I. Introduction to PDEs

WHY PDEs? The old Galileo Galilei answer still holds in this case: Because

"Mathematics (read PDEs) is the Language in which God has written the Book of Nature"

We will concentrate on the

HEAT Eq. (describes diffusion of heat in a body)

WAVE Eq. (describes oscillations of a string)

but you'll see more equations (Projects, etc.)

Book:

Ian Stewart: In Pursuit of the Unknown – 17 Equations that Changed the World

1. PDEs from Mathematical Modelling

Question: How do we reach certain PDEs?

IDEA: discrete $\to$ cont.

for a different IDEA check [Precup. Sec 2.1.]

§ 1.1. The Heat Equation (from Brownian Motion)

IDEA: $\underbrace{\text{goes}}_{\text{back to}}$ FELLER

Random Vars. Let X_n be random vars.
which can take only 3 values

h	$-h$	0	with probabilities
p	q	r	$(p+q+r=1.)$

Understand X_n as independent trials so
$S_n = X_1 + \dots + X_n$ accumul. result of n trials.

Brownian Motion Interpretation (1D gas)

A particle is exposed to random collisions that occur at fixed times $\tau, 2\tau, 3\tau$ etc. and produce displacements $h, -h,$ or 0.

S_n = the position of the particle after n collisions (at time $t = n\tau$)

Dynamics: If the current position of the particle is $x = kh$ then the next position will be

$x+h$ $\quad$ $x-h$ $\quad$ or x $\quad$ with probabs.

p $\quad$ q $\quad$ r

Define $u_{n,k} = \text{probab}\,(S_n = kh)$

"probab that the particle occupies pos $x = kh$ after $t = n\tau$"

$S_{n+1} = kh$ can occur in only 3 <u>mutually exclusive</u> (!) ways

$$S_n = (k-1)h, \quad X_{n+1} = h \qquad p$$
$$S_n = kh, \quad X_{n+1} = 0 \qquad r$$
$$S_n = (k+1)h, \quad X_{n+1} = -h \qquad q$$

By adding probabilities we have

$$(1) \qquad u_{n+1,k} = p\,u_{n,k-1} + q\,u_{n,k+1} + r\,u_{n,k}$$

<u>The continuum limit</u> (Ass. that $h, \tau \ll 1$ very small)

$u_{n,k}$ = average of cont. quantity $u(t,x)$ over an h interval

$= u(n\tau, kh)\,h \qquad t = n\tau \quad x = kh$

We can rewrite (1) as

$$(1c) \qquad u(t+\tau, x) = p\,u(t, x-h) + q\,u(t, x+h) + r\,u(t,x)$$

using $r = 1-p-q$ we have that

$$\frac{u(t+\tau, x) - u(t,x)}{\tau} = -\frac{p}{\tau}\left[u(t,x) - u(t,x-h)\right] + \frac{q}{\tau}\left[u(t,x+h) - u(t,x)\right]$$

By Taylor's formula to

$$u(t, x+h) - u(t,x) = \frac{\partial u}{\partial x}(t,x)\, h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x)\, h^2 + \dots$$

...

combining all these

$$\text{(FP)} \quad \frac{\partial u}{\partial t} = -\frac{(p-q)h}{\tau}\frac{\partial u}{\partial x} + \frac{1}{2}\frac{(p+q)h^2}{\tau}\frac{\partial^2 u}{\partial x^2} + \dots$$

After discarding HOT we are left with the advection-diffusion (Fokker-Planck) equation

If fwd & bwd collisions are equally probable (no advection) $p - q = 0$, then we have the HEAT (DIFFUSION) eq.

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}$$

Shorthand Notation $u_t = c\, u_{xx}$.

§ 1.2. The Wave Eq.

$u(t,x) =$ position of the x-particle at time t

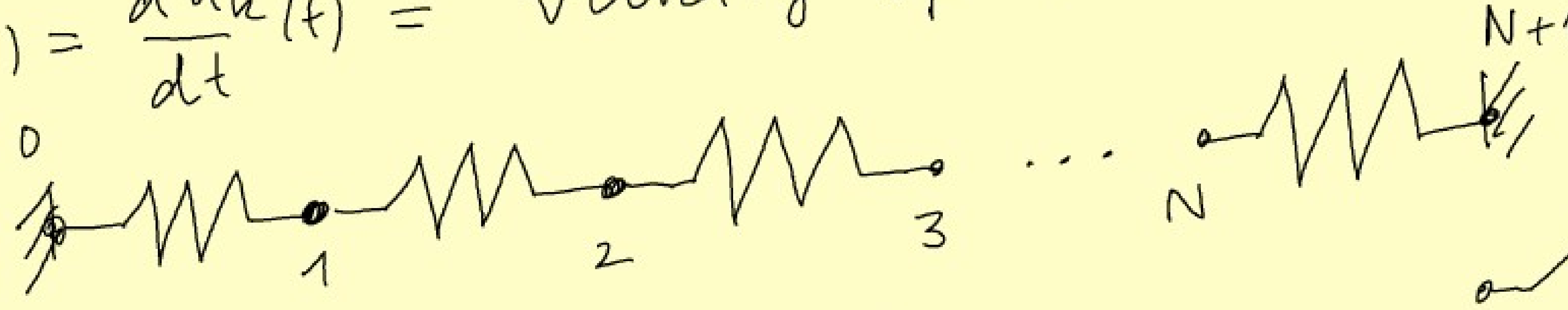
Transversal vs. Longitudinal oscillations:

similar math.
$\sin\theta \approx \theta$ limit

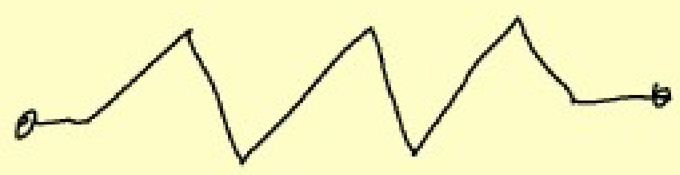
Discrete model: Chain of pointmasses connected by elastic springs

$u_k(t) =$ deviation from equil. of the mass "k"

$v_k(t) = \dfrac{du_k(t)}{dt} =$ velocity of mass "k"



"streched"



Start with "Energy"

$$K(t) = \sum_{k=1}^{N} \frac{1}{2} m\, v_k(t)^2 \qquad E(t) = \sum_{k=0}^{N} \frac{1}{2} c \left(u_{k+1}(t) - u_k(t)\right)^2$$

kinetic energy — potential / elastic energy

The equations of motion are given by Hamilton's system

with $H = K + E$ total energy

$$\text{"}\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \\[2mm] \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} \end{cases}\text{"}$$

$$\frac{du_k}{dt} = v_k(t)$$

$$m\,\frac{dv_k}{dt} = -c\left(u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)\right)$$

The continuum limit

h = length of one spring $\qquad h \sim \frac{L}{N}$

$m = \frac{M}{N}$ M = mass of chain/string $\qquad m \sim \frac{M}{N}$

"$N \to \infty$"

$$u(t,x) = u(t, kh) = u_k(t)$$

$$\underbrace{u(t, x+h)}_{u_{k+1}(t)} - \underbrace{u(t,x)}_{u_k(t)} = \frac{\partial u}{\partial x}(t,x)\, h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x)\, h^2 + \dots$$

...

Plug all Taylor expansions in

$$(2) \qquad m \frac{d^2 u_k}{dt^2} = -c\,(u_{k+1} - 2u_k + u_{k-1})$$

Hamilt. syst. written as 2^{nd} order eq.

then we get the WAWE eq.

$$\frac{\partial^2 u}{\partial t^2} = (ch)\frac{h}{m}\frac{\partial^2 u}{\partial x^2} + \dots$$

ch = elastic modulus, m/h = mass density

$$u_{tt} = a\, u_{xx} \qquad \text{Shorthand notation.}$$

2. Solving PDEs: simple examples

Lecture 1: How can we obtain certain PDEs from discrete models?

Now: How can we find the solutions of those PDEs?

§2.1. The Heat eq. on $\mathbb{R}$

(1) $u_t = u_{xx}$, $t \geq 0$, $x \in \mathbb{R}$, $u(t,x) = ?$

SCALING: if $u(t,x)$ solves (1) then so does $u(\lambda^2 t, \lambda x)$

↑ change units in which you measure

So, the ratio $\frac{x}{t^{1/2}}$ is interesting!

IDEA: look for solution of the form

$$u(t,x) = \frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right) \qquad \text{"dilation scaling"}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right)\right) = -\frac{1}{2}\frac{1}{t^{3/2}} v - \frac{1}{t^{1/2}} v' \frac{1}{2}\frac{x}{t^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{t^{1/2}}\cdot\frac{1}{t^{1/2}} v'\right)$$

$$= \frac{1}{t^{3/2}} v''\left(\frac{x}{t^{1/2}}\right)$$

$r := \frac{x}{t^{1/2}}$, $v = v(r)$, $' = \frac{d}{dr}$

so we get (1) $\iff$ $v''(r) + \frac{1}{2} r v'(r) + \frac{1}{2} v = 0$

$$(1) \iff v''(r) + \underbrace{\tfrac{1}{2} r v'(r) + \tfrac{1}{2} v}_{\frac{1}{2}(rv)'} = 0 \quad (1_v) \,\Big|\, \int dr$$

$r = \dfrac{x}{t^{1/2}}$,

$$v' + \tfrac{1}{2} r v = C_0$$

Take $C_0 = 0$ (Argument $\lim_{r\to\infty} v, \lim_{r\to\infty} v' = 0$ because it makes sense to have $u(t,x) \longrightarrow 0$ as $|x| \to \infty$ t = fixed)

[Evans, p. 46]

then $v(r) = C_1 e^{-\frac{r^2}{4}}$ Gaussian

[see Exercise 2]

$$u(t,x) = \frac{C_1}{t^{1/2}} e^{-\frac{x^2}{4t}}$$

Rk. It is convenient to take C_1 such that

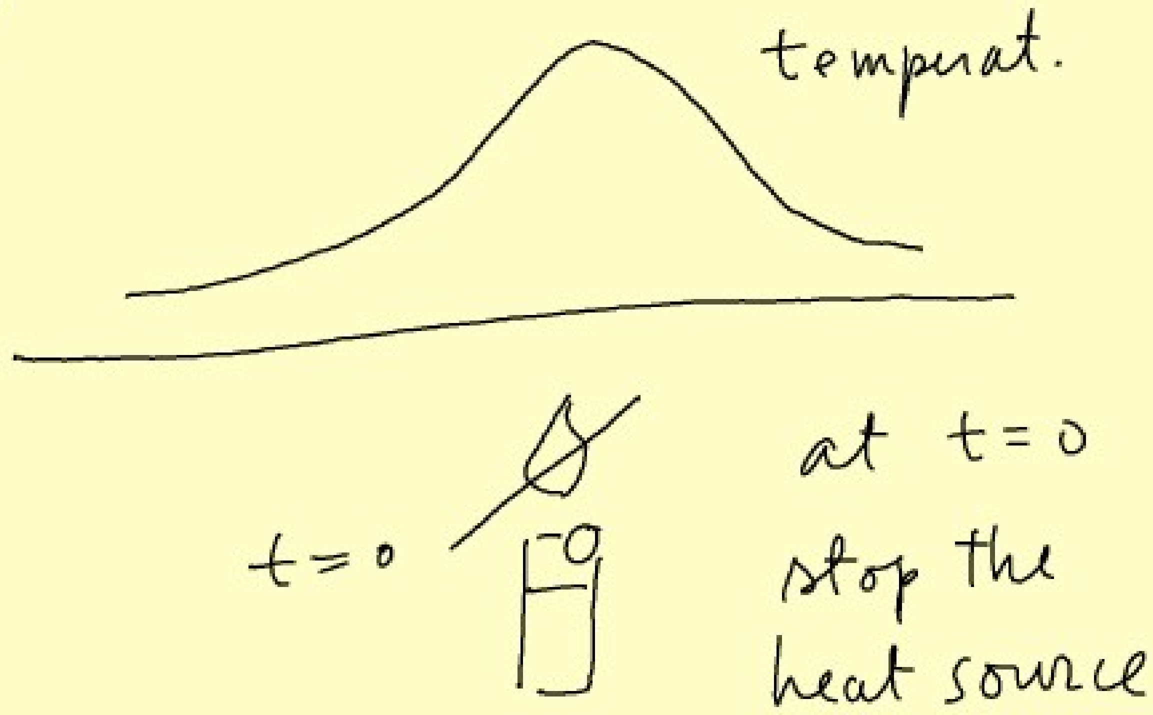
$$\int_{-\infty}^{\infty} u(t,x)\,dx = 1$$ [see; Exercise 4]

This way you get the fundamental solution of the Heat eq.

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \quad x \in \mathbb{R},\ t \geq 0$$

$$\Phi_t = \Phi_{xx}, \quad \int_{-\infty}^{\infty} \Phi(t,x)\,dx = 1 \quad \forall\, t \geq 0.$$

The Initial Value Problem associated to the Heat eq.
(IVP)



$u_t = u_{xx}$ (1)

$u(0,x) = g(x)$ (IC)

↖ given / known

<u>Claim</u> [see Exercises #2 = next week]

The solution of the Heat IVP = (1) + (IC)

is given by a <u>convolution</u> integral

$$u(t, x) = \int_{-\infty}^{\infty} \Phi(t, x-y)\, g(y)\, dy$$

$$e^{-\frac{(x-y)^2}{4t}}$$

Trick with convolution is that all

$\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ derivatives act on Φ

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \Phi(\dots)\, g(y)\, dy = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\Phi(\dots)\cdot g(y)\, dy$$

$$\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} \Phi(\dots)\, g(y)\, dy = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2}\Phi(\dots)\cdot g(y)\, dy$$

§ 2.2. The Transport eq.

(2) $\quad u_t + c u_x = 0, \quad c \in \mathbb{R}, \quad t \geq 0, \quad x \in \mathbb{R}$

TRAVELING WAVES : look for wave-sols.

$$u(t,x) = v(x-ct) \quad (*)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} v(x-ct) = -c\, v'(x-ct)$$

$$c \,|\, \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} v(x-ct) = v'(x-ct)$$

$$u_t + c u_x = \cancel{-cv'} + \cancel{cv'} = 0 \checkmark$$

The IVP for the Transport eq.

$$u_t + c u_x = 0 \quad (2)$$

$$u(0,x) = g(x) \quad (IC)$$

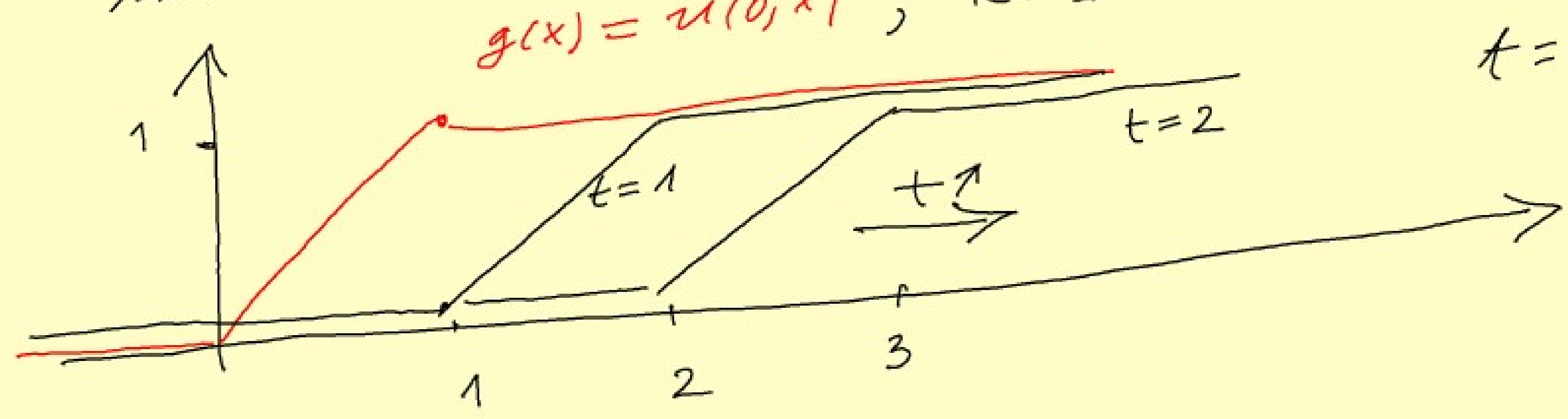
From $(*)$ $\quad g(x) = u(0,x) = v(x - \cancel{c0})$

This means that $u(t,x) = v(x-ct) = g(x-ct)$

the solution is actually $u(t,x) = g(x-ct)$

$t=1 \quad u(1,x) = g(x-1)$

$t=2 \quad u(2,x) = g(x-2)$



The nonhomogeneous Transport IVP

$$u_t + c\, u_x = f(t,x) \quad (2_f)$$
$$u(0,x) = g(x) \quad (IC)$$

To solve $(IVP) = (2_f) + (IC)$ we use the new variable $z = z(s) := u(t+s,\ x+cs)$

$$(**) \quad \frac{dz}{ds}(s) = u_t(t+s,\ x+cs) + u_x(t+s,\ x+cs)\cdot c \overset{(2_f)}{=} f(t+s,\ x+cs)$$

The trick is that we can write both u & g using z

$$u(t,x) - \underbrace{g(x-ct)}_{\substack{u(0,\,x-ct) \\ = z(-t)}} = z(0) - z(-t) = \int_{-t}^{0} \underbrace{\frac{dz}{ds}(s)}_{\overset{(**)}{=}\ f(t+s,\ x+cs)} ds$$

So

$$\underbrace{u(t,x)}_{\substack{\text{Sol of} \\ \text{nonhomog} \\ (IVP)}} = \underbrace{g(x-ct)}_{\substack{\text{Sol of} \\ \text{homog} \\ IVP}} + \underbrace{\int_0^t f(s,\ x+c(t-s))\, ds}_{\substack{\text{contrib due to} \\ \text{RHS of } (2_f)}}$$

§ 2.3. The Wave eq

$$u_{tt} = u_{xx} \quad (3) \qquad x \in \mathbb{R},\ t \geq 0$$

$$u(0,x) = u_0(x) \quad (iC_1) \qquad u_0, v_0 \text{ given}$$

$$u_t(0,x) = v_0(x) \quad (iC_2)$$

To solve the iVP = (3) + (iC₁) + (iC₂)

we use the transport eq. / wave solutions.

Notice that based on properties of $\frac{\partial}{\partial \ldots}$ we have

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)$$

$a^2 - b^2 = (a+b)(a-b)$

Transport eq.

$$\text{So (3)} \iff \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u}_{= v} = 0 \iff v_t + v_x = 0$$

where $v(t,x) = u_t(t,x) - u_x(t,x) \quad (**)$

This means that v solves a transport iVP

$$\begin{cases} v_t + v_x = 0 \\ v(0,x) = a(x) \overset{(**)}{=} v_0(x) - \frac{\partial}{\partial x} u_0(x) \end{cases} \quad (iC_1 \ \& \ iC_2)$$

sol is $v(t,x) = a(x-t)$

Now returning to the def. of v which leads to a transport iVP for u:

$$\begin{cases} u_t - u_x = v(t,x) = a(x-t) \\ u(0,x) = u_0(x) \quad (iC_1) \end{cases}$$

$$\hookrightarrow u(t,x) = u_0(x+t) + \int_0^t a(x + (t-s) - s)\, ds$$

and we reach d'Alembert's Formula for the sol of (3) + (iC₁) + (iC₂):

$$u(t,x) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\, dy.$$

II. Linear IVPs and the Fourier Transform

Before we start: A brief hystory of the exponential

EULER 18^{th} century

$$\begin{cases} \frac{d}{dt} u(t) = -a\,u(t) \\ u(0) = u_0 \end{cases} \qquad a \in \mathbb{R},\ u(t) \in \mathbb{R} \qquad \text{Sol } u(t) = e^{-at} u_0$$

PEANO ~ 1900

$$\begin{cases} \frac{d}{dt} u = -A\,u(t) \\ u(0) = u_0 \end{cases} \qquad u(t) = \begin{pmatrix} u_1(t) \\ \cdots \\ u_d(t) \end{pmatrix} \qquad \text{Sol } u(t) = e^{-tA} u_0$$

$A = (a_{ij})_{i,j=\overline{1,d}}$ matrix $d \times d$

exp. of a matrix $e^A = I + A + \frac{1}{2} A^2 + \dots$ (Taylor)

YOSIDA ~ 1930

$$\begin{cases} \frac{d}{dt} u = -Au \\ u(0) = u_0 \end{cases} \qquad u(t) \in X \text{ Banach sp.} \qquad \text{Sol } u(t) = e^{-tA} u_0$$

$A \in \mathcal{L}(X)$ i.e.

