Local Gauge Anomalies and The Standard Model

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Abstract

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Contents

1 Introduction

The following report is aimed at undergraduate students familiar with lagrangian mechanics, hamiltonian mechanics, quantum mechanics, and topology. For a guide to the expected prerequisite knowledge see [1–3].

[The aim of this report is to offer a gentle introduction to quantum field theory, and offer insight into one example of where topology shows up in theoretical physics.] - CHANGE THIS

Explain structure of report here.

2 Quantum States

One should be familiar with the notion of a quantum wavefunction, such as $\psi(x)$, which describes the position of a particle. However, in quantum field theory we deal with a more abstract concept called a quantum state.

Definition 2.1. Define Hilbert space

Notation 2.2. We use the Bra-Ket notation for vectors in Hilbert space. A vector is written as $|\psi\rangle$ and a covector is written as $\langle\phi|$. We define

$$\langle\phi|\psi\rangle := \langle\phi|(|\psi\rangle)$$

Given a vector $|\psi\rangle$ the corresponding hermition transpose of this vector is $\langle\psi|$, meaning $|\psi|^2 = \langle\psi|\psi\rangle$.

A quantum state is a mathematical entity that embodies all the known information of some given quantum system. There are two types of quantum states, pure and mixed. We will only explain pure quantum states in this section.

A (pure) quantum state is an abstract vector in a complex Hilbert space, denoted by $|\psi\rangle$. This Hilbert space is called our state space and often has infinite dimensions.

Observables of our quantum system correspond to Hermition operators on our state space. The eigenvalues of such an operator are real and correspond to possible observed values. It can be shown that the eigenstates of any Hermition operator form a basis for our state space. This allows us to relate quantum states to the more familiar notion of a quantum wavefunction.

Let $\langle\psi|$ be a quantum state and $\hat{X}$ be the position operator with eigenstates $|x\rangle$. Then $\psi(x) = \langle x|\psi\rangle$. WHY???

3 The Heisenburg Picture of Quantum Mechanics

One is usually introduced to quantum mechanics in the Schrödinger picture, however, in quantum field theory, it is more natural to consider the Heisenburg picture.

In the Schrödinger picture, the quantum states vary with time. For example, our quantum state may be represented by a wave function $\psi(x,t)$, which evolves in time. Conversely, the operators that act on our state space, such as momentum, position, etc, are usually fixed with respect to time. The only exception to this is that the Hamiltonian may include a potential energy term that varies with time.

Consider a time dependent quantum state $|\psi(t)\rangle$. This state evolves in time according the Schrodinger equation. One can represent this via a unitary time-evolution operator $U(t,t_0)$ as follows;

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle.$$

In the Schrödinger picture, the expectation of a potentially time-dependent hermition operator, $A(t)$, in the state $|\psi(t)\rangle$, can be written as

$$\langle\hat{A}(t)\rangle = \langle\psi(t)|\hat{A}(t)|\psi(t)\rangle.$$

Choosing some reference time t_0 we can rewrite this as

$$\langle\psi(t_0)|\hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)|\psi(t_0)\rangle.$$

If we define

$$\hat{A_H}(t) = \hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)$$

then we have

$$\langle\hat{A}(t)\rangle = \langle\psi(t_0)|\hat{A_H}(t)|\psi(t_0)\rangle.$$

So we notice that we could have obtained this same expectation by considering a constant quantum state $|\psi(t_0)\rangle$ and a new operator that evolves with time. This is the idea behind the Heisenberg picture of quantum mechanics. One interprets the quantum states as being fixed in time, and it is the operators that evolve in time.

In the Schrödinger picture, the Schrödinger equation governs the time evolution of quantum states. Hence, a natural question is, in the Heisenberg picture, how do the operators evolve in time?

One can derive the following equation

$$\frac{d}{dt}\hat{A}_H(t) = \frac{1}{i\hbar}[\hat{A}_H(t),\hat{H}_H(t)] + \left[\frac{d}{dt}\hat{A}_S(t)\right]_H$$

where the subscripts S and H denote whether we are considering the element in the Schrodinger or Heisenberg picture respectively.

4 Classical field theory & Noether's theorem

4.1 Classical field theory

In classical mechanics we consider a countable set of particles each with finitely many degrees of freedom and generalised coordinates $q_i(t)$. These generalised coordinates $\{q_i(t)\}_i$ specify the system's configuration (position in configuration space) and together with the generalised conjugate momenta $\left\{p_i(t) := \frac{\partial L}{\partial \dot{q}_i}\right\}_i$ define the system's position in phase space.

In classical field theory we generalise this notion of configuration space to a continuum with infinite degrees of freedom. The scalar field $\phi(t)$ can be seen as the generalised coordinates of a continuum $\left(q_i(t) \xrightarrow{i \to \vec{x}} \phi(\vec{x}, t)\right)$ and for a system of continua the set of scalar fields $\{\phi_k(\vec{x}, t)\}_k$ specifies the system's configuration.

