13) The Heat equation with diffusion coefficient $\mu \neq 1$.
Find the representation formula for the solution of

$$\begin{cases} u_t = \mu u_{xx}, \ \mu > 0, x \in \mathbb{R}, t > 0 \\ u(0) = u_0 \end{cases}$$

and give an interpretation to the $\mu \ll 1$ small diffusion case

We scale time by $\mu \Rightarrow \begin{cases} \frac{d}{dt} u(t) = \mu \Delta u(t) \\ u(0) = u_0 \end{cases}$ becomes

Let $t' = \mu . t$

$\Rightarrow$

$$\begin{cases} \frac{d}{dt'} u(t') = \Delta u(t') \\ u(0) = u_0 \end{cases}$$

Following the $\mathcal{F}$ approach, we will get

$$N(t')(x) = \begin{cases} \frac{1}{(4\pi t')^{n/2}} e^{-\frac{|x|^2}{4t'}} & , t > 0 \\ 0 & , t \leq 0 \end{cases}$$

$$\mathcal{F}(N(t'))(y) = e^{-t'|y|^2}$$

$$\Rightarrow u(t')(x) = (N(t') * u_0)(x) = \frac{1}{(4\pi t')^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t'}} \cdot u_0(y)\, dy$$

$$\Rightarrow u(\mu t)(x) = \frac{1}{(4\pi \mu t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\mu t}} \cdot u_0(y)\, dy$$

$$u(t)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \cdot u_0(y) dy = \mu^{n/2} \text{ something} \cdot u(\mu t)(x)$$

For $u(\mu t)(x)$

$\mu << 1$ (very small) $\Rightarrow \frac{1}{(4\pi\mu t)^{n/2}}$ becomes large
$\mu > 0$

$-\frac{|x-y|^2}{4\mu t}$ gets close to $-\infty \Rightarrow e^{-\frac{|x-y|^2}{4\mu t}}$ gets closer to 0

$\Rightarrow u$ is big ?

Potop Oana
Group 832

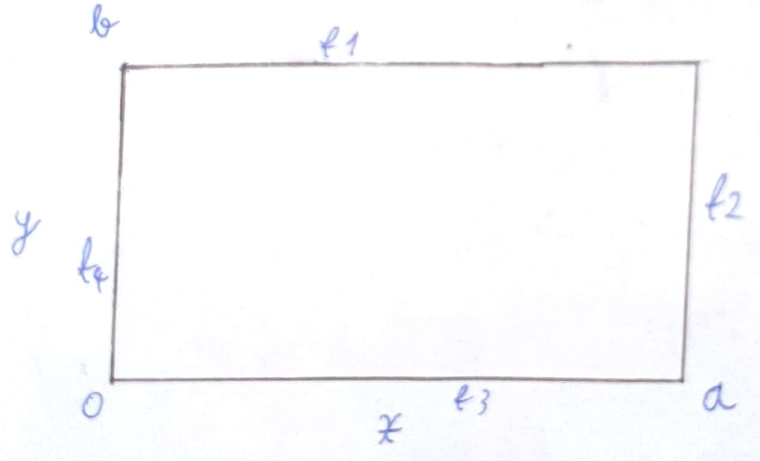
30. Separation of variables II: the Laplace eq. on a ~~...~~ rectangle.

Let $\Omega = (0,a) \times (0, b)$.

$$\begin{cases} \Delta u = 0, \text{ in } \Omega \\ u|_{x=0} = u|_{x=a} = u|_{y=b} = 0, \Rightarrow \\ u|_{y=0} = g(x) \end{cases}$$

$$\Rightarrow \begin{aligned} u|_{x=0} &= u(0,y) = 0 \\ u|_{x=a} &= u(a,y) = 0 \\ u|_{y=b} &= u(x,b) = 0 \\ u|_{y=0} &= u(x,0) = g(x) \end{aligned}$$

$f_1(x) = u(x,b)$, $y = b$
$f_2(y) = u(a,y)$, $x = a$
$f_3(x) = u(x,0)$, $y = 0$
$f_4(y) = u(0,y)$, $x = 0$.



We use the Laplace equation on a rectangle.

$\Delta u = U_{xx} + U_{yy}$
$\Delta u = 0$

$$\left.\begin{aligned} u_{xx} &= \frac{\delta}{\delta x^2} u(x,y) \\ u_{yy} &= \frac{\delta}{\delta y^2} u(x,y) \end{aligned}\right\} \Longrightarrow$$

$u(x,y) = A(x) \cdot B(y)$
(Fourier's separation of variables)
$\Downarrow$

$$\Rightarrow u_{xx} = \frac{\delta}{\delta x^2} A(x) B(y) = A''(x) B(y)$$
$$u_{yy} = \frac{\delta}{\delta y^2} A(x) B(y) = A(x) B''(y)$$

$\oplus$

$\Delta u = A''(x) B(y) + A(x) B''(y) = 0$ / divide by $A(x) B(x)$

(eigenvalue pb.!

