Local Gauge Anomalies and The Standard Model

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Abstract

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Contents

1 Introduction

The following report is aimed at undergraduate students familiar with lagrangian mechanics, hamiltonian mechanics, quantum mechanics, and topology. For a guide to the expected prerequisite knowledge see [1–3].

[The aim of this report is to offer a gentle introduction to quantum field theory, and offer insight into one example of where topology shows up in theoretical physics.] - CHANGE THIS

Explain structure of report here.

2 Quantum States

One should be familiar with the notion of a quantum wavefunction, such as $\psi(x)$, which describes the position of a particle. However, in quantum field theory we deal with a more abstract concept called a quantum state.

Definition 2.1. Define Hilbert space

Notation 2.2. We use the Bra-Ket notation for vectors in Hilbert space. A vector is written as $|\psi\rangle$ and a covector is written as $\langle\phi|$. We define

$$\langle\phi|\psi\rangle := \langle\phi|(|\psi\rangle)$$

Given a vector $|\psi\rangle$ the corresponding hermition transpose of this vector is $\langle\psi|$, meaning $|\psi|^2 = \langle\psi|\psi\rangle$.

A quantum state is a mathematical entity that embodies all the known information of some given quantum system. There are two types of quantum states, pure and mixed. We will only explain pure quantum states in this section.

A (pure) quantum state is an abstract vector in a complex Hilbert space, denoted by $|\psi\rangle$. This Hilbert space is called our state space and often has infinite dimensions.

Observables of our quantum system correspond to Hermition operators on our state space. The eigenvalues of such an operator are real and correspond to possible observed values. It can be shown that the eigenstates of any Hermition operator form a basis for our state space. This allows us to relate quantum states to the more familiar notion of a quantum wavefunction.

Let $\langle\psi|$ be a quantum state and $\hat{X}$ be the position operator with eigenstates $|x\rangle$. Then $\psi(x) = \langle x|\psi\rangle$. WHY???

3 The Heisenburg Picture of Quantum Mechanics

One is usually introduced to quantum mechanics in the Schrödinger picture, however, in quantum field theory, it is more natural to consider the Heisenburg picture.

In the Schrödinger picture, the quantum states vary with time. For example, our quantum state may be represented by a wave function $\psi(x,t)$, which evolves in time. Conversely, the operators that act on our state space, such as momentum, position, etc, are usually fixed with respect to time. The only exception to this is that the Hamiltonian may include a potential energy term that varies with time.

Consider a time dependent quantum state $|\psi(t)\rangle$. This state evolves in time according the Schrodinger equation. One can represent this via a unitary time-evolution operator $U(t,t_0)$ as follows;

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle.$$

In the Schrödinger picture, the expectation of a potentially time-dependent hermition operator, $A(t)$, in the state $|\psi(t)\rangle$, can be written as

$$\langle\hat{A}(t)\rangle = \langle\psi(t)|\hat{A}(t)|\psi(t)\rangle.$$

Choosing some reference time t_0 we can rewrite this as

$$\langle\psi(t_0)|\hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)|\psi(t_0)\rangle.$$

If we define

$$\hat{A_H}(t) = \hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)$$

then we have

$$\langle\hat{A}(t)\rangle = \langle\psi(t_0)|\hat{A_H}(t)|\psi(t_0)\rangle.$$

So we notice that we could have obtained this same expectation by considering a constant quantum state $|\psi(t_0)\rangle$ and a new operator that evolves with time. This is the idea behind the Heisenberg picture of quantum mechanics. One interprets the quantum states as being fixed in time, and it is the operators that evolve in time.

In the Schrödinger picture, the Schrödinger equation governs the time evolution of quantum states. Hence, a natural question is, in the Heisenberg picture, how do the operators evolve in time?

One can derive the following equation

$$\frac{d}{dt}\hat{A}_H(t) = \frac{1}{i\hbar}[\hat{A}_H(t),\hat{H}_H(t)] + \left[\frac{d}{dt}\hat{A}_S(t)\right]_H$$

where the subscripts S and H denote whether we are considering the element in the Schrodinger or Heisenberg picture respectively.

4 Classical field theory & Noether's theorem

In classical mechanics we consider a countable set of particles each with finitely many degrees of freedom and generalised coordinates $q_i(t)$. These generalised coordinates $\{q_i(t)\}_i$ specify the system's configuration (position in configuration space) and together with the generalised conjugate momenta $\left\{p_i(t) := \frac{\partial L}{\partial \dot{q}_i}\right\}_i$ define the system's position in phase space.

In classical field theory we generalise this notion of configuration space to a continuum with infinite degrees of freedom. The scalar field $\phi(t)$ can be seen as the generalised coordinates of a continuum $\left(q_i(t) \xrightarrow{i\to\vec{x}} \phi(\vec{x},t)\right)$ and for a system of continua the set of scalar fields $\{\phi_k(\vec{x},t)\}_k$ specifies the system's configuration.

