Knoepfle (2017) demonstrate key results about the behavior of the ride-sharing economy using traditional economic methods. Here we study this economy using Algorithmic Game Theory Techniques. The fundamental concepts of this topic will be introduced in the background section. In the following sections we break down results that have been found throughout this project. The primary purpose of these sections are to inform the reader of the process from which our results come.

3 Background

We first introduce the idea of a normal-form game which we will in general just refer to as a game. The first formulation of a "game" as a mathematical object was by Von Neumann and Morgenstern (1966). However, we will use the formulation of a normal form game provided by Nash (1950, p. 2) whose thesis is the major foundation upon which modern Game Theory is based.

3.1 Games

We take the definition specifically as presented by Cheung (lec. 1 2024)

Definition 3.1.1 (Normal form finite game)**.** *A normal form game Γ consists of n players, each player i with a finite action set (sometimes strategy set) A_i containing there possible actions. Let*

$$A := \bigtimes_{i=1}^{n} A_i$$

be the space of joint action profiles (sometimes joint strategy profiles). The game also consists of a payoff function

$$\gamma : A \to \mathbb{R}^n$$

Given a player i and a joint action profile $s \in A$ the payoff of i will be the ith component of $\gamma(s)$, written as $\gamma_i(s)$

This definition does not really seek to be a way to encapsulate traditional games mathematically. Take chess, the implementation of which into this framework is possible but not easy. The main reason is that a normal form game expresses just one round of play. There are means to think about multiple round games in this way, by thinking about γ as a heuristic function that evaluates a players potential to win after and before each round, or by thinking of an entire sequence of play (with programmed responses) as a single action in A_i. As we will see the point of the above definition is to create a framework to analyse non-cooperative competition.

3.2 Solving Games

Consider for a moment how bloated the above definition may seem, we may be inclined to view a game just as a function $\gamma : A \to \mathbb{R}^n$ and this approach is not strictly incorrect. That said our definition is much more informative about how we seek to study a game. Consider that traditional analysis if given some nebulous function γ, concerns itself with things like the continuity, solutions or bounds of γ. These are not very relevant to understanding a game. This is to highlight that game theory applies mathematics to studying competition using the tools of analysis in a unique way. Before we can even begin to think how to analyse a game, we need to consider what we want to analyse. It is natural and convenient to imagine that every player is competitive, non-cooperative and completely logical. This is a fundamental assumption of game theory, even if it may not always be true. That is a player i only cares about her personal payoff γ_i. Now of course every player will wish to maximise their personal payoff, but it is immediate that the point

$$(\max_{s\in A}\gamma_1(s), \dots, \max_{s\in A}\gamma_n(s))$$

where every player has realised her maximum personal payoff need not be in the image of γ, so it does not assist us in understanding the general notion of a game.

Instead consider that a player has no control over other players actions and thus may be satisfied simply by receiving the highest she can with respect to all other players not changing their actions. To formulate this mathematically I will introduce the following potentially unfamiliar notation, if s is our action profile let s_i be the ith component of this, which is player i's action. Let s_{-i} consist of all components of s except the ith, so every other players action. We say that $s = s_i \oplus s_{-i}$ which is in abuse of the notion of a direct sum, since there is some implicit reordering. In general $a \oplus s_{-i}$ will be the joint action profile where player i chooses action $a \in A_i$ and every other player chooses their action from s. To convenience ourselves even further we may often just write $s = s_i, s_{-i}$ to do away with direct sum notation all together, we still often use direct sum notation as this much clearer. Then we can formulate this idea as so

Definition 3.2.1 (Pure Nash Equilibrium)**.** *$s^* \in A$ in a Pure Nash Equilibrium (PNE), if and only if for every player i, $\gamma_i(s^*) = max_{a\in A_i}\gamma_i(a \oplus s^*_{-i})$*

Clearly PNE are a super set of our first attempt at finding a solution to a game. This solution in a way is also more useful in a pedagogical sense as it directs our thoughts toward notions on equilibrium, a point of stability between our players, whereas our initial attempt ignored the inherit competitiveness of a game and thus is hardly reflected in reality. Let us consider an example to get used to this concept

Example 3.2.2. *Suppose that Alice and Bob find themselves in a four way intersection with no traffic lights. Both can either Go or Stop. If either stops they receive a pay out of 0 and if they Go and successfully cross the intersection they will receive a payoff of 1. However, if they both go they will crash and both receive a payoff of -100. It is convenient to express our payoff function as a table:*

		Go	*Stop*
		A	
Go	B	$(-100, -100)$	$(1, 0)$
Stop		$(0, 1)$	$(0, 0)$

Here the columns correspond to an action Alice can take and the Rows an action Bob can take. The payoffs are tuples with the left value corresponding to Bob's payoff and the right Alice's. Now we would like to find a PNE. We think about our definition in the context of this table. If a cell is a PNE, that corresponds to a point where Bob cannot increase his payoff assuming Alice does not change her strategy, and likewise for Alice. This is equivalent to a cell on this table being a PNE iff Bob cannot increase his payoff by choosing another cell in column (because Alice's move is assumed fixed), and likewise Alice cannot increase her payoff by choosing a different cell in the row. Then $(0,0)$ *is not a PNE because both of them would - assuming the other remains stopped - want to Go instead of Stop here. For the same reason* $(-100,-100)$ *is not a PNE because Bob and Alice would both wish that they stoped. However* $(1,0)$ *(and* $(0,1)$*) is a PNE, because if Bob stops instead we move to the cell* $(0,0)$ *which is a decrease in payoff for Bob, and if Alice decided to Go we would move to the cell* $(-100,-100)$ *so she would also be decreasing her payoff.*

That said a PNE does not always exist, and thus is also flawed as a solution concept. Famously the prolific rock-paper-scissors game used by Cheung (2024, lec. 2) highlights this.

Example 3.2.3 (Rock-Paper-Scissors)**.** *Suppose Alice and Bob play a round of rock paper scissors, we wish to show that there is no PNE. In order to apply a payoff function to this game, just say that Alice or Bob get* 1 *if they win,* 0 *for a draw and* -1 *for a lose. Since we only have two players and three actions we may represent the game as atable.*

		R	P	S
			A	
R		$(0,0)$	$(-1,1)$	$(1,-1)$
P	B	$(1,-1)$	$(0,0)$	$(-1,1)$
S		$(-1,1)$	$(1,-1)$	$(0,0)$

The table is set up the same as before. Except this time no matter what cell we choose, either Bob can choose another cell in the column where he gets a higher payoff, or Alice can choose another cell in the row where she gets a higher payoff. Thus there is no PNE.

We are then challenged (or atleast those who came before us) to think of a solution concept with the contextual relevancy of a PNE that is more robust so that we can study more games.

3.3 Expanding our solutions to Games

In order to find a more robust solution, we need to do some more conventional analysis. we wish to consider a domain extension of γ. Given a set S let $\Delta(S)$ be the set of random variables on S. For example if $S = \{a, b\}$ then

$$\Delta(S) = \{X \text{ a random variable on } S |\ Pr(X = a) = x\ \& Pr(X = b) = 1 - x\}$$

It is not hard to see that $\Delta(S)$ can be realised as a subset of $\mathbb{R}^{|S|}$ which is convex and compact under the euclidean topology. Furthermore, consider the fact an element $x \in S$ can itself be thought of as a random variable that is equal to x with certainty. Then $S \subset \Delta(S)$ and we are permitted to formally extend the domain of γ and we do this quite naturally using expectation. We define our extension $\overline{\gamma}$ as

$$\begin{aligned} \overline{\gamma} &: \Delta(A) \to \mathbb{R}^n \\ \overline{\gamma}(\sigma) &= \mathop{\mathbb{E}}_{s \backsim \sigma} \gamma(s) \end{aligned}$$

Here $s \backsim \sigma$ are just the values s that may be realised by σ. Now of course for convenience we will not bother to distinguish between γ and its extension $\overline{\gamma}$ unless it is particularly helpful for us conceptually. Within $\Delta(A)$ we can consider more robust solution concepts, in particular I will describe 2 of these but we will first cover the conceptual groundwork for both fo them.

Consider what we often call the space of joint mixed profiles denoted as

$$\bigtimes_{i=1}^{n} \Delta(A_i)$$

This can also be thought of as a subset of $\Delta(A)$, if $\sigma_i \in \Delta(A_i)$ is a random variable governing the strategy that player i chooses then we define our inclusion as[1]

$$\bigtimes_{i=1}^{n} \Delta(A_i) \hookrightarrow \Delta(A)$$
$$(\sigma_1, \ldots, \sigma_n) \mapsto \sigma$$

Where σ is equal to a joint strategy s with probability

$$\prod_{i=1}^{n} Pr(\sigma_i = s_i)$$

The image of this inclusion has the special property that if σ is in this image for any $i \neq j$ then the i and j component of σ are independent. We will call this (actual) subspace the product distributions of $\Delta(A)$. Now we introduce a new type of solution

Definition 3.3.1 (CE)**.** *A Correlated Equilibrium (CE) is $\sigma \in \Delta(A)$ s.t. for all players $i \in \{1, \ldots, n\}$ all strategies and all unilateral deviations $s_i, s_i' \in A_i$*

$$\overline{\gamma}_i(\sigma|s_i) \geq \overline{\gamma}_i(s_i', \sigma_{-i}|s_i)$$

To make sense of this it usually best to frame it as below (I give it as above to demonstrate the importance of the domain extension)

$$\mathop{\mathbb{E}}_{\boldsymbol{s}\backsim\sigma}[\gamma_i(\boldsymbol{s})|s_i] \geq \mathop{\mathbb{E}}_{\boldsymbol{s}\backsim\sigma}[\gamma_i(s_i', \boldsymbol{s}_{-i})|s_i]$$

The best way to informally describe this definition is that a distribution of joint strategy profiles is a correlated equilibrium if for some player i, it is best for them to play the action that the other players are anticipating them to.
The benefit of this notion of an equilibrium is that it always exists and is easy to compute in general, in part because the set of all CE for a game γ is a convex subset of $\Delta(A)$. As remarked by Roughgarden (2012, lec. 13) the computational simplicity of this equilibrium is incredibly important within the context of game theory; how can we expect players of the game in reality to reach a computationally difficult equilibrium. The downside of a CE is that to make sense of an arbitrary distribution of joint we require either some form of coordination between players or some independent party that coordinates players. We can see this by considering some two player game where both players can choose between action a or b. If σ is equal to player 1 plays a and player 2 plays b with probability $\frac{1}{2}$ and vise versa with probability $\frac{1}{2}$ then our players have either agreed never to play the same strategy or they following the instructions of a third party. It would be desirable to imagine that our players do not coordinate between themselves, certainly a solution that does not require inter-player coordination would be more reflective of reality. Indeed we shall find such a notion but before we do so we consider this example to demonstrate that such a requirement is not completely detached from reality.

