# Local Gauge Anomalies and The Standard Model

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## Abstract

Abstract here

## Contents

# 1 Introduction

The following report is aimed at undergraduate students familiar with lagrangian mechanics, hamiltonian mechanics, quantum mechanics, and topology. For a guide to the expected prerequisite knowledge see [**? ? ?** ].

[The aim of this report is to offer a gentle introduction to quantum field theory, and offer insight into one example of where topology shows up in theoretical physics.] - CHANGE THIS

Explain structure of report here.

# 2 Quantum States

One should be familiar with the notion of a quantum wavefunction, such as $\psi(x)$, which describes the position of a particle. However, in quantum field theory we deal with a more abstract concept called a quantum state.

**Definition 2.1.** Define Hilbert space

**Notation 2.2.** We use the Bra-Ket notation for vectors in Hilbert space. A vector is written as $|\psi\rangle$ and a covector is written as $\langle\phi|$. We define

$$\langle\phi|\psi\rangle := \langle\phi|(|\psi\rangle)$$

Given a vector $|\psi\rangle$ the corresponding hermition transpose of this vector is $\langle\psi|$, meaning $|\psi|^2 = \langle\psi|\psi\rangle$.

A quantum state is a mathematical entity that embodies all the known information of some given quantum system. There are two types of quantum states, pure and mixed. We will only explain pure quantum states in this section.

A (pure) quantum state is an abstract vector in a complex Hilbert space, denoted by $|\psi\rangle$. This Hilbert space is called our state space and often has infinite dimensions.

Observables of our quantum system correspond to Hermition operators on our state space. The eigenvalues of such an operator are real and correspond to possible observed values. It can be shown that the eigenstates of any Hermition operator form a basis for our state space. This allows us to relate quantum states to the more familiar notion of a quantum wavefunction.

Let $\langle\psi|$ be a quantum state and $\hat{X}$ be the position operator with eigenstates $|x\rangle$. Then $\psi(x) = \langle x|\psi\rangle$. WHY???

## 3 The Heisenburg Picture of Quantum Mechanics

One is usually introduced to quantum mechanics in the Schrödinger picture, however, in quantum field theory, it is more natural to consider the Heisenburg picture.

In the Schrödinger picture, the quantum states vary with time. For example, our quantum state may be represented by a wave function $\psi(x,t)$, which evolves in time. Conversely, the operators that act on our state space, such as momentum, position, etc, are usually fixed with respect to time. The only exception to this is that the Hamiltonian may include a potential energy term that varies with time.

Consider a time dependent quantum state $|\psi(t)\rangle$. This state evolves in time according the Schrodinger equation. One can represent this via a unitary time-evolution operator $U(t,t_0)$ as follows;

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle.$$

In the Schrödinger picture, the expectation of a potentially time-dependent hermition operator, $A(t)$, in the state $|\psi(t)\rangle$, can be written as

$$\langle \hat{A}(t)\rangle = \langle \psi(t)|\hat{A}(t)|\psi(t)\rangle.$$

Choosing some reference time $t_0$ we can rewrite this as

$$\langle \psi(t_0)|\hat{U}^{\dagger}(t,t_0)\hat{A}(t)\hat{U}(t,t_0)|\psi(t_0)\rangle.$$

If we define

$$\hat{A_H}(t) = \hat{U}^{\dagger}(t,t_0)\hat{A}(t)\hat{U}(t,t_0)$$

then we have

$$\langle \hat{A}(t)\rangle = \langle \psi(t_0)|\hat{A_H}(t)|\psi(t_0)\rangle.$$

So we notice that we could have obtained this same expectation by considering a constant quantum state $|\psi(t_0)\rangle$ and a new operator that evolves with time. This is the idea behind the Heisenberg picture of quantum mechanics. One interprets the quantum states as being fixed in time, and it is the operators that evolve in time.

In the Schrödinger picture, the Schrödinger equation governs the time evolution of quantum states. Hence, a natural question is, in the Heisenberg picture, how do the operators evolve in time?

