Lecture 7

Linear optimization; Duality; Primal/dual feasible bases and optimal bases

Definition 7.1 By a *constrained linear optimization problem in standard form* we mean any minimization problem of type:

$$(P)\quad \begin{cases} \text{Minimize } f(x) = \langle c, x\rangle \\ Ax = b \\ x \geq 0_n, \end{cases} \tag{7.1}$$

where $m, n \in \mathbb{N}^*$, $A \in \mathcal{M}_{m,n}(\mathbb{R})$ with $\operatorname{rank} A = m < n$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Of course, such a problem may be written explicitly as:

$$(P)\quad \begin{cases} \text{Minimize } f(x) = c_1x_1 + \cdots + c_nx_n \\ a_{11}x_1 + \cdots + a_{1n}x_n \quad = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \quad = b_m \\ x_1, \ldots, x_n \geq 0. \end{cases}$$

The objective function of problem (P) is a linear function, namely $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$f(x) := \langle c, x\rangle = c_1x_1 + \cdots + c_nx_n,\ \forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{7.2}$$

while the set of feasible points of (P) is the standard polyhedral set (see Definition **??**):

$$S := S(A, b) = \{x \in \mathbb{R}^n \mid Ax = b \text{ and } x \geqq 0_n\}. \tag{7.3}$$

Thus the set of optimal solutions of problem (P) is

$$\operatorname*{argmin}_{x\in S} f(x) = \{x^0 \in S \mid f(x^0) \leq f(x),\ \forall\, x \in S\}.$$

Definition 7.2 The optimization problem

$$(D)\ \begin{cases} \text{Maximize } f^*(y) := \langle b, y\rangle \\ \\ A^\top y \leqq c \end{cases} \tag{7.4}$$

is called the *dual of problem* (P). Its explicit form is:

$$(D)\ \begin{cases} \text{Maximize } f^*(y) = b_1y_1 + \cdots + b_my_m \\ \\ a_{11}y_1 + \cdots + a_{m1}y_m \quad \leq c_1 \\ \vdots \\ a_{1n}y_1 + \cdots + a_{mn}y_n \quad \leq c_n. \end{cases}$$

The objective function of the dual problem (D) is a linear function, namely $f^* : \mathbb{R}^m \to \mathbb{R}$, defined by

$$f^*(y) := b_1y_1 + \cdots + b_my_m,\ \forall\, y = (y_1, \ldots, y_m) \in \mathbb{R}^m, \tag{7.5}$$

while the set of feasible points of (D) is the polyhedral set

$$S^* := \{y \in \mathbb{R}^m \mid A^\top y \leqq c\}. \tag{7.6}$$

Thus the set of optimal solutions of the dual problem (D) is

$$\operatorname*{argmax}_{y\in S^*} f^*(y) = \{y^0 \in S^* \mid f^*(y^0) \geq f^*(y),\ \forall\, y \in S^*\}.$$

Remark 7.3 In order to unify the terminology, in what follows the initial problem (P) will be called the *primal problem.*

Theorem 7.4 (Weak Duality) *For any feasible points $x \in S$ and $y \in S^*$ of the problems (P) and (D), the following inequality holds:*

$$f(x) \geq f^*(y).$$

Proof. Let $x \in S$ and $y \in S^*$. By (7.2), (7.3), (7.5) and (7.6), it is easy to deduce that:

$$f^*(y) = \langle b, y\rangle = \langle Ax, y\rangle = \langle x, A^\top y\rangle \leq \langle x, c\rangle = f(x). \qquad \square$$

Theorem 7.5 (Strong Duality) *If some feasible points $x^0 \in S$ and $y^0 \in S^*$ of the problems (P) and (D) satisfy the property*

$$f(x^0) = f^*(y^0),$$

then the following assertions hold:

1° *x^0 is an optimal solution of the primal problem (P), i.e., $x^0 \in \operatorname*{argmin}_{x \in S} f(x)$.*

2° *y^0 is an optimal solution of the dual problem (D), i.e., $y^0 \in \operatorname*{argmax}_{y \in S^*} f^*(y)$.*

Proof. Let $x^0 \in S$ and $y^0 \in S^*$ be two feasible points such that $f(x^0) = f^*(y^0)$.

In order to prove 1°, we have to show that

$$f(x^0) \leq f(x), \ \forall\, x \in S.$$

To this aim, consider an arbitrary point $x \in S$. By applying the Weak Duality Theorem 7.4 (for y^0 in the role of y), we deduce that

$$f(x) \geq f^*(y^0) = f(x^0).$$

Similarly, in order to prove 2° we can show that

$$f^*(y^0) \geq f^*(y), \ \forall\, y \in S^*.$$

Indeed, for any $y \in S^*$ we can apply again the Weak Duality Theorem 7.4 (this time for x^0 in the role of x), which yields

$$f^*(y) \leq f(x^0) = f^*(y^0).$$

□

Corollary 7.6 (on the unboundedness of objective functions) *The following hold:*

1° *If the objective function f of the primal problem (P) is unbounded from below on the set of feasible points S, then the dual problem (D) has no feasible points, i.e., $S^* = \emptyset$. In other words,*

$$\inf f(S) = -\infty \quad \Longrightarrow \quad S^* = \emptyset.$$

2° *If the objective function f^* of the dual problem (D) is unbounded from above on the set of feasible points S^*, then the primal problem (P) has no feasible points, i.e., $S = \emptyset$. In other words,*

$$\sup f^*(S^*) = +\infty \quad \Longrightarrow \quad S = \emptyset.$$

Definition 7.7 An ordered set B consisting of m linearly independent columns of A is called a *basis* of problem (P) (or a basis of the polyhedral set S). For such a set B, we define the *set of basic indices*

$$\mathcal{B}_B := \{j \in \{1, \ldots, n\} \mid A^j \in B\}$$

and the *set of nonbasic indices*

$$\mathcal{N}_B := \{j \in \{1, \ldots, n\} \mid A^j \notin B\} = \{1, \ldots, n\} \setminus \mathcal{B}_B.$$