$A : X \to X$, lin, bdd.

everything works just as in $\mathbb{R}^d$. Use Taylor for e^A.

HILLE & YOSIDA late 1940s

$$\begin{cases} \frac{d}{dt} u = -Au \\ u(0) = u_0 \end{cases} \qquad u(t) \in X \text{ Banach sp.} \qquad \text{Sol } u(t) = \underbrace{\text{“}e^{-tA}\text{”}}_{S(t)} u_0$$

$A : D(A) \subsetneq X \to X$

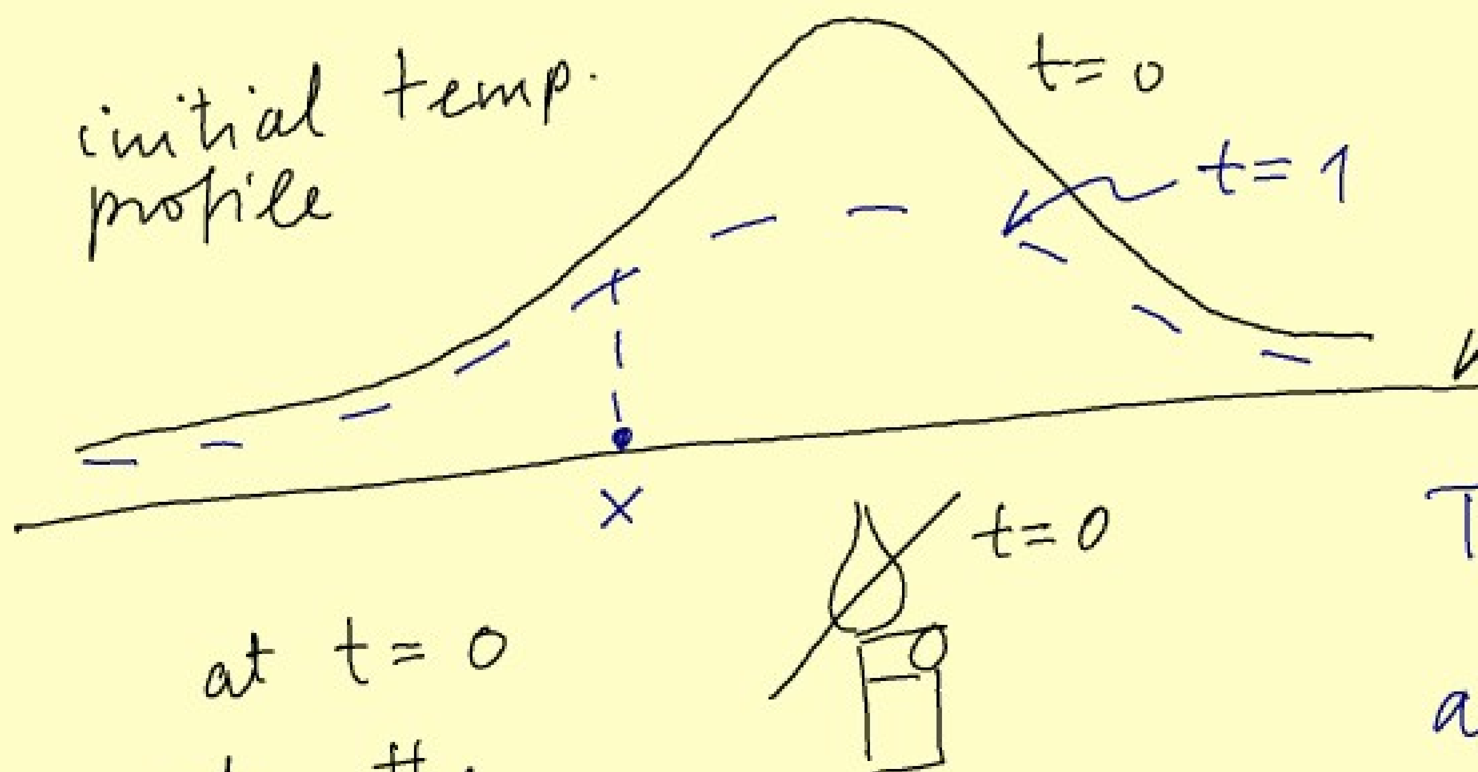
lin. <u>un</u>bounded op. Semigroup.

The issue: $A, A^2, \dots$ don't have the same domain

Key Idea: treat evol. PDE as ODE in Banach or Hilbert sp.

(modern evol. PDEs)

Example: Heat eq. IVP $(*)\begin{cases} u_t = u_{xx} & t>0,\ x\in\mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$



at $t=0$ stop the heat source

The point is that at each time t you can describe the temp. profile by a function of x

$$u(t,x) = \underbrace{u(t)}_{\text{a function of } x \text{ for every } t}(x)$$

rewrite

rewrite $(*)$ $\begin{cases} \underbrace{\frac{d}{dt}u(t)} = \frac{\partial^2}{\partial x^2}u(t) & \text{in } L^2(\mathbb{R}) \\ u(0) = u_0 & \text{in } L^2(\mathbb{R}) \end{cases}$

$u : [0,T) \to L^2(\mathbb{R})$, $\quad u(t) \in L^2(\mathbb{R})$

$$L^2(\mathbb{R}) = \left\{ u : \mathbb{R} \to \mathbb{R} : \begin{array}{l} u \text{ measurable} \\ \int_{\mathbb{R}} u(x)^2\,dx < \infty \end{array} \right\}$$

§ 3.1. L^p spaces

$$\|u\|_{L^p} = \left(\int |u(x)|^p dx \right)^{1/p}$$

$1 \le p < \infty$ $\quad L^p(\underset{\Omega}{\mathbb{R}^n}) = \{ u : \underset{\Omega}{\mathbb{R}^n} \to \mathbb{R} : \text{meas.} \ \int_{\underset{\Omega}{\mathbb{R}^n}} |u(x)|^p dx < \infty \}$

L^∞ is different

$L^\infty(\underset{\Omega}{\mathbb{R}}) = \{ u : \underset{\Omega}{\mathbb{R}^n} \to \mathbb{R} : \text{meas} \ \& \ \exists C \text{ with } |u(x)| \le C \text{ a.e. } \underset{\Omega}{\mathbb{R}^n} \}$

frequentey used are L^1, L^2, L^∞

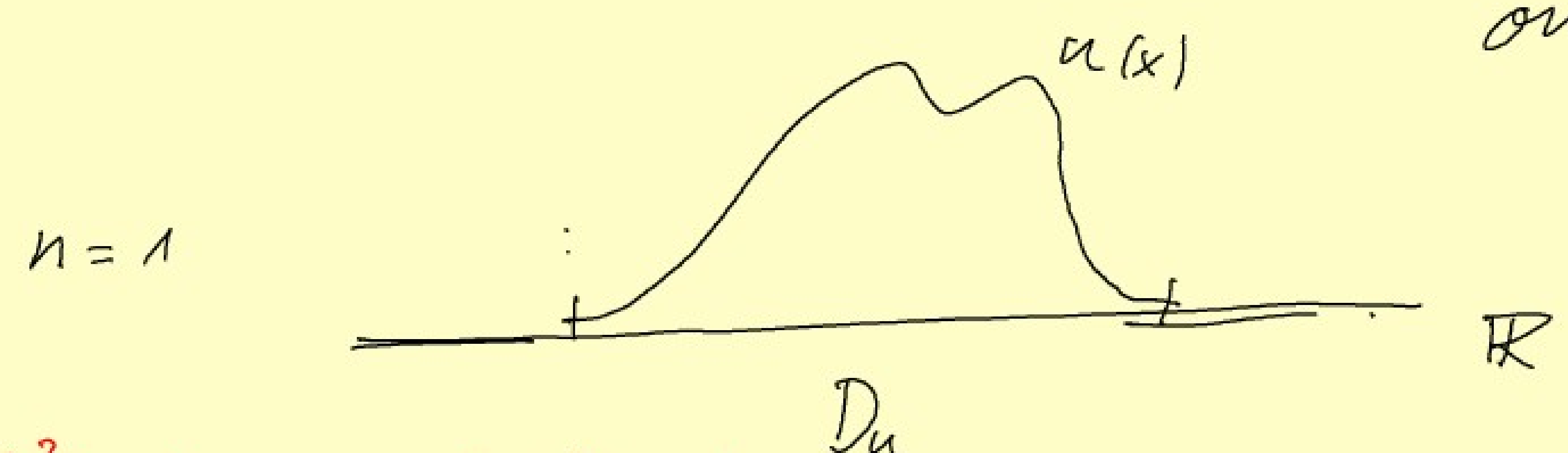
All L^p spaces ($p = \infty$ included) are Banach spaces (See [Brezis, Th 4.8])

We will use quite often the Approx. result

Th 1 ([Brezis, Th 4.12]) $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $\quad 1 \le p < \infty$

"any L^p function can be approx. by a sequence of cont. functions that vanish outside a compact subset of $\mathbb{R}^n$"

$C_c(\mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{R}$ cont. : u vanishes (is zero) outside a compact $D_u \}$



— L^2 is a Hilbert space

§ 3.2. Convolution product and the Fourier Transform

Convolution: $f, g : \mathbb{R}^n \to \mathbb{R}$

$$(f * g)(x) \overset{\text{Def.}}{=} \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Th 2. (fundam prop. of convol.)
If $f \in L^1$, $g \in L^p$ $(1 \le p \le \infty)$ then $f * g \in L^p$
and $\| f * g \|_{L^p} \le \| f \|_{L^1} \| g \|_{L^p}$

The Fourier Transform ass. for the moment that $u \in L^1 \cap L^2$

$$\mathcal{F}(u)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i x \cdot y} u(x)\, dx$$

$x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ $\quad i = \sqrt{-1} \in \mathbb{C}$

$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$|x|^2 = x \cdot x = x_1^2 + x_2^2 + \dots + x_n^2$ Euclidean Norm

The $\mathcal{F}$ transforms a function $u(x)$ into a new function $\hat{u}(y) = \mathcal{F}(u)(y)$

The inverse Fourier Transform

$$\mathcal{F}^{-1}(\hat{u})(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{+i x \cdot y} \hat{u}(y)\, dy$$

✓

Fundamental properties of $\mathcal{F}$

(1) $\mathcal{F}(f * g) = (2\pi)^{n/2} \mathcal{F}(f)\mathcal{F}(g)$

(2) $\mathcal{F}(D^\alpha u)(y) = (iy)^\alpha \mathcal{F}(u)(y)$

$\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex

$|\alpha| := \alpha_1 + \dots + \alpha_n$, $\quad D^\alpha = \dfrac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}$

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ for $x = (x_1, \dots, x_n)$

$\mathcal{F}(\Delta u)(y) = -|y|^2 \mathcal{F}(u)(y)$

$\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dots + \dfrac{\partial^2}{\partial x_n^2}$ Laplace operator

this was wrong in 1st version of these Notes!

Recording contains ERROR at time ~ 1:15:00.

Further properties:

(PLANCHEREL 'TH') $\quad \|u\|_{L^2} = \|\mathcal{F}(u)\|_{L^2}$

(SHIFT) $\quad \mathcal{F}(u(x-x_0))(y) = e^{-ix_0 \cdot y} \mathcal{F}(u)(y)$

(SCALE) $\quad \mathcal{F}(u(\lambda x))(y) = \dfrac{1}{|\lambda|^n} \mathcal{F}(u)\left(\dfrac{y}{\lambda}\right)$

($\mathbb{C}$ CONJUGATE) $\quad \mathcal{F}(\overline{u(-x)})(y) = \overline{\mathcal{F}(u(x))(y)}$

(INVARIANT) $\quad \mathcal{F}(e^{-\frac{1}{2}|x|^2})(y) = e^{-\frac{1}{2}|y|^2}$

here (& in (2)) $\quad |x|^2 = x_1^2 + \dots + x_n^2, \quad |y|^2 = y_1^2 + \dots + y_n^2$

§ 3.3. The $\mathcal{F}$ Approach

Heat IVP
$$\begin{cases} \frac{d}{dt} u(t) = \Delta u(t) & t > 0 \text{ in } L^2(\mathbb{R}^n) \\ u(0) = u_0 & \text{in } L^2(\mathbb{R}^n) \end{cases}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}$$

The $\mathcal{F}$ Approach consists of several steps

Step 1. Apply $\mathcal{F}$ to the IVP

$$\mathcal{F}\left(\frac{d}{dt} u(t)\right)(y) = \frac{d}{dt} \mathcal{F}(u)(y)$$

$$\mathcal{F}(\Delta u(t))(y) = -|y|^2 \mathcal{F}(u)(y)$$

$$\mathcal{F}(u(0))(y) = \mathcal{F}(u_0)(y)$$

with the convenient notation $\hat{u}(t) := \mathcal{F}(u(t))(y)$

Heat IVP $\xrightarrow{\mathcal{F}}$ ODE IVP $\begin{cases} \frac{d}{dt} \hat{u}(t) = -|y|^2 \hat{u}(t) \\ \hat{u}(0) = \hat{u}_0 \end{cases}$

here y is a parameter

Step 2. Solve ODE IVP $\hat{u}(t) = e^{-|y|^2 t} \hat{u}_0$

$$\mathcal{F}(u(t))(y) = e^{-|y|^2 t} \mathcal{F}(u_0)(y)$$

Step 3. Apply $\mathcal{F}^{-1}$ (reverse engineering)

" Find ? : $\mathcal{F}(?)(y) = e^{-|y|^2 t} \mathcal{F}(u_0)(y)$ "

nontrivial!

$\mathcal{F}^{-1} \Big| \quad \mathcal{F}(f * g) = (2\pi)^{n/2} \mathcal{F}(f)\mathcal{F}(g)$

$$(**) \quad \mathcal{F}^{-1}(\mathcal{F}(f)\,\mathcal{F}(g)) = \frac{1}{(2\pi)^{n/2}}\, f * g$$

in our case $\quad \underbrace{e^{-|y|^2 t}}_{\mathcal{F}(f)} \quad \underbrace{\mathcal{F}(u_0)(y)}_{\mathcal{F}(g)}$ ✓

Find ? : $\quad \mathcal{F}(?)(y) = e^{-t|y|^2}$

Recall "invariant" $\quad \mathcal{F}(e^{-\frac{1}{2}|x|^2})(y) = e^{-\frac{1}{2}|y|^2}$

"scaling" $\quad \mathcal{F}(u(\lambda x))(y) = \ldots$

So, after computations,

finally $\quad ? = N(t)(x) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$

HEAT KERNEL

$$\mathcal{F}(N(t))(y) = e^{-t|y|^2}$$

So, back to $(**)$ we have

$$u(t)(x) = (N(t) * u_0)(x)$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy$$

4. The Fourier Transform: Rigorous Results.

$L^p(\mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{C} : u \text{ meas. } \int_{\mathbb{R}^n} |u(x)|^p < \infty \}$

$1 \le p < \infty$

$L^\infty(\mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{C} : u \text{ meas } \& \ \exists C \text{ s.t. } |u(x)| \le C \text{ a.e. } x \in \mathbb{R}^n \}$

Th 1. ([Brezis, Th 4.12]) (Approx of L^p functions by continuous functions that vanish outside a compact)

$C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$

(see also Lecture 3.)

§ 3.1. Convolution and regularization

$f, g : \mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}$)

$$\underbrace{(f * g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy}_{\text{Def of convol.}} \overset{\boxed{HW}\text{ check}}{=} (g * f)(x)$$

Thm 2. (Fundam prop. of $*$)

If $f \in L^1$, $g \in L^p$ $(1 \le p \le \infty)$

Then $f * g \in L^p$ and we have

$$\| f * g \|_{L^p} \le \| f \|_{L^1} \| g \|_{L^p} .$$

Proof: Three cases $p = 1$, $p = \infty$, $1 < p < \infty$

$\boxed{p = \infty}$ $\quad \| f * g \|_{L^\infty} \text{"="} \max_x | f * g (x) |$

$$| f * g (x) | = \left| \int_{\mathbb{R}^n} f(x-y) g(y) \, dy \right| \le \int_{\mathbb{R}^n} |f(x-y)| \underbrace{|g(y)|}_{\le \|g\|_{L^\infty}} dy$$

$$\le \|g\|_{L^\infty} \underbrace{\int_{\mathbb{R}} |f(x-y)| \, dy}_{= \|f\|_{L^1}} \quad \checkmark$$

$\boxed{p = 1}$ follows from $\begin{cases} \text{Thm Tonelli} \\ \text{Thm Fubini} \end{cases}$

$\boxed{HW}$

(see [Brezis Thm 4.4 & Thm 4.5.])

$\boxed{1 < p < \infty}$ based on Hölder's Inequality!

conjug exponent q : $\frac{1}{p} + \frac{1}{q} = 1$

Trick: $fg = f^{\frac{1}{p}+\frac{1}{q}} g = f^{\frac{1}{q}} (f^{\frac{1}{p}} g)$

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y) g(y) \, dy \right|$$

$$\leq \int_{\mathbb{R}} |f(x-y)| \, |g(y)| \, dy$$

$$\leq \int_{\mathbb{R}} |f(x-y)|^{1/q} \left(|f(x-y)|^{1/p} |g(y)| \right) dy$$

$$\overset{(*)}{\leq} \left(\int_{\mathbb{R}^n} |f(x-y)| \, dy \right)^{1/q} \left(\int_{\mathbb{R}^n} |f(x-y)| \, |g(y)|^p \right)^{1/p}$$

Young's Inequality: $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$

$$\| f * g \|_{L^p} = \left(\int_{\mathbb{R}^n} |(f * g)(x)|^p \, dx \right)^{1/p}$$

use $\frac{p}{q} = p-1$

$$\overset{(*)}{\leq} \left(\int_{\mathbb{R}^n} \left[\left(\int_{\mathbb{R}^n} |f(x-y)| \, dy \right)^{\frac{p}{q} = p-1} \left(\int_{\mathbb{R}^n} |f \cdots| \, |g|^p \right)^{\cancel{p/p}} \right] dx \right)$$

$$\leq \| f \|_{L^1}^{p-1} \int_{\mathbb{R}^n} \underbrace{|f(x-y)|}_{L^1} \underbrace{|g(y)|^p}_{L^1} \, dy$$

apply $p=1$ case ✓

$$\|f*g\|_{L^p} \le \Big(\|f\|_{L^1}^{p-1} \underbrace{\int_{\mathbb{R}^n} \underbrace{|f(x-y)|}_{L^1} \underbrace{|g(y)|^p}_{L^1} dy}\Big)^{1/p}$$

$p = 1$ case $\longrightarrow$ $\le \|f\|_{L^1} \underbrace{\|g^p\|_{L^1}}_{\|g\|_{L^p}^p}$

$$\le \Big(\|f\|_{L^1}^{p-\cancel{1}} \cancel{\|f\|_{L^1}} \|g\|_{L^p}^p \Big)^{1/p} //$$

$$\le \|f\|_{L^1} \|g\|_{L^p} \quad ✓$$

$\square$ 3. If $f \in C_c^k(\mathbb{R}^n)$, $g \in L^1(\mathbb{R})$ and

then $f*g \in C_c^k(\mathbb{R}^n)$ and

$$D^\alpha(f*g) = (D^\alpha f)*g$$

"The $f*g$ product has the regularity of the better function (here f) and derivatives can be put on the better function."