One can derive the following equation

$$\frac{d}{dt}\hat{A}_H(t) = \frac{1}{i\hbar}[\hat{A}_H(t),\hat{H}_H(t)] + \left[\frac{d}{dt}\hat{A}_S(t)\right]_H$$

where the subscripts $S$ and $H$ denote whether we are considering the element in the Schrodinger or Heisenberg picture respectively.

# 4 Classical field theory & Noether's theorem

## 4.1 Classical field theory

In classical mechanics we consider a countable set of particles each with finitely many degrees of freedom and generalised coordinates $q_i(t)$. These generalised coordinates $\{q_i(t)\}_i$ specify the system's configuration (position in configuration space) and together with the generalised conjugate momenta $\left\{p_i(t) := \frac{\partial L}{\partial \dot{q}_i}\right\}_i$ define the system's position in phase space.

In classical field theory we generalise this notion of configuration space to a continuum with infinite degrees of freedom. The scalar field $\phi(t)$ can be seen as the generalised coordinates of a continuum $\left(q_i(t) \xrightarrow{i \to \vec{x}} \phi(\vec{x}, t)\right)$ and for a system of continua the set of scalar fields $\{\phi_k(\vec{x}, t)\}_k$ specifies the system's configuration.

Subsequently, we generalise the classical Lagrangian $L(q_i(t), \dot{q}_i(t), t)$ via

$$L(t) = \int d^3\vec{x}\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian density. With respect to some path in configuration space, $\vec{\Phi}(\vec{x}, t)$, the classical action becomes

$$S\left[\vec{\Phi}(\vec{x}, t)\right] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_k(\vec{x}, t), \partial_\mu \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.2}$$

Using Hamilton's principle $\left(\frac{\delta S}{\delta \vec{\Phi}} = 0\right)$ we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right] \tag{4.3}$$

In classical mechanics we define the hamiltonian, $H(p_i(t), q_i(t), t)$, via the Legendre transform (IS THIS ACTUALLY THE DEFINITION OF H? ASK INAKI)

$$H = \sum_i p_i(t)\dot{q}_i(t) - L \tag{4.4}$$

to generalise this to field theory we introduce the momentum field conjugate to $\phi_k(\vec{x}, t)$

$$\pi_k(\vec{x}, t) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} \tag{4.5}$$

$$p_i(t) := \frac{\partial L}{\partial \dot{q}_i} \xrightarrow{i \to \vec{x}} \frac{\partial}{\partial \dot{\phi}_k}\left[\int d^3\vec{x}\, \mathcal{L}\right] = \int d^3\vec{x}\, \pi_k(\vec{x}, t) \tag{4.6}$$

Thus the classical Hamiltonian is generalised to

$$H(t) = \int d^3\vec{x}\, \mathcal{H}(\pi_k(\vec{x}, t), \phi_k(\vec{x}, t), \vec{x}, t) \tag{4.7}$$

$$\mathcal{H} = \sum_k \pi_k(\vec{x},t)\dot{\phi}_k(\vec{x},t) - \mathcal{L} \tag{4.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and Hamilton's equations become the Hamiltonian field equations

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial \pi_k} - \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \pi_k)}\right], \; \dot{\pi}_k = \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \phi_k)}\right] - \frac{\partial \mathcal{H}}{\partial \phi_k} \tag{4.9}$$

[I tried not to use this notation but the rest will be unreadable if I don't, so I hope this does not contradict following variational deriv notation, also use this notation earlier]

$$\frac{\delta \mathcal{F}}{\delta g} := \frac{\partial \mathcal{F}}{\partial g} - \partial_\mu \left[\frac{\partial \mathcal{F}}{\partial(\partial_\mu g)}\right], \tag{4.10}$$

To begin second quantisation we will also require the notion of a field theoretic poisson bracket. Given two functionals, $F$ and $G$, of the dynamical fields given by

$$F = \int d^3\vec{x}\, \mathcal{F}(\pi_k, \phi_k, \vec{x}, t), \; G = \int d^3\vec{x}\, \mathcal{G}(\pi_k, \phi_k, \vec{x}, t) \tag{4.11}$$

