# Seminar 4

**Exercise 1** Solve the following optimization problem by means of the Primal SIMPLEX Algorithm.

$$
(P)\left\{\begin{array}{l}
\text{Minimize } \; f(x) = x_2 - 3x_3 + 2x_4 \\[1em]
\left.\begin{array}{rrrrrrcr}
x_1 & +3x_2 & -x_3 & +2x_4 & & & = & 7 \\
 & -2x_2 & +4x_3 & & +x_5 & & = & 12 \\
 & -4x_2 & +3x_3 & +7x_4 & & +x_6 & = & 10 \\
\multicolumn{8}{l}{x_1, \dots, x_6 \geq 0.}
\end{array}\right] (S)
\end{array}\right.
$$

*Solution.* Problem $(P)$ is already written in standard form (see Definition 6.1 of Lecture 6). We have:

$$
A = \begin{pmatrix} 1 & 3 & -1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & -4 & 3 & 7 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}, \quad c = (0, 1, -3, 2, 0, 0).
$$

Observe that the canonical basis of $\mathbb{R}^3$,

$$
B := \{A^1, A^5, A^6\}
$$

is a primal feasible basis of $S$, since $\alpha_0 j = b_j \geq 0$ for all $j \in \mathcal{B}_B = \{1, 5, 6\}$. Thus we can use it as the starting basis within the Primal SIMPLEX Algorithm.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 0 | 0 | | |
| | 1 | $A^1$ | $A^5$ | $A^6$ | Test d.f.b. | |
| 1 | $A^2$ | 3 | $-2$ | $-4$ | $-1$ | |
| $-3$ | $A^3$ | $-1$ | $\boxed{4}$ | 3 | 3 | * |
| 2 | $A^4$ | 2 | 0 | 7 | $-2$ | |
| | Test p.f.b. | 7 | 12 | 10 | 0 | |
| | | / | $\frac{12}{4}$ * | $\frac{10}{3}$ | | |

| | | | | | |
|---|---|---|---|---|---|
| 2 | $A^1$ | $A^3$ | $A^6$ | Test d.f.b. | |
| $A^2$ | $5/2$ | $\boxed{-1/2}$ | $-5/2$ | $1/2$ | * |
| $A^5$ | $1/4$ | $1/4$ | $-3/4$ | $-3/4$ | |
| $A^4$ | 2 | 0 | 7 | $-2$ | |
| Test p.f.b. | 10 | 3 | 1 | $-9$ | |
| | $\frac{10}{5/2}$ * | / | / | | |

| | | | | |
|---|---|---|---|---|
| 3 | $A^2$ | $A^3$ | $A^6$ | Test d.f.b. |
| $A^1$ | | | | $-1/5$ |
| $A^5$ | | | | $-4/5$ |
| $A^4$ | | | | $-12/5$ |
| Test p.f.b. | 4 | 5 | 11 | $-11$ |

Since $\alpha_{i0} \leq 0$ for all $i \in \mathcal{N}_B = \{1, 5, 4\}$ (i.e., $-1/5 \leq 0, -4/5 \leq 0$ and $-12/5 \leq 0$), we conclude that

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0) = (0, 4, 5, 0, 0, 11)$$

is an optimal solution of problem $(P)$ and

$$\alpha_{00} = -11$$

is the optimal (minimal) value of the objective function $f$ on the feasible set $S$. $\square$

**Exercise 2** Using the Primal SIMPLEX Algorithm prove that the following constrained linear optimization problem has no optimal solutions (more precisely, the objective function is unbounded from below on the feasible set).

$$
(P)\quad \begin{cases} \text{Minimize } \ f(x) = -x_2 + x_4 \\[1em] \left.\begin{array}{lllllll} 2x_1 & -x_2 & -x_3 & +x_4 & & & = 6 \\ x_1 & -2x_2 & & & +x_5 & & = 2 \\ -2x_1 & +2x_2 & & & & +x_6 & = 3 \\ x_1, \dots, x_6 \geq 0. & & & & & & \end{array}\right] (S) \end{cases} \tag{1}
$$

*Solution.* We have

$$
A = \begin{pmatrix} 2 & -1 & -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}, \quad c = (0, -1, 0, 1, 0, 0).
$$

