I. Introduction to PDEs

WHY PDEs? The old Galileo Galilei answer still holds in this case: Because

"Mathematics (read PDEs) is the Language in which God has written the Book of Nature"

We will concentrate on the

HEAT Eq. (describes diffusion of heat in a body)

WAVE Eq. (describes oscillations of a string)

but you'll see more equations (Projects, etc.)

Book:

Ian Stewart: In Pursuit of the Unknown – 17 Equations that Changed the World

1. PDEs from Mathematical Modelling

Question: How do we reach certain PDEs?

IDEA: discrete $\to$ cont.

for a different IDEA check [Precup. Sec 2.1.]

§ 1.1. The Heat Equation (from Brownian Motion)

IDEA: $\underbrace{\text{goes}}_{\text{back to}}$ FELLER

Random Vars. Let X_n be random vars.
which can take only 3 values

values	h	$-h$	0
with probabilities	p	q	r

$(p+q+r=1.)$

Understand X_n as independent trials so
$S_n = X_1 + \dots + X_n$ accumul. result of n trials.

Brownian Motion Interpretation (1D gas)

A particle is exposed to random collisions that occur at fixed times $\tau, 2\tau, 3\tau$ etc. and produce displacements $h, -h$, or 0.

S_n = the position of the particle after n collisions (at time $t = n\tau$)

Dynamics: If the current position of the particle is $x = kh$ then the next position will be

$x+h \quad x-h \quad$ or $\quad x \quad$ with probabs.

$p \quad q \quad r$

Define $u_{n,k} = \text{probab}\,(S_n = kh)$

"probab that the particle occupies pos $x = kh$ after $t = n\tau$"

$S_{n+1} = kh$ can occur in only 3 <u>mutually exclusive</u> (!) ways

$$
\begin{array}{llc}
S_n = (k-1)h, & X_{n+1} = h & p \\
S_n = kh, & X_{n+1} = 0 & r \\
S_n = (k+1)h, & X_{n+1} = -h & q
\end{array}
$$

By adding probabilities we have

$$(1) \quad u_{n+1,k} = p\,u_{n,k-1} + q\,u_{n,k+1} + r\,u_{n,k}$$

<u>The continuum limit</u> (Ass. that $h, \tau \ll 1$ very small)

$u_{n,k}$ = average of cont. quantity $u(t,x)$ over an h interval

$= u(n\tau, kh)\,h \qquad t = n\tau \quad x = kh$

We can rewrite (1) as

$$(1c) \quad u(t+\tau, x) = p\,u(t, x-h) + q\,u(t, x+h) + r\,u(t,x)$$

using $r = 1-p-q$ we have that

$$\frac{u(t+\tau,x) - u(t,x)}{\tau} = -\frac{p}{\tau}\left[u(t,x) - u(t,x-h)\right] + \frac{q}{\tau}\left[u(t,x+h) - u(t,x)\right]$$

By Taylor's formula to

$$u(t,x+h) - u(t,x) = \frac{\partial u}{\partial x}(t,x)\,h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x)\,h^2 + \dots$$

...

combining all these

$$\text{(FP)}\quad \frac{\partial u}{\partial t} = -\underline{\frac{(p-q)h}{\tau}\frac{\partial u}{\partial x}} + \underline{\frac{1}{2}\frac{(p+q)h^2}{\tau}\frac{\partial^2 u}{\partial x^2}} + \dots$$

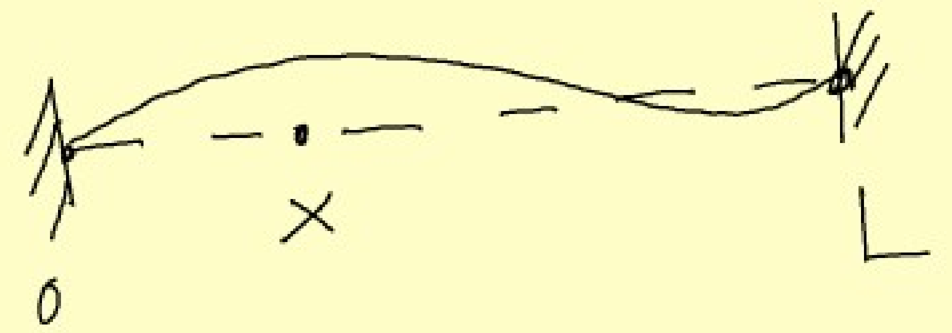
After discarding HOT we are left with the <u>advection</u>-<u>diffusion</u> (Fokker-Planck) equation

If fwd & bwd collisions are equally probable (no advection) $p-q=0$, then we have the HEAT (DIFFUSION) eq.

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}$$

Shorthand Notation $u_t = c\,u_{xx}$.

§ 1.2. The Wave Eq.



$u(t,x)$ = position of the x-particle at time t

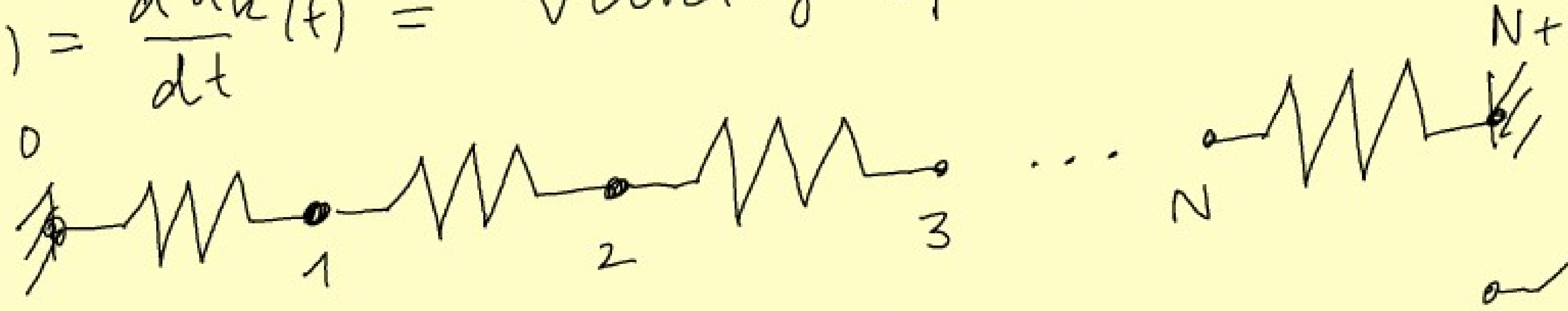
Transversal vs. Longitudinal oscillations:

similar math. $\sin\theta \approx \theta$ limit

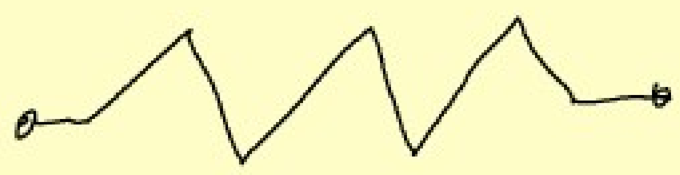
Discrete model: Chain of pointmasses connected by elastic springs

$u_k(t)$ = deviation from equil. of the mass "k"

$v_k(t) = \dfrac{du_k(t)}{dt}$ = velocity of mass "k"



"streched"



Start with "Energy"

$$K(t) = \sum_{k=1}^{N} \frac{1}{2} m v_k(t)^2 \qquad E(t) = \sum_{k=0}^{N} \frac{1}{2} c \left(u_{k+1}(t) - u_k(t)\right)^2$$

kinetic energy — potential / elastic energy

The equations of motion are given by Hamilton's system

with $H = K + E$ total energy

$$\text{"}\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} \end{cases}\text{"}$$

$$\frac{du_k}{dt} = v_k(t)$$

$$m\,\frac{dv_k}{dt} = -c\left(u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)\right)$$

The continuum limit

h = length of one spring $\qquad h \sim \dfrac{L}{N}$

$m = \dfrac{M}{N}$ M = mass of chain/string $\qquad m \sim \dfrac{M}{N}$

"$N \longrightarrow \infty$"

$$u(t, x) = u(t, kh) = u_k(t)$$

$$\underbrace{u(t, x+h)}_{u_{k+1}(t)} - \underbrace{u(t, x)}_{u_k(t)} = \frac{\partial u}{\partial x}(t,x)\, h + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t,x)\, h^2 + \dots$$

...

Plug all Taylor expansions in

$$(2) \qquad m \frac{d^2 u_k}{dt^2} = -c\,(u_{k+1} - 2u_k + u_{k-1})$$

Hamilt. syst. written as 2^{nd} order eq.

then we get the WAWE eq.

$$\frac{\partial^2 u}{\partial t^2} = (ch)\frac{h}{m} \frac{\partial^2 u}{\partial x^2} + \dots$$

ch = elastic modulus, m/h = mass density

$$u_{tt} = a\, u_{xx} \qquad \text{Shorthand notation.}$$

2. Solving PDEs: simple examples

Lecture 1: How can we obtain certain PDEs from discrete models?

Now: How can we find the solutions of those PDEs?

§2.1. The Heat eq. on $\mathbb{R}$

(1) $u_t = u_{xx}$, $t \geq 0$, $x \in \mathbb{R}$, $u(t,x) = ?$

SCALING: if $u(t,x)$ solves (1) then so does $u(\lambda^2 t, \lambda x)$

↑ change units in which you measure

So, the ratio $\frac{x}{t^{1/2}}$ is interesting!

IDEA: look for solution of the form

$$u(t,x) = \frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right) \quad \text{"dilation scaling"}$$

r

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right)\right) = -\frac{1}{2}\frac{1}{t^{3/2}} v - \frac{1}{t^{1/2}} v' \frac{1}{2}\frac{x}{t^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{t^{1/2}} \cdot \frac{1}{t^{1/2}} v'\right)$$

$$= \frac{1}{t^{3/2}} v''\left(\frac{x}{t^{1/2}}\right)$$

$r := \frac{x}{t^{1/2}}$, $v = v(r)$, $' = \frac{d}{dr}$

So we get (1) $\Leftrightarrow$ $v''(r) + \frac{1}{2} r v'(r) + \frac{1}{2} v = 0$

$$(1) \iff v''(r) + \underbrace{\tfrac{1}{2} r v'(r) + \tfrac{1}{2} v}_{\tfrac{1}{2}(r v)'} = 0 \quad (1_v) \,\Big|\, \int dr$$

$r = \frac{x}{t^{1/2}}$,

$$v' + \tfrac{1}{2} r v = C_0$$

Take $C_0 = 0$ (Argument $\lim_{r\to\infty} v, \lim_{r\to\infty} v' = 0$ because it makes sense to have $u(t,x) \longrightarrow 0$ as $|x| \to \infty$ t = fixed)