Given a basis B of problem (P), the coordinates of $b \in \mathbb{R}^m$ w.r.t. the basis B are denoted by $\alpha_{0j} \in \mathbb{R}$ $(j \in \mathcal{B}_B)$, i.e.,

$$b = \sum_{j \in \mathcal{B}_B} \alpha_{0j} A^j. \tag{7.7}$$

Similarly, every column of the matrix A that does not belong to the basis B, i.e., A^i with $i \in \mathcal{N}_B$, admits a unique representation as a linear combination of the columns of B, namely

$$A^i = \sum_{j \in \mathcal{B}_B} \alpha_{ij} A^j, \ i \in \mathcal{N}_B, \tag{7.8}$$

where the numbers $\alpha_{ij} \in \mathbb{R}$ with $j \in \mathcal{B}_B$ represent the coordinates of A^i w.r.t. B.

By means of these coordinates $\alpha_{ij} \in \mathbb{R}$, $(i, j) \in \mathcal{N}_B \times \mathcal{B}_B$, we introduce some technical numbers α_{i0}, $i \in \mathcal{N}_B$, as follows:

$$\alpha_{i0} := \sum_{j \in \mathcal{B}_B} \alpha_{ij} c_j - c_i, \ i \in \mathcal{N}_B. \tag{7.9}$$

Also, we define the number

$$\alpha_{00} := \sum_{j \in \mathcal{B}_B} \alpha_{0j} c_j. \tag{7.10}$$

We associate to B a point $x^B = (x_1^B, \ldots, x_n^B) \in \mathbb{R}^n$, given by

$$x_j^B := \begin{cases} \alpha_{0j} & \text{dacă} \quad j \in \mathcal{B}_B \\ 0 & \text{dacă} \quad j \in \mathcal{N}_B, \end{cases} \tag{7.11}$$

as well as a point $y^B = (y_1^B, \ldots, y_m^B) \in \mathbb{R}^m$, defined as the unique solution of a Cramer type linear system of equations:

$$\langle A^j, y^B \rangle = c_j, \ \forall\, j \in \mathcal{B}_B. \tag{7.12}$$

Definition 7.8 A basis B of the standard polyhedral set $S = S(A, b)$ is called:

- *primal feasible basis* (*p.f.b.*), if

$$\alpha_{0j} \geq 0, \ \forall j \in \mathcal{B}_B.$$

- *dual feasible basis* (*d.f.b.*), if

$$\alpha_{i0} \leq 0, \ \forall i \in \mathcal{N}_B.$$

- *optimal basis*, if B is both primal feasible and dual feasible, i.e.,

$$\begin{cases} \alpha_{0j} \geq 0, & \forall j \in \mathcal{B}_B \\ \alpha_{i0} \leq 0, & \forall i \in \mathcal{N}_B. \end{cases}$$

This terminology is motivated by the following three results.

Theorem 7.9 (Characterization of primal feasible bases) *For any basis B of the standard polyhedral set $S = S(A, b)$ the following assertions are equivalent:*

1° *B is a primal feasible basis.*

2° *x^B is a feasible point of the primal problem (P), i.e., $x^B \in S$.*

Theorem 7.10 (Characterization of dual feasible bases) *For any basis B of the standard polyhedral set $S = S(A, b)$ the following assertions are equivalent:*

1° *B is a a dual feasible basis.*

2° *y^B is a feasible point of the dual problem (D), i.e., $y^B \in S^*$.*

Corollary 7.11 (on the optimal solutions generated by optimal bases) *If B is an optimal basis of the standard polyhedral set $S = S(A, b)$, then the following assertions hold:*

1° $f(x^B) = f^*(y^B) = \alpha_{00}$.

2° x^B *is an optimal solution of the primal problem* (P), *i.e.,* $x^B \in \underset{x \in S}{\operatorname{argmin}} f(x)$.

3° y^B *is an optimal solution of the dual problem* (D), *i.e.,* $y^B \in \underset{y \in S^*}{\operatorname{argmax}} f^*(y)$.

Since in general the polyhedral set $S = S(A, b)$ possesses several bases, we associate to each basis B of $S = S(A, b)$ a so-called *Simplex tableau*, that contains all numbers defined by (7.7), (7.8), (7.9) and (7.10):

		$\ldots$	c_j	$\ldots$	
	Current basis no.	$\ldots$	A^j	$\ldots$	Test d.f.b.
$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_i	A^i	$\ldots$	α_{ij}	$\ldots$	α_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
	Test p.f.b.	$\ldots$	α_{0j}	$\ldots$	α_{00}

Lecture 8

Primal SIMPLEX Algorithm

The Primal SIMPLEX Algorithm is an implementable numerical method conceived to solve the primal optimization problem

$$(P)\begin{cases} f(x) := \langle c, x\rangle \longrightarrow \min \\ Ax = b \\ x \geq 0_n \end{cases}$$

under the assumptions of Definition 7.1. We will study two variants of the SIMPLEX algorithm. The so-called *Primal SIMPLEX Algorithm* consists in changing succesively certain primal feasible bases of the feasible set $S := S(A, b)$, such that after a finite number of iterations we obtain:

• an optimal solution of problem (P) and the optimal value of the objective function f, i.e., its minimal value on S.

or

• the conclusion that f is unbounded from below on S, hence the primal problem (P) has no optimal solutions.

