Heat equation => ut=c*uxx=c*d^2/dx^2

2 - Heat Equation from Brownian motion, Brownian Motion Interpretation (1D gas)

3 - The continuum limit (Heat equation)

4 - Focker Planck Equation, Heat (Diffusion) Equation

5 - Wave Equation (discrete model), Hamilton’s System

6 - The continuum limit (Wave Equation)

7 - Solving PDEs (simple examples), Heat Equation on R (Scaling,IDEA), dilation scaling

8 - Fundamental solution of the Heat Equation

9 - IVP for Heat Equation ( ut=uxx; u(0,x)=g(x) ), Convolution integral (trick with convolution)

10 - The (Hom) Transport Equation (ut+c*ux=0), Traveling waves, IVP for transport equation

11 - The nonhomogeneous Transport IVP (how to solve + trick)

12 - Wave Equation ( utt=uxx; u(0,x)=u0(x); ut(0,x)=v0(x) ) + how to solve, d'Alambert's Formula (jos)

13 - Linear IVPs and the Fourier Transform

14 – (probabil pentru pagina 13) IDEA: treat evol PDE as ODE in Banach or Hilbert spaces; Heat Equation IVP Example

15 - Lp spaces, All Lp spaces are Banach Spaces, Aprox result C_c dense in Lp(R), L2 is a Hilbert space

16 - Convolution, Fundamental property of convolution, The Fourier Transform, Inverse Fourier transform, chestii cu F(u)(y)

17 - Fundamental properties of Fourier (Plancherel, Shift/ing, Scale/ing, Conjugate, Invariant)

18 - The Fourier Approach, Heat IVP cu delta pentru ca L2, How to apply Fourier

19 - Heat Kernel N(t)(x)={

20 - Linear PDEs in bounded (spatial) domains, Heat Initial Boundary Value Problem (IBVP), Dirichlet Boundary Condition, Heatflow Model

21 - Fourier Approach with BC (Insight 1, Insight 2+ Aim)

22 - Insight 3

23 – Fourier Approach for dimension > 1 (Helmsholz or Dirichlet)

24 - Classical Dirichlet BVP (AIM) vector field scalar field, Laplacian divergence gradient? We call an open set OMEGA of class C if

25 - Classical Integral Calculus Results: Divergence (Gauss-Ostrogradski), Green's Formulae + idea of proof, Gauss

26 - Harmonic functions with radial symmetry EX:21, The fundamental solution of Laplace’s equation, Riemann-Green Formula

27 - (Consequences of Riemann-Green)Mean Value Theorem For Harmonic functions, The Strong Max Principle, The Weak Max Principle

28 - Boundary Value, Dirichlet, Poisson Formula, Dirichlet fresh idea

29 - The Dirichlet Energy Functional, The Dirichlet Principle

30 – The problem with classical solutions, Nonexistance of solutions for optimizing problems, Courant's CounterExample EX:28, Modern Theory IDEA 3 reasons

31 - Weak(Generalized) solutions IF VI HOLDS, Sobolev, H01(omega), H01 Hilbert, The energy norm

32 - H01 analysis possible because Poincare Inequality proof EX:25, C01 dense by construction in H01 Dirichlet Principle in H01, Existence & Uniqueness of a weak sol

33 - Idea of proof for EXISTENCE & UNIQUENESS (Riesz)

34 - List of models: Wave, Damped Wave, Visco-elasticity(linear), Visco-capillarity(1D), Euler-Bernoulli Beam (matrici)

35 - Fisher's Reaction Diffusion (it deals with spatially distributed populations), Fisher KPP, Allen-Cahn, Fisher's model (spatially distributed population),classical population models: Malthus,Verhulst

36 - Fisher's new model (1937) Fisher's Eq, Frick’s law, Fisher’s original question, KPP(Kolmogrov-Petrovsky-P)

37 - Travelling wave sols for fisher eq (TWS) (how to..), Logan (there exists a unique wave profile?)

39 - Reaction diffusion Eq & systems, Examples:Allen-Cahn Eq; Brusselator system, General Reaction diffusion Equilibria(GRDE), Equilibria

40 - Lyapunov Theory, X State Space(Hilbert Space),Dynamical system on X if… , Equilibrium point for , Strict Lyapunov function

41 - La Salle's Invariance Principle, Strict Lyapunov + compact trajectory => convergence to equilibrium

42 - Allen-Cahn Model ( Neumann BC ) + norm

43 - Allen-Cahn Energy is a Lyapunov function

44 - Allen-Cahn convergence to equilibrium

45 - Turing instability, Turing Model(TM), General Reaction Diffusion System(GRDS), stationary solutions, homogeneous stationary solutions

