Local Gauge Anomalies and The Standard Model

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Abstract

Abstract here

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1 Introduction

The following report is aimed at undergraduate students familiar with lagrangian mechanics, hamiltonian mechanics, quantum mechanics, and topology. For a guide to the expected prerequisite knowledge see [1–3].

[The aim of this report is to offer a gentle introduction to quantum field theory, and offer insight into one example of where topology shows up in theoretical physics.] - CHANGE THIS

Explain structure of report here.

2 Quantum States

One should be familiar with the notion of a quantum wavefunction, such as $\psi(x)$, which describes the position of a particle. However, in quantum field theory we deal with a more abstract concept called a quantum state.

Definition 2.1. Define Hilbert space

Notation 2.2. We use the Bra-Ket notation for vectors in Hilbert space. A vector is written as $|\psi\rangle$ and a covector is written as $\langle\phi|$. We define

$$\langle\phi|\psi\rangle := \langle\phi|(|\psi\rangle)$$

Given a vector $|\psi\rangle$ the corresponding hermition transpose of this vector is $\langle\psi|$, meaning $|\psi|^2 = \langle\psi|\psi\rangle$.

A quantum state is a mathematical entity that embodies all the known information of some given quantum system. There are two types of quantum states, pure and mixed. We will only explain pure quantum states in this section.

A (pure) quantum state is an abstract vector in a complex Hilbert space, denoted by $|\psi\rangle$. This Hilbert space is called our state space and often has infinite dimensions.

Observables of our quantum system correspond to Hermition operators on our state space. The eigenvalues of such an operator are real and correspond to possible observed values. It can be shown that the eigenstates of any Hermition operator form a basis for our state space. This allows us to relate quantum states to the more familiar notion of a quantum wavefunction.

Let $\langle\psi|$ be a quantum state and $\hat{X}$ be the position operator with eigenstates $|x\rangle$. Then $\psi(x) = \langle x|\psi\rangle$. WHY???

3 The Heisenburg Picture of Quantum Mechanics

One is usually introduced to quantum mechanics in the Schrödinger picture, however, in quantum field theory, it is more natural to consider the Heisenburg picture.

In the Schrödinger picture, the quantum states vary with time. For example, our quantum state may be represented by a wave function $\psi(x,t)$, which evolves in time. Conversely, the operators that act on our state space, such as momentum, position, etc, are usually fixed with respect to time. The only exception to this is that the Hamiltonian may include a potential energy term that varies with time.

Consider a time dependent quantum state $|\psi(t)\rangle$. This state evolves in time according the Schrodinger equation. One can represent this via a unitary time-evolution operator $U(t,t_0)$ as follows;

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle.$$

In the Schrödinger picture, the expectation of a potentially time-dependent hermition operator, $A(t)$, in the state $|\psi(t)\rangle$, can be written as

$$\langle \hat{A}(t)\rangle = \langle \psi(t)|\hat{A}(t)|\psi(t)\rangle.$$

Choosing some reference time t_0 we can rewrite this as

$$\langle \psi(t_0)|\hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)|\psi(t_0)\rangle.$$

If we define

$$\hat{A_H}(t) = \hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)$$

then we have

$$\langle \hat{A}(t)\rangle = \langle \psi(t_0)|\hat{A_H}(t)|\psi(t_0)\rangle.$$

So we notice that we could have obtained this same expectation by considering a constant quantum state $|\psi(t_0)\rangle$ and a new operator that evolves with time. This is the idea behind the Heisenberg picture of quantum mechanics. One interprets the quantum states as being fixed in time, and it is the operators that evolve in time.

In the Schrödinger picture, the Schrödinger equation governs the time evolution of quantum states. Hence, a natural question is, in the Heisenberg picture, how do the operators evolve in time?

One can derive the following equation

$$\frac{d}{dt}\hat{A}_H(t) = \frac{1}{i\hbar}[\hat{A}_H(t),\hat{H}_H(t)] + \left[\frac{d}{dt}\hat{A}_S(t)\right]_H$$

where the subscripts S and H denote whether we are considering the element in the Schrodinger or Heisenberg picture respectively.

4 Classical field theory & Noether's theorem

4.1 Classical field theory

In classical mechanics we consider a countable set of particles each with finitely many degrees of freedom and generalised coordinates $q_i(t)$. These generalised coordinates $\{q_i(t)\}_i$ specify the system's configuration (position in configuration space) and together with the generalised conjugate momenta $\left\{p_i(t) := \frac{\partial L}{\partial \dot{q}_i}\right\}_i$ define the system's position in phase space.

In classical field theory we generalise this notion of configuration space to a continuum with infinite degrees of freedom. The scalar field $\phi(t)$ can be seen as the generalised coordinates of a continuum $\left(q_i(t) \xrightarrow{i \to \vec{x}} \phi(\vec{x}, t)\right)$ and for a system of continua the set of scalar fields $\{\phi_k(\vec{x}, t)\}_k$ specifies the system's configuration.