Pseudocode of the Primal SIMPLEX Algorithm

`Step 1.` `Choose a primal feasible basis` B `of the feasible set` $S = S(A,b)$ `of the primal problem` (P) `introduced in (7.1).`

`Step 2.` `Determine the numbers` $\alpha_{ij} \in \mathbb{R}$, $(i,j) \in \mathcal{N}_B \times \mathcal{B}_B$ `and` $\alpha_{0j} \in \mathbb{R}$, $j \in \mathcal{B}_B$ `of` b, `given by (7.8) and (7.7).`

`Step 3.` `Compute the numbers` α_{i0}, $i \in \mathcal{N}_B$ `defined by (7.9) and` α_{00} `defined by (7.10).`

`Step 4.` `Study the sign of` α_{i0} `for all` $i \in \mathcal{N}_B$.

(a) `If` $\alpha_{i0} \leq 0$ `for all` $i \in \mathcal{N}_B$, `then conclude that the point` $x^0 := x^B$, `given by (7.11), is an optimal solution of the primal problem` (P), `and` α_{00} `is the optimal (minimal) value of` f `on` S. `STOP.`

(b) If there exists $i \in \mathcal{N}_B$ such that $\alpha_{i0} > 0$, then generate the set of indices

$$\mathcal{N}_B^+ = \{i \in \mathcal{N}_B \mid \alpha_{i0} > 0\}.$$

Step 5. For every $i \in \mathcal{N}_B^+$ study the sign of α_{ij} for all $j \in \mathcal{B}_B$.

(a) If there exists $i \in \mathcal{N}_B^+$ such that $\alpha_{ij} \leq 0$ for all $j \in \mathcal{B}_B$, then conclude function f is unbounded from below on S, hence the primal problem (P) has no optimal solutions. STOP.

(b) If for each $i \in \mathcal{N}_B^+$ there is $j \in \mathcal{B}_B$ such that $\alpha_{ij} > 0$, then choose an index $h \in \mathcal{N}_B^+$. For instance, h can be chosen such that $h \in \mathcal{N}_B$ and

$$\alpha_{h0} = \max\{\alpha_{i0} \mid i \in \mathcal{N}_B\}.$$

Step 6. Choose an index $k \in \mathcal{B}_B$ such that $\alpha_{hk} > 0$ and

$$\frac{\alpha_{0k}}{\alpha_{hk}} = \min\left\{ \frac{\alpha_{0j}}{\alpha_{hj}} \,\middle|\, j \in \mathcal{B}_B \text{ s.t. } \alpha_{hj} > 0 \right\}.$$

Step 7. Consider a new basis $B := (B \setminus \{A^k\}) \cup \{A^h\}$, obtained from the previous one by replacing A^k by A^h, and go to Step 2.

Remark 8.1 Due to the specific choice of the indices h and k at `Step 5 (b)` and `Step 6`, it can be shown that the new basis B, constructed at `Step 7`, is a primal feasible basis of S. Moreover, the objective function's outcome at the new point x^B, namely $\alpha_{00} = f(x^B)$, decreases in comparison to the previous one.

In order to solve small size optimization problems by applying the Primal SIMPLEX Algorithm (without implementing it on a computer), the numbers $\alpha_{ij} \in \mathbb{R}$ with $i \in \mathcal{N}_B \cup \{0\}$ and $j \in \mathcal{B}_B \cup \{0\}$, involved at `Steps` 2 and 3, will be arranged in a Simplex tableau (as described at the end of Lecture 7):

		…	c_j	…	
	Current basis no.	…	A^j	…	Test d.f.b.
$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_i	A^i	…	α_{ij}	…	α_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
	Test p.f.b.	…	α_{0j}	…	α_{00}

Consider the particular case when the initial basis B, chosen at `Step 1`, is the canonical basis of $\mathbb{R}^m$. Then, the numbers α_{ij} at α_{0j} involved in `Step 2` are already know, since they are the Cartesian coordinates of the non-basic columns A^i and of b, respectively. Notice that the canonical basis B is a primal feasible basis (p.f.b.) if and only if the coordinates of b satisfy the condition $\alpha_{0j} = b_j \geq 0$ for every basic index j (i.e., all numbers located in the horizontal section "Test p.f.b."of the Simplex tableau are greater than or equal to zero).

At `Step 3` all numbers α_{i0} şi α_{00} can be computed easily by writing the coefficients $c_1, \ldots, c_n$ of the objective function f outside the Simplex tableau, next to the corresponding columns $A^1, \ldots, A^n$ of matrix A. More precisely, the coefficients c_j labelled by basic indices are written above the Simplex tableau, while the coefficients c_i labelled by nonbasic indices are written on the left side of the Simplex tableau. In this way, the formulae (7.9) and (7.10) admit a comprehensive visual interpretation.

The aim of `Step 4` is to test whether B is a dual feasible basis (d.f.b.). The following wo cases may occur:

`(a)` If $\alpha_{i0} \leq 0$ for all nonbasic indices i (i.e., all numbers located in the vertical section "Test d.f.b." are less than or equal to zero), then B is a dual feasible basis, hence an optimal one (since it is already primal feasible, by `Step 1`). In this case, we conclude that an optimal solution of the primal problem (P) is the point

$$x^0 = (\ldots, x_i^0 = 0, \ldots, x_j^0 = \alpha_{0j}, \ldots),$$

whose coordinates x_j^0 labelled by basic indices can be recoverd from the horizontal section "Test p.f.b." while the optimal (minimal) value of the objective function f on S is

$$\min f(S) = \alpha_{00},$$

that can be found in the last (bottom, right) cell of the Simplex tableau.