[Evans, p. 46]

then $v(r) = C_1 e^{-\frac{r^2}{4}}$ Gaussian

[see Exercise 2]

$$u(t,x) = \frac{C_1}{t^{1/2}} e^{-\frac{x^2}{4t}}$$

Rk. It is convenient to take C_1 such that

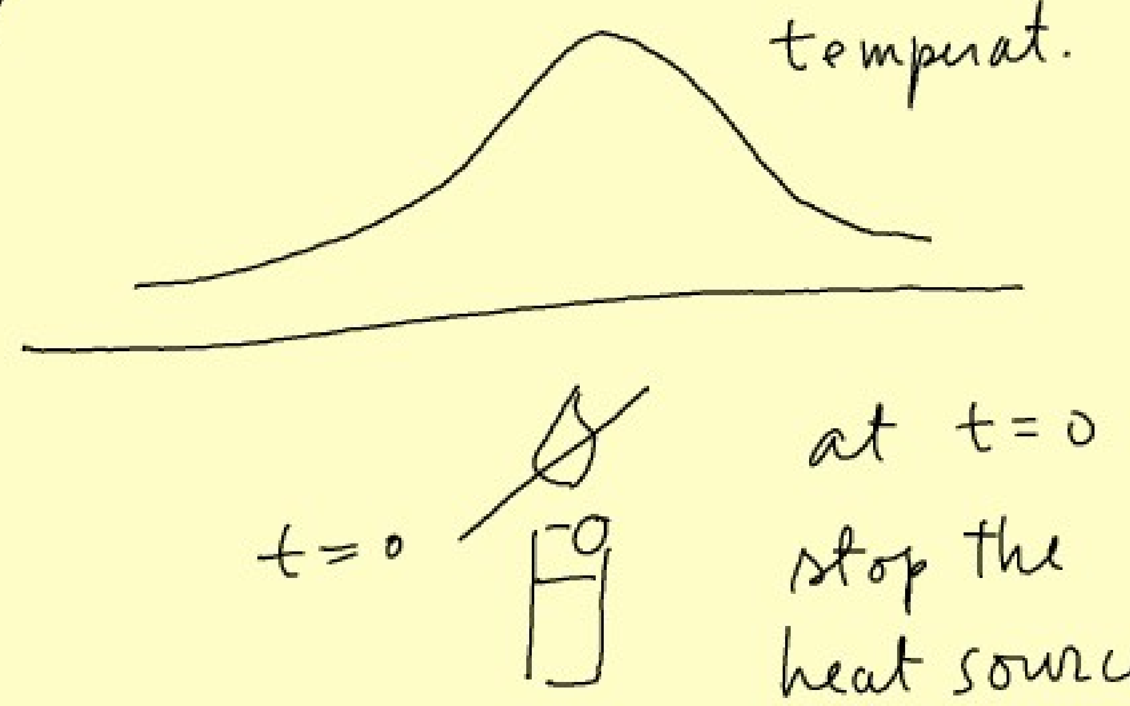
$$\int_{-\infty}^{\infty} u(t,x)\, dx = 1 \qquad \text{[see; Exercise 4]}$$

This way you get the fundamental solution of the Heat eq.

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \qquad x \in \mathbb{R},\ t \geq 0$$

$$\Phi_t = \Phi_{xx}, \qquad \int_{-\infty}^{\infty} \Phi(t,x)\, dx = 1 \quad \forall\, t \geq 0.$$

The **I**nitial **V**alue **P**roblem associated to the Heat eq. (IVP)



$$u_t = u_{xx} \qquad (1)$$

$$u(0,x) = g(x) \qquad (\text{ic})$$

↖ given / known

<u>Claim</u> [see Exercises #2 = next week]

The solution of the Heat IVP = (1) + (ic)

is given by a <u>convolution</u> integral

$$u(t, x) = \int_{-\infty}^{\infty} \Phi(t, x-y)\, g(y)\, dy$$

$$e^{-\frac{(x-y)^2}{4t}}$$

Trick with convolution is that all

$\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ derivatives act on Φ

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \Phi(\dots)\, g(y)\, dy = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \Phi(\dots) \cdot g(y)\, dy$$

$$\frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \Phi(\dots)\, g(y)\, dy = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} \Phi(\dots) \cdot g(y)\, dy$$

§2.2. The Transport eq.

(2) $\quad u_t + c\,u_x = 0, \quad c \in \mathbb{R}, \quad t \geq 0, \quad x \in \mathbb{R}$

TRAVELING WAVES: look for wave-sols.

$$u(t,x) = v(x - ct) \qquad (*) \quad \leftarrow$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} v(x-ct) = -c\,v'(x-ct)$$

$$c \,\Big|\; \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} v(x-ct) = v'(x-ct)$$

$$u_t + c\,u_x = -\cancel{cv'} + \cancel{cv'} = 0 \ \checkmark$$

The IVP for the Transport eq.

$$u_t + c\,u_x = 0 \qquad (2)$$
$$u(0,x) = g(x) \qquad (IC)$$

From $(*)$ $\quad g(x) = u(0,x) = v(x - \cancel{c0})$

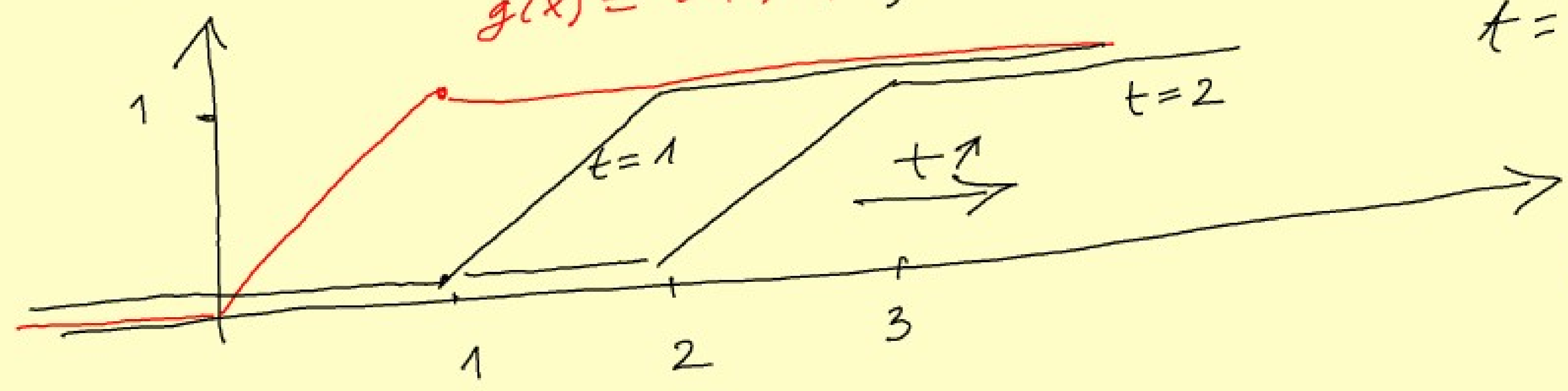
This means that $u(t,x) = v(x-ct) = g(x-ct)$

the solution is actually $\quad u(t,x) = g(x-ct)$

$g(x) = u(0,x)$, $c=1$

$t=1 \quad u(1,x) = g(x-1)$

$t=2 \quad u(2,x) = g(x-2)$



The nonhomogeneous Transport IVP

$$u_t + c\,u_x = f(t,x) \qquad (2f)$$
$$u(0,x) = g(x) \qquad (iC)$$

To solve $(IVP) = (2f) + (iC)$ we use the new variable $z = z(s) := u(t+s,\ x+cs)$

$$\frac{dz}{ds}(s) = u_t(t+s,\ x+cs) + u_x(t+s,\ x+cs)\cdot c$$

$$(**) \qquad \overset{(2f)}{=} f(t+s,\ x+cs)$$

The trick is that we can write both u & g using z

$$u(t,x) - \underbrace{g(x-ct)}_{\substack{u(0,\,x-ct)\\ = z(-t)}} = z(0) - z(-t) = \int_{-t}^{0} \underbrace{\frac{dz}{ds}(s)}_{\overset{(**)}{=}\ f(t+s,\ x+cs)} ds$$

So

$$\underbrace{u(t,x)}_{\substack{\text{Sol of}\\ \text{nonhomog}\\ (IVP)}} = \underbrace{g(x-ct)}_{\substack{\text{Sol of}\\ \text{homog}\\ IVP}} + \underbrace{\int_0^t f(s,\ x+c(t-s))\,ds}_{\substack{\text{contrib due to}\\ \text{RHS of } (2f)}}$$

§ 2.3. The Wave eq

$$u_{tt} = u_{xx} \quad (3) \qquad x \in \mathbb{R},\ t \geq 0$$

$$u(0,x) = u_0(x) \quad (iC_1) \qquad u_0, v_0 \text{ given}$$

$$u_t(0,x) = v_0(x) \quad (iC_2)$$

To solve the iVP $= (3) + (iC_1) + (iC_2)$

we use the transport eq. / wave solutions.

Notice that based on properties of $\frac{\partial}{\partial \dots}$ we have

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)$$

$a^2 - b^2 = (a+b)(a-b)$

Transport eq.

$$\text{So } (3) \iff \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u}_{=v} = 0 \iff v_t + v_x = 0$$

where $v(t,x) = u_t(t,x) - u_x(t,x) \quad (**)$

This means that v solves a transport iVP

$$\begin{cases} v_t + v_x = 0 \\ v(0,x) = a(x) \overset{(**)}{=} v_0(x) - \frac{\partial}{\partial x} u_0(x) \quad (iC_1 \,\&\, iC_2) \end{cases}$$

sol is $v(t,x) = a(x-t)$

Now returning to the def. of v which leads to a transport iVP for u:

$$\begin{cases} u_t - u_x = v(t,x) = a(x-t) \\ u(0,x) = u_0(x) \quad (iC_1) \end{cases}$$

$$\hookrightarrow u(t,x) = u_0(x+t) + \int_0^t a(x + (t-s) - s)\, ds$$

and we reach d'Alembert's Formula for the sol of $(3) + (iC_1) + (iC_2)$:

$$u(t,x) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\, dy.$$

II. Linear IVPs and the Fourier Transform

Before we start: A brief hystory of the exponential

EULER 18^{th} century

$$\begin{cases} \frac{d}{dt} u(t) = -a u(t) \\ u(0) = u_0 \end{cases} \qquad a \in \mathbb{R},\ u(t) \in \mathbb{R} \qquad \text{Sol } u(t) = e^{-at} u_0$$

PEANO ~ 1900

$$\begin{cases} \frac{d}{dt} u = -A u(t) \\ u(0) = u_0 \end{cases} \qquad u(t) = \begin{pmatrix} u_1(t) \\ \cdots \\ u_d(t) \end{pmatrix} \qquad \text{Sol } u(t) = e^{-tA} u_0$$

matrix $d \times d$

$$A = (a_{ij})_{i,j=\overline{1,d}}$$

exp. of a matrix $e^A = I + A + \frac{1}{2} A^2 + \dots$ (Taylor)

YOSIDA ~ 1930

$$\begin{cases} \frac{d}{dt} u = -A u \\ u(0) = u_0 \end{cases} \qquad u(t) \in X \text{ Banach sp.} \qquad \text{Sol } u(t) = e^{-tA} u_0$$

$A \in \mathcal{L}(X)$ i.e.

$A : X \to X$, lin, bdd.

everything works just as in $\mathbb{R}^d$. Use Taylor for e^A.

