4 Random Vectors

Everything that holds for random *variables* (one-dimensional case) can be easily generalized to any dimension, i.e. to random *vectors*. We restrict our discussion to two-dimensional random vectors $(X, Y) : S \to \mathbb{R}^2$.

Let $(S, \mathcal{K}, P)$ be a probability space. A **random vector** is a function $(X, Y) : S \to \mathbb{R}^2$ satisfying the following the condition

$$(X \le x, Y \le y) = \{e \in S \mid X(e) \le x, Y(e) \le y\} \in \mathcal{K},$$

for all $(x, y) \in \mathbb{R}^2$.

- if the set of values that it takes, $(X, Y)(S)$, is at most countable in $\mathbb{R}^2$, then (X, Y) is a **discrete random vector**,
- if $(X, Y)(S)$ is a continuous subset of $\mathbb{R}^2$, then (X, Y) is a **continuous random vector**.
- the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

  $$F(x, y) \;=\; P(X \le x, Y \le y)$$

  is called the **joint cumulative distribution function (joint cdf)** of the vector (X, Y).

The properties of the cdf of a random variable translate very naturally for a random vector, as well: Let (X, Y) be a random vector with joint cdf $F : \mathbb{R}^2 \to \mathbb{R}$ and let $F_X, F_Y : \mathbb{R} \to \mathbb{R}$ be the cdf's of X and Y, respectively. Then following properties hold:

- If $a_k < b_k,\ k = \overline{1, 2}$, then

  $$\begin{aligned} P(a_1 < X \le b_1, a_2 < Y \le b_2) \;&=\; F(b_1, b_2) \;-\; F(b_1, a_2) \\ &-\; F(a_1, b_2) \;+\; F(a_1, a_2). \end{aligned}$$

- $\lim\limits_{x,y\to\infty} F(x, y) = 1,$
  $\lim\limits_{y\to-\infty} F(x, y) = \lim\limits_{x\to-\infty} F(x, y) = 0,\ \forall x, y \in \mathbb{R},$
  $\lim\limits_{y\to\infty} F(x, y) = F_X(x),\ \forall x \in \mathbb{R},$
  $\lim\limits_{x\to\infty} F(x, y) = F_Y(y),\ \forall y \in \mathbb{R}.$

4.1 Discrete Random Vectors

Let $(X,Y): S \to \mathbb{R}^2$ be a two-dimensional discrete random vector. The **joint probability distribution (function)** of (X,Y) is a two-dimensional array of the form

$$\begin{array}{c|ccccc|c}
X \setminus Y & y_1 & \dots & y_j & \dots & & \\
\hline
x_1 & & & & & & \\
\vdots & & & \vdots & & & \\
x_i & & \dots & p_{ij} & \dots & & p_i \\
\vdots & & & \vdots & & & \\
 & & & & & & \\
\hline
 & & & q_j & & &
\end{array} \tag{4.1}$$

where $(x_i, y_j) \in \mathbb{R}^2,\ (i,j) \in I \times J$ are the values that (X,Y) takes and $p_{ij} = P(X = x_i, Y = y_j)$.

An important property is that

$$\sum_{j \in J} p_{ij} = p_i,\ \sum_{i \in I} p_{ij} = q_j \ \text{ and } \ \sum_{i \in I} \sum_{j \in J} p_{ij} = \sum_{j \in J} \sum_{i \in I} p_{ij} = 1,$$

where $p_i = P(X = x_i),\ i \in I$ and $q_j = P(Y = y_j),\ j \in J$. The probabilities p_i and q_j are called **marginal** pdf's.

For discrete random vectors, the computational formula for the cdf is

$$F(x,y) \ = \ \sum_{x_i \le x} \sum_{y_j \le y} p_{ij},\ x, y \in \mathbb{R}.$$

Operations with discrete random variables

Let X and Y be two discrete random variables with pdf's

$$X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I} \ \text{ and } \ Y \begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j \in J}.$$

Sum. The sum of X and Y is the random variable with pdf given by

$$X + Y \begin{pmatrix} x_i + y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}. \tag{4.2}$$

Product. The product of X and Y is the random variable with pdf given by

$$X \cdot Y \begin{pmatrix} x_i y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}. \tag{4.3}$$

Scalar Multiple. The random variable αX, $\alpha \in \mathbb{R}$, with pdf given by

$$\alpha X \begin{pmatrix} \alpha x_i \\ p_i \end{pmatrix}_{i \in I}. \tag{4.4}$$

Quotient. The quotient of X and Y is the random variable with pdf given by

$$X/Y \begin{pmatrix} x_i/y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}, \tag{4.5}$$

provided that $y_j \neq 0$, for all $j \in J$.

