Local Gauge Anomalies and The Standard Model

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Abstract

Abstract here

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1 Introduction

The following report is aimed at undergraduate students familiar with lagrangian mechanics, hamiltonian mechanics, quantum mechanics, and topology. For a guide to the expected prerequisite knowledge see [1–3].

[The aim of this report is to offer a gentle introduction to quantum field theory, and offer insight into one example of where topology shows up in theoretical physics.] - CHANGE THIS

Explain structure of report here.

2 Quantum States

One should be familiar with the notion of a quantum wavefunction, such as $\psi(x)$, which describes the position of a particle. However, in quantum field theory we deal with a more abstract concept called a quantum state.

Definition 2.1. Define Hilbert space

Notation 2.2. We use the Bra-Ket notation for vectors in Hilbert space. A vector is written as $|\psi\rangle$ and a covector is written as $\langle\phi|$. We define

$$\langle\phi|\psi\rangle := \langle\phi|(|\psi\rangle)$$

Given a vector $|\psi\rangle$ the corresponding hermition transpose of this vector is $\langle\psi|$, meaning $|\psi|^2 = \langle\psi|\psi\rangle$.

A quantum state is a mathematical entity that embodies all the known information of some given quantum system. There are two types of quantum states, pure and mixed. We will only explain pure quantum states in this section.

A (pure) quantum state is an abstract vector in a complex Hilbert space, denoted by $|\psi\rangle$. This Hilbert space is called our state space and often has infinite dimensions.

Observables of our quantum system correspond to Hermition operators on our state space. The eigenvalues of such an operator are real and correspond to possible observed values. It can be shown that the eigenstates of any Hermition operator form a basis for our state space. This allows us to relate quantum states to the more familiar notion of a quantum wavefunction.

Let $\langle\psi|$ be a quantum state and $\hat{X}$ be the position operator with eigenstates $|x\rangle$. Then $\psi(x) = \langle x|\psi\rangle$. WHY???

3 The Heisenburg Picture of Quantum Mechanics

One is usually introduced to quantum mechanics in the Schrödinger picture, however, in quantum field theory, it is more natural to consider the Heisenburg picture.

In the Schrödinger picture, the quantum states vary with time. For example, our quantum state may be represented by a wave function $\psi(x,t)$, which evolves in time. Conversely, the operators that act on our state space, such as momentum, position, etc, are usually fixed with respect to time. The only exception to this is that the Hamiltonian may include a potential energy term that varies with time.

Consider a time dependent quantum state $|\psi(t)\rangle$. This state evolves in time according the Schrodinger equation. One can represent this via a unitary time-evolution operator $U(t,t_0)$ as follows;

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle.$$

In the Schrödinger picture, the expectation of a potentially time-dependent hermition operator, $A(t)$, in the state $|\psi(t)\rangle$, can be written as

$$\langle \hat{A}(t)\rangle = \langle \psi(t)|\hat{A}(t)|\psi(t)\rangle.$$

Choosing some reference time t_0 we can rewrite this as

$$\langle \psi(t_0)|\hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)|\psi(t_0)\rangle.$$

If we define

$$\hat{A_H}(t) = \hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)$$

then we have

$$\langle \hat{A}(t)\rangle = \langle \psi(t_0)|\hat{A_H}(t)|\psi(t_0)\rangle.$$

So we notice that we could have obtained this same expectation by considering a constant quantum state $|\psi(t_0)\rangle$ and a new operator that evolves with time. This is the idea behind the Heisenberg picture of quantum mechanics. One interprets the quantum states as being fixed in time, and it is the operators that evolve in time.

In the Schrödinger picture, the Schrödinger equation governs the time evolution of quantum states. Hence, a natural question is, in the Heisenberg picture, how do the operators evolve in time?