Subsequently, we generalise the classical Lagrangian $L(q_i(t), \dot{q}_i(t), t)$ via

$$L(t) = \int d^3\vec{x}\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian density. With respect to some path in configuration space, $\vec{\Phi}(\vec{x}, t)$, the classical action becomes

$$S\left[\vec{\Phi}(\vec{x}, t)\right] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.2}$$

Using Hamilton's principle $\left(\frac{\delta S}{\delta \vec{\Phi}} = 0\right)$ we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right] \tag{4.3}$$

In classical mechanics we define the hamiltonian, $H(p_i(t), q_i(t), t)$, via the Legendre transform (IS THIS ACTUALLY THE DEFINITION OF H? ASK INAKI)

$$H = \sum_i p_i(t)\dot{q}_i(t) - L \tag{4.4}$$

to generalise this to field theory we introduce the momentum field conjugate to $\phi_k(\vec{x}, t)$

$$\pi_k(\vec{x}, t) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} \tag{4.5}$$

$$p_i(t) := \frac{\partial L}{\partial \dot{q}_i} \xrightarrow{i \to \vec{x}} \frac{\partial}{\partial \dot{\phi}_k}\left[\int d^3\vec{x}\, \mathcal{L}\right] = \int d^3\vec{x}\, \pi_k(\vec{x}, t) \tag{4.6}$$

Thus the classical Hamiltonian is generalised to

$$H(t) = \int d^3\vec{x}\, \mathcal{H}(\pi_k(\vec{x}, t), \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.7}$$

$$\mathcal{H} = \sum_k \pi_k(\vec{x},t)\dot{\phi}_k(\vec{x},t) - \mathcal{L} \tag{4.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and Hamilton's equations become the Hamiltonian field equations

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial \pi_k} - \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \pi_k)}\right],\ \dot{\pi}_k = \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \phi_k)}\right] - \frac{\partial \mathcal{H}}{\partial \phi_k} \tag{4.9}$$

[I tried not to use this notation but the rest will be unreadable if I don't, so I hope this does not contradict following variational deriv notation, also use this notation earlier]

$$\frac{\delta \mathcal{F}}{\delta g} := \frac{\partial \mathcal{F}}{\partial g} - \partial_\mu \left[\frac{\partial \mathcal{F}}{\partial(\partial_\mu g)}\right], \tag{4.10}$$

To begin second quantisation we will also require the notion of a field theoretic poisson bracket. Given two functionals, F and G, of the dynamical fields given by

$$F = \int d^3\vec{x}\, \mathcal{F}(\pi_k, \phi_k, \vec{x}, t),\ G = \int d^3\vec{x}\, \mathcal{G}(\pi_k, \phi_k, \vec{x}, t) \tag{4.11}$$

We define the poisson bracket

$$\{F, G\}_f = \int d^3\vec{x} \sum_k \left[\frac{\delta \mathcal{F}}{\delta \phi_k}\frac{\delta \mathcal{G}}{\delta \pi_k} - \frac{\delta \mathcal{G}}{\delta \phi_k}\frac{\delta \mathcal{F}}{\delta \pi_k}\right] \tag{4.12}$$

Note that $K(x) = \int dy\, [K(y)\delta(x-y)]$

4.2 Noether's first Theorem

[From here we should somehow state that $\vec{x}$ is a spatial vector and x is a four-vector]

In classical mechanics Noether's first theorem shows the correspondence between global symmetries of the lagrangian and conserved quantities called Noether charges. This generalises to global symmetries of the lagrangian density corresponding to conserved Noether currents in classical field theory. Consider an infinitesimal field transformation, $\vartheta(\epsilon)$, such that

$$\phi_k \mapsto \phi_k + \epsilon \vartheta_k(\phi) \tag{4.13}$$

where the generators, ϑ_k, may be a functions of an arbitrary number of fields ϕ_k but are independent of spacetime, and we suppress high order terms in ϵ. Such a transformation is called a global symmetry if its effect on the Lagrangian density is

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon \partial_\mu \Lambda^\mu(\phi_k, x) \tag{4.14}$$

This change in the Lagrangian density leaves the Euler-Lagrange equations invariant (add proof L8r). Noether's theorem for fields states that a transformation $\vartheta(\epsilon)$ is a symmetry of the lagrangian if and only if

$$j^\mu := \sum_k \vartheta_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} - \Lambda^\mu,\ \partial_\mu j^\mu = 0 \tag{4.15}$$

and the divergence free quantity j^μ is called the conserved Noether current. This easily generalises to local symmetries: spacetime dependent generators $\vartheta_k(\phi, x)$ (add proof L8r).

5 The Dirac Lagrangian

[Introduce dirac eqn as relativistic QM, show U(1) global symm. and that gauging this symm leads to QED lagrangian. Show we now wish to gauge the axial symm. because of the prev. success but at the quantum level there is an anomaly acting as an obstruction to gauging. From here we use natural units.]