Proof: $k = 0$ $\quad f \in C_c(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n) \overset{?}{\Longrightarrow} f*g \in C(\mathbb{R}^n)$

$\forall x \quad y \mapsto f(x-y)g(y)$ is integrable (f cont on compact)

so $(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$ integral exists & is finite cont? $\quad x_k \to x \quad$ take $K \subset \mathbb{R}^n$ compact such that $\forall k$ $f(x_k - y) = 0$ $\forall y \notin K$

f unif. cont (cont. on compact)

$$|f(x_k - y) - f(x-y)| \le \varepsilon_k \chi_K(y), \quad \varepsilon_k \xrightarrow{k \to \infty} 0$$

$$\chi_K(y) = \begin{cases} 1 & y \in K \\ 0 & y \notin K \end{cases}$$

indicator function

$$|(f*g)(x_k) - (f*g)(x)| \le \int |f(x_k - y)g(y) - f(x-y)g(y)|\, dy$$

$$= \int_{\mathbb{R}^n} |f(x_k - y) - f(x-y)|\, |g(y)|\, dy$$

$\in L^1$

$$\le \varepsilon_K \int_{\mathbb{R}^n} \chi_K(y)\, |g(y)|\, dy = \varepsilon_k \underbrace{\int_K |g(y)|\, dy}_{\le C} \xrightarrow[\varepsilon_k \to 0]{} 0$$

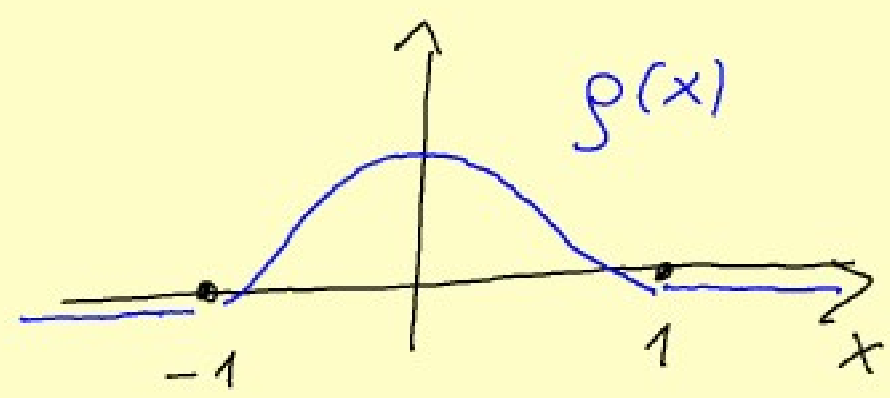
so $f*g$ is cont. at x.

IDEA (K. O. Friedrichs) :
~ '40s

regularization by convolution

using **Mollifiers**

$n = 1$



$$\rho(x) = \begin{cases} 0, & |x| > 1 \\ e^{\frac{1}{|x|^2 - 1}}, & |x| < 1 \end{cases}$$

$\in C_c^\infty(\mathbb{R}^n)$

$|x| = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

SCALE ρ : $\rho_k(x) = k^n \rho(kx)$ $\quad k = 1, 2, \dots$

$\rho_k(0) = 0$ outside the $1/k$ radius Ball of $\mathbb{R}^n$

Th 4 (regularized mollifier-based approximation)

If $f \in L^p(\mathbb{R}^n)$ $\quad 1 \le p < \infty$

then $\underbrace{\rho_k * f}_{\in C_c^\infty(\mathbb{R}^n)} \xrightarrow{k \to \infty} f$ (based on Th 3)

Rk: This is a constructive approximation as ρ_k are all known!

To prove $\boxed{T}$ 4 you need some auxil. results.

$\boxed{T}$ 5 (shifted functions)

If $f \in L^p(\mathbb{R}^n)$ $1 \le p < \infty$ then

$$\| f(x-h) - f(x) \|_{L^p} \to 0 \quad \text{as } h \to 0.$$

Sketch of the Proof:

$f \in C_c^\infty(\mathbb{R}^n)$ then $f(x-h) \xrightarrow{h \to 0} f(x)$ uniformly

if $f \in L^p$, $\varepsilon > 0$ choose $g \in C_c^\infty$: $\|f-g\|_{L^p} < \frac{\varepsilon}{3}$

obviously $\| f(x-h) - g(x-h) \|_{L^p} < \frac{\varepsilon}{3}$

and

$$\| f(x-h) - f(x) \|_{L^p} \le \| f(x-h) - g(x-h) \|_{L^p} + \| g(x-h) - g(x) \|_{L^p} + \| g(x) - f(x) \|_{L^p}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$\boxed{T}$ 6. Let $g \in L^1(\mathbb{R}^n)$, $a = \int_{\mathbb{R}^n} g(x)\,dx$. Let $f \in L^p(\mathbb{R}^n)$

Define $g_k(x) = k^n g(kx)$.

a) for $1 \le p < \infty$ $\quad f * g_k \xrightarrow[k \to \infty]{L^p} af$

b) $p = \infty$ and f is unif. cont. $\quad f * g_k \xrightarrow[k \to \infty]{\text{unif}} af$

— Sketch of Proof ($\boxed{P}$ 6.)

a) change vars "$kx = y$"

$$(f * g_k)(x) - a f(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x)) g_k(y)\, dy$$

$$= \int_{\mathbb{R}} (f(x-k^{-1}y) - f(x)) g(y))\, dy$$

$$\int \big| |(f * g_k)(x) - a f(x)|^p \le \|g\|_{L^1}^{p/q} \int_{\mathbb{R}^n} |f(x-k^{-1}y) - f(x)|^p |g|\, dy$$

$$\|f * g_k - a f\|_{L^p}^p \le \|g\|_{L^1}^{p/q} \int_{\mathbb{R}^n} \|f(x-k^{-1}y) - f(x)\|_{L^p}^p |g(y)|\, dy$$

$k^{-1} \to 0$ as $k \to \infty$

The seq. of functions $y \mapsto \|f(x-k^{-1}y) - f(x)\|_{L^p}^p |g(y)|$

- is dominated by $2\|f\|_{L^p}^p |g(y)| \in L^1$
- and tends to zero ($\boxed{L}$ 5)

So Lebesgue's dominated convergence Thm guarantees that it tends to zero in L^1.

b) see [Precup, p. 111]

— Proof of $\boxed{T}$ 4 : Apply $\boxed{P}$ 6 to $g = \rho$ the mollifiers

5. The Fourier Transform: Rigorous results

$u \in L^1(\mathbb{R}^n)$ $\qquad \mathcal{F}(u)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)\, e^{-ix\cdot y}\, dx$

- obvious (linearity)

  $\mathcal{F}(u+v) = \mathcal{F}(u) + \mathcal{F}(v)$ and $\mathcal{F}(\alpha u) = \alpha\, \mathcal{F}(u)$

- L^∞ vs L^1 estimate

  $$|\mathcal{F}(u)(y)| \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} |u(x)|\, \underbrace{|e^{-ix\cdot y}|}_{=1}\, dx = \frac{1}{(2\pi)^{n/2}} \|u\|_{L^1}$$

  so $$\underbrace{\|\mathcal{F}(u)\|_{L^\infty}}_{\text{"amplit. of } \mathcal{F}(u)\text{"}} \le \frac{1}{(2\pi)^{n/2}} \underbrace{\|u\|_{L^1}}_{\text{"area under } u\text{"}}$$

  dim 1 $(n=1)$

- continuity ...

Th 1. (invariant) $G(x) = e^{-\frac{1}{2}|x|^2}, \; x \in \mathbb{R}^n$

$\mathcal{F}(G) = G$ (Gaussian)

$x = (x_1, x_2, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $i^2 = -1$

$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$, $|x|^2 = x_1^2 + \dots + x_n^2$

Outline of Proof: $\mathcal{F}(G)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2 - i x \cdot y} \, dx$

Step 1: Force $e^{-\frac{1}{2}|y|^2}$ ($\Longleftrightarrow$ create square)

$$-\frac{1}{2}|x|^2 - i x \cdot y = -\frac{1}{2} \sum_{j=1}^n x_j^2 - i \sum_{j=1}^n x_j y_j$$

$$= -\frac{1}{2} \sum_{j=1}^n (x_j + i y_j)^2 - \frac{1}{2} \overbrace{\sum_{j=1}^n y_j^2}^{|y|^2}$$

square

So $\mathcal{F}(G)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum (x_j + i y_j)^2} e^{-\frac{1}{2}|y|^2} \, dx$

$$= G(y) \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum (x_j + i y_j)^2} \, \underbrace{dx}_{= dx_1 \dots dx_n}$$

Step 2: $e^{-\frac{1}{2}\sum (x_j + i y_j)^2} = \prod_{j=1}^n e^{-\frac{1}{2}(x_j + i y_j)^2}$

So (Fubini)

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum (x_j + i y_j)^2} \, dx = \frac{1}{(2\pi)^{n/2}} \prod_{j=1}^n \int_{\mathbb{R}} e^{-\frac{1}{2}(x_j + i y_j)^2} \, d x_j$$

Step 3: use $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(\dots)} \, d x_j = 1$ (see exercise sheet #1)!

—

[See Bonus Point Exercise]

T2 (Plancherel) $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then

$$\|u\|_{L^2} = \|\mathcal{F}(u)\|_{L^2}$$

For simplicity $n=1$ (see $n>1$ [Evans])

Rk: Here we consider $\mathbb{C}$-valued functions "& spaces"

$$\left(\|u\|^2_{L^2} := \int_{\mathbb{R}} u(x)\, \overline{u(x)}\, dx \right)$$ $\mathbb{C}$ conjugate

Proof:

Step 1.: introduce $v(x) = \overline{u(-x)}$, $w = u * v$ (convolution), observe that

- $w(0) = u * v(0) = \int_{\mathbb{R}} u(0-y)\, \overline{u(-y)}\, dy = \|u\|^2_{L^2}$

$$\boxed{(1) \quad \|u\|_{L^2} = w(0)}$$

- $\mathcal{F}(w) = \mathcal{F}(u * v) = \sqrt{2\pi}\, \mathcal{F}(u)\mathcal{F}(v) = \sqrt{2\pi}\, |\mathcal{F}(u)|^2 \Big| \int_{\mathbb{R}}$

  a straigthford comput shows that $\mathcal{F}(v) = \overline{\mathcal{F}(u)}$

$$\boxed{(2) \quad \|\mathcal{F}(u)\|^2_{L^2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}(w)(y)\, dy}$$

Step 2: Create connection between (1) & (2)

use $$\int_{\mathbb{R}} w(y) \underbrace{\mathcal{F}(h)(y)}_{g_\varepsilon} dy = \int_{\mathbb{R}} \mathcal{F}(w)(x) \underbrace{h(x)}_{h_\varepsilon} dx$$

with $h(x) = h_\varepsilon(x) = e^{-\varepsilon|x|}$, $g_\varepsilon(y) = \mathcal{F}(h_\varepsilon)(y) = \sqrt{\frac{2}{\pi}} \frac{\varepsilon}{\varepsilon^2 + y^2}$

$$\underbrace{\int_{\mathbb{R}} w(y)\, g_\varepsilon(y)\, dy}_{\xrightarrow{\varepsilon \to 0}\ w(0) \overset{(1)}{=} \|u\|^2} = \underbrace{\int_{\mathbb{R}} \mathcal{F}(w)(x)\, h_\varepsilon(x)\, dx}_{\xrightarrow{\varepsilon \to 0}\ \int_{\mathbb{R}} \mathcal{F}(w)(x)\, dx \overset{(2)}{=} \|\mathcal{F}(u)\|^2}$$

(Lebesgue's dom. conv. Thm.) ✓

Consequence of Plancherel's Thm

you can define $\mathcal{F} : L^2 \to L^2$ (not $\underline{L^1 \cap L^2}$)

Idea: "use Cauchy sequences"

$u_k \xrightarrow[L^2]{k\to\infty} u \Rightarrow (u_k)$ Cauchy sequence, but

$$\| \mathcal{F}(u_k) - \mathcal{F}(u) \|_{L^2} \overset{\mathcal{F}\text{ lin}}{=} \| \mathcal{F}(u_k - u) \|_{L^2}$$

$$\overset{\text{Planch.}}{=} \| u_k - u \|_{L^2} \xrightarrow[\text{(ass.)}]{k\to\infty} 0$$

So $(\mathcal{F}(u_k))$ is L^2 Cauchy sequence $\Rightarrow$

$\Rightarrow (\mathcal{F}(u_k))$ converges (limit exists)

$$\mathcal{F}(u_k) \xrightarrow[L^2]{k\to\infty} \hat{u} =: \mathcal{F}(u)$$

(orig. def may not work because $\int$ could diverge for $u \in L^2$)

So $\mathcal{F}$ is an $L^2 \to L^2$ isometry

(actually also bijective as $u = \mathcal{F}^{-1}(u)$)

inverse $\mathcal{F}$

§5.1. Rigorous Results for the Heat iVP

$$\begin{pmatrix}\text{Heat}\\ \text{iVP}\end{pmatrix} \quad \begin{cases} \frac{d}{dt} u(t) = \Delta u(t), & \text{in } L^2(\mathbb{R}^n), \\ u(0) = u_0, & u_0 \in L^2(\mathbb{R}^n). \end{cases}$$

Th 3 (Heat iVP) If $u_0 \in L^2(\mathbb{R}^n)$ there exist a unique "solution" $u \in C([0,\infty), L^2(\mathbb{R}^n)) \cap C^1((0,\infty), L^2(\mathbb{R}^n))$ such that iVP holds in $L^2(\mathbb{R}^n)$. Furthermore

a) immediate regularization
$u(t) \in C^\infty(\mathbb{R}^n)$ for $t > 0$

b) positivity
$u_0(x) \geq 0$ a.e. $x \in \mathbb{R}^n \Rightarrow u(t)(x) \geq 0$ a.e. $x \in \mathbb{R}^n$, $t \geq 0$

c) maximum principle (if $u_0 \in L^2 \cap L^\infty(\mathbb{R}^n)$)
$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$$

d) L^2 contractivity
$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$$

e) "energy" decay (along evolution solutions)
$$\frac{d}{dt} E(u(t)) \leq 0, \quad E(u) = \int_{\mathbb{R}} |\nabla u(x)|^2 dx$$

Ex. 14.

Approach: use $\mathcal{F}$ to get "candidate" of sol. Then prove it has right properties $u(t)(x) = (N(t) * u_0)(x)$

§5.2. Rigorous Results for the Wave IVP.

$$\text{(Wave IVP)}\quad \begin{cases} \dfrac{d^2}{dt^2} u(t) = \Delta u(t) \\ u(0) = u_0 \quad \text{and} \quad \dfrac{d}{dt} u(0) = v_0 \end{cases}$$

Thm 4. (Wave IVP) Let

$u_0 \in H^1(\mathbb{R}^n)$ and $v_0 \in L^2(\mathbb{R})$

There exist a unique "generalized" solution

$u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}))$ not C^2

($\Leftrightarrow$ $u \in C([0,\infty), H^1)$ and $v = \frac{d}{dt} u \in C([0,\infty), L^2)$)

such that (equation holds in a weak sense) $\forall t \geq 0$

$$\left\langle \frac{d^2}{dt^2} u(t), \varphi \right\rangle_{L^2} = \langle u_x(t), \varphi_x \rangle_{L^2} \quad \forall \varphi \in H^1(\mathbb{R}^n)$$

and furthermore energy is conserved

$$E(t) := \frac{1}{2} \|v(t)\|^2_{L^2} + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx = E_0 \quad \forall t \geq 0$$

total = kinetik + potential energy

$H^1(\mathbb{R}^n)$ = function with derivatives in L^2

$= \{ u \in L^2 : D^\alpha u \in L^2(\mathbb{R}^n), \ |\alpha| \leq 1 \}$

equiv defs $\overset{\mathcal{F}}{=} \{ u \in L^2 : (1+|y|^2)^{1/2} \mathcal{F}(u) \in L^2 \}$

Representation Formulae for the Wave IVP

$n = 1$ (dim $= 1$) $\qquad \hat{u} := \mathcal{F}(u)$

1. Apply $\mathcal{F}$ (to Wave IVP) $\qquad \begin{cases} \frac{d^2}{dt^2}\hat{u} = -|y|^2 \hat{u} \\ \hat{u}(0) = \hat{u}_0, \quad \frac{d\hat{u}}{dt} = \hat{v}_0 \end{cases}$

2. Solve the ODE IVP

$$\hat{u}(t) = \hat{u}_0 \cos(t|y|) + \frac{\hat{v}_0}{|y|} \sin(t|y|)$$

3. Apply $\mathcal{F}^{-1}$

$$u(t)(x) = \underbrace{\mathcal{F}^{-1}(\hat{u}_0 \cos(t|y|))(x)}_{(A)} + \underbrace{\mathcal{F}^{-1}\left(\hat{v}_0 \frac{\sin t|y|}{|y|}\right)(x)}_{(B)}$$

for (A) use $\quad \cos t|y| = \frac{1}{2}\left(e^{it|y|} + e^{-it|y|}\right) \quad (*)$

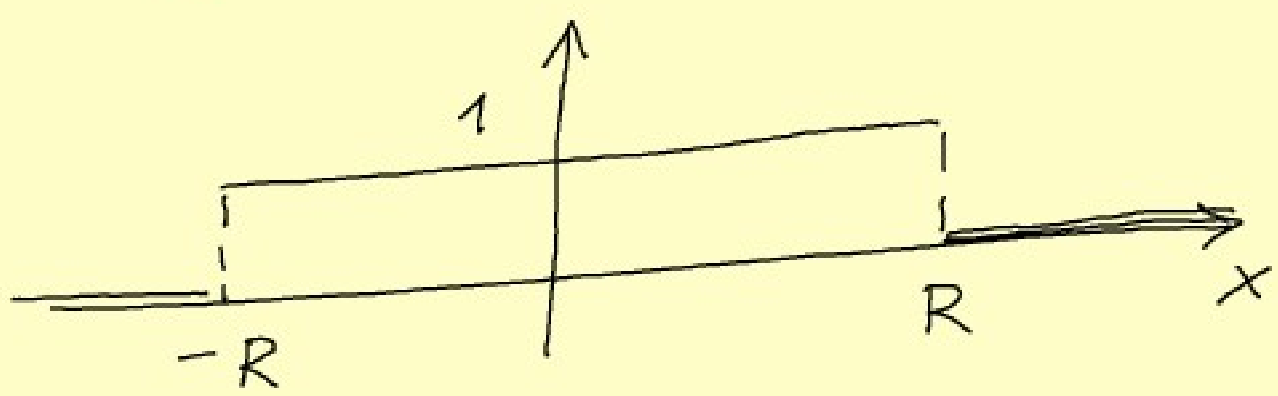
and (SHIFT) $\quad \mathcal{F}(u(x-x_0))(y) = e^{-ix_0 \cdot y} \mathcal{F}(u)(y) \quad (**)$

$$\hat{u}_0 \cos(t|y|) \overset{\text{def } \hat{u}_0}{=} \mathcal{F}(u_0)(y) \cos(t|y|) =$$

$$\overset{(*)}{=} \frac{1}{2}\mathcal{F}(u_0)(y)\, e^{it|y|} + \frac{1}{2}\mathcal{F}(u_0)(y)\, e^{-it|y|}$$

$$\overset{(**)}{=} \frac{1}{2}\left[\mathcal{F}(u_0(x-t))(y) + \mathcal{F}(u_0(x+t))(y)\right]$$

for (B) use $\quad \mathcal{F}(H(R-|x|)) = \sqrt{\frac{2}{\pi}} \frac{\sin Ry}{y}$



so

$$\hat{v}_0 \frac{\sin t|y|}{|y|} \overset{?}{=} \mathcal{F}(H(t-|x|) * v_0(x))$$

Finally from (A) and (B) we have

$$u(t)(x) = \mathcal{F}^{-1}\left(\frac{1}{2}\left[\mathcal{F}(u_0(x-t)) + \mathcal{F}(u_0(x+t))\right]\right)$$

$\boxed{n=1}$

$$+ \mathcal{F}^{-1}\left(\mathcal{F}\left(\frac{1}{2} H(t-|x|) * v_0(x)\right)\right)$$

$$= \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right] + \frac{1}{2}\int_{\mathbb{R}} H(t-|x-y|)\, v_0(y)\, dy$$

d'Alembert Formula (see Lecture #2)

In higher dimensions we have

$\boxed{n=2}$ Poisson Formula

$$u(t)(x) = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{B_t(x)} \frac{u_0(y)\, dy}{\sqrt{t^2-|x-y|^2}} + \frac{1}{2\pi}\int_{B_t(x)} \frac{v_0(y)\, dy}{\sqrt{t^2-|x-y|^2}}$$

$\boxed{n=3}$ Kirchoff Formula

$$u(t)(x) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{\partial B_t(x)} u_0(y)\, d\sigma\right) + \frac{1}{4\pi t}\int_{\partial B_t(x)} v_0(y)\, d\sigma$$

$d\sigma$ = surface element.