`(b)` If there exists a number $\alpha_{i0} > 0$ labelled by a non-basic index i (i.e., the vertical section "Test d.f.b." contains at least one positive number), then we consider the set $\mathcal{N}_B^+$ (consisting of all non-basic indices that correspond to positive numbers located in the vertical section "Test d.f.b.").

At `Step 5` we study whether for each $\alpha_{i0} > 0$ with $i \in \mathcal{N}_B$ there exists some $\alpha_{ij} > 0$ with $j \in \mathcal{B}_B$ (i.e, we investigate if on the left side of each positive number α_{i0} from the vertical section "Test d.f.b." there is at least one positive number α_{ij}). Two cases may occur:

(a) If there exists a number $\alpha_{i0} > 0$ with $i \in \mathcal{N}_B$ such that $\alpha_{ij} \leq 0$ for all $j \in \mathcal{B}_B$ (i.e., there exists a positive number α_{i0} in the vertical section "Test d.f.b." on the left side of which none of the numbers α_{ij} with $j \in \mathcal{B}_B$ is positive), then we conclude that the objective function f is unbounded from below on S, hence the primal problem (7.1) has no optimal solutions. In other words, in this case we have

$$\inf f(S) = -\infty \quad \text{and} \quad \operatorname*{argmin}_{x \in S} f(x) = \emptyset.$$

(b) If for each $i \in \mathcal{N}_B^+$ there exists $j \in \mathcal{B}_B$ such that $\alpha_{ij} > 0$ (i.e, on the left side of each positive number α_{i0} from the vertical section "Test d.f.b." there exists at least one positive number α_{ij}), then we choose a non-basic index h that corresponds to any positive number α_{h0} from the vertical section "Test d.f.b.", as for instance the largest number of this section:

$$\alpha_{h0} = \max\{\alpha_{i0} \mid i \in \mathcal{N}_B\}.$$

The Simplex tableau's row that corresponds to h is called the *pivot row* and is marked by an asterisk $*$ on the right side of the tableau.

Within `Step 6` we choose a basic index k such that $\alpha_{hk} > 0$ and

$$\frac{\alpha_{0k}}{\alpha_{hk}} = \min\left\{\frac{\alpha_{0j}}{\alpha_{hj}} \;\middle|\; j \in \mathcal{B}_B \text{ and } \alpha_{hj} > 0\right\}.$$

Notice that all numerators of the ratios $\dfrac{\alpha_{0j}}{\alpha_{hj}}$ involved in the formula above are located in the horizontal section "Test p.f.b." while the corresponding denomitors are the positive numbers in the pivot row. In practice, it will be convenient to write these ratios below the Simplex tableau. The smallest ratio $\dfrac{\alpha_{0k}}{\alpha_{hk}}$ will be marked by an asterisk $*$ and the corresponding tabeau's column will be called the *pivot column*.

Before going to the next step, it is recommended to highlight (encircle) the number α_{hk}, which represents the so-called *pivot*.

		$\dots$	c_j	$\dots$	c_k	$\dots$		
	Current basis	$\dots$	A^j	$\dots$	A^k	$\dots$	Test d.f.b.	
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
c_i	A^i	$\dots$	α_{ij}	$\dots$	α_{ik}	$\dots$	α_{i0}	
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
c_h	A^h	$\dots$	α_{hj}	$\dots$	$\boxed{\alpha_{hk}}$	$\dots$	α_{h0}	$*$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
	Test p.f.b.	$\dots$	α_{0j}	$\dots$	α_{0k}	$\dots$	α_{00}	
			$\dfrac{\alpha_{0j}}{\alpha_{hj}}$		$\dfrac{\alpha_{0k}}{\alpha_{hk}}$ $*$			

At `Step 7` we construct a new basis, obtained from the previous one by replacing A^k with A^h (notice that these elements belong to the pivot column and the pivot row, respectively). Then we go to `Step 2`. The Simplex tableau associated to the new basis can be obtained from the previous tableau by performing a so-called Gauss-Jordan transform:

Current basis	$\dots$	A^j	$\dots$	A^h	$\dots$	Test d.f.b.
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^i	$\dots$	$\frac{\alpha_{ij}\alpha_{hk} - \alpha_{hj}\alpha_{ik}}{\alpha_{hk}}$	$\dots$	$\frac{\alpha_{ik}}{\alpha_{hk}}$	$\dots$	$\frac{\alpha_{i0}\alpha_{hk} - \alpha_{ik}\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^k	$\dots$	$-\frac{\alpha_{hj}}{\alpha_{hk}}$	$\dots$	$\frac{1}{\alpha_{hk}}$	$\dots$	$-\frac{\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
Test p.f.b.	$\dots$	$\frac{\alpha_{0j}\alpha_{hk} - \alpha_{0k}\alpha_{hj}}{\alpha_{hk}}$	$\dots$	$\frac{\alpha_{0k}}{\alpha_{hk}}$	$\dots$	$\frac{\alpha_{00}\alpha_{hk} - \alpha_{0k}\alpha_{h0}}{\alpha_{hk}}$

Lecture 9

Dual SIMPLEX Algorithm

The *Dual SIMPLEX Algorithm* is an implementable numerical method conceived for solving the primal optimization problem

$$(P) \begin{cases} f(x) := \langle c, x\rangle \longrightarrow \min \\ Ax = b \\ x \geq 0_n. \end{cases}$$

In contrast to the Primal SIMPLEX Algorithm, presented in Lecture 8, the Dual SIMPLEX Algorithm consists in changing succesively certain dual feasible bases of the feasible set $S := S(A, b)$, such that after a finite number of iterations we obtain::

• an optimal solution of problem (P) and the optimal (minimal) value of f on S

or

• the conclusion that problem (P) has no feasible points, i.e., the set S is empty, hence problem (P) has no optimal solutions.

