Dobre Delia
Grupa 131

# Tema 6

1) Notăm $S := \left\{ \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{1} \\ \hat{0} & \hat{1} \end{pmatrix}, \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{1} & \hat{1} \end{pmatrix} \right\}$

$S \leq (GL_2(\mathbb{Z}_3), \cdot) \iff \begin{cases} a) & S \neq \emptyset \\ b) & S_1, S_2 \in S \Rightarrow S_1 \cdot S_2 \in S \\ c) & S_1 \in S \Rightarrow S_1^{-1} \in S \end{cases}$

a) $S \neq \emptyset$ evident

b) $\underbrace{\begin{pmatrix} \hat{1} & \hat{1} \\ \hat{0} & \hat{1} \end{pmatrix}}_{\in S} \cdot \underbrace{\begin{pmatrix} \hat{1} & \hat{0} \\ \hat{1} & \hat{1} \end{pmatrix}}_{\in S} = \begin{pmatrix} \hat{1}+\hat{1} & \hat{0}+\hat{1} \\ \hat{0}+\hat{1} & \hat{0}+\hat{1} \end{pmatrix} = \begin{pmatrix} \hat{2} & \hat{1} \\ \hat{1} & \hat{1} \end{pmatrix} \notin S$

$\Rightarrow \exists S_1, S_2 \in S$ a.î. $S_1 \cdot S_2 \notin S$

$\Rightarrow S \not\leq (GL_2(\mathbb{Z}_3), \cdot)$

2) $7 + 5\sqrt{2}$ inversabil în $\mathbb{Z}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Z} \} \iff$

$\exists\, a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ a.î. $\underbrace{(7+5\sqrt{2})(a+b\sqrt{2})}_{(1)} = \underbrace{(a+b\sqrt{2})(7+5\sqrt{2})}_{(2)} = 1$

(1): $(7+5\sqrt{2})(a+b\sqrt{2}) = 7a + 7b\sqrt{2} + 5a\sqrt{2} + 10b = 1 \iff$

$\begin{cases} 7a + 10b = 1 & |\cdot(-5) \\ 5a + 7b = 0 & |\cdot 7 \end{cases} \iff \begin{cases} -35a - 50b = -5 \\ 35a + 49b = 0 \end{cases} \xrightarrow[(+)]{} -b = -5$

$\Rightarrow b = 5 \in \mathbb{Z}$

$\Rightarrow a = -\frac{7}{5} \cdot 5 \Rightarrow a = -7 \in \mathbb{Z}$

$\Rightarrow a + b\sqrt{2} = -7 + 5\sqrt{2}$.

$\Rightarrow 7 + 5\sqrt{2}$ e inversabil în $\mathbb{Z}[\sqrt{2}]$ și $(7+5\sqrt{2})^{-1} = -7 + 5\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$

Verificare în (2): $(-7+5\sqrt{2})(7+5\sqrt{2}) = -49 - 35\sqrt{2} + 35\sqrt{2} + 50 = 1$

3) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x+y, x-y)$ transformare liniară

$f(e_1) = f(1,0) = (1,1)$
$f(e_2) = f(0,1) = (1,-1)$
$\Rightarrow [f]_e = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$(f \circ f)(x,y) = f(f(x,y)) = f(x+y, x-y) = (x+y+x-y, x+y-x+y) = (2x, 2y)$

$(f \circ f)(e_1) = (f \circ f)(1,0) = (2,0)$
$(f \circ f)(e_2) = (f \circ f)(0,1) = (0,2)$
$\Rightarrow [f \circ f]_e = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$