5.1 Gauging the vector symmetry

In quantum mechanics the Schrodinger equation is not Lorentz invariant and is thus incompatible with special relativity. A naive generalisation of the relativistic dispersion relation using Hamiltonian and momentum operators for free particles leads to the Klein-Gordon equation. However, the Klein-Gordon equation is a second order partial differential equation and subsequently does not uniquely determine time-evolution of the wavefunction. A more complicated treatment produces the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{5.1}$$

Which corresponds to the Lagrangian density

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.2}$$

Where the adjoint bispinor is defined as $\bar{\psi} \coloneqq \psi^\dagger \gamma^0$. This Lagrangian admits a global U(1) symmetry called the vector symmetry

$$\psi \mapsto e^{i\vartheta}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{-i\vartheta} \tag{5.3}$$

This symmetry corresponds to a conserved vector current via Noether's theorem for fields [add proof L8R]

$$j^\mu = \bar{\psi}\gamma^\mu \psi \tag{5.4}$$

We now attempt to 'gauge' this symmetry by adding spacetime dependence to the parameter

$$\psi \mapsto e^{i\vartheta(x)}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{-i\vartheta(x)} \tag{5.5}$$

However, the Lagrangian is not invariant under such a transformation so we introduce the derivative operator which transforms covariantly

$$D_\mu \coloneqq \partial_\mu + ie\Pi_\mu \tag{5.6}$$

$$D_\mu \psi \mapsto D'_\mu \left[e^{i\vartheta(x)}\psi \right] = e^{i\vartheta(x)} D_\mu \psi \tag{5.7}$$

To satisfy this the gauge field must transform as

$$\Pi_\mu \mapsto \Pi_\mu - \frac{1}{e}\partial_\mu \vartheta(x) \tag{5.8}$$

Thus we have a modified Lagrangian which admits a local U(1) gauge symmetry

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\Pi_\mu j^\mu \tag{5.9}$$

where $\not{D} := \gamma^\mu D_\mu$ is the Dirac operator. This is strikingly similar to the electromagnetic Lagrangian density which also has a U(1) gauge symmetry

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \tag{5.10}$$

where A_μ, J^μ, and $F^{\mu\nu}$ are the electromagnetic four-potential, four-current, and field tensor respectively. Thus we identify $\Pi_\mu = A_\mu$ and $ej^\mu = J^\mu$ and arrive at the Lagrangian for quantum electrodynamics.

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu j^\mu + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.11}$$

[Speak about how QED is an incredibly successful theory of photon-electron interactions]

5.2 The Axial symmetry

However, this is not the end of the story. The massless Dirac Lagrangian admits another symmetry better seen in the Weyl basis.

$$\psi = \begin{pmatrix} \psi_{\text{L}} \\ \psi_{\text{R}} \end{pmatrix}, \quad \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix} \tag{5.12}$$

Thus the Dirac Lagrangian can be seen as the interaction between two Weyl fermions of opposite chirality

$$\mathcal{L}_{\text{D}} = \psi_{\text{L}}^\dagger (i\sigma^\mu \partial_\mu)\psi_{\text{L}} + \psi_{\text{R}}^\dagger (i\bar{\sigma}^\mu \partial_\mu)\psi_{\text{R}} - m(\psi_{\text{L}}^\dagger \psi_{\text{R}} + \psi_{\text{R}}^\dagger \psi_{\text{L}}) \tag{5.13}$$

In the massless case the Lagrangian gains a global U(1) symmetry known as the axial symmetry

$$\psi_{\text{L}} \mapsto e^{i\vartheta}\psi_{\text{L}}, \quad \psi_{\text{R}} \mapsto e^{-i\vartheta}\psi_{\text{R}} \tag{5.14}$$

which is equivalent to

$$\psi \mapsto e^{i\vartheta\gamma^5}\psi \tag{5.15}$$

and a Noether current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j_5^\mu = 2im\bar{\psi}\gamma^5\psi \tag{5.16}$$

We now proceed as before, gauging the axial symmetry by promoting ∂_μ to a covariant derivative $(D_\mathrm{A})_\mu$ as in (5.6) and determining the required gauge transformation

$$\mathcal{L}_D|_{m=0} = \bar{\psi}(i\gamma^\mu\partial_\mu)\psi - q\bar{\psi}\gamma^\mu\Pi_\mu\psi \tag{5.17}$$

$$\Pi_\mu \mapsto e^{i\vartheta\gamma^5}\Pi_\mu e^{-i\vartheta\gamma^5} - \frac{1}{q}\partial_\mu\vartheta\gamma^5 \tag{5.18}$$