In general, if $h : \mathbb{R} \to \mathbb{R}$ is a function, then we can define the random variable $h(X)$, with pdf given by

$$h(X) \begin{pmatrix} h(x_i) \\ p_i \end{pmatrix}_{i \in I}. \tag{4.6}$$

Variables X and Y are said to be **independent** if

$$p_{ij} = P\left(X = x_i, Y = y_j\right) = P\left(X = x_i\right) P\left(Y = y_j\right) = p_i q_j, \tag{4.7}$$

for all $(i, j) \in I \times J$.

If X and Y are independent, then in (4.2), (4.3) and (4.5), $p_{ij} = p_i q_j$, for all $(i, j) \in I \times J$.

4.2 Continuous Random Vectors

Let (X, Y) be a continuous random vector with joint cdf $F : \mathbb{R}^2 \to \mathbb{R}$. Then F is *absolutely continuous*, i.e. there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$, such that

$$F(x, y) \quad = \quad \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv, \tag{4.8}$$

for all $x, y \in \mathbb{R}$. The function f is called the **joint probability density function (joint pdf)** of (X, Y).

The usual properties of continuous pdf's (and their relationship with cdf's) hold for the two-dimensional case, as well: Let (X, Y) be a continuous random vector with joint cdf $F$and joint density function f. Let $F_X, F_Y : \mathbb{R} \to \mathbb{R}$ be the cdf's of X and Y and $f_X, f_Y : \mathbb{R} \to \mathbb{R}$ be the pdf's of X and Y, respectively. Then the following properties hold:

- $\dfrac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$, for all $(x,y) \in \mathbb{R}^2$.
- $\displaystyle\iint\limits_{\mathbb{R}^2} f(x,y)\, dxdy = 1.$
- for any domain $D \subseteq \mathbb{R}^2$, $P\big((X,Y) \in D\big) = \displaystyle\iint\limits_{D} f(x,y)\, dxdy.$
- $f_X(x) = \displaystyle\int\limits_{\mathbb{R}} f(x,y)\, dy,\ \forall x \in \mathbb{R}$ and $f_Y(y) = \displaystyle\int\limits_{\mathbb{R}} f(x,y)\, dx,\ \forall y \in \mathbb{R}.$

When obtained from the vector (X, Y), the pdf's f_X and f_Y are called *marginal* densities.
The continuous random variables X and Y are said to be **independent** if

$$f_{(X,Y)}(x,y) \quad = \quad f_X(x) f_Y(y), \tag{4.9}$$

for all $(x,y) \in \mathbb{R}^2$.

5 Common Distributions

5.1 Common Discrete Distributions

Bernoulli Distribution $Bern(p)$

A random variable X has a Bernoulli distribution with parameter $p \in (0,1)$ $(q = 1-p)$, if its pdf is

$$X \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}. \tag{5.1}$$

Then

$$\begin{aligned} E(X) &= p, \\ V(X) &= pq. \end{aligned}$$

A Bernoulli r.v. models the occurrence or nonoccurrence of an event.

Discrete Uniform Distribution $U(m)$

A random variable X has a Discrete Uniform distribution ($\boxed{\text{unid}}$) with parameter $m \in \mathbb{N}$, if its pdf is

$$X \begin{pmatrix} k \\ \frac{1}{m} \end{pmatrix}_{k=\overline{1,m}}, \tag{5.2}$$

with mean and variance

$$\begin{aligned} E(X) &= \frac{m+1}{2}, \\ V(X) &= \frac{m^2-1}{12}. \end{aligned}$$

The random variable that denotes the face number shown on a die when it is rolled, has a Discrete Uniform distribution $U(6)$.

