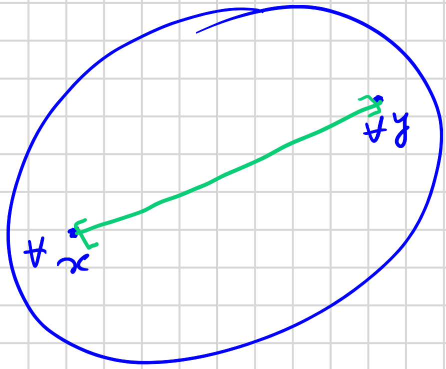
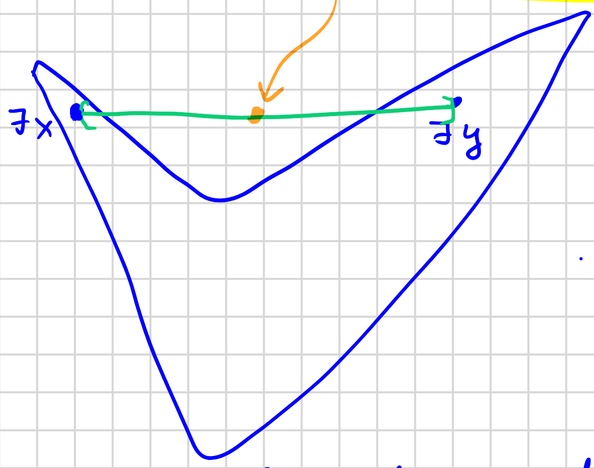
Seminar 2, G832, 15 Mar 2022

A set $S \subseteq \mathbb{R}^n$ is called convex if

$$\boxed{\forall x, y \in S,\ \forall t \in [0,1],\ \text{we have } (1-t)\cdot x + t\cdot y \in S}$$





S is convex, since

$\forall x, y \in S,\ [x,y] \subseteq S$

S is not convex, because

$\exists\, x, y \in S$ s.t. $[x,y] \nsubseteq S$

The notation

$$[x,y] := \{(1-t)x + ty \mid t \in [0,1]\} =$$
$$= \{x + t(y-x) \mid t \in [0,1]\}$$

is introduced in L4

Ex 1. For every $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we define the sets:

$$H(c,\lambda) := \{x \in \mathbb{R}^n \mid \langle c, x\rangle = \lambda\},$$ where $\langle c, x\rangle = c_1x_1 + \dots + c_nx_n$ inner product

$$H^{\leq}(c,\lambda) := \{x \in \mathbb{R}^n \mid \langle c, x\rangle \leq \lambda\}$$

$$H^{\geq}(c,\lambda) := \{x \in \mathbb{R}^n \mid \langle c, x\rangle \geq \lambda\}$$

$$H^{<}(c,\lambda) := \{x \in \mathbb{R}^n \mid \langle c, x\rangle < \lambda\}$$

$$H^{>}(c,\lambda) := \{x \in \mathbb{R}^n \mid \langle c, x\rangle > \lambda\}$$

Prove that these sets are convex.

Solution: • Let us show first that $H^>(\kappa,\lambda)$ is convex.

Indeed, let $x, y \in H^>(\kappa,\lambda)$ and $t \in [0,1]$. We are going to prove that $(1-t)\cdot x + t\cdot y \in H^>(\kappa,\lambda)$.

Since $x, y \in H^>(\kappa,\lambda)$ we have

$$\begin{cases} \langle \kappa, x\rangle > \lambda & | \cdot (1-t) \geq 0 \\ \langle \kappa, y\rangle > \lambda & | \cdot t \geq 0 \end{cases}$$ at least inequality is strict $\Rightarrow$

$$\Rightarrow (1-t)\cdot\langle \kappa, x\rangle + t\cdot\langle \kappa, y\rangle > (1-t)\lambda + t\lambda$$

$$\Rightarrow \langle \kappa, (1-t)x\rangle + \langle \kappa, ty\rangle > \lambda$$

$$\Rightarrow \langle \kappa, (1-t)x + ty\rangle > \lambda$$

$$\Rightarrow (1-t)x + ty \in H^>(\kappa,\lambda).$$

by def of $H^>(\kappa,\lambda)$

$$\alpha\cdot\langle u, v\rangle = \alpha\cdot(u_1v_1 + \dots + u_nv_n) =$$
$$= (\alpha u_1)\cdot v_1 + \dots + (\alpha u_n)v_n = \langle \alpha u, v\rangle$$
$$= u_1\cdot(\alpha v_1) + \dots + u_n(\alpha v_n) = \langle u, \alpha v\rangle$$

$$\langle u, v\rangle + \langle u, w\rangle = (u_1v_1 + \dots + u_nv_n) + (u_1w_1 + \dots + u_nw_n)$$
$$= u_1(v_1 + w_1) + \dots + u_n(v_n + w_n)$$
$$= \langle u, v + w\rangle$$

• Observe that

$$H^<(\kappa,\lambda) = \underbrace{H^>(-\kappa,-\lambda)}_{\text{is a convex set}}$$

$\Rightarrow H^<(\kappa,\lambda)$ is convex.