We define the poisson bracket

$$\{F, G\}_f = \int d^3\vec{x} \sum_k \left[\frac{\delta \mathcal{F}}{\delta \phi_k}\frac{\delta \mathcal{G}}{\delta \pi_k} - \frac{\delta \mathcal{G}}{\delta \phi_k}\frac{\delta \mathcal{F}}{\delta \pi_k}\right] \tag{4.12}$$

Note that $K(x) = \int dy\, [K(y)\delta(x-y)]$

### 4.2 Noether's Theorem

[From here we should somehow state that $\vec{x}$ is a spatial vector and $x$ is a four-vector]

In classical mechanics Noether's theorem shows the correspondence between global symmetries of the lagrangian and conserved quantities called Noether charges. This generalises to global symmetries of the lagrangian density corresponding to conserved Noether currents in classical field theory. Consider an infinitesimal field transformation, $\vartheta(\epsilon)$, such that

$$\phi_k \mapsto \phi_k + \epsilon \vartheta_k(\phi_k) \tag{4.13}$$

where each $\vartheta_k$ may be a function of an arbitrary number of fields $\phi_k$ but is independent of spacetime, and we suppress high order terms in $\epsilon$. Such a transformation is called a global symmetry if its effect on the Lagrangian density is

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon \partial_\mu \Lambda^\mu(\phi_k, x) \tag{4.14}$$

This change in the Lagrangian density leaves the Euler-Lagrange equations invariant (add proof L8r). Noether's theorem for fields states that a transformation $\vartheta_k(\epsilon)$ is a symmetry of the lagrangian if and only if

$$j^\mu := \sum_k \vartheta_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} - \Lambda^\mu, \; \partial_\mu j^\mu = 0 \tag{4.15}$$

and the divergence free quantity $j^\mu$ is called the conserved Noether current (add proof L8r).

### 4.3 Second Quantisation

In classical mechanics, the canonical transformations are exactly those which preserve the symplectic structure; invariance of poisson brackets of the dynamical variables $(p_i, q_i)$ characterises transformations which leave Hamilton's equations unchanged. A system's position in phase space specifies its classical state.

However, in quantum mechanics all properties of a system are included in a quantum state, $|\psi\rangle$, inside of a Hilbert space upon which operators corresponding to observables act. Dirac's famous canonical quantisation rule ($\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$) allows us quantise the canonical structure of classical mechanics (although Groenewold's theorem shows us that such a rule cannot hold for all functions of the dynamical variables). To obtain a quantised field theory we consider the field theoretic poisson brackets

$$\{\phi_n(z), \phi_m(w)\}_f = 0 = \{\pi_n(z), \pi_m(w)\}_f \tag{4.16}$$

$$\{\phi_n(z), \pi_m(w)\}_f = \delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{4.17}$$

Which are quantised to the canonical commutation relations for the dynamical field operators acting on a Fock space

$$[\hat{\phi}_n(z), \hat{\phi}_m(w)] = 0 = [\hat{\pi}_n(z), \hat{\pi}_m(w)] \tag{4.18}$$

$$[\hat{\phi}_n(z), \hat{\pi}_m(w)] = i\hbar\delta_{nm}\delta^{(3)}(\vec{z} - \vec{w}) \tag{4.19}$$

However, just like Dirac's rule this procedure does not always produce a consistent quantum theory and in the case of quantising the Dirac Lagrangian we require the analogous anti-commutation relations because of the spin-statistics theorem. In principle, commutation relations are axioms of any quantum theory chosen to reproduce experimental observations.

## 5 The Dirac Lagrangian

[Introduce dirac eqn as relativistic QM, show U(1) global symm. and that gauging this symm leads to QED lagrangian. Show we now wish to gauge the axial symm. because of the prev. success but at the quantum level there is an anomaly acting as an obstruction to gauging. From here we use natural units.]