Binomial Distribution $B(n, p)$

A random variable X has a Binomial distribution ($\boxed{\text{bino}}$) with parameters $n \in \mathbb{N}$ and $p \in (0,1)$ ($q = 1-p$), if its pdf is

$$X \begin{pmatrix} k \\ C_n^k p^k q^{n-k} \end{pmatrix}_{k=\overline{0,n}}, \tag{5.3}$$

with

$$\begin{aligned} E(X) &= np, \\ V(X) &= npq. \end{aligned}$$

This distribution corresponds to the Binomial model. Given n Bernoulli trials with probability of success p, let X denote the number of successes. Then $X \in B(n, p)$. Also, notice that the Bernoulli distribution is a particular case of the Binomial one, for $n = 1$, $Bern(p) = B(1, p)$.

Hypergeometric Distribution $H(N, n_1, n)$

A random variable X has a Hypergeometric distribution ($\boxed{\text{hyge}}$) with parameters $N, n_1, n \in \mathbb{N}$ ($n, n_1 \leq N$), if its pdf is

$$X \left(\begin{array}{c} k \\ \dfrac{C_{n_1}^k C_{N-n_1}^{n-k}}{C_N^n} \end{array} \right)_{k=\overline{0,n}}, \tag{5.4}$$

with

$$\begin{aligned} E(X) &= \frac{nn_1}{N}, \\ V(X) &= \frac{nn_1(N-n_1)(N-n)}{N^2(N-1)}. \end{aligned}$$

This distribution corresponds to the Hypergeometric model. If X is the number of successes in a Hpergeometric model, then $X \in H(N, n_1, n)$.

Geometric Distribution $Geo(p)$

A random variable X has a Geometric distribution ($\boxed{\text{geo}}$) with parameter $p \in (0,1)$ ($q = 1-p$), if its pdf is given by

$$X \left(\begin{array}{c} k \\ pq^k \end{array} \right)_{k=0,1,\ldots}. \tag{5.5}$$

Its expectation and variance are given by

$$\begin{aligned} E(X) &= \frac{q}{p}, \\ V(X) &= \frac{q}{p^2}. \end{aligned}$$

If X denotes the number of failures that occurred before the occurrence of the 1^{st} success in a Geometric model, then $X \in Geo(p)$.

Negative Binomial (Pascal) Distribution $NB(n, p)$

A random variable X has a Negative Binomial (Pascal) ($\boxed{\text{nbin}}$) distribution with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ ($q = 1 - p$), if its pdf is

$$X \left(\begin{array}{c} k \\ C_{n+k-1}^{k} p^n q^k \end{array} \right)_{k=0,1,\dots}. \tag{5.6}$$

Then

$$\begin{aligned} E(X) &= \frac{nq}{p}, \\ V(X) &= \frac{nq}{p^2}. \end{aligned}$$

This distribution corresponds to the Negative Binomial model. If X denotes the number of failures that occurred before the occurrence of the n^{th} success in a Negative Binomial model, then $X \in NB(n, p)$. It is a generalization of the Geometric distribution, $Geo(p) = NB(1, p)$.

Poisson Distribution $\mathcal{P}(\lambda)$

A random variable X has a Poisson distribution ($\boxed{\text{poiss}}$) with parameter $\lambda > 0$, if its pdf is

$$X \left(\begin{array}{c} k \\ \dfrac{\lambda^k}{k!} e^{-\lambda} \end{array} \right)_{k=0,1,\dots} \tag{5.7}$$

with

$$E(X) = V(X) = \lambda.$$

A Poisson r.v. **<u>does not</u>** come from the Poisson model! Poisson random variables arise in connection with so-called Poisson *processes,* processes that involve observing discrete events in a continuous interval of <u>time</u>, length, space, etc. The variable of interest in a Poisson process, X, represents the number of occurrences of the discrete event in a fixed interval of time, length, space. For instance, the number of gas emissions taking place at a nuclear plant in a 3-month period, the number of earthquakes hitting a certain area in a year, the number of white blood cells in a drop of blood, all these are modeled by Poisson random variables. The parameter λ of a Poisson distribution represents the average number of occurrences of the event in question in that interval (of time, length, space, etc.).

Poisson's distribution is also known as the "law of rare events", the name coming from the fact that

$$\lim_{k\to\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 0,$$

i.e. as k gets larger, the event $(X = k)$ becomes less probable, more "rare". The discrete events that are counted in a Poisson process are also called "rare events".

