11.01.2022

Seminar W13 - 831

H_0 : the defendant is not guilty

H_1: the defendant is guilty

error of type I : rejecting H_0, while H_0 is true
"sending an innocent person to jail"

error of type II : accepting H_0, while H_1 is true
"setting a guilty person free"

$$\alpha = P(\text{error of type I}) = P(\text{reject } H_0 \mid H_0)$$

$$\beta = P(\text{error of type II}) = P(\text{accept } H_0 \mid H_1)$$

Theorem (Neyman-Pearson). *Let X be a characteristic with pdf $f(x;\theta)$, with $\theta \in A \subseteq \mathbb{R}$ unknown. Suppose we test on θ the simple hypotheses:*

$$H_0 : \theta = \theta_0$$

$$H_1 : \theta = \theta_1$$

based on a random sample $X_1, X_2, \dots, X_n$. Then for a fixed $\alpha \in (0,1)$, a ***most powerful test*** *is the test with rejection region given by*

$$RR = \left\{ \frac{L(\theta_1)}{L(\theta_0)} \geq k_\alpha \right\}$$

where the constant $k_\alpha > 0$ depends only on α and the sample variables.

Exercise 1. Let X be a characteristic that has a normal distribution $N(m, 1)$, where $m \in \mathbb{R}$ is unknown.

(a) For a random sample of size 9 for X, find a most powerful test with the significance level 5% for the hypothesis $H_0 : m = 0$ against $H_1 : m = 1$. Find the power of this test.

(b) For the following sample data of X:

$$-1, 0.5, -0.25, -0.75, 0.75, 1.25, 0.5, 1, 0.25$$

accept or reject the hypothesis $H_0 : m = 0$ against $H_1 : m = 1$, using the obtained test.

Sol.:

(a) $n = 9$ $\alpha = 0.05$ $H_0 : m = 0$

$H_1 : m = 1$

$X \sim N(m, 1)$

By the (NPL) we know that the most powerful test for this parameter has the rejection region:

$$RR = \left\{ (x_1, \dots, x_9) \;\middle|\; \frac{L(x_1, x_2, \dots, x_9 ; 1)}{L(x_1, x_2, \dots, x_9 ; 0)} \geq k_\alpha \right\}$$

$$L(x_1, x_2, \dots, x_9 ; m) = \prod_{i=1}^{9} f_X(x_i)$$

$$X \sim N(m, 1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-m)^2}{2}}$$

$$L(x_1, \dots, x_9 ; m) = \left(\frac{1}{\sqrt{2\pi}}\right)^9 \cdot e^{-\frac{1}{2} \sum_{i=1}^{9} (x_i - m)^2}$$

$$RR = \left\{ (x_1, \ldots, x_9) \;\middle|\; \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^9 \cdot e^{-\frac{1}{2}\sum_{i=1}^{9}(x_i-1)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^9 \cdot e^{-\frac{1}{2}\sum_{i=1}^{9} x_i^2}} \geq k_\alpha \right\} =$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; e^{\frac{1}{2}\sum_{i=1}^{9}\left(x_i^2 - (x_i-1)^2\right)} \geq k_\alpha \right\} =$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; \frac{1}{2}\sum_{i=1}^{9}(2x_i - 1) \geq \ln(k_\alpha) \right\} =$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; x_1 + \ldots + x_9 - \frac{9}{2} \geq \ln(k_\alpha) \right\} =$$

$$= \left\{ (x_1, \ldots, x_9) \;\middle|\; x_1 + \ldots + x_9 \geq \underbrace{\frac{9}{2} + \ln(k_\alpha)}_{= K_\alpha} \right\}$$

$$\alpha = P\left((X_1, \ldots, X_9) \in RR \mid H_0 \right).$$

$$X_i \sim N(m, 1) \Rightarrow \underbrace{X_1 + X_2 + \ldots + X_9}_{= Y} \sim N(9m, 1 \cdot \sqrt{9})$$

$$\alpha = P\left((X_1, \ldots, X_9) \in RR \mid H_0 \right) = P\left(Y \geq K_\alpha \mid m = 0 \right) =$$

