Seminar 2

Exercise 1 Let $\mathscr{F}$ be a family of convex sets in $\mathbb{R}^n$, which is directed w.r.t. inclusion, i.e.,

$$\forall A, B \in \mathscr{F}, \exists C \in \mathscr{F} \quad : \quad A \cup B \subseteq C.$$

Prove that the union of the family $\mathscr{F}$, i.e., the set $\bigcup_{S \in \mathscr{F}} S$, is convex.

Solution. Let $x, y \in \bigcup_{S \in \mathscr{F}} S$ and $t \in [0, 1]$. Then there exist $X, Y \in \mathscr{F}$ such that $x \in X$ and $y \in Y$. The family $\mathscr{F}$ being directed, we can find $Z \in \mathscr{F}$ such that $X \cup Y \subseteq Z$. Since S is convex and $x, y \in Z$, it follows that $(1-t)x + ty \in Z \subseteq \bigcup_{S \in \mathscr{F}} S$. Thus $\bigcup_{S \in \mathscr{F}} S$ is convex. $\square$

Exercise 2 Let $(M_i)_{i \in \mathbb{N}}$ be a sequence of convex sets in $\mathbb{R}^n$, which is ascending, i.e.,

$$M_i \subseteq M_{i+1}, \ \forall i \in \mathbb{N}.$$

Prove that $\bigcup_{i=1}^{\infty} M_i$ is a convex set.

Solution. Obviously, the family $\mathscr{F} := \{M_i \mid i \in \mathbb{N}\}$ is directed. Thus, the conclusion follows by Exercise 1. $\square$

Exercise 3 Prove that for any set $S \subseteq \mathbb{R}^n$ the following assertions are equivalent:

1° S is convex.

2° $(\alpha + \beta)S = \alpha S + \beta S$ for all $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$.

Solution. 1° $\Rightarrow$ 2°. Under the assumption that S is convex, let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$. If $\alpha + \beta = 0$, then $\alpha = 0$ and $\beta = 0$, thus the equality $(\alpha + \beta)S = \alpha S + \beta S$ obviously holds. Suppose now that $\alpha + \beta \neq 0$. The inclusion $(\alpha + \beta)S \subseteq \alpha S + \beta S$ is obvious. For the converse inclusion consider any points $x, y \in S$. Since S is convex and $\frac{\alpha}{\alpha+\beta} \in [0, 1]$, it follows that

$$\alpha x + \beta y = (\alpha + \beta) \left(\tfrac{\alpha}{\alpha+\beta} x + \tfrac{\beta}{\alpha+\beta} y \right) \in (\alpha + \beta)S.$$

Hence $\alpha S + \beta S \subseteq (\alpha + \beta)S$. This yields finally that $(\alpha + \beta)S = \alpha S + \beta S$.

1° $\Rightarrow$ 2°. Assume that for any $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$ the equality $(\alpha + \beta)S = \alpha S + \beta S$ holds. It follows that for any $x, y \in S$ and $t \in [0, 1]$, we have $(1-t)x + ty \in (1-t)S + tS = S$. Thus S is convex. $\square$

Exercise 4 Give an example of a non-convex set $S \subseteq \mathbb{R}^n$ satisfying the condition $2S = S+S$.

Solution. Let $S := \mathbb{Q}^n$ be the set consisting of all vectors in $\mathbb{R}^n$ having all the coordinates rational numbers. Clearly $2\mathbb{Q}^n \subseteq \mathbb{Q}^n + \mathbb{Q}^n \subseteq \mathbb{Q}^n = 2(\frac{1}{2}\mathbb{Q}^n) \subseteq 2\mathbb{Q}^n$, so $S + S = 2S$. On the other hand, for $x := 0_n$, $y := e^1$, and $t := 1/\sqrt{2}$, we have $x, y \in S$ and $t \in [0,1]$, but $(1-t)x + ty = (1/\sqrt{2}, 0, \dots, 0) \notin \mathbb{Q}^n = S$. Thus S is not convex. $\square$

Exercise 5 Let $C \subseteq \mathbb{R}^n$ be a cone, that is, a set that is stable under multiplication by nonnegative real numbers:

$$\mathbb{R}_+ \cdot C := \{\alpha x \mid \alpha \in \mathbb{R},\ \alpha \geq 0, x \in C\} \subseteq C.$$

Prove that the following assertions are equivalent:

1° C is convex.

2° $C + C \subseteq C$.

Solution. 1° $\Rightarrow$ 2°. Assume that 1° holds, i.e., C is convex. Then we have $\frac{1}{2}C + \frac{1}{2}C \subseteq C$. Since C is a cone, we can deduce that $C + C = 2(\frac{1}{2}C + \frac{1}{2}C) \subseteq 2C \subseteq \mathbb{R}_+ \cdot C \subseteq C$.

2° $\Rightarrow$ 1°. Assume that 2° holds and consider any $x, y \in C$ and $t \in [0,1]$. Since C is a cone, it follows that $\{(1-t)x, ty\} \subseteq \mathbb{R}_+ \cdot C \subseteq C$, hence $(1-t)x + ty \in C + C$. By 2° we obtain that $(1-t)x + ty \in C$. Thus C is convex. $\square$