[1] This and the inclusion $S \subset \Delta(S)$ is of course is not a true inclusion in the context of set theory, however, we use this language to highlight our injection is very natural within our context

Example 3.3.2 (Applying CE to the traffic light game)**.** *Recall the traffic light game that Alice and Bob played, where we looked for PNE. While the PNEs we found avoided a car crash we can only choose one. If we chose Alice Go and Bob Stop then what if instead of one Alice their was a whole stream of traffic behind her choosing the same strategy as Alice, Bob would be waiting at the intersection untill atleast all the traffic has cleared. It is impossible for Alice and Bob to reach a sensible arrangement without some form of communication with each other, or a traffic light acting as an instructing 3rd party. Indeed if there was a traffic light that told Alice Go and Bob Stop half the time and Bob Go and Alice Stop the other times this seems much more sensible (and very realistic). We verify that this is actually a CE. Let us realise this formally. Let A be Alice's action set and B Bob's and let G represent the action go and S stop. Then $(G,S) \in A \times B$ is the PNE we discussed just before and if $\sigma \in \Delta(A \times B)$ is equal to (S,G) with probability $\frac{1}{2}$ and (G,S) with probability $\frac{1}{2}$ as well σ is the proposed CE. Indeed* [2]

$$
\begin{gathered}
\underset{s \backsim \sigma}{\mathbb{E}}\left[\gamma_{Alice}(s) | Alice\ go\right] = \gamma_{Alice}((G,S)) = 1 \\
\underset{s \backsim \sigma}{\mathbb{E}}\left[\gamma_{Alice}(S, s_{-i}) | Alice\ go\right] = \gamma_{Alice}((S,S)) = 0 \\
and\ 1 \geq 0 \\
\underset{s \backsim \sigma}{\mathbb{E}}\left[\gamma_{Alice}(s) | Alice\ stop\right] = \gamma_{Alice}((S,G)) = 0 \\
\underset{s \backsim \sigma}{\mathbb{E}}\left[\gamma_{Alice}(G, s_{-i}) | Alice\ Stop\right] = \gamma_{Alice}((G,G)) = -100 \\
and\ 0 \geq -100
\end{gathered}
$$

So we have shown that the condition for σ to hold applies to every possible action and deviation Alice may make. We of course would normally have to check this for Bob, but since Alice and Bob are identical up to renaming we already know that the same holds for Bob. Thus we see that σ is a CE.

In order to find a notion equilibrium that does not require coordination between players we require that the strategy of each player realised by σ should be independent. But we have already done the work to understand this, we just require that σ be a product distribution.

Definition 3.3.3 (MNE 1)**.** *A Mixed Nash Equilibrium (MNE) is a CE σ s.t. σ is a product distribution or*

$$\sigma \in \mathop{\times}_{i=1}^{n} \Delta(A_i)$$

This is not the conventional way to describe an MNE usually we introduce it as

Definition 3.3.4 (MNE 2)**.** *A MNE is some*

$$\sigma \in \mathop{\times}_{i=1}^{n} \Delta(A_i)$$

s.t. for all players $i \in \{1, \ldots, n\}$ and deviation $s'_i \in A_i$

$$\underset{s \backsim \sigma}{\mathbb{E}}\left[\gamma_i(s)\right] \geq \underset{s \backsim \sigma}{\mathbb{E}}\left[\gamma_i(s'_i, \boldsymbol{s}_{-i})\right]$$

this may also be expressed as

$$\gamma_i(\sigma) \geq \gamma_i(s'_i, \sigma_{-i})$$

We often refer of MNE and PNE together as NE. It is a very famouse result by Nash, 1950 that every game permits a MNE. This is the exact reason why every game permits a CE since of course all MNE are CE. However, Nash's proof of this is not constructive and in general solving for a MNE in an arbituary game is computationally hard. So in exchange for a more natural equilibrium concept we find that it is harder to compute and thus harder to reach. Let us demonstrate the use of a MNE

[2]see previous table to help make sense of these calculations

Example 3.3.5 (Applying MNE to Rock-Paper-Scissor)**.** *It is perhaps incredibly unsatisfying that a game seemingly as simple as Rock-Paper-Scissors inspired us to go through so much effort in defining a more robust equilibrium concept. However our work has payed off. Considering the intuition that no move in Rock-Paper-Scissors is better than the other we propose that each player choosing every move uniformly is a MNE. Again we will need to formally describe this. Let* A *and* B *be Alice and Bobs action sets and let* R, P *and* S *refer to Rock, Paper and Scissors respectively. Now let* $\boldsymbol{a} \in \Delta(A)$ *be equal to* R*,* P *and* S *with probability of* $\frac{1}{3}$ *each, let* $\boldsymbol{b}$ *be the same but an element of* $\Delta(B)$*. Then* $(\boldsymbol{a}, \boldsymbol{b})$ *is the propsed MNE. we check that this indeed an MNE*

$$\begin{aligned}
\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(s)] &= \frac{1}{3}\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(R, s_{-i})] + \frac{1}{3}\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(P, s_{-i})] + \frac{1}{3}\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(S, s_{-i})] \\
\frac{1}{3}\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(R, s_{-i})] &= \frac{1}{9}(\gamma_{Alice}(R, R) + \gamma_{Alice}(R, S) + \gamma_{Alice}(R, P)) \\
&= \frac{1}{9}(1 - 1 + 0) = 0 \\
\frac{1}{3}\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(P, s_{-i})] &= \frac{1}{9}(\gamma_{Alice}(S, R) + \gamma_{Alice}(S, S) + \gamma_{Alice}(S, P)) \\
&= \frac{1}{9}(1 - 1 + 0) = 0 \\
\frac{1}{3}\mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(S, s_{-i})] &= \frac{1}{9}(\gamma_{Alice}(P, R) + \gamma_{Alice}(P, S) + \gamma_{Alice}(P, P)) \\
&= \frac{1}{9}(1 - 1 + 0) = 0 \\
\implies \mathop{\mathbb{E}}_{s \backsim (\boldsymbol{a},\boldsymbol{b})}[\gamma_{Alice}(s)] &= 0
\end{aligned}$$

Note that first line is possible because in a product distribution like $(\boldsymbol{a}, \boldsymbol{b})$ *Alice and Bob act independently. Now to check this is MNE we need to compare it to each deviation, except that we did that on the way to calculating the above, they are all equal to* 0*, and* $0 \geq 0$*. Thus we have checked the necessary conditions with respect to Alice, but since once again Bob and Alice are identical up to relabelling then it holds for Bob as well and indeed* $(\boldsymbol{a}, \boldsymbol{b})$ *is an MNE*

Noting the potentially inefficient calculations above leads us to the best response lemma sometimes called the best response condition. It says that a product distribution $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ is an MNE iff for every player i, $\mathbf{s}_i = s_i$ with non-zero probability only when $s_i \in \operatorname{argmax}_{s'_i \in A_i} \gamma(s'_i, \mathbf{s}_{-i})$. see (Cheung, 2024, lec. 2) for proof of this. Informally the idea is that a player should never play a less than optimal action with non zero probability.

Now we can give the PNE the same treatment as we did MNE in our first definition: a PNE is a CE σ s.t. $\sigma = \mathbf{s} \in A$ with probability 1. This is exactly why we bothered to draw attention to our domain extension and inclusions while it is apparent that $PNE \subset MNE \subset CE$ (thought of as sets of these equilibrium concept for a given game) understanding the finer points of their differentiation is mostly a matter of understanding

$$\mathop{\times}_{i=1}^{n} A_i \subset \mathop{\times}_{i=1}^{n} \Delta(A_i) \subset \Delta(\mathop{\times}_{i=1}^{n} A_i)$$

3.4 Price of Anarchy

We now discus Price of Anarchy a concept first introduced by Koutsoupias and Papadimitriou, 1999. To unpack this recall that Game Theory is differentiated from traditional analytic methods in how examine our payoff function. The notions that we unpacked above are the key aspect in this theory that ties us to the context of multiple agents serving only there own interests. The above reasonably assumes that these players are content when an equilibrium is reached. However, if we imagine ourselves to be a government body, we might be concerned in how "efficient" these equilibrium are, or indeed any distribution of joint strategies is.

To achieve such an analysis mathematically, assume there is some real valued function to evaluate this, we will call this V, we will only define V on A, and use the same trick as we did with γ to naturally extend its domain to $\Delta(A)$. The action of V is often defined from γ as well. For example in a game concerning the personal welfare of individuals our notion of efficiency may be about "fairness" and thus we take V the be the minimum over the individual payoffs of our players. When analysis the competition between companies we may concern ourselves with economic efficiency and wish to define V as the sum of the personal payoffs.

Now suppose that we have chosen a definition for $V : A \to \mathbb{R}$. Evaluating the efficiency of an equilibrium $\mathbf{s}^*$, is a matter of first identifying the maximum of V on $\Delta(A)$. Of course we need to guarantee that a maximum exists and in doing so we find a beautiful aspect of this theory. Our extension of V from A to $\Delta(A)$ (though of as a subset of $\mathbb{R}^{|A|}$, recalling that A is finite) via expectation must be continuous on $\Delta(A)$ and since $\Delta(A)$ is a compact subset of $\mathbb{R}^{|A|}$, a maximum must exist. Let $\mathbf{s} \in \Delta(A)$ be an element upon which V realises its maximum. Then we may quantify the efficiency of any equilibrium in any game just by $\frac{V(\mathbf{s}^*)}{V(\mathbf{s})}$, this is the comparative efficiency of an equilibrium to the ideal case. However we are often interested in studying the game as a whole not just one equilibrium, since we may often have no guarantee what equilibrium our players may reach, thus we define the notion of Price of Anarchy which we think of as the loss of efficiency when there is no intervention in the game.