Subsequently, we generalise the classical Lagrangian $L(q_i(t), \dot{q}_i(t), t)$ via

$$L(t) = \int d^3\vec{x}\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian density. With respect to some path in configuration space, $\vec{\Phi}(\vec{x}, t)$, the classical action becomes

$$S\left[\vec{\Phi}(\vec{x}, t)\right] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.2}$$

Using Hamilton's principle $\left(\frac{\delta S}{\delta \vec{\Phi}} = 0\right)$ we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right] \tag{4.3}$$

In classical mechanics we define the hamiltonian, $H(p_i(t), q_i(t), t)$, via the Legendre transform (IS THIS ACTUALLY THE DEFINITION OF H? ASK INAKI)

$$H = \sum_i p_i(t)\dot{q}_i(t) - L \tag{4.4}$$

to generalise this to field theory we introduce the momentum field conjugate to $\phi_k(\vec{x}, t)$

$$\pi_k(\vec{x}, t) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} \tag{4.5}$$

$$p_i(t) := \frac{\partial L}{\partial \dot{q}_i} \xrightarrow{i \to \vec{x}} \frac{\partial}{\partial \dot{\phi}_k}\left[\int d^3\vec{x}\, \mathcal{L}\right] = \int d^3\vec{x}\, \pi_k(\vec{x}, t) \tag{4.6}$$

Thus the classical Hamiltonian is generalised to

$$H(t) = \int d^3\vec{x}\, \mathcal{H}(\pi_k(\vec{x}, t), \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.7}$$

$$\mathcal{H} = \sum_k \pi_k(\vec{x},t)\dot{\phi}_k(\vec{x},t) - \mathcal{L} \tag{4.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and Hamilton's equations become the Hamiltonian field equations

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial \pi_k} - \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \pi_k)}\right],\ \dot{\pi}_k = \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \phi_k)}\right] - \frac{\partial \mathcal{H}}{\partial \phi_k} \tag{4.9}$$

[I tried not to use this notation but the rest will be unreadable if I don't, so I hope this does not contradict following variational deriv notation, also use this notation earlier]

$$\frac{\delta \mathcal{F}}{\delta g} := \frac{\partial \mathcal{F}}{\partial g} - \partial_\mu \left[\frac{\partial \mathcal{F}}{\partial(\partial_\mu g)}\right], \tag{4.10}$$

To begin second quantisation we will also require the notion of a field theoretic poisson bracket. Given two functionals of the dynamical fields F and G given by

$$F = \int d^3\vec{x}\,\mathcal{F}(\pi_k, \phi_k, \vec{x}, t),\ G = \int d^3\vec{x}\,\mathcal{G}(\pi_k, \phi_k, \vec{x}, t) \tag{4.11}$$

We define the poisson bracket

$$\{F, G\}_{\phi,\pi} = \int d^3\vec{x} \sum_k \left[\frac{\delta \mathcal{F}}{\delta \phi_k}\frac{\delta \mathcal{G}}{\delta \pi_k} - \frac{\delta \mathcal{G}}{\delta \phi_k}\frac{\delta \mathcal{F}}{\delta \pi_k}\right] \tag{4.12}$$

Note that $K(x) = \int dy\,[K(y)\delta(x-y)]$

4.2 Noether's Theorem

[From here we should somehow state that $\vec{x}$ is a spatial vector and x is a four-vector]

In classical mechanics Noether's theorem shows the correspondence between global symmetries of the lagrangian and conserved quantities called Noether charges. This generalises to global symmetries of the lagrangian density corresponding to conserved Noether currents in classical field theory. Consider an infinitesimal field transformation, $\vartheta(\epsilon)$, such that

$$\phi_k \mapsto \phi_k + \epsilon\vartheta_k(\phi_k) \tag{4.13}$$

where each ϑ_k may be a function of an arbitrary number of fields ϕ_k but is independent of spacetime, and we suppress high order terms in ϵ. Such a transformation is called a global symmetry if its effect on the Lagrangian density is

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\partial_\mu\Lambda^\mu(\phi_k, x) \tag{4.14}$$

This change in the Lagrangian density leaves the Euler-Lagrange equations invariant (add proof L8r). Noether's theorem for fields states that a transformation $\vartheta_k(\epsilon)$ is a symmetry of the lagrangian if and only if

$$j^\mu := \sum_k \vartheta_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} - \Lambda^\mu,\ \partial_\mu j^\mu = 0 \tag{4.15}$$

and the divergence free quantity j^μ is called the conserved Noether current (add proof L8r).

4.3 Second Quantisation

In classical mechanics, the canonical transformations are those which preserve the symplectic structure; poisson brackets of the dynamical variables (p_i, q_i) are invariant under such transformations. A system's position in phase space specifies its classical state.