Pseudocode of the Dual SIMPLEX Algorithm

Step 1. Choose a dual feasible basis B of the feasible set $S = S(A, b)$ of the primal problem (P).

Step 2. Determine the numbers $\alpha_{ij} \in \mathbb{R}$, $(i, j) \in \mathcal{N}_B \times \mathcal{B}_B$, and $\alpha_{0j} \in \mathbb{R}$, $j \in \mathcal{B}_B$, given by (7.8) and (7.7).

Step 3. Compute the numbers α_{i0}, $i \in \mathcal{N}_B$, defined by (7.9), and the number α_{00} defined by (7.10).

Step 4. Study the sign of α_{0j} for all $j \in \mathcal{B}_B$.

(a) If $\alpha_{0j} \geq 0$ for all $j \in \mathcal{B}_B$, then conclude that the point $x^0 := x^B$, given by (7.11), is an optimal solution of problem (7.1), and α_{00} is the optimal (minimal) value of f on S. STOP.

(b) If there exists $j \in \mathcal{B}_B$ such that $\alpha_{0j} < 0$, then generate the set of indices

$$\mathcal{B}_B^- = \{j \in \mathcal{B}_B \mid \alpha_{0j} < 0\}.$$

Step 5. For every $j \in \mathcal{B}_B^-$ study the sign of α_{ij} for all $i \in \mathcal{N}_B$.

(a) If there exists $j \in \mathcal{B}_B^-$ such that $\alpha_{ij} \geq 0$ for all $i \in \mathcal{N}_B$, then conclude that the primal problem (P) has no feasible points, i.e., $S = \emptyset$, hence (P) has no optimal solutions. STOP.

(b) If for each $j \in \mathcal{B}_B^-$ there is $i \in \mathcal{N}_B$ such that $\alpha_{ij} < 0$, then choose an index $k \in \mathcal{B}_B^-$. For instance, k can be chosen such that $\mathcal{B}_B$ and

$$\alpha_{0k} = \min\{\alpha_{0j} \mid j \in \mathcal{B}_B\}.$$

Step 6. Choose an index $h \in \mathcal{N}_B$ such that $\alpha_{hk} < 0$ and

$$\frac{\alpha_{h0}}{\alpha_{hk}} = \min\left\{ \frac{\alpha_{i0}}{\alpha_{ik}} \,\middle|\, i \in \mathcal{N}_B \;\; \texttt{s.t.}\; \alpha_{ik} < 0 \right\}.$$

Step 7. Consider a new basis $B := (B \setminus \{A^k\}) \cup \{A^h\}$, obtained from the previous one by replacing A^k by A^h, and go to Step 2.

Remark 9.1 Due to the specific choice of the indices k and h at Step 5 (b) and Step 6, it can be shown that the new basis B, constructed at Step 7, is a dual feasible basis of S.

In order to solve small size optimization problems by means of the Dual SIMPLEX Algorithm (without implementig it on a computer) in all exercises it will be assumed that the initial basis at Step 1 is the canonical basis of $\mathbb{R}^m$. The numbers α_{ij} involved in Steps 2 and 3 will be arranged in a SIMPLEX tableau, as we did for the Dual SIMPLEX Algorithm in Lecture 8:

		$\ldots$	c_j	$\ldots$	
	Current basis no.	$\ldots$	A^j	$\ldots$	Test d.f.b.
$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_i	A^i	$\ldots$	α_{ij}	$\ldots$	α_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
	Test p.f.b.	$\ldots$	α_{0j}	$\ldots$	α_{00}

Notice that, according to Definition 7.8, the canonical basis B is a dual feasible basis (d.f.b.) if and only if $\alpha_{i0} \leq 0$ for any non-basic index i (i.e., all numbers located in the vertical section "Test d.f.b." are less than or equal to zero).

At `Step 4` we test whether B is a a primal feasible basis (p.f.b.). The following two cases may occur:

`(a)` If $\alpha_{0j} \geq 0$ for all basic indices j (i.e., all numbers in the horizontal section "Test p.f.b" are greater than or equal to zero), then B is a primal feasible basis, hence an optimal one (since it is already dual feasible, by `Step 1`). In this case, we conclude that an optimal solution of the primal problem (P) is the point

$$x^0 = (\ldots, x_i^0 = 0, \ldots, x_j^0 = \alpha_{0j}, \ldots)$$

whose coordinates x_j^0 labelled by basic indices j can be recovered from the horizontal section "Test p.f.b" while the optimal (minimal) value of f on S is

$$\min f(S) = \alpha_{00},$$

that can be found in the last (bottom, right) cell of the Simplex tableau.

`(b)` If there exists a number $\alpha_{0j} < 0$ labelled by a basic index j (i.e., the horizontal section "Test p.f.b." contains at least one negative number), then we consider the set $\mathcal{B}_B^-$ (consisting of all basic indices that correspond to negative numbers located in the horizontal section "Test p.f.b.").