- In order to show that $H^{\geq}(r,\lambda)$ is convex, we can follow the main lines of the proof presented above for the set $H^{>}(r,\lambda)$ or we can use the fact that

$$
\begin{aligned}
H^{\geq}(r,\lambda) &= \{x \in \mathbb{R}^n \mid \langle r,x\rangle \geq \lambda\} \\
&= \{x \in \mathbb{R}^n \mid \langle r,x\rangle > \lambda-\varepsilon, \forall \varepsilon>0\} \\
&= \bigcap_{\varepsilon>0} \{x \in \mathbb{R}^n \mid \langle r,x\rangle > \lambda-\varepsilon\} \\
&= \underbrace{\bigcap_{\varepsilon>0} \underbrace{H^{>}(r,\lambda-\varepsilon)}_{\text{convex}}}_{\text{convex, as being an intersection of a family of convex sets (see L4)}}
\end{aligned}
$$

- Observe that

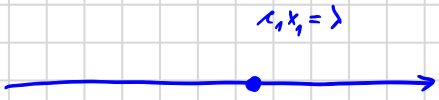
$$H^{\leq}(r,\lambda) = \underbrace{H^{\geq}(-r,-\lambda)}_{\text{convex}}$$

$\Rightarrow H^{\leq}(r,\lambda)$ is convex.

- Finally, observe that

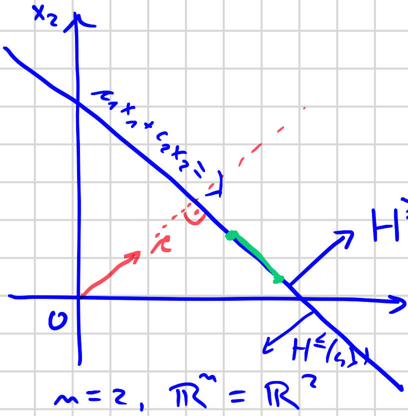
$$H(r,\lambda) = \underbrace{\underbrace{H^{\leq}(r,\lambda)}_{\text{convex}} \cap \underbrace{H^{\geq}(r,\lambda)}_{\text{convex}}}_{\text{convex, as an intersection of convex sets.}}$$

<u>Remark:</u> when $r \neq 0_n$, that is, $\exists\, i \in \{1,\dots,n\}$ s.t. $r_i \neq 0$, then $H(r,\lambda)$ represents a hyperplane in $\mathbb{R}^n$



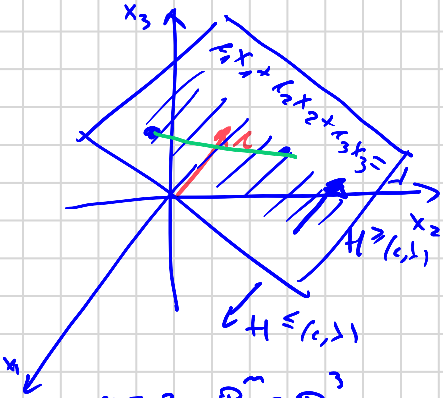
$n=1$, $\mathbb{R}^n=\mathbb{R}$

hyperplanes are points



$n=2$, $\mathbb{R}^n=\mathbb{R}^2$

hyperplanes are straight lines



$n=3$, $\mathbb{R}^n=\mathbb{R}^3$

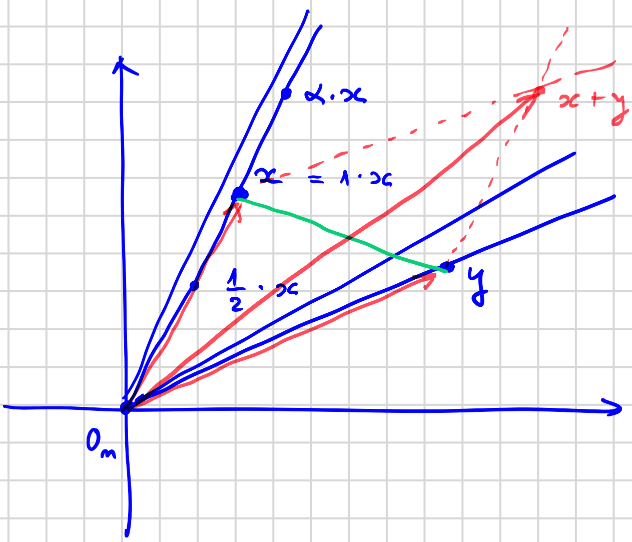
hyperplanes are planes

Also, the sets $H^{\leq}(c,\lambda)$ and $H^{\geq}(c,\lambda)$ represent closed half-spaces, while $H^{<}(c,\lambda)$ and $H^{>}(c,\lambda)$ are their interiors, called open half-spaces

Cones in $\mathbb{R}^n$ are nonempty sets that are stable under multiplication by nonnegative real numbers:

$$C\neq\emptyset \text{ and } \mathbb{R}_+\cdot C\subseteq C \iff$$

$$\iff \begin{cases} 0_n\in C \\ \forall\,\alpha\geq 0 \text{ and } x\in C \Rightarrow \alpha\cdot x\in C \end{cases}$$



Cones are, in general, not convex (as C in the picture above)

As shown by Ex5 in OT-S2.pdf, a cone is convex if and only if it is stable under addition of vectors.

1° C convex

2° $C+C\subseteq C \iff \forall\, x,y\in C,\ x+y\in C$.