Now we must add a kinetic term to the Lagrangian to determine the dynamics of the matrix-valued gauge field. Such a term must be gauge-invariant to preserve the U(1) symmetry. We find a Lagrangian analogous to the Maxwell Lagrangian

$$\mathcal{L}_\Pi = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) \tag{5.19}$$

with field strength

$$\Gamma_{\mu\nu} = \partial_\mu\Pi_\nu - \partial_\nu\Pi_\mu + q[\Pi_\mu, \Pi_\nu] \tag{5.20}$$

has the required gauge-invariance and is Lorentz invariant. Thus we have constructed a classical theory describing the interaction between massless dirac fermions and an axial gauge boson

$$\mathcal{L}_\mathrm{A} = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) + i\bar{\psi}\not{D}_\mathrm{A}\psi \tag{5.21}$$

$$\mathcal{L}_\mathrm{A} = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) - q\bar{\psi}\gamma^\mu\Pi_\mu\psi + \bar{\psi}(i\gamma^\mu\partial_\mu)\psi \tag{5.22}$$

6 Quantum Field Theory

So far we have studied relativistic quantum mechanics, making use of classical fields to describe the dynamics of quantum states. However, such a quantum theory is inconsistent subsequently predicting negative energy states which require peculiar interpretations to reconcile (Dirac's hole theory). To proceed to a quantum theory of particle interactions we must promote the dynamical fields to field operators; second quantisation. We shall see how this shift to a quantised field theory affects our calculation of expectation values and probabilities.

6.1 Second Quantisation

In classical mechanics, the canonical transformations are exactly those which preserve the symplectic structure; invariance of poisson brackets of the dynamical variables (p_i, q_i) characterises transformations which leave Hamilton's equations unchanged. A system's position in phase space specifies its classical state.

However, in quantum mechanics all properties of a system are included in a quantum state, $|\psi\rangle$, inside of a Hilbert space upon which operators corresponding to observables act. Dirac's famous canonical quantisation rule ($\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$) allows us quantise the canonical structure of classical mechanics (although Groenewold's theorem shows us that such a rule cannot hold for all functions of the dynamical variables). To obtain a quantised field theory we consider the field theoretic poisson brackets

$$\{\phi_n(z), \phi_m(w)\}_f = 0 = \{\pi_n(z), \pi_m(w)\}_f \tag{6.1}$$

$$\{\phi_n(z), \pi_m(w)\}_f = \delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{6.2}$$

Which are quantised to the canonical commutation relations for the dynamical field operators acting on a Fock space

$$[\hat{\phi}_n(z), \hat{\phi}_m(w)] = 0 = [\hat{\pi}_n(z), \hat{\pi}_m(w)] \tag{6.3}$$

$$[\hat{\phi}_n(z), \hat{\pi}_m(w)] = i\hbar\delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{6.4}$$

However, just like Dirac's rule this procedure does not always produce a consistent quantum theory and in the case of quantising the Dirac Lagrangian we require the analogous anti-commutation relations because of the spin-statistics theorem. In principle, commutation relations are axioms of any quantum theory chosen to reproduce experimental observations.

6.2 Fock Space

After second quantisation in general we obtain an expression for the Hamiltonian operator in terms of various creation and annihilation operators (e.g. $a^{r\dagger}_{\vec{p}_1}, b^{s\dagger}_{\vec{p}_2}, \ldots$ and $a^{r}_{\vec{p}_1}, b^{s}_{\vec{p}_2}, \ldots$ respectively). Subsequently we define the vacuum state to be annihilated by all particle annihilation operators

$$a^r_{\vec{p}_1}|0\rangle = b^r_{\vec{p}_1}|0\rangle = \cdots = 0 \tag{6.5}$$

The indistinguishable n-particle state is given by

$$a^{r_1\dagger}_{\vec{p}_1} a^{r_2\dagger}_{\vec{p}_2} \cdots a^{r_n\dagger}_{\vec{p}_n}|0\rangle = |r_1, \vec{p}_1; r_2, \vec{p}_2; \ldots; r_n, \vec{p}_n\rangle \in H^{\otimes n} \tag{6.6}$$

$$H^{\otimes n} := \bigotimes_{k=1}^{n} H \tag{6.7}$$

Where $H^{\otimes n}$ is the n-particle Hilbert space ($H^0 = \mathbb{C}$ corresponds to the vacuum), and an arbitrary quantum state is an element of the Fock space

$$F(H) := \bigoplus_{n=0}^{\infty} H^{\otimes n} \tag{6.8}$$

Thus the probability amplitude of a state transitioning from one particle to another is

$$\langle r_2, \vec{p}_2 | r_1, \vec{p}_1\rangle = \left\langle 0 \,\middle|\, a^{r_2}_{\vec{p}_2} a^{r_1\dagger}_{\vec{p}_1} \,\middle|\, 0 \right\rangle := \left\langle a^{r_2}_{\vec{p}_2} a^{r_1\dagger}_{\vec{p}_1} \right\rangle \tag{6.9}$$

which generalises to transitions between arbitrary quantum states. Hence in quantum field theory the study of probabilities is characterised by vacuum expectation values of operators and their compositions.