One can derive the following equation

$$\frac{d}{dt}\hat{A}_H(t) = \frac{1}{i\hbar}[\hat{A}_H(t),\hat{H}_H(t)] + \left[\frac{d}{dt}\hat{A}_S(t)\right]_H$$

where the subscripts S and H denote whether we are considering the element in the Schrodinger or Heisenberg picture respectively.

4 Classical field theory & Noether's theorem

4.1 Classical field theory

In classical mechanics we consider a countable set of particles each with finitely many degrees of freedom and generalised coordinates $q_i(t)$. These generalised coordinates $\{q_i(t)\}_i$ specify the system's configuration (position in configuration space) and together with the generalised conjugate momenta $\left\{p_i(t) := \frac{\partial L}{\partial \dot{q}_i}\right\}_i$ define the system's position in phase space.

In classical field theory we generalise this notion of configuration space to a continuum with infinite degrees of freedom. The scalar field $\phi(t)$ can be seen as the generalised coordinates of a continuum $\left(q_i(t) \xrightarrow{i \to \vec{x}} \phi(\vec{x}, t)\right)$ and for a system of continua the set of scalar fields $\{\phi_k(\vec{x}, t)\}_k$ specifies the system's configuration.

Subsequently, we generalise the classical Lagrangian $L(q_i(t), \dot{q}_i(t), t)$ via

$$L(t) = \int d^3\vec{x}\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian density. With respect to some path in configuration space, $\vec{\Phi}(\vec{x}, t)$, the classical action becomes

$$S\left[\vec{\Phi}(\vec{x}, t)\right] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.2}$$

Using Hamilton's principle $\left(\frac{\delta S}{\delta \vec{\Phi}} = 0\right)$ we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right] \tag{4.3}$$

In classical mechanics we define the hamiltonian, $H(p_i(t), q_i(t), t)$, via the Legendre transform (IS THIS ACTUALLY THE DEFINITION OF H? ASK INAKI)

$$H = \sum_i p_i(t)\dot{q}_i(t) - L \tag{4.4}$$

to generalise this to field theory we introduce the momentum field conjugate to $\phi_k(\vec{x}, t)$

$$\pi_k(\vec{x}, t) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} \tag{4.5}$$

$$p_i(t) := \frac{\partial L}{\partial \dot{q}_i} \xrightarrow{i \to \vec{x}} \frac{\partial}{\partial \dot{\phi}_k}\left[\int d^3\vec{x}\, \mathcal{L}\right] = \int d^3\vec{x}\, \pi_k(\vec{x}, t) \tag{4.6}$$

Thus the classical Hamiltonian is generalised to

$$H(t) = \int d^3\vec{x}\, \mathcal{H}(\pi_k(\vec{x}, t), \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.7}$$

$$\mathcal{H} = \sum_k \pi_k(\vec{x}, t)\dot{\phi}_k(\vec{x}, t) - \mathcal{L} \tag{4.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and Hamilton's equations become the Hamiltonian field equations

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial \pi_k} - \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \pi_k)}\right],\ \dot{\pi}_k = \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \phi_k)}\right] - \frac{\partial \mathcal{H}}{\partial \phi_k} \tag{4.9}$$

[I tried not to use this notation but the rest will be unreadable if I don't, so I hope this does not contradict following variational deriv notation, also use this notation earlier]

$$\frac{\delta \mathcal{F}}{\delta g} := \frac{\partial \mathcal{F}}{\partial g} - \partial_\mu \left[\frac{\partial \mathcal{F}}{\partial(\partial_\mu g)}\right], \tag{4.10}$$

To begin second quantisation we will also require the notion of a field theoretic poisson bracket. Given two functionals, F and G, of the dynamical fields given by

$$F = \int d^3\vec{x}\, \mathcal{F}(\pi_k, \phi_k, \vec{x}, t),\ G = \int d^3\vec{x}\, \mathcal{G}(\pi_k, \phi_k, \vec{x}, t) \tag{4.11}$$

