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Optimization Techniques

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Lecture 1

Optimization problems in general setting

Let $f : D \to \mathbb{R}$ be a function defined on a nonempty set D and let $S \subseteq D$, $S \neq \emptyset$.

Definition 1.1 An element $x^0 \in S$ is called:

- *minimum point* of f w.r.t. S, if
$$f(x^0) \leq f(x),\ \forall\, x \in S.$$
- *maximum point* of f w.r.t. S, if
$$f(x^0) \geq f(x),\ \forall\, x \in S.$$

The sets of all minimum points and maximum points of f w.r.t. S will be denoted by

$$\begin{aligned}\operatorname*{argmin}_{x \in S} f(x) &:= \{x^0 \in S \mid f(x^0) \leq f(x),\ \forall\, x \in S\}\\ \operatorname*{argmax}_{x \in S} f(x) &:= \{x^0 \in S \mid f(x^0) \geq f(x),\ \forall\, x \in S\}.\end{aligned}$$

Remark 1.2 Since S is nonempty, its image by f, i.e., the set

$$f(S) := \{f(x) \mid x \in S\}$$

is a nonempty subset of $\mathbb{R}$, hence the following (extended-)real numbers

$$\inf f(S) \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad \sup f(S) \in \mathbb{R} \cup \{+\infty\}$$

are well-defined. It is easily seen that

$$\begin{aligned}\operatorname*{argmin}_{x \in S} f(x) &:= \{x^0 \in S \mid f(x^0) = \inf f(S)\}\\ &= \{x^0 \in S \mid f(x^0) = \min f(S)\};\\ \operatorname*{argmax}_{x \in S} f(x) &:= \{x^0 \in S \mid f(x^0) = \sup f(S)\}\\ &= \{x^0 \in S \mid f(x^0) = \max f(S)\}.\end{aligned}$$

Remark 1.3 $\inf f(S) \in \mathbb{R}$ if and only if $f(S)$ is bounded from below. However, the lower boundedness of $f(S)$ does not guarantee the existence of the least element $\min f(S)$ of $f(S)$. For instance, if $S = D = \mathbb{R}$ and $f : D \to \mathbb{R}$ is the exponential function

$$f(x) = e^x, \ \forall\, x \in D,$$

we have $f(S) =]0, +\infty[$, hence $\inf f(S) = 0 \in \mathbb{R}$, but $f(S)$ does not possess a least element. In this case we have $\operatorname*{argmin}_{x \in S} f(x) = \emptyset$.

Remark 1.4 $\sup f(S) \in \mathbb{R}$ if and only if $f(S)$ is bounded from above. However, the upper boundedness of $f(S)$ does not guarantee the existence of the largest element $\max f(S)$ of $f(S)$. For instance, if $S = [0, +\infty[\subseteq D = \mathbb{R}$ and $f : D \to \mathbb{R}$ is defined by

$$f(x) = \arctan x, \ \forall\, x \in D,$$

we have $f(S) = [0, \pi/2[$, hence $\sup f(S) = \pi/2 \in \mathbb{R}$, but $f(S)$ does not possess a largest element. In this case we have $\operatorname*{argmax}_{x \in S} f(x) = \emptyset$.

Definition 1.5 The problem of finding the value $\inf\ f(S) \in \mathbb{R} \cup \{-\infty\}$ and the set $\operatorname*{argmin}_{x \in S} f(x)$ (or, in practice, at least one element of this set, if any) is called *minimization problem* with *objective function* f and *feasible set* S. We denote this problem by

$$\begin{cases} \text{Minimize } f(x) \\ \\ x \in S. \end{cases} \tag{1.1}$$

The elements of S are called *feasible points* (or *feasible solutions*) of problem (1.1) while the elements of the set $\operatorname*{argmin}_{x \in S} f(x)$ are called *optimal solutions* of problem (1.1).

Definition 1.6 The problem of finding the value $\sup\ f(S) \in \mathbb{R} \cup \{+\infty\}$ and the set $\operatorname*{argmax}_{x \in S} f(x)$ (or, in practice, at least one element of this set, if any) is called *maximization problem* with *objective function* f and *feasible set* S. We denote this problem by

$$\begin{cases} \text{Maximize } f(x) \\ \\ x \in S. \end{cases} \tag{1.2}$$

The elements of S are called *feasible points* (or *feasible solutions*) of problem (1.2) while the elements of the set $\operatorname*{argmax}_{x \in S} f(x)$ are called *optimal solutions* of problem (1.2).

Remark 1.7 It is easy to check that

$$\begin{aligned}\operatorname*{argmax}_{x\in S} f(x) &= \operatorname*{argmin}_{x\in S}(-f)(x);\\ \operatorname*{argmin}_{x\in S} f(x) &= \operatorname*{argmax}_{x\in S}(-f)(x).\end{aligned}$$

These realtions show that any maximization problem of type (1.2) can be transformed into a minimization problem of type (1.1) and vice-versa.

Remark 1.8 A minimization problem (1.1) has no optimal solutions, i.e.,

$$\operatorname*{argmin}_{x\in S} f(x) = \emptyset,$$

in one and only one of the following situations:

- f is not bounded from below on S, i.e.,

$$\inf f(S) = -\infty;$$

- f is bounded from below on S, but f does not attain its minimal value, i.e.,

$$\inf f(S) \in \mathbb{R} \setminus f(S).$$

Remark 1.9 A maximization problem (1.2) has no optimal solutions, i.e.,

$$\operatorname*{argmax}_{x\in S} f(x) = \emptyset,$$

in one and only one of the following situations:

- f is not bounded from above on S, i.e.,

$$\sup f(S) = +\infty;$$

- f is bounded from above on S, but f does not attain its maximal value, i.e.,

$$\sup f(S) \in \mathbb{R} \setminus f(S).$$

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Lecture 2

Level sets; characterizations of optimal solutions

Let $f : D \to \mathbb{R}$ be a function defined on a nonempty set D and let $S \subseteq D$, $S \neq \emptyset$.

Definition 2.1 For any $\lambda \in \mathbb{R}$, the following sets are called *level sets* of f (w.r.t. S and λ):

$$
\begin{aligned}
S_f(\lambda) &:= \{x \in S \mid f(x) = \lambda\}, \\
S_f^{\leqslant}(\lambda) &:= \{x \in S \mid f(x) \leq \lambda\}, \\
S_f^{<}(\lambda) &:= \{x \in S \mid f(x) < \lambda\} = S_f^{\leqslant}(\lambda) \setminus S_f(\lambda), \\
S_f^{>}(\lambda) &:= \{x \in S \mid f(x) > \lambda\} = S \setminus S_f^{\leqslant}(\lambda), \\
S_f^{\geqslant}(\lambda) &:= \{x \in S \mid f(x) \geq \lambda\} = S \setminus S_f^{<}(\lambda).
\end{aligned}
$$

Proposition 2.2 *The following characterizations of optimal solutions hold:*

$$
\operatorname*{argmin}_{x \in S} f(x) = \{x^0 \in S \mid S \subseteq D_f^{\geqslant}(f(x^0))\}, \tag{2.1}
$$

$$
\operatorname*{argmax}_{x \in S} f(x) = \{x^0 \in S \mid S \subseteq D_f^{\leqslant}(f(x^0))\}. \tag{2.2}
$$

Proposition 2.4 *Let $\lambda \in \mathbb{R}$. If $A \in \{S_f^{\leqslant}(\lambda), S_f^{<}(\lambda)\}$ and $B \in \{S_f^{\geqslant}(\lambda), S_f^{>}(\lambda)\}$ are nonempty sets, then the following relations hold:*

$$\operatorname*{argmin}_{x \in S} f(x) = \operatorname*{argmin}_{x \in A} f(x), \tag{2.3}$$

$$\operatorname*{argmax}_{x \in S} f(x) = \operatorname*{argmax}_{x \in B} f(x). \tag{2.4}$$

Lecture 3

Existence and unicity of optimal solutions

Let $f : D \to \mathbb{R}$ be a function defined on a nonempty set D and let $S \subseteq D$, $S \neq \emptyset$.

Theorem 3.1 *The minimization problem* (1.1) *has at least one optimal solution, i.e.,*

$$\operatorname*{argmin}_{x \in S} f(x) \neq \emptyset,$$

if one of the following conditions is fulfilled:

(C1) *There exists $\mu \in \mathbb{R}$ such that $S_f^{\leqslant}(\mu)$ is nonempty and bounded, and $S_f^{\leqslant}(\lambda)$ is closed for every $\lambda \in]-\infty, \mu]$.*

(C2) *S is closed, f is continuous, and there exists $\mu \in \mathbb{R}$ such that $S_f^{\leqslant}(\mu)$ is nonempty and bounded.*

(C3) *S is compact and $S_f^{\leqslant}(\lambda)$ is closed for each $\lambda \in \mathbb{R}$.*

Corollary 3.2 *The maximization problem* (1.2) *has at least one optimal solution, i.e.,*

$$\operatorname*{argmax}_{x \in S} f(x) \neq \emptyset,$$

if one of the following conditions is fulfilled:

(C1) *There exists $\mu \in \mathbb{R}$ such that $S_f^{\geqslant}(\mu)$ is nonempty and bounded, and $S_f^{\geqslant}(\lambda)$ is closed for every $\lambda \in [\mu, +\infty[$.*

(C2) *S is closed, f is continuous, and there exists $\mu \in \mathbb{R}$ such that $S_f^{\geqslant}(\mu)$ is nonempty and bounded.*

(C3) *S is compact and $S_f^{\geqslant}(\lambda)$ is closed for each $\lambda \in \mathbb{R}$.*

Remark 3.3 In the particular case when f is continuous on the nonempty compact set S, then both level sets $S_f^{\leqslant}(\lambda)$ and $S_f^{\geqslant}(\lambda)$ are closed for every $\lambda \in \mathbb{R}$. In this case we recover from $(C3)$ in Theorem 3.1 and Corollary 3.2 the conclusion of the classical Weierstrass Theorem.

Theorem 3.4 *Let $f\colon S \to \mathbb{R}$ be a function defined on a nonempty set $S \subseteq \mathbb{R}^n$. The following assertions are equivalent*:

1° *The minimization problem* (1.1) *has at most one optimal solution.*

2° *For all $x^1,\ x^2 \in S,\ x^1 \neq x^2,$ there exists $x^* \in S$ such that $f(x^*) < \max\{f(x^1), f(x^2)\}$.*

Proof. 1° $\Rightarrow$ 2°. Assume that $\operatorname{card}(\operatorname{argmin}_{x\in S} f(x)) \leq 1$ and suppose to the contrary that there exist two distinct points $x^1, x^2 \in S$ satisfying the inequality $f(x) \geq \max\{f(x^1), f(x^2)\}$ for every $x \in S$. We infer that $f(x^1) \leq f(x)$ and $f(x^2) \leq f(x)$ for any point $x \in S$, i.e., $x^1, x^2 \in \operatorname{argmin}_{x\in S} f(x)$, contradicting the hypothesis.

2° $\Rightarrow$ 1°. Assume that for every distinct points $x^1, x^2 \in S$ there exists $x^* \in S$ such that $f(x^*) < \max\{f(x^1), f(x^2)\}$, and suppose to the contrary that $\operatorname{card}(\operatorname{argmin}_{x\in S} f(x)) > 1$. Then we can choose $x^1, x^2 \in \operatorname{argmin}_{x\in S} f(x)$, $x^1 \neq x^2$. By hypothesis, we can find $x^* \in S$ such that $f(x^*) < \max\{f(x^1), f(x^2)\}$. We infer that $f(x^*) < f(x^1) = f(x^2) = \inf f(S) \leq f(x^*)$, a contradiction. □

Corollary 3.5 *Let $f\colon S \to \mathbb{R}$ be a function defined on a nonempty set $S \subseteq \mathbb{R}^n$. The following assertions are equivalent*:

1° *The maximization problem* (1.2) *has at most one optimal solution.*

2° *For all $x^1,\ x^2 \in S,\ x^1 \neq x^2,$ there exists $x^* \in S$ such that $f(x^*) > \min\{f(x^1), f(x^2)\}$.*

Proof. Follows by Theorem 3.4 applied to $-f$ in the role of f. □

Lecture 4

Convex sets and their extreme subsets/points

For any points $x, y \in \mathbb{R}^n$ we denote

$$\begin{aligned} [x,y] &:= \{(1-t)x + ty \mid t \in [0,1]\}; \\ ]x,y[&:= \{(1-t)x + ty \mid t \in]0,1[\}. \end{aligned}$$

Notice that if $x \neq y$, then $]x,y[= [x,y] \setminus \{x,y\}$; otherwise,if $x = y$, then $[x,y] =]x,y[= \{x\}$.

Definition 4.1 A set $S \subseteq \mathbb{R}^n$ is said to be *convex* if

$$[x,y] \subseteq S \quad \text{for all} \quad x, y \in S.$$

Remark 4.2 For any set $S \subseteq \mathbb{R}^n$ the following assertions are equivalent:

1° S is convex.

2° $(1-t)x + ty \in S,\ \forall\, x, y \in S,\ \forall\, t \in [0,1]$.

Example 4.3 By a *hyperplane* in $\mathbb{R}^n$ we mean any set of type

$$H(c, \lambda) := \{x \in \mathbb{R}^n \mid \langle c, x \rangle = \lambda\},$$

where $c \in \mathbb{R}^n \setminus \{0_n\}$ and $\lambda \in \mathbb{R}$. The sets

$$\begin{aligned} H^{\leqslant}(c,\lambda) &:= \{x \in \mathbb{R}^n \mid \langle c, x\rangle \leq \lambda\} \\ H^{\geqslant}(c,\lambda) &:= \{x \in \mathbb{R}^n \mid \langle c, x\rangle \geq \lambda\} \end{aligned}$$

are called *closed half-spaces* while the sets

$$\begin{aligned} H^{<}(c,\lambda) &:= \{x \in \mathbb{R}^n \mid \langle c, x\rangle < \lambda\} \\ H^{>}(c,\lambda) &:= \{x \in \mathbb{R}^n \mid \langle c, x\rangle > \lambda\} \end{aligned}$$

are called *open half-spaces*. It is a simple exercise to check that all hyperplanes, as well as closed and open half-spaces, are convex sets.

Proposition 4.4 *If $\mathscr{F}$ is a family of convex sets in $\mathbb{R}^n$, then the set $\bigcap_{S\in\mathscr{F}} S$ is also convex.*

Proof. Let $t \in [0,1]$. For any $S' \in \mathscr{F}$ we have

$$(1-t)\bigcap_{S\in\mathscr{F}} S + t\bigcap_{S\in\mathscr{F}} S \subseteq (1-t)S' + tS' \subseteq S',$$

hence $(1-t)\bigcap_{S\in\mathscr{F}} S + t\bigcap_{S\in\mathscr{F}} S \subseteq \bigcap_{S'\in\mathscr{F}} S' = \bigcap_{S\in\mathscr{F}} S$. Thus $\bigcap_{S\in\mathscr{F}} S$ is a convex set. $\square$

Definition 4.5 The *convex hull* of an arbitrary set $M \subseteq \mathbb{R}^n$ is defined by

$$\operatorname{conv} M := \bigcap\{S \subseteq \mathbb{R}^n \mid S \text{ is convex and } M \subseteq S\}.$$

Remark 4.6 $\operatorname{conv} M$ is a convex set as an intersection of a family of convex sets. Therefore, M is convex if and only if $M = \operatorname{conv} M$.

