Seminar W13 — 832

H_0 : not guilty
H_1: guilty

accepting H_0 = the defendant goes free
rejecting H_0 = the defendant goes to jail

type I error = H_0 true, but we reject it
= sending an innocent to jail

type II error = H_1 true, but we reject it
= freeing a guilty person

$$\alpha = P(\text{reject } H_0 \mid H_0)$$

$$\beta = P(\text{accept } H_0 \mid H_1)$$

Theorem (Neyman-Pearson). *Let X be a characteristic with pdf $f(x;\theta)$, with $\theta \in A \subseteq \mathbb{R}$ unknown. Suppose we test on θ the simple hypotheses:*

$$H_0 : \theta = \theta_0$$

$$H_1 : \theta = \theta_1$$

*based on a random sample $X_1, X_2, \ldots, X_n$. Then for a fixed $\alpha \in (0,1)$, a **most powerful test** is the test with rejection region given by*

$$RR = \left\{ \frac{L(\theta_1)}{L(\theta_0)} \geq k_\alpha \right\}$$

where the constant $k_\alpha > 0$ depends only on α and the sample variables.

Exercise 1. Let X be a characteristic that has a normal distribution $N(m, 1)$, where $m \in \mathbb{R}$ is unknown.

(a) For a random sample of size 9 for X, find a most powerful test with the significance level 5% for the hypothesis $H_0 : m = 0$ against $H_1 : m = 1$. Find the power of this test.

Neyman-Piersa

(b) For the following sample data of X:

$$-1, 0.5, -0.25, -0.75, 0.75, 1.25, 0.5, 1, 0.25$$

accept or reject the hypothesis $H_0 : m = 0$ against $H_1 : m = 1$, using the obtained test.

Sol: $n = 9 \qquad \alpha = 0.05$

We apply NPL: the most powerful test has:

$$RR = \left\{ (x_1, \dots, x_n) \;\middle|\; \frac{L(x_1, \dots, x_n ; 1)}{L(x_1, \dots, x_n ; 0)} \geq k_\alpha \right\}$$

$$k_\alpha \in \mathbb{R}$$

$$L(x_1, \dots, x_n ; m) = \prod_{i=1}^{n} f_{X_i}(x_i, m)$$

$$X \sim N(m, 1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-(x-m)^2}{2}}$$

$$\Rightarrow L(x_1, \dots, x_n ; m) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - m)^2} =$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^9 \cdot e^{-\frac{1}{2}\sum_{i=1}^{9}(x_i - m)^2}$$

NPL

$$\Rightarrow \quad RR = \left\{ (x_1, \ldots, x_n) \;\middle|\; \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^9 \cdot e^{-\frac{1}{2}\sum_{i=1}^{9}(x_i-1)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^9 \cdot e^{-\frac{1}{2}\sum_{i=1}^{9} x_i^2}} \geq k_\alpha \right\}$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; e^{-\frac{1}{2}\sum_{i=1}^{9}\left(x_i^2 - (x_i-1)^2\right)} \geq k_\alpha \right\} =$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; -\frac{1}{2}\sum_{i=1}^{9}(2x_i - 1) \geq \ln(k_\alpha) \right\} =$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; \sum_{i=1}^{9} x_i \geq \underbrace{\ln(k_\alpha) + \frac{9}{2}}_{=: K_\alpha} \right\}$$

We know that $X_i \sim N(\overset{\mu}{m}, \overset{\sigma}{1})$

$$\Rightarrow \quad X_1 + X_2 + \ldots + X_9 \sim N(9m, 3)$$

$$\alpha = P(\text{reject } H_0 \mid H_0) = P((X_1, \ldots, X_n) \in RR \mid H_0) =$$

$$= P(X_1 + X_2 + \ldots + X_9 \geq K_\alpha \mid H_0)$$

If H_0 is true, then $m = 0$, so $\underbrace{X_1 + X_2 + \ldots + X_9}_{= Y} \sim N(0, 3)$

$$\Rightarrow \quad \alpha = P(Y \geq K_\alpha) = 1 - P(Y \leq K_\alpha) =$$

$$= 1 - \text{cdf}(K_\alpha) \Rightarrow \text{cdf}(K_\alpha) = 1 - \alpha \Rightarrow$$

$$\Rightarrow \quad K_\alpha = \text{norminv}(1-\alpha, 0, 3) = 4.9346$$

$$\Rightarrow \quad RR = \{ (x_1, x_2, \ldots, x_9) \mid x_1 + x_2 + \ldots + x_9 \geq 4.9346 \}$$