However, in quantum mechanics all properties of a system are included in a quantum state, $|\psi\rangle$, inside of a Hilbert space upon which operators corresponding to observables act. Dirac's famous canonical quantisation rule ($\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$) allows us quantise the canonical structure of classical mechanics (although Groenewold's theorem shows us that such a rule cannot hold for all functions of the dynamical variables). To obtain a quantised field theory we consider the field theoretic poisson brackets

$$\{\phi_n(z), \phi_m(w)\} = 0 = \{\pi_n(z), \pi_m(w)\} \tag{4.16}$$

$$\{\phi_n(z), \pi_m(w)\} = \delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{4.17}$$

Which are canonically quantised to the commutator and anti-commutator relations for bosonic and fermionic fields operators respectively (consequence of spin-statistics) acting on a Fock space

$$[\hat{\phi}_n(z), \hat{\phi}_m(w)]_\pm = 0 = [\hat{\pi}_n(z), \hat{\pi}_m(w)]_\pm \tag{4.18}$$

$$[\hat{\phi}_n(z), \hat{\pi}_m(w)]_\pm = i\hbar\delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{4.19}$$

5 QED Lagrangian

Introduce dirac eqn as relativistic QM, show U(1) global symm. and that gauging this symm leads to QED lagrangian. Show we now wish to gauge the axial symm. because of the prev. success but at the quantum level there is an anomaly acting as an obstruction to gauging. From here we use natural units.

In quantum mechanics the Schrodinger equation is not Lorentz invariant and is thus incompatible with special relativity. A naive generalisation of the relativistic dispersion relation using the quantum mechanical Hamiltonian and momentum operators for free particles leads to the Klein-Gordon equation. However, the Klein-Gordon is a second order partial differential equation and subsequently does not uniquely determine time-evolution of the wavefunction. A more complicated treatment produces the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{5.1}$$

Which corresponds to the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.2}$$

Where the adjoint spinor is defined as $\bar{\psi} \coloneqq \psi^\dagger \gamma^0$. This Lagrangian admits a global U(1) symmetry

$$\psi \mapsto e^{i\vartheta}\psi, \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\vartheta} \tag{5.3}$$

This symmetry corresponds to a conserved current via Noether's theorem for fields [add proof L8R]

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{5.4}$$

We now attempt to 'gauge' this symmetry by adding spacetime dependence to the parameter

$$\psi \mapsto e^{i\vartheta(x)}\psi, \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\vartheta(x)} \tag{5.5}$$

However, the Lagrangian is not invariant under such a transformation so we introduce the derivative operator which transforms covariantly

$$D_\mu \coloneqq \partial_\mu + ieA_\mu \tag{5.6}$$

$$D_\mu\psi \mapsto D'_\mu \left[e^{i\vartheta(x)}\psi\right] = e^{i\vartheta(x)} D_\mu\psi \tag{5.7}$$

This implies the gauge field transforms as

$$A_\mu \mapsto A_\mu - \frac{1}{e}\partial_\mu \vartheta(x) \tag{5.8}$$

***NOT SURE IF THIS WILL BE NEEDED

The fields in the Lagrangian density are relativistic, meaning under Lorentz tranformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

they transform as

$$\phi'_i(x') = R^j_l \phi_i(x)$$

**what is i? Does it matter?

We use the Minkowski metric

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$$

in these notes.

***DOWN TO HERE

Define a symmetry

State Noether's thm

6 Path Integral & Generating Functional

Given a particle position x at time t the probability amplitude of a measurement at time t' observing a position x' is

$$\begin{aligned} M(x',t';x,t) &= {}_H\langle x',t'|x,t\rangle_H \\ &= \langle x'|\exp\left[\frac{-i}{\hbar}\hat{H}(t-t')\right]|x\rangle \end{aligned}$$

now we note that if the states are normalised

$$\int dy|y,t\rangle\langle y,t| = \mathbb{I} \tag{6.1}$$

we can write

$$M(x',t';x,t) = \sum_n \psi_n(x')\psi_n^*(x)\exp\left[\frac{-i}{\hbar}\hat{H}(t-t')\right] \tag{6.2}$$

which acts as our propagator

$$\psi(x',t') = \int dx\ M(x',t';x,t)\psi(x,t)\partial_\mu^x \tag{6.3}$$

7 Scalar Fields & Green's Functions

8 Wess-Zumino Consistency & Ward Identity

9 Stora-Zumino Descent

10 Dai-Freed & Index Theorems

References

[1] Iñaki Etxebarria. Lagrangian and hamiltonian mechanics, june 2025.

[2] Casper Peeters. Introduction to quantum mechanics, Janurary 2025.

[3] Sophie Darwin. Topology ii, Janurary 2025.