07.12.2021

Seminar W 10 – 832

Exercise 1. Consider the following sample data for the weight (in kg) of the people in a certain city:

67.6 84.7 88.1 68.0 64.2 75.9 69.2 71.3 82.4 78.6

Assume that the weight is a characteristic that follows the normal distribution. Find 95% confidence intervals for:

(a) the mean value of the weight, given that the standard deviation of the weight is 10 (kg);

(b) the mean value of the weight, given that the standard deviation of the weight is unknown;

(c) the standard deviation of the weight.

Sol: $W \sim N(\mu, \sigma)$

$1-\alpha = 95\%$, $\alpha = 0.05$

$\sigma = 10$ is known, so the confidence interval for the mean is:

$$\mu \in \left[\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \ \overline{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \right]$$

$$\overline{X} = \frac{1}{10} \cdot (67.6 + 84.7 + 88.1 + 68.0 + 64.2 + 75.9 + 69.2 + 71.3 + 82.4 + 78.6) = 75.01$$

$$z_{\frac{\alpha}{2}} = z_{\frac{0.05}{2}} = z_{0.025} = \text{norminv}(0.025) = -1.96$$

$$z_{1-\frac{\alpha}{2}} = \text{norminv}(0.975) = 1.96$$

$$\mu \in \left[\bar{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \right] =$$

$$= \left[75.01 - 1.96 \cdot \frac{\sqrt{10}}{\sqrt{10}},\ 75.01 + 1.96 \cdot \frac{\sqrt{10}}{\sqrt{10}} \right]$$

$\Rightarrow$ the CI is $[75.01 - 1.96,\ 75.01 + 1.96] = [73.05,\ 76.97]$

(b) $$\mu \in \left[\bar{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}},\ \bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} \right]$$

$z_{\frac{\alpha}{2}} = \text{tinv}(0.025, 9) = -2.2622$

$z_{1-\frac{\alpha}{2}} = 2.2622$

$s = \text{std}(v) = 8.1832$

$$\Rightarrow \mu \in \left[75.01 - 2.2622 \cdot \frac{8.1832}{\sqrt{10}},\ 75.01 + 2.2622 \cdot \frac{8.1832}{\sqrt{10}} \right]$$

$$\left[z_\beta = P(Z \le \beta) = F_Z(\beta) \right]$$

(c) $$\sigma \in \left[\sqrt{\frac{(n-1) \cdot s^2}{z_{1-\frac{\alpha}{2}}}},\ \sqrt{\frac{(n-1) s^2}{z_{\frac{\alpha}{2}}}} \right]$$

$z_{\frac{\alpha}{2}} = \text{chi2inv}\left(\frac{0.05}{2}, 9\right) = \text{chi2inv}(0.025, 9) = 2.70$

$$z_{1-\frac{\alpha}{2}} = chi2inv(0.975, 9) = 19.02$$

$$s = 8.7833$$

$$\Rightarrow \sigma \in \left[\underbrace{\frac{\overset{3}{\sqrt{9}} \cdot 8.7833}{\sqrt{19.08}}}_{=5.6203}, \underbrace{\frac{\overset{3}{\sqrt{9}} \cdot 8.7833}{\sqrt{2.7}}}_{14.941} \right]$$

Exercise 2. In a pre-election poll, we are interested in the proportion p of people who plan to vote for candidate A against candidate B.

(a) Find a 95% confidence interval for p, given that 64 persons out of a random sample of 100 persons support A;

(b) Estimate the minimum number of persons polled to obtain a confidence interval for p with a marginal error less than 2.5% and a confidence level at least 95%.

$$95\% = 100 \cdot (1-\alpha)\%$$

$$1-\alpha = \frac{95}{100} = 0.95$$

$$p = \frac{\text{card}\{\text{people that vote for } A\}}{n}$$

$$p \in \left[\overline{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}, \overline{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}\right], \; z_\beta = norminv(\beta)$$

$$\overline{p} = \text{sample proportion} = \frac{64}{100} = 0.64$$

$$z_{\frac{\alpha}{2}} = -1.96 \qquad z_{1-\frac{\alpha}{2}} = 1.96$$

$$p \in \left[0.64 - 1.96 \cdot \sqrt{\frac{0.64 \cdot 0.36}{100}}, 0.64 + 1.96 \cdot \sqrt{\frac{0.64 \cdot 0.36}{100}}\right] =$$

$$= \left[0.64 - 1.96 \cdot \frac{0.8 \cdot 0.6}{10}, 0.64 + 1.96 \cdot \frac{0.8 \cdot 0.6}{10}\right] =$$

$$= \left[0.64 - 1.96 \cdot \frac{48}{10^3}, 0.64 + 1.96 \cdot \frac{48}{10^3}\right] = [0.54, 0.73]$$

marginal error = half of the lenght of the confidence interval

$$p \in \left[\overline{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}, \overline{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}\right],\ z_\beta = norminv(\beta)$$

$$p \in \left[0.64 - 1.96 \cdot \frac{48}{10^2} \cdot \frac{1}{\sqrt{n}},\ 0.64 + 1.96 \cdot \frac{48}{10^2} \cdot \frac{1}{\sqrt{n}}\right]$$

marginal error : $\frac{1}{2} \cdot 2 \cdot 1.96 \cdot \frac{48}{10^2} \cdot \frac{1}{\sqrt{n}} = \frac{1.96 \cdot 48}{10^2} \cdot \frac{1}{\sqrt{n}}$

$$\Rightarrow \frac{1.96 \cdot 48}{10^2} \cdot \frac{1}{\sqrt{n}} < \frac{2.5}{100} \Rightarrow \sqrt{n} > \frac{1.96 \cdot 48}{10^2} \cdot \frac{100}{2.5} =$$

$$\simeq \frac{196 \cdot 48}{10^2 \cdot 2.5} = 37.6 \Rightarrow n > 3.76^2 \Rightarrow n > 1413.8.$$