Observe that the canonical basis of $\mathbb{R}^3$,

$$
B := \{A^4, A^5, A^6\}
$$

is a primal feasible basis of $S$, since $\alpha_0 j = b_j \geq 0$ for all $j \in \mathcal{B}_B = \{4, 5, 6\}$. Thus we can use it as the starting basis within the Primal SIMPLEX Algorithm.

| | | 1 | 0 | 0 | | |
|---|---|---|---|---|---|---|
| | 1 | $A^4$ | $A^5$ | $A^6$ | Test d.f.b. | |
| 0 | $A^1$ | 2 | $\boxed{1}$ | $-2$ | 2 | * |
| $-1$ | $A^2$ | $-1$ | $-2$ | 2 | 0 | |
| 0 | $A^3$ | $-1$ | 0 | 0 | $-1$ | |
| | Test p.f.b. | 6 | 2 | 3 | 6 | |
| | | $\frac{6}{2}$ | $\frac{2}{1}$ | / | | |
| | | | * | | | |

| 2 | $A^4$ | $A^1$ | $A^6$ | Test d.f.b. | |
|---|---|---|---|---|---|
| $A^5$ | $-2$ | $1$ | $2$ | $-2$ | |
| $A^2$ | $3$ | $\boxed{-2}$ | $-2$ | $4$ | $*$ |
| $A^3$ | $-1$ | $0$ | $0$ | $-1$ | |
| Test p.f.b. | $2$ | $2$ | $7$ | $2$ | |
| | $\frac{2}{3}$ | / | / | | |
| | $*$ | | | | |

| 3 | $A^2$ | $A^1$ | $A^6$ | Test d.f.b. |
|---|---|---|---|---|
| $A^5$ | $-2/3$ | $-1/3$ | $2/3$ | $2/3$ |
| $A^4$ | $1/3$ | $2/3$ | $2/3$ | $-4/3$ |
| $A^3$ | $-1/3$ | $-2/3$ | $-2/3$ | $1/3$ |
| Test p.f.b. | $2/3$ | $10/3$ | $25/3$ | $-2/3$ |

Since for $i = 3 \in \mathcal{N}_B$ we have $\alpha_{i0} = 1/3 > 0$ and $\alpha_{3j} \leq 0$ for all $j \in \mathcal{B}_B = \{2, 1, 6\}$ (i.e., $\alpha_{32} = -1/3 < 0$, $\alpha_{31} = -2/3 < 0$ and $\alpha_{36} = -2/3 < 0$), we inder that $f$ is unbounded from below on $S$. Thus problem $(P)$ has no optimal solutions. $\square$

# Seminar 5

**Exercise 1** Solve the following optimization problem by the Dual SIMPLEX Algorithm.

$$(P)\ \begin{cases} \text{Minimize} \ \ f(x) = 3x_1 + 5x_2 \\ \left.\begin{array}{rrrrrcr} -x_1 & -2x_2 & +x_3 & & & = & -6 \\ & -x_2 & & +x_4 & & = & -2 \\ -2x_1 & & & & +x_5 & = & -4 \\ \multicolumn{7}{l}{x_1, \ldots, x_5 \geq 0.} \end{array}\right] (S) \end{cases}$$

*Solution.* Problem $(P)$ is given in standard form and we have:

$$A = \begin{pmatrix} -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ -2 \\ -4 \end{pmatrix}, \quad c = (3, 5, 0, 0, 0).$$

Consider the canonical basis

$$B := \{A^3, A^4, A^5\}.$$

It is easily seen that the set of basic indices is $\mathcal{B}_B = \{3, 4, 5\}$ while the set of non-basic indices is $\mathcal{N}_B = \{1, 2\}$. Since $b$ has at least one negative coordinate (i.e., the horizontal section "Test p.f.b." of the Simplex tableau associated to $B$ contains at least one negative number $b_1 = \alpha_{0j} = -6 < 0$ with $j = 3 \in \mathcal{B}_B$), the basis $B$ is not primal feasible, hence we can not use the Primal SIMPLEX Algorithm for this initial basis. However, since all numbers in the vertical section "Test d.f.b." of the Simplex tableau are less than or equal to zero (i.e., $\alpha_{i0} \leq 0$ for all $i \in \mathcal{N}_B$), the canonical basis $B$ is dual feasible, hence we can use the Dual SIMPLEX Algorithm.