### 5.1 Gauging the vector symmetry

In quantum mechanics the Schrodinger equation is not Lorentz invariant and is thus incompatible with special relativity. A naive generalisation of the relativistic dispersion relation using Hamiltonian and momentum operators for free particles leads to the Klein-Gordon equation. However, the Klein-Gordon equation is a second order partial differential equation and subsequently does not uniquely determine time-evolution of the wavefunction. A more complicated treatment produces the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{5.1}$$

Which corresponds to the Lagrangian density

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.2}$$

Where the adjoint bispinor is defined as $\bar{\psi} \coloneqq \psi^\dagger \gamma^0$. This Lagrangian admits a global U(1) symmetry called the vector symmetry

$$\psi \mapsto e^{i\vartheta}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{-i\vartheta} \tag{5.3}$$

This symmetry corresponds to a conserved vector current via Noether's theorem for fields [add proof L8R]

$$j^\mu = \bar{\psi}\gamma^\mu \psi \tag{5.4}$$

We now attempt to 'gauge' this symmetry by adding spacetime dependence to the parameter

$$\psi \mapsto e^{i\vartheta(x)}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{-i\vartheta(x)} \tag{5.5}$$

However, the Lagrangian is not invariant under such a transformation so we introduce the derivative operator which transforms covariantly

$$D_\mu \coloneqq \partial_\mu + ie\Pi_\mu \tag{5.6}$$

$$D_\mu \psi \mapsto D'_\mu \left[ e^{i\vartheta(x)}\psi \right] = e^{i\vartheta(x)} D_\mu \psi \tag{5.7}$$

To satisfy this the gauge field must transform as

$$\Pi_\mu \mapsto \Pi_\mu - \frac{1}{e}\partial_\mu \vartheta(x) \tag{5.8}$$

Thus we have a modified Lagrangian which admits a local U(1) gauge symmetry

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\Pi_\mu j^\mu \tag{5.9}$$

This is strikingly similar to the electromagnetic Lagrangian density which also has a U(1) gauge symmetry

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \tag{5.10}$$

where $A_\mu$, $J^\mu$, and $F^{\mu\nu}$ are the electromagnetic four-potential, four-current, and field tensor respectively. Thus we identify $\Pi_\mu = A_\mu$ and $ej^\mu = J^\mu$ and arrive at the Lagrangian for quantum electrodynamics.

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu j^\mu + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{5.11}$$

[Speak about how QED is an incredibly successful theory of photon-electron interactions]

### 5.2 The Axial symmetry

However, this is not the end of the story. The massless Dirac Lagrangian admits another symmetry better seen in the Weyl basis.

$$\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}, \quad \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix} \tag{5.12}$$

Thus the Dirac Lagrangian can be seen as the interaction between two Weyl fermions of opposite chirality

$$\mathcal{L}_{\mathrm{D}} = \psi_{\mathrm{L}}^\dagger (i\sigma^\mu \partial_\mu)\psi_{\mathrm{L}} + \psi_{\mathrm{R}}^\dagger (i\bar{\sigma}^\mu \partial_\mu)\psi_{\mathrm{R}} - m(\psi_{\mathrm{L}}^\dagger \psi_{\mathrm{R}} + \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{L}}) \tag{5.13}$$

In the massless case the Lagrangian gains a global U(1) symmetry known as the axial symmetry

$$\psi_{\mathrm{L}} \mapsto e^{i\vartheta}\psi_{\mathrm{L}}, \quad \psi_{\mathrm{R}} \mapsto e^{-i\vartheta}\psi_{\mathrm{R}} \tag{5.14}$$

which is equivalent to

$$\psi \mapsto e^{i\vartheta\gamma^5}\psi \tag{5.15}$$

and a Noether current

$$j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j^\mu = 2im\bar{\psi}\gamma^5\psi \tag{5.16}$$

We now proceed as before, gauging the axial symmetry by promoting $\partial_\mu$ to a covariant derivative as in (5.6) and determining the required gauge transformation

$$\mathcal{L}_D|_{m=0} = \bar{\psi}(i\gamma^\mu\partial_\mu)\psi - q\bar{\psi}\gamma^\mu\Pi_\mu\psi \tag{5.17}$$

$$\Pi_\mu \mapsto e^{i\vartheta\gamma^5}\Pi_\mu e^{-i\vartheta\gamma^5} - \frac{1}{q}\partial_\mu\vartheta\gamma^5 \tag{5.18}$$