$$= P\left(Y \geq K_\alpha \mid Y \sim N(0, 3) \right) =$$

$$= 1 - P\left(Y < K_\alpha \mid Y \sim N(0,3) \right) =$$

$$= 1 - F_Y(K_\alpha)$$

$$\Rightarrow F_Y(K_\alpha) = 1 - \alpha \Rightarrow K_\alpha = \mathrm{norminv}(1-\alpha, 0, 3)$$

$$= \mathrm{norminv}(0.95, 0, 3) = 4.9346$$

$$\Rightarrow RR = \{ (x_1, \dots, x_9) \mid x_1 + x_2 + \dots + x_9 \geq 4.9346 \}$$

$$\pi(1) = 1 - \beta(1) = 1 - P(\text{accept } H_0 \mid H_1) = P(\text{reject } H_0 \mid H_1) =$$

$$= P((X_1, \dots, X_9) \in RR \mid H_1)$$

If H_1 is true, then $m = 1 \Rightarrow Y \sim N(9, 3)$

$$\Rightarrow \pi(1) = P(Y \geq 4.9346 \mid Y \sim N(9,3)) =$$

$$= 1 - P(Y < 4.9346 \mid Y \sim N(9,3)) =$$

$$= 1 - F_Y(4.9346) = 1 - \text{normcdf}(4.9346, 9, 3) =$$

$$\approx 1 - 0.087 = 0.913$$

(b) $y = x_1 + x_2 + \dots + x_9 = 2.25$

$$RR = \{ (x_1, \dots, x_9) \mid x_1 + \dots + x_9 \geq 4.9346 \}$$

$2.25 \ngeq 4.9346 \Rightarrow (x_1, \dots, x_9) \notin RR \Rightarrow$ we accept H_0

Exercise 2. Let X be a characteristic that has a Bernoulli distribution with parameter p, where $p \in (0,1)$ is unknown.

(a) For a random sample of (large) size n for X, find a most powerful test with the significance level 5% for the hypothesis $H_0 : p = 0.5$ against $H_1 : p = 0.45$.

(b) Let p be the probability of heads for a coin. Assume we toss the coin 900 times and we get 441 heads. Accept or reject the hypothesis $H_0 : p = 0.5$ against $H_1 : p = 0.45$, using the obtained test.

Sol:

$$X \sim \text{Bern}(p) \qquad X \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

$$f_X(x) = (1-p)^{1-x} \cdot p^x$$

n large $\qquad \alpha = 0.05$

$$H_0 : p = 0.5$$
$$H_1 : p = 0.45$$

$$L(x_1, x_2, \dots, x_n ; p) = \prod_{i=1}^{n} f_X(x_i) = \prod_{i=1}^{n} (1-p)^{1-x_i} \cdot p^{x_i} =$$

$$= (1-p)^{n-(x_1+x_2+\dots+x_n)} \cdot p^{x_1+x_2+\dots+x_n}$$

CLT (central limit theorem): For large:

$$\overline{X_n} := \frac{X_1+X_2+\dots+X_n}{n} \Rightarrow \frac{\overline{X_n} - E(X)}{\frac{\sigma_X}{\sqrt{n}}} \sim N(0,1)$$

$$L(X_1,\dots,X_n;p) = (1-p)^{n\cdot(1-\overline{X_n})} \cdot p^{n\cdot\overline{X_n}}$$

From NPL:

$$RR = \left\{(x_1,\dots,x_n) \;\middle|\; \frac{L(x_1,\dots,x_n;0.45)}{L(x_1,\dots,x_n;0.5)} \geq k_\alpha\right\} =$$

$$= \left\{(x_1,\dots,x_n) \;\middle|\; \frac{0.55^{n(1-\overline{x_n})}\cdot 0.45^{n\cdot\overline{x_n}}}{0.5^{n(1-\overline{x_n})}\cdot 0.5^{n\overline{x_n}}} \geq k_\alpha\right\} =$$

$$= \left\{(x_1,\dots,x_n) \;\middle|\; \frac{0.55^n\cdot\left(\frac{0.45}{0.55}\right)^{n\overline{x_n}}}{0.5^n} \geq k_\alpha\right\} =$$

$$= \left\{(x_1,\dots,x_n) \;\middle|\; n\ln(0.55) + n\cdot\overline{x_n}\cdot\ln\left(\frac{0.45}{0.55}\right) - n\ln(0.5) \geq k_\alpha\right\}$$

