12.10.2021

# Seminar W3 - 832

**Recap.** *If $X \sim Beta(a,b)$, $a, b > 0$, then its pdf is:*

$$f_X(x) = \frac{1}{B(a,b)} \cdot x^{a-1}(1-x)^{b-1}$$

*for $x \in [0,1]$ (and 0 otherwise). We have:*

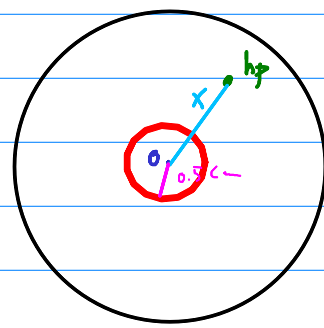
$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \ a, b > 0$$

**Exercise 3.** A darts player aims at the bullseye, which has a diameter of 1 cm. The distance from the center of the dartboard to the hit point of an arrow thrown by the player follows the $Beta(a,b)$ distribution, where $a, b > 0$, with a mean value of $\frac{3}{4}$ cm and a standard deviation of $\frac{\sqrt{15}}{20}$ cm. We assume that the throws are independent.

(a) Find the probability that the player hits the bullseye on a throw.

(b) Find the mean value of the number of throws before the player hits the bullseye the first time.

(c) Find the probability that the player hits the bullseye 2 times out of 10 throws.




"the player hits the bullseye" = $= (X \leq 0.5)$

$$E(X) = \int_{\mathbb{R}} x \cdot f_X(x) \, dx$$

$$\sigma = \sqrt{V(X)} = \sqrt{E(X^2) - E(X)^2}$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f_X(x) \, dx$$

a) We have to find $P(X \leq 0.5)$, knowing that

$X \sim Beta(a, b)$

We know that $E(X) = \frac{3}{4}$ and that $\sigma = \frac{\sqrt{15}}{20}$

Using the formulas for $E(X)$ and $V(X)$, we must find $a$ and $b$.

$$E(X) = \int_{\mathbb{R}} x \cdot f_X(x)\, dx$$

**Recap.** *If $X \sim Beta(a,b)$, $a,b > 0$, then its pdf is:*

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*for $x \in [0,1]$ (and 0 otherwise). We have:*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx, \ a,b > 0$$

$$E(X) = \int_0^1 x \cdot f_X(x)\, dx = \int_0^1 x \cdot \frac{1}{B(a,b)} \cdot x^{a-1} \cdot (1-x)^{b-1} dx =$$

$$= \frac{1}{B(a,b)} \cdot \underbrace{\int_0^1 x^a \cdot (1-x)^{b-1} dx}_{B(a+1,b)} = \frac{B(a+1,b)}{B(a,b)}$$

$$B(a,b) = \frac{a-1}{a+b-1} B(a-1,b)$$

$$\Rightarrow \quad E(X) = \frac{B(a+1,b)}{B(a,b)} = \frac{a}{a+b}$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f_X(x)\, dx = \int_0^1 x^2 \cdot \frac{1}{B(a,b)} \cdot x^{a-1} \cdot (1-x)^{b-1} dx$$

$$= \frac{1}{B(a,b)} \underbrace{\int_0^1 x^{a+1} \cdot (1-x)^{b-1} dx}_{B(a+2,b)} = \frac{B(a+2,b)}{B(a,b)} = \frac{B(a+2,b)}{B(a+1,b)} \cdot \frac{B(a+1,b)}{B(a,b)} =$$

$$= \frac{a+1}{a+b+1} \cdot \frac{a}{a+b} = \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2 =$$

$$= \frac{a(a+1)\cdot(a+b)}{(a+b)^2(a+b+1)} - \frac{a^2\cdot(a+b+1)}{(a+b)^2(a+b+1)} = \frac{a(a^2+a+ab+b-a^2-ab-a)}{(a+b)^2(a+b+1)} =$$

$$= \frac{ab}{(a+b)^2\cdot(a+b+1)}$$

$$\Rightarrow \begin{cases} \frac{a}{a+b} = \frac{3}{4} \\ \frac{ab}{(a+b)^2(a+b+1)} = \frac{15}{400} \end{cases} \Rightarrow \begin{cases} 4a = 3a+3b \\ \frac{ab}{(a+b)^2(a+b+1)} = \frac{15}{400} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = 3b \\ \frac{ab}{(a+b)^2(a+b+1)} = \frac{15}{400} \end{cases} \Rightarrow \begin{cases} a = 3b \\ \frac{3b^2}{16b^2\cdot(4b+1)} = \frac{15}{400} \end{cases}$$

$$\Rightarrow \frac{3}{16(4b+1)} = \frac{15}{400} \Rightarrow \cancel{400}^{\cancel{50}\,10}\cdot\cancel{3} = \cancel{15}^{2}\cdot\cancel{16}\cdot(4b+1) \Rightarrow$$

$$\Rightarrow 4b+1 = 5 \Rightarrow b = 1 \Rightarrow a = 3$$

$$\Rightarrow X \sim Beta(3,1)$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{B(3,1)}\cdot x^2, & \text{for } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$B(3,1) = \int_0^1 x^{3-1}\cdot(1-x)^{1-1}\,dx =$$

$$= \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\Rightarrow f_X(x) = \begin{cases} 3x^2, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$P(\text{"the player hits the bullseye"}) = P(X \le \frac{1}{2}) =$