Definition 4.7 For any $k \in \mathbb{N}$, the set

$$\Delta_k := \{(t_1,\dots,t_k) \in (\mathbb{R}_+)^k \mid t_1 + \cdots + t_k = 1\}$$

is called the *standard simplex* of $\mathbb{R}^k$. It is easily seen that Δ_k is convex.

Definition 4.8 Given an arbitrary nonempty set $M \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is said to be a *convex combination* of elements of $M \subseteq \mathbb{R}^n$, if there exist $k \in \mathbb{N}^*$, $x^1,\dots,x^k \in M$, and $(t_1,\dots,t_k) \in \Delta_k$, such that $x = t_1x^1 + \cdots + t_kx^k$.

Theorem 4.9 (Characterization of the convex hull via convex combinations) *The convex hull of a nonempty set $M \subseteq \mathbb{R}^n$ admits the following representation:*

$$\operatorname{conv} M = \left\{\textstyle\sum_{i=1}^k t_i x^i \mid k \in \mathbb{N}^*,\, x^1,\dots,x^k \in M,\, (t_1,\dots,t_k) \in \Delta_k\right\}.$$

Proof. Denote by

$$C(M) := \left\{ \sum_{i=1}^{k} t_i x^i \mid k \in \mathbb{N}^*,\ x^1, \dots, x^k \in M,\ (t_1, \dots, t_k) \in \Delta_k \right\}. \tag{4.1}$$

For the equality $\operatorname{conv} M = C(M)$ it suffices to show that the following conditions are fulfilled:

(i) $M \subseteq C(M)$;

(ii) $C(M)$ is convex;

(iii) $C(M) \subseteq S$ for every convex set $S \subseteq \mathbb{R}^n$ with $M \subseteq S$.

Condition (i) holds, since one obtains in $C(M)$ the elements of M considering $k = 1$.

In order to prove (ii) pick $x, y \in C(M)$ and $\alpha \in [0, 1]$. Then there exist $k, \ell \in \mathbb{N}^*$, $x^1, \dots, x^k, y^1, \dots, y^\ell \in M$, $(t_1, \dots, t_k) \in \Delta_k$ and $(s_1, \dots, s_\ell) \in \Delta_\ell$ such that $x = \sum_{i=1}^{k} t_i x^i$ and $y = \sum_{i=1}^{\ell} s_i y^i$. Thus

$$(1-\alpha)x + \alpha y = \sum_{i=1}^{k} (1-\alpha) t_i x^i + \sum_{i=1}^{\ell} \alpha s_i y^i.$$

Since $\sum_{i=1}^{k}(1-\alpha)t_i + \sum_{i=1}^{\ell} \alpha s_i = (1-\alpha)\sum_{i=1}^{k} t_i + \alpha \sum_{i=1}^{\ell} s_i = 1 - \alpha + \alpha = 1$, it follows that $(1-\alpha)x + \alpha y$ is also a convex combination of elements of M, that is, it belongs to $C(M)$. Thus (ii) holds.

For proving (iii) consider a convex subset $S \subseteq \mathbb{R}^n$ such that $M \subseteq S$. We get the inclusion $C(M) \subseteq S$ by performing an induction argument. We prove that proposition

$\mathscr{P}(k)$: "$\sum_{i=1}^{k} t_i x^i \in S,\ \forall\, x^1, \dots, x^k \in M,\ \forall\, (t_1, \dots, t_k) \in \Delta_k$"

is true for every $k \in \mathbb{N}^*$. Obviously $\mathscr{P}(1)$ is true (since $M \subseteq S$). Assume now that $\mathscr{P}(h)$ is true for a natural number $h \in \mathbb{N}^*$. We are going to prove that $\mathscr{P}(h+1)$ is also true. Let $x^1, \dots, x^h, x^{h+1} \in M$ and $(t_1, \dots, t_h, t_{h+1}) \in \Delta_{h+1}$. Without loss of generality we may assume that $t := \sum_{i=1}^{h} t_i > 0$ (otherwise $\mathscr{P}(h+1)$ obviously would be true). Then $t + t_{h+1} = 1$ and $\frac{1}{t}(t_1 + \cdots + t_h) = 1$. By the induction hypothesis and using the convexity of S, we get

$$\sum_{i=1}^{h+1} t_i x^i = t\left(\sum_{i=1}^{h} \frac{t_i}{t} x^i\right) + (1-t)x^{h+1} \in S.$$

Hence $\mathscr{P}(h+1)$ is true. It follows that $\mathscr{P}(k)$ is true for every $k \in \mathbb{N}^*$. Thus $C(M) \subseteq S$. □

Theorem 4.12 (Carathéodory) *If S is a nonempty subset of $\mathbb{R}^n$, then every point $x \in \operatorname{conv} S$ can be expressed as a convex combination of at most $n+1$ elements of S.*

Definition 4.13 Let $S \subseteq \mathbb{R}^n$ be a convex set. We say that $E \subseteq S$ is an *extremal subset* (or *face*) of S if E is convex and

$$\forall\ x, y \in S\ :\ ]x, y[\cap E \neq \emptyset \quad \Rightarrow \quad [x, y] \subseteq E.$$

An element $v \in S$ is called *extreme point* (or *vertex*) of S if the singleton $\{v\}$ is an extremal subset of S. The set of all extreme points of of S is denoted by $\mathrm{ext}S$, i.e.,

$$\mathrm{ext}S := \{v \in S \mid \forall\ x, y \in S\ :\ v \in]x, y[\Rightarrow\ x = y = v\}.$$

Theorem 4.14 (Characterization of extreme points) *Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set . For any point $v \in S$ the following assertions are equivalent*:

1° $v \in \mathrm{ext}S$.

2° *There are no distinct points $x, y \in S$ such that $v = \frac{1}{2}(x + y)$.*

Lecture 5

Convex functions

Definition 5.1 Let $f : S \to \mathbb{R}$ be a function defined o a nonempty set $S \subseteq \mathbb{R}^n$. We say that f is a *convex function* if its domain S is a convex set and

$$f((1-t)x^1 + tx^2) \leq (1-t)f(x^1) + t\,f(x^2),\ \forall\, x^1, x^2 \in S,\ \forall\, t \in [0,1].$$

Example 5.2 (Distance function) Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. Consider the distance function w.r.t. C, $d_C \colon \mathbb{R}^n \to \mathbb{R}$, defined by

$$d_C(x) \coloneqq \inf\{||x - c|| \ : \ c \in C\},\ \forall\, x \in \mathbb{R}^n.$$

We will prove that this function is convex. In particular, for any $a \in \mathbb{R}^n$, the distance function $d(\cdot, a) \colon \mathbb{R}^n \to \mathbb{R}$, defined by

$$d(x, a) \coloneqq \|x - a\|,\ \forall\, x \in \mathbb{R}^n,$$

is convex (in this case the set $C ::= \{a\}$ is a singleton).

Indeed, let $x^1, x^2 \in \mathbb{R}^n$ and $t \in [0,1]$. Consider two sequences $(c^k)_{k\in\mathbb{N}}$ and $(\tilde{c}^k)_{k\in\mathbb{N}}$ of points of C such that

$$\lim_{k\to\infty} \|x^1 - c^k\| = d_C(x^1), \quad \lim_{k\to\infty} \|x^2 - \tilde{c}^k\| = d_C(x^2). \tag{5.1}$$

Since C is a convex set, it follows that for any $k \in \mathbb{N}$ we have $(1-t)c^k + t\tilde{c}^k \in C$, hence

$$\begin{aligned} d_C((1-t)x^1 + tx^2) &\leq \|(1-t)x^1 + tx^2 - (1-t)c^k - t\tilde{c}^k\| \\ &\leq (1-t)\|x^1 - c^k\| + t\|x^2 - \tilde{c}^k\|. \end{aligned}$$

Letting $k \to \infty$ and recalling (5.1), we infer

$$d_C((1-t)x^1 + tx^2) \leq (1-t)d_C(x^1) + td_C(x^2).$$

Thus d_C is a convex function.

Definition 5.4 Let $f\colon M \to \mathbb{R}$ be a function defined on a nonempty set $M \subset \mathbb{R}^n$. The set

$$\mathrm{epi}f \;\; := \;\; \{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R} \mid x \in M,\ f(x) \leq \lambda\}$$

is called the *epigraph* of$f$.

Theorem 5.5 (Characterization of convex functions by means of their epigraph) *Let $f : S \to \mathbb{R}$ be a function defined on a nonempty convex set $S \subseteq \mathbb{R}^n$. Then the following assertions are equivalent:*

1° *Function f is convex.*

2° *The epigraph of f (i.e., $\mathrm{epi}f$) is a convex set (in the space $\mathbb{R}^n \times \mathbb{R}$).*

Proof. $1° \Rightarrow 2°$. Assume that 1° holds and consider any points $(x^1, \lambda_1), (x^2, \lambda_2) \in \mathrm{epi}f$ and any number $t \in [0, 1]$. Then we have $f(x^1) \leq \lambda_1$ and $f(x^2) \leq \lambda_2$. By 1° it follows that

$$f((1-t)x^1 + tx^2) \leq (1-t)f(x^1) + tf(x^2) \leq (1-t)\lambda_1 + t\lambda_2$$

which shows that

$$((1-t)x^1 + tx^2, (1-t)\lambda_1 + t\lambda_2) = (1-t)(x^1, \lambda_1) + t(x^2, \lambda_2) \in \mathrm{epi} f.$$

Thus epi$f$ is a convex set, i.e., 3° holds.

2° $\Rightarrow$ 1°. Assume that 2° holds and consider any $x^1, x^2 \in S$ and $t \in [0, 1]$. Since $(x^1, f(x^1)), (x^2, f(x^2)) \in \mathrm{epi} f$, we have

$$((1-t)x^1 + tx^2, (1-t)f(x^1) + tf(x^2)) = (1-t)(x^1, f(x^1)) + t(x^2, f(x^2)) \in \mathrm{epi} f,$$

hence $f((1-t)x^1 + tx^2) \le (1-t)f(x^1) + tf(x^2)$. Thus function f is convex, i.e., 1° holds. □

Theorem 5.6 (Characterization of differentiable convex functions) *Let $S \subseteq \mathbb{R}^n$ be a nonempty convex set and let $f : S \to \mathbb{R}$ be a differentiable function. The following assertions are equivalent*:

1° *Function f is convex.*

2° $\langle x - x^0, \nabla f(x^0)\rangle \le f(x) - f(x^0)$ *for any $x, x^0 \in S$.*

3° $\langle x^1 - x^2, \nabla f(x^1) - \nabla f(x^2)\rangle \ge 0$ *for any $x^1, x^2 \in S$.*

Theorem 5.7 (Characterization of twice differentiable convex functions) *Assume that $S \subseteq \mathbb{R}^n$ is a nonempty open convex set and let $f : S \to \mathbb{R}$ be a twice differentiable function. Then the following assertions are equivalent:*

1° *Function f is convex.*

2° *For any point $x^0 \in S$, the Hessian matrix $\nabla^2 f(x^0) = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x^0)\right)_{1 \le i,j \le n}$ is positive semidefinite, i.e.,*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) x_i x_j \ge 0, \ \forall \, x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

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Lecture 6

Local/global extrema of convex functions

Definition 6.1 Let $f : S \to \mathbb{R}$ be a function defined on a nonempty set $S \subseteq \mathbb{R}^n$. An element $x^0 \in S$ is said to be a:

- *local minimum point* of f, if there exists a neighborhood $V \in \mathcal{V}(x^0)$ of the point x^0 such that
$$f(x^0) \le f(x),\ \forall\, x \in V \cap S.$$
- *local maximum point* of f, if there exists a neighborhood $V \in \mathcal{V}(x^0)$ of the point x^0 such that
$$f(x^0) \ge f(x),\ \forall\, x \in V \cap S.$$
- *global minimum point* of f (or, simply, *minimum point* of f), if x^0 is a minimum point of f w.r.t. S in the sense of Definition 1.1, i.e., $x^0 \in \operatorname*{argmin}_{x \in S} f(x)$, which actually means that
$$f(x^0) \le f(x),\ \forall\, x \in S\, (= V \cap S \text{ with } V := \mathbb{R}^n \in \mathcal{V}(x^0)).$$
- *global maximum point* of f (or, simply, *maximum point* of f), if x^0 is a maximum point of f w.r.t. S in the sense of Definition 1.1, i.e., $x^0 \in \operatorname*{argmax}_{x \in S} f(x)$, which actually means that
$$f(x^0) \ge f(x),\ \forall\, x \in S\, (= V \cap S \text{ with } V := \mathbb{R}^n \in \mathcal{V}(x^0)).$$

The local/global minimum points and maximum points are generically called *local/global extremum points* (or *local/global extrema*) of f.

Remark 6.2 For any point $a \in \mathbb{R}$ and any real number $r > 0$, we denote by $B(a, r)$ the open Euclidean ball centered at a with radius r, i.e.,
$$B(a, r) := \{x \in \mathbb{R}^n \mid \|x - a\| < r\}.$$
It is easy to see that:

- x^0 is a local minimum point of f if and only if there exists $\varepsilon > 0$ such that

$$f(x^0) \leq f(x), \ \forall\, x \in B(x^0, \varepsilon) \cap S.$$

- x^0 is a local maximum point of f if and only if there exists $\varepsilon > 0$ such that

$$f(x^0) \geq f(x), \ \forall\, x \in B(x^0, \varepsilon) \cap S.$$

Remark 6.3 Every global minimum (resp. maximum) point of f is a local minimum (resp. maximum) point of f. The converse is not true, as the following example shows.

Example 6.4 Consider the function $f : S = [0, 3] \to \mathbb{R}$, defined by

$$f(x) := \lfloor x \rfloor, \ \forall\, x \in [0, 3],$$

where $\lfloor x \rfloor$ denotes the integer part (floor) of x. It is a simple exercise to check that:

- The set of all global minimum points of f is $\underset{x \in S}{\operatorname{argmin}} f(x) = [0, 1[$.
- The set of all global maximum points of f is $\underset{x \in S}{\operatorname{argmax}} f(x) = \{3\}$.
- The set of all local minimum points of f is $[0, 1[\cup]1, 2[\cup]2, 3[$.
- The set of all local maximum points of f is $[0, 3]$.

