1st Semester, 2021-2022

3rd Year, Math & CompSci

# Mathematical Statistics

## Seminar Exercises: Week 9

**Recap.** *Throughout this class, $X_1, X_2, \dots, X_n, \dots$ will be i.i.d. random variables that follow the distribution of a given characteristic $X$.*

$$\overline{\nu}_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\overline{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

- *If $X \sim Bino(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$, then:*

  $$X \begin{pmatrix} k \\ C_n^k p^k (1-p)^{n-k} \end{pmatrix}_{k \in \overline{0,n}}$$

  $E(X) = np$, $V(X) = np(1-p)$

- *If $X \sim NBin(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$, then:*

  $$X \begin{pmatrix} k \\ C_{n+k-1}^k p^n (1-p)^k \end{pmatrix}_{k \in \mathbb{N}}$$

  $E(X) = \dfrac{n(1-p)}{p}$, $V(X) = \dfrac{n(1-p)}{p^2}$

- *If* $X \sim Geo(p)$, $p \in (0,1)$, *then*

$$X \begin{pmatrix} k \\ p(1-p)^k \end{pmatrix}_{k \in \mathbb{N}}$$

$E(X) = \dfrac{1-p}{p}$, $V(X) = \dfrac{1-p}{p^2}$

- *If* $X \sim Unif[a,b]$, $a, b \in \mathbb{R}$, $a < b$, *then:*

$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$

$E(X) = \dfrac{a+b}{2}$, $V(X) = \dfrac{(b-a)^2}{12}$

- *If* $X \sim Gamma(a,b)$, $a, b > 0$, *then:*

$$f_X(x) = \frac{1}{\Gamma(a) b^a} \cdot x^{a-1} e^{\frac{-x}{b}} \cdot 1_{(0,\infty)}(x)$$

*where:*

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, \ a > 0$$

$E(X) = ab$, $V(X) = ab^2$

- *If* $X \sim Exp(\lambda)$, $\lambda > 0$ *then:*

$$f_X(x) = \lambda e^{-\lambda x} \cdot 1_{[0,\infty)}(x)$$

$E(X) = \dfrac{1}{\lambda}$, $V(X) = \dfrac{1}{\lambda^2}$

- *If* $X \sim Pareto(\alpha, \beta)$, $\alpha, \beta > 0$, *then:*

$$f_X(x) = \alpha \beta^\alpha \frac{1}{x^{\alpha+1}} \cdot 1_{[\beta,\infty)}(x)$$

$$E(X) = \begin{cases} \infty, \text{ for } \alpha \leq 1 \\ \dfrac{\alpha\beta}{\alpha - 1}, \text{ for } \alpha > 1 \end{cases}, \ V(X) = \begin{cases} \infty, \text{ for } \alpha \leq 2 \\ \dfrac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}, \text{ for } \alpha > 2 \end{cases}$$

- *The* ***likelihood function*** *of the sample* $X_1, X_2, \ldots, X_n$*:*

$$L(X_1, X_2, \ldots, X_n; \theta) = \prod_{i=1}^{n} f(X_i; \theta);$$

**Exercise 1.** Using the method of moments, find estimators for the parameter(s) of the distribution of the characteristic $X$, if:

(a) $X \sim Geo(p)$, $p \in (0,1)$;

(b) $X \sim Exp(\lambda)$, $\lambda > 0$;

(c) $X \sim Bino(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$;

(d) $X \sim Unif[a,b]$, $a < b$;

(e) $X \sim NBin(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$;

(f) $X \sim Gamma(a,b)$, $a,b > 0$;

(g) $X \sim Pareto(\alpha,\beta)$, $\alpha > 2$, $\beta > 0$;

**Exercise 2.** Using the maximum likelihood method, estimate the parameters of the distribution of the characteristic $X$, if:

(a) $X \sim Unif[a,b]$, $a < b$, for the sample data: $x_1, \ldots, x_n \in [a,b]$.

(b) $X \sim Bino(10,p)$, $p \in (0,1)$, for the sample data: $x_1, \ldots, x_n \in \{0,1,\ldots,10\}$ not all zero and not all ten.

(c) $X \sim Exp(\lambda)$, $\lambda > 0$, for the sample data: $x_1, \ldots, x_n > 0$.

(d) $X \sim Geo(p)$, $p \in (0,1)$, for the sample data: $x_1, \ldots, x_n \in \mathbb{N}$ not all zero.