We define the poisson bracket

$$\{F, G\}_f = \int d^3\vec{x} \sum_k \left[\frac{\delta \mathcal{F}}{\delta \phi_k}\frac{\delta \mathcal{G}}{\delta \pi_k} - \frac{\delta \mathcal{G}}{\delta \phi_k}\frac{\delta \mathcal{F}}{\delta \pi_k}\right] \tag{4.12}$$

Note that $K(x) = \int dy\, [K(y)\delta(x-y)]$

4.2 Noether's Theorem

[From here we should somehow state that $\vec{x}$ is a spatial vector and x is a four-vector]

In classical mechanics Noether's theorem shows the correspondence between global symmetries of the lagrangian and conserved quantities called Noether charges. This generalises to global symmetries of the lagrangian density corresponding to conserved Noether currents in classical field theory. Consider an infinitesimal field transformation, $\vartheta(\epsilon)$, such that

$$\phi_k \mapsto \phi_k + \epsilon\vartheta_k(\phi_k) \tag{4.13}$$

where each ϑ_k may be a function of an arbitrary number of fields ϕ_k but is independent of spacetime, and we suppress high order terms in ϵ. Such a transformation is called a global symmetry if its effect on the Lagrangian density is

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\partial_\mu \Lambda^\mu(\phi_k, x) \tag{4.14}$$

This change in the Lagrangian density leaves the Euler-Lagrange equations invariant (add proof L8r). Noether's theorem for fields states that a transformation $\vartheta_k(\epsilon)$ is a symmetry of the lagrangian if and only if

$$j^\mu := \sum_k \vartheta_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} - \Lambda^\mu,\ \partial_\mu j^\mu = 0 \tag{4.15}$$

and the divergence free quantity j^μ is called the conserved Noether current (add proof L8r).

5 The Dirac Lagrangian

[Introduce dirac eqn as relativistic QM, show U(1) global symm. and that gauging this symm leads to QED lagrangian. Show we now wish to gauge the axial symm. because of the prev. success but at the quantum level there is an anomaly acting as an obstruction to gauging. From here we use natural units.]

5.1 Gauging the vector symmetry

In quantum mechanics the Schrodinger equation is not Lorentz invariant and is thus incompatible with special relativity. A naive generalisation of the relativistic dispersion relation using Hamiltonian and momentum operators for free particles leads to the Klein-Gordon equation. However, the Klein-Gordon equation is a second order partial differential equation and subsequently does not uniquely determine time-evolution of the wavefunction. A more complicated treatment produces the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{5.1}$$

Which corresponds to the Lagrangian density

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{5.2}$$

Where the adjoint bispinor is defined as $\bar{\psi} := \psi^\dagger\gamma^0$. This Lagrangian admits a global U(1) symmetry called the vector symmetry

$$\psi \mapsto e^{i\vartheta}\psi, \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\vartheta} \tag{5.3}$$

This symmetry corresponds to a conserved vector current via Noether's theorem for fields [add proof L8R]

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{5.4}$$

We now attempt to 'gauge' this symmetry by adding spacetime dependence to the parameter

$$\psi \mapsto e^{i\vartheta(x)}\psi, \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\vartheta(x)} \tag{5.5}$$

However, the Lagrangian is not invariant under such a transformation so we introduce the derivative operator which transforms covariantly

$$D_\mu := \partial_\mu + ie\Pi_\mu \tag{5.6}$$

$$D_\mu\psi \mapsto D'_\mu\left[e^{i\vartheta(x)}\psi\right] = e^{i\vartheta(x)}D_\mu\psi \tag{5.7}$$

To satisfy this the gauge field must transform as

$$\Pi_\mu \mapsto \Pi_\mu - \frac{1}{e}\partial_\mu \vartheta(x) \tag{5.8}$$

Thus we have a modified Lagrangian which admits a local U(1) gauge symmetry

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\Pi_\mu j^\mu \tag{5.9}$$

This is strikingly similar to the electromagnetic Lagrangian density which also has a U(1) gauge symmetry

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \tag{5.10}$$

where A_μ, J^μ, and $F^{\mu\nu}$ are the electromagnetic four-potential, four-current, and field tensor respectively. Thus we identify $\Pi_\mu = A_\mu$ and $ej^\mu = J^\mu$ and arrive at the Lagrangian for quantum electrodynamics.