$= F_X(\frac{1}{2}) = \int_{-\infty}^{1/2} f_X(x)\,dx = \int_0^{1/2} f_X(x)\,dx =$

$= \int_0^{1/2} 3x^2\,dx = x^3 \Big|_0^{1/2} = \frac{1}{8}$

(b) $Y$ = "no. of throws before the player hits the bullseye first"

$Y \sim Geo(p)$, where $p = \frac{1}{8}$

$\Rightarrow Y \begin{pmatrix} k \\ pq^k \end{pmatrix}_{k \in \mathbb{N}}$ $\quad q = \frac{7}{8}$ $\quad E(Y) = \frac{q}{p} = \frac{7/8}{1/8} =$

$= 7$

$\Rightarrow P(Y = k) = \frac{1}{8} \cdot \frac{7^k}{8^k}$

$E(Y) = \sum_{k \in \mathbb{N}} k \cdot f_Y(k) = \sum_{k \in \mathbb{N}} k \cdot pq^k = \frac{1}{8} \sum_{k \in \mathbb{N}} k \cdot \left(\frac{7}{8}\right)^k =$

$= \frac{1}{8} \sum_{k=1}^{\infty} k \cdot \left(\frac{7}{8}\right)^k = \frac{1}{8} \cdot \frac{7}{8} \cdot \sum_{k=1}^{\infty} k \cdot \left(\frac{7}{8}\right)^{k-1}$.

$f(\alpha) := \sum_{k=0}^{\infty} \alpha^k = \lim_{n \to \infty} \sum_{k=0}^{n} \alpha^k = \lim_{n \to \infty} \frac{\alpha^{n+1} - \alpha}{\alpha - 1} =$

$$= \frac{0-\alpha}{\alpha-1} = \frac{\alpha}{1-\alpha}$$

$$\Rightarrow f'(\alpha) = \sum_{k=1}^{\infty} k\alpha^{k-1} = \left(\frac{\alpha}{1-\alpha}\right)'_\alpha = \left(-1+\frac{1}{1-\alpha}\right)'_\alpha =$$

$$= (-1)\cdot\frac{1}{(1-\alpha)^2}\cdot(-1) = \frac{1}{(1-\alpha)^2}$$

$$E(Y) = \frac{7}{64}\cdot \underbrace{\sum_{k=1}^{\infty} k\cdot\left(\frac{7}{8}\right)^{k-1}}_{f'\left(\frac{7}{8}\right) = \frac{1}{\left(1-\frac{7}{8}\right)^2} = 64}$$

$$\Rightarrow E(Y) = \frac{7}{64}\cdot 64 = 7$$

(c) 10 throws

"throw" = Bernoulli trial

$p$ = proba of success $= \frac{1}{8}$

$P(\text{"2 successes in 10 throws"}) = P(Z=2)$

$Z$ = "no of successes in 10 throws"

$$Z \sim \text{Bino}(\underset{10}{\underset{\downarrow}{n}}, \underset{\frac{1}{8}}{\underset{\downarrow}{p}}) \Rightarrow Z\begin{pmatrix} k \\ C_{10}^k\cdot p^k\cdot q^{10-k}\end{pmatrix}_{k=\overline{0,10}}$$

$$\Rightarrow P(Z=2) = C_{10}^2\cdot\frac{1}{8^2}\cdot\frac{7}{8^8} = \frac{10\cdot 9}{2}\cdot\frac{1}{8^2}\cdot\frac{7^8}{8^8} \simeq 0.2416$$

**Exercise 4.** An electronics store purchases a certain type of motherboard in boxes of 100 pieces each. It is known that 40% of the purchased boxes have 3 defective motherboards each, while 60% of the purchased boxes have 2 defective motherboards each. An employee of the store tests 2 motherboards from a random box. Find the probability that at least one of the tested motherboards is defective.

<u>Sol.</u>: Type I : 3 defective / 100

Type II : 2 defective / 100

P("at least one of the MB is defective") = 1 − P("none of the two is defective")

A = "both MB that we chose are not defective"

$B_i$ = "we chose a box of type $i$", $i = 1,2$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) =$$

$$= P(A \mid B_1) \cdot P(B_1) + P(A \mid B_2) \cdot P(B_2)$$

$P(A \mid B_1)$ = P("selecting 2 non-defective MB's out of a type I box") =

$$= \frac{\text{\# favourable cases}}{\text{\# possible cases}} = \frac{C_{97}^2}{C_{100}^2}$$

$$P(A \mid B_2) = \frac{C_{98}^2}{C_{100}^2}$$

$P(B_1) = \frac{4}{10}$ $\qquad P(B_2) = \frac{6}{10}$

$$\Rightarrow \quad P(A) = \frac{4}{10} \cdot \frac{C_{97}^{2}}{C_{100}^{2}} + \frac{6}{10} \cdot \frac{C_{98}^{2}}{C_{100}^{2}} \approx 0.95236$$

$P($"having at least 1 defective"$) =$

$= 1 - P(A) \approx 0.05$