HILLE & YOSIDA late 1940s

$$\begin{cases} \frac{d}{dt} u = -A u \\ u(0) = u_0 \end{cases} \qquad u(t) \in X \text{ Banach sp.} \qquad \text{Sol } u(t) = \text{"}e^{-tA}\text{"} u_0$$

$A : D(A) \subsetneq X \to X$ $\qquad S(t)$

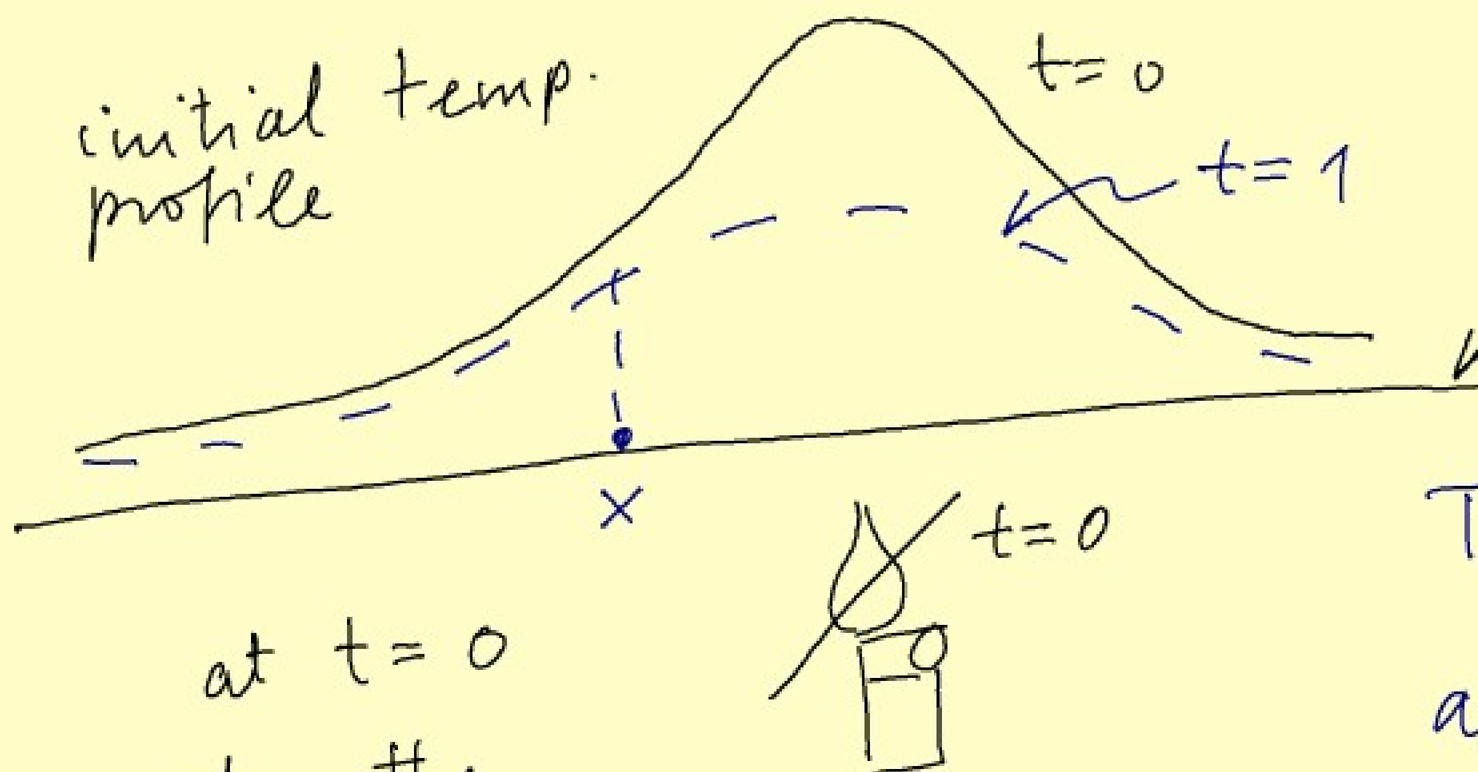
lin. <u>un</u>bounded op. $\qquad$ semigroup.

The issue: $A, A^2, \dots$ don't have the same domain

Key Idea: treat evol. PDE as ODE in Banach or Hilbert sp.

(modern evol. PDEs)

Example: Heat eq. IVP $(*)\begin{cases} u_t = u_{xx} & t>0,\ x\in\mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$



at $t=0$ stop the heat source

The point is that at each time t you can describe the temp. profile by a function of x

$$u(t,x) = \underbrace{u(t)}_{\text{a function of } x \text{ for every } t}(x)$$

rewrite

rewrite $(*)$
$$\begin{cases} \underbrace{\frac{d}{dt} u(t)} = \frac{\partial^2}{\partial x^2} u(t) & \text{in } L^2(\mathbb{R}) \\ u(0) = u_0 & \text{in } L^2(\mathbb{R}) \end{cases}$$

$$u : [0,T) \to L^2(\mathbb{R}), \quad u(t) \in L^2(\mathbb{R})$$

$$L^2(\mathbb{R}) = \left\{ u : \mathbb{R} \to \mathbb{R} : \begin{array}{l} u \text{ measurable} \\ \int_{\mathbb{R}} u(x)^2\, dx < \infty \end{array} \right\}$$

§ 3.1. L^p spaces

$$\|u\|_{L^p} = \left(\int |u(x)|^p dx \right)^{1/p}$$

$1 \le p < \infty$ $\quad L^p(\underset{\Omega}{\mathbb{R}^n}) = \left\{ u : \underset{\Omega}{\mathbb{R}^n} \to \mathbb{R} : \text{meas.} \; \int_{\underset{\Omega}{\mathbb{R}^n}} |u(x)|^p dx < \infty \right\}$

L^∞ is different

$L^\infty(\underset{\Omega}{\mathbb{R}}) = \left\{ u : \underset{\Omega}{\mathbb{R}^n} \to \mathbb{R} : \text{meas } \& \; \exists C \text{ with } |u(x)| \le C \text{ a.e. } \underset{\Omega}{\mathbb{R}^n} \right\}$

frequentey used are L^1, L^2, L^∞

All L^p spaces ($p = \infty$ included) are Banach spaces (See [Brezis, Th 4.8])

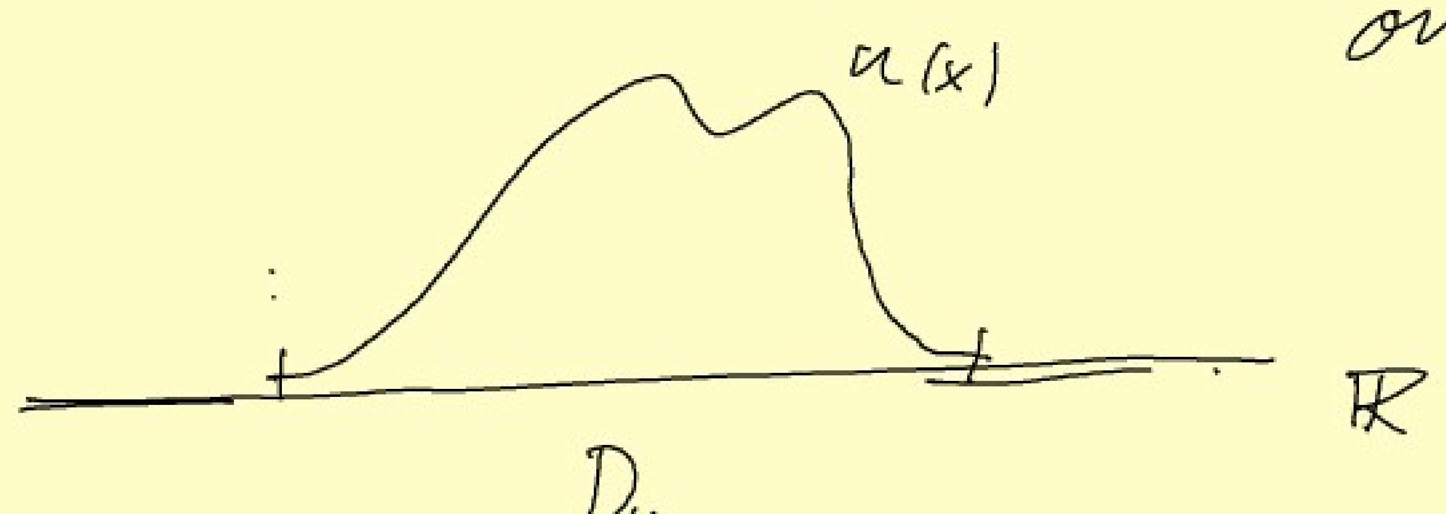
We will use quite often the Approx. result

Th 1 ([Brezis, Th 4.12]) $\quad C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $\quad 1 \le p < \infty$

" any L^p function can be approx. by a sequence of cont. functions that vanish outside a compact subset of $\mathbb{R}^n$ "

$C_c(\mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{R}$ cont. : u vanishes (is zero) outside a compact $D_u \}$

$n = 1$



— L^2 is a Hilbert space

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§ 3.2. Convolution product and the Fourier Transform

Convolution: $f, g : \mathbb{R}^n \to \mathbb{R}$

$$(f * g)(x) \overset{\text{Def.}}{=} \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Th 2. (fundam prop. of convol.)
If $f \in L^1$, $g \in L^p$ $(1 \le p \le \infty)$ then $f * g \in L^p$
and $\| f * g \|_{L^p} \le \| f \|_{L^1} \| g \|_{L^p}$

The Fourier Transform ass. for the moment that $u \in L^1 \cap L^2$

$$\mathcal{F}(u)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i x \cdot y} u(x)\, dx$$

$x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ $\quad i = \sqrt{-1} \in \mathbb{C}$

$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$|x|^2 = x \cdot x = x_1^2 + x_2^2 + \dots + x_n^2$ Euclidean Norm

The $\mathcal{F}$ transforms a function $u(x)$ into a new function $\hat{u}(y) = \mathcal{F}(u)(y)$

The inverse Fourier Transform

$$\mathcal{F}^{-1}(\hat{u})(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{+i x \cdot y}\, \hat{u}(y)\, dy$$

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✓

Fundamental properties of $\mathcal{F}$

(1) $\mathcal{F}(f * g) = (2\pi)^{n/2} \mathcal{F}(f)\, \mathcal{F}(g)$

(2) $\mathcal{F}(D^\alpha u)(y) = (iy)^\alpha\, \mathcal{F}(u)(y)$

$\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex

$|\alpha| := \alpha_1 + \dots + \alpha_n$, $\quad D^\alpha = \dfrac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}$

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ for $x = (x_1, \dots, x_n)$

$$\mathcal{F}(\Delta u)(y) = -|y|^2\, \mathcal{F}(u)(y)$$

$\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dots + \dfrac{\partial^2}{\partial x_n^2}$ Laplace operator

this was wrong in 1st version of these Notes!

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Further properties:

(PLANCHEREL 'T') $\quad \|u\|_{L^2} = \|\mathcal{F}(u)\|_{L^2}$

(SHIFT) $\quad \mathcal{F}(u(x - x_0))(y) = e^{-i x_0 \cdot y}\, \mathcal{F}(u)(y)$

(SCALE) $\quad \mathcal{F}(u(\lambda x))(y) = \dfrac{1}{|\lambda|^n}\, \mathcal{F}(u)\left(\dfrac{y}{\lambda}\right)$

($\mathbb{C}$ CONJUGATE) $\quad \mathcal{F}(\overline{u(-x)})(y) = \overline{\mathcal{F}(u(x))(y)}$

(INVARIANT) $\quad \mathcal{F}\left(e^{-\frac{1}{2}|x|^2}\right)(y) = e^{-\frac{1}{2}|y|^2}$

here (& in (2)) $\quad |x|^2 = x_1^2 + \dots + x_n^2$, $\quad |y|^2 = y_1^2 + \dots + y_n^2$

§ 3.3. The $\mathcal{F}$ Approach

Heat IVP $\begin{cases} \frac{d}{dt} u(t) = \Delta u(t) & t > 0 \text{ in } L^2(\mathbb{R}^n) \\ u(0) = u_0 & \text{in } L^2(\mathbb{R}^n) \end{cases}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}$$

The $\mathcal{F}$ Approach consists of several steps

Step 1. Apply $\mathcal{F}$ to the IVP

$$\mathcal{F}\left(\frac{d}{dt} u(t)\right)(y) = \frac{d}{dt} \mathcal{F}(u)(y)$$

$$\mathcal{F}(\Delta u(t))(y) = -|y|^2 \mathcal{F}(u)(y)$$

$$\mathcal{F}(u(0))(y) = \mathcal{F}(u_0)(y)$$

with the convenient notation $\hat{u}(t) := \mathcal{F}(u(t))(y)$

Heat IVP $\xrightarrow{\mathcal{F}}$ ODE IVP $\begin{cases} \frac{d}{dt} \hat{u}(t) = -|y|^2 \hat{u}(t) \\ \hat{u}(0) = \hat{u}_0 \end{cases}$

here y is a parameter

Step 2. Solve ODE IVP $\quad \hat{u}(t) = e^{-|y|^2 t} \hat{u}_0$

$$\mathcal{F}(u(t))(y) = e^{-|y|^2 t} \mathcal{F}(u_0)(y)$$

Step 3. Apply $\mathcal{F}^{-1}$ (reverse engineering)

" Find ? : $\quad \mathcal{F}(?)(y) = e^{-|y|^2 t} \mathcal{F}(u_0)(y)$ "

nontrivial!