46 – Alan Turing system NEUMAN BC: @math @biology @cs ahead of its time

47 - Turing instability what it means, with diffusion, without diffusion

48 - Turing instability (even more details) ODE System

49 - Turing instability PDE System

FORMULE

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}$$

Heat

$$u_{tt} = a\,u_{xx}$$

Wave

$$u_t + c\,u_x = 0$$

Transport Homogeneous

$$\int_\Omega (\nabla u\cdot\nabla v - f v)\,dx = 0$$

Weak/Generalized solutions

$$\|u\|^2_{H_0^1} = \int_\Omega |\nabla u|^2\,dx$$

Energy norm

$$u_t = \mu_D\,u_{xx} + r u\left(1-\frac{u}{K}\right)$$

Fisher's Equation

$$u_t = u_{xx} + f(u)$$

Reaction Diffusion General Form

$$\begin{cases} u_t = u_{xx} + f(u,v) \\ v_t = \mu\, v_{xx} + g(u,v) \end{cases}$$

System Reaction Diffusion

$$u_t + c\,u_x = f(t,x)$$

Transport Nonhomogeneous

$$u(t,x) = \int_{-\infty}^{\infty} \Phi(t, x-y)\, g(y)\,dy \qquad e^{-\frac{(x-y)^2}{4t}}$$

Solution IVP Heat

and we reach *d'Alembert's Formula* for the sol

(3) + (IC$_1$) + (IC$_2$):

$$u(t,x) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy.$$

Solution IVP Wave / d'Alambert

Allen-Cahn eq. $u_t = u_{xx} + u(1-u^2)$

Allen-Cahn

Brusselator syst. $\begin{cases} u_t = u_{xx} + 1 + u^2 v - 2u \\ v_t = \mu\, v_{xx} + u - u^2 v \end{cases}$

Brusselator System

$$(f * g)(x) \overset{\text{Def.}}{=} \int_{\mathbb{R}^n} f(x-y)\, g(y)\,dy$$

Convolution

$$\mathcal{F}(u)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i x\cdot y}\, u(x)\,dx$$

Fourier Transform

$$-\pi^2 k^2 \quad e^{-\pi^2 k^2 t} \sin \pi k x$$

Solution IBVP Heat

$$\int_{\partial\Omega} V\cdot\nu\, d\sigma = \int_\Omega \operatorname{div} V\,dx$$

Divergence/Gauss-Ostrogradski

$$\int_{\partial\Omega} u\,\frac{\partial v}{\partial \nu}\,d\sigma = \int_\Omega (u\Delta v + \nabla u\cdot\nabla v)\,dx$$

Green's formula

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,d\sigma = 0.$$

Gauss

T4. (RIEMANN - GREEN

$$u(x) = \int_\Omega \Delta u(y)\, N(x-y)\,dy + \int_{\partial\Omega}\left( u(y)\,\frac{\partial N(x-y)}{\partial \nu_y} - N(x-y)\,\frac{\partial u(y)}{\partial \nu}\right) d\sigma_y$$

in $y \in \partial\Omega$.

Riemann-Green Formula

$$\begin{cases} u_t = u_{xx} + (u - u^3) & \text{for } (t,x) \in (0,\infty) \\ u_x(t,0) = u_x(t,1) = 0 & \text{(Neumann BC)} \\ u(0,x) = u_0(x) & \text{(IC)} \end{cases}$$

Allen-Cahn model + Newmann

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\,d\sigma$$

Mean value for harmonic functions

T6. (The Strong Max Principle)

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ open, connected} \\ u \in C^2(\Omega),\ \Delta u(x) \geq 0\ \forall x \in \Omega \\ \exists x_0 \in \Omega :\ u(x_0) = \sup_\Omega u \end{array}\right\}$$

Strong Max Principle

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ open, bounded} \\ u \in C^2(\Omega) \cap C(\bar\Omega) \\ \Delta u \geq 0 \text{ in } \Omega \\ u \leq 0 \text{ on } \partial\Omega \end{array}\right\}$$

Weak Max Principle

Poisson

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad f \neq 0$$

Poisson

$$E(u) = \frac{1}{2}\|u_x\|^2_{L^2} + \int_\Omega W(u)\,dx$$

strict Lyapunov function for the Dyn Syst generated by the AC eq.

Strict Lyapunov function

Def:

$$N: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad N(x) = \begin{cases} \frac{1}{2\pi} \ln|x|, & n = 2 \\ -\frac{1}{(n-2)\,\omega_n}\,\frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases} \quad \text{const}$$

$$\Delta N(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

of unit sphere in $\mathbb{R}^n$

Fundamental Solution of Laplace Equation

The (Dirichlet) Energy functional

$$E: C_0^1(\bar\Omega) \to \mathbb{R}, \quad E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - f u\right) dx$$

see Lecture #7

Dirichlet Energy Functional

associated to the inner product

$$\langle u, v\rangle_{H_0^1} = \int_\Omega \nabla u\cdot\nabla v\,dx \qquad \left(\|u\|^2_{H_0^1} = \langle u, u\rangle_{H_0^1}\right)$$

Inner product

$\Omega$ open and bounded

$$(*) \quad \int_\Omega u^2\,dx \leq c^2 \int_\Omega |\nabla u|^2\,dx$$

Poincare Inequality

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

operator

Wave

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & -I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Damped Wave

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & \Delta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Visco-Elasticity

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\partial_x^4 + \partial_x^2 & \partial_x^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Visco-capillarity

Euler-Bernoulli Beam $u_{tt} = -u_{xxxx}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\partial_x^4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Euler-Bernoulli Beam