Definition 3.4.1 (POA)**.** *The Price of Anarchy (POA) of a game Γ with respect to a given notion of equilibrium is equal to*

$$min_{\boldsymbol{s}^* \in E} \frac{V(\boldsymbol{s}^*)}{V(\boldsymbol{s})}$$

where E is the set of all such equilibrium.

Often the equilibrium concept that this is in respect to is left to context. However a surprising amount of results about the POA of games are very robust and are sharp at PNE and still holds for CE. Note that for the same reason we can maximise V, the POA of a particular game always exists. However, we often consider (non-fintie) classes of games and in such a case the POA lower bound may just be 0, even though as is often the case our potential function is always strictly positive. To see this consider some class of games. Suppose within this class there is a sequence of games $\{\Gamma\}_n$ s.t. the POA of $\Gamma_i < \frac{1}{i}$, then we need to bound this sequence by below and the best we can do of course is 0. We consider an example that highlights this.

Example 3.4.2 (Extended Pigou's Example (Pigou, 1920))**.** *We consider a simplified form of the routing game. Here suppose that our players are infinitely divisible* [3] *and wish to traverse from a to b in the following graph.*

$$\begin{tikzcd} A \arrow[r, bend left, "1"] \arrow[r, bend right, "x"'] & B \end{tikzcd}$$

They can choose either edge, but both edges have an associated cost function as written above and below

[3] We are of course concerning ourselves with finite games where infinite division does not make sense, however this is such a classic example for illustrating POA it bears mentioning.

the edges. Players will incur a cost equal to the function from their chosen edge applied to the proportion of players that chose this edge. Now we note that here players incur a cost instead of receive a payoff. Nothing important changes however in an equilibrium players seek to minimise their cost and POA instead uses the minimum of our efficiency function as the optimal benchmark which compare to the maximum equilibrium.
Let us compute the POA. First we will do this by identifying all MNE. There is actually only one. Suppose y players choose the top edge they will incur a cost of 1, but if one player where to switch to the bottom edge they would incur a cost of $(1-y) \leq 1$ (since our players are infinitely divisible a single player has no quantity in the proportion). Now this actually tells us that no matter what other players are doing the bottom path will have a cost less than or equal to the top path. And thus by the best response condition our only MNE is the PNE when all players always choose the bottom edge, we will call this PNE $\boldsymbol{s}^*$. Now let our efficiency function V just be the expected cost over all the players. Let us minimise this with some basic calculus and note that V can be realised just as function on $x \in [0,1]$ where x is the proportion of players that choose the bottom edge, since if we know the proportion of players on the bottom edge we know it for the top edge:

$$\begin{aligned}
V(x) &= (1-x)(1) + x(x) \\
V'(x) &= 2x - 1 \\
2x - 1 = 0 &\implies x = \frac{1}{2} \\
V(\frac{1}{2}) &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}
\end{aligned}$$

Now we consider the value of V at $\boldsymbol{s}^*$ which is just $V(1) = 1$. Thus the POA of this game is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$.
Now let us think about the broader class of games in which the functions associated with the edges can be anything. Because this is a cost minimization game (as opposed to our framework of a payoff maximisation game), a POA is always greater than or equal to 1 and thus it is possible for the POA of a class of games to be infinite/undefined. Indeed such is the case for this class of games. Let Γ_p be the game.

$$\begin{tikzcd} A \arrow[r, bend left, "1"] \arrow[r, bend right, "x^p"'] & B \end{tikzcd}$$

As before the only PNE is when players choose just the bottom edge but consider the optimal value of $V(x)$ again by using basic calculus to minimize it.

$$\begin{aligned}
V(x) &= (1-x) + x^{p+1} \\
V'(x) &= (p+1)x^p - 1 \\
V'(x) = 0 &\implies x = \frac{1}{(p+1)^{\frac{1}{p}}} =: \mu_p \\
V(\mu_p) &= (1-\mu) + \mu\frac{1}{p+1}
\end{aligned}$$

And as before the effeciency of our PNE is just $V(1) = 1$. Our POA then is just $\frac{1}{V(\mu_p)}$. However it can be shown that $\lim_{p\to\infty} V(\mu_p) = 0$ and thus $\lim_{p\to\infty} \frac{1}{V(\mu_p)}$ is not defined. Thus we cannot bound the POA of these games by above and we cannot prescribe the class of games specified above a POA.

POA is a very useful tool. When we model something as a game, it is often because of the inherent competition or lack of cooperation within the system. We assume that the system naturally arranged itself into a possible equilibrium, and thus the POA gives us quantification of the inefficiencies of the system in its realised form. This is of course rarely sufficient as the next step is to identify and mitigate these inefficiencies, however this falls out of the mathematical framework of game theory. It is also a philosophically potent notion, relating back to the work of Smith (1723-1790) who posed that an individual acting in their best interest can provide some all be it limited benefit to society i.e. the price of anarchy can be expected to be bounded.

3.5 Potential function

To conclude our background we introduce to the reader the notion of a potential function for a game.

Definition 3.5.1 (Potential function)**.** *A potential function $\Psi : A \to \mathbb{R}$ for a game Γ is a function that has local minimum at $s \in A \iff s$ is a PNE*

A potential function may also be defined on $\Delta(A)$ or $\times_{i=1}^{n} \Delta(A_i)$ and solve for CE or MNE respectively, but this is less common since the use of a potential function is usually to show the existence of a PNE (the only type of equilibrium we are not guaranteed). On that point one could define a potential function to be 1 when $s \in A$ is not a PNE and 0 when it is, and this would be a well-defined potential function, but it would be of little use. Usually what is done is that a candidate function is identified, and it is proven to be a potential function. We also often hope our candidate function is relatively simple and has some connection to our efficiency function. This is because not only is a potential function used to demonstrate the existence/non existence of a PNE, but also to compute such a PNE, and if it is similar to our efficiency function assist us in bounding or identifying the POA of Γ which is difficult in general.

4 The Class of Games

We now introduce the class game from which our results come in the most general sense, however our results centre around specific sub classes of this. We will refer to this game as the Uber game. We will build up this definition in pieces. The game will have d players (our drivers), and it will have a finite graph G. The player's action set consists just of the vertices of G which we will call G_0. The payoff of each player is a little more complicated. In order to establish it we imbue the game further with the following properties. Each vertex j of G will have associated with it a discrete, positive random variable X_j. This is to represent the amount of clients that may request a ride beginning at j. Then the arrangement of clients within the board is itself a single random variable which I will just depict as $(X_j)_{j\in G_0}$. The edges of G, which I will call G_1 are all assigned values. We will say that the value of edge $e \in G_1$ is l_e, these represent the cost (e.i. time cost or fuel cost) of traversing this edge.

We now actually think about the payoff of our drivers. Consider a realisation of $(X_j)_{j\in G_0}$. Drivers seek to pick-up only 1 client, they will be assigned to a client by an driver-client pairing algorithm which we will assume seeks to assign to the client the closest driver it may. The driver will pick up this client and pay a tax equal to the sum of the values of the edges they traversed to get to the client (we assume they take the shortest path). Specifically we express the payoff of player i given a realised value τ of

$(X_j)_{j \in G_0}$ and a joint strategy profile of drivers s as[4]

$$\gamma_i(s|\tau) = \sum_{j \in G_0} Pr(i \text{ is assigned to a client at } j) \cdot \left(1 - \sum_{e \in P} l_e\right)$$

Where P is the path with smallest sum of edges from the vertex i where the driver is at to the vertex j where the client is at.
Note the case that i may fail to collect any clients at all. This comes from the last point of complexity of the game (and our source of competition) is that not only is a driver limited to a single client, but a client cannot not be collected by more than 1 driver. And thus if a realisation τ of $(X_j)_{j \in G_0}$ consists of lest total clients than d (our number of drivers) then some drivers of course will not be assigned a client. Finally if we know γ_i given any τ we can interpret γ_i with a little abuse of notation just to be its own expectation

$$\gamma_i(s) = \mathop{\mathbb{E}}_{\tau \backsim (X_j)_{j \in G_0}} [\gamma_i(s|\tau)]$$

Informally then, drivers seek to position themselves s.t. they maximise there chance of collecting a client that is as close as possible.

To expand on the fact that drivers will be assigned the closest client that they can collect. The distance from a driver at vertex j to a client at j' is the minimum over all paths P from j to j' of $\sum_{e \in P} l_e$. If for two drivers the closest client to them is the same but driver A is closer to this client then driver B, then A will always collect that client instead of B. If instead several drivers are all equally close to a vertex with clients they will all have equal probability of collecting a client from that vertex. It is apparent that a program that can most efficiently assign drivers in this way is almost definitely not trivial, however, we assume such exists as this is not the focus of the work that has been done. We also keep in mind that we analyse specific sub classes of this where such a matching algorithm is much clearer to us.

Describing an explicit description of the payoff function for the general case is unfeasible, hence the fact that we examine in closer details specific cases of this game. However the key attributes of this game; the limit to 1 driver per client and 1 client per driver, the random nature of client distribution and the dynamic nature between driver client pairing are all necessary aspects in order to represent local competitive interactions between drivers, and in general any form of micro-competition between mobile service providers.

That said consider the following result that can assist us in general. This result is rather simple but is very useful so we will go through the effort to consider it closely, especially since it gives us good insight into what we expect our driver-client assignment algorithm to be.

