29. $\begin{cases} \Delta u = 0, \text{ for } \rho < 2 \\ u|_{\rho=2} = (\sin\alpha)^2 \end{cases}$

Using the separation method

We know that $u = u(\rho, \alpha)$;

Fourier's idea states that we can rewrite u as a product of 2 function;

so: $u(\rho,\alpha) = U(\rho)\cdot V(\alpha)$. Now we plug this in our equation (Laplace equation):

$\frac{\partial^2}{\partial \rho^2} u + \frac{\partial^2}{\partial \alpha^2} u = 0$; So, we will get

$$U''(\rho)\cdot V(\alpha) + U(\rho)\cdot V''(\alpha) = 0 \quad \Big| \frac{1}{UV}$$

$$\underbrace{-\frac{U''(\rho)}{U(\rho)}}_{f(\rho)} = \underbrace{+\frac{V''(\alpha)}{V(\alpha)}}_{g(\alpha)}$$

Since this is an equality of functions they need to be equal with a constant λ

$$f(\rho) = g(\alpha) = \lambda$$

Now we have 2 equations that only depend on ρ and α.

$$-\frac{U''(\rho)}{U(\rho)} = \lambda \text{ and } \frac{V''(\alpha)}{V(\alpha)} = \lambda$$

$$U''(\rho) = \lambda \cdot U(\rho) \text{ and } V''(\alpha) = \lambda \cdot V(\alpha)$$

Having this and knowing that $u(\rho, \alpha)$:

and $\Delta u = \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \alpha^2}$ (1)

and that $\begin{cases} \Delta u = 0 ; \ \alpha \leq 2 \\ u_{|\alpha = 2} = (\sin \rho)^2 \end{cases}$

We denote (1) as: $u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} \cdot u_{\alpha\alpha}$

By knowing all this and $u(\rho, \alpha) = U(\rho)V(\alpha) \Rightarrow 0 = \Delta u = U''(\rho)V(\alpha) + \frac{1}{\rho} U'(\rho)V(\alpha) + \frac{1}{\rho^2} U(\rho)V''(\alpha)$

$$\rho^2 U'' + \rho U' + \lambda U = 0$$

$$\begin{cases} V'' - \lambda V = 0 \\ V(\alpha + 2\pi) = V(\alpha) \ (*) \end{cases} \quad r^2 + \lambda = 0 \begin{cases} \lambda > 0 \ (1) \\ \lambda < 0 \ (2) \\ \lambda = 0 \ (3) \end{cases}$$

(1) $V(\alpha) = c_1 \cos\sqrt{\pi}\,\alpha + c_2 \sin\sqrt{\lambda}\,\alpha$

(2) $V(\alpha) = c_1 e^{\sqrt{-\lambda}\,\alpha} + c_2 e^{-\sqrt{-\lambda}\,\alpha}$, not good for us

(3) $V(\alpha) = c_1\alpha + c_2$, good only if $c_1 = 0$

From $(*) \Rightarrow 2\pi = k \cdot \dfrac{2\pi}{\sqrt{\lambda}} \Rightarrow \sqrt{\lambda} = k \Rightarrow$

$\Rightarrow \lambda = k^2$; $\forall k = 1, 2, 3 \ldots \Rightarrow V(\alpha) = c_1 \cos k\alpha +$
$+ c_2 \sin k\alpha$

$$\rho^2 U'' + \rho U' - k^2 U = 0 \text{ - Euler eq.}$$

$k = 0 \Rightarrow \rho^2 U'' + \rho U' = 0$. We substitute U' with $x \Rightarrow$
$\Rightarrow \rho^2 x' + \rho x = 0 \Rightarrow \dfrac{x'}{x} = -\dfrac{1}{\rho}$ since $U' = V = \dfrac{1}{\rho}$

$U = \ln\rho + c$

$k = 1, 2, 3 \ldots \Rightarrow U(\rho) = \rho^\alpha \Rightarrow \rho^2 \alpha(\alpha - 1)\rho^{\alpha-2} +$
$\rho \cdot \alpha \cdot \rho^{\alpha-1} - k^2 \rho^\alpha = 0$ By reducing to common factor we get $\alpha^2 + k^2 = 0 \Rightarrow \alpha = \pm k \Rightarrow$
$\Rightarrow U(\rho) = c_1 \rho^k + c_2 \rho^{-k}$

So for $\lambda = 0 \Rightarrow U_0 = c_1 \lambda\rho + c_2$

$\lambda = k^2 \Rightarrow u_k = (c_1\rho^k + c_2\rho^{-k})(c_3 \cos k\alpha + c_4 \sin k\alpha)$

and since we know that our case is

$\rho < 2$, we get the general formula for

$\rho < n$; where $n \in \mathbb{N}$

$$\frac{a_0}{2} + \sum_{k=1}^{\alpha} (a_1 \cos k\alpha + b_1 \sin k\alpha)$$

Now knowing this lets value our exercise:

$$\begin{cases} \Delta u = 0; & \rho < 2 \\ u_{,\rho=2} = (\sin\alpha)^2 \end{cases}$$

$R = 2 \Rightarrow \rho = 2 \Rightarrow$ for $u|\rho-2 =$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\alpha + b_k \sin k\alpha = (\sin\alpha)^2$$

We identify the coef.

$a_0 = 0$; $a_1 = 0$; $b_1 = \sin\alpha$, $b_{k\neq 1} = 0$;

$u = \frac{\rho^2}{4} (\sin\alpha)^2$ $\quad a_{k\neq 1} = 0$;