$\mathcal{F}^{-1} \Big| \quad \mathcal{F}(f * g) = (2\pi)^{n/2} \mathcal{F}(f)\mathcal{F}(g)$

(**) $\quad \mathcal{F}^{-1}(\mathcal{F}(f)\,\mathcal{F}(g)) = \frac{1}{(2\pi)^{n/2}} f * g$

in our case $\quad \underbrace{e^{-|y|^2 t}}_{\mathcal{F}(f)} \; \underbrace{\mathcal{F}(u_0)(y)}_{\mathcal{F}(g)}$ ✓

Find ? : $\quad \mathcal{F}(?)(y) = e^{-t|y|^2}$

Recall "invariant" $\quad \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\cancel{y}) = e^{-\frac{1}{2}|y|^2}$

"scaling" $\quad \mathcal{F}(u(\lambda x))(y) = ..$

So, after computations,

finally $\quad ? = N(t)(x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } t>0 \\ 0 & \text{for } t \le 0 \end{cases}$

HEAT KERNEL

$$\mathcal{F}(N(t))(y) = e^{-t|y|^2}$$

So, back to (**) we have

$$u(t)(x) = (N(t) * u_0)(x)$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy$$

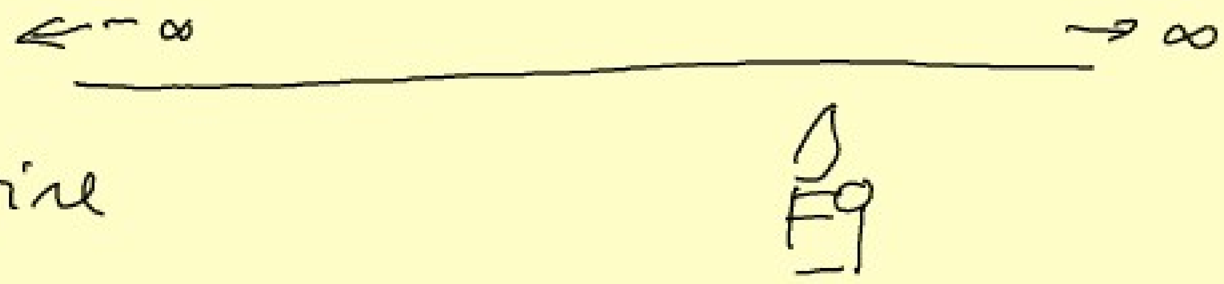
II. Linear PDEs in bounded (spatial) domains

7. Boundary Value Problems I: Motivation & Classical Theory

§7.1. Motivation

Until now: equations on unbounded domains

e.g. Heat in infinite wire



From now on: bounded domains

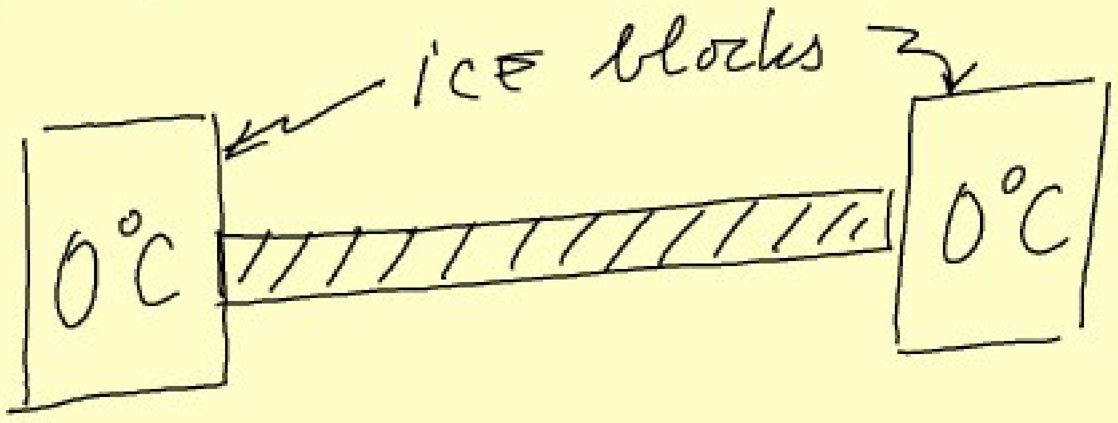
So, consider $\Omega = (0,1) \subset \mathbb{R}$ and the Heat Initial Boundary Value Problem

$$(\text{IBVP}) \quad \begin{cases} u_t = u_{xx} & t \geq 0, \quad x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \quad (\text{IC}) \\ u(t,0) = u(t,1) = 0 & \forall t \geq 0 \quad (\text{BC}) \end{cases}$$

$\Leftrightarrow \; u = 0$ on $\partial\Omega$

(DIRICHLET) Boundary condition

Model: Heatflow in a finite bar of lenghth $= 1$ and with fixed temp. at ends ($= 0°C$)



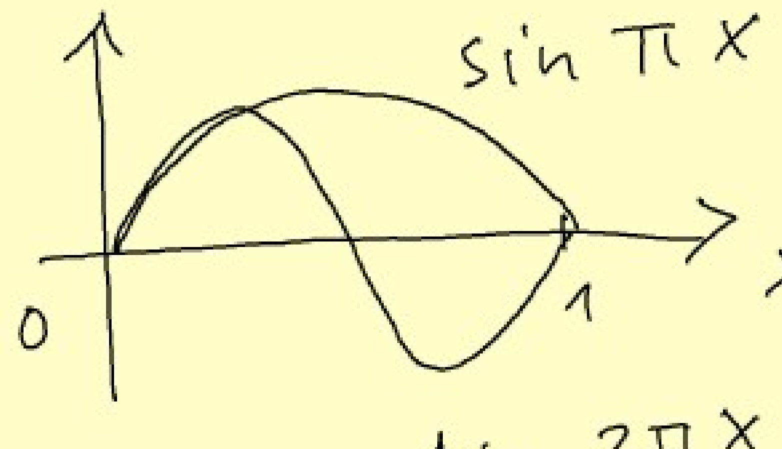
J. FOURIER: Analytic Theory of Heat (1822)

AIM: Solve Heat (IBVP)

Fourier's Approach

(Important and <u>new</u>: the (BC)!)

Insight 1: Which functions satisfy (BC)?



Consider $\sin(\pi x)$

$\sin(\pi 2 x)$

$\dots$

$k \in \mathbb{N}^*$ $\sin(\pi k x)$

all these satisfy the (BC).

Insight 2: You also have to consider the eq...

• Let's compute $(\underline{\sin \pi k x})_{xx} = (\pi k \cos \pi k x)_x$

$= -\pi^2 k^2 \underline{\sin \pi k x}$

• Consider also time-dependence

Consider $u(t, x) = e^{-\pi^2 k^2 t} \sin \pi k x$

$$\underbrace{\left(e^{-\pi^2 k^2 t} \sin \pi k x\right)_t}_{u_t \,=\, u_{xx}} = \left(e^{-\pi^2 k^2 t}\right)_t \sin \pi k x$$

$$= \underline{-\pi^2 k^2 e^{-\pi^2 k^2 t} \sin \pi k x}$$

$$\overbrace{\left(e^{-\pi^2 k^2 t} \sin \pi k x\right)_{xx}} = e^{-\pi^2 k^2 t} (\sin \pi k x)_{xx}$$

$$= \underline{-\pi^2 k^2 e^{-\pi^2 k^2 t} \sin \pi k x}$$

This is a sol of Heat eq. & satisfies (BC).

Furthermore: eq is linear so if $u_1, \dots, u_k$ sols.
then so is any lin. comb. $\sum c_k u_k$ (sol.)
$(c_k \in \mathbb{R})$

This means that

(*) $$u(t,x) = \sum_{k=1}^{\infty} c_k e^{-\pi^2 k^2 t} \sin \pi k x$$ is sol & satisf (BC).

Insight 3: What about the (IC)?

Choose c_k such that (IC) is satisf. by (*).

Fourier's glorious insight is that you can compute all c_k using the (simple) fact

$$(\perp) \quad \int_0^1 \sin \pi k x \, \sin \pi j x \, dx = \begin{cases} 0, & \text{if } k \neq j, \\ 1/2, & \text{if } k = j. \end{cases}$$

orthogonality w.r.t. $L^2(\Omega)$ scalar product

Want: $$u_0(x) = u(0,x) \overset{(*)}{=} \sum_{k=1}^{\infty} c_k \cancel{e^{-\pi^2 k^2 0}}^{\,1} \sin \pi k x$$

$$u_0(x) = \sum c_k \sin \pi k x \quad \Big| \ \sin \pi j x, \int_0^1$$

$$\int_0^1 u_0(x) \sin \pi j x \, dx = \frac{1}{2} c_j \quad \text{(only one nonzero term!)}$$

So, we can compute all c_k (since $u_0(x)$ is given),

and $$u(t,x) = \sum_{k=1}^{\infty} c_k e^{-\pi^2 k^2 t} \sin \pi k x$$ is a sol of the (IBVP)!

We can solve the Heat (IBVP) for a 1D bounded (spatial) domain.

However, what happens in dim > 1 ?

$$u_t = \Delta u \qquad t \geq 0, \quad x \in \Omega \subset \mathbb{R}^n$$

$$u(0,x) = u_0(x) \qquad x \in \Omega \qquad \text{(IC)}$$

$$u(t,x) = 0 \qquad x \in \partial\Omega, \ t \geq 0 \qquad \text{(BC)}$$

with $\overbrace{}^{\text{notations}}$ $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$\Delta u(x) = \frac{\partial^2 u}{\partial x_1^2}(x) + \dots + \frac{\partial^2 u}{\partial x_n^2}(x)$$

How can you choose appropriate replacements for $\sin \pi k x$ (in a general domain Ω)?

You need to solve the eigenvalue (Helmholz) problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

or more generally the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \text{(BVP)}$$

No time-dep.
No (IC)

These are surprisingly hard problems because they depend on the geometry of Ω (esp. $\partial\Omega$)

§7.2. Classical Dirichlet BVP Theory

Details: [Premp]

IDEA of classical theory: this is a diff eq. So integrate to solve it.

AIM: Solve $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

$\Omega \subsetneq \mathbb{R}^n$, $\quad u : \Omega \to \mathbb{R}$ ("scalar field")

$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_n}(x)\right) \in \mathbb{R}^n$ ("vector field")

gradient

$V = (V_1, \dots, V_n)$; $\quad V_i : \Omega \to \mathbb{R}$ ("vector field")

$\operatorname{div} V(x) = \frac{\partial V_1}{\partial x_1}(x) + \dots + \frac{\partial V_n}{\partial x_n}(x)$ ("scalar field")

divergence

$\Delta u(x) = \operatorname{div} \nabla u(x) = \frac{\partial^2 u}{\partial x_1}(x) + \dots + \frac{\partial^2 u}{\partial x_n}(x)$

Laplacian

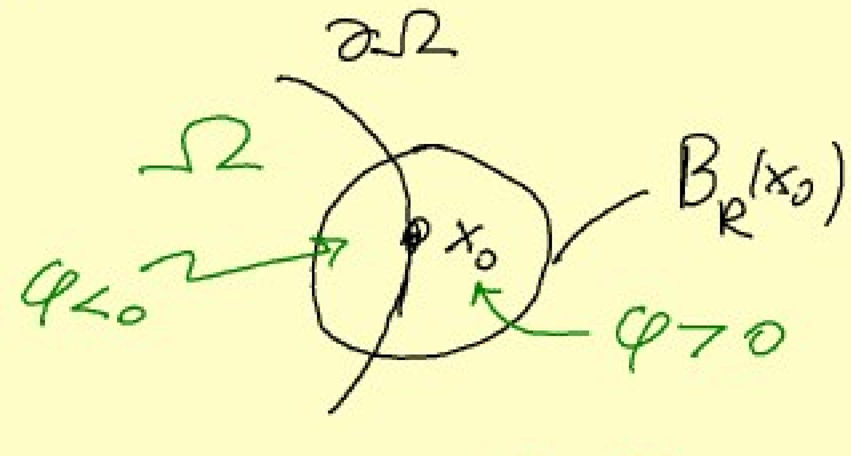
directional derivative

$v \in \mathbb{R}^n$, $|v| = 1$ $\qquad \frac{\partial u}{\partial v}(x) = \nabla u(x) \cdot v$

$\partial\Omega$ boundary of Ω while $\overline{\Omega}$ closure of Ω.