Now we must add a kinetic term to the Lagrangian to determine the dynamics of the matrix-valued gauge field. Such a term must be gauge-invariant to preserve the U(1) symmetry. We find a Lagrangian analogous to the Maxwell Lagrangian

$$\mathcal{L}_\Pi = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) \tag{5.19}$$

with field strength

$$\Gamma_{\mu\nu} = \partial_\mu\Pi_\nu - \partial_\nu\Pi_\mu + q[\Pi_\mu, \Pi_\nu] \tag{5.20}$$

has the required gauge-invariance. Thus we have a theory describing the coupling of massless dirac fermions to a mysterious gauge field

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma^{\mu\nu}) - q\bar{\psi}\gamma^\mu\Pi_\mu\psi + \bar{\psi}(i\gamma^\mu\partial_\mu)\psi \tag{5.21}$$

## 6 Quantum Field Theory

So far we have studied relativistic quantum mechanics, making use of classical fields to describe the dynamics of quantum states. However, such a quantum theory is inconsistent subsequently predicting negative energy states which require peculiar interpretations to reconcile (Dirac's hole theory). To proceed to a quantum theory of particle interactions we must promote the dynamical fields to field operators; second quantisation. We shall see how this shift to a quantised field theory affects our calculation of expectation values and probabilities.

### 6.1 Quantising the dirac field/ Fock Space

[You're gonna need to rewrite most of this to make sense]

For the Dirac equation we have the momentum conjugate to $\psi_\mu$ is

$$\pi_\mu := \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\mu} = i\psi_\mu^\dagger \tag{6.1}$$

Subsequently we impose the anti-commutation relations for the field operators

$$[\hat{\psi}_\mu(z), \hat{\psi}_\nu(w)]_+ = 0 = [\hat{\psi}_\mu^\dagger(z), \hat{\psi}_\nu^\dagger(w)]_+ \tag{6.2}$$

$$[\hat{\psi}_\mu(z), \hat{\psi}_\nu^\dagger(w)]_+ = \delta_{\mu\nu}\delta^{(3)}(\vec{z} - \vec{w}) \tag{6.3}$$

This implies the creation and annihilation operators must satisfy

$$[\hat{b}^r_{\vec{p}}, \hat{b}^{s\dagger}_{\vec{q}}]_+ = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p} - \vec{q}) = [\hat{c}^r_{\vec{p}}, \hat{c}^{s\dagger}_{\vec{q}}]_+ \tag{6.4}$$

We define the vacuum state by

$$\hat{b}^r_{\vec{p}}\,|0\rangle = 0 = \hat{c}^s_{\vec{q}}\,|0\rangle \tag{6.5}$$

and the creation operators add particles to the vacuum

$$\hat{b}^{r\dagger}_{\vec{p}}\,|0\rangle = |\vec{p}, r\rangle \tag{6.6}$$

***NOT SURE IF THIS WILL BE NEEDED

The fields in the Lagrangian density are relativistic, meaning under Lorentz tranformations

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

they transform as

$$\phi'_i(x') = R^j_l \phi_i(x)$$

**what is i? Does it matter?

We use the Minkowski metric

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$$

in these notes.

***DOWN TO HERE

Define a symmetry

State Noether's thm

# 7 Path Integral Formulation of Quantum Mechanics

## 7.1 The Propagator

In quantum mechanics, the propagator is the probability amplitude of observing a position $x_n$ at time $t_n$, given the particle is at postion $x_1$ at time $t_1$. This can be written as the scalar product of the two quantum states,

$$K(x', t'; x, t) := \langle x't'|xt\rangle. \tag{7.1}$$

We can rewrite this propagator as an integral over all possible paths the particle can take. We call this the Feynman path integral.

First, we discretise the time between $t'$ and $t$ as

$$t_{n+1} := t' > t_n > t_{n-1} > ... > t_2 > t_1 > t =: t_0,$$

giving $n + 1$ equal peices $\Delta t = t_j - t_{j-1}$ for $1 \leq j \leq n + 1$.

Since position eigenstates are complete, we have

$$\int dx_j |x_j t_j\rangle\langle x_j t_j| = \mathbb{I}, \tag{7.2}$$

meaning we can rewite (7.1) as

$$K(x',t';x,t) = \int dx_n dx_{n-1}...dx_1 \langle x_{n+1}t_{n+1}|x_n t_n\rangle\langle x_n t_n|x_{n-1}t_{n-1}\rangle...$$
$$...\langle x_2 t_2|x_1 t_1\rangle\langle x_1 t_1|x_0 t_0\rangle. \tag{7.3}$$

This can be thought of as integrating over all possible paths, as depicted in Figure 1.

