02.11.2021

Seminar WG-832

$(X_n)_{n\in\mathbb{N}}$ sequence of random variable that are i.i.d

$X_n \sim X$, X random variable

Fix $n\in\mathbb{N}$

$h: \mathbb{R}^n \to \mathbb{R}$ measurable function

$$h = h(X_1, X_2, \dots, X_n)$$

Ex:

$$\overline{X} = \frac{1}{n}(X_1 + \dots + X_n)$$

$$\overline{\mu_k} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^k$$

$$\overline{\nu_k} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

Exercise 1. Let $X_1, X_2, \dots, X_n, \dots$ be i.i.d. (independent identically distributed) random variables that follow the normal distribution, $X \sim \mathcal{N}(\mu, \sigma)$. Find the constant k_n such that the sampling function

$$\overline{s} = k_n \sum_{j=1}^{n} |X_j - \overline{X}|$$

verifies $E(\overline{s}) = \sigma$.

Sol: $E(\overline{s}) = E\left(k_n \cdot \sum_{j=1}^{n}|X_j - \overline{X}|\right) = k_n \cdot \sum_{j=1}^{n} E(|X_j - \overline{X}|) =$

$= k_n \cdot n \cdot E(|X_j - \overline{X}|)$

Let $Y := X_j - \overline{X} = X_j - \frac{1}{n}(X_1 + \dots + X_n) =$

$= \frac{n-1}{n} X_j + \left(-\frac{1}{n}\right)X_1 + \left(-\frac{1}{n}\right)X_2 + \dots + \left(-\frac{1}{n}\right)X_{j-1} + \left(-\frac{1}{n}\right)X_{j+1} +$
$+ \dots + \left(-\frac{1}{n}\right)X_n$

$$X_j \sim \mathcal{N}(\mu, \sigma)$$

$$\overline{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\Rightarrow \quad Y \sim \mathcal{N}(\mu', \sigma')$$

$$E(\overline{S}) = k_n \cdot n \cdot E(|Y|)$$

$$E(Y) = E(X_j - \overline{X}) = E(X_j) - E(\overline{X}) = E(X) - E(\overline{X}) =$$

$$= \mu - \mu = 0$$

$$Var(Y) = Var(X_j - \overline{X}) = Var\left(X_j - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)\right) =$$

$$= Var\left(X_j - \frac{1}{n}X_j - \frac{1}{n}\sum_{\substack{i=1 \\ i \neq j}}^{n} X_i\right) = Var\left(\frac{n-1}{n} \cdot X_j - \frac{1}{n}\sum_{\substack{i=1 \\ i \neq j}}^{n} X_i\right) =$$

$$= Var\left(\frac{n-1}{n} X_j\right) + \left(-\frac{1}{n}\right)^2 \cdot \sum_{\substack{i=1 \\ i \neq j}}^{n} \underbrace{Var(X_i)}_{= Var(X) = \sigma^2} =$$

$$= \left(\frac{n-1}{n}\right)^2 \cdot \underbrace{Var(X_j)}_{=\sigma^2} + \frac{1}{n^2} \cdot (n-1) \cdot \sigma^2 =$$

$$= \sigma^2 \cdot \frac{n-1}{n^2}(n-1+1) = \sigma^2 \cdot \frac{n-1}{n}$$

$$\Rightarrow \quad \sigma_Y = \sigma\sqrt{\frac{n-1}{n}} \quad \Rightarrow \quad Y \sim \mathcal{N}\left(0, \sigma\sqrt{\frac{n-1}{n}}\right)$$

$$E(|Y|) = \int_{\mathbb{R}} |y| \cdot f_Y(y)\, dy$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \; x \in \mathbb{R} \qquad X \sim \mathcal{N}(\mu, \sigma)$$

$$Y \sim \mathcal{N}\left(0, \sigma\sqrt{\tfrac{n-1}{n}}\right)$$

$$f_Y(y) = \frac{1}{\sigma\sqrt{\frac{n-1}{n}} \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-0)^2}{2 \cdot \sigma^2 \cdot \frac{n-1}{n}}} = \frac{1}{\sigma\sqrt{2\pi}} \cdot \sqrt{\frac{n}{n-1}} \cdot e^{-\frac{x^2}{2\sigma^2} \cdot \frac{n}{n-1}}$$

$$E(|Y|) = \int_{\mathbb{R}} |y| \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \sqrt{\frac{n}{n-1}} \cdot e^{-\frac{y^2}{2\sigma^2} \cdot \frac{n}{n-1}} \cdot dy =$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \sqrt{\frac{n}{n-1}} \left(\underbrace{\int_{-\infty}^{0} (-y) \cdot e^{-\frac{y^2}{2\sigma^2} \cdot \frac{n}{n-1}} dy}_{y=-z \quad \int_0^\infty z\, e^{\frac{-z^2}{2\sigma^2} \cdot \frac{n}{n-1}} \cdot dz} + \int_0^\infty y\, e^{-\frac{y^2}{2\sigma^2} \cdot \frac{n}{n-1}} dy \right) =$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \sqrt{\frac{n}{n-1}} \cdot 2 \cdot \int_0^\infty \underbrace{y \cdot e^{\frac{-y^2}{2\sigma^2} \cdot \frac{n}{n-1}}} dy$$

$$\left(e^{-\frac{y^2}{2\sigma^2} \cdot \frac{n}{n-1}}\right)' = e^{\frac{-y^2}{2\sigma^2} \cdot \frac{n}{n-1}} \cdot \left(-\frac{2y}{2\sigma^2}\right) \cdot \frac{n}{n-1}$$

$$E(|Y|) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \sqrt{\frac{n}{n-1}} \cdot 2 \cdot \left(\underbrace{e^{\frac{-y^2}{2\sigma^2} \cdot \frac{n}{n-1}} \Big|_0^\infty}_{=-1} \right) \cdot \frac{n-1}{n} \cdot (-\sigma^2) =$$

$$= \frac{\sqrt{2}\,\sigma \cdot \sqrt{\frac{n-1}{n}}}{\sqrt{\pi}}$$

$$E(\bar{S}) = k_n \cdot n \cdot \frac{\sqrt{2} \cdot \sigma \cdot \sqrt{\frac{n-1}{n}}}{\sqrt{\pi}}$$

$$E(\bar{S}) = \sigma \Rightarrow k_n = \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{n}{n-1}} \cdot \frac{1}{n}$$