We call an open set $\Omega \subset \mathbb{R}^n$ of <u>class C^k</u> if $\forall x_0 \in \partial\Omega$ $\exists B_R(x_0)$ and $\exists \varphi \in C^k(B_R(x_0))$ with:

1. $\nabla\varphi(x) \neq 0_{\mathbb{R}^n}$ $\quad \forall x \in B_R(x_0)$
2. $\varphi(x) < 0$ $\quad \forall x \in \Omega \cap B_R(x_0)$
3. $\varphi(x) > 0$ $\quad \forall x \in (\mathbb{R}^n \setminus \overline{\Omega}) \cap B_R(x_0)$



"$\varphi = 0$" is the "equation" of the boundary $\partial\Omega$

—

Classical integral calculus results:

Thm 1 (Divergence (or GAUSS-OSTROGRADSKI) Thm)

$\Omega \subset \mathbb{R}^n$ open, bounded and C^1

V vector field, ν normal to $\partial\Omega$

$$\int_{\partial\Omega} V \cdot \nu \, d\sigma = \int_{\Omega} \operatorname{div} V \, dx$$

Thm 2. (GREEN'S Formulae)

(G1) $$\int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, d\sigma = \int_{\Omega} (u \Delta v + \nabla u \cdot \nabla v) \, dx$$

(G2) $$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\sigma = \int_{\Omega} (u \Delta v - v \Delta u) \, dx$$

Idea of Proof: apply Gauss-O. to get (G1)

then (G2) follows directly from (G1) [Recap]

Def: $u : \Omega \to \mathbb{R}$ is called <u>harmonic</u> if

$\Delta u(x) = 0 \quad \forall x \in \Omega$

Thm 3. (GAUSS) $\Omega \subset \mathbb{R}^n$ open, bdd, C^1

while u is harmonic (on Ω) then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\sigma = 0.$$

Rk (see Ex. 21)

Harmonic functions with radial symmetry

$u(x) = u(|x|)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$|x| = \sqrt{x_1^2 + \dots + x_n^2}$ Euclidean norm

are

$$u(x) = C_1 \ln |x| + C_2 \quad x \neq 0 \text{ in } \mathbb{R}^2$$

$$u(x) = C_1 |x|^{2-n} + C_2 \quad x \neq 0 \text{ in } \mathbb{R}^n, \ n \geq 3$$

Def: The fundam. sol. of Laplace's eq

$$N : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad N(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & n = 2 \\ -\underbrace{\frac{1}{(n-2)\omega_n}}_{\text{const}} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

$\Delta N(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

ω_n = measure of unit sphere in $\mathbb{R}^n$

($\omega_2 = 2\pi$, $\omega_3 = 4\pi$, etc.)

Th 4. (RIEMANN - GREEN Formula) Ω open, bdd, C^1

$$\text{(RG)} \quad u(x) = \int_\Omega \Delta u(y)\, N(x-y)\, dy + \int_{\partial\Omega} \left(u(y) \frac{\partial N(x-y)}{\partial \nu_y} - N(x-y) \frac{\partial u(y)}{\partial \nu} \right) d\sigma_y$$

$u \in C^2$

for $x \in \Omega$, ν_y normal in $y \in \partial\Omega$.

[Precup] for Proof.

Consequences of (RG)

Th 5. (Mean Value Thm for Harmonic functions)

$\Omega \subset \mathbb{R}^n$ open, u harmonic on Ω

For any closed ball $\overline{B}_r(x) \subset \Omega$ we have

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, d\sigma$$

("value at center of ball = avg of values on sphere")

Th 6. (The Strong Max Principle)

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ open, connected} \\ u \in C^2(\Omega),\quad \Delta u(x) \geq 0 \ \forall x \in \Omega \\ \exists x_0 \in \Omega : \ u(x_0) = \sup_\Omega u \end{array}\right\} \Rightarrow u = \text{const on } \Omega$$

Th 7. (The Weak Max Princ.)

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ open, bounded} \\ u \in C^2(\Omega) \cap C(\overline{\Omega}) \\ \Delta u \geq 0 \ \text{ in } \Omega \\ u \leq 0 \ \text{ on } \partial\Omega \end{array}\right\} \Rightarrow u \leq 0 \ \text{ on } \overline{\Omega}$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Applications:
- positivity of solutions
- uniqueness of sol.
- cont data dependence of sol. w.r.t. f, g.

But you can't prove existence...

8. Boundary Value Problems II : Modern Theory

$$(\text{BVP}) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

given

Dirichlet BC (but you could also have $u = g$)

Lecture 7 : Classical Idea (Riemann – Green)

"integrate in order to solve the BVP"

BAD News : It doesn't work

Remarkable example where integral calculus works

"Poisson Formula" in dim = 3 !

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

$f \equiv 0$

with $\Omega = B_R(0)$ ball of radius R

$$u(x) = \frac{R^2 - |x|^2}{R\,\omega_n} \int_{|x|=R} \frac{g(y)}{|x-y|^3}\, d\sigma_y \qquad |x| < R$$

(integral over sphere $) = \partial\Omega$

DIRICHLET's fresh IDEA :

$\{$ Optimization (rewrite BVP as Optimiz. Probl)

$\{$ + Approximation (approximate solution)

intimately related to the <u>concept</u> of <u>solution</u> !

§ 8.1. The Dirichlet Principle (for classical sols)

Consider the Dirichlet (BVP) $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ (D)

Def: u is classical sol of (D) if $u \in C^2(\bar{\Omega})$ and (D) holds.

With the (BC) in mind: $C_0^1(\bar{\Omega}) = \{ u \in C^1(\bar{\Omega}) ; \ u = 0 \text{ on } \partial\Omega \}$

functions that vanish on the boundary

The (Dirichlet) Energy functional

$$E : C_0^1(\bar{\Omega}) \to \mathbb{R}, \quad E(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - f u \right) dx$$

see Lecture #7

Th 1. (Dirichlet's Princ.) $\Omega \subset \mathbb{R}^n$ open, bdd and C^1

$f \in C(\bar{\Omega})$ and $u \in C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$

Then the following statements are equivalent

(i) u is classical sol. of (D)

(ii) u satisfies the variational identity

$$\int_\Omega (\nabla u \cdot \nabla v - f v)\, dx = 0 \quad \forall v \in C_0^1(\bar{\Omega}) \quad \text{(Vi)}$$

Dirichlet Energy

(iii) u is a strict global minimum of E

$$E(u) < E(w) \quad \forall w \in C_0^1(\bar{\Omega}), \ w \neq u.$$

Proof: (i) $\Leftrightarrow$ (ii) Exercise 24 see also [Precup].

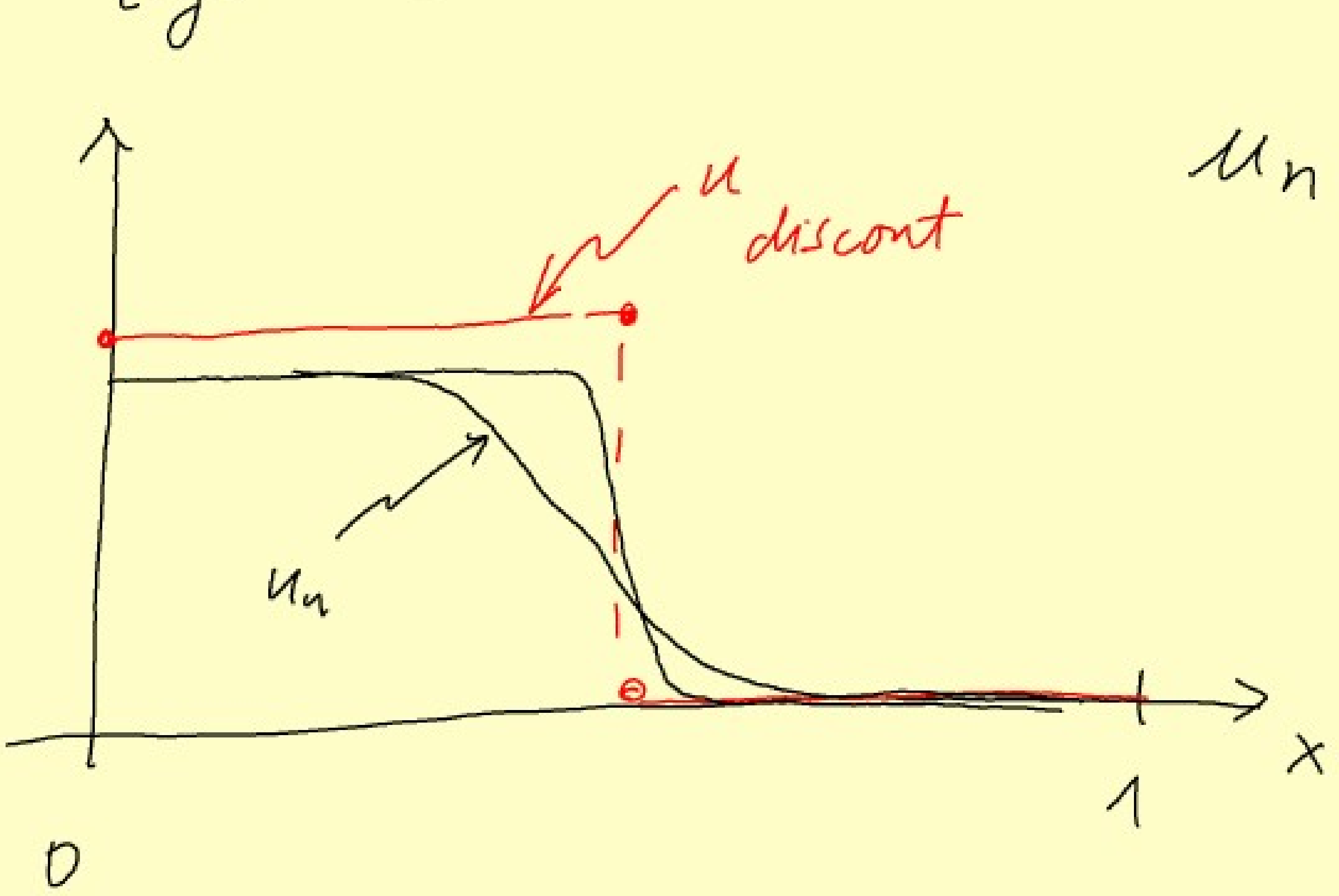
The Problem with classical solutions:

$C^2(\bar{\Omega})$ is NOT complete w.r.t. the L^2 norm

you can construct a sequence of $C^2(\bar{\Omega})$ functions which converge to a discont. function

e.g. $\Omega = (0,1) \subset \mathbb{R}$

$u_n \xrightarrow{L^2} u$



Nonexistence of sols for optimiz. problems:

take $E(x) = \frac{1}{2}(x-\sqrt{2})^2$, $E \to \min!$

this minimization probl. has NO solution in $\mathbb{Q}$

(because the sol is $x^* = \sqrt{2}$)

See also COURANT's Counterexample (Exercise #28)

Modern Theory, the IDEA: move away from C^2 classical solution

3 reasons:
- more general data (i.e. $f \in L^2(\Omega)$)
- existence
- want to work with E-minimizing sequences (approximation)

✓

§ 8.2. Moving away from classical solutions: weak (generalized) solutions

SOBOLEV: consider the smallest complete subspace of $L^2(\Omega)$ which contains $C_0^1(\bar{\Omega})$.