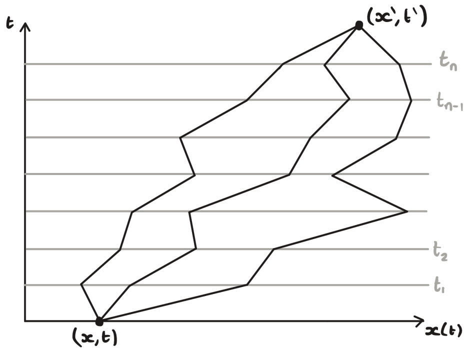


**Figure 1**: Three possible paths from $(x,t)$ to $(x',t')$ with discretised time interval.

Dirac and Feynman cite(dirac, feynman) showed that for small $\Delta t$ the path of a particle is determined by the classical action, S, via the following,

$$\langle x_j t_j|x_{j-1}t_{j-1}\rangle = A\ exp\left\{\frac{i}{\hbar}S(j,j-1)\right\} \tag{7.4}$$

where $A$ is a normalisation constant.

$S$ is the classical action determined by the time integral of the Lagrangian. As we let $\Delta t \longrightarrow 0$ we can approximate the path as a straight line, allowing us to calculate the Lagrangian, and hence the action.

$$\begin{aligned} S(j,j-1) &= \int_{t_{j-1}}^{t_j} dt L_{classical}(x,\dot{x}) \\ &= \int_{t_{j-1}}^{t_j} dt \left[\frac{\mu\dot{x}^2}{2} - V(x)\right] \\ &= \Delta t\left[\frac{\mu}{2}\left(\frac{x_j - x_{j-1}}{\Delta t}\right)^2 - V\left(\frac{x_j + x_{j-1}}{2}\right)\right] \end{aligned} \tag{7.5}$$

$$arethesesecondandthirdlinesneccessary?$$

Note that one can show $A = \left(\frac{\mu}{2\pi\hbar i\Delta t}\right)^{\frac{1}{2}}$.

Subbing (7.4) into (7.3) and taking the limit as $N \longrightarrow \infty$ and $\Delta t \longrightarrow 0$ gives

$$\begin{aligned}
K(x',t';x,t) &= \lim_{\substack{n\to\infty \\ \Delta t\to 0}} \left[ \left(\frac{\mu}{2\pi\hbar i\Delta t}\right)^{\frac{n+1}{2}} \int dx_n dx_{n-1}...dx_1 \prod_{j=1}^{n} exp\left\{\frac{i}{h}S(j,j-1)\right\}\right] \\
&= \lim_{\substack{n\to\infty \\ \Delta t\to 0}} \left[ \left(\frac{\mu}{2\pi\hbar i\Delta t}\right)^{\frac{n+1}{2}} \int dx_n dx_{n-1}...dx_1 \; exp\left\{\frac{i}{h}\sum_{j=1}^{n} S(j,j-1)\right\}\right] \\
&= \lim_{\substack{n\to\infty \\ \Delta t\to 0}} \left[ \left(\frac{\mu}{2\pi\hbar i\Delta t}\right)^{\frac{n+1}{2}} \int dx_n dx_{n-1}...dx_1 \; exp\left\{\frac{i}{h}\sum_{j=1}^{n} \int_{j-1}^{j} L_{clasical}(x,\dot{x})\right\}\right] \\
&= \lim_{\substack{n\to\infty \\ \Delta t\to 0}} \left(\frac{\mu}{2\pi\hbar i\Delta t}\right)^{\frac{n+1}{2}} \int dx_n dx_{n-1}...dx_1 \; exp\left\{\frac{i}{h}\sum_{j=1}^{n} \int_{t_0}^{n+1} L_{clasical}(x,\dot{x})\right\}
\end{aligned} \tag{7.6}$$

$$needtofixthisequation \tag{7.7}$$

**Proposition 7.1.** *The Feynman path integral is equivalent to Schrödinger's theory.*

### 7.2 The Partition Function

[IS THIS QFT OR QM?] We consider a system that evolves from a vacuum state at $t \to -\infty$ back to a vacuum state at $t \to \infty$, in the presence of a source. A vacuum state is a groundstate with zero quantum number (what?).