Remark 5.1.
1. The sum of n independent $Bern(p)$ random variables is a $B(n,p)$ variable.
2. The sum of n independent $Geo(p)$ random variables is a $NB(n,p)$ variable.

5.2 Common Continuous Distributions

Uniform Distribution $U(a, b)$

A random variable X has a Uniform distribution ($\boxed{\text{unif}}$)with parameters $a, b \in \mathbb{R},\ a < b$, if its pdf is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } x \in [a,b] \\ 0, & \text{if } x \notin [a,b]. \end{cases} \tag{5.8}$$

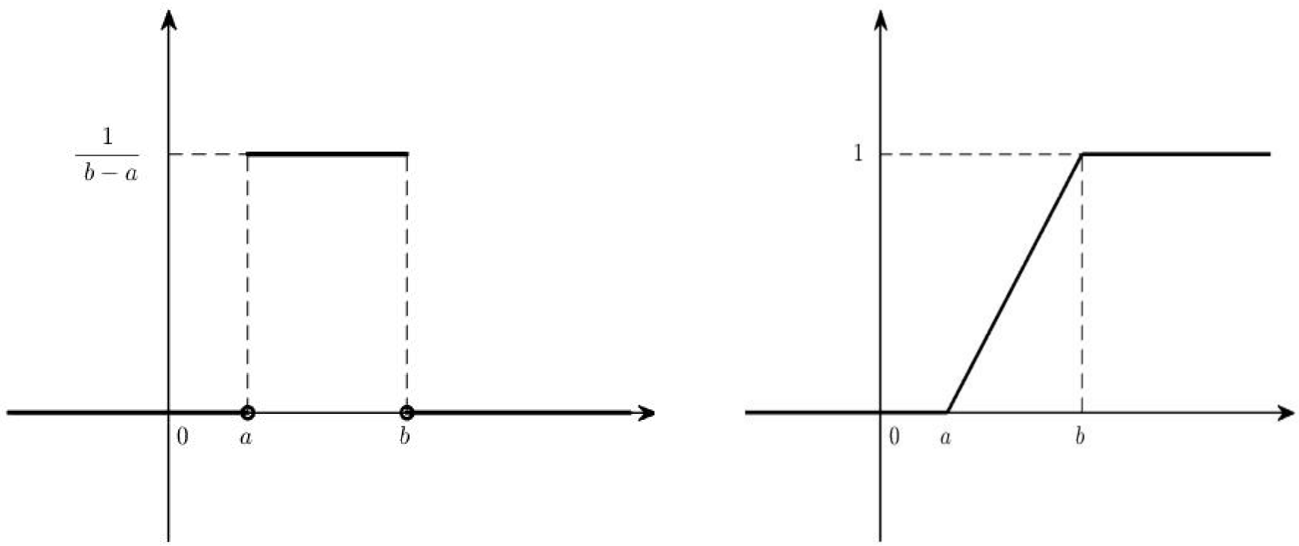
Then its cdf is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & \text{if } x \le a \\ \dfrac{x-a}{b-a}, & \text{if } a < x \le b \\ 1, & \text{if } x \ge b \end{cases} \tag{5.9}$$

and its numerical characteristics are

$$\begin{aligned} E(X) &= \frac{a+b}{2}, \\ V(X) &= \frac{(b-a)^2}{12}. \end{aligned}$$

The Uniform distribution is used when a variable can take *any* value in a given interval, equally probable. For example, locations of syntax errors in a program, birthdays throughout a year, arrival times of customers, etc.



(a) Density Function (pdf) (b) Cumulative Distribution Function (cdf)

Fig. 1: Uniform Distribution

A special case is that of a **Standard Uniform Distribution**, where $a = 0$ and $b = 1$. The pdf and cdf are given by

$$f_U(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases} \quad , \quad F_U(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x \leq 1 \\ 1, & x \geq 1 \,. \end{cases} \tag{5.10}$$

Standard Uniform variables play an important role in stochastic modeling; in fact, *any* random variable, with any thinkable distribution (discrete or continuous) can be generated from Standard Uniform variables.