7 Path Integral Formulation of Quantum Mechanics

7.1 The Propagator

In quantum mechanics, the propagator is the probability amplitude of observing a position x_n at time t_n, given the particle is at postion x_1 at time t_1. This can be written as the scalar product of the two quantum states,

$$K(x', t'; x, t) := \langle x't'|xt\rangle. \tag{7.1}$$

We can rewrite this propagator as an integral over all possible paths the particle can take. We call this the Feynman path integral.

First, we discretise the time between t_n and t as

$$t_{n+1} := t' > t_n > t_{n-1} > ... > t_2 > t_1 > t =: t_0,$$

giving $n+1$ equal peices $\Delta t = t_j - t_{j-1}$ for $1 \leq j \leq n+1$.

Since position eigenstates are complete, we have

$$\int dx_j |x_j t_j\rangle\langle x_j t_j| = \mathbb{I}, \tag{7.2}$$

meaning we can rewite (7.1) as

$$\begin{aligned} K(x', t'; x, t) = \int dx_n dx_{n-1} ... dx_1 \langle x_{n+1} t_{n+1}|x_n t_n\rangle\langle x_n t_n|x_{n-1} t_{n-1}\rangle ... \\ ... \langle x_2 t_2|x_1 t_1\rangle\langle x_1 t_1|x_0 t_0\rangle. \end{aligned} \tag{7.3}$$

This can be thought of as integrating over all possible paths, as depicted in Figure 1.

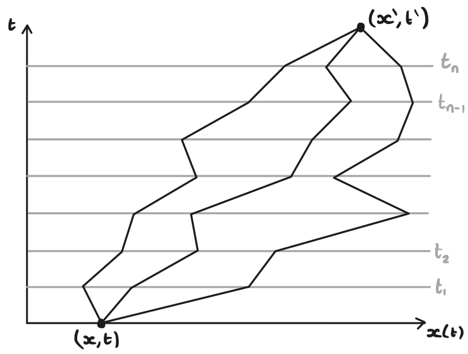

Figure 1: Three possible paths from (x, t) to (x', t') with discretised time interval.

Dirac and Feynman cite(dirac, feynman) showed that for small Δt the path of a particle is determined by the classical action, S, via the following,

$$\langle x_j t_j | x_{j-1} t_{j-1} \rangle = A\ exp\left\{ \frac{i}{h} S(j, j-1) \right\} \tag{7.4}$$

where A is a normalisation constant.

S is the classical action determined by the time integral of the Lagrangian. As we let $\Delta t \longrightarrow 0$ we can approximate the path as a straight line, allowing us to calculate the Lagrangian, and hence the action.

$$\begin{aligned} S(j, j-1) &= \int_{t_{j-1}}^{t_j} dt L_{classical}(x, \dot{x}) \\ &= \int_{t_{j-1}}^{t_j} dt \left[\frac{\mu \dot{x}^2}{2} - V(x) \right] \\ &= \Delta t \left[\frac{\mu}{2} \left(\frac{x_j - x_{j-1}}{\Delta t} \right)^2 - V\left(\frac{x_j + x_{j-1}}{2} \right) \right] \end{aligned} \tag{7.5}$$

$$arethesesecondandthirdlinesneccessary?$$

Note that one can show $A = \left(\frac{\mu}{2\pi\hbar i \Delta t} \right)^{\frac{1}{2}}$.

Subbing (7.4) into (7.3) and taking the limit as $N \longrightarrow \infty$ and $\Delta t \longrightarrow 0$ gives

$$\begin{aligned} K(x', t'; x, t) &= \lim_{\substack{n \to \infty \\ \Delta t \to 0}} \left[\left(\frac{\mu}{2\pi\hbar i \Delta t} \right)^{\frac{n+1}{2}} \int dx_n dx_{n-1} ... dx_1 \prod_{j=1}^{n} exp\left\{ \frac{i}{h} S(j, j-1) \right\} \right] \\ &= \lim_{\substack{n \to \infty \\ \Delta t \to 0}} \left[\left(\frac{\mu}{2\pi\hbar i \Delta t} \right)^{\frac{n+1}{2}} \int dx_n dx_{n-1} ... dx_1\ exp\left\{ \frac{i}{h} \sum_{j=1}^{n} S(j, j-1) \right\} \right] \\ &= \lim_{\substack{n \to \infty \\ \Delta t \to 0}} \left[\left(\frac{\mu}{2\pi\hbar i \Delta t} \right)^{\frac{n+1}{2}} \int dx_n dx_{n-1} ... dx_1\ exp\left\{ \frac{i}{h} \sum_{j=1}^{n} \int_{j-1}^{j} L_{clasical}(x, \dot{x}) \right\} \right] \\ &= \lim_{\substack{n \to \infty \\ \Delta t \to 0}} \left(\frac{\mu}{2\pi\hbar i \Delta t} \right)^{\frac{n+1}{2}} \int dx_n dx_{n-1} ... dx_1\ exp\left\{ \frac{i}{h} \sum_{j=1}^{n} \int_{t_0}^{n+1} L_{clasical}(x, \dot{x}) \right\} \end{aligned} \tag{7.6}$$

$$needtofixthisequation \tag{7.7}$$

Proposition 7.1. *The Feynman path integral is equivalent to Schrödinger's theory.*

7.2 The Partition Function

We consider a system that evolves from a vacuum state at $t \to -\infty$ back to a vacuum state at $t \to \infty$, in the presence of a source. A vacuum state is a groundstate with zero quantum number (what?).