The source is introduced in the Lagrangian by

$$L \to L + \hbar J(t)x(t), \tag{7.8}$$

where $J(t)$ is the source and $x(t)$ is the path.

We define the partition function as the transition amplitude in the presence of this source,

$$Z[J] := \langle 0,-\infty;0,\infty\rangle^{J}, \tag{7.9}$$

which one can prove is equivalent to

$$Z[J] = \int d[x(t)] exp\left\{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt(L + \hbar J(t)x(t) + \frac{1}{2}i\epsilon x^2)\right\}. \tag{7.10}$$

what is epsilon?

### 7.3 Time Ordered Products

AKA greens functions

Consider inserting two position operators within the propagator formula as follows,

$$K = \langle x_f x_f | \hat{x}(t_q) \hat{x}(t_s) | x_i t_i \rangle \tag{7.11}$$

where $t_f > t_q > t_s > t_i$. One can think of this as the probability amplitude of propagating from the state $|x_i t_i\rangle$ to $|x_f t_f\rangle$ whilst passing via the states $|x_1 t_1\rangle$ and $|x_2 t_2\rangle$. [IS THIS ACTUALLY CORRECT INTERPRETATION?]

We calculate this in the same way we derived the path integral. We first discretise the time, ensuring that two of our time 'slices' match the times $t_s$ and $t_k$. We then use the completeness of eigenstates to rewrite $K$ as

$$\begin{aligned} K = \int dx_n dx_{n-1}...dx_1 \langle x_f t_f | x_n t_n \rangle \langle x_n t_n | x_{n-1} t_{n-1} \rangle ... \\ ...\hat{x}(t_q) | x_q t_q \rangle ... \hat{x}(t_s) | x_s t_s \rangle ... \\ ...\langle x_2 t_2 | x_1 t_1 \rangle \langle x_1 t_1 | x_i t_i \rangle. \end{aligned} \tag{7.12}$$

The operators act on their corresponding eigenstates and give

$$\begin{aligned} K = \int dx_n dx_{n-1}...dx_1 x(t_q) x(t_s) \langle x_f t_f | x_n t_n \rangle \langle x_n t_n | ... \\ ...| x_1 t_1 \rangle \langle x_1 t_1 | x_i t_i \rangle. \end{aligned} \tag{7.13}$$

We again use Dirac and Feynman's Lagrangian formula for $\langle x_j t_j | x_{j-1} t_{j-1} \rangle$ with small time differences cite(dirac, feynman), and take the limit as $n \to \infty$ and $\Delta t \to \infty$. This gives the path integral

$$\langle x_f x_f | \hat{x}(t_q) \hat{x}(t_s) | x_i t_i \rangle = \int_{x_i}^{x_f} d[x(t)] x(t_s) x(t_q) exp \left\{ \frac{i}{h} \int_{t_i}^{t_f} dt L_{classical}(x, \dot{x}) \right\}. \tag{7.14}$$

Note how important it was in our derivation that $t_q > t_s$. In fact, if $t_q < t_s$ then the left hand side of 7.14 wouldn't hold much meaning. This allows us to consider the right hand side as giving the 'time ordered product' on the left hand side.

Generalising to more operators than two gives the following,

$$\begin{aligned} &\langle x_f x_f | T\{\hat{x}(t_1) \hat{x}(t_2) ... \hat{x}(t_n)\} | x_i t_i \rangle \\ &= \int_{x_i}^{x_f} d[x(t)] x(t_1) x(t_2) ... x(t_n) exp \left\{ \frac{i}{h} \int_{t_i}^{t_f} dt L_{classical}(x, \dot{x}) \right\}, \end{aligned} \tag{7.15}$$

where $T$ is a function that reorderes the operators to give an order where their time arguments are decreasing. More formally, in the case of two operators we would have,

$$T\{A(t_1) \tag{7.16}$$

GENERALISE THIS TO N OPERATORS - Construct a formula

## 7.4 Functional Derivatives of the Partition Function

This is why the partition function is aolso called generating functional...