pg 1

$$\frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} = 0 \Rightarrow \frac{A''(x)}{A(x)} = -\frac{B''(y)}{B(y)} = 0$$

$$\begin{cases} \frac{-A''(x)}{A(x)} = \lambda \\ \frac{B''(y)}{B(y)} = \lambda \end{cases}$$

$A''(x) = -A(x)\lambda \Leftrightarrow r^2 = -\lambda$

$r^2 = -\lambda \Rightarrow r = \pm i\sqrt{\lambda}$

$\Rightarrow A(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$(A(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x))$

$\alpha = 0$

Lo: $u(0,y) = 0 = A(0)B(y) \Rightarrow A(0) = 0$

$u(a,y) = 0 = A(a)B(y) \Rightarrow A(a) = 0$

$u(x,b) = 0 = A(x)B(b) \Rightarrow B(b) = 0$

$u(x,0) = g(x) = A(x)B(0) = g(x)$

Then we replace $\sqrt{\lambda} x$ above and

$A(0) = C_1 \cos(0) + C_2 \sin(0) = 0.$ $\underline{C_1 = 0}$

$A(x) = C_2 \sin(\sqrt{\lambda} x). \Rightarrow$

$\Rightarrow A(a) = C_2 \sin(\sqrt{\lambda} a) = 0 \Rightarrow$

$\Rightarrow \sqrt{\lambda} a = k\pi, \forall k \in N \Rightarrow$

$\Rightarrow \lambda_k = \left(\frac{k\pi}{a}\right)^2, \forall k \in N$

$\Rightarrow A_k(x) = C_2 \sin\left(\frac{k\pi}{a} x\right)$

$\hookrightarrow A''(x) = -A(x)\lambda$

We need to find the solution for $B''(y) = B(y)\lambda$

$B''(y) = B(y)\lambda \Rightarrow r^2 = \lambda_k$

$r^2 = \left(\frac{k\pi}{a}\right)^2 \Rightarrow r = \pm \frac{k\pi}{a} \Rightarrow$

$\Rightarrow B_k(y) = C_3 e^{\frac{k\pi}{a}y} + C_4 e^{-\frac{k\pi}{a}y}$

We replace in the above eq.

$\cos h = \frac{e^x + e^{-x}}{2}$ $\quad \cos h' = \sin h \quad \cos h(0) = 1$

$\sin h = \frac{e^x - e^{-x}}{2}$ $\quad \sin h' = \cos h \quad \sin h(0) = 0$

s.t.

$B_k(y) = C_3 \cos h\left(\frac{k\pi}{a}y\right) + C_4 \sin\left(\frac{k\pi}{a}y\right), \forall k \in \mathbb{N}$

$u(x,y) = A(x)B(y) = \sum_{k=1}^{\infty} A_k(x)B(y) =$

$= \sum_{k=1}^{\infty} C_2 \sin\left(\frac{k\pi}{a}x\right)\left(C_3 \cos h\left(\frac{k\pi}{a}y\right) + C_3 \sin\left(\frac{k\pi}{a}y\right)\right)$

We need to check if our fct. respects the initial boundary conditions.

I. $u(0,y) = 0$, yes, b/c we now know that $u(x,y) = 0$

II. $u(a,y) = 0$, yes, b/c in the above sum the sine $= 0 \Rightarrow$ sum is 0

III $u(x,b) = 0 \Leftrightarrow B(b) = 0$.

$u(x,b) = \sum_{k=1}^{\infty} C_2 \sin\left(\frac{k\pi}{a}x\right)\left(C_3 \cos h\left(\frac{k\pi}{a}b\right) + C_3 \sin h\left(\frac{k\pi}{a}b\right)\right) =$

$= \sum_{k=1}^{\infty} C_2 \sin\left(\frac{k\pi}{a}x\right) \cdot 0 = \underline{0}$

IV $u(x,0) = g(x)$.

$u(x,0) = \sum_{k=1}^{\infty} C_2 \sin\left(\frac{k\pi}{a}x\right)\left(C_3 \cos h\left(\frac{k\pi}{a} \cdot 0\right) + C_4 \sin h\left(\frac{k\pi}{a} \cdot 0\right)\right) \Rightarrow$

$\Rightarrow u(x,0) = \sum_{k=1}^{\infty} C_2 \sin\left(\frac{k\pi}{a}x\right)(C_3 \cdot 1 + C_4 \cdot 0) \Rightarrow u(x,0) = \sum_{k=1}^{\infty} C_2 \cdot C_3 \cdot \sin\left(\frac{k\pi}{a}x\right)$

$= \underline{g(x)}$

Barbu Andreea Maria - 831

14. Positivity, contractivity and energy decay for the Heat eq. Consider the IVP for the Heat eq. (with $\mu = 1$) and assume that the inital temperature profile is continuous $u_0 \in L^2(\mathbb{R}) \cap C(\mathbb{R})$