$$= \left\{(x_1,x_2,\dots,x_n) \;\middle|\; n\overline{x_n}\cdot\ln\left(\frac{45}{55}\right) \geq k_\alpha + n\ln(0.5) - n\ln(0.55)\right\}$$

$$= \left\{(x_1,\dots,x_n) \;\middle|\; \overline{x_n} \geq \underbrace{\frac{1}{n\ln\frac{45}{55}}\left(k_\alpha + n\ln(0.5) - n\ln(0.55)\right)}_{=:K_\alpha}\right\}$$

$$X \sim Bern(p) \Rightarrow E(X) = 0\cdot(1-p) + 1\cdot p = p$$

$$\sigma_X = \sqrt{V(X)} = \sqrt{E(X^2) - E(X)^2}$$

$$X\begin{pmatrix} 0 & 1 \\ 1-p & p\end{pmatrix} \Rightarrow X^2\begin{pmatrix} 0 & 1 \\ 1-p & p\end{pmatrix} \Rightarrow E(X^2) = p$$

$$\Rightarrow \sigma_X = \sqrt{p-p^2} = \sqrt{p(1-p)}$$

$$\frac{\overline{X}_n - E(X)}{\frac{\sigma_X}{\sqrt{n}}} = \frac{\overline{X}_n - p}{\frac{\sqrt{p(1-p)}}{\sqrt{n}}} \overset{\text{by the CLT (central limit theorem)}}{\sim} \mathcal{N}(0,1)$$

$$RR = \{(x_1, \dots, x_n) \mid \overline{x}_n \geq K_\alpha\} =$$

$$= \left\{(x_1, \dots, x_n) \;\middle|\; \frac{\overline{x}_n - p}{\frac{\sqrt{p(1-p)}}{\sqrt{n}}} \geq \underbrace{\frac{K_\alpha - p}{\frac{\sqrt{p(1-p)}}{\sqrt{n}}}}_{=:\, K_\alpha'}\right\}$$

$$Z := \frac{\overline{X}_n - p}{\frac{\sqrt{p(1-p)}}{\sqrt{n}}} \sim \mathcal{N}(0,1)$$

$$RR = \{(x_1, \dots, x_n) \mid Z \geq K_\alpha'\}$$

$$\alpha = P((x_1, \dots, x_n) \in RR \mid H_0)$$

If H_0 true $\Rightarrow$ $p = 0.5$

$$\Rightarrow \alpha = P(Z \geq K_\alpha' \mid p = 0.5) =$$

$$= 1 - P(Z < K_\alpha') =$$

$$= 1 - F_Z(K_\alpha') \Rightarrow F_Z(K_\alpha') = 1 - \alpha$$

$$\Rightarrow \underset{= 1.6449}{K_\alpha'} = F_Z^{-1}(1-\alpha) = \text{normcdf}(0.95, 0, 1)$$

$$\Rightarrow RR = \{ (x_1, \dots, x_n) \mid Z \geq 1.649 \}$$

$$\pi(0.45) = 1 - \beta(0.45) = 1 - P(\text{accept } H_0 \mid H_1) =$$

$$= P(\text{reject } H_0 \mid H_1) = P((x_1, \dots, x_n) \in RR \mid p = 0.45) =$$

$$= P(Z \geq 1.649 \mid p = 0.45) =$$

$$= 1 - P(Z < 1.649) = 1 - F_Z(1.649) =$$

$$= 1 - \text{normcdf}(1.649, 0, 1) = 1 - 0.95 = 0.05$$

(b) $n = 900$ $\qquad \overline{X_n} = \dfrac{x_1 + \dots + x_n}{n} = \dfrac{441}{900}$

$$z = \frac{\overline{x_n} - p}{\frac{\sqrt{p(1-p)}}{\sqrt{n}}} = \frac{\frac{441}{900} - 0.5}{\frac{0.5}{\sqrt{900}}} = \frac{\frac{441}{900} - \frac{450}{900}}{\frac{0.5}{30}} =$$

$$= \frac{\frac{-9}{900}}{\frac{1}{60}} = \frac{-9}{15} = -\frac{3}{5} = -\frac{6}{10} = -0.6$$

$-0.6 \not\geq 1.649 \Rightarrow (x_1, \dots, x_n) \notin RR \Rightarrow$ we accept $H_0 \Rightarrow p = 0.5$