The way waves propagate "depends on space dimension".

6. What everyone should know about ODEs.

ODEs (diff eqns) = very diverse Field

Trade-off:

General Theory vs Detailed Example

ODE References

- V. Barbu "general theory"
- S. Strogatz "detailed examples" [videos on youtube]
- Smale, Hirsch & Devaney "in between" US textbook

Dichotomies

Linear vs Nonlinear

$u' = au$ Malthus

$u' = au(1-u)$ Verhulst (Logistic growth)

Lin. const. coefs vs. Lin. variable coefs.

Evolution Equations vs. Equilibrium

Explicit sols vs Qualit. Analysis

Dimension: 1, 2, 3, ..., finite, ... ∞

No periodic sols, No chaos (1) | YES periodic, No chaos (2) | YES periodic, YES chaos (3) [Strogatz]

My fav. approach: structure of eq $\Longrightarrow$ properties

latin "to build"

§6.1. Evolution equations

IDEA (& ingredients): NEWTON

You have: a state space X

$u(t)$ = state of syst. at time t

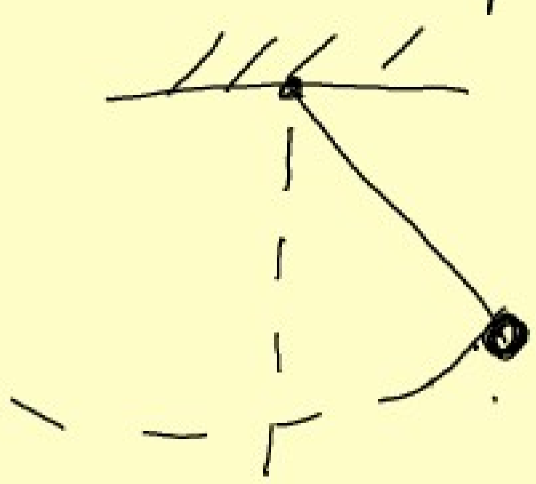
Law (of motion) $\frac{d}{dt} u(t) = F(u(t))$

infinitezimal change of state = "forces"

cause change

CAUSALITY (Causal evolution)

First question: Find correct state space



Math (idealized) pendulum

$u'' = -\omega^2 u \qquad (u = u(t))$

does not seem to fit $\frac{d}{dt} u = F(u)$ form

but after rewriting

$$(*) \begin{cases} u' = v & \text{velocity = time-deriv of position} \\ v' = -\omega^2 u & \text{Newton's law} \end{cases}$$

$X = \mathbb{R}^2$ state space $\qquad U = \begin{bmatrix} u \\ v \end{bmatrix}$ states $\qquad (*) \iff \frac{d}{dt} U(t) = F(U(t))$

$$F(U) = AU = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

sol $u(t) = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$

where $u_0 = u(0)$, $v_0 = v(0)$

General Theory for (1) $\begin{cases} \frac{d}{dt} u(t) = F(u(t)) \\ u(0) = u_0 \end{cases}$

Th 1. X state space (= Banach space)

$F: X \to X$ locally Lipschitz cont.

Then for each $u_0 \in X$ there exists a $\underbrace{\text{sol.}}_{\text{local in time}}$ $u(t; u_0)$ defined on $[0, T_{max})$ and <u>either</u>

a) sol. global in time ($T_{max} = \infty$) <u>or</u>

b) blow-up ($\exists\, T_{max} < \infty$ and $\|u(t;u_0)\| \xrightarrow{t \nearrow T_{max}} \infty$.)

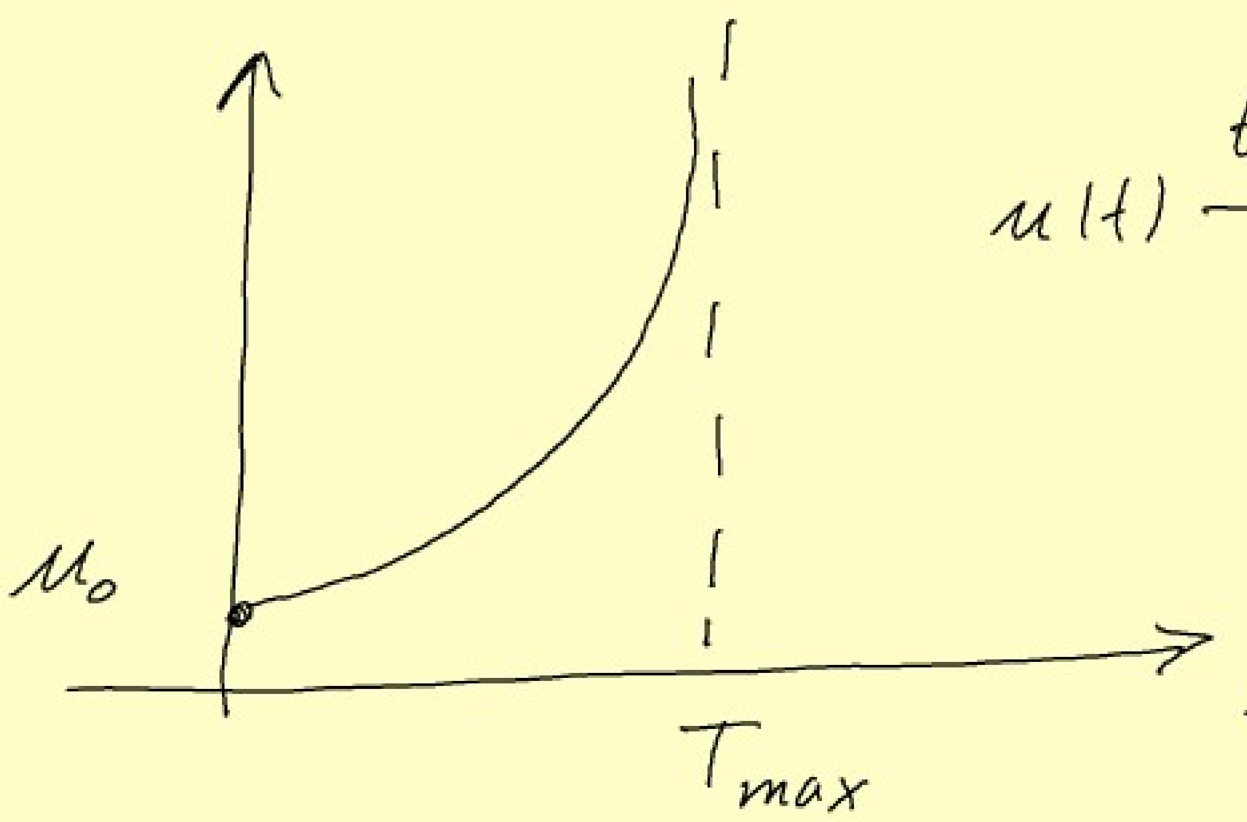
Examples:

- locally Lip. functions
  Polynomials
  $F(u) = u - u^2$

counterexample
$F(u) = \sqrt{u}$ at $u_0 = 0$
$\left(F'(u) = \frac{1}{2\sqrt{u}} \xrightarrow{u \to 0} \infty \right)$

- blow-up

$u' = u^2$
$u(0) = u_0 > 0$
nonlinear positive feed-back



$u(t) \xrightarrow{t \to T_{max}} \infty$

Th 2. Linear evol. equs. always have global in time solutions. ($F(u) = Au$).

HW: Try to prove this
use Gronwall Ineq.

Rk : The Simplest solutions are stationary (or equilibrium) sols

i.e. $\frac{d}{dt} u = 0 \iff \underline{F(u) = 0}$

equil. equation

Examples

- $u' = au$, $a \in \mathbb{R}$, $a \neq 0$ equil $u^* = 0$
- $U' = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A} U$ $AU = 0$ (eq. for Ker A)

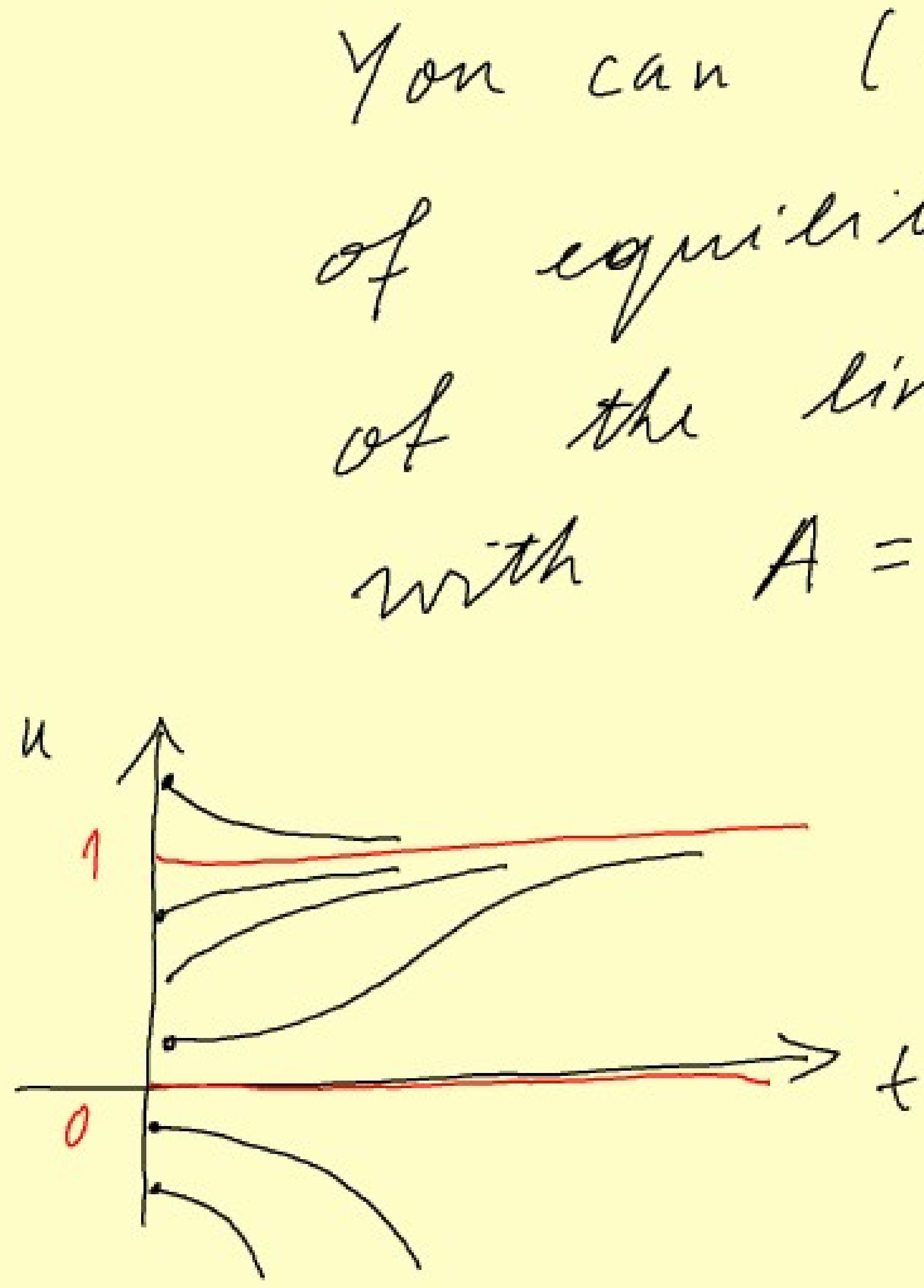
  Ker A $= \{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2 : u \in \mathbb{R}, v = 0 \}$

  a continuum of equilibria

- $u' = u(1-u)$

  $F(u) = 0 \iff u(1-u) = 0$ $u_0^* = 0$, $u_1^* = 1$

Th 3. (Hartman - Grobman)
You can (in some cases) study the stability of equilibria in terms of eigenvalues of the linearized equation $\frac{du}{dt} = Au$ with $A = J_F(u^*)$ (u^* equil, J_F Jacobian of F)



$u' = u(1-u)$, $\underbrace{F(u) = u - u^2}$

$J_F(u) = F'(u) = 1 - 2u$

0 repells solutions
1 attracts sols

$F'(0) = 1$ instab. $F'(1) = -1$ as. stab.

Asymptotic ($t \to \infty$) behavior and its connection to equilibria (The simplest sols.)

LYAPUNOV Theory for $\frac{d}{dt} u = F(u)$, $F: X \to X$

$E: X \to \mathbb{R}$ ("energy", every state has an "energy")

<u>strict</u> Lyapunov function if

(i) $E(u(t;u_0)) \leq E(u_0)$ decrease along traject.

(ii) $E(u(t;u_0)) = E(u_0) \implies u_0$ equilibrium ($F(u_0) = 0$)

Th 4. (LASALLE's Invariance Principle)

If E is strict Lyapunov function for (1) and $u_0 \in X$ is such that the traj (as a set) $\bigcup_{t \geq 0} \{u(t;u_0)\}$ is relatively compact in X then

(i) $\lim_{t\to\infty} E(u(t;u_0)) = \ell$ exists

(ii) $\operatorname{dist}(u(t;u_0), \mathcal{E}) \xrightarrow{t\to\infty} 0$ $\mathcal{E}$ set of all equilibria

(iii) if $\mathcal{E}$ is discrete then $\exists u^* = u^*(u_0) \in \mathcal{E}$ such that $u(t;u_0) \xrightarrow{t\to\infty} u^*$

"strict Lyap fct. $\implies$ conv. to equil"

§ 6.2. Models with (automatic) Energy / Lyapunov structure

$E: X \to \mathbb{R}$

A. Gradient Flows

$$\frac{d}{dt} u = -\nabla E(u) \quad \text{(GF)}$$

(need inner product space)

CHAIN rule gives

$$\frac{d}{dt} E(u(t)) \overset{\text{CHAIN rule}}{=} \left\langle \nabla E(u(t)), \frac{d}{dt} u(t) \right\rangle \overset{\text{(GF)}}{=} - \| \nabla E(u(t)) \|^2$$

E ~~decreasing~~ non increasing along solutions (Lyapunov)

B. Hamiltonian Systems

$$\text{(H)} \begin{cases} \frac{d}{dt} u = v \\ \frac{d}{dt} v = -\nabla E(u) \end{cases}$$

check that

$$\frac{d}{dt} \Big(\underbrace{\tfrac{1}{2} \|v(t)\|^2 + E(u(t))}_{\tilde{E}} \Big) = 0$$

$\tilde{E}$ is conserved along traj (Lyap. but not strict!)

C. Damped / Dissipative Systems

$$\text{(D)} \begin{cases} \frac{d}{dt} u = v \\ \frac{d}{dt} v = -av - \nabla E(u) \end{cases}$$

$a > 0$

$\tilde{E}$ strict Lyap.

$$\frac{d}{dt} \Big(\overbrace{\tfrac{1}{2} \|v(t)\|^2 + E(u(t))}^{\tilde{E}} \Big) = -a \|v(t)\|^2$$

kin. Energy, potential Energy, dissipated energy

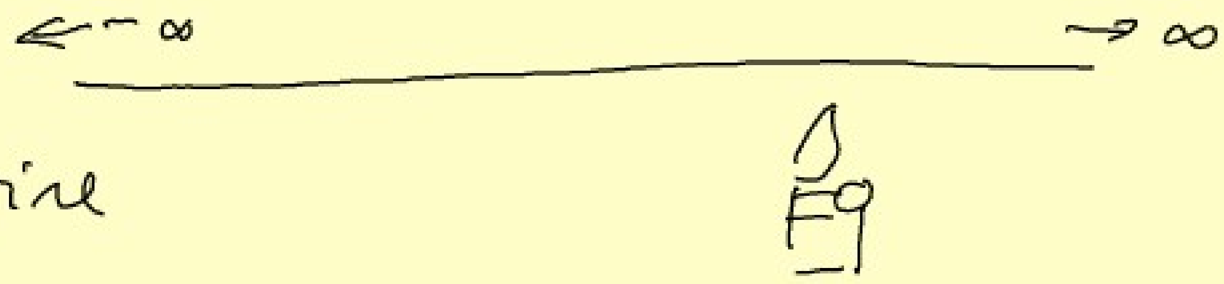
II. Linear PDEs in bounded (spatial) domains

7. Boundary Value Problems I: Motivation & Classical Theory

§7.1. Motivation

Until now: equations on unbounded domains

e.g. Heat in infinite wire



From now on: bounded domains

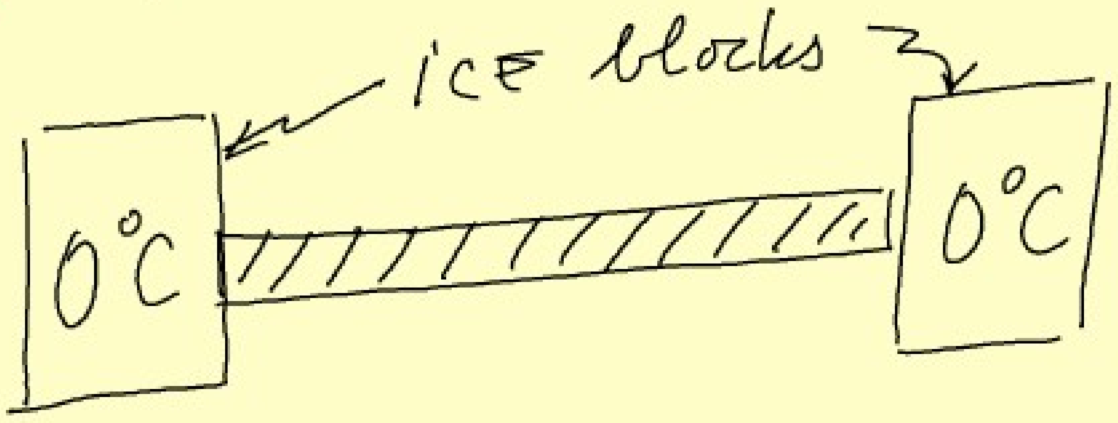
So, consider $\Omega = (0,1) \subset \mathbb{R}$ and the Heat Initial Boundary Value Problem

$$(IBVP) \quad \begin{cases} u_t = u_{xx} & t \geq 0, \quad x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \quad (IC) \\ u(t,0) = u(t,1) = 0 & \forall t \geq 0 \quad (BC) \end{cases}$$

$\Leftrightarrow$ $u = 0$ on $\partial\Omega$

(DIRICHLET) Boundary condition

Model: Heat flow in a finite bar of length $= 1$ and with fixed temp. at ends ($= 0°C$)



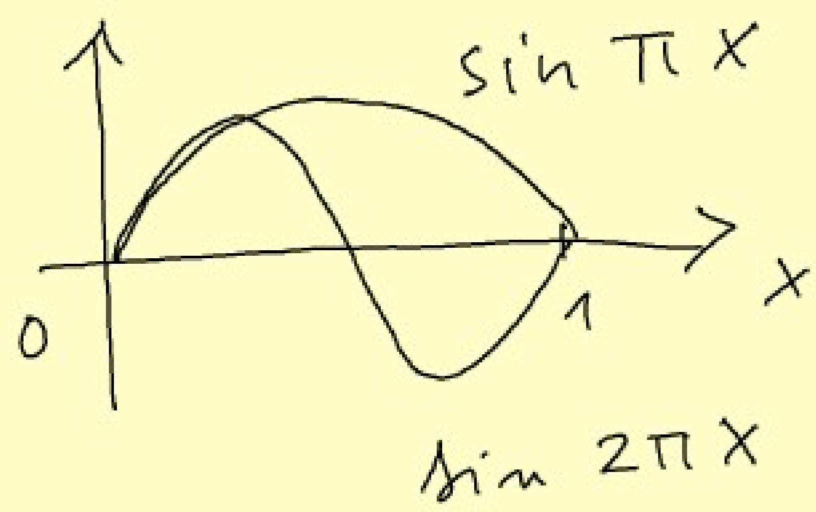
J. FOURIER: Analytic Theory of Heat (1822)

AIM: Solve Heat (IBVP)

Fourier's Approach
(Important and <u>new</u>: the (BC)!)