Lemma 6.5 (Structure of lower level sets of convex functions) *Let $f : S \to \mathbb{R}$ be a convex function, defined on a nonempty convex set $S \subseteq \mathbb{R}^n$. Then, for any $\lambda \in \mathbb{R}$, the level set $S_f^{\leqslant}(\lambda)$ is convex.*

Proof. Let $\lambda \in \mathbb{R}$. Recall that

$$S_f^{\leqslant}(\lambda) := \{x \in S \mid f(x) \leq \lambda\},$$

according to Definition 2.1 of Lecture 2. Let $x^1, x^2 \in S_f^{\leqslant}(\lambda)$ and $t \in [0, 1]$. It follows that $f(x^1) \leq \lambda$ and $f(x^2) \leq \lambda$, hence

$$(1 - t) f(x^1) + t f(x^2) \leq (1 - t)\lambda + t\lambda = \lambda.$$

Since function f is convex, the following inequality also holds:

$$f((1 - t)x^1 + tx^2) \leq (1 - t) f(x^1) + t f(x^2).$$

Therefore, we have $f((1 - t)x^1 + tx^2) \leq \lambda$. Taking into account that S is a convex set, we deduce that $(1 - t)x^1 + tx^2 \in S_f^{\leqslant}(\lambda)$. Thus the level set $S_f^{\leqslant}(\lambda)$ is convex. □

Theorem 6.6 (Structure of the set of minimum points of convex functions) *Let $f : S \to \mathbb{R}$ be a convex function, defined on a nonempty convex set $S \subseteq \mathbb{R}^n$. Then, the set of all global minimum points of f, i.e., $\underset{x \in S}{\operatorname{argmin}}\, f(x)$, is convex.*

Proof. In view of Remark 1.8, we distinguish two cases.

Case 1: $\inf f(S) = -\infty$.

In this case, the set $\operatorname*{argmin}_{x \in S} f(x) = \emptyset$ is obviously convex.

Case 2: $\inf f(S) =: \lambda \in \mathbb{R}$.

In this case, the set $\operatorname*{argmin}_{x \in S} f(x) = S_f^{\leqslant}(\lambda)$ is convex, by Lemma 6.5. □

Theorem 6.7 (Coincidence of local and global minimum points of convex functions) *Let $f : S \to \mathbb{R}$ be a convex function, defined on a nonempty convex set $S \subseteq \mathbb{R}^n$. For any point $x^0 \in S$, the following assertions are equivalent:*

1° *x^0 is a global minimum point of f.*

2° *x^0 is a local minimum point of f.*

Lemma 6.8 (Fermat's necessary optimality condition) *Let $f: S \to \mathbb{R}$ be a function, defined on a nonempty set $S \subseteq \mathbb{R}^n$ and let $x^0 \in S$ be a local extremum point of f. If $x^0 \in \operatorname{int} S$ and f is partially derivable at x^0, then x^0 is a stationary point of f, i.e., $\nabla f(x^0) = 0_n$.*

Theorem 6.9 (Characterization of minimum points of differentiable convex functions) *Let $f : S \to \mathbb{R}$ be a differentiable convex function, defined on a nonempty open convex set $S \subseteq \mathbb{R}^n$. For any point $x^0 \in S$ the following assertions are equivalent:*

1° *x^0 is a global minimum point of f.*

2° *x^0 is a stationary point of f.*

Proof. The implication $1^\circ \Rightarrow 2^\circ$ holds by Fermat's optimality condition (Lemma 6.8) even when f is not convex.

For proving the implication $2^\circ \Rightarrow 1^\circ$ we will use Theorem 5.6. More precisely, since function f is differentiable and convex, we have $\langle x - x^0, \nabla f(x^0)\rangle \leq f(x) - f(x^0)$ for all $x \in S$. On the other hand, by hypothesis 2° we have $\nabla f(x^0) = 0_n$. Thus we have $0 \leq f(x) - f(x^0)$, i.e., $f(x^0) \leq f(x)$, for all $x \in S$, which actually means 1°. $\square$

Lecture 7

Linear optimization; Duality; Primal/dual feasible bases and optimal bases

Definition 7.1 By a *constrained linear optimization problem in standard form* we mean any minimization problem of type:

$$(P)\begin{cases} \text{Minimize } f(x) = \langle c, x\rangle \\ Ax = b \\ x \geq 0_n, \end{cases} \tag{7.1}$$

where $m, n \in \mathbb{N}^*$, $A \in \mathcal{M}_{m,n}(\mathbb{R})$ with $\operatorname{rank} A = m < n$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Of course, such a problem may be written explicitly as:

$$(P)\begin{cases} \text{Minimize } f(x) = c_1x_1 + \cdots + c_nx_n \\ a_{11}x_1 + \cdots + a_{1n}x_n \quad = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \quad = b_m \\ x_1, \dots, x_n \geq 0. \end{cases}$$

The objective function of problem (P) is a linear function, namely $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$f(x) := \langle c, x\rangle = c_1x_1 + \cdots + c_nx_n,\ \forall\, x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{7.2}$$

while the set of feasible points of (P) is the standard polyhedral set (see Definition **??**):

$$S := S(A, b) = \{x \in \mathbb{R}^n \mid Ax = b \text{ and } x \geqq 0_n\}. \tag{7.3}$$

Thus the set of optimal solutions of problem (P) is

$$\operatorname*{argmin}_{x\in S} f(x) = \{x^0 \in S \mid f(x^0) \leq f(x),\ \forall\, x \in S\}.$$

Definition 7.2 The optimization problem

$$(D)\ \begin{cases} \text{Maximize } f^*(y) := \langle b, y\rangle \\ \\ A^\top y \leqq c \end{cases} \tag{7.4}$$

is called the *dual of problem* (P). Its explicit form is:

$$(D)\ \begin{cases} \text{Maximize } f^*(y) = b_1y_1 + \cdots + b_my_m \\ \\ a_{11}y_1 + \cdots + a_{m1}y_m \quad \leq c_1 \\ \vdots \\ a_{1n}y_1 + \cdots + a_{mn}y_n \quad \leq c_n. \end{cases}$$

The objective function of the dual problem (D) is a linear function, namely $f^* : \mathbb{R}^m \to \mathbb{R}$, defined by

$$f^*(y) := b_1y_1 + \cdots + b_my_m,\ \forall\, y = (y_1, \ldots, y_m) \in \mathbb{R}^m, \tag{7.5}$$

while the set of feasible points of (D) is the polyhedral set

$$S^* := \{y \in \mathbb{R}^m \mid A^\top y \leqq c\}. \tag{7.6}$$

Thus the set of optimal solutions of the dual problem (D) is

$$\operatorname*{argmax}_{y\in S^*} f^*(y) = \{y^0 \in S^* \mid f^*(y^0) \geq f^*(y),\ \forall\, y \in S^*\}.$$

Remark 7.3 In order to unify the terminology, in what follows the initial problem (P) will be called the *primal problem*.

Theorem 7.4 (Weak Duality) *For any feasible points $x \in S$ and $y \in S^*$ of the problems (P) and (D), the following inequality holds:*

$$f(x) \geq f^*(y).$$

Proof. Let $x \in S$ and $y \in S^*$. By (7.2), (7.3), (7.5) and (7.6), it is easy to deduce that:

$$f^*(y) = \langle b, y\rangle = \langle Ax, y\rangle = \langle x, A^\top y\rangle \leq \langle x, c\rangle = f(x). \qquad \square$$

Theorem 7.5 (Strong Duality) *If some feasible points $x^0 \in S$ and $y^0 \in S^*$ of the problems (P) and (D) satisfy the property*

$$f(x^0) = f^*(y^0),$$

then the following assertions hold:

1° *x^0 is an optimal solution of the primal problem (P), i.e., $x^0 \in \operatorname*{argmin}_{x \in S} f(x)$.*

2° *y^0 is an optimal solution of the dual problem (D), i.e., $y^0 \in \operatorname*{argmax}_{y \in S^*} f^*(y)$.*

Proof. Let $x^0 \in S$ and $y^0 \in S^*$ be two feasible points such that $f(x^0) = f^*(y^0)$.

In order to prove 1°, we have to show that

$$f(x^0) \leq f(x), \ \forall\, x \in S.$$

To this aim, consider an arbitrary point $x \in S$. By applying the Weak Duality Theorem 7.4 (for y^0 in the role of y), we deduce that

$$f(x) \geq f^*(y^0) = f(x^0).$$

Similarly, in order to prove 2° we can show that

$$f^*(y^0) \geq f^*(y), \ \forall\, y \in S^*.$$

Indeed, for any $y \in S^*$ we can apply again the Weak Duality Theorem 7.4 (this time for x^0 in the role of x), which yields

$$f^*(y) \leq f(x^0) = f^*(y^0). \qquad \square$$

Corollary 7.6 (on the unboundedness of objective functions) *The following hold:*

1° *If the objective function f of the primal problem (P) is unbounded from below on the set of feasible points S, then the dual problem (D) has no feasible points, i.e., $S^* = \emptyset$. In other words,*

$$\inf f(S) = -\infty \quad \Longrightarrow \quad S^* = \emptyset.$$

2° *If the objective function f^* of the dual problem (D) is unbounded from above on the set of feasible points S^*, then the primal problem (P) has no feasible points, i.e., $S = \emptyset$. In other words,*

$$\sup f^*(S^*) = +\infty \quad \Longrightarrow \quad S = \emptyset.$$

Definition 7.7 An ordered set B consisting of m linearly independent columns of A is called a *basis* of problem (P) (or a basis of the polyhedral set S). For such a set B, we define the *set of basic indices*

$$\mathcal{B}_B := \{j \in \{1, \ldots, n\} \mid A^j \in B\}$$

and the *set of nonbasic indices*

$$\mathcal{N}_B := \{j \in \{1, \ldots, n\} \mid A^j \notin B\} = \{1, \ldots, n\} \setminus \mathcal{B}_B.$$

Given a basis B of problem (P), the coordinates of $b \in \mathbb{R}^m$ w.r.t. the basis B are denoted by $\alpha_{0j} \in \mathbb{R}$ $(j \in \mathcal{B}_B)$, i.e.,

$$b = \sum_{j \in \mathcal{B}_B} \alpha_{0j} A^j. \tag{7.7}$$

Similarly, every column of the matrix A that does not belong to the basis B, i.e., A^i with $i \in \mathcal{N}_B$, admits a unique representation as a linear combination of the columns of B, namely

$$A^i = \sum_{j \in \mathcal{B}_B} \alpha_{ij} A^j, \ i \in \mathcal{N}_B, \tag{7.8}$$

where the numbers $\alpha_{ij} \in \mathbb{R}$ with $j \in \mathcal{B}_B$ represent the coordinates of A^i w.r.t. B.

By means of these coordinates $\alpha_{ij} \in \mathbb{R}$, $(i, j) \in \mathcal{N}_B \times \mathcal{B}_B$, we introduce some technical numbers α_{i0}, $i \in \mathcal{N}_B$, as follows:

$$\alpha_{i0} := \sum_{j \in \mathcal{B}_B} \alpha_{ij} c_j - c_i, \ i \in \mathcal{N}_B. \tag{7.9}$$

Also, we define the number

$$\alpha_{00} := \sum_{j \in \mathcal{B}_B} \alpha_{0j} c_j. \tag{7.10}$$

We associate to B a point $x^B = (x_1^B, \ldots, x_n^B) \in \mathbb{R}^n$, given by

$$x_j^B := \begin{cases} \alpha_{0j} & \text{dacă} \quad j \in \mathcal{B}_B \\ 0 & \text{dacă} \quad j \in \mathcal{N}_B, \end{cases} \tag{7.11}$$

as well as a point $y^B = (y_1^B, \ldots, y_m^B) \in \mathbb{R}^m$, defined as the unique solution of a Cramer type linear system of equations:

$$\langle A^j, y^B \rangle = c_j, \ \forall j \in \mathcal{B}_B. \tag{7.12}$$

Definition 7.8 A basis B of the standard polyhedral set $S = S(A, b)$ is called:

- *primal feasible basis* (*p.f.b.*), if

$$\alpha_{0j} \geq 0, \ \forall\, j \in \mathcal{B}_B.$$

- *dual feasible basis* (*d.f.b.*), if

$$\alpha_{i0} \leq 0, \ \forall\, i \in \mathcal{N}_B.$$

- *optimal basis*, if B is both primal feasible and dual feasible, i.e.,

$$\begin{cases} \alpha_{0j} \geq 0, & \forall\, j \in \mathcal{B}_B \\ \alpha_{i0} \leq 0, & \forall\, i \in \mathcal{N}_B. \end{cases}$$

This terminology is motivated by the following three results.

Theorem 7.9 (Characterization of primal feasible bases) *For any basis B of the standard polyhedral set $S = S(A, b)$ the following assertions are equivalent:*

1° *B is a primal feasible basis.*

2° *x^B is a feasible point of the primal problem (P), i.e., $x^B \in S$.*

Theorem 7.10 (Characterization of dual feasible bases) *For any basis B of the standard polyhedral set $S = S(A, b)$ the following assertions are equivalent:*

1° *B is a a dual feasible basis.*

2° *y^B is a feasible point of the dual problem (D), i.e., $y^B \in S^*$.*

Corollary 7.11 (on the optimal solutions generated by optimal bases) *If B is an optimal basis of the standard polyhedral set $S = S(A, b)$, then the following assertions hold:*

1° $f(x^B) = f^*(y^B) = \alpha_{00}$.

2° x^B *is an optimal solution of the primal problem* (P), *i.e.,* $x^B \in \underset{x \in S}{\operatorname{argmin}} f(x)$.

3° y^B *is an optimal solution of the dual problem* (D), *i.e.,* $y^B \in \underset{y \in S^*}{\operatorname{argmax}} f^*(y)$.

Since in general the polyhedral set $S = S(A, b)$ possesses several bases, we associate to each basis B of $S = S(A, b)$ a so-called *Simplex tableau*, that contains all numbers defined by (7.7), (7.8), (7.9) and (7.10):

		$\dots$	c_j	$\dots$	
	Current basis no.	$\dots$	A^j	$\dots$	Test d.f.b.
$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_i	A^i	$\dots$	α_{ij}	$\dots$	α_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
	Test p.f.b.	$\dots$	α_{0j}	$\dots$	α_{00}

Lecture 8

Primal SIMPLEX Algorithm

The Primal SIMPLEX Algorithm is an implementable numerical method conceived to solve the primal optimization problem

$$(P) \begin{cases} f(x) := \langle c, x\rangle \longrightarrow \min \\ Ax = b \\ x \geq 0_n \end{cases}$$

under the assumptions of Definition 7.1. We will study two variants of the SIMPLEX algorithm. The so-called *Primal SIMPLEX Algorithm* consists in changing succesively certain primal feasible bases of the feasible set $S := S(A, b)$, such that after a finite number of iterations we obtain:

• an optimal solution of problem (P) and the optimal value of the objective function f, i.e., its minimal value on S.

or

• the conclusion that f is unbounded from below on S, hence the primal problem (P) has no optimal solutions.

Pseudocode of the Primal SIMPLEX Algorithm

`Step 1.` Choose a primal feasible basis B of the feasible set $S = S(A, b)$ of the primal problem (P) introduced in (7.1).