At `Step 5` we study whether for each α_{0j} with $j \in \mathcal{B}_B^-$ there exists some $\alpha_{ij} < 0$ with $i \in \mathcal{N}_B$ (i.e., we investigate if above each positive number α_{0j} from the horizontal section "Test p.f.b." there is at least one negative number α_{ij}). Two cases may occur:

`(a)` If there exists $j \in \mathcal{B}_B^-$ such that $\alpha_{ij} \geq 0$ for all $i \in \mathcal{N}_B$ (i.e., there exists a negative number α_{0j} in the horizontal section "Test p.f.b." above which none of the numbers α_{ij} is negative), then we conclude that problem (P) has no feasible points, hence (P) has no optimal solutions. In other words, in this case we have:

$$S = \emptyset = \operatorname*{argmin}_{x \in S} f(x).$$

`(b)` If for each $j \in \mathcal{B}_B^-$ there exists $i \in \mathcal{N}_B$ such that $\alpha_{ij} < 0$ (i.e., above each negative number α_{0j} from the horizontal section "Test p.f.b." there exists at least one negative number α_{ij}), then we can choose a basic index k that corresponds to any negative number α_{0k} from the horizontal section "Test p.f.b.", as for instance the smallest number of this section:

$$\alpha_{0k} = \min\{\alpha_{0j} \mid j \in \mathcal{B}_B\}.$$

The Simplex tableau's column that corresponds to k represents the *pivot column* and it is marked by an asterisk $*$ below the tableau.

Within `Step 6` we choose a non-basic index h such that $\alpha_{hk} < 0$ and

$$\frac{\alpha_{h0}}{\alpha_{hk}} = \min\left\{\frac{\alpha_{i0}}{\alpha_{ik}} \;\middle|\; i \in \mathcal{N}_B \text{ and } \alpha_{ik} < 0\right\}.$$

Notice that all numerators of the ratios $\dfrac{\alpha_{i0}}{\alpha_{ik}}$ involved in the formula above are located in the vertical section "Test d.f.b." while the corresponding denominators are the negative numbers in the pivot column. In practice, it will be convenient to write these ratios on the right side of the Simplex tableau. The smallest ratio $\frac{\alpha_{h0}}{\alpha_{hk}}$ will be marked by an asterisk $*$ and the corresponding row represents the Simplex tableau's *pivot row*. Before going to the next step, it is recommended to highlight (encircle) the *pivot*, i.e., the number α_{hk}.

		$\dots$	c_j	$\dots$	c_k	$\dots$			
	Current basis no.	$\dots$	A^j	$\dots$	A^k	$\dots$	Test d.f.b.		
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		
c_i	A^i	$\dots$	α_{ij}	$\dots$	α_{ik}	$\dots$	α_{i0}	$\frac{\alpha_{i0}}{\alpha_{ik}}$	
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		
c_h	A^h	$\dots$	α_{hj}	$\dots$	$\boxed{\alpha_{hk}}$	$\dots$	α_{h0}	$\frac{\alpha_{h0}}{\alpha_{hk}}$	$*$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		
	Test p.f.b.	$\dots$	α_{0j}	$\dots$	α_{0k}	$\dots$	α_{00}		
					$*$				

At `Step 7` we construct a new basis, obtained from the previous one by replacing A^k with A^h (notice that these elements belong to the pivot column and the pivot row, respectively). Then we go to `Step 2`. The Simplex tableau associated to the new basis can be obtained from the previous tableau by performing a Gauss-Jordan transform:

Current basis no.	$\dots$	A^j	$\dots$	A^h	$\dots$	Test d.f.b.
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^i	$\dots$	$\dfrac{\alpha_{ij}\alpha_{hk} - \alpha_{hj}\alpha_{ik}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{ik}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{i0}\alpha_{hk} - \alpha_{ik}\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
A^k	$\dots$	$-\dfrac{\alpha_{hj}}{\alpha_{hk}}$	$\dots$	$\dfrac{1}{\alpha_{hk}}$	$\dots$	$-\dfrac{\alpha_{h0}}{\alpha_{hk}}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
Test p.f.b.	$\dots$	$\dfrac{\alpha_{0j}\alpha_{hk} - \alpha_{0k}\alpha_{hj}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{0k}}{\alpha_{hk}}$	$\dots$	$\dfrac{\alpha_{00}\alpha_{hk} - \alpha_{0k}\alpha_{h0}}{\alpha_{hk}}$

Lecture 10

Matrix games

Definition 10.1 By a (*two-person zero-sum finite*) *matrix game* we mean a triple $(\mathcal{A}^1, \mathcal{A}^2, C)$ consisting of two finite sets,

$$\mathcal{A}^1 = \left\{a_1^1, \ldots, a_m^1\right\} \quad \text{and} \quad \mathcal{A}^2 = \left\{a_1^2, \ldots, a_n^2\right\},$$

and a matrix of real numbers,

$$C = \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \ldots & c_{mn} \end{pmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}),$$

where $m, n \in \mathbb{N}$. For any

$$i \in I := \{1, \ldots, m\} \quad \text{and} \quad j \in J := \{1, \ldots, n\},$$

we say that

- a_i^1 is an *action of the first player* (also called raw player);
- a_j^2 is an *action of the second player* (also called column player);
- (a_i^1, a_j^2) is a *turn of the game*;
- c_{ij} is the *gain/loss (outcome) of the first player* while $-c_{ij}$ is *the loss/gain of the second player* w.r.t. (a_i^1, a_j^2).

Thus, $\mathcal{A}^1$ and $\mathcal{A}^2$ represent the sets of all possible actions of the players while C represents the so-called *payoff matrix* of the game.