Insight 1: Which functions satisfy (BC)?

Consider $\sin(\pi x)$
$\sin(\pi 2 x)$
$\dots$
$k \in \mathbb{N}^*$ $\quad \sin(\pi k x)$



all these satisfy the (BC).

Insight 2: You also have to consider the eq...

• Let's compute $(\sin \pi k x)_{xx} = (\pi k \cos \pi k x)_x$
$= -\pi^2 k^2 \sin \pi k x$

• Consider also time-dependence

Consider $u(t,x) = e^{-\pi^2 k^2 t} \sin \pi k x$

$$\underbrace{\left(e^{-\pi^2 k^2 t} \sin \pi k x\right)_t}_{u_t \;=\; u_{xx}} = \left(e^{-\pi^2 k^2 t}\right)_t \sin \pi k x$$
$$= -\pi^2 k^2 e^{-\pi^2 k^2 t} \sin \pi k x$$

$$\overbrace{\left(e^{-\pi^2 k^2 t} \sin \pi k x\right)_{xx}} = e^{-\pi^2 k^2 t} (\sin \pi k x)_{xx}$$
$$= -\pi^2 k^2 e^{-\pi^2 k^2 t} \sin \pi k x$$

This is a sol of Heat eq. & satisfies (BC).

Furthermore: eq is linear so if $u_1, \dots, u_k$ sols. then so is any lin. comb. $\sum c_k u_k$ (sol.)

$(c_k \in \mathbb{R})$

This means that

$$(*) \quad u(t,x) = \sum_{k=1}^{\infty} c_k e^{-\pi^2 k^2 t} \sin \pi k x$$

is sol & satisf (BC).

Insight 3: What about the (IC)?

Choose c_k such that (IC) is satisf. by (*).

Fourier's glorious insight is that you can compute all c_k using the (simple) fact

$$(\perp) \quad \int_0^1 \sin \pi k x \sin \pi j x \, dx = \begin{cases} 0, & \text{if } k \neq j, \\ 1/2, & \text{if } k = j. \end{cases}$$

orthogonality w.r.t. $L^2(\Omega)$ scalar product

Want: $$u_0(x) = u(0,x) \overset{(*)}{=} \sum_{k=1}^{\infty} c_k \cancel{e^{-\pi^2 k^2 0}}^{1} \sin \pi k x$$

$$u_0(x) = \sum c_k \sin \pi k x \quad \Big| \sin \pi j x, \int_0^1$$

$$\int_0^1 u_0(x) \sin \pi j x \, dx = \frac{1}{2} c_j$$ (only one nonzero term!)

So, we can compute all c_k (since $u_0(x)$ is given),

and $$u(t,x) = \sum_{k=1}^{\infty} c_k e^{-\pi^2 k^2 t} \sin \pi k x$$ is a sol of the (IBVP)!

We can solve the Heat (IBVP) for a 1D bounded (spatial) domain.

However, what happens in dim > 1 ?

$$u_t = \Delta u \qquad t \geq 0, \quad x \in \Omega \subset \mathbb{R}^n$$

$$u(0,x) = u_0(x) \qquad x \in \Omega \qquad \text{(IC)}$$

$$u(t,x) = 0 \qquad x \in \partial\Omega, \quad \text{(BC)} \quad t \geq 0$$

with $\overbrace{x = (x_1, \dots, x_n) \in \mathbb{R}^n}^{\text{notations}}$

$$\Delta u(x) = \frac{\partial^2 u}{\partial x_1^2}(x) + \dots + \frac{\partial^2 u}{\partial x_n^2}(x)$$

How can you choose appropriate replacements for $\sin \pi k x$ (in a general domain Ω)?

You need to solve the eigenvalue (Helmholtz) problem
$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

or more generally the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases} \qquad \text{(BVP)}$$

No time-dep.

No (IC)

These are surprisingly hard problems because they depend on the geometry of Ω (esp. $\partial\Omega$)

§ 7.2. Classical Dirichlet BVP Theory

Details: [Prenup]

IDEA of classical theory:
this is a diff eq.
So integrate
to solve it.

AIM: Solve $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

$\Omega \subsetneq \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ ("scalar field")

$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_n}(x) \right) \in \mathbb{R}^n$ ("vector field")

gradient

$V = (V_1, \dots, V_n)$; $V_i : \Omega \to \mathbb{R}$ ("vector field")

$\operatorname{div} V(x) = \frac{\partial V_1}{\partial x_1}(x) + \dots + \frac{\partial V_n}{\partial x_n}(x)$ ("scalar field")

divergence

$\Delta u(x) = \operatorname{div} \nabla u(x) = \frac{\partial^2 u}{\partial x_1}(x) + \dots + \frac{\partial^2 u}{\partial x_n}(x)$

Laplacian

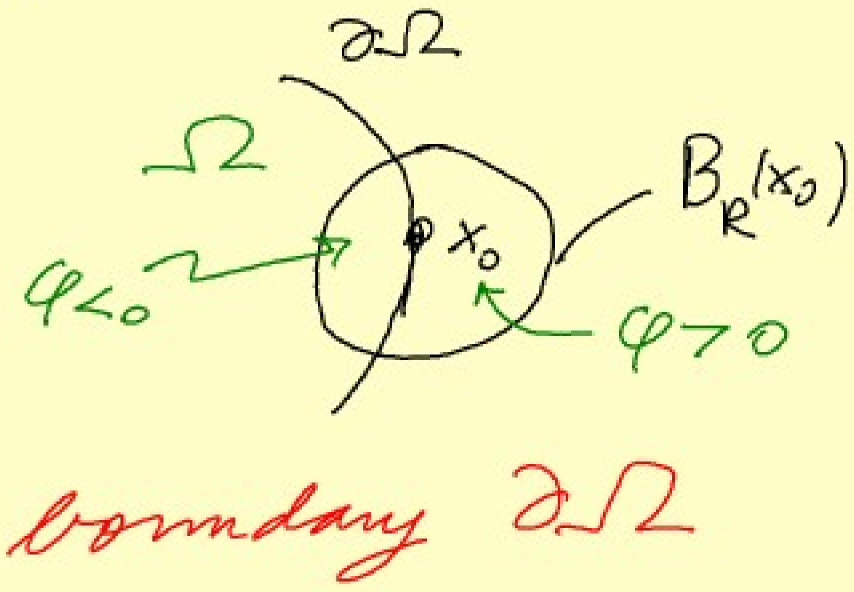
directional derivative

$v \in \mathbb{R}^n$, $|v| = 1$ $\quad \frac{\partial u}{\partial v}(x) = \nabla u(x) \cdot v$

$\partial\Omega$ boundary of Ω while $\overline{\Omega}$ closure of Ω.

We call an open set $\Omega \subset \mathbb{R}^n$ of <u>class C^k</u> if
$\forall x_0 \in \partial\Omega$ $\exists B_R(x_0)$ and $\exists \varphi \in C^k(B_R(x_0))$ with:

1. $\nabla\varphi(x) \neq 0_{\mathbb{R}^n}$ $\forall x \in B_R(x_0)$
2. $\varphi(x) < 0$ $\quad \forall x \in \Omega \cap B_R(x_0)$
3. $\varphi(x) > 0$ $\quad \forall x \in (\mathbb{R}^n \setminus \overline{\Omega}) \cap B_R(x_0)$



"$\varphi = 0$" is the "equation" of the boundary $\partial\Omega$

Classical integral calculus results:

Th 1 (Divergence (or GAUSS-OSTROGRADSKI) Thm)
$\Omega \subset \mathbb{R}^n$ open, bounded and C^1
V vector field, ν normal to $\partial\Omega$

$$\int_{\partial\Omega} V\cdot\nu \, d\sigma = \int_\Omega \operatorname{div} V \, dx$$

Th 2. (GREEN'S Formulae)

(G1) $$\int_{\partial\Omega} u \frac{\partial v}{\partial \nu} d\sigma = \int_\Omega (u\Delta v + \nabla u\cdot\nabla v)\,dx$$

(G2) $$\int_{\partial\Omega} \left(u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right) d\sigma = \int_\Omega (u\Delta v - v\Delta u)\,dx$$

Idea of Proof: apply Gauss-O. to get (G1)
then (G2) follows directly from (G1) [Precap]

Def: $u:\Omega\to\mathbb{R}$ is called <u>harmonic</u> if
$\Delta u(x) = 0 \quad \forall x\in\Omega$

Th 3. (GAUSS) $\Omega\subset\mathbb{R}^n$ open, bdd, C^1
while u is harmonic (on Ω) then

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,d\sigma = 0.$$

Rk (see Ex. 21)

Harmonic functions with radial symmetry

$u(x) = u(|x|)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$|x| = \sqrt{x_1^2 + \dots + x_n^2}$ Euclidean norm

are

$u(x) = C_1 \ln |x| + C_2$ $x \neq 0$ in $\mathbb{R}^2$

$u(x) = C_1 |x|^{2-n} + C_2$ $x \neq 0$ in $\mathbb{R}^n$, $n \geq 3$

Def: The fundam. sol. of Laplace's eq

$N : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $N(x) = \begin{cases} \frac{1}{2\pi} \ln|x|, & n=2 \\ -\frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$ const

$\Delta N(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

ω_n = measure of unit sphere in $\mathbb{R}^n$

($\omega_2 = 2\pi$, $\omega_3 = 4\pi$, etc.)

Th 4. (RIEMANN-GREEN Formula) Ω open, bdd, C^1

(RG) $$u(x) = \int_\Omega \Delta u(y)\, N(x-y)\, dy + \int_{\partial\Omega} \left(u(y) \frac{\partial N(x-y)}{\partial \nu_y} - N(x-y) \frac{\partial u(y)}{\partial \nu} \right) d\sigma_y$$

for $x \in \Omega$, ν_y normal in $y \in \partial\Omega$.

[Precup] for Proof.

Consequences of (RG)

Th 5. (Mean Value Thm for Harmonic functions)

$\Omega \subset \mathbb{R}^n$ open, u harmonic on Ω

For any closed ball $\overline{B}_r(x) \subset \Omega$ we have

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, d\sigma$$

("value at center of ball = avg of values on sphere")

Th 6. (The Strong Max Principle)

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ open, connected} \\ u \in C^2(\Omega), \quad \Delta u(x) \geq 0 \ \forall x \in \Omega \\ \exists x_0 \in \Omega : \quad u(x_0) = \sup_{\Omega} u \end{array}\right\} \Rightarrow \begin{array}{l} u = \text{const} \\ \text{on } \Omega \end{array}$$

Th 7. (The Weak Max Princ.)

$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^n \text{ open, bounded} \\ u \in C^2(\Omega) \cap C(\overline{\Omega}) \\ \Delta u \geq 0 \quad \text{in } \Omega \\ u \leq 0 \quad \text{on } \partial\Omega \end{array}\right\} \Rightarrow u \leq 0 \quad \text{on } \overline{\Omega}$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Applications:
- positivity of solutions
- uniqueness of sol.
- cont data dependence of sol. w.r.t. f, g.

But you can't prove existence...

8. Boundary Value Problems II : Modern Theory

$$(BVP) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Dirichlet BC (but you could also have $u = g$) given

Lecture 7 : Classical Idea (Riemann – Green)

"integrate in order to solve the BVP"

BAD News : It doesn't work

Remarkable example where integral calculus works in dim = 3 !

"Poisson Formula"

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad f \equiv 0 \qquad \text{with } \Omega = B_R(0) \text{ ball of radius } R$$

$$u(x) = \frac{R^2 - |x|^2}{R\,\omega_n} \int_{|x|=R} \frac{g(y)}{|x-y|^3}\, d\sigma_y \qquad |x| < R$$

(integral over sphere $) = \partial\Omega$

DIRICHLET's fresh IDEA :

$\begin{cases} \text{Optimization} & \text{(rewrite BVP as Optimiz. Probl)} \\ + \text{Approximation} & \text{(approximate solution)} \end{cases}$

intimately related to the concept of solution !

§ 8.1. The Dirichlet Principle (for classical sols)

Consider the Dirichlet (BVP) $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ (D)

Def: u is classical sol of (D) if $u \in C^2(\bar{\Omega})$ and (D) holds.

With the (BC) in mind: $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$

functions that vanish on the boundary

The (Dirichlet) Energy functional

$$E : C_0^1(\bar{\Omega}) \to \mathbb{R}, \quad E(u) = \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 - f u\right) dx$$

see Lecture #7

Th 1. (Dirichlet's Princ.) $\Omega \subset \mathbb{R}^n$ open, bdd and C^1

$f \in C(\bar{\Omega})$ and $u \in C^2(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$

Then the following statements are equivalent:

(i) u is classical sol. of (D)

(ii) u satisfies the variational identity

$$\int_\Omega (\nabla u \cdot \nabla v - f v)\, dx = 0 \quad \forall v \in C_0^1(\bar{\Omega}) \quad \text{(VI)}$$

(iii) u is a strict global minimum of E (Dirichlet Energy)

$$E(u) < E(w) \quad \forall w \in C_0^1(\bar{\Omega}),\ w \neq u.$$

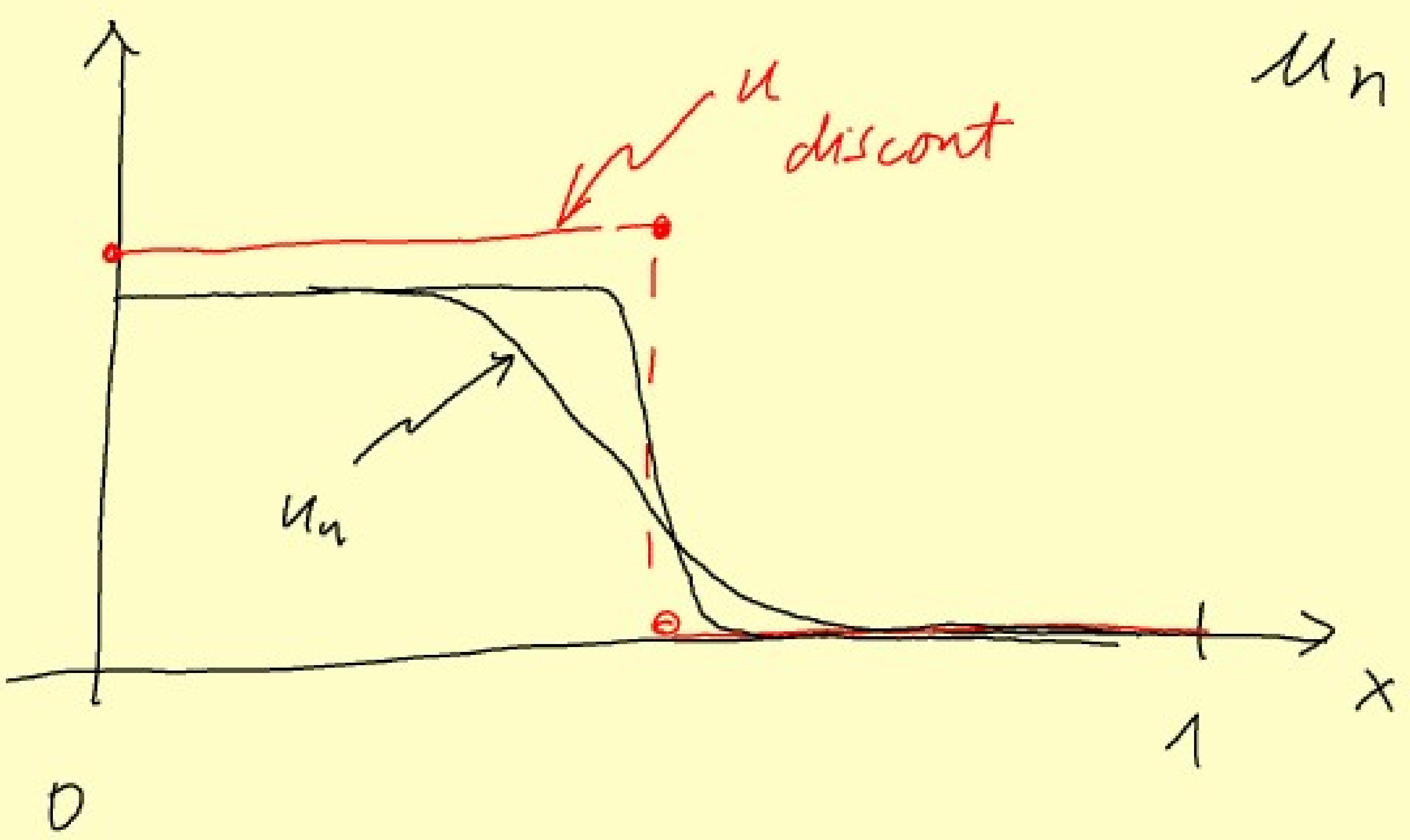
Proof: (i) $\Leftrightarrow$ (ii) Exercise 24 see also [Precup].

The Problem with classical solutions:

$C^2(\bar{\Omega})$ is NOT complete w.r.t. the L^2 norm

you can construct a sequence of $C^2(\bar{\Omega})$ functions which converge to a discont. function

e.g. $\Omega = (0,1) \subset \mathbb{R}$



$u_n \xrightarrow{L^2} u$

Nonexistence of sols for optimiz. problems:

take $E(x) = \frac{1}{2}(x-\sqrt{2})^2$, $E \to \min!$

this minimization probl. has NO solution in $\mathbb{Q}$

(because the sol is $x^* = \sqrt{2}$)

See also COURANT's Counterexample (Exercise #28)

Modern Theory, the IDEA: move away from C^2 classical solution

3 reasons:
- more general data (i.e. $f \in L^2(\Omega)$)
- existence
- want to work with E-minimizing sequences (approximation)

✓

§ 8.2. Moving away from classical solutions: weak (generalized) solutions

SOBOLEV: consider the smallest complete subspace of $L^2(\Omega)$ which contains $C_0^1(\bar{\Omega})$.
This turns out to be $H_0^1(\Omega)$.

$$C_0^1(\bar{\Omega}) \subset H_0^1(\Omega) \subset L^2(\Omega)$$

$H_0^1(\Omega)$ consists of all limits of $C_0^1(\bar{\Omega})$ Cauchy (w.r.t. $L^2(\Omega)$ norm) sequences

Th 2. $H_0^1(\Omega)$ is a Hilbert space with resp. to (complete inner product space)

the energy norm $\|u\|_{H_0^1}^2 = \int_\Omega |\nabla u|^2 dx$

associated to the inner product

$$\langle u, v\rangle_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \left(\|u\|_{H_0^1}^2 = \langle u, u\rangle_{H_0^1}\right)$$

without Proof (see [Preasp], [Brezis]) recom.

Def: $u \in H_0^1(\Omega)$ is a weak (generalized) sol. of (D) if (VI) holds, namely

$$\text{(VI)} \quad \int_\Omega (\nabla u \cdot \nabla v - f v)\, dx = 0 \qquad \forall v \in H_0^1(\Omega)$$

$$\iff \quad \langle u, v\rangle_{H_0^1} - \langle f, v\rangle_{L^2} = 0$$

The H_0^1 analysis is possible only due to the critically important Poincaré (-Friedrichs) inequality.