`Step 2.` Determine the numbers $\alpha_{ij} \in \mathbb{R}$, $(i, j) \in \mathcal{N}_B \times \mathcal{B}_B$ and $\alpha_{0j} \in \mathbb{R}$, $j \in \mathcal{B}_B$ of b, given by (7.8) and (7.7).

`Step 3.` Compute the numbers α_{i0}, $i \in \mathcal{N}_B$ defined by (7.9) and α_{00} defined by (7.10).

`Step 4.` Study the sign of α_{i0} for all $i \in \mathcal{N}_B$.

(a) If $\alpha_{i0} \leq 0$ for all $i \in \mathcal{N}_B$, then conclude that the point $x^0 := x^B$, given by (7.11), is an optimal solution of the primal problem (P), and α_{00} is the optimal (minimal) value of f on S. STOP.

(b) If there exists $i \in \mathcal{N}_B$ such that $\alpha_{i0} > 0$, then generate the set of indices

$$\mathcal{N}_B^+ = \{i \in \mathcal{N}_B \mid \alpha_{i0} > 0\}.$$

Step 5. For every $i \in \mathcal{N}_B^+$ study the sign of α_{ij} for all $j \in \mathcal{B}_B$.

(a) If there exists $i \in \mathcal{N}_B^+$ such that $\alpha_{ij} \leq 0$ for all $j \in \mathcal{B}_B$, then conclude function f is unbounded from below on S, hence the primal problem (P) has no optimal solutions. STOP.

(b) If for each $i \in \mathcal{N}_B^+$ there is $j \in \mathcal{B}_B$ such that $\alpha_{ij} > 0$, then choose an index $h \in \mathcal{N}_B^+$. For instance, h can be chosen such that $h \in \mathcal{N}_B$ and

$$\alpha_{h0} = \max\{\alpha_{i0} \mid i \in \mathcal{N}_B\}.$$

Step 6. Choose an index $k \in \mathcal{B}_B$ such that $\alpha_{hk} > 0$ and

$$\frac{\alpha_{0k}}{\alpha_{hk}} = \min\left\{ \frac{\alpha_{0j}}{\alpha_{hj}} \,\middle|\, j \in \mathcal{B}_B \text{ s.t. } \alpha_{hj} > 0 \right\}.$$

Step 7. Consider a new basis $B := (B \setminus \{A^k\}) \cup \{A^h\}$, obtained from the previous one by replacing A^k by A^h, and go to Step 2.

Remark 8.1 Due to the specific choice of the indices h and k at Step 5 (b) and Step 6, it can be shown that the new basis B, constructed at Step 7, is a primal feasible basis of S. Moreover, the objective function's outcome at the new point x^B, namely $\alpha_{00} = f(x^B)$, decreases in comparison to the previous one.

In order to solve small size optimization problems by applying the Primal SIMPLEX Algorithm (without implementing it on a computer), the numbers $\alpha_{ij} \in \mathbb{R}$ with $i \in \mathcal{N}_B \cup \{0\}$ and $j \in \mathcal{B}_B \cup \{0\}$, involved at `Steps` 2 and 3, will be arranged in a Simplex tableau (as described at the end of Lecture 7):

		$\dots$	c_j	$\dots$	
	Current basis no.	$\dots$	A^j	$\dots$	Test d.f.b.
$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_i	A^i	$\dots$	α_{ij}	$\dots$	α_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
	Test p.f.b.	$\dots$	α_{0j}	$\dots$	α_{00}

Consider the particular case when the initial basis B, chosen at `Step 1`, is the canonical basis of $\mathbb{R}^m$. Then, the numbers α_{ij} at α_{0j} involved in `Step 2` are already know, since they are the Cartesian coordinates of the non-basic columns A^i and of b, respectively. Notice that the canonical basis B is a primal feasible basis (p.f.b.) if and only if the coordinates of b satisfy the condition $\alpha_{0j} = b_j \geq 0$ for every basic index j (i.e., all numbers located in the horizontal section "Test p.f.b."of the Simplex tableau are greater than or equal to zero).

At `Step 3` all numbers α_{i0} şi α_{00} can be computed easily by writing the coefficients $c_1, \dots, c_n$ of the objective function f outside the Simplex tableau, next to the corresponding columns $A^1, \dots, A^n$ of matrix A. More precisely, the coefficients c_j labelled by basic indices are written above the Simplex tableau, while the coefficients c_i labelled by nonbasic indices are written on the left side of the Simplex tableau. In this way, the formulae (7.9) and (7.10) admit a comprehensive visual interpretation.

The aim of `Step 4` is to test whether B is a dual feasible basis (d.f.b.). The following wo cases may occur:

`(a)` If $\alpha_{i0} \leq 0$ for all nonbasic indices i (i.e., all numbers located in the vertical section "Test d.f.b." are less than or equal to zero), then B is a dual feasible basis, hence an optimal one (since it is already primal feasible, by `Step 1`). In this case, we conclude that an optimal solution of the primal problem (P) is the point

$$x^0 = (\dots, x_i^0 = 0, \dots, x_j^0 = \alpha_{0j}, \dots),$$

whose coordinates x_j^0 labelled by basic indices can be recoverd from the horizontal section "Test p.f.b." while the optimal (minimal) value of the objective function f on S is

$$\min f(S) = \alpha_{00},$$

that can be found in the last (bottom, right) cell of the Simplex tableau.

`(b)` If there exists a number $\alpha_{i0} > 0$ labelled by a non-basic index i (i.e., the vertical section "Test d.f.b." contains at least one positive number), then we consider the set $\mathcal{N}_B^+$ (consisting of all non-basic indices that correspond to positive numbers located in the vertical section "Test d.f.b.").

At `Step 5` we study whether for each $\alpha_{i0} > 0$ with $i \in \mathcal{N}_B$ there exists some $\alpha_{ij} > 0$ with $j \in \mathcal{B}_B$ (i.e, we investigate if on the left side of each positive number α_{i0} from the vertical section "Test d.f.b." there is at least one positive number α_{ij}). Two cases may occur:

(a) If there exists a number $\alpha_{i0} > 0$ with $i \in \mathcal{N}_B$ such that $\alpha_{ij} \leq 0$ for all $j \in \mathcal{B}_B$ (i.e., there exists a positive number α_{i0} in the vertical section "Test d.f.b." on the left side of which none of the numbers α_{ij} with $j \in \mathcal{B}_B$ is positive), then we conclude that the objective function f is unbounded from below on S, hence the primal problem (7.1) has no optimal solutions. In other words, in this case we have

$$\inf f(S) = -\infty \quad \text{and} \quad \operatorname*{argmin}_{x \in S} f(x) = \emptyset.$$

(b) If for each $i \in \mathcal{N}_B^+$ there exists $j \in \mathcal{B}_B$ such that $\alpha_{ij} > 0$ (i.e, on the left side of each positive number α_{i0} from the vertical section "Test d.f.b." there exists at least one positive number α_{ij}), then we choose a non-basic index h that corresponds to any positive number α_{h0} from the vertical section "Test d.f.b.", as for instance the largest number of this section:

$$\alpha_{h0} = \max\{\alpha_{i0} \mid i \in \mathcal{N}_B\}.$$

The Simplex tableau's row that corresponds to h is called the *pivot row* and is marked by an asterisk $*$ on the right side of the tableau.

Within `Step 6` we choose a basic index k such that $\alpha_{hk} > 0$ and

$$\frac{\alpha_{0k}}{\alpha_{hk}} = \min\left\{\frac{\alpha_{0j}}{\alpha_{hj}} \;\middle|\; j \in \mathcal{B}_B \text{ and } \alpha_{hj} > 0\right\}.$$

Notice that all numerators of the ratios $\dfrac{\alpha_{0j}}{\alpha_{hj}}$ involved in the formula above are located in the horizontal section "Test p.f.b." while the corresponding denomitors are the positive numbers in the pivot row. In practice, it will be convenient to write these ratios below the Simplex tableau. The smallest ratio $\dfrac{\alpha_{0k}}{\alpha_{hk}}$ will be marked by an asterisk $*$ and the corresponding tabeau's column will be called the *pivot column*.

Before going to the next step, it is recommended to highlight (encircle) the number α_{hk}, which represents the so-called *pivot*.

		$\dots$	c_j	$\dots$	c_k	$\dots$		
	Current basis	$\dots$	A^j	$\dots$	A^k	$\dots$	Test d.f.b.	
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
c_i	A^i	$\dots$	α_{ij}	$\dots$	α_{ik}	$\dots$	α_{i0}	
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
c_h	A^h	$\dots$	α_{hj}	$\dots$	$\boxed{\alpha_{hk}}$	$\dots$	α_{h0}	$*$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
	Test p.f.b.	$\dots$	α_{0j}	$\dots$	α_{0k}	$\dots$	α_{00}	
			$\dfrac{\alpha_{0j}}{\alpha_{hj}}$		$\dfrac{\alpha_{0k}}{\alpha_{hk}}$ $*$			

At `Step 7` we construct a new basis, obtained from the previous one by replacing A^k with A^h (notice that these elements belong to the pivot column and the pivot row, respectively). Then we go to `Step 2`. The Simplex tableau associated to the new basis can be obtained from the previous tableau by performing a so-called Gauss-Jordan transform:

Current basis	$\dots$	A^j	$\dots$	A^h	$\dots$	Test d.f.b.
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^i	$\dots$	$\dfrac{\alpha_{ij}\alpha_{hk} - \alpha_{hj}\alpha_{ik}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{ik}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{i0}\alpha_{hk} - \alpha_{ik}\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^k	$\dots$	$-\dfrac{\alpha_{hj}}{\alpha_{hk}}$	$\dots$	$\dfrac{1}{\alpha_{hk}}$	$\dots$	$-\dfrac{\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
Test p.f.b.	$\dots$	$\dfrac{\alpha_{0j}\alpha_{hk} - \alpha_{0k}\alpha_{hj}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{0k}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{00}\alpha_{hk} - \alpha_{0k}\alpha_{h0}}{\alpha_{hk}}$

Lecture 9

Dual SIMPLEX Algorithm

The *Dual SIMPLEX Algorithm* is an implementable numerical method conceived for solving the primal optimization problem

$$(P)\begin{cases} f(x) := \langle c, x\rangle \longrightarrow \min \\ Ax = b \\ x \geq 0_n. \end{cases}$$

In contrast to the Primal SIMPLEX Algorithm, presented in Lecture 8, the Dual SIMPLEX Algorithm consists in changing succesively certain dual feasible bases of the feasible set $S := S(A, b)$, such that after a finite number of iterations we obtain::

- an optimal solution of problem (P) and the optimal (minimal) value of f on S

or

- the conclusion that problem (P) has no feasible points, i.e., the set S is empty, hence problem (P) has no optimal solutions.

Pseudocode of the Dual SIMPLEX Algorithm

`Step 1.` `Choose a dual feasible basis` B `of the feasible set` $S = S(A, b)$ `of the primal problem` (P).

`Step 2.` `Determine the numbers` $\alpha_{ij} \in \mathbb{R}$, $(i, j) \in \mathcal{N}_B \times \mathcal{B}_B$, `and` $\alpha_{0j} \in \mathbb{R}$, $j \in \mathcal{B}_B$, `given by (7.8) and (7.7).`

`Step 3.` `Compute the numbers` α_{i0}, $i \in \mathcal{N}_B$, `defined by (7.9), and the number` α_{00} `defined by (7.10).`

`Step 4.` `Study the sign of` α_{0j} `for all` $j \in \mathcal{B}_B$.

(a) `If` $\alpha_{0j} \geq 0$ `for all` $j \in \mathcal{B}_B$, `then conclude that the point` $x^0 := x^B$, `given by (7.11), is an optimal solution of problem (7.1), and` α_{00} `is the optimal (minimal) value of` f `on` S. `STOP.`

(b) If there exists $j \in \mathcal{B}_B$ such that $\alpha_{0j} < 0$, then generate the set of indices

$$\mathcal{B}_B^- = \{j \in \mathcal{B}_B \mid \alpha_{0j} < 0\}.$$

Step 5. For every $j \in \mathcal{B}_B^-$ study the sign of α_{ij} for all $i \in \mathcal{N}_B$.

(a) If there exists $j \in \mathcal{B}_B^-$ such that $\alpha_{ij} \geq 0$ for all $i \in \mathcal{N}_B$, then conclude that the primal problem (P) has no feasible points, i.e., $S = \emptyset$, hence (P) has no optimal solutions. STOP.

(b) If for each $j \in \mathcal{B}_B^-$ there is $i \in \mathcal{N}_B$ such that $\alpha_{ij} < 0$, then choose an index $k \in \mathcal{B}_B^-$. For instance, k can be chosen such that $\mathcal{B}_B$ and

$$\alpha_{0k} = \min\{\alpha_{0j} \mid j \in \mathcal{B}_B\}.$$

Step 6. Choose an index $h \in \mathcal{N}_B$ such that $\alpha_{hk} < 0$ and

$$\frac{\alpha_{h0}}{\alpha_{hk}} = \min\left\{ \frac{\alpha_{i0}}{\alpha_{ik}} \,\middle|\, i \in \mathcal{N}_B \ \texttt{s.t.}\ \alpha_{ik} < 0 \right\}.$$

Step 7. Consider a new basis $B := (B \setminus \{A^k\}) \cup \{A^h\}$, obtained from the previous one by replacing A^k by A^h, and go to Step 2.

Remark 9.1 Due to the specific choice of the indices k and h at Step 5 (b) and Step 6, it can be shown that the new basis B, constructed at Step 7, is a dual feasible basis of S.

In order to solve small size optimization problems by means of the Dual SIMPLEX Algorithm (without implementig it on a computer) in all exercises it will be assumed that the initial basis at Step 1 is the canonical basis of $\mathbb{R}^m$. The numbers α_{ij} involved in Steps 2 and 3 will be arranged in a SIMPLEX tableau, as we did for the Dual SIMPLEX Algorithm in Lecture 8:

		$\dots$	c_j	$\dots$	
	Current basis no.	$\dots$	A^j	$\dots$	Test d.f.b.
$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_i	A^i	$\dots$	α_{ij}	$\dots$	α_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
	Test p.f.b.	$\dots$	α_{0j}	$\dots$	α_{00}

Notice that, according to Definition 7.8, the canonical basis B is a dual feasible basis (d.f.b.) if and only if $\alpha_{i0} \leq 0$ for any non-basic index i (i.e., all numbers located in the vertical section "Test d.f.b." are less than or equal to zero).

At `Step 4` we test whether B is a a primal feasible basis (p.f.b.). The following two cases may occur:

`(a)` If $\alpha_{0j} \geq 0$ for all basic indices j (i.e., all numbers in the horizontal section "Test p.f.b" are greater than or equal to zero), then B is a primal feasible basis, hence an optimal one (since it is already dual feasible, by `Step 1`). In this case, we conclude that an optimal solution of the primal problem (P) is the point

$$x^0 = (\dots, x_i^0 = 0, \dots, x_j^0 = \alpha_{0j}, \dots)$$

whose coordinates x_j^0 labelled by basic indices j can be recovered from the horizontal section "Test p.f.b" while the optimal (minimal) value of f on S is

$$\min f(S) = \alpha_{00},$$

that can be found in the last (bottom, right) cell of the Simplex tableau.