The source is introduced in the Lagrangian by

$$L \to L + \hbar J(t)x(t), \tag{7.8}$$

where $J(t)$ is the source and $x(t)$ is the path.

We define the partition function as the transition amplitude in the presence of this source,

$$Z[J] := \langle 0, -\infty; 0, \infty \rangle^J, \tag{7.9}$$

which one can prove is equivalent to

$$Z[J] = \int d[x(t)] exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt (L + \hbar J(t)x(t) + \frac{1}{2} i\epsilon x^2) \right\}. \tag{7.10}$$

what is epsilon?

7.3 Time Ordered Products

AKA greens functions

7.4 Functional Derivatives of the Partition Function

This is why the partition function is aolso called generating functional

8 Generalising the Path Integral to Quantum Field Theory

In the same way we generalised classical mechanics to classical field theory in section —, we can generalise quantum mechanics to quantum field theory. That is, instead of considering trajectories of a particle x(t), we

maybe talk about why deriving the PI approach to QFT is difficult

talk about how generating functional generalises and how time ordered products generalise to operators I think?

9 Anomalies in QFT

9.1 Symmetries & Slavnov-Taylor identities

We now extend our investigation of symmetries from classical fields to a quantised theory. Suppose the classical action admits a local symmetry such that the path integral measure is invariant

$$\phi_\mu \mapsto \phi_\mu + \epsilon \vartheta_\mu(\phi, x) \tag{9.1}$$

$$\mathcal{D}\phi \mapsto \mathcal{D}\phi \, \det\left(\frac{\partial \phi'_\mu(x)}{\partial \phi_\nu(y)}\right) = \mathcal{D}\phi \tag{9.2}$$

where $K_{\mu\nu} = \frac{\partial \phi'_\mu(x)}{\partial \phi_\nu(y)}$ is the transformation Jacobian. Thus the partition function is invariant and for a source J^μ corresponding to ϕ_μ we can write [ADD PROOF L8R]

$$\int d^4x \, J^\mu(x) \langle \vartheta_\mu(\phi, x) \rangle_J = 0 \tag{9.3}$$

The quantum action is $W[J] = -i \ln Z[J]$, and we define the effective quantum action as the Legendre transform

$$\Gamma[\varphi] = W[J] - \int d^4x \, J^\mu(x) \varphi_\mu(x) \quad ; \quad \varphi_\mu := \langle \phi_\mu \rangle_J \tag{9.4}$$

Taking the functional derivative we obtain

$$J^\mu(x) = -\frac{\delta \Gamma[\varphi]}{\delta \varphi_\mu(x)} \tag{9.5}$$

Subsequently using (9.3) and the definition of a functional derivative

$$\delta \Gamma[\varphi_\mu; \langle \vartheta_\mu(\phi, x) \rangle_J] = 0 \tag{9.6}$$

Thus the effective quantum action is invariant under the transformations

$$\varphi_\mu \mapsto \varphi_\mu + \epsilon \langle \vartheta_\mu(\phi, x) \rangle_J \tag{9.7}$$

such a relation is called a Slavnov-Taylor identity. If the transformation is a linear symmetry such that

$$\vartheta_\mu = \Theta_\mu[\phi, x] = \alpha_\mu(x) + \int d^4y \, \beta_\mu^\nu(x, y) \phi_\nu(y) \tag{9.8}$$

the transformation Jacobian becomes field independent

$$K_{\mu\nu} = \frac{\delta}{\delta \phi_\nu(y)} [\phi_\mu + \epsilon \Theta_\mu(\phi, x)] = \delta_{\mu\nu} \delta^{(4)}(x - y) + \epsilon \beta_\mu^\nu(x, y) \tag{9.9}$$

so the normalised correlators are invariant. Moreover, we have $\langle \Theta_\mu(\phi, x) \rangle_J = \Theta_\mu(\varphi, x)$ which implies the effective quantum action is invariant under the same transformation as the classical action, namely,

$$\varphi_\mu \mapsto \varphi_\mu + \epsilon \Theta_\mu(\varphi, x) \tag{9.10}$$

is a symmetry. In this case the quantum theory inherits the classical symmetry and it is impossible to violate the symmetry via quantum effects granted a regularisation procedure which respects the effective quantum action's symmetry is employed. However, it is possible that no regularisation can preserve a classical symmetry, such a symmetry is called an anomaly and is characterised using the Atiyah–Singer index theorem. Returning to the Dirac Lagrangian, the vector and axial symmetries are linear in the fields but without identifying symmetry preserving regularisations their accession to a quantum theory is undetermined.