This turns out to be $H_0^1(\Omega)$.

$$C_0^1(\bar{\Omega}) \subset H_0^1(\Omega) \subset L^2(\Omega)$$

$H_0^1(\Omega)$ consists of all limits of $C_0^1(\bar{\Omega})$ Cauchy (w.r.t. $L^2(\Omega)$ norm) sequences

T 2. $H_0^1(\Omega)$ is a Hilbert space with resp. to (complete inner product space)

the energy norm $\|u\|^2_{H_0^1} = \int_\Omega |\nabla u|^2 dx$

associated to the inner product

$$\langle u, v \rangle_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \left(\|u\|^2_{H_0^1} = \langle u, u \rangle_{H_0^1} \right)$$

without Proof (see [Precup], [Brezis]) recom.

Def: $u \in H_0^1(\Omega)$ is a weak (generalized) sol. of (D) if (VI) holds, namely

$$\text{(VI)} \quad \int_\Omega (\nabla u \cdot \nabla v - f v) \, dx = 0 \quad \forall v \in H_0^1(\Omega)$$

$$\iff \quad \langle u, v \rangle_{H_0^1} - \langle f, v \rangle_{L^2} = 0$$

The H_0^1 analysis is possible only due to the critically important Poincaré (–Friedrichs) inequality.

Th 3. (POINCARÉ Inequality)

Ω open and bounded $\Rightarrow$ $\exists\, c = c(\Omega) = \text{const.}$ such that

$$(*) \quad \int_\Omega u^2\,dx \leq c^2 \int_\Omega |\nabla u|^2\,dx \quad \forall u \in C_0^1(\bar\Omega)$$

Proof: [Prump] or Exercise # 25.

Rk $\quad C_0^1(\bar\Omega)$ is dense by construction in $H_0^1(\Omega)$

This means (by extension to H_0^1) that

actually $\quad (*) \quad \|u\|_{L^2}^2 \leq c^2 \|u\|_{H_0^1}^2 \quad \forall u \in H_0^1(\Omega)$

$\hookrightarrow$ Poincaré Ineq. in H_0^1

Th 4. (Dirichlet's Princ. in H_0^1) Ω open, bounded

$f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ then

def of weak sol. (ii) $\langle u, v\rangle_{H_0^1} - \langle f, v\rangle_{L^2} = 0 \quad \forall v \in H_0^1(\Omega) \quad$ (VI)

$\Updownarrow$

(iii) u minimizes E over $H_0^1(\Omega)$

Dirichlet energy

Th 5. (EXISTENCE & UNIQUENESS of a weak. sol)

Ω open, bounded and $f \in L^2(\Omega)$

Then there exist a unique weak sol $u \in H_0^1(\Omega)$ of (D).

iDEA of Proof:

Apply the RIESZ Representation Thm.

Th 6. (RIESZ) Let $(X, \langle \cdot, \cdot \rangle_X)$ Hilbert space.

For any linear continuous functional

$F: X \to \mathbb{R}$, there exist a unique $u_F \in X$ such that we have $F(v) = \langle u_F, v \rangle_X \quad \forall v \in X$

(any linear functional can be represented as product)

To prove Th 5 we check that $F: H_0^1(\Omega) \to \mathbb{R}$, $F(v) = \langle f, v \rangle_{L^2}$ is a linear (obvious) and continuous? functional.

$$|F(v)| = |\langle f, v \rangle_{L^2}| \underset{\text{Schwarz}}{\leq} \|f\|_{L^2} \|v\|_{L^2} \underset{\text{Poincaré Ineq.}}{\leq} \text{Const} \|f\|_{L^2} \|v\|_{H_0^1}$$

So $\|v\| \xrightarrow{H_0^1} 0 \implies |F(v)| \to 0$ (cont ✓)

We can apply Th RIESZ on $(H_0^1(\Omega), \langle \cdot, \cdot \rangle_{H_0^1})$ Hilbert sp

By Riesz, there exists $u_F \in H_0^1(\Omega)$ such that

$$\langle f, v \rangle_{L^2} \overset{\text{Def}}{=} F(v) = \langle u_F, v \rangle_{H_0^1} \quad \forall v \in H_0^1(\Omega)$$

But this is just (VI) for u_F ((VI) = def weak sol

Hence this u_F is the desired unique weak sol. of (D

More PDEs can be approached like this!

- Wave $u_{tt} = \Delta u \iff \begin{cases} u_t = v \\ v_t = \Delta u \end{cases}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

operator matrix

- Damped Wave Eq $u_{tt} = -u_t + \Delta u$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & -I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Visco-elasticity (linear) $u_{tt} = \Delta u_t + \Delta u$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & \Delta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Visco-capillarity (1D) $u_{tt} = -u_{xxxx} + u_{txx} + u_{xx}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\partial_x^4 + \partial_x^2 & \partial_x^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Euler-Bernoulli Beam $u_{tt} = -u_{xxxx}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\partial_x^4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

12. Reaction-diffusion equations I: Fisher-KPP and Allen-Cahn eqns.

Fisher (1937): The wave of advance of advantageous genes
(rel. to population dynamics)

§ 12.1. Fisher's reaction-diffusion model

It deals with a spatially distributed popul. (New!)

Classical population models ($y(t)$ = size of pop. at time t)

Malthus: $y' = ry$, $r > 0$ (growth rate)
(1798, 1826) unrealistic exp. growth

Verhulst: $y' = ry\left(1 - \frac{y}{K}\right)$
(1838, 1847) $K > 0$ capacity of environment to sustain a popul. ($\sim$ scarce resources)
self-limiting (logistic) behavior

$y(t) \equiv 0$ is unstable equil.

Neither of these models account for spatial distrib. of the pop.

Fisher's (new, 1937) model: assumes a spatial distribution of the population.

IDEA: Superimpose two effects

- a logistic (Verhulstian) population growth (at every point in space, the pop. grows according to Verhulst)
- population migrates according to <u>Fick's Law</u> (moves from areas with high population density to areas of low density)

$u(t,x)$ = pop. at time t located at x

Fisher's Eq. $\quad u_t = \mu_D u_{xx} + ru\left(1-\frac{u}{K}\right)$

Dimensionless form $\quad \boxed{u_t = u_{xx} + u(1-u)}$

can be obtained by a change of vars.

$$t_{new} = \frac{t}{r^{-1}}, \quad x_{new} = \frac{x}{\sqrt{\mu_D/r}}, \quad u_{new} = \frac{u}{K}$$

Fisher's original question: are waves in the pop possible/reasonable?

$\rightarrow$ YES [Fisher, 1937]

KOLMOGOROV - Petrovsky - Piskunov 1937 (KPP)

study Fisher's eq.. They study initial profiles that will approach a travelling wave.

§ 12.2. Travelling wave sols (TWS) for Fisher eq.

We look for $u(t,x) = U(x-ct)$, $c>0$ (TWS)

U = wave front, c = wave speed (two unknowns)

plug (TWS) in (Fisher) $\quad {}' = \frac{d}{dz}$

(FTWS) $\quad -cU' = U'' + U(1-U)$, $U = U(z)$

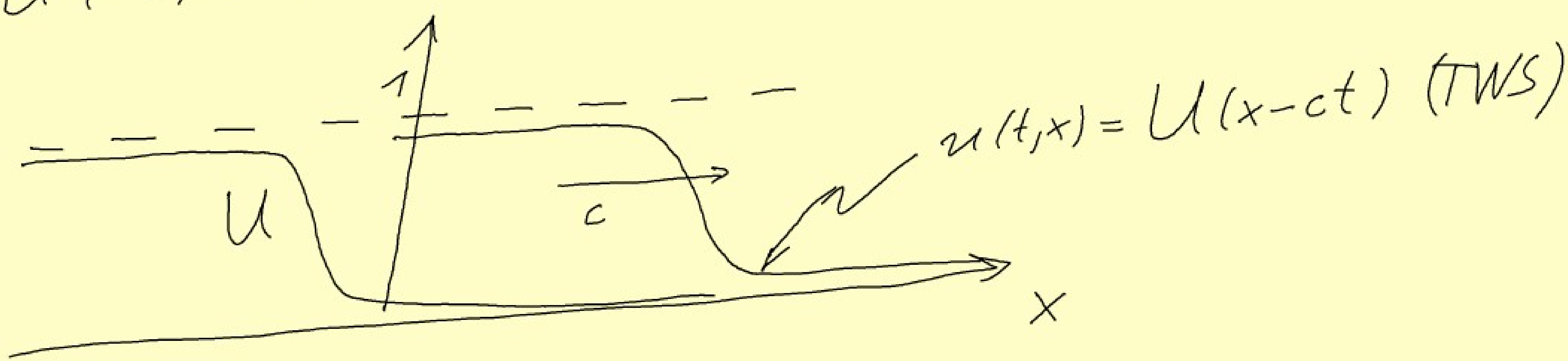
2nd order (nonlinear!) ODE but not explicitly solvable

However, standard (qualitative) ODE / Dyn. Syst. techniques (equilibrium points / stability) allow a complete understanding of (FTWS).

Th 1. ([Logan: An introduction to nonlinear PDEs])

For each $c \geq 2$ there exists a unique wave profile U (and conseq. a unique TWS u) with U monot. decreasing and

$U(-\infty) = 1$, $U(\infty) = 0$ and $U'(\pm\infty) = 0$



Although no explicit formula is available there exists a very good approx. via <u>perturbation theory</u> (details [Logan])