`(b)` If there exists a number $\alpha_{0j} < 0$ labelled by a basic index j (i.e., the horizontal section "Test p.f.b." contains at least one negative number), then we consider the set $\mathcal{B}_B^-$ (consisting of all basic indices that correspond to negative numbers located in the horizontal section "Test p.f.b.").

At `Step 5` we study whether for each α_{0j} with $j \in \mathcal{B}_B^-$ there exists some $\alpha_{ij} < 0$ with $i \in \mathcal{N}_B$ (i.e., we investigate if above each positive number α_{0j} from the horizontal section "Test p.f.b." there is at least one negative number α_{ij}). Two cases may occur:

`(a)` If there exists $j \in \mathcal{B}_B^-$ such that $\alpha_{ij} \geq 0$ for all $i \in \mathcal{N}_B$ (i.e., there exists a negative number α_{0j} in the horizontal section "Test p.f.b." above which none of the numbers α_{ij} is negative), then we conclude that problem (P) has no feasible points, hence (P) has no optimal solutions. In other words, in this case we have:

$$S = \emptyset = \operatorname*{argmin}_{x \in S} f(x).$$

`(b)` If for each $j \in \mathcal{B}_B^-$ there exists $i \in \mathcal{N}_B$ such that $\alpha_{ij} < 0$ (i.e., above each negative number α_{0j} from the horizontal section "Test p.f.b." there exists at least one negative number α_{ij}), then we can choose a basic index k that corresponds to any negative number α_{0k} from the horizontal section "Test p.f.b.", as for instance the smallest number of this section:

$$\alpha_{0k} = \min\{\alpha_{0j} \mid j \in \mathcal{B}_B\}.$$

The Simplex tableau's column that corresponds to k represents the *pivot column* and it is marked by an asterisk $*$ below the tableau.

Within `Step 6` we choose a non-basic index h such that $\alpha_{hk} < 0$ and

$$\frac{\alpha_{h0}}{\alpha_{hk}} = \min\left\{\frac{\alpha_{i0}}{\alpha_{ik}} \;\middle|\; i \in \mathcal{N}_B \text{ and } \alpha_{ik} < 0\right\}.$$

Notice that all numerators of the ratios $\dfrac{\alpha_{i0}}{\alpha_{ik}}$ involved in the formula above are located in the vertical section "Test d.f.b." while the corresponding denominators are the negative numbers in the pivot column. In practice, it will be convenient to write these ratios on the right side of the Simplex tableau. The smallest ratio $\frac{\alpha_{h0}}{\alpha_{hk}}$ will be marked by an asterisk $*$ and the corresponding row represents the Simplex tableau's *pivot row*. Before going to the next step, it is recommended to highlight (encircle) the *pivot*, i.e., the number α_{hk}.

			c_j		c_k				
	Current basis no.	$\ldots$	A^j	$\ldots$	A^k	$\ldots$	Test d.f.b.		
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		
c_i	A^i	$\ldots$	α_{ij}	$\ldots$	α_{ik}	$\ldots$	α_{i0}	$\frac{\alpha_{i0}}{\alpha_{ik}}$	
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		
c_h	A^h	$\ldots$	α_{hj}	$\ldots$	$\boxed{\alpha_{hk}}$	$\ldots$	α_{h0}	$\frac{\alpha_{h0}}{\alpha_{hk}}$	*
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		
	Test p.f.b.	$\ldots$	α_{0j}	$\ldots$	α_{0k}	$\ldots$	α_{00}		
					*				

At `Step 7` we construct a new basis, obtained from the previous one by replacing A^k with A^h (notice that these elements belong to the pivot column and the pivot row, respectively). Then we go to `Step 2`. The Simplex tableau associated to the new basis can be obtained from the previous tableau by performing a Gauss-Jordan transform:

Current basis no.	$\ldots$	A^j	$\ldots$	A^h	$\ldots$	Test d.f.b.
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^i	$\ldots$	$\dfrac{\alpha_{ij}\alpha_{hk} - \alpha_{hj}\alpha_{ik}}{\alpha_{hk}}$	$\ldots$	$\dfrac{\alpha_{ik}}{\alpha_{hk}}$	$\ldots$	$\dfrac{\alpha_{i0}\alpha_{hk} - \alpha_{ik}\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^k	$\ldots$	$-\dfrac{\alpha_{hj}}{\alpha_{hk}}$	$\ldots$	$\dfrac{1}{\alpha_{hk}}$	$\ldots$	$-\dfrac{\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
Test p.f.b.	$\ldots$	$\dfrac{\alpha_{0j}\alpha_{hk} - \alpha_{0k}\alpha_{hj}}{\alpha_{hk}}$	$\ldots$	$\dfrac{\alpha_{0k}}{\alpha_{hk}}$	$\ldots$	$\dfrac{\alpha_{00}\alpha_{hk} - \alpha_{0k}\alpha_{h0}}{\alpha_{hk}}$

Lecture 10

Matrix games

Definition 10.1 By a (*two-person zero-sum finite*) *matrix game* we mean a triple $(\mathcal{A}^1, \mathcal{A}^2, C)$ consisting of two finite sets,

$$\mathcal{A}^1 = \left\{a_1^1, \dots, a_m^1\right\} \quad \text{and} \quad \mathcal{A}^2 = \left\{a_1^2, \dots, a_n^2\right\},$$

and a matrix of real numbers,

$$C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mn} \end{pmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}),$$

where $m, n \in \mathbb{N}$. For any

$$i \in I := \{1, \dots, m\} \quad \text{and} \quad j \in J := \{1, \dots, n\},$$

we say that

- a_i^1 is an *action of the first player* (also called raw player);
- a_j^2 is an *action of the second player* (also called column player);
- (a_i^1, a_j^2) is a *turn of the game*;
- c_{ij} is the *gain/loss (outcome) of the first player* while $-c_{ij}$ is *the loss/gain of the second player* w.r.t. (a_i^1, a_j^2).

Thus, $\mathcal{A}^1$ and $\mathcal{A}^2$ represent the sets of all possible actions of the players while C represents the so-called *payoff matrix* of the game.

Example 10.2 (*Rock-Scissors-Paper*) Assume that two players choose simultaneously and independently one of the words (symbols) *Rock* (R), *Scissors* (S) or *Paper* (P), i.e., the sets of actions are

$$\begin{aligned} \mathcal{A}^1 &= \left\{a_1^1, a_2^1, a_3^1\right\} = \{R, S, P\}, \\ \mathcal{A}^2 &= \left\{a_1^2, a_2^2, a_3^2\right\} = \{R, S, P\}. \end{aligned}$$

If both players choose the same word, then the game is tied. Otherwise, the rock beats the scissors and the paper beats the rock. The payoff matrix of this game is:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Definition 10.3 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. For every $i \in I$ let

$$\alpha_i := \min_{j \in J} c_{ij}.$$

The number

$$\underline{w} := \max_{i \in I} \alpha_i = \max_{i \in I} \min_{j \in J} c_{ij}$$

is called *the game's lower value.* Similarly, for every $j \in J$ let

$$\beta_j = \max_{i \in I} c_{ij}.$$

The number

$$\overline{w} := \min_{j \in J} \beta_j = \min_{j \in J} \max_{i \in I} c_{ij}$$

is called *the game's upper value.*

Remark 10.4 It is a simple exercise to prove that in any matrix game the following inequality holds:

$$\underline{w} \leq \overline{w}.$$

Definition 10.5 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. We say that $c_{i_0 j_0}$ is a *saddle point* of the payoff matrix C if

$$c_{ij_0} \leq c_{i_0 j_0} \leq c_{i_0 j}, \ \forall (i, j) \in I \times J,$$

which means that $c_{i_0 j_0}$ is the largest element on its column as well as the smallest element on its row. We say that $(\mathcal{A}^1, \mathcal{A}^2, C)$ is a

- *game with saddle points*, if its payoff matrix C has at least one saddle point;
- *game without saddle points*, if its payoff matrix C has no saddle points.

Remark 10.6 It is easy to show that $(\mathcal{A}^1, \mathcal{A}^2, C)$ is a game with saddle points if anf only if

$$\underline{w} = \overline{w}.$$

Example 10.7 For the Rock-Scissors-Paper game we have

$$\underline{w} = -1 < \overline{w} = 1,$$

hence this is a game without saddle points.

Example 10.8 Consider a game whose payoff matrix is

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 1 & 0 & 3 \end{pmatrix}.$$

In this case we have

$$\underline{w} = 0 = \overline{w},$$

hence this game has saddle points. More precisely, its saddle points are

$$c_{12} \quad \text{and} \quad c_{32}.$$

Definition 10.9 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. The elements of the set

$$X := \Delta_m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1, \ldots, x_m \geq 0,\ x_1 + \cdots + x_m = 1\}$$

are called *strategies of the first player* while the elements of the set

$$Y := \Delta_n = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1, \ldots, y_n \geq 0,\ y_1 + \ldots + y_n = 1\}$$

are called *strategies of the second player*, where Δ_k represents the standard simplex of $\mathbb{R}^k$, introduced in Definition 4.8.

Definition 10.10 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. The number

$$F(x, y) := \sum_{(i,j) \in I \times J} c_{ij} x_i y_j = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i y_j$$

is called *the average gain* of the first player for a given pair of strategies $(x, y) \in X \times Y$.

Definition 10.11 By a solution of the matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ we mean a pair of strategies $(x^0, y^0) \in X \times Y$ satisfying the property

$$F(x, y^0) \leq F(x^0, y^0) \leq F(x^0, y),\ \forall\, (x, y) \in X \times Y.$$

In this case we say that:

- x^0 is an *optimal strategy of the first player*;
- y^0 is an *optimal strategy of the second player.*

Theorem 10.12 (John Von Neumann) *Every matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ has at least one solution. Moreover, if $(x^0, y^0) \in X \times Y$ is a solution of the game, then*

$$\max_{x \in X} \min_{y \in Y} F(x, y) = F(x^0, y^0) = \min_{y \in Y} \max_{x \in X} F(x, y).$$

Definition 10.13 The real number

$$w := \max_{x \in X} \min_{y \in Y} F(x, y) = \min_{y \in Y} \max_{x \in X} F(x, y)$$

(which is well-defined by Von Neumann's Theorem) is called *the value of the game* $(\mathcal{A}^1, \mathcal{A}^2, C)$.

Proposition 10.14 *For any matrix game* $(\mathcal{A}^1, \mathcal{A}^2, C)$ *w ehave*

$$\underline{w} \leq w \leq \overline{w}.$$

To solve a matrix game from mathematical point of view we have to determine the game's value w and one of its solutions (x^0, y^0), i.e., a pair of optimal strategies of the two players.

Lecture 11

The relationship between matrix games and linear optimization problems

Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game whose payoff matrix is

$$C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mn} \end{pmatrix}.$$

Theorem 11.1 *If the matrix C has a saddle point $c_{i_0 j_0}$, then the following assertions hold:*

1° *The game's value is* $w = \underline{w} = \overline{w}$.

2° *The canonical vector* $x^0 = (x_1^0, \dots, x_m^0) = (0, \dots, x_{i_0}^0 = 1, \dots, 0) = e^{i_0}$ *of* $\mathbb{R}^m$ *is an optimal strategy of Player 1.*

3° *The canonical vector* $y^0 = (y_1^0, \dots, y_n^0) = (0, \dots, y_{j_0}^0 = 1, \dots, 0) = e^{j_0}$ *of* $\mathbb{R}^n$ *is an optimal strategy of Player 2.*

In general (if the payoff matrix has no saddle points), then we consider the following optimization problems:

$$(P_1) \quad \begin{cases} \text{Minimize } u_1 + \dots + u_m \\ c_{11}u_1 \quad + \dots \quad + c_{m1}u_m \quad \geq 1 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ c_{1n}u_1 \quad + \dots \quad + c_{mn}u_m \quad \geq 1 \\ u_1, \dots, u_m \geq 0 \end{cases}$$

and

$$
(P_2) \begin{cases} \text{Maximize } v_1 + \cdots + v_n \\[2ex] \begin{array}{ccccc} c_{11}v_1 & + \ldots & +c_{1n}v_n & \leq 1 \\ \vdots & & & \vdots \\ c_{m1}v_1 & + \ldots & +c_{mn}v_n & \leq 1 \end{array} \\[2ex] v_1, \ldots, v_n \geq 0. \end{cases}
$$

Theorem 11.2 *If a matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ has a positive value, i.e.,*

$$w > 0,$$

then the problems $(P1)$ and (P_2) have optimal solutions. Moreover, for any optimal solutions $u^0 = (u_1^0, \ldots, u_m^0)$ and $v^0 = (v_1^0, \ldots, v_n^0)$ of these problems, the following assertions hold:

1° *The game's value is* $w = \dfrac{1}{u_1^0 + \cdots + u_m^0} = \dfrac{1}{v_1^0 + \cdots + v_n^0}$.

2° *The vector* $x^0 = w \cdot u^0$ *is an optimal strategy of Player 1.*

3° *The vector* $y^0 = w \cdot v^0$ *is an optimal strategy of Player 2.*

Remark 11.3 If a matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ has a positive lower value, i.e.,

$$\underline{w} > 0,$$

then its value is positive, since

$$w \geq \underline{w} > 0,$$

by Proposition 10.14. Thus, in this case we can apply Theorem 11.2.

Remark 11.4 If $\underline{w} \leq 0$, then w might be not positive. In this case, we can modify the payoff matrix C by adding a suitable constant $k \in \mathbb{R}$ to all entries of C, such that

$$\underline{w} + k > 0.$$

In this way we obtain a new payoff matrix

$$\widehat{C} := C + (k)$$

whose lower value is positive:

$$\underline{\widehat{w}} = \underline{w} + k > 0.$$

Therefore we can apply Theorem 11.2 in order to solve the game with the payoff matrix $\widehat{C}$. Of course, this new game has the same solutions as the initial one, while their values are related by

$$w = \widehat{w} - k.$$

Remark 11.5 Under the hypothesis that $w > 0$, the problems (P_1) and (P_2) should be written in standard before applying the SIMPLEX algorithm:

$$(P_{1,\mathrm{st}}) \begin{cases} \text{Minimize } u_1 + \cdots + u_m \\ \begin{array}{cccccccc} -c_{11}u_1 & -\dots & -c_{m1}u_m & +u_{m+1} & & & = -1 \\ \dots & & & & \ddots & & \vdots \\ -c_{1n}u_1 & -\dots & -c_{mn}u_m & & & +u_{m+n} & = -1 \end{array} \\ u_1, \dots, u_{m+n} \geq 0 \end{cases}$$

and

$$(P_{2,\mathrm{st}}) \begin{cases} \text{Minimize } -v_1 - \cdots - v_n \\ \begin{array}{cccccccc} c_{11}v_1 & +\dots & +c_{1n}v_n & +v_{n+1} & & & = 1 \\ \dots & & & & \ddots & & \vdots \\ c_{m1}v_1 & +\dots & +c_{mn}v_n & & & +v_{n+m} & = 1 \end{array} \\ v_1, \dots, v_{n+m} \geq 0 \end{cases}$$

Actually, problem $(P_{1,\mathrm{st}})$ can be solved by the Primal SIMPLEX algorithm, while problem $(P_{2,\mathrm{st}})$ can be solved by the Dual SIMPLEX algorithm.