Prove that:

c) $\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}$

Sol: We know that $u(t) = N(t) \cdot u_0$ and we use the fundamental propertie of convolution:

Let $f \in L^1$, $g \in L^p$ $(1 \leq p \leq \infty)$ then $f * g \in L^p$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$$

So $u(t) = N(t) \cdot u_0 \Rightarrow \|u(t)\|_{L^2(\mathbb{R})} = \|N(t) \cdot u_0\|_{L^2(\mathbb{R})}$

$$\|N(t) \cdot u_0\|_{L^2(\mathbb{R})} \leq \underbrace{\|N(t)\|_{L^1}}_{=1} \cdot \|u_0\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}$$

d) $\frac{\partial}{\partial t} E(u(t)) \leq 0$ for $t > 0$, where $E(u) = \|u_x\|^2_{L^2(\mathbb{R})}$

(Hint: use the fact that $\frac{\partial}{\partial t} \|w(t)\|^2_{L^2(\mathbb{R})} = 2\langle w(t), \frac{\partial w}{\partial t}(t)\rangle_{L^2(\mathbb{R})}$ with $\langle v, w\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} v(x) w(x)\, dx$ the standard L^2 inner product)

Sol: $u_t = u_{xx} \xrightarrow{\cdot u_{xx}} u_t \cdot u_{xx} = \underbrace{(u_{xx})^2}_{\geq 0} \Rightarrow$

$\Rightarrow \int u_t \cdot u_{xx}\, dx \geq \int (u_{xx})^2\, dx \geq 0 \Rightarrow$

$\Rightarrow \int u_t \cdot u_{xx}\, dx \geq 0 \Rightarrow \cancel{\frac{\partial}{\partial t} \int u_{xx}\, dx \geq 0}$

$\Rightarrow \frac{\partial}{\partial t} \int u_{xx} \cdot u\, dx \geq 0 \xrightarrow[\text{by parts}]{\text{integration}} \frac{\partial}{\partial t} u_x u - \frac{\partial}{\partial t} \int (u_x(x))^2\, dx \geq 0$

where $\frac{\partial}{\partial t} u_x u = 0 \Rightarrow -\frac{\partial}{\partial t} \underbrace{\int (u_x(x))^2\, dx}_{E(u(t))} \geq 0 \overset{\cdot(-1)}{\Rightarrow}$ $\quad E(u) = \|u_x\|^2_{L^2(\mathbb{R})}$

$\Rightarrow \boxed{\frac{\partial}{\partial t} E(u(t)) \leq 0}$

(12)

$u_t = -u_{xxx}$

Airy function: $Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{z^3}{3} + xz\right) dz$

The Airy function is the solution of the equation:

$v_{xx} = x \cdot v$

$\Rightarrow \quad \frac{\partial^2}{\partial x} Ai(x) = x \cdot Ai(x)$

We will apply the Fourier transform.

$\mathcal{F}\left(\frac{\partial^2}{\partial x^2} u(t)\right)(y) = \frac{\partial^2}{\partial x^2} \mathcal{F}(u)(y) = -|y|^2 \mathcal{F}(u)(y)$

$\Rightarrow -|y|^2 \mathcal{F}(Ai(x)) = \mathcal{F}(x\, Ai(x))$

$\mathcal{F}(x\, Ai(x)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} Ai(x) \cdot x \cdot e^{-ixy} dy$

$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} Ai(x) \left(-i \frac{\partial}{\partial y} \cdot e^{-ixy}\right) dy$

$= -i \frac{\partial}{\partial y} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} Ai(x) \cdot e^{-ixy} dy}_{= \mathcal{F}(Ai(x))}$

$= -i \frac{\partial}{\partial y} \mathcal{F}(Ai(x))$

$\Rightarrow -|y|^2 \mathcal{F}(A_i(x)) = -i \frac{d}{dy} \mathcal{F}(A_i(x))$

$u_t = -u_{xxx}$

We apply the Fourier Transform

$\frac{\partial}{\partial t} \mathcal{F}(u) = -|y|^3 \mathcal{F}u \Leftrightarrow \frac{\partial}{\partial t} \hat{u} = -|y|^3 \hat{u}$

This has the relation $\hat{u} = e^{-y^3 \cdot t} \hat{u}_0$ and we get:

$\mathcal{F}(u) = e^{-y^3 \cdot t} \mathcal{F}(u_0)$

1/2 p $|-y|^2 \mathcal{F}(A_i(x)) = -i \frac{d}{dy} \mathcal{F}(A_i(x))$

Lulencō Adrian

10. The transport equation

Compute the solution of the transport IVP with $c \in \mathbb{R}$

$$\begin{cases} u_t + c u_x = 0, & x \in \mathbb{R}, t \geq 0 \\ u(0,x) = u_0(x) \end{cases}$$