9.2 The Path Integral measure

To further investigate the effect of transformations on the partition function we must introduce a more precise notion a path integral for fermionic fields.

We decompose the Dirac spinor and its adjoint using an orthonormal basis $\{\phi_n(x)\}$ of the Dirac operator's eigenfunctions and the independent Grassmann variables θ_n and $\bar{\xi}_m$

$$\psi = \sum_n \theta_n \phi_n(x) = \sum_n \theta_n \langle x|n\rangle \tag{9.11}$$

$$\bar{\psi} = \sum_m \bar{\xi}_m \phi_m^\dagger(x) = \sum_m \bar{\xi}_m \langle m|x\rangle \tag{9.12}$$

Hence the path integral measure can be seen as a transformation of a Grassmann measure under a change of variables

$$\theta \mapsto \theta_n \langle x|n\rangle , \quad \bar{\xi} \mapsto \bar{\xi}_m \langle m|x\rangle \tag{9.13}$$

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = [\det \langle x|n\rangle \det \langle m|x\rangle]^{-1} \lim_{\substack{N\to\infty \\ M\to\infty}} \left\{ \prod_n^N d\theta_n \prod_m^M d\bar{\xi}_m \right\} = \lim_{N\to\infty} \prod_n^N d\theta_n d\bar{\xi}_n \tag{9.14}$$

Under the infinitesimal vector transformation the Dirac spinor becomes

$$\psi' = \sum_n \theta_n' \phi_n(x); \quad \theta_n' = e^{i\vartheta(x)}\theta_n \tag{9.15}$$

and similarly for the adjoint spinor. Thus the path integral measure transforms as

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \mapsto \left[\det\left(e^{i\vartheta(x)\mathbb{I}}\right)\det\left(e^{-i\vartheta(x)\mathbb{I}}\right)\right]^{-1}\mathcal{D}\psi\mathcal{D}\bar{\psi} \tag{9.16}$$

and using the identity [move this bcz it doesnt apply here anymore]

$$\det\left(e^A\right) = e^{\mathrm{Tr}(A)} \tag{9.17}$$

we see the measure is invariant; the vector symmetry is retained by the quantum theory and the correlators are gauge invariant. However, this is not the case for the axial transformation where the Grassmann variables transform as

$$\theta_n' = \sum_m C_{nm}\theta_m, \quad \bar{\xi}_n' = \sum_m C_{nm}\bar{\xi}_m \tag{9.18}$$

$$C_{nm} = \delta_{nm} + i\int d^4x\, \vartheta(x)\phi_n^\dagger(x)\gamma^5\phi_m(x) \tag{9.19}$$

and the path integral measure changes by the Jacobian

$$J[\vartheta] = (\det C)^{-2} = \exp\left[-2\mathrm{Tr}(\ln C)\right] \tag{9.20}$$

Expanding the matrix logarithm to first order in the infinitesimal parameter

$$\ln\left(\mathbb{I} + \epsilon A\right) = \epsilon A \tag{9.21}$$

we obtain

$$J[\vartheta] = \exp\left[-2\mathrm{Tr}\left(i\int d^4x\,\vartheta(x)\phi_n^\dagger(x)\gamma^5\phi_m(x)\right)\right] \tag{9.22}$$

$$J[\vartheta] = \exp\left[-2i\int d^4x\,\vartheta(x)\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)\right] \tag{9.23}$$

The sum appearing in the Jacobian is clearly divergent (by orthonormality) and requires the introduction of a regulator or ultraviolet cut-off to obtain a non-singular Jacobian. However, there is a much more elegant approach available involving the Atiyah–Singer index theorem but first we show the relation between the axial anomaly and the axial current. Henceforth, we define the axial anomaly $\mathcal{A}$ via the Jacobian of the path integral measure

$$J[\vartheta] = \exp\left[-\int d^4x\,\vartheta(x)\mathcal{A}\right] \tag{9.24}$$

under the axial transformation the partition function becomes

$$Z'[A_\mu,\vartheta] = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\,\exp\left[\int d^4x\,\bar{\psi}'(i\not{D}-m)\psi' - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{9.25}$$

but this is just a relabeling of the integration variables and thus the partition function is invariant. Considering the change in the QED lagrangian we have

$$Z[A_\mu,\vartheta] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,J[\vartheta]\exp\left[\int d^4x\,\mathcal{L}_{\mathrm{QED}} + \vartheta(x)(\partial_\mu j_5^\mu - 2im\bar{\psi}\gamma^5\psi)\right] \tag{9.26}$$

Thus to first order in $\vartheta(x)$ we have

$$Z[A_\mu,\vartheta] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\left\{1+\int d^4x\,\vartheta(x)(\partial_\mu j_5^\mu - 2im\bar{\psi}\gamma^5\psi - \mathcal{A})\right\}\exp\left[\int d^4x\,\mathcal{L}_{\mathrm{QED}}\right] \tag{9.27}$$

by the fundamental lemma of the calculus of variations we attain the anomalous divergence for the axial current

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma^5\psi + \mathcal{A} \tag{9.28}$$

9.3 Atiyah–Singer index Theorem

[Here our task is to show (maybe just state) what the Jacobians of the vector and axial symms are and show that we can always write a relationship between the index of a differential operator and the divergence of a current. By using the AST we should find the analytical index and topological index of a differential operator coincide. My guess is the topological index should be the same using any 'sensible' regularisation thus the anomaly should persist.]