Example 10.2 (*Rock-Scissors-Paper*) Assume that two players choose simultaneously and independently one of the words (symbols) *Rock* (R), *Scissors* (S) or *Paper* (P), i.e., the sets of actions are

$$\begin{aligned} \mathcal{A}^1 &= \left\{a_1^1, a_2^1, a_3^1\right\} = \{R, S, P\}, \\ \mathcal{A}^2 &= \left\{a_1^2, a_2^2, a_3^2\right\} = \{R, S, P\}. \end{aligned}$$

If both players choose the same word, then the game is tied. Otherwise, the rock beats the scissors and the paper beats the rock. The payoff matrix of this game is:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Definition 10.3 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. For every $i \in I$ let

$$\alpha_i := \min_{j \in J} c_{ij}.$$

The number

$$\underline{w} := \max_{i \in I} \alpha_i = \max_{i \in I} \min_{j \in J} c_{ij}$$

is called *the game's lower value.* Similarly, for every $j \in J$ let

$$\beta_j = \max_{i \in I} c_{ij}.$$

The number

$$\overline{w} := \min_{j \in J} \beta_j = \min_{j \in J} \max_{i \in I} c_{ij}$$

is called *the game's upper value.*

Remark 10.4 It is a simple exercise to prove that in any matrix game the following inequality holds:

$$\underline{w} \leq \overline{w}.$$

Definition 10.5 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. We say that $c_{i_0 j_0}$ is a *saddle point* of the payoff matrix C if

$$c_{ij_0} \leq c_{i_0 j_0} \leq c_{i_0 j}, \ \forall (i, j) \in I \times J,$$

which means that $c_{i_0 j_0}$ is the largest element on its column as well as the smallest element on its row. We say that $(\mathcal{A}^1, \mathcal{A}^2, C)$ is a

- *game with saddle points*, if its payoff matrix C has at least one saddle point;
- *game without saddle points*, if its payoff matrix C has no saddle points.

Remark 10.6 It is easy to show that $(\mathcal{A}^1, \mathcal{A}^2, C)$ is a game with saddle points if anf only if

$$\underline{w} = \overline{w}.$$

Example 10.7 For the Rock-Scissors-Paper game we have

$$\underline{w} = -1 < \overline{w} = 1,$$

hence this is a game without saddle points.

Example 10.8 Consider a game whose payoff matrix is

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 1 & 0 & 3 \end{pmatrix}.$$

In this case we have

$$\underline{w} = 0 = \overline{w},$$

hence this game has saddle points. More precisely, its saddle points are

$$c_{12} \quad \text{and} \quad c_{32}.$$

Definition 10.9 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. The elements of the set

$$X := \Delta_m = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1, \dots, x_m \geq 0,\ x_1 + \dots + x_m = 1\}$$

are called *strategies of the first player* while the elements of the set

$$Y := \Delta_n = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n \mid y_1, \dots, y_n \geq 0,\ y_1 + \ldots + y_n = 1\}$$

are called *strategies of the second player*, where Δ_k represents the standard simplex of $\mathbb{R}^k$, introduced in Definition 4.8.

Definition 10.10 Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game. The number

$$F(x, y) := \sum_{(i,j) \in I \times J} c_{ij} x_i y_j = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i y_j$$

is called *the average gain* of the first player for a given pair of strategies $(x, y) \in X \times Y$.

Definition 10.11 By a solution of the matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ we mean a pair of strategies $(x^0, y^0) \in X \times Y$ satisfying the property

$$F(x, y^0) \leq F(x^0, y^0) \leq F(x^0, y),\ \forall\, (x, y) \in X \times Y.$$

In this case we say that:

- x^0 is an *optimal strategy of the first player*;
- y^0 is an *optimal strategy of the second player.*

Theorem 10.12 (John Von Neumann) *Every matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ has at least one solution. Moreover, if $(x^0, y^0) \in X \times Y$ is a solution of the game, then*

$$\max_{x \in X} \min_{y \in Y} F(x, y) = F(x^0, y^0) = \min_{y \in Y} \max_{x \in X} F(x, y).$$

Definition 10.13 The real number

$$w := \max_{x \in X} \min_{y \in Y} F(x, y) = \min_{y \in Y} \max_{x \in X} F(x, y)$$

(which is well-defined by Von Neumann's Theorem) is called *the value of the game* $(\mathcal{A}^1, \mathcal{A}^2, C)$.

Proposition 10.14 *For any matrix game* $(\mathcal{A}^1, \mathcal{A}^2, C)$ *w ehave*

$$\underline{w} \leq w \leq \overline{w}.$$

To solve a matrix game from mathematical point of view we have to determine the game's value w and one of its solutions (x^0, y^0), i.e., a pair of optimal strategies of the two players.

Lecture 11

The relationship between matrix games and linear optimization problems

Let $(\mathcal{A}^1, \mathcal{A}^2, C)$ be a matrix game whose payoff matrix is

$$C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mn} \end{pmatrix}.$$

Theorem 11.1 *If the matrix C has a saddle point $c_{i_0 j_0}$, then the following assertions hold:*

1° *The game's value is* $w = \underline{w} = \overline{w}$.