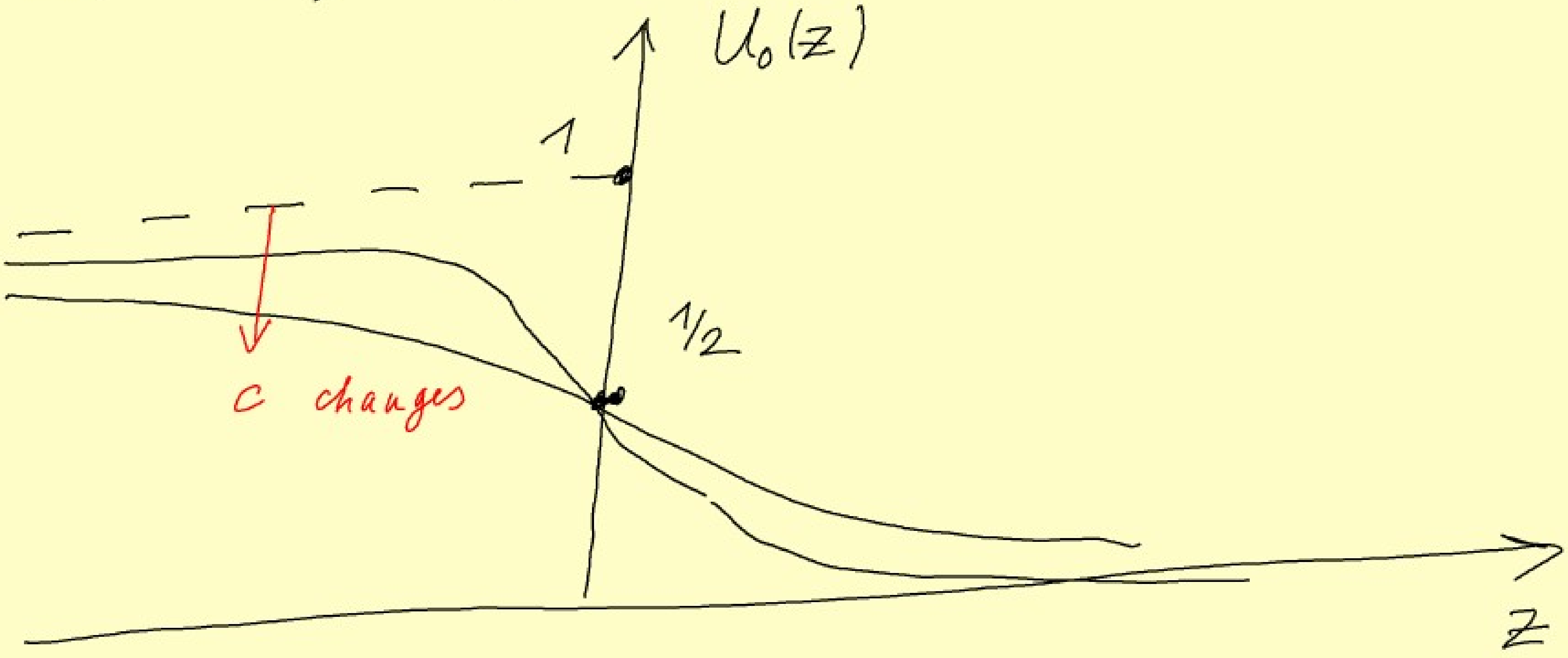
(works for large $c \gg 1$)

$$U(z) = \boxed{\frac{1}{1+e^{z/c}}} + \frac{1}{c^2} e^{z/c}\left(1+e^{z/c}\right)^{-2} \ln \frac{4e^{z/c}}{\left(1+e^{z/c}\right)^2} + O\left(\frac{1}{c^4}\right)$$

Actually, the first term already gives a very good approximation of the profile!

$$U_0(z) = \frac{1}{1+e^{z/c}}$$

$$U_0(-\infty) = 1, \quad U_0(\infty) = 0, \quad U_0(0) = \frac{1}{2}$$



§12.3. Reaction-diffusion equations & systems

General form $\quad u_t = \underbrace{u_{xx}}_{\text{diffusion}} + \underbrace{f(u)}_{\text{reaction}}$

System (RD) $\quad \begin{cases} u_t = u_{xx} + f(u,v) \\ v_t = \mu v_{xx} + g(u,v) \end{cases}$

Rk: f, g are nonlinear (coupling) terms and depend on both reactants.

Examples: Allen-Cahn eq. $\quad u_t = u_{xx} + u(1-u^2)$

Brusselator syst. $\quad \begin{cases} u_t = u_{xx} + 1 + u^2 v - 2u \\ v_t = \mu v_{xx} + u - u^2 v \end{cases}$

Two types of questions:

1. Travelling Waves: existence & stability
2. Equilibria and their stability? (Asymptotic $t \to \infty$ behavior of the system?)

(GRDE) $\quad w_t = D w_{xx} + F(w)$, $\quad w = (w_1, \ldots, w_n)$, D = diffusion matrix (diagonal)

(Equil) $\quad 0 = D w^*_{xx} + F(w^*)$

- $w^*(x)$ stationary / equilibrium solution
- stationary homogeneous solutions $\bar{w}^*(x) = \bar{w}^* \ (\forall x)$ i.e. $\bar{w}^*_x = 0$

(these are sols. of $F(\bar{w}^*) = 0$)

§ 12.4. Lyapunov Theory (for evolutionary PDEs)

- X state space (Hilbert space)
- $\{S(t)\}_{t\geq 0}$ is a dynamical system on X if

(i) $S(t): X \to X$ cont. (for all $t \geq 0$)

(ii) $S(0) = I$ (identity)

(iii) $S(t+s) = S(t) \circ S(s) \quad \forall t, s \geq 0$

(iv) $t \longmapsto S(t)x$ is $C([0,\infty); X)$

Idea: pick $x_0 \in X$ initial state

$S(t)\, x_0$ the state at time t which has evolved from x_0

$\bigcup_{t\geq 0} \{S(t)x_0\}$ the trajectory originating from x_0

- $x^* \in X$ is an equilibrium for $\{S(t)\}_{t\geq 0}$ if $S(t)\, x^* = x^* \quad \forall t \geq 0$ (stationary state)

The set of all equilibria is denoted by $\mathcal{E}$.

- $\Phi: X \to \mathbb{R}$ cont. is a strict Lyapunov function if

(i) $\Phi(S(t)x) \leq \Phi(x) \quad \forall x \in X \ \forall t \geq 0$

decrease along trajectories

(ii) $\Phi(S(t)x) = \Phi(x) \implies x \in \mathcal{E}$.

("(i) is strict "$<$" for $x \notin \mathcal{E}$)

"Th" (La Salle's Invariance Principle)

If Φ is a strict Lyapunov function for $\{S(t)\}$ and $x_0 \in X$ is such that the trajectory $\bigcup_{t \geq 0} \{S(t)x_0\}$ is relatively compact subset of X

then

(i) $\lim_{t\to\infty} \Phi(S(t)x_0) = \ell$ exists

(ii) $\operatorname{dist}(S(t)x_0, \mathcal{E}) \xrightarrow{t\to\infty} 0$

("convergence to equilibrium")

(iii) if $\mathcal{E}$ is discrete then there exists $x^* = x^*(x_0) \in \mathcal{E}$ such that $S(t)x_0 \xrightarrow{t\to\infty} x^*$

Meaning: $\left.\begin{array}{c}\text{strict Lyapunov function} \\ + \\ \text{rel. compact trajectories}\end{array}\right\} \Rightarrow$

$\Rightarrow$ convergence to equilibrium

(the long-time behavior of the system is predetermined

(simple = no chaos, no periodic orbits etc.))

§ 12.5. Application: The Allen–Cahn model

$$\begin{cases} u_t = u_{xx} + (u - u^3) & \text{for } (t,x) \in (0,\infty) \times \Omega \\ u_x(t,0) = u_x(t,1) = 0 & \text{(Neumann BC)} \\ u(0,x) = u_0(x) & \text{(IC)} \end{cases}$$

$\Omega = (0,1) \subset \mathbb{R}$

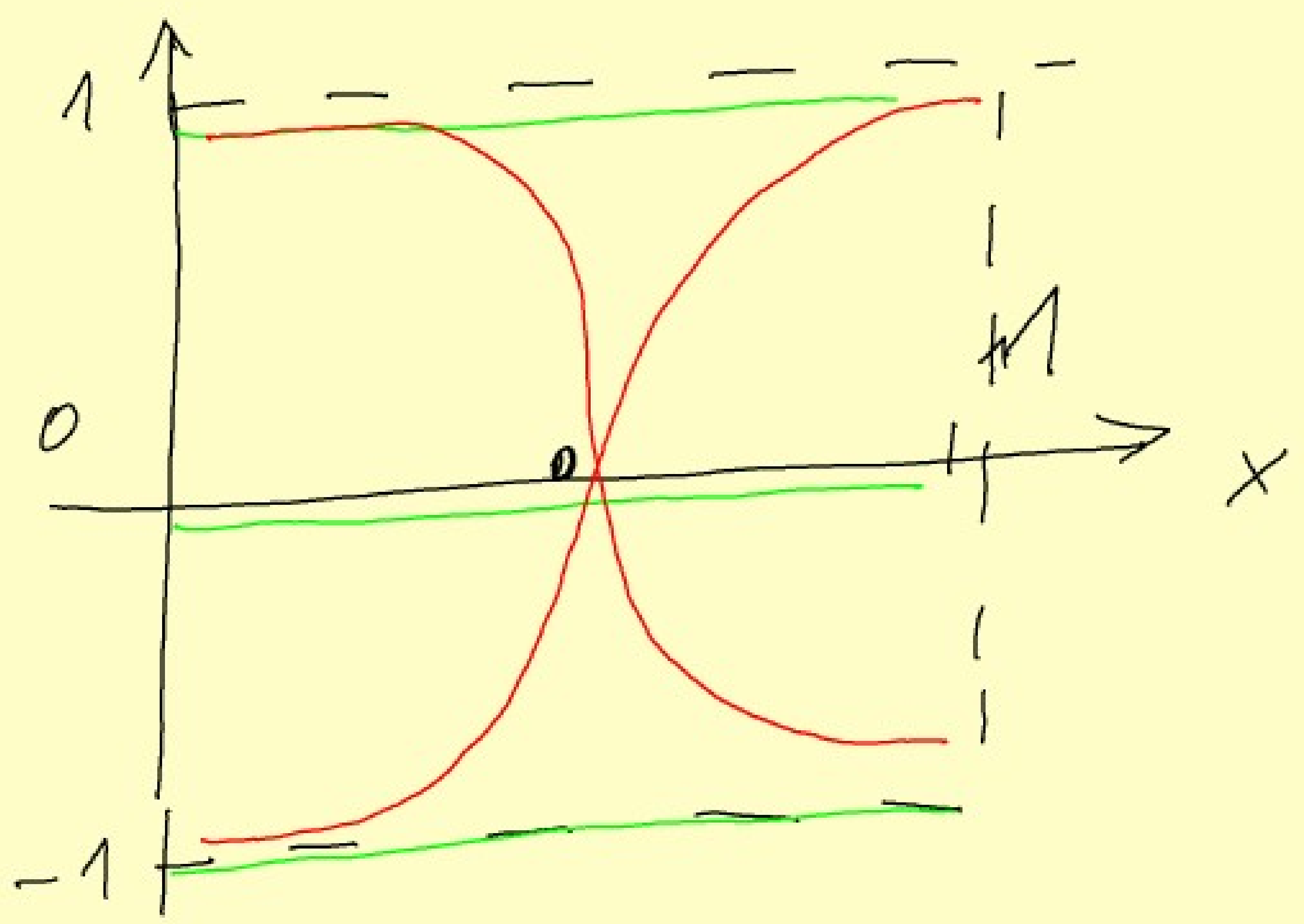
equil. eq is
$$\begin{cases} u^*_{xx} = {u^*}^3 - u^* \\ u^*_x(0) = u^*_x(1) = 0 \end{cases}$$

One can show that this has 5 solutions

$u^*(x) = 0, \quad u^*(x) = 1, \quad u^*(x) = -1 \qquad \forall x \in \Omega$

stationary homogeneous solutions $(u^*_x = 0)$

& two stationationary sols with $\boxed{u^*_x \neq 0}$



The Allen-Cahn Energy is a Lyapunov fct. (strict!)