| | | 0 | 0 | 0 | | | |
|---|---|---|---|---|---|---|---|
| | 1 | $A^3$ | $A^4$ | $A^5$ | Test d.f.b. | | |
| 3 | $A^1$ | $-1$ | 0 | $-2$ | $-3$ | $\frac{-3}{-1}$ | |
| 5 | $A^2$ | $-2$ | $-1$ | 0 | $-5$ | $\frac{-5}{-2}$ | * |
| | Test p.f.b. | $-6$ | $-2$ | $-4$ | 0 | | |
| | | * | | | | | |

| 2 | $A^2$ | $A^4$ | $A^5$ | Test d.f.b. | | |
|---|---|---|---|---|---|---|
| $A^1$ | $1/2$ | $1/2$ | $-2$ | $-1/2$ | $\frac{-1/2}{-2}$ | * |
| $A^3$ | $-1/2$ | $-1/2$ | 0 | $-5/2$ | | |
| Test p.f.b. | 3 | 1 | $-4$ | 15 | | |
| | | | * | | | |

| 3 | $A^2$ | $A^4$ | $A^1$ | Test d.f.b. |
|---|---|---|---|---|
| $A^5$ | | | | |
| $A^3$ | | | | |
| Test p.f.b. | 2 | 0 | 2 | 16 |

Since in the 3rd Simplex Tableau all numbers in the horizontal section "Test p.f.b." are greater than or equal to zero, we conclude that

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0) = (\alpha_{01}, \alpha_{02}, 0, \alpha_{04}, 0) = (2, 2, 0, 0, 0)$$

is an optimal solution of problem $(P)$ and

$$\alpha_{00} = 16$$

is the optimal (minimal) value of the objective function $f$ on the feasible set $S$. $\square$

**Exercise 2** Using the Dual SIMPLEX Algorithm prove that the following optimization problem has no optimal solutions (more precisely, it has no feasible points, i.e., $S = \emptyset$).

$$(P)\begin{cases} \text{Minimize} \;\; f(x) = x_2 - x_3 \\ \left.\begin{array}{lllll} & -x_2 & +x_3 & +x_4 & = -1 \\ x_1 & +3x_2 & & & = \;\; 0 \\ x_1, \ldots, x_4 \geq 0. & & & & \end{array}\right] (S) \end{cases} \tag{1}$$

*Solution.* We have

$$A = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 3 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad c = (0, 1, -1, 0).$$

First observe that $b$ has a negative coordinate, hence the canonical bases $\{A^3, A^1\}$ and $\{A^4, A^1\}$ are not primal feasible. Therefore we cannot use them as starting bases for the Primal SIMPLEX Algorithm. Consider the canonical basis

$$B := \{A^3, A^1\}.$$

Since all numbers since all numbers in the vertical section "Test d.f.b." of the Simplex tableau are less than or equal to zero (i.e., $\alpha_{i0} \leq 0$ for all $i \in \mathbb{N}_B$), the canonical basis $B$ is dual feasible, hence we can use the Dual SIMPLEX Algorithm.

| | | $-1$ | $0$ | | | |
|---|---|---|---|---|---|---|
| | 1 | $A^3$ | $A^1$ | Test d.f.b. | | |
| 1 | $A^2$ | $\boxed{-1}$ | 3 | 0 | $\frac{0}{-1}$ | * |
| 0 | $A^4$ | 1 | 0 | $-1$ | / | |
| | Test p.f.b. | $-1$ | 0 | 1 | | |
| | | * | | | | |

| 2 | $A^2$ | $A^1$ | Test d.f.b. |
|---|---|---|---|
| $A^3$ | $-1$ | 3 | 0 |
| $A^4$ | $-1$ | 3 | $-1$ |
| Test p.f.b. | 1 | $-3$ | 1 |

Since in the horizontal section "Test p.f.b." of the 2nd Simplex Tableau there exists a negative number (namely $\alpha_{01} = -3 < 0$) above which none of the numbers is negative (more precisely, $\alpha_{31} = 3 \not< 0$ and $\alpha_{41} = 3 \not< 0$), we inder that problem $(P)$ has no feasible points, hence it has no optimal solutions. $\square$