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu j^\mu + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.11}$$

[Speak about how QED is an incredibly successful theory of photon-electron interactions]

5.2 The Axial symmetry

However, this is not the end of the story. The massless Dirac Lagrangian admits another symmetry better seen in the Weyl basis.

$$\psi = \begin{pmatrix} \psi_{\text{L}} \\ \psi_{\text{R}} \end{pmatrix}, \quad \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix} \tag{5.12}$$

Thus the Dirac Lagrangian can be seen as the interaction between two Weyl fermions of opposite chirality

$$\mathcal{L}_{\text{D}} = \psi_{\text{L}}^\dagger (i\sigma^\mu \partial_\mu)\psi_{\text{L}} + \psi_{\text{R}}^\dagger (i\bar{\sigma}^\mu \partial_\mu)\psi_{\text{R}} - m(\psi_{\text{L}}^\dagger \psi_{\text{R}} + \psi_{\text{R}}^\dagger \psi_{\text{L}}) \tag{5.13}$$

In the massless case the Lagrangian gains a global U(1) symmetry known as the axial symmetry

$$\psi_{\text{L}} \mapsto e^{i\vartheta}\psi_{\text{L}}, \quad \psi_{\text{R}} \mapsto e^{-i\vartheta}\psi_{\text{R}} \tag{5.14}$$

which is equivalent to

$$\psi \mapsto e^{i\vartheta\gamma^5}\psi \tag{5.15}$$

and a Noether current

$$j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j^\mu = 2im\bar{\psi}\gamma^5\psi \tag{5.16}$$

We now proceed as before, gauging the axial symmetry by promoting ∂_μ to a covariant derivative as in (5.6) and determining the required gauge transformation

$$\mathcal{L}_D|_{m=0} = \bar{\psi}(i\gamma^\mu \partial_\mu)\psi - q\bar{\psi}\gamma^\mu \Pi_\mu \psi \tag{5.17}$$

$$\Pi_\mu \mapsto e^{i\vartheta\gamma^5}\Pi_\mu e^{-i\vartheta\gamma^5} - \frac{1}{q}\partial_\mu\vartheta\gamma^5 \tag{5.18}$$

Now we must add a kinetic term to the Lagrangian to determine the dynamics of the matrix-valued gauge field. Such a term must be gauge-invariant to preserve the U(1) symmetry. We find a Lagrangian analogous to the Maxwell Lagrangian

$$\mathcal{L}_\Pi = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) \tag{5.19}$$

with field strength

$$\Gamma_{\mu\nu} = \partial_\mu\Pi_\nu - \partial_\nu\Pi_\mu + q[\Pi_\mu, \Pi_\nu] \tag{5.20}$$

has the required gauge-invariance. Thus we have a theory describing the coupling of massless dirac fermions to a mysterious gauge field

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) - q\bar{\psi}\gamma^\mu\Pi_\mu\psi + \bar{\psi}(i\gamma^\mu\partial_\mu)\psi \tag{5.21}$$

6 Quantum Field Theory

So far we have studied relativistic quantum mechanics, making use of classical fields to describe the dynamics of quantum states. However, such a quantum theory is inconsistent subsequently predicting negative energy states which require peculiar interpretations to reconcile (Dirac's hole theory). To proceed to a quantum theory of particle interactions we must promote the dynamical fields to field operators; second quantisation. We shall see how this shift to a quantised field theory affects our calculation of expectation values and probabilities.

6.1 Second Quantisation

In classical mechanics, the canonical transformations are exactly those which preserve the symplectic structure; invariance of poisson brackets of the dynamical variables (p_i, q_i) characterises transformations which leave Hamilton's equations unchanged. A system's position in phase space specifies its classical state.