lets assume

$$Z[J] = \lim_{T' \to -i\infty, T \to i\infty} \int d[x(t)] exp \left\{ \frac{i}{\hbar} \int_T^{T'} dt(L + \hbar J(t)x(t)) \right\}. \tag{7.17}$$

and work out the derivative

The derivative of a functional F of a function f can be defined as

$$\frac{\delta F[f](x)}{\delta f(y)} := \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \{F[f(x) + \delta(x-y)] - F[f](x)\}. \tag{7.18}$$

We use this definition to find the derivative of the functional $Z[J]$ with respect to the source at a given time, $J(t_1)$. To do this we assume the $\epsilon$ limit commutes with the $T$ and $T'$ limits in our formula for $Z[J]$. We omit the $T$ and $T'$ limit for clarity.

$$\begin{aligned}\frac{\delta Z[J](t)}{\delta J(t_1)} = \lim_{\substack{T' \to -i\infty \\ T \to i\infty}} \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \int d[x(t)] \Bigg[ & exp \left\{ \frac{i}{\hbar} \int_T^{T'} dt(L + \hbar(J(t) + \delta(t - t_1))x(t)) \right\} \\ & - exp \left\{ \frac{i}{\hbar} \int_T^{T'} dt(L + \hbar J(t)x(t)) \right\} \Bigg]\end{aligned} \tag{7.19}$$

Splitting the first exponential and factoring gives

$$\begin{aligned}\frac{\delta Z[J](t)}{\delta J(t_1)} = \lim_{\substack{T' \to -i\infty \\ T \to i\infty}} \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \int d[x(t)] \Bigg[ & exp \left\{ \frac{i}{\hbar} \int_T^{T'} dt(L + \hbar J(t)x(t)) \right\} \\ & \cdot \left( exp \left\{ \frac{i}{\hbar} \int_T^{T'} dt(L + \hbar \delta(t - t_1)x(t)) \right\} - 1 \right) \Bigg]\end{aligned} \tag{7.20}$$

Expanding the last exponential to first order and using integration property of the delta function gives

$$\frac{\delta Z[J](t)}{\delta J(t_1)} = \lim_{\substack{T' \to -i\infty \\ T \to i\infty}} \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \int d[x(t)] \left[ \hbar \cdot x(t_1) \cdot exp \left\{ \frac{i}{\hbar} \int_T^{T'} dt(L + \hbar J(t)x(t)) \right\} \right] \tag{7.21}$$

which you might notice, when evaluated at $J = 0$ is the formula for our time ordered product. A more general derivation leads to

$$\left. \frac{\delta Z[J]}{\delta J(t_1)...\delta J(t_n)} \right|_{J=0} \sim \langle 0, \infty | T\{\hat{x}(t_1)\hat{x}(t_2)...\hat{x}(t_n)\} | 0, -\infty \rangle. \tag{7.22}$$

This is why the partition function is also called the generating functional, its derivatives generate the time ordered products.

## 8 Generalising the Path Integral to Quantum Field Theory

So far we have considered the path integral in quantum mechanics, that is, we have considered trajectories of particles $x(t)$ and Lagrangians, $L$. In quantum field theory we consider fields on spacetime, $\phi(x)$, and Lagrangian densities $\mathcal{L}$ .

The path integral in quantum mechanics can be formally extended to quantum field theory. However, a rigourous approach to path integrals in quantum field theory involves a process called renormalisation, which is incredibly complex and has only been achieved for a few quantum field theories.

The generating functional still represents the vacuum-to-vacuum transition amplitude in the presence of a source $J$, only now it becomes an integral over all possible fields, rather than all possible paths.

$$Z[J] = \int d\phi \; exp\left\{ i \int d^4x \left[ \mathcal{L}(\phi, \partial_\mu \phi) + J\phi + \frac{1}{2}\epsilon\phi^2 \right] \right\} \tag{8.1}$$

where the $\epsilon$ term [IS WHAT?]

The time ordered products, also referred to as the Greens functions now become time ordered products of field operators? and we have 3.143