12.10.2023

Seminar W3 – 837

Exercise 2. A user spends X minutes on a certain website, where

$$X \sim Gamma(a,b),\ a,b>0$$

with a mean value of 12 minutes and a standard deviation of 6 minutes. Find the probability that the user spends at most 10 minutes on the website, given that he spends at least 1 minute.

Sol.: $E(X) = 12 \qquad \sigma(X) = 6$

We want to find $E(X)$ and $\sigma(X)$ in terms of a and b, so that we can then find a and b

Recap. *If $X \sim Gamma(a,b)$, $a,b>0$, then its pdf is:*

$$f_X(x) = \frac{1}{\Gamma(a)b^a}\cdot x^{a-1}e^{\frac{-x}{b}}$$

for $x>0$ (and 0 otherwise). We have:

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx,\ a>0$$

$$E(X) = \int_{\mathbb{R}} x \cdot f_X(x)\, dx$$

$$\sigma(X) = \sqrt{V(X)}$$

$$V(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_{\mathbb{R}} x^2 \cdot f_X(x)\, dx$$

$$E(X) = \int_{\mathbb{R}} x f_X(x)\, dx = \int_0^\infty x f_X(x)\, dx = \int_0^\infty x \cdot \frac{1}{\Gamma(a)\cdot b^a} \cdot x^{a-1} \cdot e^{-\frac{x}{b}} dx =$$

$$= \frac{1}{\Gamma(a)\cdot b^a} \cdot \int_0^\infty x^a e^{-\frac{x}{b}} dx \overset{\frac{x}{b}=y}{=} \frac{1}{\Gamma(a) b^a} \cdot \int_0^\infty (yb)^a \cdot e^{-y} \cdot b\, dy =$$

$$= \frac{1}{\Gamma(a)\cdot b^a} \cdot b^a \cdot b \underbrace{\int_0^\infty y^a e^{-y} \cdot dy}_{\Gamma(a+1)} = \frac{b\, \overset{= a\Gamma(a)}{\Gamma(a+1)}}{\Gamma(a)} = ab$$

$$E(X^2) = \int_0^\infty x^2 \cdot \frac{1}{\Gamma(a)\cdot b^a} \cdot x^{a-1} \cdot e^{-\frac{x}{b}} \, dx =$$

$$= \frac{1}{\Gamma(a)\cdot b^a} \int_0^\infty x^{a+1} \cdot e^{-\frac{x}{b}} \, dx \overset{x=by}{=} \frac{1}{\Gamma(a)\cdot b^a} \cdot \int_0^\infty (yb)^{a+1} \cdot e^{-y} \cdot b \, dy =$$

$$= \frac{b^2 \cdot \Gamma(a+2)}{\Gamma(a)} = \frac{b^2 \cdot a(a+1)\,\Gamma(a)}{\Gamma(a)} = ab^2(a+1)$$

$$V(X) = E(X^2) - E(X)^2 = ab^2(a+1) - a^2b^2 = ab^2(a+1-a) =$$

$$= ab^2$$

$$\begin{cases} E(X) = ab = 12 \\ \sigma(X) = b\sqrt{a} = 6 \end{cases} \Rightarrow \sqrt{a} = 2 \Rightarrow a = 4 \;, b = 3$$

$$\Rightarrow \quad X \sim \text{Gamma}(4,3)$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{\Gamma(4)\cdot 3^4} \cdot x^3 \cdot e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Gamma(4) = (4-1)! = 3! = 6$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{486} \cdot x^3 e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Bayes' formula: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(X \leq 10 \mid X \geq 1) = \frac{P((X \leq 10) \cap (X \geq 1))}{P(X \geq 1)}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = P(X \leq x)$$

$\downarrow$ the cdf

(the cumulative distribution function)

$$P(X \leq 10 \mid X \geq 1) = \frac{P(1 \leq X \leq 10)}{P(X \geq 1)} =$$

$$= \frac{\int_1^{10} f_X(x)\,dx}{1 - P(X < 1)} = \frac{\int_1^{10} f_X(x)\,dx}{1 - \int_{-\infty}^{1} f_X(x)\,dx} = \frac{F_X(10) - F_X(1)}{1 - F_X(1)}$$

Exercise 6. A computer is connected to 2 printers: P_1 and P_2. A printing job is sent with probability 0.4 to P_1 and with probability 0.6 to P_2. P_1 prints an $A2$ poster in T_1 seconds, where T_1 has an exponential distribution with the mean value of 5 seconds, while P_2 handles the same job in T_2 seconds, where T_2 has a uniform distribution on the interval $[4, 6]$. An A2 poster is printed by using the computer. Find the expected value of the printing time.