Lemma 4.0.1 (Anonymity Lemma)**.** *The anonymity lemma is that we can think of action profiles in this class of games in terms of just $(d_j)_{j \in G_0}$ where d_j is equal the number of players that choose vertex j in a loss-less way or just that we only need to know how our drivers are distributed but not there identity. Formally, it is that two drivers both at vertex j will always have the same payoff, and to calculate this we only need to know how many drivers are at each vertex, we will call such a game anonymous.*
For the first it is sufficient to show that if i and i' are at the vertex in action profile $s \in \times_{i=1}^{n} A_i$ then $\gamma_i(s|\tau) = \gamma_i'(s|\tau)$. But i and i' are always equally close to any vertex then they always have equal chance of picking up a client from the vertex and will always have the same payoff if they do. So indeed

[4]It is unconventional for a payoff maximisation game such as this to have the possibility of a negative payoff. Thus care is taken to ensure that $1 - \sum_{e \in P} l_e$ cannot be negative

$\gamma_i(s|\tau) = \gamma_i'(s|\tau)$.
Now we assess if we can calculate $\gamma_i(s|\tau)$ while only knowing how many drivers are at each vertex. We need only consider the probability that i will get a client at vertex j, since the payoff of collecting this client is independent of where any other drivers are. However the above is actually just an assumption we make about our client and driver pairing algorithm not just that it exists, but that it is unbiased about the identity of the driver, and thus only consider how close each driver is to a given vertex to make its assignments. Since it reasons that a good algorithm should only assign a driver to a client they will certainly get, and thus the probability a driver gets this client is solely based on the algorithm that we reason only takes into our count that number of drivers at other vertex.[5].

This gives us a simpler way to talk to about action profiles in our game, just as elements within $\mathbb{N}_0^{|G_0|}$, however we want to narrow this down to a subset of this

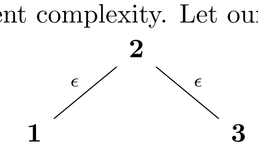
$$S = \left\{ (d_1, \ldots, d_{|G_0|}) \in \mathbb{N}_0^{|G_0|} \mid \sum_{i \in G_0} d_j = d \right\}$$

since this contains only and all, valid driver arrangements, which also makes the domain of our payoff even smaller[6]. However if we do want to do this we will need to think about γ slightly differently. Under this framework we need to change the codomain of γ since it cannot give the component wise payoffs of players, since we have no idea who is who within each vertex anymore. Instead let $\gamma : S \to \mathbb{R}^{|G_0|}$ and let γ_j be the payoff of a player at vertex j. Note that by the same technique we used when taking equilibrium concepts in general we can take $\Delta(S)$ so that we can talk about MNE as well, and the anonymity extends to this nicely, since computing $\gamma(\sigma)$ for $\sigma \in \Delta(S)$, just requires us to compute $\gamma(s)$ for each s that may be realised by σ, and then take a weighted sum of these. CE do exist on this space, however important information is lost when we make a non-product distribution anonymous Our first example CE where players swap between a and b would appear as a fixed strategy when anonymous. Furthermore, it does not reason to consider a distribution in this space that is dependent on certain players, since of course these players are anonymous. Hence while CE are an important part of the theory we will not be concerning ourselves with them going forward.

One of the goals in studying this model was to find some results relating to the POA of this game. We think about efficiency from the perspective of the clients. However we need to be careful about this. If we were to consider any τ s.t. there are more clients than drivers then under our model these clients would be waiting forever to be picked up. The most reasonable option then is to frame our efficiency function as a function V equal to 1 minus the expected waiting time, that is equal to 0 when the client is not picked up. It is then imperative that we ensure $1 - \sum_{e \in P} l_e$ is never less than 0.

5 A Dependent Model

The approach has been to consider two sub classes of our general game model in order to develop some insight into this system given its apparent complexity. Let our graph G be just as depicted below:



[5]This assumption makes sense in our context however it is likely that economy of mobile service providers 3rd parties like Uber may favour certain drivers.

[6]To see this consider that if we swap the position of two drivers that have chosen distinct vertices in a traditional action profile this corresponds to the same element in S, so our mapping into S is not injective.

We have labelled our vertex **1** to **3** and each edge has a value of ϵ. Now in the general construction of the game there is once again d drivers and $(X_j)_{j\in G_0}$ will be equal to $(c,0,0)$ with probability $\frac{1}{2}$ and $(0,0,c)$ also with probability $\frac{1}{2}$. That is to say that no clients will ever appear at **2** and there will either be c clients at **1** or at **3** (but never both) each with a probability of $\frac{1}{2}$. The complexity of this game comes from the dependency between $X_{\mathbf{1}}$ and $X_{\mathbf{3}}$. Let us consider some initial observations.

5.1 Symmetries

We see the inherent symmetry of the Graph and associated distribution, in that **1** and **3** are completely identical. However this does not apply to our payoff function, $\gamma_{\mathbf{1}}(\mathbf{s})$ and $\gamma_{\mathbf{3}}(\mathbf{s})$ are not necessarily equal because **s** itself need not have this symmetry. Let us observe this with some computation, which will also help us get a feel of this subclass of games.

Example 5.1.1. *Suppose that $d=10$ and $c=6$, let us not fix ϵ. Recall that we consider our action profiles to be within S, by the ordering we chose for our vertices consider the profile $s=(7,2,1)$ that is 7 drivers at **1**, 2 and **2** and 1 at **3**. Now we compute our payoffs. Consider $\gamma_{\mathbf{1}}(s)$. we consider first when $(X_j)_{j\in G_0}=(6,0,0)$. Then the algorithm will only ever assign drivers from **1** to our clients. Since there are 7 drivers at **1** they each have a $\frac{6}{7}$ chance of getting a client at all (all the clients at 1), and in this case they get a payoff of 1. When $(X_j)_{j\in G_0}=(0,0,6)$ the program will first match drivers from **3** to our clients and then drivers from **2**, leaving 3 unmatched clients. Thus drivers from 1 have a $\frac{3}{7}$ chance of collecting a client and get a payoff of $(1-2\epsilon)$ is this happens. Then $\gamma_{\mathbf{1}}=\frac{1}{2}(\frac{6}{7}+\frac{3}{7}(1-2\epsilon))=\frac{9-6\epsilon}{14}$. Now we consider $\gamma_{\mathbf{3}}(s)$. In the case $(X_j)_{j\in G_0}=(6,0,0)$ the driver at **3** will never be assigned a client. When $(X_j)_{j\in G_0}=(0,0,6)$ the driver will always be assigned a client and receive a payoff of 1. Thus $\gamma_{\mathbf{3}}(s)=\frac{1}{2}$. Now of course we see that $\gamma_{\mathbf{3}}(s)=\gamma_{\mathbf{1}}(s)$ only when $\epsilon=\frac{2}{6}$ but not in general.*

We consider then how we might utilise this result. Observe what happens if we where to switch the vertices that players choose. Before I begin this proof I introduce the function $[a,b](x)=\min(b,\max(a,x))$ since we will come across this a lot going forward and this will make the work more readable. Let $[-,b](x)=\min(b,x)$ and let $[a,-]=\max(a,x)$.

Theorem 5.1.2 (Symmetry theorem for dependant model)**.** *Consider the permutation*

$$\begin{gathered}\phi:\{\mathbf{1},\mathbf{2},\mathbf{3}\}\to\{\mathbf{1},\mathbf{2},\mathbf{3}\}\\ \mathbf{1}\mapsto\mathbf{3}\\ \mathbf{3}\mapsto\mathbf{1}\\ \mathbf{2}\mapsto\mathbf{2}\end{gathered}$$

And consider its extension onto S (where it still acts as a permutation)

$$\phi((a,b,c))=(c,b,a)$$

If $s\in S$ we argue that $\gamma_{\phi(v)}(\phi(s))=\gamma_v(s)$ Consider the map:

$$Pr_{v'}:V\times S\to\mathbb{R}$$

*Where $Pr_{v'}(v,s)$ gives the probability that a driver at vertex v, given joint profile s will be assigned a client in the event that there c clients at v', then of course $Pr_{v'}$ only makes sense when $v'=$ **1** or **2**. We wish to show that $Pr_{v'}(v,s)=Pr_{\phi(v')}(\phi(v),\phi(s))$.*

$$Pr_{v'}(v,s)=[0,1]\left(\frac{c-\sum_{u\in U_{v',v}}s_u}{s_v}\right)$$

where $U_{v',v}$ is the set of vertex closer to v' than v. It is apparent that

$$U_{v',v} = \phi(U_{\psi(v'),\phi(v)})$$

That is vertices closer to v' then v is the image under ϕ of the vertices closer to $\phi(v')$ then $\phi(v)$. Thus

$$\begin{aligned}
Pr_{v'}(v,s) &= [0,1]\left(\frac{c - \sum_{u \in U_{v',v}} s_u}{s_v}\right) \\
&= [0,1]\left(\frac{c - \sum_{u \in \phi(U_{\phi(v'),\phi(v)})} s_u}{s_v}\right) \\
&= [0,1]\left(\frac{c - \sum_{u \in U_{\phi(v'),\phi(v)}} s_{\phi(u)}}{s_\phi(v)}\right) \\
&= Pr_{\phi(v')}(\phi(v), \phi(s))
\end{aligned}$$

*Thus we see that for a player at v when v is **1** or **3***

$$\begin{aligned}
\gamma_v(s) &= \frac{1}{2}(Pr_v(v,s) + (1-2\epsilon)Pr_{\phi(v)}(v,s)) \\
&= \frac{1}{2}(Pr_{\psi(v)}(\phi(v),\psi(s)) + (1-2\epsilon)Pr_v(\psi(v),\psi(s)) \\
&= \gamma_{\phi(v)}(\phi(s))
\end{aligned}$$

and if $v = \boldsymbol{2}$ then

$$\begin{aligned}
\gamma_v(s) &= \frac{1}{2}(1-\epsilon)(Pr_{\boldsymbol{1}}(v,s) + Pr_{\boldsymbol{2}}(v,s)) \\
&= \frac{1}{2}(1-\epsilon)(Pr_{\boldsymbol{2}}(\phi(v),\phi(s)) + Pr_{\boldsymbol{1}}(\phi(v),\phi(s))) \\
&= \gamma_{\phi(v)}(\phi(s))
\end{aligned}$$

So in general the statement holds. Note it may appear that only one direction of the statement was shown however since $\phi \circ \phi(s) = s$ we have actually shown both directions of the statement.