Th 3. (POINCARÉ Inequality)

Ω open and bounded $\Longrightarrow \exists\ c = c(\Omega) = \text{const.}$ such that

$$(*) \quad \int_\Omega u^2\,dx \le c^2 \int_\Omega |\nabla u|^2\,dx \quad \forall u \in C_0^1(\bar\Omega)$$

Proof: [Pinchover-Rubinstein] or Exercise # 25.

Rk $\quad C_0^1(\bar\Omega)$ is dense by construction in $H_0^1(\Omega)$

This means (by extension to H_0^1) that

actually $\quad (*) \quad \|u\|_{L^2}^2 \le c^2 \|u\|_{H_0^1}^2 \quad \forall u \in H_0^1(\Omega)$

$\hookrightarrow$ Poincaré Ineq. in H_0^1

Th 4. (Dirichlet's Princ. in H_0^1) Ω open, bounded

$f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ then

def of weak sol.

(ii) $\quad \langle u, v\rangle_{H_0^1} - \langle f, v\rangle_{L^2} = 0 \quad \forall v \in H_0^1(\Omega) \quad$ (VI)

$\Updownarrow$

(iii) $\quad u$ minimizes E over $H_0^1(\Omega)$

Dirichlet energy

Th 5. (EXISTENCE & UNIQUENESS of a weak sol)

Ω open, bounded and $f \in L^2(\Omega)$

Then there exist a unique weak sol $u \in H_0^1(\Omega)$ of (D).

IDEA of Proof:

Apply the RIESZ Representation Thm.

Th 6. (RIESZ) Let $(X, \langle \cdot,\cdot \rangle_X)$ Hilbert space.

For any linear continuous functional $F: X \to \mathbb{R}$, we have $F(v) = \langle u_F, v \rangle_X \ \forall v \in X$

(any linear functional can be represented as product)

there exist a unique $u_F \in X$ such that

To prove Th 5 we check that $F: H_0^1(\Omega) \to \mathbb{R}$, $F(v) = \langle f, v \rangle_{L^2}$ is a linear (obvious) and continuous? functional.

$$|F(v)| = |\langle f, v \rangle_{L^2}| \underset{\text{Schwarz}}{\leq} \|f\|_{L^2} \|v\|_{L^2} \underset{\text{Poincaré Ineq.}}{\leq} \text{Const} \|f\|_{L^2} \|v\|_{H_0^1}$$

So $\|v\| \xrightarrow{H_0^1} 0 \implies |F(v)| \to 0$ (cont ✓)

We can apply Th RIESZ on $(H_0^1(\Omega), \langle \cdot,\cdot \rangle_{H_0^1})$ Hilbert sp

By Riesz, there exists $u_F \in H_0^1(\Omega)$ such that

$$\langle f, v \rangle_{L^2} \overset{\text{Def}}{=} F(v) = \langle u_F, v \rangle_{H_0^1} \quad \forall v \in H_0^1(\Omega)$$

But this is just (VI) for u_F ((VI) = def weak sol

Hence this u_F is the desired unique weak sol. of (D

9. Boundary Value Problems III: Approx. Solutions of BVP – The Finite Element Method (FEM)

Recap Lecture 8.

Dirichlet (in $H_0^1(\Omega)$)

Ω open, bounded and if $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ then $\boxed{(ii) \Leftrightarrow (iii)}$

(i) $\langle u, v \rangle_{H_0^1} - \langle f, v \rangle_{L^2} = 0 \quad \forall v \in H_0^1(\Omega)$

(VI) variational identity = def of weak sol. of the (D) $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

(ii) u minimizes the Dirichlet Energy over $H_0^1(\Omega)$

(OPT) $$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx$$

$$= \frac{1}{2}\|u\|^2_{H_0^1} - \langle f, u \rangle$$

Thm (Exist & uniqueness of weak sol.)

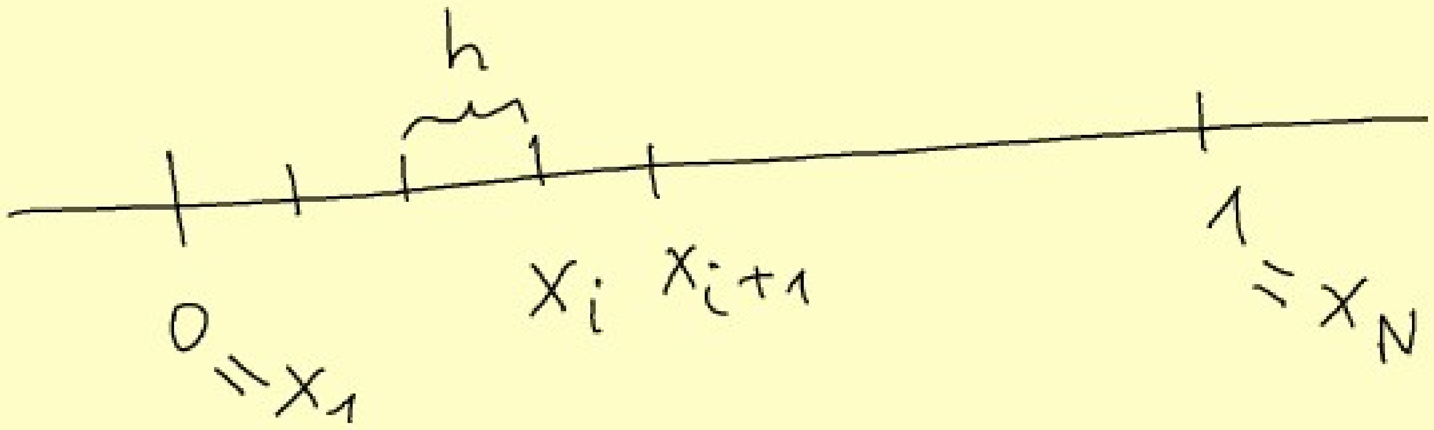
Ω bounded, open, $f \in L^2(\Omega)$

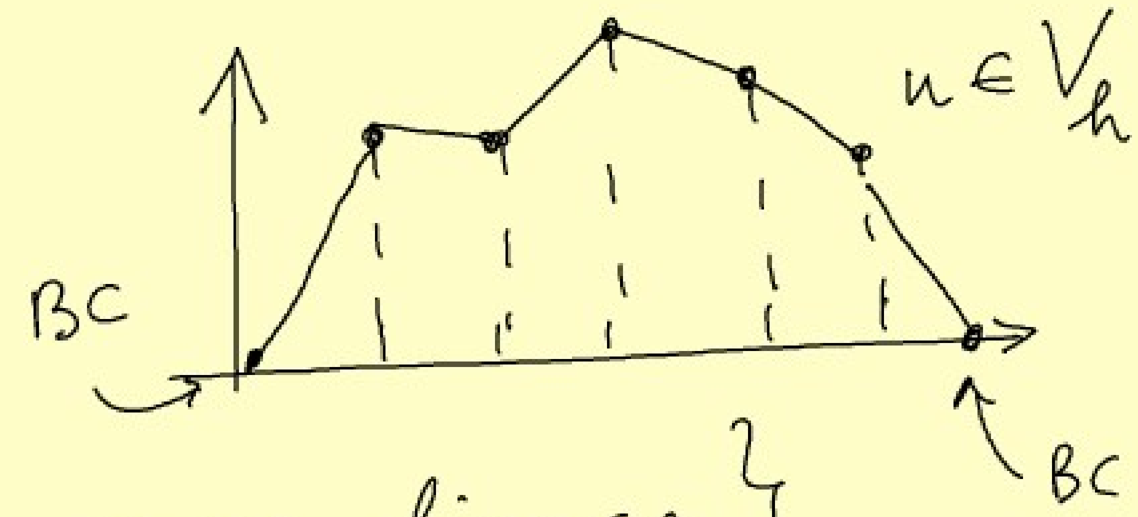
Then $\exists!$ weak! sol u of (D) (Proof based Thm Riesz)

§9.1. The FEM with piece-wise linear funct.

1D domain $\Omega = (0,1) \subset \mathbb{R}$

with a grid $x_i = ih$, $|x_{i+1} - x_i| = h$





$$V_h = \{ u \in H_0^1(\Omega) \; ; \; u \text{ is p-w linear} \}$$

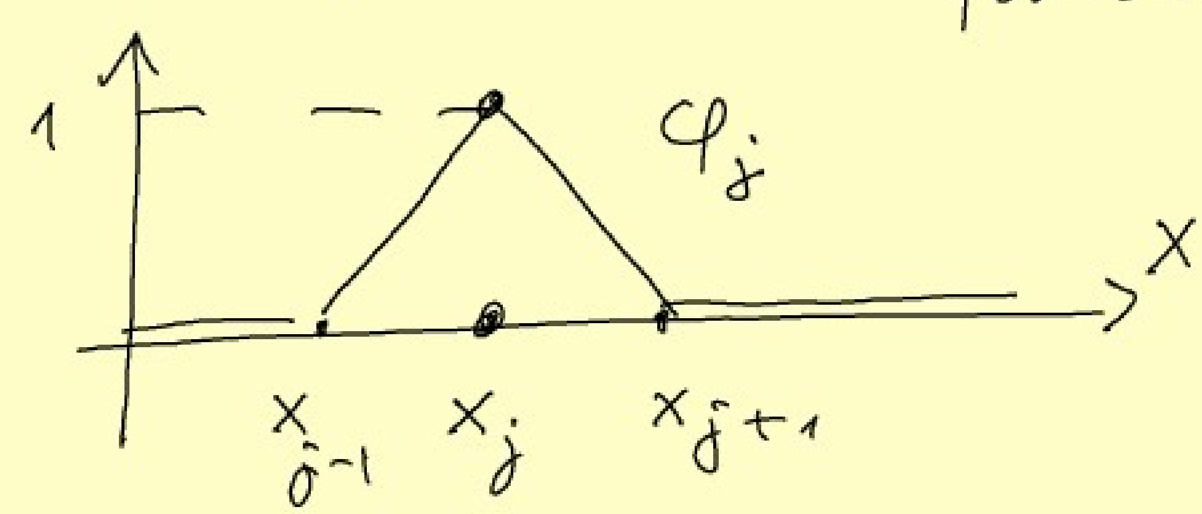
← replaces $H_0^1(\Omega)$ at discrete level

$u \in V_h$ is characterized by its values on the grid

i.e. $u_j = u(x_j)$ (it suffices to know u_j's)

Rk. Unsurprisingly, V_h is a finite-dim vector space ($\dim = N-1$) and admits a basis of convenient functions (TENT functions)

$$\varphi_j(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$



$$\forall u \in V_h \qquad u(x) = \sum_j u(x_j)\, \varphi_j(x)$$

We have the following discrete versions of the variational identity and optimality prop

(VI_h) $\quad u_h \in V_h \quad \langle (u_h)_x, (v_h)_x \rangle_{L^2} - \langle f, v_h \rangle_{L^2} = 0 \quad \forall v_h \in V_h$

(OPT_h) $\quad E(u_h) \leq E(v_h)$

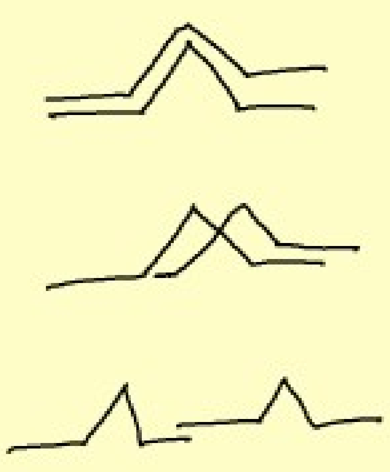
We can show $(VI_h) \Leftrightarrow (OPT_h)$ and also that (VI_h) has a unique sol. $\underset{\text{(called)}}{u_h}$

$$u_h(x) = \sum_j u_h(x_j)\, \varphi_j(x) \quad \longleftarrow \text{insert this is } (VI_h) \text{ with } v_h = \varphi_i$$

$$(*) \quad \sum_j \overbrace{u_h(x_j)}^{\text{unknowns}} \underbrace{\langle (\varphi_j)_x, (\varphi_i)_x \rangle_{L^2}} = \overbrace{\langle f, \varphi_i \rangle_{L^2}}^{b_i} \qquad i = \overline{2, N}$$

can be computed

$$\langle (\varphi_j)_x, (\varphi_i)_x \rangle_{L^2} = \begin{cases} \frac{2}{h}, & i = j \\ -\frac{1}{h}, & i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$



lin syst $N-1$ eq $N-1$ unknowns

$$\frac{1}{h} \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_h(x_2) \\ u_h(x_3) \\ \vdots \\ u_h(x_N) \end{bmatrix} = \begin{bmatrix} b_2 \\ \vdots \\ b_N \end{bmatrix}$$

$\exists!$ sol because matrix is symm & pos. def.

What we've seen is just a discrete version of the existence Thm. in Lecture 8.

Thm 1 ("Thm 5$_h$") There exists a unique u_h minimizing E over V_h (and satisfies (VI_h), thus being a "discrete weak solution" of the Dirichlet problem).

§ 9.2. Error estimates

Question: Let u_h the sol of (VI_h) and u the sol of (D) (both with the same given f)

exact sol $\rightarrow u(x) - u_h(x) = ?$ $\leftarrow$ approx (finite dim.)

IDEA: (VI) holds for any $v \in H_0^1(\Omega)$, so it it holds for any $v \in V_h \subset H_0^1(\Omega)$

and we can write

$$(VI) \quad \langle u_x, v_x \rangle_{L^2} - \langle f, v \rangle_{L^2} = 0 \quad \forall v \in V_h$$

$$(VI_h) \quad \langle (u_h)_x, v_x \rangle_{L^2} - \langle f, v \rangle_{L^2} = 0 \quad \forall v \in V_h$$

$$\text{Subtract:} \quad \langle (u - u_h)_x, v_x \rangle_{L^2} - \diagup = 0 \quad \forall v \in V_h \quad (**)$$

$\boxed{T}$ 2. (Best. approx, and estimate)

a) u_h is the best approx of u in V_h

(i.e. $\|u-u_h\|_{H_0^1} \leq \|u-v\|_{H_0^1} \quad \forall v \in V_h$) ✓

b) $|u(x) - u_h(x)| \leq C h^2$

pointwise estimate

Proof: a) recall (**)

(**) $\langle (u-u_h)_x, v_x \rangle_{L^2} = 0 \quad \forall v \in V_h$

Take arbitrary ~~v~~ $\in V_h$ and denote $w = u_h - v$

$$\|u-u_h\|^2_{H_0^1} \overset{\text{Def}}{=} \langle (u-u_h)_x, (u-u_h)_x \rangle_{L^2} + \underbrace{0}$$

$$\underset{v=w}{\overset{(**)}{=}} \langle (u-u_h)_x, (u-u_h)_x \rangle_{L^2} + \langle (u-u_h)_x, w_x \rangle_{L^2}$$

$$= \langle (u-u_h)_x, \underbrace{(u-u_h+w)_x}_{u - \cancel{u_h} + \cancel{u_h} - v = u-v} \rangle_{L^2}$$

Cauchy-Schwarz

$$\leq \underbrace{\|(u-u_h)_x\|_{L^2}}\,\underbrace{\|(u-v)_x\|_{L^2}}$$

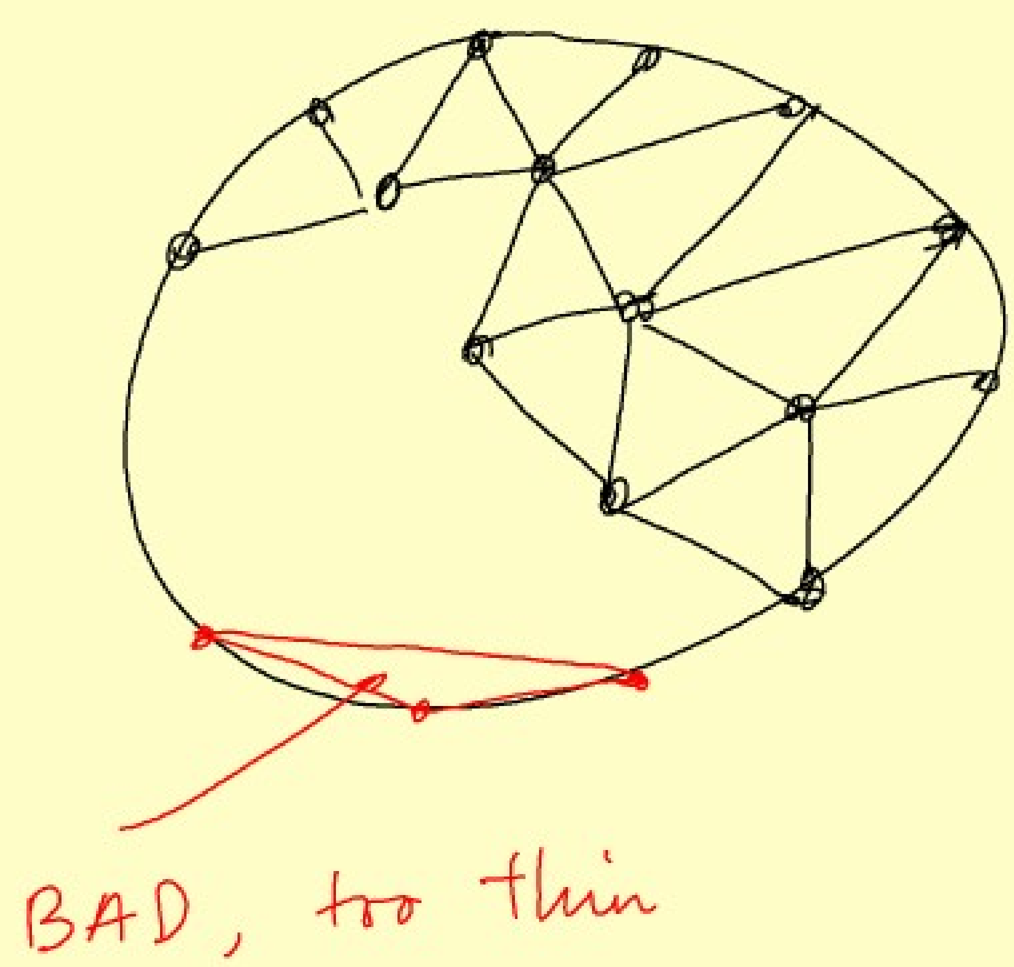
$$\|u-u_h\|^{\cancel{2}}_{H_0^1} \leq \cancel{\|u-u_h\|_{H_0^1}} \cdot \|u-v\|_{H_0^1}$$ ✓

§ 9.3. The FEM in dim > 1

Things get complicated ... (but IDEA remains the same)

The grid is the problem

"triangulation"



This will not result in a 3-diagonal matrix anymore ...

The main issue (& cost) =

= constructing the grid.

10. From BVPs to IBVPs

(stationary) (evolutionary)

$$(BVP) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$(iBVP) \begin{cases} u_t = \Delta u & \text{in } [0,T]\times\Omega \\ u = 0 & \text{on } \partial\Omega \quad \forall t \\ u(0) = u_0(x) & \text{in } \Omega \quad (\text{for } t=0) \end{cases}$$

Heat

§10.1. Spectral Thm for compact symmetric operators on a separable Hilbert space

Th 1. X separable ($=\exists$ countable base) Hilbert space.

$T: X \to X$ linear, bounded ($=$cont.) symmetric ($\langle Tx,y\rangle = \langle x, Ty\rangle$) compact operator

Then $\exists \{\overset{\in\mathbb{R}}{\lambda_n}, \overset{\in X}{x_n}\}$: $T x_n = \lambda_n x_n$

eigvals eigvectors

$\{x_n\}$ orthonormal base in X and $\lambda_n \xrightarrow{n\to\infty} 0$

Ex. $X = \ell^2 = \{ x = \{x^k\}_{k\in\mathbb{N}^*} : \sum_{k=1}^{\infty} (x^k)^2 < \infty \}$

$$T = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \ddots \end{bmatrix}$$ "infinite" matrix defines the multiplic op.