<u>Sol.</u> :

$$T_1 \sim \mathcal{E}xp(\lambda), \quad E(T_1) = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{5}$$

$$f_{T_1}(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{5} e^{-\frac{1}{5}x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$T_2 \sim U(4, 6) \qquad f_{T_2}(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases} =$$

$$= \begin{cases} \frac{1}{2}, & x \in [4,6] \\ 0, & \text{otherwise} \end{cases}$$

X = "printing time", $f_X = ?$, $E(X) = \int_{\mathbb{R}} x f_X(x)\,dx = ?$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = P(X \leq x)$$

$$P(X \leq x) = P\big((X \leq x) \cap \overbrace{(\text{we used } P_1)}^{\stackrel{\text{not}}{=} A_1}\big) +$$

$$+ P\big((X \leq x) \cap \underbrace{(\text{we used } P_2)}_{\stackrel{\text{not}}{=} A_2}\big) = P(X \leq x \mid A_1) \cdot P(A_1) +$$

$+ P(X \leq x \mid A_2) \cdot P(A_2)$.

$$P(X \leq x \mid A_1) = P(T_1 \leq x) = F_{T_1}(x)$$

$$P(X \leq x \mid A_2) = P(T_2 \leq x) = F_{T_2}(x)$$

$$P(A_1) = 0.4 \qquad P(A_2) = 0.6$$

$$F_X(x) = P(X \leq x) = P(X \leq x \mid A_1) \cdot P(A_1) + P(X \leq x \mid A_2) \cdot P(A_2) =$$

$$= P(T_1 \leq x) \cdot 0.4 + P(T_2 \leq x) \cdot 0.6$$

$$= F_{T_1}(x) \cdot 0.4 + F_{T_2}(x) \cdot 0.6$$

We differentiate:

$$f_X(x) = f_{T_1}(x) \cdot 0.4 + f_{T_2}(x) \cdot 0.6$$

$$f_{T_1}(x) = \begin{cases} \frac{1}{5} e^{-\frac{1}{5}x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad f_{T_2}(x) = \begin{cases} \frac{1}{2}, & x \in [4, 6] \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 0, & x < 0 \\ 0.4 \cdot \frac{1}{5} \cdot e^{-\frac{1}{5}x}, & x \in [0, 4) \\ 0.4 \cdot \frac{1}{5} e^{-\frac{1}{5}x} + 0.6 \cdot \frac{1}{2}, & x \in [4, 6] \\ 0.4 \cdot \frac{1}{5} \cdot e^{-\frac{1}{5}x}, & x \in (6, \infty) \end{cases}$$

$$E(X) = \int_{\mathbb{R}} f_X(x) \, dx$$

$$E(X) = \int_{-\infty}^{0} \underbrace{f_X(x)}_{0}\,dx + \int_{0}^{4} f_X(x)\,dx + \int_{4}^{6} f_X(x)\,dx + \int_{6}^{\infty} f_X(x)\,dx =$$

$$= \int_0^4 (0.4)\cdot\frac{1}{5}\cdot e^{-\frac{1}{5}x}\,dx + \int_4^6 \left((0.4)\cdot\frac{1}{5}\cdot e^{-\frac{1}{5}x} + (0.6)\cdot\frac{1}{2}\right)dx +$$

$$+ \int_6^{\infty} (0.4)\cdot\frac{1}{5}\,e^{-\frac{1}{5}x}\,dx = \int_0^{\infty} (0.4)\cdot\frac{1}{5}\,e^{-\frac{1}{5}x}\,dx + \int_4^6 \frac{0.6}{2}\,dx$$

$$= 0.08\cdot\left(-5e^{-\frac{1}{5}x}\right)\Big|_0^{\infty} + (0.3)(6-4) =$$

$$= (0.08)\cdot 5 + 0.6 = 1$$

Exercise 5. We consider the following game: two dice are rolled n times, where n is a positive integer.

(a) Find the probability that the sum of the two dice is a prime number 3 times out of 9 throws.

(b) Find the probability of getting the first double after 5 throws.

.

Sol.: X = "no of points on the first die"

Y = "no of points on the second die"

$X, Y \sim \mathcal{U}(\{1,6\})$

We have to find $P(X+Y = \text{prime})$

$(X+Y = \text{prime}) = (X+Y = 2) \cup (X+Y = 3) \cup (X+Y = 5) \cup (X+Y = 7) \cup (X+Y = 11)$

$$P(X+Y=2) = \frac{1}{36}$$

$$P(X+Y=3) = \frac{2}{36}$$

$$P(X+Y=5) = \frac{4}{36}$$

$$P(X+Y=7) = \frac{6}{36}$$

$$P(X+Y=11) = \frac{2}{36}$$

$$\Rightarrow P(X+Y = \text{prime}) = \frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36}$$

Bernoulli trial = throwing the dice

success = getting a sum that is a prime number

$$\Rightarrow p = \frac{15}{36}, \quad n = 9, \quad Z = X+Y \sim B\left(9, \frac{15}{36}\right)$$

$\Rightarrow$ we need to find $P(Z=3)$

$$Z \left(\begin{array}{c} k \\ C_9^k \cdot \left(\frac{15}{36}\right)^k \cdot \left(\frac{21}{36}\right)^{9-k} \end{array} \right)$$

$$\Rightarrow P(Z=3) = C_9^3 \cdot \frac{15^3 \cdot 21^6}{36^9} = 7 \cdot 4 \cdot 3 \cdot \frac{15^3 \cdot 21^6}{36^9}.$$

≈ 0.23941

(b) U = no of throws before the first double

$U \sim Geo(p_2)$, $p_2 = P(\text{"you get a double"}) = \frac{6}{36} = \frac{1}{6}$

$P(U=5)=?$

$$U\begin{pmatrix} k \\ p_2 q_2^{k} \end{pmatrix} \Rightarrow U\begin{pmatrix} k \\ \frac{1}{6}\cdot\left(\frac{5}{6}\right)^{k} \end{pmatrix}$$

$$P(U=5)=\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{5}\approx 0{,}066$$