To obtain the power of the test:

$$\pi(1) = 1 - \beta(1) = 1 - P(\text{reject } H_0 | H_1) =$$

$$= 1 - P((x_1, \ldots, x_9) \in RR \mid H_1)$$

If H_1 is true, then $Y = X_1 + \ldots + X_9 \sim N(9, 3)$

$$\Rightarrow \pi(1) = 1 - P(Y \geq K_\alpha) = P(Y < K_\alpha) = F_Y(K_\alpha) =$$

$$= \text{normcdf}(4.9346, 9, 3) = 0.087$$

b) $x_1 + x_2 + \ldots + x_9 = 2.25 \not\geq 4.9346 \Rightarrow (x_1, \ldots, x_9) \notin RR$

$\Rightarrow$ accept H_0

Exercise 2. Let X be a characteristic that has a Bernoulli distribution with parameter p, where $p \in (0, 1)$ is unknown.

(a) For a random sample of (large) size n for X, find a most powerful test with the significance level 5% for the hypothesis $H_0 : p = 0.5$ against $H_1 : p = 0.45$.

(b) Let p be the probability of heads for a coin. Assume we toss the coin 900 times and we get 441 heads. Accept or reject the hypothesis $H_0 : p = 0.5$ against $H_1 : p = 0.45$, using the obtained test.

Sol. : $X \sim \text{Bern}(p)$: $X \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

$$\Rightarrow f_X(x) = (1-p)^{1-x} \cdot p^x$$

By (NPL):

$$RR = \left\{ (x_1, \dots, x_n) \;\middle|\; \frac{L(x_1, x_2, \dots, x_n; 0.45)}{L(x_1, x_2, \dots, x_n; 0.5)} \geq k_\alpha \right\}$$

$$L(x_1, \dots, x_n, p) = \prod_{i=1}^{n} f_X(x_i) = \prod_{i=1}^{n} (1-p)^{1-x_i} \cdot p^{x_i} =$$

$$= (1-p)^{n-(x_1+\dots+x_n)} \cdot p^{x_1+\dots+x_n}$$

$$\overline{X_n} := \frac{X_1 + \dots + X_n}{n}$$

$$\Rightarrow RR = \left\{ (x_1, \dots, x_n) \;\middle|\; \frac{0.55^{n(1-\overline{x_n})} \cdot 0.45^{n \cdot \overline{x_n}}}{0.5^{n(1-\overline{x_n})} \cdot 0.5^{n \cdot \overline{x_n}}} \geq k_\alpha \right\}$$

$$= \left\{ (x_1, \dots, x_n) \;\middle|\; n(1-\overline{x_n}) \cdot \ln(0.55) + n \cdot \overline{x_n} \cdot \ln(0.45) - \right.$$

$$\left. - n \cdot (1-\overline{x_n}) \cdot \ln(0.5) - n \cdot \overline{x_n} \cdot \ln(0.5) \geq \ln k_\alpha \right\}$$

$$= \left\{ (x_1, \dots, x_n) \;\middle|\; \ln(0.55) - \overline{x_n} \ln(0.55) + \overline{x_n} \cdot \ln(0.45) + \right.$$

$$\left. + \ln(0.5) \geq \frac{1}{n} \ln k_\alpha \right\} =$$

$$= \left\{ (x_1, \dots, x_n) \;\middle|\; \overline{x_n} \cdot \ln \frac{0.45}{0.55} \geq \frac{1}{n} \ln k_\alpha - \ln(0.55 \cdot 0.5) \right\}$$

$$= \left\{ (x_1, \dots, x_n) \;\middle|\; \overline{x_n} \geq \underbrace{\frac{1}{\ln \frac{0.45}{0.55}} \left(\frac{1}{n} \ln k_\alpha - \ln(0.55 \cdot 0.5) \right)}_{=: K_\alpha} \right\}$$

$$\Rightarrow RR = \{ (x_1, \dots, x_n) \mid \overline{x}_n \geq k_\alpha \}$$

We know that $\dfrac{\overline{X}_n - E(X)}{\frac{\sigma_X}{\sqrt{n}}} \sim \mathcal{N}(0,1)$

$$E(X) = 0 \cdot (1-p) + 1 \cdot p = p \qquad X^2 \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

$$\sigma_X = \sqrt{E(X^2) - E(X)^2} = \sqrt{p - p^2}$$

$$RR = \left\{ (x_1, \dots, x_n) \;\middle|\; \underbrace{\frac{\overline{x}_n - p}{\sqrt{\frac{p-p^2}{n}}}}_{Z \sim \mathcal{N}(0,1)} \geq \underbrace{\frac{k_\alpha - p}{\sqrt{\frac{p-p^2}{n}}}}_{= k_\alpha'} \right\}$$

$$\alpha = P(\text{reject } H_0 \mid H_0) = P((x_1, \dots, x_n) \in RR \mid H_0)$$