$T: \{x^k\} \longmapsto \{\lambda_k x^k\}$

The main application of the Spectral Thm is to Helmholtz problems (= eigenvalue probl for Laplace op.)

$$(H) \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

IDEA: the inverse of a differential operator (i.e. solution operator of BVP) is symmetric and compact (Spectral Th holds)

dim = 1 for simplicity

$$(H_{dim=1}) \begin{cases} -u_{xx} = \lambda u & \text{in } \Omega = (0,1) \\ u = 0 & \text{on } \partial\Omega \quad (u(0) = u(1) = 0) \end{cases}$$

Look at $\begin{cases} -u_{xx} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$. which has unique sol (for $f \in L^2$)

and which defines a solution operator $T : \underset{\in L^2}{f} \mapsto \underset{\in H^1_0}{u}$

- T is compact due to the compact embedding of $H^1_0(\Omega)$ in $L^2(\Omega)$ (see [Precup] or [Brezis])

$$\|u_x\|^2_{L^2} = \|u\|^2_{H^1_0} < \infty \quad \text{vs.} \quad \|f\|^2_{L^2} < \infty$$

- T is symmetric (next page)

Symmetry $\langle Tf, g \rangle_{L^2} = \langle f, Tg \rangle_{L^2}$ (want)

Sol op

$$\begin{cases} -u_{xx} = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \Big| v, \int \qquad \begin{cases} -v_{xx} = g & \Omega \\ v = 0 & \partial\Omega \end{cases} \Big| u, \int$$

$Tf = u$ $\qquad\qquad\qquad\qquad$ $Tg = v$

$$\overbrace{-\int_0^1 u_{xx} v \, dx}^{\int_0^1 u_x v_x} = \int_0^1 f v \, dx = \langle f, Tg \rangle_{L^2}$$

$\|$

$$\underbrace{-\int_0^1 v_{xx} u \, dx}_{\text{by parts}} = \int_0^1 \underbrace{g u}_{= ug} \, dx = \langle Tf, g \rangle_{L^2} \quad \text{want}$$

$$\int_0^1 v_x u_x$$

T 2. (Solution of eigenproblem) Ω bounded, open

There exist a basis $\{e_n\}_{n \in \mathbb{N}^*}$ of $L^2(\Omega)$ and a sequence $\{\lambda_n\}_{n \in \mathbb{N}^*}$ of reals $\lambda_n > 0$, $\lambda_n \xrightarrow{n \to \infty} 0$ such that $-\Delta e_n = \lambda_n e_n$ in Ω

and $e_n \in H_0^1(\Omega) \cap C^\infty(\Omega)$

BC $\qquad\qquad$ better than "expected"

We have eigenbase for $-\Delta$ with Dirichlet BC.

Recall Fourier:

1D $(\Omega = (0,1))$ $\quad (\sin k\pi x)_{xx} = -\pi^2 k^2 \sin \pi k x$

$$\text{"} -\Delta e_k = \lambda_k e_k \text{"}$$

$\begin{cases} \text{eigenfunctions are} & \sin k\pi x \\ \text{eigenvalues are} & \pi^2 k^2 \end{cases}$

What we've seen in the general result is precisely the desired generalization of Fourier's insight!

$\boxed{\text{☐}_2 \quad \text{Allows us to study virtually any (linear) evolutionary PDE involving } -\Delta \text{ \& BC.}}$

That's the main IDEA.

§ 10.2. IBVPs for Heat & Wave eqns.

Initial Boundary Value Problems are also called "Mixed Problems".

Th 3. (Heat Eq. on bounded Ω)

$$\text{(IBVP)} \quad \begin{cases} u_t = \Delta u & t > 0,\ x \in \Omega \\ u = 0 \quad \text{on } \partial\Omega & \forall t \geq 0 \qquad \text{BC} \\ u(0) = u_0(x) \quad \text{in } \Omega & \qquad\qquad \text{IC} \end{cases}$$

If $\underline{u_0 \in L^2(\Omega)}$ then $\exists!$ solution u of IBVP

$u \in C([0,\infty), L^2(\Omega)) \cap C((0,\infty), H^2 \cap H_0^1)$

regularity | eq holds in L^2

$u \in C^1((0,\infty), L^2)$ ← $\frac{du}{dt}$ makes sense

$u \in L^2(0,\infty; H_0^1)$ ← only a finite amount of heat/energy is dissipated

Furthermore

$$\text{(EB)} \quad \frac{1}{2}\|u(t)\|_{L^2}^2 + \underbrace{\int_0^t \|u(\tau)\|_{H_0^1}^2 \, d\tau}_{\int_0^t \int_\Omega |\nabla u(\tau)|^2 \, dx \, d\tau} = \frac{1}{2}\|u_0\|_{L^2}^2$$

"energy" balance

Proof: work with $\{e_k\}$ from Thm 2.

- base $\Rightarrow$ $u(t) = \sum_{k=1}^{\infty} \overbrace{u_k(t)}^{\in \mathbb{R}} e_k$ $\quad (*)$

- plug $(*)$ into $u_t = \Delta u$ to get
$$\sum_k \frac{d}{dt} u_k(t)\, e_k = \sum_k u_k(t) \underbrace{\Delta e_k}_{-\lambda_k e_k}$$

- one ODE for each u_k: $\frac{d}{dt} u_k(t) = -\lambda_k u_k(t)$

solvable (easy) $u_k(t) = e^{-\lambda_k t} u_k(0)$

$u_k(0) = \langle \overset{ic}{u_0}, e_k \rangle_{L^2}$ (plug $(*)$ into IC)

- $u(t) = \sum_k e^{-\lambda_k t} \underbrace{\cdot}_{u_k(0)} e_k(x)$ repres. formula for the sol.

From here you can "read" all the regularity properties

To get (EB), multiply $u_t = \Delta u$ by u (in L^2), $\Big| \int_0^t$

$$\underbrace{\langle u_t, u \rangle_{L^2}}_{\frac{1}{2}\frac{d}{dt}\|u(t)\|^2} = \underbrace{\langle \Delta u, u \rangle_{L^2}}_{BC} - \|u\|^2_{H_0^1} = -\int_\Omega |\nabla u|^2 dx$$

T4. (Wave Eq on Ω bounded)

$$\text{(IBVP)} \begin{cases} u_{tt} = \Delta u & t > 0,\ x \in \Omega \\ u = 0 & \text{on } \partial\Omega\ \forall t \geq 0 \ \text{BC} \\ u(0) = u_0(x) \\ u_t(0) = v_0(x) \end{cases} \Big\} \text{ two IC}$$

If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H_0^1(\Omega)$

then there exist a unique solution

$u(t) \in H^2 \cap H_0^1\, ?$

$$u \in C([0,\infty); H^2 \cap H_0^1) \cap C^1([0,\infty), H_0^1) \cap C^2([0,\infty), L^2)$$

$u_t(t) = v(t) \in H_0^1 \ \forall t \geq 0$

$u_{tt}(t) = v_t(t) \in L^2\, ?$

and energy conservation holds

$$\underbrace{\frac{1}{2}\|u_t(t)\|_{L^2}^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|u(t)\|_{H_0^1}^2}_{\text{potential energy}} = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|u_0\|_{H_0^1}^2$$

Rk: The wave eq. is conservative!

Proof: again $u(t) = \sum_k u_k(t)\, e_k$ plug in eq.

$$\sum_k \frac{d^2}{dt^2} u_k(t)\, e_k = \sum u_k(t) \underbrace{\Delta e_k}_{-\lambda_k e_k}$$

ODE $\quad \dfrac{d^2}{dt^2} u_k(t) = -\lambda_k u_k(t)$

$$u_k(t) = u_k(0)\cos\sqrt{\lambda_k}\, t + v_k(0)\frac{1}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}\, t.$$

More PDEs can be approached like this!

- Wave $u_{tt} = \Delta u \iff \begin{cases} u_t = v \\ v_t = \Delta u \end{cases}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

↖ operator matrix

- Damped Wave Eq $u_{tt} = -u_t + \Delta u$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & -I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Visco-elasticity (linear) $u_{tt} = \Delta u_t + \Delta u$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & \Delta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Visco-capillarity (1D) $u_{tt} = -u_{xxxx} + u_{txx} + u_{xx}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\partial_x^4 + \partial_x^2 & \partial_x^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Euler-Bernoulli Beam $u_{tt} = -u_{xxxx}$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\partial_x^4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

12. Reaction-diffusion equations I: Fisher-KPP and Allen-Cahn eqns.

Fisher (1937): The wave of advance of advantageous genes
(rel. to population dynamics)

§ 12.1. Fisher's reaction-diffusion model

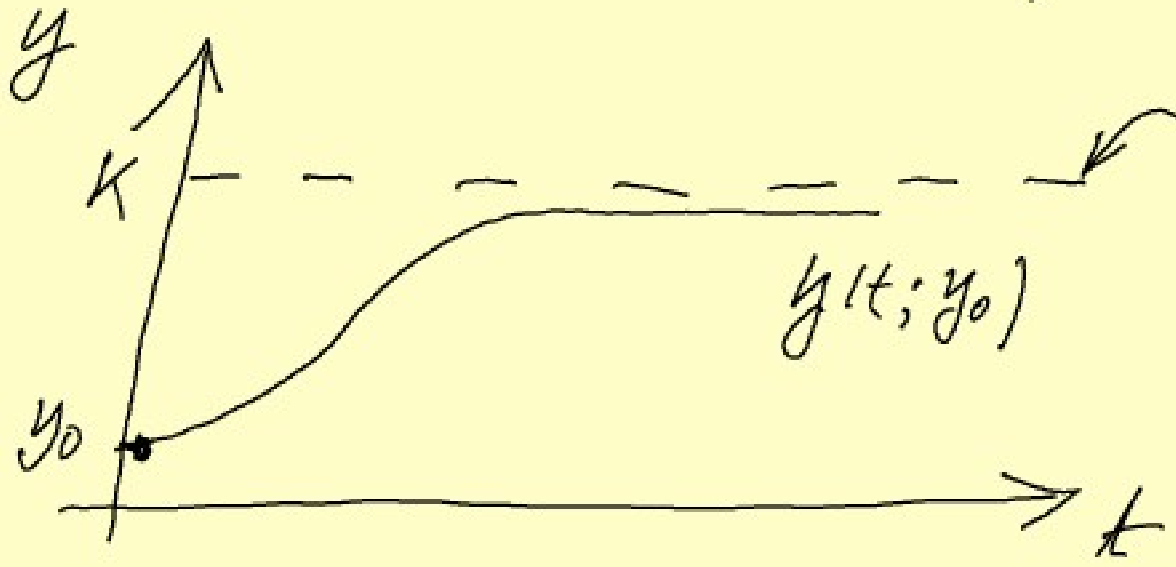
It deals with a spatially distributed popul. (New!)

Classical population models ($y(t)$ = size of pop. at time t)

Malthus: $y' = ry$, $r > 0$ (growth rate)
(1798, 1826) unrealistic exp. growth

Verhulst: $y' = ry\left(1 - \frac{y}{K}\right)$
(1838, 1847) $K > 0$ capacity of environment to sustain a popul. ($\sim$ scarce resources)

self-limiting (logistic) behavior



stable equilibrium (stationary sol.)

$y(t) \equiv 0$ is unstable equil.

Neither of these models account for spatial distrib. of the pop.

Fisher's (new, 1937) model: assumes a spatial distribution of the population.

iDEA: Superimpose two effects

- a logistic (Verhulstian) population growth (at every point in space, the pop. grows according to Verhulst)
- population migrates according to <u>Fick's Law</u> (moves from areas with high population density to areas of low density)

$u(t,x)$ = pop. at time t located at x

Fisher's Eq. $u_t = \mu_D u_{xx} + ru(1-\frac{u}{K})$

Dimensionless form $\boxed{u_t = u_{xx} + u(1-u)}$

can be obtained by a change of vars.

$t_{new} = \frac{t}{r^{-1}}$, $x_{new} = \frac{x}{\sqrt{\mu_D/r}}$, $u_{new} = \frac{u}{K}$

Fisher's original question: are waves in the pop possible/reasonable?

→ YES [Fisher, 1937]

KOLMOGOROV - Petrovsky - Piskunov 1937 (KPP)
study Fisher's eq. They study initial profiles that will approach a travelling wave.

§ 12.2. Travelling wave sols (TWS) for Fisher eq.

We look for $u(t,x) = U(x-ct)$, $c>0$ (TWS)

U = wave front, c = wave speed (two unknowns)

plug (TWS) in (Fisher) $\quad ' = \frac{d}{dz}$

(FTWS) $\quad -cU' = U'' + U(1-U)$, $U = U(z)$

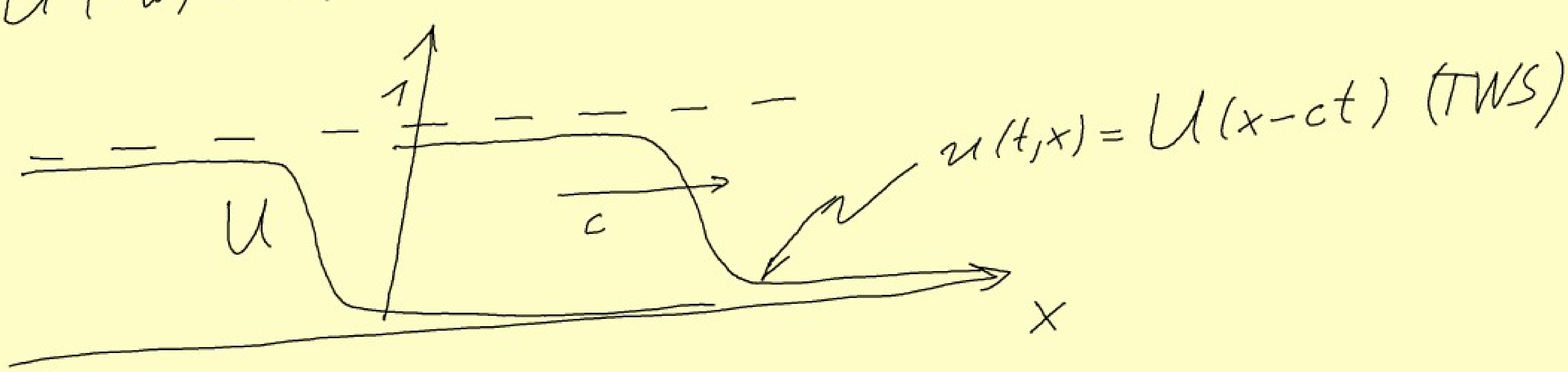
2nd order (nonlinear!) ODE but not explicitly solvable

However, standard (qualitative) ODE / Dyn. Syst. techniques (equilibrium points / stability) allow a complete understanding of (FTWS).

Th 1. ([Logan: An introduction to nonlinear PDEs])

For each $c \geq 2$ there exists a unique wave profile U (and conseq. a unique TWS u) with U monot. decreasing and

$U(-\infty) = 1$, $U(\infty) = 0$ and $U'(\pm\infty) = 0$



Although no explicit formula is available there exists a very good approx. via <u>perturbation theory</u> (details [Logan])

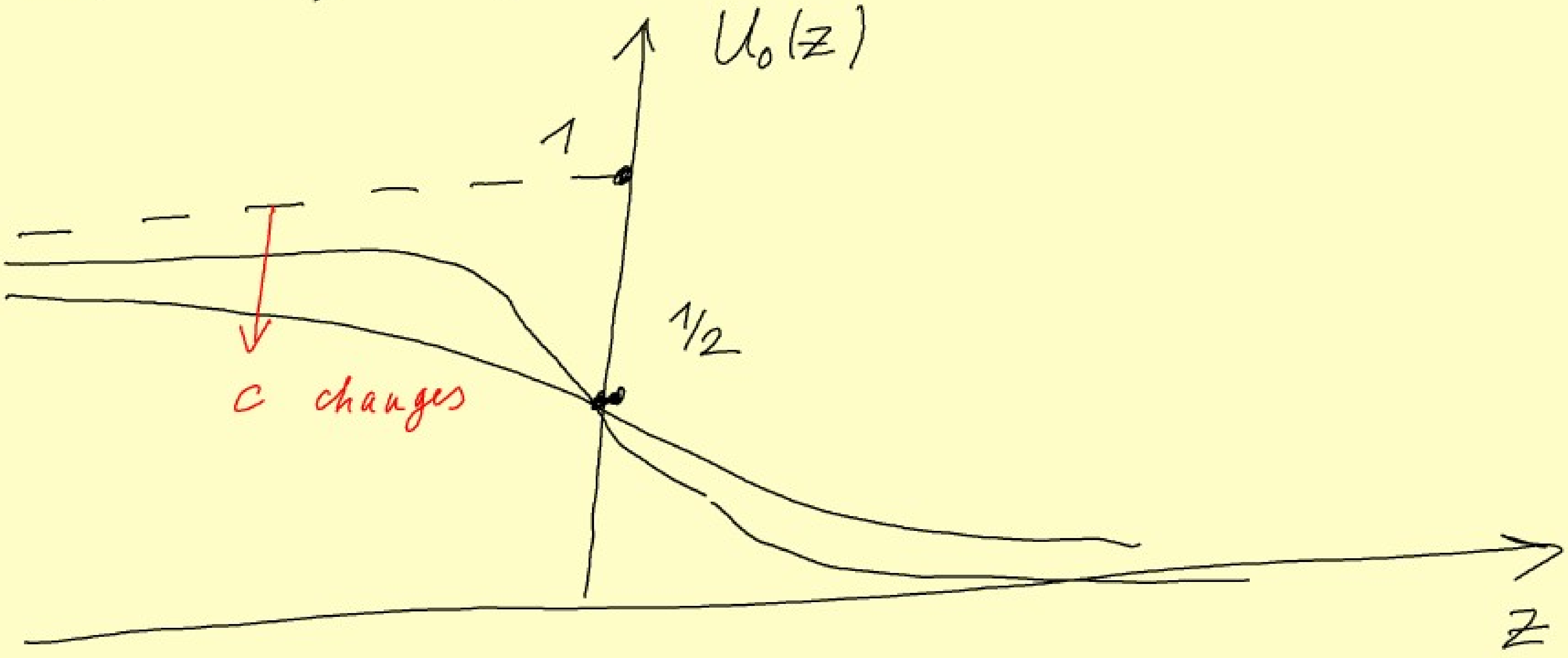
(works for large $c \gg 1$)

$$U(z) = \boxed{\frac{1}{1+e^{z/c}}} + \frac{1}{c^2} e^{z/c}\left(1+e^{z/c}\right)^{-2} \ln \frac{4e^{z/c}}{\left(1+e^{z/c}\right)^2} + O\left(\frac{1}{c^4}\right)$$

Actually, the first term already gives a very good approximation of the profile!

$$U_0(z) = \frac{1}{1+e^{z/c}}$$

$$U_0(-\infty) = 1, \quad U_0(\infty) = 0, \quad U_0(0) = \frac{1}{2}$$



§ 12.3. Reaction-diffusion equations & systems

General form $\quad u_t = \underbrace{u_{xx}}_{\text{diffusion}} + \underbrace{f(u)}_{\text{reaction}}$

System (RD) $\quad \begin{cases} u_t = u_{xx} + f(u,v) \\ v_t = \mu v_{xx} + g(u,v) \end{cases}$

Rk: f, g are nonlinear (coupling) terms and depend on both reactants.

Examples: Allen-Cahn eq. $\quad u_t = u_{xx} + u(1-u^2)$

Brusselator syst. $\quad \begin{cases} u_t = u_{xx} + 1 + u^2 v - 2u \\ v_t = \mu v_{xx} + u - u^2 v \end{cases}$

Two types of questions:

1. Travelling Waves: existence & stability
2. Equilibria and their stability? (Asymptotic $t \to \infty$ behavior of the system?)