However, in quantum mechanics all properties of a system are included in a quantum state, $|\psi\rangle$, inside of a Hilbert space upon which operators corresponding to observables act. Dirac's famous canonical quantisation rule ($\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$) allows us quantise the canonical structure of classical mechanics (although Groenewold's theorem shows us that such a rule cannot hold for all functions of the dynamical variables). To obtain a quantised field theory we consider the field theoretic poisson brackets

$$\{\phi_n(z), \phi_m(w)\}_f = 0 = \{\pi_n(z), \pi_m(w)\}_f \tag{6.1}$$

$$\{\phi_n(z), \pi_m(w)\}_f = \delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{6.2}$$

Which are quantised to the canonical commutation relations for the dynamical field operators acting on a Fock space

$$[\hat{\phi}_n(z), \hat{\phi}_m(w)] = 0 = [\hat{\pi}_n(z), \hat{\pi}_m(w)] \tag{6.3}$$

$$[\hat{\phi}_n(z), \hat{\pi}_m(w)] = i\hbar\delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{6.4}$$

However, just like Dirac's rule this procedure does not always produce a consistent quantum theory and in the case of quantising the Dirac Lagrangian we require the analogous anti-commutation relations because of the spin-statistics theorem. In principle, commutation relations are axioms of any quantum theory chosen to reproduce experimental observations.

6.2 Fock Space

After second quantisation in general we obtain an expression for the Hamiltonian operator in terms of various creation and annihilation operators (e.g. $a^{r\dagger}_{\vec{p}_1}, b^{s\dagger}_{\vec{p}_2}, \ldots$ and $a^r_{\vec{p}_1}, b^s_{\vec{p}_2}, \ldots$ respectively). Subsequently we define the vacuum state to be annihilated by all particle annihilation operators

$$a^r_{\vec{p}_1}|0\rangle = b^r_{\vec{p}_1}|0\rangle = \cdots = 0 \tag{6.5}$$

Subsequently an N particle state is

$$a^{r_1\dagger}_{\vec{p}_1} a^{r_2\dagger}_{\vec{p}_2} \cdots a^{r_N\dagger}_{\vec{p}_N}|0\rangle = |r_1, p_1; r_2, p_2; \ldots; r_N, p_N\rangle$$

(6.6)

***NOT SURE IF THIS WILL BE NEEDED

The fields in the Lagrangian density are relativistic, meaning under Lorentz tranformations

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

they transform as

$$\phi'_i(x') = R^j_l \phi_i(x)$$

**what is i? Does it matter?

We use the Minkowski metric

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$$

in these notes.

***DOWN TO HERE

Define a symmetry

State Noether's thm

7 Path Integral & Generating Functional

Given a particle position x at time t the probability amplitude of a measurement at time t' observing a position x' is

$$\begin{aligned} M(x', t'; x, t) &= {}_H\langle x', t' | x, t \rangle_H \\ &= \langle x' | \exp\left[\frac{-i}{\hbar} \hat{H}(t - t')\right] | x \rangle \end{aligned}$$

now we note that if the states are normalised

$$\int dy |y, t\rangle\langle y, t| = \mathbb{I} \tag{7.1}$$

we can write

$$M(x', t'; x, t) = \sum_n \psi_n(x') \psi^*_n(x) \exp\left[\frac{-i}{\hbar} \hat{H}(t - t')\right] \tag{7.2}$$

which acts as our propagator

$$\psi(x', t') = \int dx \; M(x', t'; x, t) \psi(x, t) \partial^x_\mu \tag{7.3}$$

8 Time-ordered products & ward identities

9 Anomalies in QFT & violation of Ward identities

References

[1] Iñaki Etxebarria. Lagrangian and hamiltonian mechanics, june 2025.

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[3] Sophie Darwin. Topology ii, Janurary 2025.