Normal Distribution $N(\mu, \sigma)$

The Normal distribution is one of the most important distributions, underlying many of the modern statistical methods used in data analysis. It was first described in the late 1700's by De Moivre, as a limiting case for the Binomial distribution (when n, the number of trials, becomes infinite), but did not get much attention. Half a century later, both Laplace and Gauss (independently of each other) rediscovered it in conjunction with the behavior of errors in astronomical measurements. It is also referred to as the "Gaussian" distribution.

A random variable X has a Normal distribution ($\boxed{\text{norm}}$) with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, if its pdf is

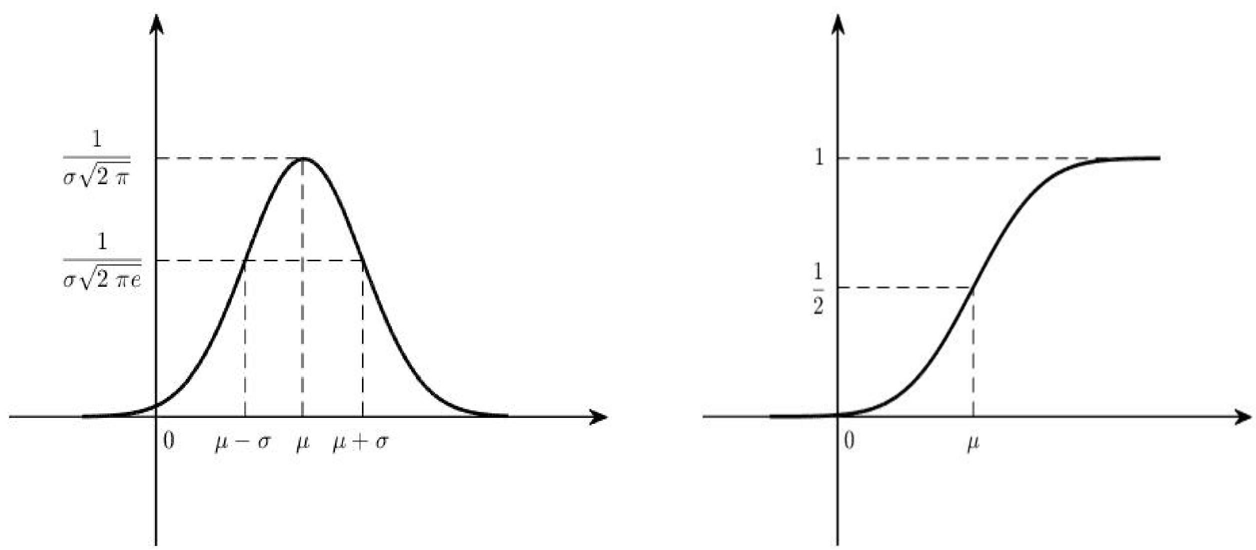
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}. \tag{5.11}$$

The cdf of a Normal variable is then given by

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\dfrac{(t-\mu)^2}{2\sigma^2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\dfrac{t^2}{2}} dt \tag{5.12}$$

and its mean and variance are

$$\begin{aligned} E(X) &= \mu, \\ V(X) &= \sigma^2. \end{aligned}$$



(a) Density Function (pdf)

(b) Cumulative Distribution Function (cdf)

Fig. 2: Normal Distribution

The graph of the Normal density is a symmetric, bell-shaped curve (known as "Gauss's bell" or "Gauss's bell curve") centered at the value of the first parameter μ, as can be seen in Figure 2(a).

The graph of the cdf of a Normally distributed random variable is given in Figure 2(b) and this is approximately what the graph of the cdf of *any* continuous random variable looks like.

There is an important particular case of a Normal distribution, namely $N(0,1)$, called the **Standard (or Reduced) Normal Distribution**. A variable having a Standard Normal distribution is usually denoted by Z. The density and cdf of Z are given by

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R} \quad \text{and} \quad F_Z(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt. \tag{5.13}$$