(GRDE) $\quad w_t = D w_{xx} + F(w)$, $\quad w = (w_1, \dots, w_n)$, $\quad D =$ diffusion matrix (diagonal)

(Equil) $\quad 0 = D w^*_{xx} + F(w^*)$

- $w^*(x)$ stationary / equilibrium solution
- stationary homogeneous solutions $\quad \bar{w}^*(x) = \bar{w}^* \ (\forall x)$ i.e. $\bar{w}^*_x = 0$

(these are sols. of $\quad F(\bar{w}^*) = 0$)

§ 12.4. Lyapunov Theory (for evolutionary PDEs)

- X state space (Hilbert space)
- $\{S(t)\}_{t \geq 0}$ is a dynamical system on X if

(i) $S(t): X \to X$ cont. (for all $t \geq 0$)

(ii) $S(0) = I$ (identity)

(iii) $S(t+s) = S(t) \circ S(s) \quad \forall t, s \geq 0$

(iv) $t \longmapsto S(t)x$ is $C([0,\infty); X)$

Idea: pick $x_0 \in X$ initial state

$S(t)x_0$ the state at time t which has evolved from x_0

$\bigcup_{t \geq 0} \{S(t)x_0\}$ the trajectory originating from x_0

- $x^* \in X$ is an equilibrium for $\{S(t)\}_{t \geq 0}$ if $S(t)x^* = x^* \quad \forall t \geq 0$ (stationary state)

The set of all equilibria is denoted by $\mathcal{E}$.

- $\Phi : X \to \mathbb{R}$ cont. is a strict Lyapunov function if

(i) $\Phi(S(t)x) \leq \Phi(x) \quad \forall x \in X \; \forall t \geq 0$

decrease along trajectories

(ii) $\Phi(S(t)x) = \Phi(x) \implies x \in \mathcal{E}$.

("(i) is strict "<" for $x \notin \mathcal{E}$)

"[]" (La Salle's Invariance Principle)

If Φ is a strict Lyapunov function for $\{S(t)\}$ and $x_0 \in X$ is such that the trajectory $\bigcup_{t \geq 0} \{S(t)x_0\}$ is relatively compact subset of X

then

(i) $\lim_{t\to\infty} \Phi(S(t)x_0) = \ell$ exists

(ii) $\mathrm{dist}(S(t)x_0, \mathcal{E}) \xrightarrow{t\to\infty} 0$

("convergence to equilibrium")

(iii) if $\mathcal{E}$ is discrete then there exists $x^* = x^*(x_0) \in \mathcal{E}$ such that $S(t)x_0 \xrightarrow{t\to\infty} x^*$

Meaning: $\left.\begin{array}{c}\text{strict Lyapunov function} \\ + \\ \text{rel. compact trajectories}\end{array}\right\} \Rightarrow$

$\Rightarrow$ convergence to equilibrium

(the long-time behavior of the system is predetermined (simple = no chaos, no periodic orbits etc.))

§ 12.5. Application: The Allen–Cahn model

$$\begin{cases} u_t = u_{xx} + (u - u^3) & \text{for } (t,x) \in (0,\infty) \times \underset{\substack{\| \\ (0,1) \subset \mathbb{R}}}{\Omega} \\ u_x(t,0) = u_x(t,1) = 0 & \text{(Neumann BC)} \\ u(0,x) = u_0(x) & \text{(IC)} \end{cases}$$

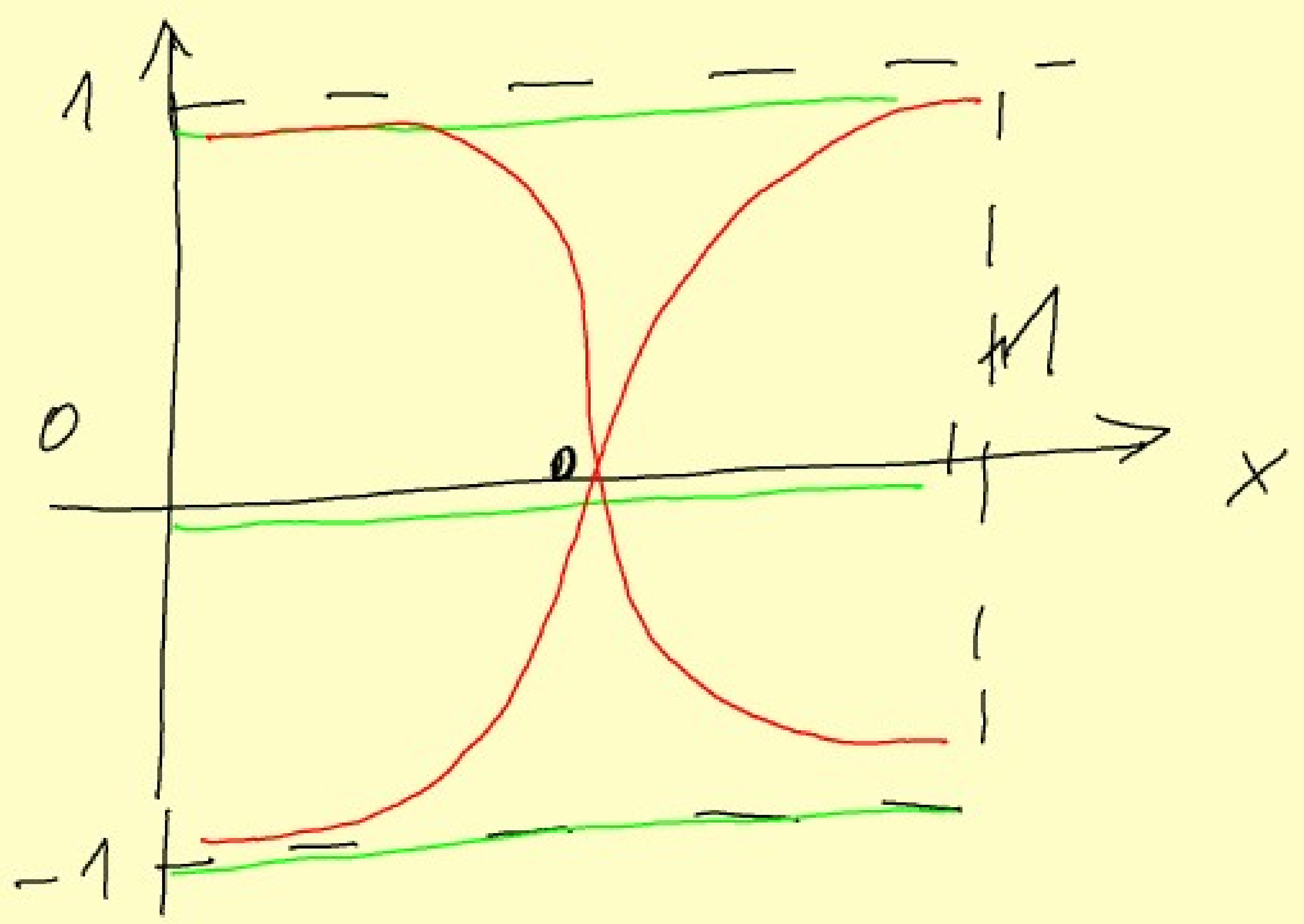
equil. eq is $\begin{cases} u^*_{xx} = u^{*3} - u^* \\ u^*_x(0) = u^*_x(1) = 0 \end{cases}$

One can show that this has 5 solutions

$u^*(x) = 0, \quad u^*(x) = 1, \quad u^*(x) = -1 \qquad \forall x \in \Omega$

stationary homogeneous solutions $(u^*_x = 0)$

& two stationationary sols with $\boxed{u^*_x \neq 0}$



The Allen–Cahn Energy is a Lyapunov fct. (strict!)

$$u_t = u_{xx} + u - u^3 \quad | \, u_t \,, \int_\Omega$$

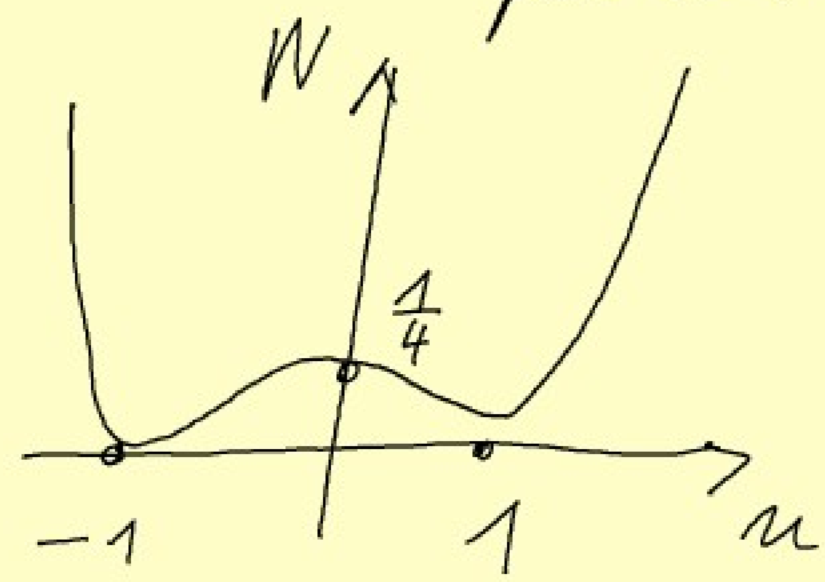
$$(*) \quad \|u_t\|_{L^2}^2 = \int_\Omega u_{xx} u_t \, dx + \int_\Omega \underbrace{(u-u^3) u_t}_{-\frac{d}{dt} W(u(t))} \, dx$$

integrate by parts !BC

$$= \underbrace{- \int_\Omega u_x u_{tx} \, dx}_{\frac{d}{dt} \frac{1}{2} \|u_x\|_{L^2}^2}$$

$$W(u) = \frac{1}{4} u^4 - \frac{1}{2} u^2 + \frac{1}{4}$$

double-well potential

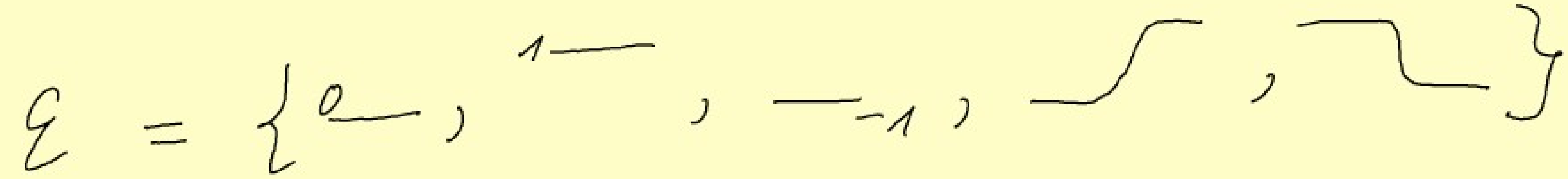


$$(*) \quad \|u_t\|_{L^2}^2 = - \frac{d}{dt} \underbrace{\frac{1}{2} \|u_x\|_{L^2}^2}_{\text{Dirichlet energy}} - \frac{d}{dt} \underbrace{\int_\Omega W(u(t)) \, dx}_{\text{Double-well energy}} \quad | \cdot (-1)$$

$$\frac{d}{dt} E(u(t)) = - \|u_t\|_{L^2}^2 \quad (E \text{ is strict Lyap. func.!})$$

$$E(u) = \frac{1}{2} \|u_x\|_{L^2}^2 + \int_\Omega W(u) \, dx \quad \text{Allen-Cahn energy}$$

Allen-Cahn convergence to equilibrium

- 

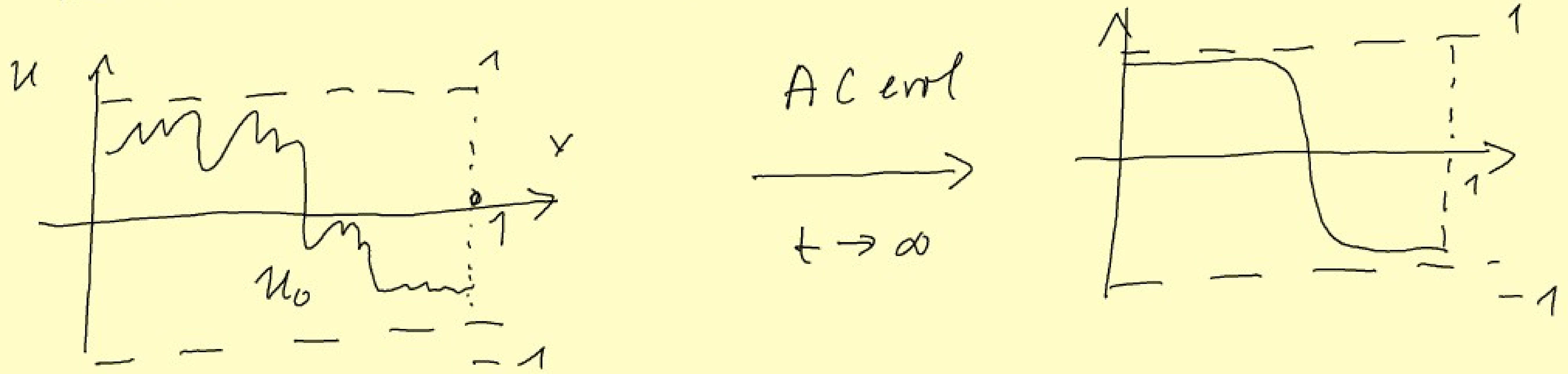
  is discrete

- $$E(u) = \frac{1}{2}\|u_x\|^2_{L^2} + \int_\Omega W(u)\,dx$$

  strict Lyapunov function for the Dyn Syst generated by the AC eq.

- trajectories are compact (technical proof which relies on "compact embeddings" [Brezis] or [Precup])

LaSalle's Thm. applies $\Longrightarrow$ conv. equil.



13. Reaction-diffusion equations II: Turing instability

"How the Leopard Got Its Spots"

Recall (from Lecture 12) the general reaction-diffusion system

$$\text{(GRDS)} \qquad w_t = D\Delta w + F(w) \qquad \text{in } (0,\infty)\times\Omega$$

diagonal

with $w = (w_1, \dots, w_m)$, D = diffusion matrix

Δ = Laplacian, $\Delta w = (\Delta w_1, \dots, \Delta w_m)$

m = nr of equations, $\Omega \subset \mathbb{R}^n$ n = spatial dim.

"You need the more complicated model in order to get the more complicated behavior."

(GRDS) has two types of equilibria (= stationary solutions)

patterns on the fur

- stationary solutions $0 = D\Delta w^* + F(w^*)$ syst of nonlin. elliptic PDEs
  (don't depend on time)
- homogeneous stationary sols $0 = 0 + F(w^*)$ algebraic eq
  (doesn't depend on space

fur in one color

Stationary sols govern the long-time behavior of the system (Lyapunov)!

(That's what you observe in nature.)

(1952) Alan TURING

$$(TM)\begin{cases} u_t = \Delta u + \gamma f(u,v) & t>0,\ x\in\Omega\ (\subset\mathbb{R}^2) \\ v_t = d\Delta v + \gamma g(u,v) \end{cases}$$

with Neumann BCs & Initial conditions

in terms of (GRDS): $w=(u,v)$, $D=\begin{pmatrix}1&0\\0&d\end{pmatrix}$

$0<\gamma\in\mathbb{R}$ is a parameter, $F(w)=\begin{pmatrix}f(u,v)\\g(u,v)\end{pmatrix}$

u, v = morphogenes (responsab. chemicals for creation of patterns)

Turing ahead of his time (by decades!):

@ biology: morphogens not exactly known even today

@ math: stability theory for PDES J.P. LaSalle ~1960s

@ computer science (simulations): simulations of animal furs Murray ~1980

What means Turing <u>instability</u>?

? Turing instability

$$(TM) \quad \begin{aligned} u_t &= \Delta u + \gamma f(u,v) \\ v_t &= d\Delta v + \gamma g(u,v) \end{aligned} \quad + \text{ BCs \& ICs}$$

IDEA (Turing instability).

Simple version: homogeneous stationary solutions (corresp. to fur in one color) are unstable and some spatially varying stationary solutions are stable (patterns)

More details: without diffusion, the homog. stat. sol. would be stable (adding diffusion is creating the patterns)

without diff-usion: ODE system

$$\begin{cases} u' = \gamma f(u,v) \\ v' = \gamma g(u,v) \end{cases} \qquad \begin{cases} \cancel{\gamma} f(u^*,v^*) = 0 \\ \cancel{\gamma} g(u^*,v^*) = 0 \end{cases}$$

Dyn. syst

equil. eqns

does not depend on γ

with diffusion: stat. sols of (TM)

$$\begin{cases} \Delta u^* + \gamma f(u^*,v^*) = 0 \\ d\Delta v^* + \gamma g(u^*,v^*) = 0 \end{cases}$$

depends essentialy on γ!

Turing instab.

<u>even more details</u>

<u>ODE system</u> $\begin{cases} u' = \gamma f(u,v) \\ v' = \gamma g(u,v) \end{cases}$ u^*, v^* are such that $\begin{cases} f(u^*, v^*) = 0 \\ g(u^*, v^*) = 0 \end{cases}$

linearize around (u^*, v^*) (Taylor expansion) $\begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = A \begin{pmatrix} u-u^* \\ v-v^* \end{pmatrix} + HOT$

neglect HOT under the assumption that $u-u^*, v-v^*$ small.

A is the Jacobian of $F = \begin{pmatrix} f \\ g \end{pmatrix}$ $\quad A = J_F(u^*, v^*)$

$$= \begin{bmatrix} \frac{\partial f}{\partial u}(u^*, v^*) & \frac{\partial f}{\partial v}(u^*, v^*) \\ \frac{\partial g}{\partial u}(u^*, v^*) & \frac{\partial g}{\partial v}(u^*, v^*) \end{bmatrix}$$

The linearized system is $w' = Aw$, $w = \begin{pmatrix} u-u^* \\ v-v^* \end{pmatrix}$

$w = 0_{\mathbb{R}^2}$ ($\Leftrightarrow u = u^*, v = v^*$) is asympt. stable $\Leftrightarrow$

$\Leftrightarrow$ A Hurwitz matrix ($\operatorname{Re}\lambda < 0$ $\forall \lambda$ eigenval. A)

$\Leftrightarrow \left[\frac{\partial f}{\partial u}(u^*, v^*) + \frac{\partial g}{\partial v}(u^*, v^*) < 0 \text{ and } \det A > 0 \right]$

stab. condition

Recall that we want: u^*, v^* is asympt stable for the ODE system (= no diffusion)

but u^*, v^* homog. stat. sol is unstable for the full diffusion model.

PDE system $\begin{cases} u_t = \Delta u + \gamma f(u,v) \\ v_t = d\,\Delta v + \gamma g(u,v) \end{cases}$

- Linearize w.r.t. the same $u^*, v^* \in \mathbb{R}$

again $w = \begin{pmatrix} u(t,x) - u^* \\ v(t,x) - v^* \end{pmatrix}$

$$w_t = D \Delta w + \gamma A w$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}, \quad A = J_F(u^*, v^*)$$

- use a Fourier series method

$$\begin{cases} -\Delta \phi_k + \mu_k \phi_k = 0 & \text{in } \Omega \\ \partial \phi_k / \partial \nu = 0 & \text{on } \partial\Omega \end{cases} \quad \text{Neumann BC}$$

μ_k eigenvals of the Laplacian $(\mu_0 = 0)$

ϕ_k eigenfunctions —"— , $k = 0, 1, \ldots$

assume that $w(t,x) = \sum_{k=0}^{\infty} e^{\lambda_k t} \phi_k(x) \begin{pmatrix} a_k \\ b_k \end{pmatrix}$

we want instability (fur not in a single color!)

$\hookrightarrow \lambda_k > 0$ (happens if $\mu_k \in (\underline{\mu}, \overline{\mu})$)

CONDS for instab are:

(1) $\operatorname{tr} A < 0$ $\quad \det A > 0$ (2)

(3) $d \frac{\partial f}{\partial u}(u^*, v^*) + \frac{\partial g}{\partial v}(u^*, v^*) > 0$

(4) $\det A < \left(d \frac{\partial f}{\partial u}(u^*, v^*) + \frac{\partial g}{\partial v}(u^*, v^*)\right)^2 / 4d$

all conds are satisf. for

$$f(u,v) = a - u + u^2 v \quad \Big| \; a > 0$$
$$g(u,v) = b - u^2 v \quad \Big| \; a + b > 0$$

Schakenberg 1979