Remark 11.6 The dual of $(P_{1,\mathrm{st}})$ is equivalent to (P_2), while the dual of $(P_{2,\mathrm{st}})$ is equivalent to (P_1).

Indeed, the dual of $(P_{1,\mathrm{st}})$ is

$$(D_1) \begin{cases} \text{Maximize } -s_1 - \cdots - s_n \\ \begin{array}{ccccc} -c_{11}s_1 & -\dots & -c_{1n}s_n & \leq 1 \\ \vdots & & & \vdots \\ -c_{11}s_1 & -\dots & -c_{1n}s_n & \leq 1 \\ s_1 & & & \leq 0 \\ & \ddots & & \leq 0 \\ & & s_n & \leq 0, \end{array} \end{cases}$$

which is equivalent to (P_2) by means of the change of variables

$$-s = (-s_1, \dots, -s_n) = v = (v_1, \dots, v_n).$$

On the other hand, the dual of $(P_{2,\mathrm{st}})$ is

$$(D_2) \begin{cases} \text{Maximize } t_1 + \cdots + t_m \\ \begin{array}{cccr} c_{11}t_1 & +\ldots & +c_{m1}t_m & \leq -1 \\ \vdots & & & \vdots \\ c_{1n}t_1 & +\ldots & +c_{m1}t_m & \leq -1 \\ t_1 & & & \leq 0 \\ & \ddots & & \leq 0 \\ & & t_m & \leq 0, \end{array} \end{cases}$$

which is equivalent to (P_1) by means of the change of variables

$$t = (t_1, \ldots, t_m) \coloneqq -u = (-u_1, \ldots, -u_m).$$

Lecture 12

Dual problems solved by applying the SIMPLEX algorithm to the primal problems

Seminar 4 Primal simplex algorithm (PSA)

Ex 1 Solve the following problem by using (PSA)

$$(P)\quad \begin{cases} \text{Minimize } f(x) = 2x_1 - x_2 + x_3 \\ \text{s.t} \\ -x_1 + 2x_3 + x_4 = 0 \\ 3x_1 + x_2 - x_3 = 2 \\ x_1, \dots, x_4 \geq 0 \end{cases} \quad (S)$$

Solution: Problem (P) is given in standard form
(see Lecture 7)

$$(P)\quad \begin{cases} \text{Minimize } f(x) = \langle c, x \rangle = c_1 x_1 + \dots + c_n x_n \\ Ax = b \\ x \geq 0_n \text{ (componentwise)}, \quad x = (x_1, \dots, x_n) \end{cases}$$

$$0_n = (0, \dots, 0)$$

We have

$n = 4$

$$c = (\underset{\substack{\| \\ 2}}{c_1}, \underset{\substack{\| \\ -1}}{c_2}, \underset{\substack{\| \\ 1}}{c_3}, \underset{\substack{\| \\ 0}}{c_4})$$

$$A = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 3 & 1 & -1 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\quad A^1 \quad A^2 \quad A^3 \quad A^4$$

$$\text{rank } A = \underset{\substack{\| \\ 2}}{m} < \underset{\substack{\| \\ 4}}{n}$$

Consider the basis

$B = (A^4, A^2)$

Then the det $[A^4, A^2] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Hence rank $A = 2$

The simplex tableau associated to B is:

B is a dual feasible basis (d.f.b)

$A^1 = (-1)A^4 + 3A^2$

		0	-1	
	1	A^4	A^2	test d.f.b
2	A^1	-1	3	-5
1	A^3	2	-1	0
	b	$0 \geq 0$	$2 \geq 0$	-2

$[(-1)\cdot 0 + 3\cdot(-1)] - 2$

$[2\cdot 0 - 1(-1)] - 1 = 0$

Test p.f.b || primal fesability of the basis

B is a primal feasible basis (p.f.b) (bază primal admisibilă)

$\Rightarrow$ we can use P.S.A

$$d_{i,0} = \sum_{j \in B_B} \alpha_{ij} c_j - c_i$$

$$d_{0,b} = \sum_{j \in B_B} \alpha_{0,j} c_j$$

Since b is both p.f.b and d.f.b, it follows that B is an optimal basis

An optimal solution of problem (P) is

$x^0 = x^B = (x_1^0, x_2^0, x_3^0, x_4^0)$

if L7 $\quad x_1^0 = 0,\ x_2^0 = 2,\ x_3^0 = 0,\ x_4^0 = 0$

The optimal value (min) of f on S is:

$\alpha_{0,0} = -2$
of L7

Checking : $f(x^0) = 2x_1^0 - x_2^0 + x_3^0 = 2\cdot 0 - 2 + 0 = -2$

$f(x) = 2x_1 - x_2 + x_3$

$x^0 = (0, 2, 0, 0)$

Solve the Resource allocation Problem with the following data (see Seminar 1) by means of P.S.A.

Resources \ Products	P1	P2	~~The requested amount of nutrients~~ Available amount of resources
R1	1	2	6
R2	0	1	4
R3	3	0	9
Unit cost	1	3	

$$\begin{cases} \text{Maximize } f(x) = 1x_1 + 3x_2 \\ \text{s.t.} \\ 1x_1 + 2x_2 \geq 6 \\ 0x_1 + 1x_2 \geq 4 \\ 3x_1 + 0x_2 \geq 9 \\ x_1, x_2 \geq 0 \end{cases}$$

This problem has no standard form (because of "≤" type constraints)

In order to apply the P.S.A. we will transform it into an equivalent problem whose constraints are in standard form

$$(P)\quad \begin{cases} \text{Minimize } g(x) = -f(x) = -x_1 - 3x_2 \\ \left.\begin{array}{l} x_1 + 2x_2 + x_3 = 6 \\ x_2 + x_4 = 4 \\ 3x_1 + x_5 = 9 \\ x_1, \dots, x_5 \geq 0 \end{array}\right] (S) \end{cases}$$

$n = 5 \qquad m = 3, \quad c = (c_1, c_2, c_3, c_4, c_5) = (-1, -3, 0, 0, 0)$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccccc} A^1 & A^2 & A^3 & A^4 & A^5 \end{array} \end{array} ; \quad b = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$$

Let $B = (A^3, A^4, A^5)$

		0	0	0	
	1	A^3	A^4	A^5	Δf.b
-1	A^1	$1>0$	0	$3>0$	$1>0$
-3	A^2	$2>0$	$1>0$	0	$3>0$ *
	r.f.b	$6 \geq 0$	$4 \geq 0$	$9 \geq 0$	0

$\frac{6}{2}$ * $\quad \frac{4}{2}$ ⟶ B is a r.f.b, hence we apply P.S.A.

2	A^2	A^4	A^5	dfb	
A^1	$\frac{1}{2}$			$-\frac{1}{2}$	≤ 0
A^3	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{3}{2}$	≤ 0
Δfb	3	1	9	-9	

$\frac{1\cdot 2 - 1\cdot 3}{2}$ Gauss Jordan

$\Rightarrow (A^2, A^4, A^5)$ is d.f.b.

Hence an optimal basis

[illegible] - muzen

An optimal sol of (P) is $x^0_{\text{standard}} = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0) = (0, 3, 0, 1, 9)$

$\min g = \alpha_{00} = 9 \Rightarrow \max f = 9$

The sol of R.A.P is

$x^0 = (x_1^0, x_2^0) = (0, 3)$

We observe that (P) is given in standard form

$$(P)\begin{cases} \text{Minimize } f(x) = 3x_1 + 2x_2 + x_3 \\ \left.\begin{array}{l} -x_1 - 2x_3 + x_5 = -6 \\ -3x_2 - x_3 + x_4 = -4 \\ x_1, \dots x_3 \geq 0 \end{array}\right] (S) \end{cases}$$

$n = 5$, $m = 2$

$$c = (c_1, c_2, c_3, c_4, c_5)$$
$$\quad\;\; \begin{array}{ccccc} \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ 3 & 2 & 1 & 0 & 0 \end{array}$$

$$A = \underbrace{\begin{pmatrix} -1 & 0 & -2 & 0 & 1 \\ 0 & -3 & -1 & 1 & 0 \end{pmatrix}}_{A^1 \quad A^2 \quad A^3 \quad A^4 \quad A^5} ; \; b = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

Let $B = (A^5, A^4$

		1	A^5 (0)	A^4 (0)	Test d.f.b
3		A^1	$-1<0$	0	$-3 \leq 0$
2		A^2	0	$-3<0$	$-2 \leq 0$
1		A^3	$-2<0$	$-1<0$	$-1 \leq 0$
		Test r.f.b	$-6<0$	$-4<0$	0
			$*$		

$0\cdot(-1)+0\cdot 0-3=-3 \quad \frac{-3}{-1}$

$0\cdot 0+0\cdot(-3)-2=-2$

$-2\cdot 0-1\cdot 0-1=-1 \quad -\frac{1}{2}*$

$-6\cdot 0-4\cdot 0=0$

If no negative $\Rightarrow$ set empty

Calculate d.f.b or r.f.b

$\Rightarrow$ B is d.f.b $\Rightarrow$ we apply the Dual Simplex Alg

2	A^3	A^4	Dist d.f.b
A^1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$
A^2	0	$-3 \leftarrow$	-2
A^5	$-\frac{1}{2}$	$-\frac{1}{2} \leftarrow$	$-\frac{1}{2}$
f.f.b	3	-1	3

$\frac{-2}{-3}$

$\frac{-1/2}{-1/2} = 1$

- Divide everything by pivot on same column as pivot
- Divide everything by the opposite of pivot on the same row

3	A^3	A^2	d.f.b
A^1			
A^4			
$A5$			
p.f.b	3	$\frac{1}{3}$	$\frac{11}{3}$

d.f.b, according to DSA

(A^3, A^2) is p.f.b, hence optimal basis

An optimal sol of (P) is

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0) = (0, \tfrac{1}{3}, 3, 0, 0)$$

The optimal value (min) of f on S is:

Remark $f(x^0) = 3x_1^0 + 2x_2^0 + x_3^0$

$= 3.0 + 2\frac{1}{3} + 3 = \frac{2}{3} + 3 = \frac{11}{3}$

Ex 2 Solve the foll

$$(P)\begin{cases} \text{Minimize } f(x) = \alpha^2 x_1 + (\alpha-1)^2 x_2 \\ \left.\begin{array}{l} -x_1 \qquad + x_3 = 0 \\ -2x_1 - x_2 \qquad + x_4 = -5 \\ x_1, \dots, x_5 \geq 0 \end{array}\right] (S) \end{cases}$$

α is a parameter

Solution :

(P) is given in standard form

$n = 4$, $m = 2$

$$c = (\underset{\alpha^2}{\overset{\shortparallel}{c_1}}, \underset{(\alpha-1)^2}{\overset{\shortparallel}{c_2}}, \underset{0}{\overset{\shortparallel}{c_3}}, \underset{0}{\overset{\shortparallel}{c_4}})$$

$$A \begin{pmatrix} -1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \quad b \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$A^1 \quad A^2 \quad A^3 \quad A^4$

$B = (A^3, A^4)$

	1	$\overset{0}{A^3}$	$\overset{0}{A^4}$	$\alpha R b$
α^2	A^1	-1	$-2<0$	$-\alpha^2 \le 0$
$\alpha^2-2\alpha+1$	A^2	0	$-1<0$	$-\alpha^2+2\alpha-1 \le 0$
	$1Rb$	0	-5	0

B is $\alpha R b \Rightarrow \frac{-\alpha^2}{-2} \Rightarrow$ we can apply $\frac{-\alpha^2+2\alpha-1}{-1}$ D.F.A.

Simplexul primal nu merge pt că $-5<0$

$-\alpha^2+2-1 = -(-\alpha-1)^2 \le 0$

Alegem cel mai mic nr din sectorul orizontal

$$\frac{-\alpha^2}{-2} = \frac{\alpha^2}{2}$$

$$\frac{-\alpha^2+2\alpha-1}{-1} = \alpha^2-2\alpha+1$$

$$\frac{\alpha^2}{2} \le \alpha^2-2\alpha+1$$

$$\Leftrightarrow \alpha^2 \le 2\alpha^2-4\alpha+2 \Rightarrow \alpha^2-4\alpha+2 \ge 0$$

$$\Delta = (-4)^2-4\cdot 2 = 16-8=8>0$$

$$\alpha_{1,2} = \frac{4\pm\sqrt{8}}{2} = 2\pm\sqrt{2}$$

α	$-\infty$		$2-\sqrt{2}$		$2+\sqrt{2}$		$+\infty$
	+ + +		0	− −	0	+ + +	

Case 1) $\alpha \in (-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$

1	A^3	A^4	dfb
A^1	-1	-2	$-\alpha^2$
A^2	0	-1	$-\alpha^2+\alpha-1$
rfb	0	-5	0

$\frac{\alpha^2}{2}$ * (in Case 1)

$\alpha^2-2\alpha+1$

...

2	A^3	A^1	dfb
A^4			
A^2			
rfb	$\frac{5}{2}$	$\frac{5}{2}$	
	≥ 0	≥ 0	

$\Rightarrow (A^3, A^1)$ is rfb, hence an optimal basis

An optimal sol of (P) in Case 1 is:

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$$
$$x_1^0 = \frac{5}{2},\quad x_2^0 = 0,\quad x_3^0 = \frac{5}{2},\quad x_4^0 = 0$$

and the optimal value (min) of f on S is

$$\alpha_{00} = \frac{5\alpha^2}{2}$$

<u>Case 2</u> :

1	A^3	A^4	dfb
A^1	-1	-2	$-\alpha^2$
A^2	0	-1	$\alpha^2+2\alpha-1$
rfb	0	-5	0

* Case 2

2	A^3	A^2	dfb
A^1			
A^4			
pbb	0	5	$-5\alpha^2+10\alpha-5$

$\Downarrow$

(A^3, A^2) is pbb, hence optimal basis

$\Rightarrow$ An optimal sol of (P) in case 2 is

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0) = (0, 5, 0, 0)$$

The optimal value of f is $\alpha_{00} = -5\alpha^2 + 10\alpha - 5$

Ex 3 _ Apply the dual simplex algorithm to the foll pb.

$$(P)\begin{cases} \text{Minimize } f(x) = -x_2 - x_3 + 3x_5 - 3x_6 \\ \left.\begin{array}{l} 2x_1 + x_2 + x_3 + x_5 + 3x_6 = 0 \\ x_1 + x_3 + x_4 - x_5 = -2 \\ x_1, \dots x_6 \geq 0 \end{array}\right] (S) \end{cases}$$

(P) is in std form

$n = 6$, $m = 2$

$$C = (c_1, c_2, c_3, c_4, c_5, c_6) = (0, -1, -1, 0, 3, -3)$$

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$A^1 \quad A^2 \quad A^3 \quad A^4 \quad A^5 \quad A^6$$

	1	A^2 (−1)	A^4 (0)	dfb
0	A^1	2	1	$-2 \le 0$
−1	A^3	1	1	$0 \le 0$
3	A^5	1	$-1 < 0$	$-4 \le 0$
−3	A^6	3	0	$0 \le 0$
	[illegible]	0	$-2 < 0$	0

*

=> B- is DFB

We apply D.FA

$\frac{-4}{-1}$ *

2	A^2	A^5	dfb
A^1	$3 \not< 0$		
A^3	$2 \not< 0$		
A^4	$1 \not< 0$		
A^6	$3 \not< 0$		
pfb	$-2 < 0$		

=> (P) has no feasible sol => $S = \emptyset$

Seminar 6 Matrix games

Ex 1 Solve the game whose pay-off matrix is:

$$C = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Solution First we compute $\underline{v}$ and $\overline{v}$ (the so-called lower value and upper value of the game)

$$\left.\begin{array}{l} \alpha_1 = \min\{2, 0\} = 0 \\ \alpha_2 = \min\{-1, 1\} = -1 \end{array}\right\} \Rightarrow \underline{v} = \max\{\alpha_1, \alpha_2\} = 0.$$

$$\left.\begin{array}{l} \beta_1 = \max\{2, -1\} = 2 \\ \beta_2 = \max\{0, 1\} = 1 \end{array}\right\} \Rightarrow \overline{v} = \min\{\beta_1, \beta_2\} = 1$$

According to L10 we know that:

$$0 = \underline{v} \le v \le \overline{v} = 1$$

hence $v \in [0, 1]$

Since $\underline{v} \neq \overline{v}$, the game has no saddle points.