$$u_t = u_{xx} + u - u^3 \quad \Big| \; u_t, \; \int_\Omega$$

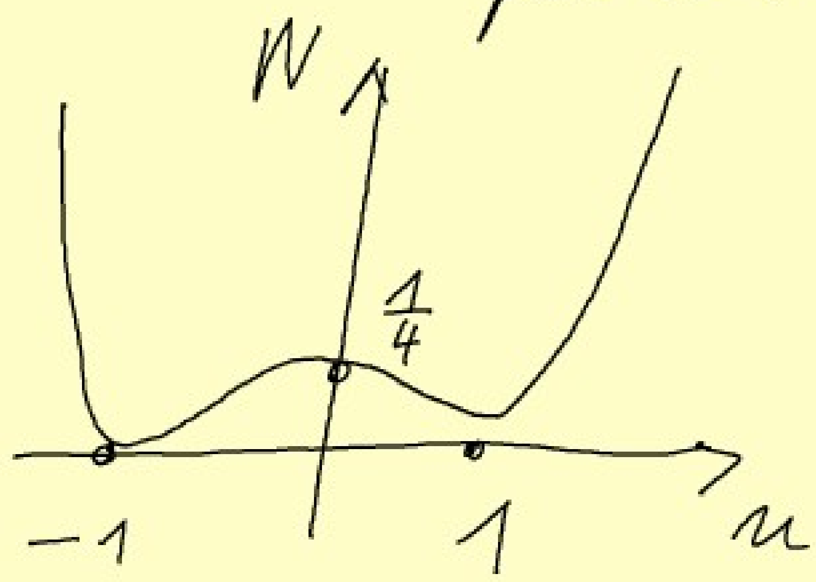
$$(*) \quad \|u_t\|^2_{L^2} = \int_\Omega u_{xx}\, u_t \, dx + \int_\Omega \underbrace{(u-u^3)\, u_t}_{-\frac{d}{dt} W(u(t))} \, dx$$

integrate by parts !BC

$$= \underbrace{- \int_\Omega u_x\, u_{tx} \, dx}_{\frac{d}{dt} \frac{1}{2} \|u_x\|^2_{L^2}}$$

$$W(u) = \frac{1}{4} u^4 - \frac{1}{2} u^2 + \frac{1}{4}$$

↑ double-well potential

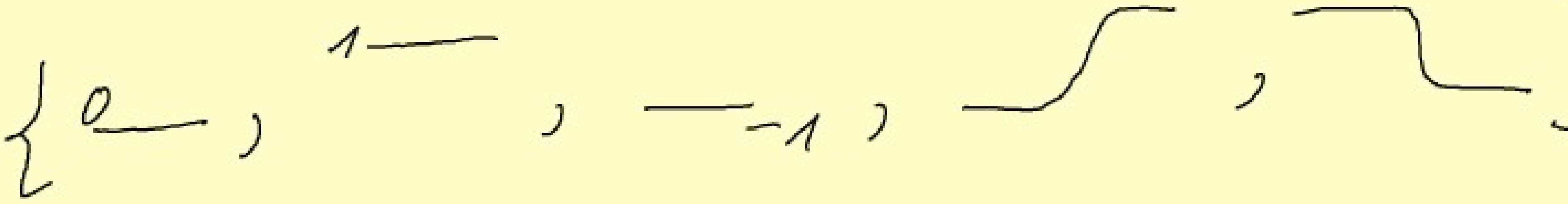


$$(*) \quad \|u_t\|^2_{L^2} = - \frac{d}{dt} \underbrace{\frac{1}{2} \|u_x\|^2_{L^2}}_{\text{Dirichlet energy}} - \frac{d}{dt} \underbrace{\int_\Omega W(u(t))\, dx}_{\text{Double-well energy}} \quad \Big| \cdot (-1)$$

$$\frac{d}{dt} E(u(t)) = - \|u_t\|^2_{L^2}$$ (E is strict Lyap. func.!)

$$E(u) = \frac{1}{2} \|u_x\|^2_{L^2} + \int_\Omega W(u)\, dx$$ Allen-Cahn energy

Allen-Cahn convergence to equilibrium

- $\mathcal{E} = \{$  $\}$

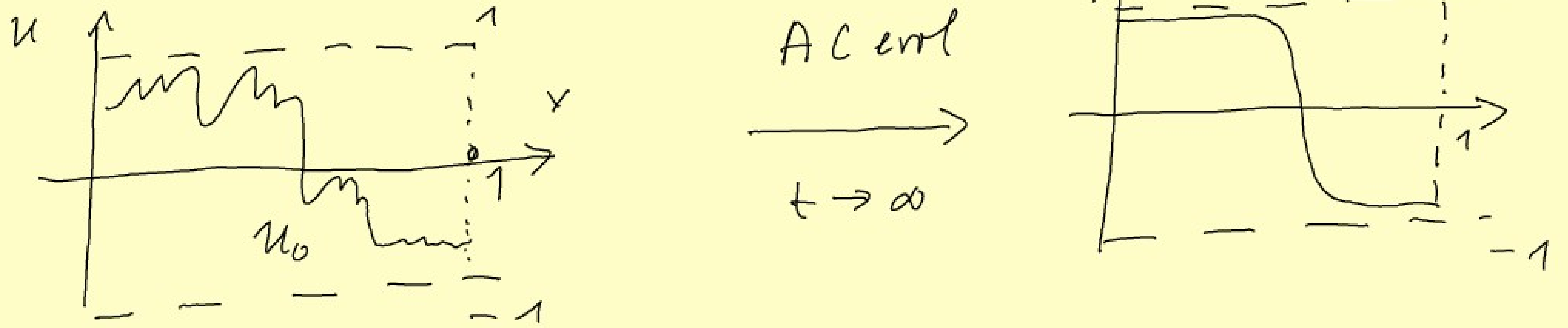
  is discrete

- $E(u) = \frac{1}{2}\|u_x\|^2_{L^2} + \int_\Omega W(u)\,dx$

  strict Lyapunov function for the Dyn Syst generated by the AC eq.

- trajectories are compact (technical proof which relies on "compact embeddings" [Brezis] or [Precup])

LaSalle's Thm. applies $\Longrightarrow$ conv. equil.



13. Reaction-diffusion equations II: Turing instability

"How the Leopard Got Its Spots"

Recall (from Lecture 12) the general reaction-diffusion system

(GRDS) $\quad w_t = D\Delta w + F(w) \quad$ in $(0,\infty)\times\Omega$

with $w = (w_1, \dots, w_m)$, D = diagonal diffusion matrix

Δ = Laplacian, $\Delta w = (\Delta w_1, \dots, \Delta w_m)$

m = nr of equations, $\Omega \subset \mathbb{R}^n$ n = spatial dim.

"You need the more complicated model in order to get the more complicated behavior."

(GRDS) has two types of equilibria (= stationary solutions)

patterns on the fur

- Stationary solutions $\quad 0 = D\Delta w^* + F(w^*)$
  (don't depend on time) $\quad$ syst of nonlin. elliptic PDEs
- homogeneous stationary sols $\quad 0 = 0 + F(w^*)$
  (doesn't depend on space $\quad$ algebraic eq

fur in one color

Stationary sols govern the long-time behavior of the system (Lyapunov)!
(That's what you observe in nature.)

(1952) Alan TURING

$$
(TM) \begin{cases} u_t = \Delta u + \gamma f(u,v) & t>0, \ x \in \Omega \ (\subset \mathbb{R}^2) \\ v_t = d\Delta v + \gamma g(u,v) & \end{cases}
$$

with Neumann BCs & Initial conditions

in terms of (GRDS): $w = (u,v)$, $D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$

$0 < \gamma \in \mathbb{R}$ is a parameter, $F(w) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$

u, v = morphogenes (chemicals responsab. for creation of patterns)

Turing ahead of his time (by decades!):

@ biology: morphogens not exactly known even today

@ math: stability theory for PDES
J.P. LaSalle ~ 1960s

@ computer science (simulations): simulations of animal furs
Murray ~ 1980

What means Turing <u>instability</u>?

? Turing instability

$$(TM)\quad \begin{aligned} u_t &= \Delta u + \gamma f(u,v) \\ v_t &= d\Delta v + \gamma g(u,v) \end{aligned} \quad + \text{ BCs \& ICs}$$

IDEA (Turing instability).

Simple version: homogeneous stationary solutions (corresp. to fur in one color) are unstable and some spatially varying stationary solutions are stable (patterns)

More details: without diffusion, the homog. stat. sol. would be stable (adding diffusion is creating the patterns)

without diff_usion: ODE system

$$\begin{cases} u' = \gamma f(u,v) \\ v' = \gamma g(u,v) \end{cases} \qquad \begin{cases} \cancel{\gamma} f(u^*,v^*) = 0 \\ \cancel{\gamma} g(u^*,v^*) = 0 \end{cases}$$

Dyn. syst

equil. eqns

does not depend on γ

with diffusion: stat. sols of (TM)

$$\begin{cases} \Delta u^* + \gamma f(u^*,v^*) = 0 \\ d\Delta v^* + \gamma g(u^*,v^*) = 0 \end{cases}$$

depends essentialy on γ!

Turing instab.

even more details

ODE system $\begin{cases} u' = \gamma f(u,v) \\ v' = \gamma g(u,v) \end{cases}$ $\quad u^*, v^*$ are such that $\begin{cases} f(u^*, v^*) = 0 \\ g(u^*, v^*) = 0 \end{cases}$

linearize around (u^*, v^*) (Taylor expansion) $\begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = A \begin{pmatrix} u-u^* \\ v-v^* \end{pmatrix} + HOT$

neglect HOT under the assumption that $u-u^*$, $v-v^*$ small.

A is the Jacobian of $F = \begin{pmatrix} f \\ g \end{pmatrix}$ $\quad A = J_F(u^*, v^*)$

$$= \begin{bmatrix} \frac{\partial f}{\partial u}(u^*, v^*) & \frac{\partial f}{\partial v}(u^*, v^*) \\ \frac{\partial g}{\partial u}(u^*, v^*) & \frac{\partial g}{\partial v}(u^*, v^*) \end{bmatrix}$$

The linearized system is $w' = Aw$, $w = \begin{pmatrix} u-u^* \\ v-v^* \end{pmatrix}$

$w = 0_{\mathbb{R}^2}$ ($\Leftrightarrow u = u^*, v = v^*$) is asympt. stable $\Leftrightarrow$

$\Leftrightarrow$ A Hurwitz matrix ($\operatorname{Re}\lambda < 0 \ \forall \lambda$ eigenval. A)

$\Leftrightarrow \left[\frac{\partial f}{\partial u}(u^*, v^*) + \frac{\partial g}{\partial v}(u^*, v^*) < 0 \text{ and } \det A > 0 \right]$

1 stab. condition

Recall that we want: u^*, v^* is asympt stable for the ODE system (= no diffusion)

but u^*, v^* homog. stat. sol is unstable for the full diffusion model.

PDE system $\begin{cases} u_t = \Delta u + \gamma f(u,v) \\ v_t = d\,\Delta v + \gamma g(u,v) \end{cases}$

- Linearize w.r.t. the same $u^*, v^* \in \mathbb{R}$

again $w = \begin{pmatrix} u(t,x) - u^* \\ v(t,x) - v^* \end{pmatrix}$

$$w_t = D \Delta w + \gamma A w$$

$D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$, $A = J_F(u^*, v^*)$

- use a Fourier series method

$\begin{cases} -\Delta \phi_k + \mu_k \phi_k = 0 & \text{in } \Omega \\ \partial \phi_k / \partial \nu = 0 & \text{on } \partial\Omega \end{cases}$ Neumann BC

μ_k eigenvals of the Laplacian $(\mu_0 = 0)$

ϕ_k eigenfunctions —ιι—, $k = 0, 1, \dots$

assume that $w(t,x) = \sum_{k=0}^{\infty} e^{\lambda_k t} \phi_k(x) \begin{pmatrix} a_k \\ b_k \end{pmatrix}$

we want instability (fun not in a single color!)

$\hookrightarrow \lambda_k > 0$ (happens if $\mu_k \in (\underline{\mu}, \overline{\mu})$)

CONDS for instab are:

(1) $\operatorname{tr} A < 0$ $\det A > 0$ (2)

(3) $d \frac{\partial f}{\partial u}(u^*, v^*) + \frac{\partial g}{\partial v}(u^*, v^*) > 0$

(4) $\det A < \left(d \frac{\partial f}{\partial u}(u^*, v^*) + \frac{\partial g}{\partial v}(u^*, v^*)\right)^2 / 4d$

all conds are satisf. for

$f(u,v) = a - u + u^2 v$
$g(u,v) = b - u^2 v$ $\Big|$ $a > 0$, $a + b > 0$

Schakenberg 1979