Look for wave solutions:

$$u(t,x) = v(x-ct) \quad (1)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} v(x-ct) = -c v'(x-ct).$$

$$\cdot c / \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} v(x-ct) = v'(x-ct) \Rightarrow$$

$$+ \quad u_t + c u_x = \cancel{-cv'} + \cancel{cv'} = 0 \Rightarrow$$

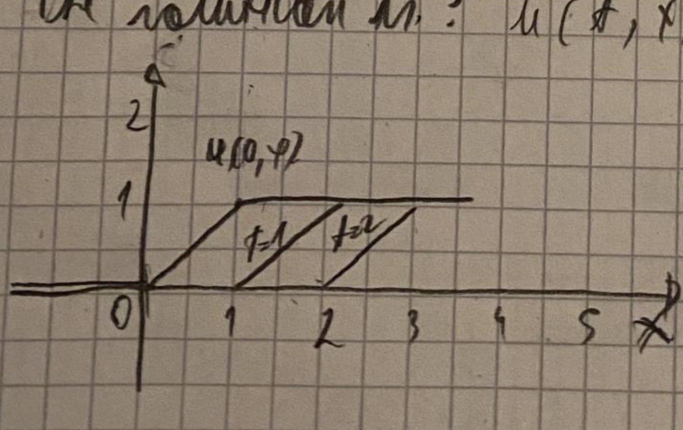
$\Rightarrow$ Solution of transport equation are travelling waves

From (1) $\Rightarrow$ $u_0(x) = u(0,x) = v(x - c \cdot 0) \Leftrightarrow$

$$\Rightarrow u_0(x) = v(x) \Rightarrow$$

$$\Rightarrow u(t,x) = v(x-ct) = u_0(x-ct)$$

The solution is: $u(t,x) = u_0(x-ct) = u_0(x)$



For $c=1$ $\quad t=0: \ u(0,x) = u_0(x)$

$t=1. \ u(1,x) = u_0(x)$

Varvari Dennis Dani gr 832/2

Date 16, Nov, 2021

Ex 25

Poincaré's Inequality:

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. There exists a positive constant C only depending on Ω such that

$$\int_\Omega u^2 dx \leq C^2 \int_\Omega |\nabla u|^2 dx$$

for all $u \in C_0^1(\overline{\Omega})$

Proof: Let $c > 0$ be chosen such that

$\Omega \subset \{x \in \mathbb{R}^n : x = (x_1, x'), |x_1| \leq \frac{c}{2}, x' \in \mathbb{R}^{n-1}\}$

Let $u \in C_0^1(\overline{\Omega})$ and extend u and ∇u to vanish outside Ω. Notice that u^2 is a class C^1 in $\{x \in \mathbb{R}^n : x = (x_1, x'), |x_1| \leq \frac{c}{2}, x' \in \mathbb{R}^{n-1}\}$

For any $x' \in \mathbb{R}^{n-1}$ and $-\frac{c}{2} \leq x_1 \leq 0$, we have

$$u^2(x_1, x') = \int_{-\frac{c}{2}}^{x_1} \frac{\partial u^2}{\partial x_1}(s, x') ds$$

$$= 2\int_{-\frac{c}{2}}^{x_1} u(s, x') \frac{\partial u}{\partial x_1}(s, x') ds$$

so, by using Hölder's inequality, we obtain

$$u^2(x_1,x') \le 2\left(\int_{-\frac{c}{2}}^{0} u^2(s,x')\,ds\right)^{1/2}$$

$$\left(\int_{-\frac{c}{2}}^{0}\left(\frac{\partial u}{\partial x_1}(s,x')\right)^2 ds\right)^{1/2}$$

right side does not depend on x, from $\frac{c}{2}$ to 0 yields:

$$\int_{-\frac{c}{2}}^{0} u^2(x_1,x')\,dx_1 \le C\left(\int_{-\frac{c}{2}}^{0} u^2(s,x')\,ds\right)^{1/2}$$

$$\left(\int_{-\frac{c}{2}}^{0}\left(\frac{\partial u}{\partial x_1}(s,x')\right)^2 ds\right)^{1/2}$$

$$\int_{-\frac{c}{2}}^{0} u^2(x_1,x')\,dx_1 \le c^2\int_{-\frac{c}{2}}^{0}\left(\frac{\partial u}{\partial x_1}(s,x')\right)^2 ds$$

we integrate over R^{n-1} w.r.t. x'

$$\int_{(-\frac{c}{2},0)\times R^{n-1}} u^2\,dx \le C^2\int_{(-\frac{c}{2},0)\times R^{n-1}}\left(\frac{\partial u}{\partial x_1}\right)^2 dx$$

with $0 \le x_1 \le \frac{c}{2}$

$$\int_\Omega u^2\,dx \le C^2\int_\Omega\left(\frac{\partial u}{\partial x_1}\right)^2 dx \le C^2\int_\Omega |\nabla u|^2\,dx$$

Holder's Inequality:

If $\frac{1}{p}+\frac{1}{q}=1$, with $p, q \geq 1$ Then:

$$\int_a^b |f(x)\cdot g(x)|\,dx \leq \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \cdot \left[\int_a^b |g(x)|^q\right]^{1/q}$$

with equality when $|g(x)| = c\,|f(x)|^{p-1}$

Merei-Mureșan Teodora
831

25) Poincaré's Inequality:

Prove Poincaré's inequality (see [Recup, Theorem 3.13]).
p. 38-39

Poincaré's inequality: Let $\Omega \subset \mathbb{R}^m$ be a bounded open set. Then there exists a positive constant c only depending on Ω such that: $\int_\Omega u^2\,dx \le c^2 \int_\Omega |\nabla u|^2\,dx$ for all $u \in C_0^1(\overline{\Omega})$

Proof: Let $c > 0$ be chosen such that:

$$\Omega \subset \{x \in \mathbb{R}^m ;\ x = (x_1, x'),\ \overbrace{|x_1|}^{\text{norm}} \le \frac{c}{2},\ x' \in \mathbb{R}^{m-1}\}$$

Let $u \in C_0^1(\overline{\Omega})$ and extend u and ∇u to vanish outside Ω.

u^2 is of class C^1 in $\{x \in \mathbb{R}^m : x = (x_1, x'),\ |x_1| < \frac{c}{2},\ x' \in \mathbb{R}^{m-1}\}$

$\Rightarrow \forall x' \in \mathbb{R}^{m-1}$ and $-\frac{c}{2} \le x_1 \le 0$, we have:

$$u^2(x_1, x') = \int_{-\frac{c}{2}}^{x_1} \underbrace{\frac{\partial u^2}{\partial x_1}(s, x')\,ds}_{\neq 0} = 2\int_{-\frac{c}{2}}^{x_1} u(s, x') \cdot \frac{\partial u}{\partial x_1}(s, x')\,ds$$

We know from Hölder's inequality that:

If $\frac{1}{p} + \frac{1}{q} = 1$, with $p, q > 1$. Then:

$$\int_a^b |f(x) \cdot g(x)|\,dx \le \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \cdot \left[\int_a^b |g(x)|^q\right]^{1/q}$$

with equality when $|g(x)| = c\,|f(x)|^{p-1}$.

$$u^2(x_1, x') = 2\int_{-\frac{c}{2}}^{x_1} \underbrace{u(s, x')}_{f(x)} \cdot \underbrace{\frac{\partial u}{\partial x_1}(s, x')}_{g(x)}\, ds \Longrightarrow \text{Holder's inequality}$$

$$u^2(x_1, x') \leq 2\left(\int_{-\frac{c}{2}}^{0} u^2(s, x')\, ds\right)^{1/2} \cdot \left(\int_{-\frac{c}{2}}^{0} \left(\frac{\partial u}{\partial x_1}(s, x')\right)^2 ds\right)^{1/2}$$

Because the right-side does not depend on x_1, we can integrate from $-\frac{c}{2}$ to 0 (with respect to x_1):

$$\int_{-\frac{c}{2}}^{0} u^2(x_1, x')\, dx_1 \leq c\left(\int_{-\frac{c}{2}}^{0} u^2(s, x')\, ds\right)^{\frac{1}{2}} \cdot \left(\int_{-\frac{c}{2}}^{0} \left(\frac{\partial u}{\partial x_1}(s, x')\right)^2 ds\right)^{\frac{1}{2}} \xrightarrow{1_2}$$

$$\left(\int_{-\frac{c}{2}}^{0} u^2(x_1, x')\, dx_1\right)^2 \leq c^2 \cancel{\int_{-\frac{c}{2}}^{0} u^2(s, x')\, ds} \cdot \int_{-\frac{c}{2}}^{0} \left(\frac{\partial u}{\partial x_1}(s, x')\right)^2 ds$$

$$\int_{-\frac{c}{2}}^{0} u^2(x, x')\, dx_1 \leq \int_{-\frac{c}{2}}^{0} \left(\frac{\partial u}{\partial x_1}(s, x')\right)^2 ds$$

Now we integrate over R^{n-1} with respect to x'

$$\int_{(-\frac{c}{2}, 0)\times R^{n-1}} u^2\, dx \leq c^2 \int_{(-\frac{c}{2}, 0)\times R^{n-1}} \left(\frac{\partial u}{\partial x_1}\right)^2 dx$$

Because we say from the start: $\forall x' \in R^{n-1}$ and $-\frac{c}{2} \leq x_1 \leq 0$

$$\Rightarrow \int_\Omega u^2\, dx \leq c^2 \int_\Omega \left(\frac{\partial u}{\partial x_1}\right)^2 dx \leq c^2 \int_\Omega |\nabla u|^2\, dx$$

$$\Rightarrow \int_\Omega u^2\, dx \leq c^2 \int_\Omega |\nabla u|^2\, dx, \quad \forall u \in C_0^1(\Omega)$$