Thus, Theorem 11.1 does not apply

On the other hand, we are not sure wether $v \overset{?}{>} 0$

(we just know that $v \in [0, 1]$)

In order to see Theorem 11.2, which requires that the game's value is positive (>0), we will add a convenient constant $k \in \mathbb{R}$ to all elements of C, such that

$\underline{v} + k = 0$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

For instance, let

$k = 1$

We obtain a new pay-off matrix

$$\hat{C} = C + (k) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

$\hat{\underline{v}} = \underline{v} + k = 0 + 1 = 1$

$\hat{\overline{v}} = \overline{v} + k = 1 + 1 = 2$

$\hat{v} = v + k = v + 1 \in [1, 2] \Rightarrow \hat{v} > 0$

La examen ne dă matrice aranjată fără translatare

$\Rightarrow$ we can apply th 11.2 to the new game, whose pay-off matrix is $\hat{C}$.

We associate to $\hat{C}$ two optimization problems.

$$(P1) \begin{cases} \text{Minimize } u_1 + u_2 \\ 3 \cdot u_1 + 0 \cdot u_2 \geq 1 \\ 1 \cdot u_1 + 2u_2 \geq 1 \\ u_1, u_2 \geq 0 \end{cases} \quad \text{and} \quad (P2) \begin{cases} \text{Maximise } v_1 + v_2 \\ 3 \cdot v_1 + 1 \cdot v_2 \leq 1 \\ 0 \cdot v_1 + 2 \cdot v_2 \leq 1 \\ v_1, v_2 \geq 0 \end{cases}$$

Next we solve (P1) and (P2)

In order to use the SIMPLEX Algorithm.
we have to transform (P1) and (P2) in standard form

$$(P_1\ st)\ \begin{cases} \text{Minimize } u_1 + u_2 \\ 3u_1 - u_3 = 1 \\ u_1 + 2u_2 - u_4 = 1 \\ u_1, \dots, u_4 \geq 0 \end{cases} \quad (=)$$

$$(=)\ \begin{cases} \text{Minimize } u_1 + u_2 \\ -3u_1 + u_3 = -1 \\ -u_1 - 2u_2 + u_4 = -1 \\ u_1, \dots, u_4 \geq 0 \end{cases}$$

and

$$(P_2\ st)\ \begin{cases} \text{Minimize } -v_1 - v_2 \\ 3v_1 + v_2 + v_3 = 1 \\ 2v_2 + v_4 = 1 \\ v_1, \dots, v_4 \geq 0 \end{cases}$$

Let us solve (P1 st)

$n=4,\ m=2,\ c=(c_1, c_2, c_3, c_4)$ with $c_1 = 1,\ c_2 = 1,\ c_3 = 0,\ c_4 = 0$

$$A = \begin{array}{c} \begin{pmatrix} -3 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix} \\ A^1 \quad A^2 \quad A^3 \quad A^4 \end{array} ; \quad b \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Let $B = (A^3, A^4)$

		0	0			
	1	A^3	A^4	dfb		
1	A^1	$-3<0$	$-1<0$	-1	≤ 0	$\frac{-1}{-3}$ *
1	A^2	0	$-2<0$	-1	≤ 0	$-\frac{1}{0}$
	dfb	-1	-1	0		
		*				

→ dfb => we apply DSA

2	A^1		dfb	dfb
dfb				

2	A^1	A^4	dfb
A^3	$-\frac{1}{3}$	$-\frac{1}{3} < 0$	$-\frac{1}{3}$
A^2	0	$-2 < 0$	-1
rfb	$\frac{1}{3}$	$-\frac{2}{3} < 0$	$\frac{1}{3}$

$$\frac{-\frac{1}{3}}{-\frac{1}{3}} = 1$$

$$\frac{-1}{-2} = \frac{1}{2} \; *$$

3	A^1	A^2	dfb
A^3			
A^4			
rfb	$\frac{1}{3} \geq 0$	$\frac{1}{3} \geq 0$	$\frac{2}{3}$

r f b, hence optimal

An optimal sol (P_1 *) is:

$$u^0{}^* = (u_1^0, u_2^0, u_3^0, u_4^0)$$
$$\quad\;\; = \tfrac{1}{3}, \quad \tfrac{1}{3}, \quad 0, \quad 0$$

The optimal value (min) is

$$\alpha_{00} = \frac{2}{3}$$

We deduce that an optimal sol of initial problem (P_1) is:

$$\mu^0 = (u^1, u^2) = \left(\frac{1}{3}, \frac{1}{3}\right)$$

and the optimal value (min) is:

$$u_1^0 + u_2^0 = \alpha_{00} = \frac{2}{3}$$

According to theorem ~~[illegible]~~ 11.2, we have:

$$\hat{w} = \frac{1}{u_1^0 + u_2^0} = \frac{1}{2/3} = \frac{3}{2}$$

Remark: $\hat{w} = \frac{3}{2} \in [1, 2]$

An optimal strategy of Player 1 is:

$$\boxed{\hat{x}^0 = \hat{w} \cdot u^0} = \frac{3}{2}\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

In what concerns the initial game with the pay-off matrix C, we have:

$$w = \hat{w} - k = \frac{3}{2} - 1 = \frac{1}{2} \in [0, 1]$$

An optimal strategy of Player 1 is:

$$x^0 = \hat{x}^0 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

– We solve P2 ~~st~~:

$n = 4, m = 2$

$$C = (c_1, c_2, c_3, c_4)$$
$$\begin{array}{cccc} \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ -1 & -1 & 0 & 0 \end{array}$$

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $B = (A^3, A^4)$

	0	0	
1	A^3	A^4	dfb
-1 A^1	$3>0$	$0>0$	$1>0$ *
-1 A^2	$1>0$	$2>0$	$5>0$
rfb	$1\geq 0$	$1\geq 0$	0

rfb ⇒ we apply P.S.A

$\frac{1}{3}$ $<\frac{1}{1}$

*

2	A^1	A^4	dfb
A^3	$\frac{1}{3}$	-0	$-\frac{1}{3}$
A^2	$\frac{1}{3}>0$	$2>0$	$\frac{2}{3}$ ≥ 0 *
rfb	$\frac{1}{3}$	1	$-\frac{1}{3}$

$\frac{\frac{1}{3}}{\frac{1}{3}}=1$

3	A^1	A^2	dfb
A^3			$-\frac{1}{3}\leq 0$
A^4			$-\frac{1}{3}\leq 0$
rfb	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{2}{3}$

} dfb
hence optimal

An optimal sol of $(P_2\ st)$ is:

$$v^0_{st} = (v_1^0, v_2^0, v_3^0, v_4^0)$$

$$v_1^0 = \frac{1}{6}, \quad v_2^0 = \frac{1}{2}, \quad v_3^0 = 0, \quad v_4^0 = 0$$

the optimal value is:

$$\alpha_{00} = -\frac{2}{3}$$

An optimal sol of (P_2) is:

$$v^0 = (v_1^0, v_2^0) = \left(\frac{1}{6}, \frac{1}{2}\right)$$

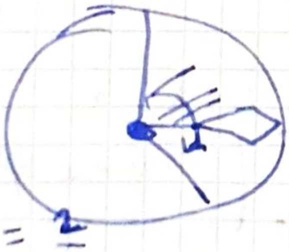
$$\hat{w} = \frac{1}{v_1^0 + v_2^0} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

An optimal strategy of Player 2 is:

$$\hat{y}^0 = \hat{w} \cdot v^0 = \frac{5}{3}\left(\frac{1}{5}, \frac{2}{5}\right) = \left(\frac{1}{4}, \frac{3}{4}\right)$$

$$w = \hat{w} - k = \frac{5}{3} - 1 = \frac{1}{2}$$

$$y^0 = \hat{y}^0 = \left(\frac{1}{4}, \frac{3}{4}\right)$$



$$y_1^0 = \frac{1}{4}$$

$$y_2^0 = \frac{2}{3}$$

Seminar 4

Exercise 1 Solve the following optimization problem by means of the Primal SIMPLEX Algorithm.

$$
(P) \left\{ \begin{array}{l}
\text{Minimize } \ f(x) = x_2 - 3x_3 + 2x_4 \\
\left. \begin{array}{rrrrrrcr}
x_1 & +3x_2 & -x_3 & +2x_4 & & & = & 7 \\
 & -2x_2 & +4x_3 & & +x_5 & & = & 12 \\
 & -4x_2 & +3x_3 & +7x_4 & & +x_6 & = & 10 \\
\multicolumn{8}{l}{x_1, \dots, x_6 \geq 0.}
\end{array} \right] (S)
\end{array} \right.
$$

Solution. Problem (P) is already written in standard form (see Definition 6.1 of Lecture 6). We have:

$$
A = \begin{pmatrix} 1 & 3 & -1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & -4 & 3 & 7 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}, \quad c = (0, 1, -3, 2, 0, 0).
$$

Observe that the canonical basis of $\mathbb{R}^3$,

$$
B := \{A^1, A^5, A^6\}
$$

is a primal feasible basis of S, since $\alpha_0 j = b_j \geq 0$ for all $j \in \mathcal{B}_B = \{1, 5, 6\}$. Thus we can use it as the starting basis within the Primal SIMPLEX Algorithm.

		0	0	0		
	1	A^1	A^5	A^6	Test d.f.b.	
1	A^2	3	-2	-4	-1	
-3	A^3	-1	$\boxed{4}$	3	3	$*$
2	A^4	2	0	7	-2	
	Test p.f.b.	7	12	10	0	
		/	$\frac{12}{4}$ $*$	$\frac{10}{3}$		

2	A^1	A^3	A^6	Test d.f.b.	
A^2	$5/2$	$\boxed{-1/2}$	$-5/2$	$1/2$	$*$
A^5	$1/4$	$1/4$	$-3/4$	$-3/4$	
A^4	2	0	7	-2	
Test p.f.b.	10	3	1	-9	
	$\frac{10}{5/2}$ $*$	/	/		

3	A^2	A^3	A^6	Test d.f.b.
A^1				$-1/5$
A^5				$-4/5$
A^4				$-12/5$
Test p.f.b.	4	5	11	-11

Since $\alpha_{i0} \leq 0$ for all $i \in \mathcal{N}_B = \{1, 5, 4\}$ (i.e., $-1/5 \leq 0, -4/5 \leq 0$ and $-12/5 \leq 0$), we conclude that

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0) = (0, 4, 5, 0, 0, 11)$$

is an optimal solution of problem (P) and

$$\alpha_{00} = -11$$

is the optimal (minimal) value of the objective function f on the feasible set S. □

Exercise 2 Using the Primal SIMPLEX Algorithm prove that the following constrained linear optimization problem has no optimal solutions (more precisely, the objective function is unbounded from below on the feasible set).

$$(P)\quad \left\{ \begin{array}{l} \text{Minimize} \;\; f(x) = -x_2 + x_4 \\ \left. \begin{array}{rrrrrrl} 2x_1 & -x_2 & -x_3 & +x_4 & & & = 6 \\ x_1 & -2x_2 & & & +x_5 & & = 2 \\ -2x_1 & +2x_2 & & & & +x_6 & = 3 \\ \multicolumn{7}{l}{x_1, \ldots, x_6 \geq 0.} \end{array} \right] (S) \end{array} \right. \tag{1}$$

Solution. We have

$$A = \begin{pmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}, \quad c = (0, -1, 0, 1, 0, 0).$$

Observe that the canonical basis of $\mathbb{R}^3$,

$$B := \{A^4, A^5, A^6\}$$

is a primal feasible basis of S, since $\alpha_0 j = b_j \geq 0$ for all $j \in \mathcal{B}_B = \{4, 5, 6\}$. Thus we can use it as the starting basis within the Primal SIMPLEX Algorithm.

		1	0	0		
	1	A^4	A^5	A^6	Test d.f.b.	
0	A^1	2	$\boxed{1}$	-2	2	$*$
-1	A^2	-1	-2	2	0	
0	A^3	-1	0	0	-1	
	Test p.f.b.	6	2	3	6	
		$\frac{6}{2}$	$\frac{2}{1}$	/		
			$*$			

2	A^4	A^1	A^6	Test d.f.b.	
A^5	-2	1	2	-2	
A^2	3	$\boxed{-2}$	-2	4	*
A^3	-1	0	0	-1	
Test p.f.b.	2	2	7	2	
	$\frac{2}{3}$	/	/		
	*				

3	A^2	A^1	A^6	Test d.f.b.
A^5	$-2/3$	$-1/3$	$2/3$	$2/3$
A^4	$1/3$	$2/3$	$2/3$	$-4/3$
A^3	$-1/3$	$-2/3$	$-2/3$	$1/3$
Test p.f.b.	$2/3$	$10/3$	$25/3$	$-2/3$

Since for $i = 3 \in \mathcal{N}_B$ we have $\alpha_{i0} = 1/3 > 0$ and $\alpha_{3j} \leq 0$ for all $j \in \mathcal{B}_B = \{2, 1, 6\}$ (i.e., $\alpha_{32} = -1/3 < 0$, $\alpha_{31} = -2/3 < 0$ and $\alpha_{36} = -2/3 < 0$), we inder that f is unbounded from below on S. Thus problem (P) has no optimal solutions. $\square$

Seminar 5

Exercise 1 Solve the following optimization problem by the Dual SIMPLEX Algorithm.

$$(P)\ \begin{cases} \text{Minimize} \ \ f(x) = 3x_1 + 5x_2 \\ \left.\begin{array}{lllllll} -x_1 & -2x_2 & +x_3 & & & = & -6 \\ & -x_2 & & +x_4 & & = & -2 \\ -2x_1 & & & & +x_5 & = & -4 \\ x_1, \ldots, x_5 \geq 0. & & & & & & \end{array}\right] (S) \end{cases}$$

Solution. Problem (P) is given in standard form and we have:

$$A = \begin{pmatrix} -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ -2 \\ -4 \end{pmatrix}, \quad c = (3, 5, 0, 0, 0).$$

Consider the canonical basis

$$B := \{A^3, A^4, A^5\}.$$

It is easily seen that the set of basic indices is $\mathcal{B}_B = \{3, 4, 5\}$ while the set of non-basic indices is $\mathcal{N}_B = \{1, 2\}$. Since b has at least one negative coordinate (i.e., the horizontal section "Test p.f.b." of the Simplex tableau associated to B contains at least one negative number $b_1 = \alpha_{0j} = -6 < 0$ with $j = 3 \in \mathcal{B}_B$), the basis B is not primal feasible, hence we can not use the Primal SIMPLEX Algorithm for this initial basis. However, since all numbers in the vertical section "Test d.f.b." of the Simplex tableau are less than or equal to zero (i.e., $\alpha_{i0} \leq 0$ for all $i \in \mathcal{N}_B$), the canonical basis B is dual feasible, hence we can use the Dual SIMPLEX Algorithm.