2° *The canonical vector* $x^0 = (x^0_1, \dots, x^0_m) = (0, \dots, x^0_{i_0} = 1, \dots, 0) = e^{i_0}$ *of* $\mathbb{R}^m$ *is an optimal strategy of Player 1.*

3° *The canonical vector* $y^0 = (y^0_1, \dots, y^0_n) = (0, \dots, y^0_{j_0} = 1, \dots, 0) = e^{j_0}$ *of* $\mathbb{R}^n$ *is an optimal strategy of Player 2.*

In general (if the payoff matrix has no saddle points), then we consider the following optimization problems:

$$(P_1)\quad \begin{cases} \text{Minimize } u_1 + \dots + u_m \\ c_{11}u_1 \; + \dots \; + c_{m1}u_m \; \geq 1 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ c_{1n}u_1 \; + \dots \; + c_{mn}u_m \; \geq 1 \\ u_1, \dots, u_m \geq 0 \end{cases}$$

and

$$
(P_2) \begin{cases} \text{Maximize } v_1 + \cdots + v_n \\ \begin{array}{llll} c_{11}v_1 & +\ldots & +c_{1n}v_n & \le 1 \\ \vdots & & & \vdots \\ c_{m1}v_1 & +\ldots & +c_{mn}v_n & \le 1 \end{array} \\ v_1, \ldots, v_n \ge 0. \end{cases}
$$

Theorem 11.2 *If a matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ has a positive value, i.e.,*

$$w > 0,$$

then the problems $(P1)$ and (P_2) have optimal solutions. Moreover, for any optimal solutions $u^0 = (u_1^0, \ldots, u_m^0)$ and $v^0 = (v_1^0, \ldots, v_n^0)$ of these problems, the following assertions hold:

1° *The game's value is* $w = \dfrac{1}{u_1^0 + \cdots + u_m^0} = \dfrac{1}{v_1^0 + \cdots + v_n^0}.$

2° *The vector* $x^0 = w \cdot u^0$ *is an optimal strategy of Player 1.*

3° *The vector* $y^0 = w \cdot v^0$ *is an optimal strategy of Player 2.*

Remark 11.3 If a matrix game $(\mathcal{A}^1, \mathcal{A}^2, C)$ has a positive lower value, i.e.,

$$\underline{w} > 0,$$

then its value is positive, since

$$w \ge \underline{w} > 0,$$

by Proposition 10.14. Thus, in this case we can apply Theorem 11.2.

Remark 11.4 If $\underline{w} \le 0$, then w might be not positive. In this case, we can modify the payoff matrix C by adding a suitable constant $k \in \mathbb{R}$ to all entries of C, such that

$$\underline{w} + k > 0.$$

In this way we obtain a new payoff matrix

$$\widehat{C} := C + (k)$$

whose lower value is positive:

$$\underline{\widehat{w}} = \underline{w} + k > 0.$$

Therefore we can apply Theorem 11.2 in order to solve the game with the payoff matrix $\widehat{C}$. Of course, this new game has the same solutions as the initial one, while their values are related by

$$w = \widehat{w} - k.$$

Remark 11.5 Under the hypothesis that $w > 0$, the problems (P_1) and (P_2) should be written in standard before applying the SIMPLEX algorithm:

$$
(P_{1,\text{st}})\left\{
\begin{array}{llllllll}
\text{Minimize } u_1 + \cdots + u_m & & & & & & & \\
-c_{11}u_1 & -\ldots & -c_{m1}u_m & +u_{m+1} & & & & = -1 \\
\quad\ldots & & & & \ddots & & & \vdots \\
-c_{1n}u_1 & -\ldots & -c_{mn}u_m & & & & +u_{m+n} & = -1 \\
u_1, \ldots, u_{m+n} \geq 0 & & & & & & &
\end{array}
\right.
$$

and

$$
(P_{2,\text{st}})\left\{
\begin{array}{llllllll}
\text{Minimize } - v_1 - \cdots - v_n & & & & & & & \\
c_{11}v_1 & +\ldots & +c_{1n}v_n & +v_{n+1} & & & & = 1 \\
\quad\ldots & & & & \ddots & & & \vdots \\
c_{m1}v_1 & +\ldots & +c_{mn}v_n & & & & +v_{n+m} & = 1 \\
v_1, \ldots, v_{n+m} \geq 0 & & & & & & &
\end{array}
\right.
$$

Actually, problem $(P_{1,\text{st}})$ can be solved by the Primal SIMPLEX algorithm, while problem $(P_{2,\text{st}})$ can be solved by the Dual SIMPLEX algorithm.

Remark 11.6 The dual of $(P_{1,\text{st}})$ is equivalent to (P_2), while the dual of $(P_{2,\text{st}})$ is equivalent to (P_1).

Indeed, the dual of $(P_{1,\text{st}})$ is

$$
(D_1)\left\{
\begin{array}{lllll}
\text{Maximize } - s_1 - \cdots - s_n & & & & \\
-c_{11}s_1 & -\ldots & -c_{1n}s_n & \leq 1 \\
\quad\vdots & & & \vdots \\
-c_{11}s_1 & -\ldots & -c_{1n}s_n & \leq 1 \\
\quad s_1 & & & \leq 0 \\
 & \ddots & & \leq 0 \\
 & & s_n & \leq 0,
\end{array}
\right.
$$

which is equivalent to (P_2) by means of the change of variables

$$-s = (-s_1, \ldots, -s_n) = v = (v_1, \ldots, v_n).$$

On the other hand, the dual of $(P_{2,\mathrm{st}})$ is

$$(D_2)\ \begin{cases} \text{Maximize } t_1 + \cdots + t_m \\ \begin{array}{cccc} c_{11}t_1 & + \dots & +c_{m1}t_m & \le -1 \\ \vdots & & & \vdots \\ c_{1n}t_1 & + \dots & +c_{m1}t_m & \le -1 \\ t_1 & & & \le 0 \\ & \ddots & & \le 0 \\ & & t_m & \le 0, \end{array} \end{cases}$$

which is equivalent to (P_1) by means of the change of variables

$$t = (t_1, \dots, t_m) := -u = (-u_1, \dots, -u_m).$$

Lecture 12

Dual problems solved by applying the SIMPLEX algorithm to the primal problems