In order to establish a use of this theorem, recall the use of (a, s_{-i}) for $s \in A$ to represent the joint action profile where player i chooses a and every other player chooses their action as they did in s let (v, s_{-i}) for $s \in S$ refer to driver i choosing vertex v and every other driver choosing the vertex they did in s. Then consider that

Corollary 5.1.2.1. *is $\sigma \in \Delta(S)$ is an MNE iff $\phi(\sigma) \in \Delta(S)$ is. Here $\phi(\sigma)$ is random variable that realises $\phi(\boldsymbol{s})$ with probability that σ realises $\boldsymbol{s}$. Again since $\phi \circ \phi(\sigma) = \sigma$ it suffices just to show one direction. This just comes from the best responce lemma so long as we can be sure that $\gamma_v(\sigma) = \gamma_{\phi(v)}(\phi(\sigma))$ but after all the work we did, this just falls out.*

$$\begin{aligned}
\gamma_v(\sigma) = \mathop{\mathbb{E}}_{s \backsim \sigma}[\gamma_v(\boldsymbol{s})] &= \sum_{\boldsymbol{s} \backsim \sigma} Pr(\sigma = \boldsymbol{s})\gamma_v(\boldsymbol{s}) \\
&= \sum_{\phi(\boldsymbol{s}) \backsim \phi(\sigma)} Pr(\phi(\sigma) = \phi(\boldsymbol{s}))\gamma_{\phi(v)}(\phi(\boldsymbol{s})) \\
&= \gamma_{\phi(v)}(\phi(\sigma))
\end{aligned}$$

Noting that we are able to change the terminals of the sum as we did since ϕ is a bijection and since S is the set of all s that σ may realise (even with potentially probability 0), then $\phi(S)$ is the set of all that $\phi(\sigma)$ can realise and $\phi(S) = S$

This is a useful result for the work done in relation to this subclass, as if we can compute γ for $S' = \{(a, b, c) \in S \mid a \leq c\}$ we know γ for all $s \in S$. This is because if $s \notin S'$, $\phi(S')$ certainly is.

5.2 Cases

We consider three classes that we can divide this subclass into. Here we will discuss them just in the context of equilibrium.
The first is when $c \geq d$. If such is the case then there is no competition which is to say it is completely trivial. By the fact that the total number of drivers realised by $(X_j)_{j \in G_0}$ is always C, every driver will then be guaranteed a client. Thus the payoff of a driver is the average of 1 minus the distance from A and from B. Which is the same at all vertex. Indeed at A or B this is $\frac{1}{2}(1 + (1 - 2\epsilon)) = 1 - \epsilon$ which is the same as it is at C. Thus for any $s \in S$ and vertex v that a driver chooses $\gamma_v(s) = 1 - \epsilon$. Then absolutely any arrangement in $\Delta(A)$ is a CE.
Now consider the case when $d \geq 2c$ these cases are also trivial, since the arrangement $s = (\text{floor}(\frac{d}{2}), 0, \text{ceil}(\frac{d}{2}))$ must always be a PNE. We see this because if we let $d'^{+} = \text{ceil}(\frac{d}{2})$ and $d'^{-} = \text{floor}(\frac{d}{2})$, we know that $d'^{+} > d'^{-} \geq c$. Then we break this down into two cases again

$d'^{-} = c$ We then consider a driver at **1**. It has no chance of being a assigned a client when the arrangement of clients is $(0, 0, c)$. When the client arrangement in $(c, 0, 0)$ it will recieve a payoff of $\frac{c}{d'^{-}} = 1$. Thus its payoff overall is $\frac{1}{2}$. If this player were to switch to **3** it will have a payoff of $\frac{1}{2}(\frac{c+(1-2\epsilon)}{d'^{+}+1})$. Since $c + (1 - 2\epsilon) \leq c + 1 \leq d'^{+} + 1$ then its payoff at **3** is less than or equal to $\frac{1}{2}$, its payoff at **1**. It it were to switch to **2** it would get a payoff of $\frac{1}{2}(1 - \epsilon)$ which is also less than or equal to its payoff at **1**. Then players at **1** have their BR condition fulfilled. Now we consider a driver i at **3**. If $d'^{+} = c$ as well then $\phi(s) = s$ and thus since the BR condition at **1** is true, it also is at **3**. So we need only consider when $d'^{+} = c + 1$. Then the payoff for a driver at **3** is $\frac{c}{2(c+1)}$. If it were to switch to **1** it would get the same payoff because in this case $(\mathbf{1}, s_{-i}) = \phi(s)$ and $\phi(\mathbf{3}) = \mathbf{1}$. If it were to switch to **2** its payoff would be 0. There are no drivers at **2**, so this is indeed a PNE.

$d'^{-} > c$ We consider a driver at **1**. Its payoff is equal to $\frac{c}{2d'^{-}}$. If it were to switch to **3** it would have a payoff of $\frac{c}{2(d'^{+}+1)}$ which is clearly less. If it were to switch to **2** it would get a payoff of 0. Now we consider the payoff a driver at **3**. it receives a payoff of $\frac{c}{2d'^{+}}$. If it were to switch to **1** it would receive a payoff of $\frac{c}{2(d'^{-}+1)}$ which is clearly less than or equal to its original payoff. If it were to switch to **2** it would once again receive a payoff of 0. So again both drivers at **1** and **3** are acting in there best response.

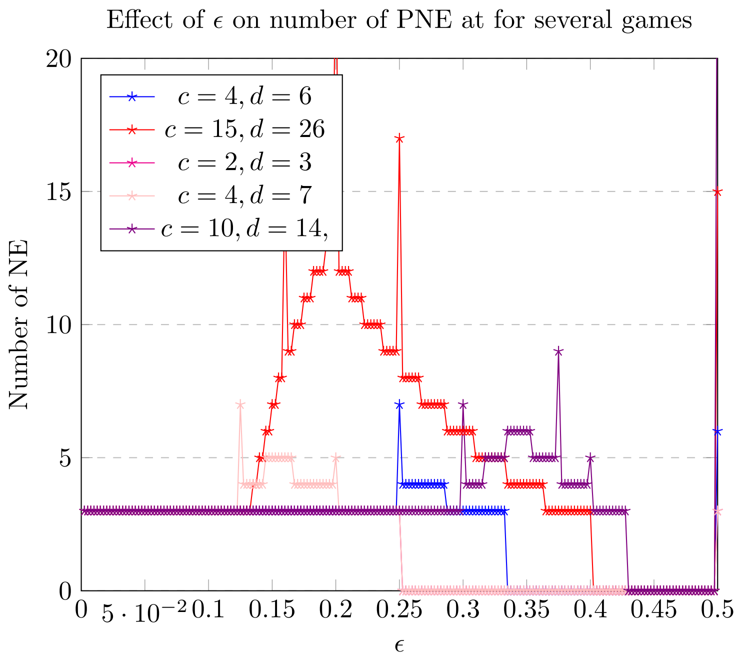
Certainly the above was a little work and goes to show the complexity of this game even in this state. Primarily because our payoff is discrete we are limited in the techniques we may employ.
The final case is when $d \in (c, 2c)$ this is in general much trickier then the other cases as we can observe.

5.3 Computational method

I now detail the means from which we gained insight into this form of our game by using code. The program is very simple it simply iterates over any arrangement $s \in S'$ and checks if each s is a PNE using the standard definition. From this we can gain comprehensive data on the PNE for any game in the subclass. I will be upfront that most of the time spent on this program, which was significant was in analysing data in order to understand the patterns of these PNE which was not successful.

5.4 Observations and Results

Consider consider the following data of how many PNE where found (including distinct values of s and $\phi(s)$) for different values of ϵ given various games.



There are some key observations to make here, about the shape of the curves, random peaks, the fact that for all of these games we see a region at the start where there are 3 NE, and a region at the end where there are 0 and finally that there is a peak when $\epsilon = \frac{1}{2}$. The peak is easiest to reason, when $\epsilon = \frac{1}{2}$ even if a driver from say **3** has a chance of being assigned a client at **1** they will not receive any payoff from this. Then take any $s \in S$ s.t. $s_1 \leq c$ and $s_3 \leq c$ and $s_2 = 0$ which must exist by the fact that $d < 2c$. This a PNE

Proof 5.4.1. *Take an s as specified above, then the payoff of a driver at v where $v = \mathbf{1}$ or $v = \mathbf{3}$ is $\frac{1}{2}$. This is because in the event of c clients at $\phi(v)$ a driver at v will not get a payoff as discussed. In the event there are c drivers at v then since there are less than c drivers at v all drivers at v will certainly be assigned a client and receive a payoff of 1. Thus the actual payoff of a driver at v is $\frac{1}{2}$. Now there is no incentive for divers to switch between v and $\phi(v)$. Consider however the maximum payoff of a driver i switching to $\mathbf{2}$, we compute this using the notation from the symmetry theorem as*

$$\frac{1}{2}\left(\frac{1}{2}Pr_{\mathbf{1}}(\mathbf{2}, (\mathbf{2} \oplus s_{-i})) + \frac{1}{2}Pr_{\mathbf{3}}(\mathbf{2}, (\mathbf{2} \oplus s_{-i}))\right)$$

This clearly maximised by maximising the probabilities which are at most 1, this optimized value is then $\frac{1}{2}$, thus there is also no insentive to switch to $\mathbf{2}$

In general the PNE of this form can be quite large, equal to $2c-d$, and as we saw in this proof s_2 need not be 0. In fact since the payoff of a driver at **2** is maximised so long as the respective probabilities are we just need to require that $s_2 \leq \min(c - s_1, c - s_2)$ which is admits more PNE still. However while this game was being analysed the major point of interest was the region in which no PNE existed. Specifically because of the pattern it makes.

Conjecture 5.4.2. *For a dependent type game will not admit a PNE only for*

$$\epsilon \in (\frac{2c-d}{4c-2d+2}, \frac{1}{2})$$

This lower bound was derived by observation only. A few key observations were made including the fact that bound does not change when c increased by k and d by $2k$. Then the lower bound could be thought of as a function on $2c-d$. By observing that the bound approached $\frac{1}{2}$ when $2c-d$ got large. This was not proved. However it is supported by data which you can see in the appendix.