# Seminar 6

**Exercise 1** Solve the Rock-Scissors-Paper game by means of linear optimization problems.

*Solution.* Since $\underline{w} \leq w \leq \overline{w}$ and

$$\underline{w} = -1 < 0,$$

we cannot guarantee that $w$ is positive. Therefore we add a suitable constant $k \in \mathbb{R}$ to all elements of the payoff matrix

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

such that the lower value of the new matrix $C + (k)$ is positive, i.e.,

$$\underline{w} + k > 0.$$

For instance, by choosing

$$k = 2$$

we obtain the new payoff matrix

$$C + (k) = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

We associate to this matrix the following two optimization problems:

$$\begin{cases} \text{Minimize } \ u_1 + u_2 + u_3 \\ 2u_1 \ + u_2 \ + 3u_3 \ \geq 1 \\ 3u_1 \ + 2u_2 \ + u_3 \ \geq 1 \\ u_1 \ + 3u_2 \ + 2u_3 \ \geq 1 \\ u_1, u_2, u_3 \geq 0 \end{cases} \tag{1}$$

and

$$\begin{cases} \text{Maximize } \; v_1 + v_2 + v_3 \\ \\ \begin{array}{rrrl} 2v_1 & +3v_2 & +v_3 & \le 1 \\ v_1 & +2v_2 & +3v_3 & \le 1 \\ 3v_1 & +v_2 & +2v_3 & \le 1 \end{array} \\ v_1, v_2, v_3 \ge 0. \end{cases} \tag{2}$$

Next we solve them by means of the SIMPLEX algorithms. To this aim we transform the problems (1) and (2) into equivalent problems in standard form, namely

$$\begin{cases} \text{Minimize } \; u_1 + u_2 + u_3 \\ \\ \begin{array}{rrrrrrl} -2u_1 & -u_2 & -3u_3 & +u_4 & & & = -1 \\ -3u_1 & -2u_2 & -u_3 & & +u_5 & & = -1 \\ -u_1 & -3u_2 & -2u_3 & & & +u_6 & = -1 \end{array} \\ u_1, \dots, u_6 \ge 0 \end{cases} \tag{3}$$

and

$$\begin{cases} \text{Minimize } \; -v_1 - v_2 - v_3 \\ \\ \begin{array}{rrrrrrl} 2v_1 & +3v_2 & +v_3 & +v_4 & & & = 1 \\ v_1 & +2v_2 & +3v_3 & & +v_5 & & = 1 \\ 3v_1 & +v_2 & +2v_3 & & & +v_6 & = 1 \end{array} \\ v_1, \dots, v_6 \ge 0. \end{cases} \tag{4}$$

It is important to notice that the dual of (3) is equivalent to (2), while the dual of (4) is equivalent to (1). Indeed, the dual of problem (3) is

$$\begin{cases} \text{Maximize} \quad -s_1 - s_2 - s_3 \\ -2s_1 \quad -3s_2 \quad -s_3 \quad \leq 1 \\ -s_1 \quad -2s_2 \quad -3s_3 \quad \leq 1 \\ -3s_1 \quad -s_2 \quad -2s_3 \quad \leq 1 \\ s_1 \quad \leq 0 \\ s_2 \quad \leq 0 \\ s_3 \quad \leq 0, \end{cases} \tag{5}$$

which, by the change of variables

$$-s = (-s_1, -s_2, -s_3) = v = (v_1, v_2, v_3),$$

becomes the initial problem (2). On the other hand, the dual of problem (4) is

$$\begin{cases} \text{Maximize} \quad t_1 + t_2 + t_3 \\ 2t_1 \quad +t_2 \quad +3t_3 \quad \leq -1 \\ 3t_1 \quad +2t_2 \quad +t_3 \quad \leq -1 \\ t_1 \quad +3t_2 \quad +2t_3 \quad \leq -1 \\ t_1 \quad \leq 0 \\ t_2 \quad \leq 0 \\ t_3 \quad \leq 0, \end{cases} \tag{6}$$

which, by the change of variables

$$t = (t_1, t_2, t_3) := -u = (-u_1, -u_2, -u_3),$$

becomes the initial problem (1).