Subsequently, we generalise the classical Lagrangian $L(q_i(t), \dot{q}_i(t), t)$ via

$$L(t) = \int d^3\vec{x}\, \mathcal{L}(\phi_k(\vec{x},t), \partial_\mu\phi_k(\vec{x},t), \vec{x}, t) \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian density. With respect to some path in configuration space, $\vec{\Phi}(\vec{x},t)$, the classical action becomes

$$S\left[\vec{\Phi}(\vec{x},t)\right] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_k(\vec{x},t), \partial_\mu\phi_k(\vec{x},t), \vec{x}, t) \tag{4.2}$$

Using Hamilton's principle $\left(\frac{\delta S}{\delta \vec{\Phi}} = 0\right)$ we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right] \tag{4.3}$$

In classical mechanics we define the hamiltonian, $H(p_i(t), q_i(t), t)$, via the Legendre transform (IS THIS ACTUALLY THE DEFINITION OF H? ASK INAKI)

$$H = \sum_i p_i(t)\dot{q}_i(t) - L \tag{4.4}$$

to generalise this to field theory we introduce the momentum field conjugate to $\phi_k(\vec{x},t)$

$$\pi_k(\vec{x},t) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} \tag{4.5}$$

$$p_i(t) := \frac{\partial L}{\partial \dot{q}_i} \xrightarrow{i\to\vec{x}} \frac{\partial}{\partial \dot{\phi}_k}\left[\int d^3\vec{x}\, \mathcal{L}\right] = \int d^3\vec{x}\, \pi_k(\vec{x},t) \tag{4.6}$$

Thus the classical Hamiltonian is generalised to

$$H(t) = \int d^3\vec{x}\, \mathcal{H}(\pi_k(\vec{x},t), \phi_k(\vec{x},t), \vec{x}, t) \tag{4.7}$$

$$\mathcal{H} = \sum_k \pi_k(\vec{x},t)\dot{\phi}_k(\vec{x},t) - \mathcal{L} \tag{4.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and Hamilton's equations become the Hamiltonian field equations

$$\dot{\phi}_k = \frac{\partial \mathcal{H}}{\partial \pi_k} - \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \pi_k)} \right], \; \dot{\pi}_k = \partial_\mu \left[\frac{\partial \mathcal{H}}{\partial(\partial_\mu \phi_k)} \right] - \frac{\partial \mathcal{H}}{\partial \phi_k} \tag{4.9}$$

[I tried not to use this notation but the rest will be unreadable if I don't, so I hope this does not contradict following variational deriv notation, also use this notation earlier]

$$\frac{\delta \mathcal{F}}{\delta g} := \frac{\partial \mathcal{F}}{\partial g} - \partial_\mu \left[\frac{\partial \mathcal{F}}{\partial(\partial_\mu g)} \right], \tag{4.10}$$

To begin second quantisation we will also require the notion of a field theoretic poisson bracket. Given two functionals of the dynamical fields F and G given by

$$F = \int d^3\vec{x}\, \mathcal{F}(\pi_k, \phi_k, \vec{x}, t), \; G = \int d^3\vec{x}\, \mathcal{G}(\pi_k, \phi_k, \vec{x}, t) \tag{4.11}$$

We define the poisson bracket

$$\{F, G\}_{\phi,\pi} = \int d^3\vec{x} \sum_k \left[\frac{\delta \mathcal{F}}{\delta \phi_k} \frac{\delta \mathcal{G}}{\delta \pi_k} - \frac{\delta \mathcal{G}}{\delta \phi_k} \frac{\delta \mathcal{F}}{\delta \pi_k} \right] \tag{4.12}$$

***NOT SURE IF THIS WILL BE NEEDED

The fields in the Lagrangian density are relativistic, meaning under Lorentz tranformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

they transform as

$$\phi'_i(x') = R^j_l \phi_i(x)$$

**what is i? Does it matter?

We use the Minkowski metric

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1)$$

in these notes.

***DOWN TO HERE

Define a symmetry

State Noether's thm

5 Path Integral & Generating Functional

Given a particle position x at time t the probability amplitude of a measurement at time t' observing a position x' is

$$\begin{aligned} M(x', t'; x, t) &= {}_H\langle x', t' | x, t\rangle_H \\ &= \langle x' | \exp\left[\frac{-i}{\hbar}\hat{H}(t-t')\right] | x\rangle \end{aligned}$$

now we note that if the states are normalised

$$\int dy |y, t\rangle\langle y, t| = \mathbb{I} \tag{5.1}$$

we can write

$$M(x', t'; x, t) = \sum_n \psi_n(x')\psi_n^*(x) \exp\left[\frac{-i}{\hbar}\hat{H}(t-t')\right] \tag{5.2}$$

which acts as our propagator

$$\psi(x', t') = \int dx\ M(x', t'; x, t)\psi(x, t)\partial_\mu^x \tag{5.3}$$

6 Scalar Fields & Green's Functions

7 Wess-Zumino Consistency & Ward Identity

8 Stora-Zumino Descent

9 Dai-Freed & Index Theorems

References

[1] Iñaki Etxebarria. Lagrangian and hamiltonian mechanics, june 2025.

[2] Casper Peeters. Introduction to quantum mechanics, Janurary 2025.

[3] Sophie Darwin. Topology ii, Janurary 2025.