If H_0 is true, then $p = 0.5$

$$\Rightarrow \alpha = P(Z \geq k_\alpha') = 1 - P(Z < k_\alpha')$$

$$\Rightarrow P(Z < k_\alpha') = 1 - \alpha \Rightarrow F_Z(k_\alpha') = 1 - \alpha$$

$$\Rightarrow k_\alpha' = \text{norminv}(1-\alpha, 0, 1) = \text{norminv}(0.95, 0, 1) =$$
$$= 1.6449$$

$$\Rightarrow RR = \left\{ (x_1, \dots, x_n) \;\middle|\; \frac{\overline{x}_n - 0.5}{\sqrt{\frac{0.5 - 0.25}{n}}} \geq 1.6449 \right\}$$

$$= \left\{ (x_1, \dots, x_n) \;\middle|\; \sqrt{n}\,\overline{x}_n \geq \frac{1}{2} \cdot 1.6449 + \frac{1}{2}\sqrt{n} \right\}$$

The power of the test: $\pi(0.45) = 1 - \beta(0.45) = 1 - P(\text{accept } H_0 \mid H_1) =$

$= P(\text{reject } H_0 \mid H_1) = P((x_1, \dots, x_n) \in RR \mid H_1)$

(b) $n = 900$ $\qquad \bar{p} = \frac{441}{900} = \overline{X_n} = \frac{X_1 + \dots + X_n}{n}$

$\overline{X_n} = 0.49$

$RR = \{ (x_1, \dots, x_n) \mid \sqrt{n}\,\overline{x_n} \geq \frac{1}{2} \cdot 1.649 + \frac{1}{2}\sqrt{n} \} =$

$= \{ (x_1, \dots, x_n) \mid 30 \cdot \overline{x_n} \geq \frac{1}{2} \cdot 1.649 + \frac{1}{2} \cdot 30 \}$

$30 \cdot 0.49 - \frac{1}{2} \cdot 1.649 - 15 \approx -1.12 \Rightarrow$

$\Rightarrow \overline{X_n} \notin RR \Rightarrow$ accept H_0, $p = 0.5$

Exercise 3. Let X be a characteristic that has an $Exp(\lambda)$ distribution, where $\lambda > 0$ is unknown.

(a) For a random sample of size 10 for X, find a most powerful test with the significance level 5% for the hypothesis $H_0 : \lambda = 1$ against $H_1 : \lambda = 3$. Find the power of this test.

(b) Assume that X is the time (in minutes) spent by a client to buy a bus ticket from a vending machine and that the total time spent by a random sample of 10 clients is 4.9 minutes. Accept or reject the hypothesis $H_0 : \lambda = 1$ against $H_1 : \lambda = 3$, using the obtained test.

Sol.: $\quad X \sim Exp(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

(a) $n = 10$ $\qquad L(x_1, \dots, x_{10}; \lambda) = \prod_{i=1}^{10} \left(\lambda e^{-\lambda x_i} \right) =$

$= \lambda^{10} \cdot e^{-\lambda (x_1 + \dots + x_{10})}$

$$RR = \left\{ (x_1, \dots, x_{10}) \;\middle|\; \frac{L(x_1, \dots, x_{10}, 3)}{L(x_1, \dots, x_{10}, 1)} \geq k_\alpha \right\} =$$

$$= \left\{ (x_1, \dots, x_{10}) \;\middle|\; \frac{3^{10} \cdot e^{-3(x_1 + \dots + x_{10})}}{e^{-(x_1 + \dots + x_{10})}} \geq k_\alpha \right\} =$$

$$= \left\{ (x_1, \dots, x_{10}) \;\middle|\; e^{-2(x_1 + \dots + x_{10})} \geq \frac{1}{3^{10}} k_\alpha \right\} =$$

$$= \left\{ (x_1, \dots, x_{10}) \;\middle|\; -2(x_1 + \dots + x_{10}) \geq -10 \ln 3 + \ln(k_\alpha) \right\}$$

$$= \left\{ (x_1, \dots, x_{10}) \;\middle|\; x_1 + x_2 + \dots + x_{10} \leq \underbrace{5 \ln 3 - \frac{1}{2} \ln(k_\alpha)}_{=: K_\alpha} \right\}$$

$$X_1 + X_2 + \dots + X_{10} \sim \text{Gamma}\left(n, \frac{1}{\lambda}\right) = \text{Gamma}\left(10, \frac{1}{\lambda}\right)$$

The rest is the same