Considering this conjecture and the explanation of the peak in PNE at $\epsilon = \frac{1}{2}$ we have a key insight into how the competition between drivers in regional areas where distances between places clients may appear (hot-spots) is large, but where there is some relationship between the two areas in order explain the negative correlation between $X_{\mathbf{1}}$ and $X_{\mathbf{3}}$. We observe that when distances are large enough it is completely nonviable for their to be more drivers in one hot-spot then potential clients. This is because payoff of a driver at one of these hot-spots (in our model **1** or **3**) if they are assigned a client in the other hots-pot is low due to the high distance between them. We see that at the extreme case ($\epsilon = \frac{1}{2}$), this severs the two hot-spot into independently function markets for service providers (drivers at **1** and **3** are not competing with each other) and at the less extreme case (region given in the conjecture) this manifests in an instability as their are no PNE. In general MNE may be practically realised as drivers changing there actions over multiple rounds of the game.

5.5 POA

Because here there is no possibility of their being more clients than drivers when $d \geq c$ which contain the more interesting cases, a client waiting for every is not something we need to worry about. Thus the naive expected waiting time (sum of edges traversed by the driver on their way to collect the client) was used to define V. If we let $t_1(s)$ be the arithmetic average waiting time of our clients in the event they appear at **1** and $t_3(s)$ for when they appear at **3** we give V by

$$V(s) = \frac{t_1(s) + t_3(s)}{2}$$

If we wish to optimise V it is sufficient to optimise $t := 2cV = ct_1 + ct_3$. Let us find an optimal s to minimise V (this our goal since V is a cost function). We argue that one is $s = (c, 0, d-c)$

Proof 5.5.1. *Observe $ct_1(s) = 0$ so this cannot be further optimised. however $ct_3(s) = 2\epsilon(c-(d-c)) = 2\epsilon(2c-d)$ so it could be smaller. Now clearly moving any drivers from **3** will not decrease ct_3. Suppose we create a new profile s' by taking δ_2 drivers from **1** and putting them at **2** and taking δ_3 from **1** and putting them at **3**. Then $c(t_1(s') - t_1(s)) = \epsilon\delta_2 + 2\epsilon\delta_3$, but at most we have decreased t_3 by this much e.i. $c(t_3(s') - t_3(s)) \geq -\epsilon\delta_2 - 2\epsilon\delta_3$. Thus $t(s') - t(s) \geq 0$. Considering that s' is the only way we can reasonably change s to decrease t_3 we observe that s must indeed be a minimum.*

Given this t is at least $2\epsilon(2c-d)$. Now we argue that t is maximised by $s^* = (d, 0, 0)$.

Proof 5.5.2. *In general consider any arrangement s' this suppose here that $ct_3(s') = k$ then naively it is clear that $ct_1(s') \leq 2c\epsilon$ but consider that $ct_3(s') = k$ must be because there are some cars ϵ and 2ϵ from **3** which is equivalent to cars at **1** or at **2** these are closer to **1** then any car at **3** and thus they will be assigned clients at **1** over drivers at **3**. Then $ct_3(s') = k = 2\epsilon k_1 + \epsilon k_2$ and $ct_1(s') \leq 2c\epsilon - k$. Then it stands to reason that $t(s') \leq 2c\epsilon$ observing that $t(s^*) = 2c\epsilon$ this is indeed the maximum of ot t.*

Considering that

$$\frac{V(s^*)}{V(s)} = \frac{t(s^*)}{t(s)} = \frac{c}{2c-d}$$

this is our bound for the POA for games when $c < d < 2c$ as in this case there is always a choose of ϵ such that s^* is a PNE

Proof 5.5.3. *We observe that the payoff for all drivers in s^* (they are all at **1**) is*

$$\gamma_1(s^*) = \frac{1}{2}(2-2\epsilon)\frac{c}{d}$$

*If one driver were to switch to **2** they would recieve a payoff of $\frac{1}{2}(1-\epsilon) = \frac{d}{2c}\gamma_1(s^*)$ and since $d < 2c$ we see that $\frac{d}{2c} < 1$ so this must be a decrease in payoff. Now if a player where to switch to **3** its payoff would $\frac{1}{2}$. Thus if this is a PNE we require that:*

$$\begin{aligned} \frac{1}{2} &\leq \frac{1}{2}(2-2\epsilon)\frac{c}{d} \\ \iff \frac{d}{2c} &\leq 1-\epsilon \\ \iff \epsilon &\leq 1 - \frac{d}{2c} \end{aligned}$$

Such a value of ϵ can be reasonably chosen since as before $\frac{d}{2c} < 1$.

Thus $\frac{c}{2c-d}$ is a bound for the POA on this subclass of games whenever $d \geq c$ that is tight on any c and d s.t. $d < 2c$.

5.6 Concluding remarks of Dependent Model

This model was analysed before thinking about the second model. As my first attempt at research I believe there is much to improve on. I think that the focus on PNE in the work was unnecessary and hindered progress, and using techniques applied to the next model to consider more specific types of MNE could have been very fruitful. I also believe that the methods through which I attempted to prove the conjecture where very coloured by techniques in pure mathematics and given the adhoc nature of the model I may have gotten results if more time where dedicated to a case by case aproach which in hindsight seems the most appropriate, allowing a more "elegant" proof to follow. I also believe that more insight could have been gleamed from closer attention to the data from the program. Finally I think that the negative correlation of $X_{\mathbf{1}}$ and $X_{\mathbf{3}}$ is not an assumption reflective of real markets and thus this model is not particularly helpful.

6 The Independent Star Model

For this model let our graph by G be of the form as below.

$$\begin{array}{ccccc} & & \cdot & & \\ & & | & & \\ & & \epsilon & & \\ & & | & & \\ \cdot & -\epsilon- & * & -\epsilon- & \cdot \\ & & | & & \\ & & \epsilon & & \\ & & | & & \\ & & \cdot & & \end{array}$$

Specifically let $G_0 = \{*, 1, \dots, n\}$ and $G_1 = \{(*, j) \mid j \in n\}$ that is there is one edge between $*$ and every other vertex. Let each edge have value ϵ. For $(X_j)_{j \in G_0}$ let each X_j be independent. Let $X_* = 0$ and let each other X_j be a binomial distribution with mean c. This game once again will have d drivers. We will call the vertex $*$ the centre and the vertices other than $*$ the terminals, these terminal vertex will belong to the set T noting that there are n of these. Going forward let j just be any terminal, and so X_j be a random variable governing the clients at a terminal j.

This model was made with more direction and a specific question in mind, under different condition, particularly the variance of X_j, how much will drivers favour $*$ in a NE.
Before I begin describing the fruitful path of analysis I will briefly mention the first method used, since it establishes some intuition for the game. Considering a driver at the centre it will either be assigned a client ϵ away or it will not be assigned a client . Then its payoff is $(1-\epsilon)Pr(\text{assigned a client})$. Note that this probability needs to be the expected probability the driver is assigned a car over every realisation τ of $(X_j)_{j \in T}$. For drivers at a terminal there are three different outcomes, the driver is assigned a client at the vertex it is at, it is assigned a client 2ϵ away or it is not assigned a client. Consider the dependencies here, each event for a driver terminal can only occur if the previous ones did not. let k_1 be the expected probability that the terminal driver is assigned a client at the terminal it is at and let k_2 be the expected probability given this does not occur that it is assigned a client 2ϵ away. Then the payoff of a terminal player is $k_1 + (1 - k_1)k_2(1 - 2\epsilon)$.

When the first path of analysis was taken the source of complexity was these payoff functions, and it was hoped that taking the limit as $n \to \infty$ but keeping the ratio $\frac{cn}{d} =: \alpha$ fixed would help untangle this. By applying the anonymity lemma we could think of action profiles as distributions of drivers over the vertices. If $s \in S$ let s_* be the drivers at the centre and s_j the drivers at the terminal indexed by j. Then for a player i at j it was formulated that its payoff should be

$$\gamma_j(s) = \mathbb{E}\left[[-,1]\left(\frac{X_j}{s_j}\right) + (1-2\epsilon)\left(1 - [-,1]\left(\frac{X_j}{s_j}\right)\right)\Xi\right]$$

where Ξ is given by

$$[-,1]\frac{[0,-]\left(\left(\sum_{j\in T}[0,-]X_j - s_j\right) - s_*\right)}{\sum_{j\in T}[0,-]s_j - X_j}$$

Computing this seemed difficult however by considering just the behaviour of very large n we could apply the central limit theorem to be able to estimate and Chebyshev's inequality we where able to bound the numerator and denominator. Taking $\beta^+ := \mathbb{E}\left[\sum_{j\in T}[0,-]X_j - s_j\right]$ we bound the numberator within

$$\left[[0,-](n\beta^+ - \Theta\sqrt{n} - s_*), [0,-](n\beta^+ + \Theta\sqrt{n} - s_*)\right]$$

And if $\beta^-+ := \mathbb{E}\left[\sum_{j\in T}[0,-]s_j - X_j\right]$ the we bound the denominator by

$$\left[n\beta^- - \Theta\sqrt{n}, n\beta^- + \Theta\sqrt{n}\right]$$

Then as $n \to \infty$ we see that $\Xi = [-,1]\frac{[0,-]\beta^- s_*}{\beta^-}$ Which is certainly far simpler. This approach was ultimately abounded for the reason that while being able to calculate the payoff is very important it was not particularly helpful for a broader and more fundamental analysis of this sub-class. The assumption was also deemed unrealistic and when attempting a broader analysis it was unhelpful.