The function F_Z given in (5.13) is known as *Laplace's function* and its values can be found in tables or can be computed by any mathematical software. One can notice that there is a relationship between the cdf of any Normal $N(\mu, \sigma)$ variable X and that of a Standard Normal variable Z, namely,

$$F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right).$$

Exponential Distribution $Exp(\lambda)$

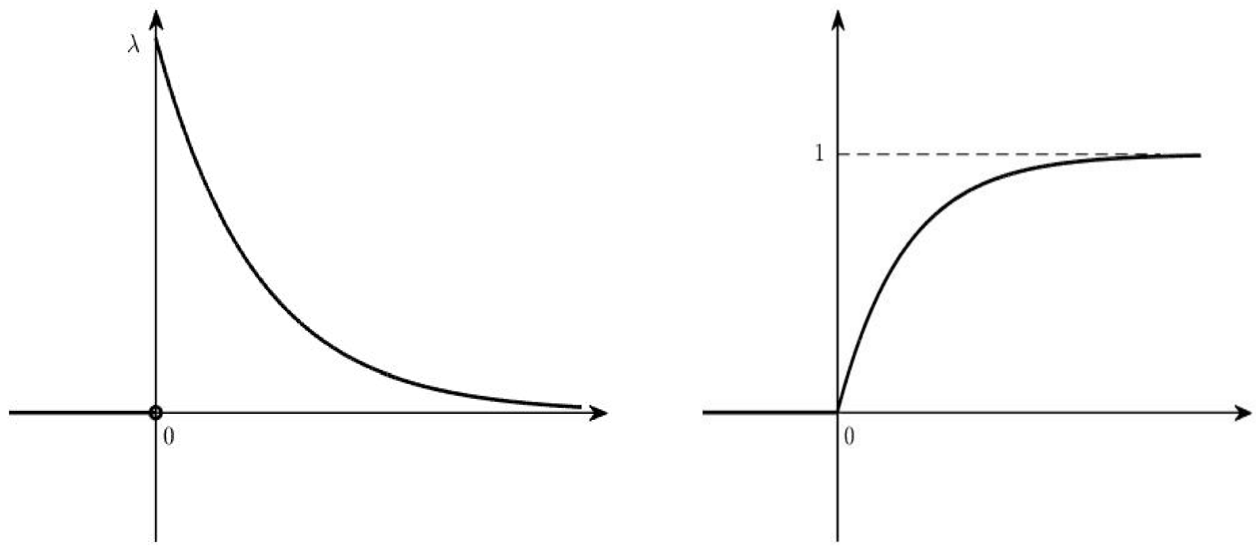
A random variable X has an Exponential distribution ($\boxed{\text{exp}}$) with parameter $\lambda > 0$, if its pdf and cdf are given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad \text{and} \quad F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \tag{5.14}$$

respectively.

Remark 5.2.
1. The Exponential distribution is often used to model *time*: lifetime, waiting time, halftime, interarrival time, failure time, time between rare events, etc. In a sequence of rare events (where the number of rare events has a Poisson distribution), the time between two consecutive rare events is Exponential. The parameter λ represents the frequency of rare events, measured in time^{-1}.
2. A word of **caution** here: The parameter μ in Matlab (where the Exponential pdf is defined as $\frac{1}{\mu} e^{-\frac{1}{\mu}x}, x \geq 0$) is actually $\mu = 1/\lambda$. It all comes from the different interpretation of the "frequency". For instance, if the frequency is "2 per hour", then $\lambda = 2/\text{hr}$, but this is equivalent to " one every half an hour"', so $\mu = 1/2$ hours. The parameter μ is measured in time units.
3. The Exponential distribution is a special case of a more general distribution, namely the $Gamma(a,b)$, $a, b > 0$, distribution ($\boxed{\text{gam}}$). The Gamma distribution models the *total* time of a

multistage scheme.
4. If $\alpha \in \mathbb{N}$, then the sum of α independent $Exp(\lambda)$ variables has a $Gamma(\alpha, 1/\lambda)$ distribution.



(a) Density Function (pdf) (b) Cumulative Distribution Function (cdf)

Fig. 3: Exponential Distribution

Remark 5.3. In Statistics, the most widely used distributions are the following:
– the Normal distribution, $N(\mu, \sigma)$, especially $N(0, 1)$ ($\boxed{\text{norm}}$),
– the Student (T) distribution, $T(n)$ ($\boxed{\text{t}}$),
– the χ^2 distribution, $\chi^2(n)$ ($\boxed{\text{chi2}}$),
– the Fisher (F) distribution, $F(m, n)$ ($\boxed{\text{f}}$).