Given an eigenfunction of the Dirac operator $i\not{D}$, ϕ_n, with an eigenvalue $\lambda_n \neq 0$,$\gamma^5\phi_n$ is also an eigenfunction

$$i\not{D}(\gamma^5\phi_n) = -i\gamma^5\not{D}(\phi_n) = -\lambda_n\gamma^5\phi_n \tag{9.29}$$

and the two are orthogonal as they have different eigenvalues. However, when the eigenvalue is zero ϕ_n and $\gamma^5\phi_n$ correspond to $\lambda_n = 0$ and are called zero modes of $i\not{D}$. We also have that γ^5 is hermitian thus by the spectral theorem there is a basis that diagonalises γ^5 and spans $\text{Ker}(i\not{D})$. The eigenvalues of γ^5 are ± 1 (because $(\gamma^5)^2 = I_4$) and we use the projection operators $P_\pm = \frac{1}{2}(I_4 \pm \gamma^5)$ to decompose spinors into their positive and negative eigenvalue (positive and negative chirality respectively) components. Ignoring the infinitesimal parameter for now, only the zero modes contribute to the integral in the Jacobian because of orthonormality and we have

$$\int d^4x \sum_n \phi_n^\dagger \gamma^5 \phi_n = \sum_n \int d^4x \, \phi_{n+}^{0\dagger} \phi_{n+}^0 - \sum_n \int d^4x \, \phi_{n-}^{0\dagger} \phi_{n-}^0 = n_+ - n_- \tag{9.30}$$

$$\int d^4x \, \mathcal{A} = 2i(n_+ - n_-) \tag{9.31}$$

where $n_\pm$ are the number of positive and negative chirality zero modes $\phi_{n\pm}^0$ respectively. This is precisely the analytical index of the Dirac operator projected onto the positive chirality subspace

$$\text{index}(i\not{D}_+) = n_+ - n_- \tag{9.32}$$

$$i\not{D}_\pm := i\not{D} P_\pm = i\not{D}|_{\{\pm\}} \tag{9.33}$$

The Atiyah–Singer index theorem states this coincides with the Dirac operator's topological index. On a 4-dimensional compact manifold Ω without boundary and Riemannian curvature R this is [This is crucial; our calculations only work using a wick transformation, also add manifolds to integrals] ("Although the index theorem should be applied only to compact manifolds, it may, nevertheless, also be used in our Euclidean 4-space problem, since the conformal invariance of the theory and the as- sumption that the gauge fields decrease rapidly at infinity allow our problem to be mapped onto the surface of a 5-dimensional hypersphere which is compact" - R. JACKIW, C. REBBI 1977)

$$\text{index}(i\not{D}_+) = \int_\Omega \hat{A}(\Omega)\text{ch}(F) \tag{9.34}$$

where the integrand, called the index density, only includes 4-form terms and the curvature form is given by $F = e\,dA = \frac{e}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. The characteristic classes $\text{ch}(F)$ and $\hat{A}(\Omega)$ are the Chern character of the curvature and Dirac genus of the manifold respectively

$$\text{ch}(F) = \text{Tr}\left(\exp\left[\frac{i}{2\pi}F\right]\right) \tag{9.35}$$

$$\hat{A}(\Omega) = \sqrt{\det\left(\frac{\frac{i}{4\pi}R}{\sinh \frac{i}{4\pi}R}\right)} \tag{9.36}$$

Now we pick the hypersphere $S^4 = \Omega$ so the Riemannian curvature is constant [put a reference to JACKIW and REBBI here] and the index becomes

$$\text{index}(\not{D}_+) = \int_{S^4} \text{ch}(F) = \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int_{S^4} \text{Tr}(F^2) \tag{9.37}$$

and the integrand coincides with the Dirac operator's index density in Euclidean space. Subsequently, we identity the axial anomaly in Euclidean space as

$$\partial_\mu j_5^\mu - 2im\bar{\psi}\gamma^5\psi = \mathcal{A}[A_\mu] = \frac{-ie^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) \tag{9.38}$$

9.4 Anomalies as obstructions to gauging

References

[1] Iñaki Etxebarria. Lagrangian and hamiltonian mechanics, june 2025.

[2] Casper Peeters. Introduction to quantum mechanics, Janurary 2025.

[3] Sophie Darwin. Topology ii, Janurary 2025.