6.1 Using the Symmetry theorem

It is not surprising that the symmetry theorem from the previous section applies here. Where before **1** and **2** were interchangeable now any terminal is interchangeable. However this time we have a related but more useful result. Consider the following equivalence class. We will say that a vertex j and j' are similar if the map $\phi_{j,j'}$ the permutes the vertices by swapping j with j' (and has an action on S as described in 5.1) has the property that for all $s \in S$, $\gamma_j(s) = \gamma_{\phi_{j,j'}(j)}(\phi_{j,j'}(s))$. This is indeed an equivalence class, $j \backsim j' \iff j' \backsim j$ by the fact that $\phi_{j,j'} = \phi_{j',j}$ and $\phi^2 = id$. Likewise if $j \backsim j'$ and $j' \backsim j''$ then $\phi_{j,j''} = \phi_{j'',j'} \circ \phi_{j,j'}$ and thus the fact that $j \backsim j''$ follows.

Then consider that if Γ is a game within our class of games let Γ' be a game where players may choose between the equivalence classes of the vertices. Let $[S]$ be the set of the set of distributions of players over these equivalence classes. Let γ' be the payoff of Γ' s.t. for $[s] \in [S]$ $\gamma'_{[j]}([s])$ is equal to $\gamma_j(\sigma)$ where $\sigma \in \Delta(S)$ given by each player that chose some $[j]$ in Γ' choosing uniformly between each $j \in [j]$. We need to check that this definition is well defined. But it is form how we defined our equivalence class. If $j, j' \in [j]$ then $\gamma_j(s) = \gamma_{j'}(\phi(s))$, and this extends to an element of $\Delta(S)$ since

$$\mathop{\mathbb{E}}_{s \backsim \sigma}[\gamma_j(s)] = \mathop{\mathbb{E}}_{s \backsim \sigma}\left[\gamma_{\phi(j)}(\phi(s))\right]$$

but since in σ each player that chose $[j]$ plays every $j \in [j]$ uniformly then σ must be invariant on ϕ. and so the above is equal to

$$\mathop{\mathbb{E}}_{s \backsim \sigma}\left[\gamma_{\phi(j)}(s)\right]$$

.

Theorem 6.1.1 (Similar vertex theorem). *The statement of the theorem is that a PNE in Γ' gives an NE in Γ by the above interpretation of $[s] \in [S]$ as some $\sigma \in \Delta(S)$. Indeed consider in Γ some driver i that chose $[j]$ in Γ'. i will choose each $j \in [j]$ with equal probability. We check is i has fulfilled their best response. It is sufficient by the best response lemma to show that Supposing i chose just j, i will receive a greater payoff then if she chose any other vertex. Indeed for any other $j' \in [j]$ we have by equivalence ϕ s.t.*

$$\mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_j(j, s_{-i}) = \mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_{\phi(j)}(\phi(j), \phi(s_{-i})) = \mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_{j'}(j', \phi(s_{-i})) =$$

but again since σ is invariant under ϕ this is just

$$\mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_{j'}(j', s_{-i})$$

Now we also know that since $[s]$ is a PNE then for i in Γ' for any other choice of equivalence class $[k]$ $\gamma'_{[j]}([j], [s]_{-i}) \geq \gamma'_{[k]}([k], [s]_{-i})$ and thus in Γ

$$\frac{1}{|[j]|} \sum_{j \in [j]} \mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_j(j, s_{-i}) \geq \frac{1}{|[k]|} \sum_{k \in [k]} \mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_k(k, s_{-i})$$

but as established each term in these sums is equal by there similarity and thus this becomes

$$\mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_j(j, s_{-i}) \geq \mathop{\mathbb{E}}_{s \backsim \sigma} \gamma_k(k, s_{-i})$$

as desired. thus σ is indeed an NE.

This theorem is extremely powerful, it tells us how we can exploit symmetries within the graph and distribution in any game of our broader class, in order to reduce the possible vertex a driver may choose and thus complexity of our game. Note that not every NE of Γ will follow from one in Γ'.
In terms of this subclass, taking for granted the trivial similarity between every terminal we can instead think of our game (if we are willing to miss out on some NE) as a game where players choose between the terminals or the centre only.

6.2 A Potential function

Note now that we have a two player game where drivers choose between terminal (T) or the centre ($*$). We can further exploit this simplicity with a potential function. However the application of the potential function that has been discovered apply to a broad class of games distinct from the one that we consider. Let Γ be a game with players $i \in \{1, \ldots, d\}$ each with an action set A_i is equal to the set $\{a, b\}$. Furthermore, if in $s \in A$ player i and i' have chosen the same action $\gamma_i(s) = \gamma_{i'}(s)$, that is Γ must be fair. We also require that given any permutation α of $\{1, \ldots, d\}$, if we change the action that i chooses to the action $\alpha(i)$ chooses, the payoff for the players that choose a and b in this new arrangement is the same as it was in the old arrangement. That is the game in anonymous in the sense that γ does not take into account the identity of a player.

The requirement that the players be limited to two moves is certainly restrictive, however the requirement that if two players choose the same move they will have the same payoff is very realistic and this is true for many novel games that are studied. Note that while they seem related a game can be fair by the above definition but not anonymous. For a counter example take n to be atleast 3 and and suppose we have $\{1, \ldots, n\}$ players. Then suppose that players who choose a get a payoff equal to the sum of the index of the players that chose a, and let players who chose b get one plus this sum if it is even and 0 if it is odd. This game is clearly fair but also not anonymous, suppose only 1 and 2 pick a they will recieve 3 if we permute this so instead only 2 and 3 pick a then they get 5. Now supposing that Γ meets these requirements, we argue that

Claim 6.2.1. *There is a potential function for* Γ

Let $f, g : \{0, \ldots, d\} \to \mathbb{R}$ let $g(x)$ be the payoff of players that choose a when x players choose a and let $f(x)$ be the payoff of the players that choose b when x players choose a. This is well defined by Γ being anonymous. In the event that no players choose an action, let the corresponding function evaluate to 0
Let us define our potential function $\Psi : \{0, \ldots, d\} \to \mathbb{R}$ as

$$\Psi(x) = \sum_{i=0}^{x-1} f(i) + \sum_{i=x+1}^{d} g(i)$$

Note that this definition has problems when x is equal to 0 or d because then in one of the sums the upper terminal will be smaller than the bottom, let us just interpret this as 0 when it occurs.

Now consider that a PNE in Γ is precisely a point s.t. no player that chooses a we can increase their payoff by choosing b instead and vice versa. This is equivalent to x represents a PNE iff, $g(x) - f(x-1) \geq 0$ and $g(x+1) - f(x) \geq 0$, note however that the former need not hold if $x = 0$ and the latter if $x = d$. Now consider (for $x \neq 0, d$) $\Psi(x) - \Psi(x+1) = g(x+1) - f(x)$ and (when $x \neq 0, d$) $\Psi(x) - \Psi(x-1) = f(x-1) - g(x)$. Thus we see for $x \in \{1, \ldots, d-1\}$, $\Psi(x)$ is a local min $\iff$ x is a PNE. Now we deal with the special cases. Suppose first that $x = 0$ then $\Psi(x) - \Psi(x+1) = g(1) - f(0) = f(x) - g(x+1)$ and $x-1$ is not defined so we need not check this side. Now we check $x = d$, $\Psi(x) - \Psi(x-1) = f(d-1) - g(d) = f(x-1) - g(x)$ and of course Ψ is not defined on $d+1$. Therefore even when $x = 0, d$, $\Psi(x)$ is a local minimum iff x is a PNE. Furthermore, since $\{0, \dots, d\}$ is finite it must admit a local minimum.

Now it is very clear that by applying the similar vertex theorem to the sub class of games we are studying, if our game is Γ, Γ' has identical action sets each with only two actions, Γ' is fair and Γ' is anonymous. Thus Γ' has the above potential function. Then we guarantee the existence of what we will call symmetric MNE for our game, and we can easily compute them. Specifically let $g(x)$ be the payoff for centre drivers when x drivers choose the centre and let $f(x)$ by the payoff for terminal drivers (those that choose each terminal with uniform probability) when x drivers choose the centre. However, we still miss a key piece of the puzzle, computing f and g.

6.3 Computing Payoffs

Now we look at methodology to compute f and g. We first reason about the driver-client assignment program. When the program seeks to assign to a client a driver it will first seek to a assign to it a driver at the terminal the client appeared at. If this fails it will try and assign the next closest driver, one that is at the centre. Only if these both fail will the program be willing to assign a driver from a different terminal. Conversely if the program is to decide which drivers are not assigned a client, it will always chose terminal drivers that have not been assigned a client at their terminal. Thus if k_1 was as before the expected probability that a driver at a terminal will be assigned a client at their terminal, let k_3 be the probability that given it was not assigned a client at its terminal it will not be assigned a client at all. Let us in general assume that there are nc clients in any realisation of the game, a reasonable assumption by the central limit theorem, however one that could be adjusted in the future. Then we know that certainly $[0,-](nc-d)$ drivers will not be assigned a client. Consider also if k_1 is the expected probability a driver succeeds in collecting a client and x the drivers at the centre, $(1-k_1)(d-x)$ should be expected not to be assigned a client at their terminal, and thus of these, each has an expected probability of $k_3 = [-,1]\frac{[0,-](nc-d)}{(1-k_1)(1-x)}$ to fail, this expression may exceed 1 if centre drivers will also fail to be assigned, hence the fact that it is guarded.

We now consider computing k_1. Let D be the probability distribution governing how many other drivers will be at the same terminal as the terminal player i. We do not distinguish between terminals because no matter what terminal i chooses D should be identical by uniformity. Consider that D is binomial and

$$Pr(D = i) = \binom{d-1}{i}\left(\frac{1}{n}\right)^i \left(1 - \frac{1}{n}\right)^{d-i}$$

We will want know the expectation and variance of $\frac{1}{D+1}$. The expectation is

$$\frac{n}{d}(1 - (1-p)^d)$$

and the variance is

$$\frac{n^2}{d^2+d}\left(1 - \left(1 - \frac{1}{n}\right)^{d+1} - (d+1)(p)(1-p)^d\right) - \left(\frac{n}{d}(1-(1-p)^d)\right)^2$$

(The derivation is not as bad the expression may imply). Now

$$k_1 = \mathbb{E}\left[[-,1]\left(\frac{X_j}{D+1}\right)\right]$$

Since X_j and $\frac{1}{D+1}$ are independent we can calculate

$$\mathbb{E}\left[\frac{X_j}{C+1}\right]$$

Just by knowing each of their expectations. However bounding this by 1 makes things slightly trickier. In general

$$\mathbb{E}\left[[-,1](X)\right] = \mathbb{E}\left[X\right] + \int_{i=1}^{\infty}(1-x)Pr(X=x)dx$$

In our case the last term is a little prickly because $\frac{X_j}{d}$ takes on rational values, of course then we take a sum instead of an integral and we need to be very careful about this. From here the Chebyshev's inequality was used to ignore values that $\frac{X_j}{d}$ may release of sufficiently low probability which is why we were interested in the variance.