		0	0	0			
	1	A^3	A^4	A^5	Test d.f.b.		
3	A^1	-1	0	-2	-3	$\frac{-3}{-1}$	
5	A^2	$\boxed{-2}$	-1	0	-5	$\frac{-5}{-2}$	*
	Test p.f.b.	-6	-2	-4	0		
		*					

2	A^2	A^4	A^5	Test d.f.b.		
A^1	$1/2$	$1/2$	$\boxed{-2}$	$-1/2$	$\frac{-1/2}{-2}$	*
A^3	$-1/2$	$-1/2$	0	$-5/2$		
Test p.f.b.	3	1	-4	15		
			*			

3	A^2	A^4	A^1	Test d.f.b.
A^5				
A^3				
Test p.f.b.	2	0	2	16

Since in the 3rd Simplex Tableau all numbers in the horizontal section "Test p.f.b." are greater than or equal to zero, we conclude that

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0) = (\alpha_{01}, \alpha_{02}, 0, \alpha_{04}, 0) = (2, 2, 0, 0, 0)$$

is an optimal solution of problem (P) and

$$\alpha_{00} = 16$$

is the optimal (minimal) value of the objective function f on the feasible set S. $\square$

Exercise 2 Using the Dual SIMPLEX Algorithm prove that the following optimization problem has no optimal solutions (more precisely, it has no feasible points, i.e., $S = \emptyset$).

$$(P)\ \begin{cases} \text{Minimize}\ \ f(x) = x_2 - x_3 \\ \qquad\qquad -x_2 \ +x_3 \ +x_4 \ = -1 \\ x_1 \ +3x_2 \qquad\qquad\qquad = \ 0 \\ x_1, \ldots, x_4 \geq 0. \end{cases} \Bigg]\ (S) \tag{1}$$

Solution. We have

$$A = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 3 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad c = (0, 1, -1, 0).$$

First observe that b has a negative coordinate, hence the canonical bases $\{A^3, A^1\}$ and $\{A^4, A^1\}$ are not primal feasible. Therefore we cannot use them as starting bases for the Primal SIMPLEX Algorithm. Consider the canonical basis

$$B := \{A^3, A^1\}.$$

Since all numbers since all numbers in the vertical section "Test d.f.b." of the Simplex tableau are less than or equal to zero (i.e., $\alpha_{i0} \leq 0$ for all $i \in \mathbb{N}_B$), the canonical basis B is dual feasible, hence we can use the Dual SIMPLEX Algorithm.

		-1	0			
	1	A^3	A^1	Test d.f.b.		
1	A^2	$\boxed{-1}$	3	0	$\frac{0}{-1}$	*
0	A^4	1	0	-1	/	
	Test p.f.b.	-1	0	1		
		*				

2	A^2	A^1	Test d.f.b.
A^3	-1	3	0
A^4	-1	3	-1
Test p.f.b.	1	-3	1

Since in the horizontal section "Test p.f.b." of the 2nd Simplex Tableau there exists a negative number (namely $\alpha_{01} = -3 < 0$) above which none of the numbers is negative (more precisely, $\alpha_{31} = 3 \nless 0$ and $\alpha_{41} = 3 \nless 0$), we inder that problem (P) has no feasible points, hence it has no optimal solutions. $\square$

Seminar 6

Exercise 1 Solve the Rock-Scissors-Paper game by means of linear optimization problems.

Solution. Since $\underline{w} \leq w \leq \overline{w}$ and

$$\underline{w} = -1 < 0,$$

we cannot guarantee that w is positive. Therefore we add a suitable constant $k \in \mathbb{R}$ to all elements of the payoff matrix

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

such that the lower value of the new matrix $C + (k)$ is positive, i.e.,

$$\underline{w} + k > 0.$$

For instance, by choosing

$$k = 2$$

we obtain the new payoff matrix

$$C + (k) = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

We associate to this matrix the following two optimization problems:

$$\begin{cases} \text{Minimize } \ u_1 + u_2 + u_3 \\ \begin{array}{rrrl} 2u_1 & +u_2 & +3u_3 & \geq 1 \\ 3u_1 & +2u_2 & +u_3 & \geq 1 \\ u_1 & +3u_2 & +2u_3 & \geq 1 \end{array} \\ u_1, u_2, u_3 \geq 0 \end{cases} \tag{1}$$

and

$$\begin{cases} \text{Maximize } \; v_1 + v_2 + v_3 \\ \\ \begin{array}{rrrl} 2v_1 & +3v_2 & +v_3 & \leq 1 \\ v_1 & +2v_2 & +3v_3 & \leq 1 \\ 3v_1 & +v_2 & +2v_3 & \leq 1 \end{array} \\ v_1, v_2, v_3 \geq 0. \end{cases} \tag{2}$$

Next we solve them by means of the SIMPLEX algorithms. To this aim we transform the problems (1) and (2) into equivalent problems in standard form, namely

$$\begin{cases} \text{Minimize } \; u_1 + u_2 + u_3 \\ \\ \begin{array}{rrrrrrl} -2u_1 & -u_2 & -3u_3 & +u_4 & & & = -1 \\ -3u_1 & -2u_2 & -u_3 & & +u_5 & & = -1 \\ -u_1 & -3u_2 & -2u_3 & & & +u_6 & = -1 \end{array} \\ u_1, \ldots, u_6 \geq 0 \end{cases} \tag{3}$$

and

$$\begin{cases} \text{Minimize } \; -v_1 - v_2 - v_3 \\ \\ \begin{array}{rrrrrrl} 2v_1 & +3v_2 & +v_3 & +v_4 & & & = 1 \\ v_1 & +2v_2 & +3v_3 & & +v_5 & & = 1 \\ 3v_1 & +v_2 & +2v_3 & & & +v_6 & = 1 \end{array} \\ v_1, \ldots, v_6 \geq 0. \end{cases} \tag{4}$$

It is important to notice that the dual of (3) is equivalent to (2), while the dual of (4) is equivalent to (1). Indeed, the dual of problem (3) is

$$\begin{cases} \text{Maximize} \quad -s_1 - s_2 - s_3 \\ \begin{array}{rrrl} -2s_1 & -3s_2 & -s_3 & \leq 1 \\ -s_1 & -2s_2 & -3s_3 & \leq 1 \\ -3s_1 & -s_2 & -2s_3 & \leq 1 \\ s_1 & & & \leq 0 \\ & s_2 & & \leq 0 \\ & & s_3 & \leq 0, \end{array} \end{cases} \tag{5}$$

which, by the change of variables

$$-s = (-s_1, -s_2, -s_3) = v = (v_1, v_2, v_3),$$

becomes the initial problem (2). On the other hand, the dual of problem (4) is

$$\begin{cases} \text{Maximize} \;\; t_1 + t_2 + t_3 \\ \begin{array}{rrrl} 2t_1 & +t_2 & +3t_3 & \leq -1 \\ 3t_1 & +2t_2 & +t_3 & \leq -1 \\ t_1 & +3t_2 & +2t_3 & \leq -1 \\ t_1 & & & \leq 0 \\ & t_2 & & \leq 0 \\ & & t_3 & \leq 0, \end{array} \end{cases} \tag{6}$$

which, by the change of variables

$$t = (t_1, t_2, t_3) := -u = (-u_1, -u_2, -u_3),$$

becomes the initial problem (1).

Lecture 9 Dual simplex algorithm

Ex Consider the diet problem

Nutrients \ Food	F_1	F_2	Req amount of nutrients
N_1	1	2	6
N_2	2	1	8
unit cost	3	5	

$$(P)\begin{cases} \text{Minimize } f(x) = 3x_1 + 5x_2 \\ 1\cdot x_1 + 2\cdot x_2 \geq 6 \\ 2x_1 + 1\cdot x_2 \geq 8 \\ x_1, x_2 \geq 0 \end{cases} \quad (S) \quad \begin{matrix} |\cdot(-1)| + x_3 \\ |\cdot(-1)| + x_4 \end{matrix}$$

Since (P) has no standard form, in order to apply the simplex algo we transform this problem in another equivalent problem written in standard form

$$(P_{stdF})\begin{cases} \text{Minimize } f_{st}(x) = 3x_1 + 5x_2 \\ -x_1 - 2x_2 + x_3 = -6 \\ -2x_1 - x_2 + x_4 = -8 \\ x_1 \ldots x_4 \geq 0 \end{cases} \quad (S_{st})$$

$x_3 = \frac{x_1 + 2x_2}{6}$

$x_4 = \frac{2x_1 + x_2}{-8}$ [illegible]

? [illegible]

$n = 4 \quad m = 2$

$c = (c_1, c_2, c_3, c_4)$
$\quad\;\; 3 \;\; 5 \;\; 0 \;\; 0$

$$A = \begin{array}{c} \\ \end{array}\begin{pmatrix} -1 & -2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$$

$$\begin{array}{cccc} A^1 & A^2 & \underbrace{A^3 \quad A^4}_{\text{basis}} \end{array}$$

Let $B = (A^3, A^4)$ (as loc basis)

$\quad 0 \quad 0 \leftrightarrow c_3, c_4 = 0$

1	A^3	A^4	Tanh sfb
3 A^1	$-1<0$	$-2<0$	$-3<0$
5 A^2	$-2<0$	$-1<0$	$-5<0$
Tanh nfb	$-6<0$	$-8<0$ *	0

B in sfb so we apply DSA

dual simplex algo

$-5\cdot 0 + -8\cdot 0$

$\dfrac{-3}{-2} < -\dfrac{5}{1}$

$= 1.5$ make it pivot

2	A^3	A^1	sfb
A^4	$-\frac{1}{2}<0$	$-\frac{1}{2}$	$-\frac{3}{2}$
A^2	$-\frac{3}{2}<0$	$\frac{1}{2}$	$-\frac{7}{2}$
nfb	$-2<0$ *	-4	12

$(4 - 1)/3 = -\frac{3}{2}$ ($-2\cdot 1 \cdot 2$)

$\dfrac{-6+4}{-2} \quad -\frac{8}{2}$

$\dfrac{-3/2}{-1/2} \quad \dfrac{-7/2}{-3/2}$ * the min $\Rightarrow$ pivot in $-\frac{3}{2}$

3	A^2	A^1	sfb
A^4			
A^3			
nfb	$\frac{4}{3}$	$\frac{10}{3}$	$\frac{25}{3}$

The optimal sol of $(P_{sf}) = x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$

$\qquad\qquad (\frac{2}{3}, \frac{4}{3}, 0, 0)$

The optimal value of f is

$\alpha_{00} = \frac{25}{3}$

$$\alpha_1 = \min\{0, 1, -1\} = -1$$

$$\alpha_2 = -1$$

$$\alpha_3 = -1$$

$$\underline{w} = -1$$

$$\overline{w} = 1$$

$$-1 = \underline{w} \leq w \leq \overline{w} = 1$$

$$w \in [-1, 1]$$

$$\hat{w} = w + 2$$

$$\hat{C} = C + 2 = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Minimize $u_1 + u_2 + u_3$

$$\begin{cases} 2u_1 + 1u_2 + 3u_3 \geq 1 \\ 3u_1 + 2u_2 + 1u_3 \geq 1 \\ 1u_1 + 3u_2 + 2u_3 \geq 1 \end{cases}$$

$$(P2)\begin{cases} \text{Maximize } v_1 + v_2 + v_3 \\ 2v_1 + 3v_2 + v_3 \leq 1 \\ 1v_1 + 2v_2 + 3v_3 \leq 1 \\ 3v_1 + 1v_2 + 2v_3 \leq 1 \end{cases}$$

$$(P1) \begin{cases} u_1 + u_2 + u_3 \\ -2u_1 - u_2 - 3u_3 + u_4 = -1 \\ -3u_1 - 2u_2 - u_3 + u_5 = -1 \\ -u_1 - 3u_2 - 2u_3 + u_6 = -1 \\ u_1, \dots, u_6 \geq 0 \end{cases}$$

$$A = \left(\begin{array}{ccc|ccc} -2 & -1 & -3 & 1 & 0 & 0 \\ -3 & -2 & -1 & 0 & 1 & 0 \\ -1 & -3 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$b = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

		0	0	0		
	1	A^4	A^5	A^6		DSA
1	A^1	-2	-3	-1	-1	$\frac{1}{2}$
1	A^2	-1	-2	-3	-1	1
1	A^3	-3	-1	-2	-1	$\frac{1}{3}$ *
		-1	-1	-1	0	
		*				

2	A^3	A^5	A^6		
A^1	$\frac{2}{3}$	$-\frac{7}{3}$	$+\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{7}$ ✗
A^2	$\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{2}{3}$	$\frac{2}{5}$
A^4	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	1
	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	

✗

3	A^3	A^1	A^6		
A^5	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	
A^2	$-\frac{1}{7}$	$\frac{5}{7}$	$-\frac{18}{7}$	$-\frac{3}{7}$	$\frac{1}{6} = 0{,}16$ ✗
A^4	$-\frac{3}{7}$	$\frac{1}{7}$	$-\frac{15}{21}$	$-\frac{2}{7}$	$\frac{6}{15} = 0{,}4$
	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	

4	A^3	A^1	A^2	
A^5				
A^6			$\frac{-7}{18}$	
A^4				
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

$$\hat{w} = \frac{1}{\frac{1}{6}+\frac{1}{6}+\frac{1}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$\hat{x}^0 = \hat{w}\cdot m^0 = 2\cdot\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) = \left(\frac{1}{3} \ \dots\right)$$

$$w = \hat{w} - h = 2 - 1 = 1$$