HW

$u(x) = v(|x|)$

$x = (x_1, \dots x_m) \in \mathbb{R}^m \setminus \{0_m\}$

$|x|^2 = x_1^2 + x_2^2 + \dots + x_m^2$

$\Delta u(x) = \sum_{i=1}^{m} u_{x_i x_i}(x)$

$u_{x_i}(x) = \frac{\partial}{\partial x_i}(v(|x|)) = v'(|x|)\frac{\partial}{\partial x_i}(|x|)$

$\frac{\partial}{\partial x_i}(|x|) = \frac{\partial}{\partial x_i}((x_1^2 + x_2^2 + \dots + x_m^2)^{\frac{1}{2}})$

$= \frac{1}{2}(x_1^2 + x_2^2 + \dots + x_m^2)^{-\frac{1}{2}} 2x_i = \frac{x_i}{(x_1^2 + x_2^2 + \dots + x_m^2)^{1/2}} = \frac{x_i}{|x|}$

$u_{x_i}(x) = \frac{x_i}{|x|} v'(|x|)$

$u_{x_i x_i}(x) = \frac{|x| - x_i \cdot \frac{x_i}{|x|}}{|x|^2} \cdot v'(|x|) + \frac{x_i}{|x|} \cdot v''(|x|)\frac{x_i}{|x|}$

$= \left(\frac{1}{|x|} - \frac{x_i^2}{|x^3|}\right) v'(|x|) + \frac{x_i^2}{|x|^2} \cdot v''(|x|)$

$\Rightarrow \sum_{i=1}^{m} u_{x_i x_i}(x) = \sum_{i=1}^{m} \left(\left(\frac{1}{|x|} - \frac{x_i^2}{|x^3|}\right) v'(|x|) + \frac{x_i^2}{|x|^2} \cdot v''(|x|) \right)$

$\Rightarrow \Delta u(x) = \left(\frac{m}{|x|} - \frac{|x|^2}{|x|^3}\right) v'(|x|) + \frac{|x|^2}{|x|^2} \cdot v''(|x|)$

$\sum_{i=1}^{m} x_i^2 = |x|^2$

$\Rightarrow \Delta u(x) = v''(|x|) + \frac{m-1}{|x|} v'(|x|)$

$\Rightarrow \Delta u(x) = v''(|x|) + \frac{n-1}{|x|} \cdot v'(x)$

$\Delta u(x) = 0 \Leftrightarrow v''(|x|) + \frac{n-1}{|x|} v'(|x|) = 0$

Let $|x| = r$, $u(x) = v(|x|) \Leftrightarrow$

$v''(r) + \frac{n-1}{r} v'(r) = 0 \qquad (1)$

We denote $z(r) = v'(r) \Rightarrow z'(r) = v''(r)$

$\Rightarrow z'(r) + \frac{n-1}{r} \cdot z(r) = 0 \;/\; r^{n-1}$

$\Rightarrow r^{n-1} z'(r) + (n-1) r^{n-2} \cdot z(r) = 0$

$\overset{n \geq 2}{\Rightarrow} r^{n-1} z'(r) + (r^{n-1})' \cdot z(r) = 0$

$\Rightarrow [r^{n-1} \cdot z(r)]' = 0$

$= r^{n-1} \cdot z(r) = c_1, \quad c_1 \in \mathbb{R}$

$\Rightarrow z(r) = \frac{c_1}{r^{n-1}} \qquad c_1 \in \mathbb{R}$

$\Rightarrow v'(r) = \frac{c_1}{r^{n-1}} \qquad c_1 \in \mathbb{R}$

$n = 2: \quad v'(r) = \frac{c_1}{r} \Rightarrow v(r) = c_1 \ln r + c_2 \quad ; c_1, c_2 \in \mathbb{R}$

$n \geq 3: v'(r) = \frac{c_1}{r^{n-1}} \Rightarrow v(r) = \frac{c}{r^{n-2}} + c_2 \quad ; c_1, c_2 \in \mathbb{R}$

So: $u(x) = c_1 \ln |x| + c_2 \qquad c_1 c_2 \in \mathbb{R}, \; n = 2$

$u(x) = \frac{c_1}{|x|^{n-2}} + c_2 \qquad c_1, c_2 \in \mathbb{R} \quad n \geq 3$

$$\Rightarrow u(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} \cdot u_0(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} \cdot e^{-ay}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t} - \overset{4t/}{ay}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2 - 4tay}{4t}}\, dy$$

Ex11 The Heat IVP. Let us still consider the IVP:

$\begin{cases} u_t = u_{xx} \\ u(0) = u_0 \end{cases}$. Find the solution of the IVP when:

a) $u_0(x) = e^{-ax^2}$, $a > 0$ b) $u_0(x) = H(1-|x|)$, H - Heaviside step functions

$$\begin{cases} u_t = u_x \\ u(0) = u_0 \end{cases} \Leftrightarrow \begin{cases} \frac{du}{dt}(t) = \frac{d^2u}{dx^2}(t) \\ u(0) = u_0 \end{cases}$$

step 1

We need to apply Fourier transform

$$\mathcal{F}\left(\frac{du}{dt}(t)\right)(y) = \frac{d\mathcal{F}}{dt}(u)(y)$$

$$\mathcal{F}\left(\frac{d^2u}{dx^2}(t)\right)(y) = \frac{d^2\mathcal{F}}{dx^2}(u)(y)$$

step 2

$$\hat{u}(t) \overset{not}{=} \mathcal{F}(u(t))(y) - |y|^2\mathcal{F}(u)(y)$$

$$\mathcal{F}(u(0))(y) = \mathcal{F}(u_0)(y)$$

$$\Rightarrow \begin{cases} \frac{d\hat{u}}{dt}(t) = \frac{d^2\hat{u}}{dx^2}(t) \\ \hat{u}(0) = \hat{u}_0 \end{cases} \Rightarrow \begin{cases} \frac{d\hat{u}}{dt}(t) = -|y|^2\hat{u}(t) \\ \hat{u}(0) = \hat{u}_0 \end{cases}$$

$u_0(x) = e^{-ax^2}$, $\hat{u}(t) = e^{-|y|^2 t}\hat{u}_0 \Rightarrow \mathcal{F}(u(t))(y) = \overbrace{e^{-|y|^2 t}}^{\mathcal{F}(f)} \cdot \overbrace{\mathcal{F}(u_0)(y)}^{\mathcal{F}(g)}$

$$\mathcal{F}(e^{-ax^2}) = \mathcal{F}(e^{-\frac{1}{2}(\sqrt{a}\,x)^2})(y) = e^{-\frac{1}{2}(\sqrt{2a}\,y)^2} = e^{-ay^2}$$

$$\Rightarrow \mathcal{F}(u(t)) = e^{(-|y|^2(t-a^2))}$$

step 3:

We apply $\mathcal{F}^{-1}$ (reverse engineering).

$$\mathcal{F}^{-1} \mid \mathcal{F}(f * g) = (2\pi)^{n/2}\mathcal{F}(f)\mathcal{F}(g)$$

$$\Rightarrow \mathcal{F}^{-1} \mid \mathcal{F}(f)\mathcal{F}(g) = \frac{1}{(2\pi)^{n/2}} f * g$$

$$\Rightarrow \mathcal{F}^{-1}(e^{-|y|^2(t-a^2)}) = \frac{1}{\sqrt{2\pi}} N(t) \cdot u_0(x) = u(t) - \underline{\text{HEAT KERNEL}}$$

MOLDOVAN CRISTIANA
GRUPA 832

Problema 9

$$\begin{cases} u_{tt} = \frac{1}{2} u_{xx}, \; t>0 \\ u(0,x) = e^{-x^2} \\ u_t(0,x) = 0 \end{cases}$$

$u_{tt} = \frac{1}{2} u_{xx}$ poate fi rezolvată prin schimbarea de variabile (aducerea la forma canonică)

$\xi = x - at \qquad \eta = x + at$

În urma schimbării de variabile ecuația devine $u_{\xi\eta} = 0$

$u(\xi, \eta) = A(\xi) + B(\eta)$

$\Rightarrow$ În variabilele inițiale x și t sol. este:

$u(t,x) = A(x - at) + B(x + at)$

Această soluție depinde de A și B și se mai numește soluția lui D'Alembert.

În general, problema Cauchy pt. ecuația omogenă a coardei:

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases}$$

Pt. problema noastră, $a^2 = \frac{1}{2} \Rightarrow a = \frac{\sqrt{2}}{2}$

Vom face substituția:

$$\xi = x - \frac{\sqrt{2}}{2} t \qquad \eta = x + \frac{\sqrt{2}}{2} t$$

$$\Rightarrow u_{\xi\eta} = 0 \Rightarrow (u_\xi)'_\eta = 0 \Rightarrow u_\xi = c_1(\xi)$$

$$u(\xi, \eta) = \int c_1(\xi)\, d\xi + c_2(\eta)$$

$$\Rightarrow u(\xi,\eta) = A(\xi) + B(\eta),\ A, B \in C^1$$

$$u(x,t) = A\left(x - \frac{\sqrt{2}}{2} t\right) + B\left(x + \frac{\sqrt{2}}{2} t\right)$$

$$u(0,x) = e^{-x^2} \Rightarrow A(x) + B(x) = e^{-x^2}$$

$$u_t(t,x) = \left(-\frac{\sqrt{2}}{2}\right) A'\left(x - \frac{\sqrt{2}t}{2}\right) + \frac{\sqrt{2}}{2} B'\left(x + \frac{\sqrt{2}}{2} t\right)$$

$$u_t(0,x) = \left(-\frac{\sqrt{2}}{2}\right) A'(x) + \frac{\sqrt{2}}{2} B'(x) = 0 \quad \Big| : \frac{\sqrt{2}}{2}$$