Finally since we know how to calculate k_1 and k_3 we can observe that the chance that a terminal driver is assigned a client on another terminal is $1-k_1-k_3$ since one of these three events must occur and they are mutually exclusive. Then we can calculate the expected payoff of a terminal driver as $k_1+(1-2\epsilon)(1-k_1-k_3)$.

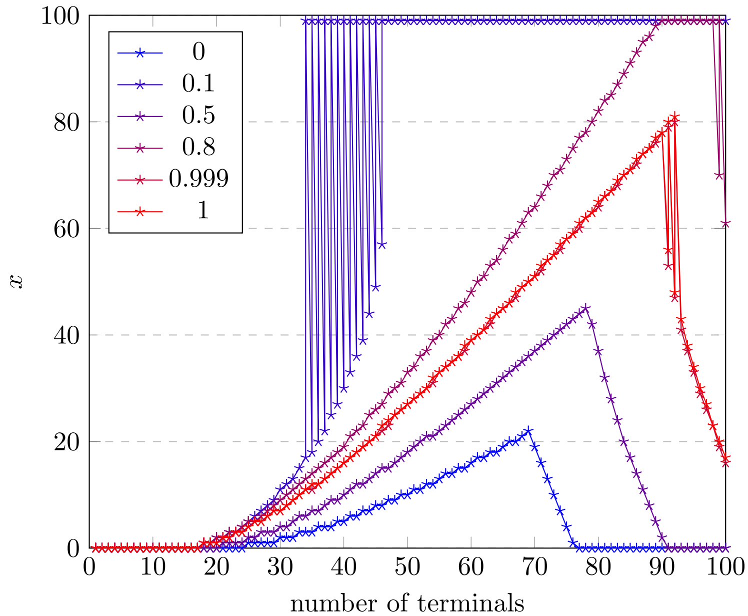
Calculating the payoff of a centre driver is much easier. We just need consider the chance they are not assigned a client. We know $[0,-](nc-d)$ total drivers are not assigned a client and that a centre driver will not be assigned a client only if all the $(1-k_1)(d-x)$ terminal drivers that were not assigned a client on their terminal are not assigned a client. Thus $[0,-](nc-d-(1-k_1)(1-x))$ centre drivers are not assigned a client. Thus the expected payoff of a centre driver is $(1-\epsilon)\left(1-\frac{[0,-](nc-d-(1-k_1)(d-x))}{x}\right)$.

6.4 Observations

Given these methods of computing the payoffs and the potential function a Python program was written in order solve for the symmetric MNE. The main class in the program is specified by n (the number of terminals), the value of ϵ, the number of drivers and a custom object representing X_j (containing its pdf, expectation and variance). This class then computes the payoff for terminal and centre drivers at each x representing how many drivers choose the centre. A secondary object then pieces together the potential function and finds any local minimums. Some experiments were performed using this program.

One such was to find the effect of variance on these MNE. Here we fix the number of drivers at 100, the expectation of X_j at 1 and ϵ at 0.4. The horizontal axis here is the number of terminals, the vertical are the values of x at which symmetric MNE occur. Each plot is a different value of the variance described by the legend.

Effect of Terminals on the Values of x When a Symmetric MNE Occurs with Different Values of Variance

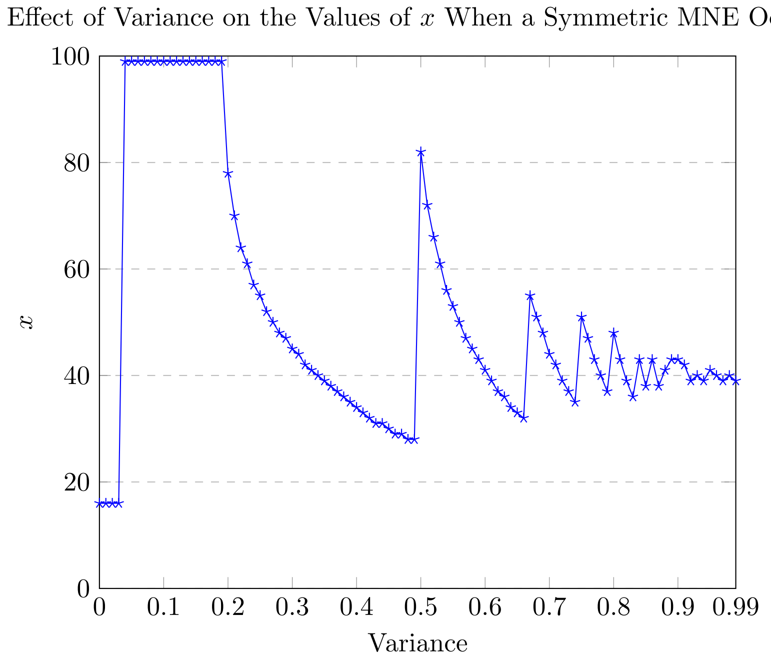


These results are surprising. Key observations are that: when the terminal count is high enough, the symmetric MNE will occur when x is smaller then when the terminal count has more moderate values; the centre is viable even when X_j is equal to c with probability of 1 (variance is 0); and that increasing variance does not necessarily make the centre more viable.

The first observation can be explained by the presence of two competing effects. There is the effect that increasing terminals then increases the chance for there to be a terminal in which there are less drivers then clients and thus a client must be assigned to a centre driver, thus increasing the viability of the centre. However as the terminals continue to increase terminal drivers also benefit as they can now expect less competition at whatever terminal they choose, but still be exposed to the same amount of clients.

The Second observation is due to the fact that even when every terminal has a consistent amount of drivers, under this model each terminal driver will still choose between terminals randomly, thus at some terminals there will be more clients than drivers, increasing the viability of centre drivers, even if this is clearly wasteful. Hence we also observe that this arrangement is not optimal.

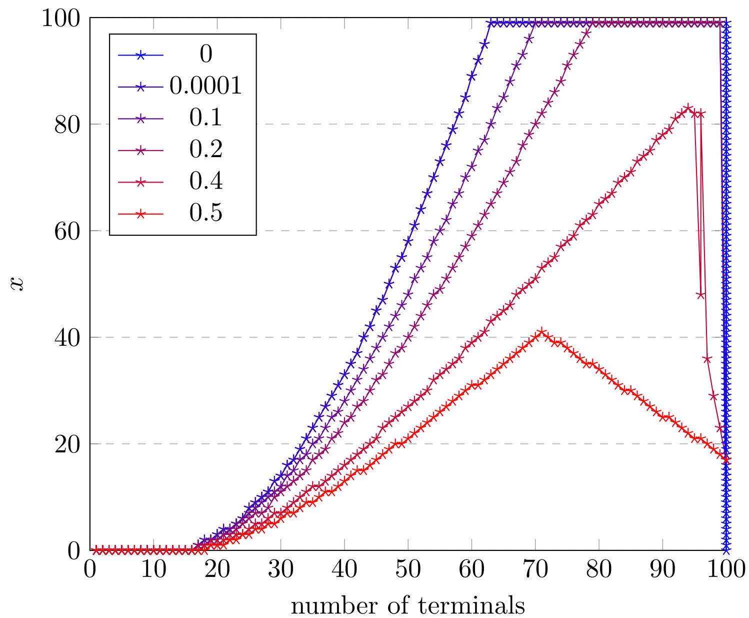
To consider the next observation we consider another experiment. Here we fix the drivers, expectation of X_j and ϵ as before. However we also fix the terminals to 60 and plot variance over the horizontal axis. The vertical axis remains as before.



In doing so we find a remarkably interesting result. I have not succeeded in finding an explanation for this.

We consider one final experiment, on the value of ϵ. This has a similar setup to our first experiment in this subsection. Let again there be 100 drives, let the expectation of X_j be 1 and let the variance be 0.999. Again the horizontal axis is the number of terminals and each plot is a different value of ϵ

Effect of Terminals on the Values of x When a Symmetric MNE Occurs with Different Values of ϵ



The key observation here is that by increasing ϵ we seem to favour the viability of the terminals. This seems reasonable. A centre player will never receive a payoff more than $(1 - \epsilon)$. This is not the case for a terminal player, in fact the event that they are assigned a client on another terminal is not very likely, and thus the event in which they receive a payoff of $(1 - 2\epsilon)$ makes up a small part of a terminal players payoff. One other thing to note that is not particularly clear from the graph is that when $\epsilon = 0$ and $n = 100$, every x is a symmetric MNE. This is due to the fact that my script in assuming that there will always be $nc = 100$ clients in this case anticipates that every driver will be assigned a client. Since $\epsilon = 0$ and every driver is assigned a client all drivers are calculated to have an expected payoff of 1, thus any arrangement is a symmetric MNE. This effect is not present even for ϵ as small as 0.0001.

6.5 Concluding remarks on the Independent Star Model

Overall analysing this sub class of games was more successful. Particularly in that a class of MNE was effectively characterised and more general results like the similar vertex theorem and a potential function for 2 action, fair and anonymous games were obtained. In future I would hope to further explore the behaviour of these symmetric MNE, since most of the effort up until now was spent building the machinery to produce the data. There are also likely worthwhile results to be found in POA of symmetric MNE which will provide useful insight into the Uber market. It may also be worthwhile trying to expand the potential function argument to games with more than 2 actions, however by the dependent model we know that it does not exist for 3 action, fair and anonymous games.

This concludes the report as a whole. I would like to thank Marco for making this ASC to happen, teaching and providing me sources for learning about Algorithmic Game Theory and guiding me through the analysis of the chosen sub-classes of games.

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