Exercise 2. (a) Let $X \sim \chi^2(n)$. Find the probability density function of the random variable $Y = \sqrt{\frac{X}{n}}$.

(b) Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. (independent identically distributed) random variables that follow the distribution $X \sim \mathcal{N}(\mu, \sigma)$. Find the constant k_n such that the sampling function

$$\overline{s} = k_n \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

verifies $E(\overline{s}) = \sigma$.

Recap.

- *If $X_1, \ldots, X_n$ are independent identically distributed random variables with distribution given by $X \sim \mathcal{N}(\mu, \sigma)$ and we define the statistic:*

$$V := \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{(n-1)s^2}{\sigma^2}$$

then

$$V \sim \chi^2(n-1)$$

- *If $X \sim \chi^2(n)$, then:*

$$\boxed{f_X(x) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \; x > 0}$$

where for every $\alpha > 0$ we have:

$$\Gamma(\alpha) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

Sol: (a) In order to find f_Y, we find F_Y first (the cdf)

$$F_Y(x) = P(Y \le x) = P\left(\sqrt{\frac{X}{n}} \le x\right) = P(X \le nx^2) =$$

$$= F_X(nx^2)$$

$$\Rightarrow \quad f_Y(x) = \begin{cases} 0, & x < 0 \\ \frac{d}{dx} F_X(nx^2), & x \ge 0 \end{cases} =$$

$$= \begin{cases} 0, & x < 0 \\ f_X(nx^2) \cdot 2nx, & x \ge 0 \end{cases}$$

$$f_X(x) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \; x > 0$$

$$\Rightarrow \quad f_Y(x) = \begin{cases} 0, & x < 0 \\ 2nx \cdot \frac{1}{2^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot (nx^2)^{\frac{n}{2}-1} \cdot e^{\frac{-nx^2}{2}}, & x \ge 0 \end{cases} =$$

$$= \begin{cases} 0, & x < 0 \\ n^{n/2} \cdot x^{2n-1} \cdot e^{\frac{-nx^2}{2}} \cdot 2^{1-\frac{n}{2}} \cdot \frac{1}{\Gamma\left(\frac{n}{2}\right)}, & x \ge 0 \end{cases}$$

(b) $\bar{S} = k_n \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (X_i - \bar{X})^2}$

Let $V = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \Rightarrow V \sim \chi^2(n-1)$

$$\bar{S} = k_n \cdot \sqrt{\frac{\sigma^2}{n-1} \cdot V} = \sigma k_n \cdot \underbrace{\sqrt{\frac{V}{n-1}}}_{W}$$

$V \sim \chi^2(n) \overset{(a)}{\Rightarrow} W$ has the density function:

$$f_W(x) = \begin{cases} 0, & x < 0 \\ (n-1)^{\frac{n-1}{2}} \cdot x^{2n-3} \cdot e^{\frac{-(n-1)x^2}{2}} \cdot 2^{1-\frac{n-1}{2}} \cdot \frac{1}{\Gamma(\frac{n-1}{2})} \end{cases}$$

$$E(\bar{S}) = E(\sigma k_n W) = \sigma k_n \cdot E(W)$$

$$E(W) = \int_{\mathbb{R}} x \cdot f_W(x)\, dx = \int_0^\infty x \cdot (n-1)^{\frac{n-1}{2}} \cdot x^{2n-3} \cdot e^{\frac{-(n-1)x^2}{2}} \cdot 2^{1-\frac{n-1}{2}} \cdot \frac{1}{\Gamma(\frac{n-1}{2})}\, dx =$$

$$= (n-1)^{\frac{n-1}{2}} \cdot 2^{1-\frac{n-1}{2}} \cdot \frac{1}{\Gamma(\frac{n-1}{2})} \underbrace{\int_0^\infty x^{2n-2} \cdot e^{\frac{-(n-1)x^2}{2}}\, dx}_{I}$$

$$\frac{(n-1) \cdot x^2}{2} = y \Rightarrow x = \sqrt{\frac{2y}{n-1}} \Rightarrow dx = \sqrt{\frac{2}{n-1}} \cdot \frac{1}{2\sqrt{y}}$$

$$I = \int_0^\infty \left(\frac{2y}{n-1}\right)^{\frac{1}{2} \cdot (2n-2)} \cdot e^{-y} \cdot \sqrt{\frac{2}{n-1}} \cdot \frac{1}{2\sqrt{y}}\, dy =$$

$$= \int_0^\infty \frac{2^{n-1} \cdot y^{n-1}}{(n-1)^{n-1}} \cdot e^{-y} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n-1}} \cdot \frac{1}{\sqrt{y}}\, dy =$$

$$= \frac{2^{n-1}}{(n-1)^{n-1}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n-1}} \cdot \underbrace{\int_0^\infty y^{n-\frac{3}{2}} \cdot e^{-y} \, dy}_{= \Gamma\left(\underbrace{n-\frac{1}{2}}_{\Gamma\left(\frac{2n-1}{2}\right)}\right)}$$