$$\begin{cases} A(x) + B(x) = e^{-x^2} \\ \\ -A'(x) + B'(x) = 0 \end{cases} \Rightarrow \begin{cases} A'(x) + B'(x) = -2xe^{-x^2} \\ \\ -A'(x) + B'(x) = 0 \end{cases}$$

$$2B'(x) = -2xe^{-x^2}$$

$$\Rightarrow B'(x) = -xe^{-x^2} \Rightarrow B(x) = \int -xe^{-x^2}dx = \frac{e^{-x^2}}{2}$$

$$A(x) + B(x) = e^{-x^2} \Rightarrow A(x) = e^{-x^2} - B(x) = \frac{e^{-x^2}}{2}$$

$$\Rightarrow u(t,x) = A\left(x - \frac{\sqrt{2}}{2}t\right) + B\left(x + \frac{\sqrt{2}}{2}t\right)$$

$$= e^{-\left(x - \frac{\sqrt{2}}{2}t\right)^2} + e^{-\left(x + \frac{\sqrt{2}}{2}t\right)^2}$$

Ex 28 Courant's counterexemple

Consider the Dirichlet problem on the unit disc $\{\Omega = x \in \mathbb{R}^2 : |x| < 1\}$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and the functions (in polar coord)

$$u_k(r,\phi) = \begin{cases} k^{1/3} & \text{for } 0 < r < e^{-2k} \\ -k^{-2/3}(k - \ln r) & \text{for } e^{-2k} < r < e^{-k} \\ 0 & \text{for } e^{-k} < r < 1 \end{cases}$$

a.) prove that $u \equiv 0$ is the unique solution of the Dirichlet problem

if · u minimizes the Dirichlet Energy ?
· Ω open, bounded and $f \in L^2(\Omega)$ ✓, $f = 0$ $\Rightarrow \exists!$ weak sol. $u \in H_0^1(\Omega)$

$$E[v(x)] = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - v \cdot f\right) dx = \int_\Omega \frac{1}{2} \underbrace{|\nabla v^2|}_{\geq 0} dx =$$

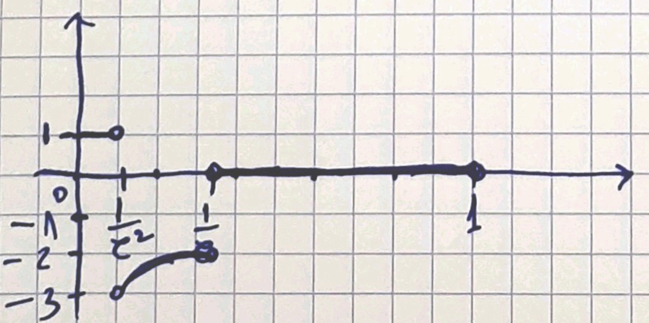
$$= \int_\Omega \frac{1}{2} \underbrace{\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)}_{=0} dx \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow v = C$$

and because u vanishes on the boundary of Ω, i.e. $u = 0$ on $\partial\Omega$

$\Rightarrow u \equiv 0$

b.) plot $f : [0,1] \to \mathbb{R}$, $f(r) = u_1(r, 0)$

$$u_1(r,0) = \begin{cases} 1^{1/3}, & 0 < r < e^{-2} \\ -1^{-2/3}(1 - \ln r), & e^{-2} < r < e^{-1} \\ 0, & e^{-1} < r < 1 \end{cases}$$



c.) compute explicity the Dirichlet energy of u_k, that is

$$E(u_k) = 2\pi \int_0^1 \left|\frac{\partial u_k}{\partial r}\right|^2 r\,dr$$

$$= 2\pi\left(\int_0^{e^{-2k}} \underbrace{\left|\frac{\partial}{\partial r} k^{1/3}\right|^2}_{=0} r\,dr + \int_{e^{-2k}}^{e^{-k}} \left|\frac{\partial}{\partial r} -k^{-2/3}(k - \ln r)\right|^2 r\,dr + \right.$$

$$\left. + \underbrace{\int_{e^{-k}}^1 \left|\frac{\partial}{\partial r} 0\right|^2 r\,dr}_{=0}\right) = 2\pi \int_{e^{-2k}}^{e^{-k}} \left|k^{-2/3} \cdot \frac{1}{r}\right|^2 r\,dr =$$

$$= 2\pi \int_{e^{-2k}}^{e^{-k}} k^{-4/3} \cdot \frac{r\,dr}{|r|^2} = 2\pi k^{-4/3} \ln r \Big|_{e^{-2k}}^{e^{-k}} = 2\pi k^{-4/3} (-k+2k)$$

$$= 2\pi \cdot k^{-4/3} \cdot k = \frac{2\pi}{k^{1/3}}$$

d.) study the limits

• $\lim\limits_{k \to \infty} u_k(0,0)$

Assuming $u_k(0,0) = \lim\limits_{r \to 0} u_k(r,0)$, so we have $\lim\limits_{k \to \infty} u_k(0,0) =$

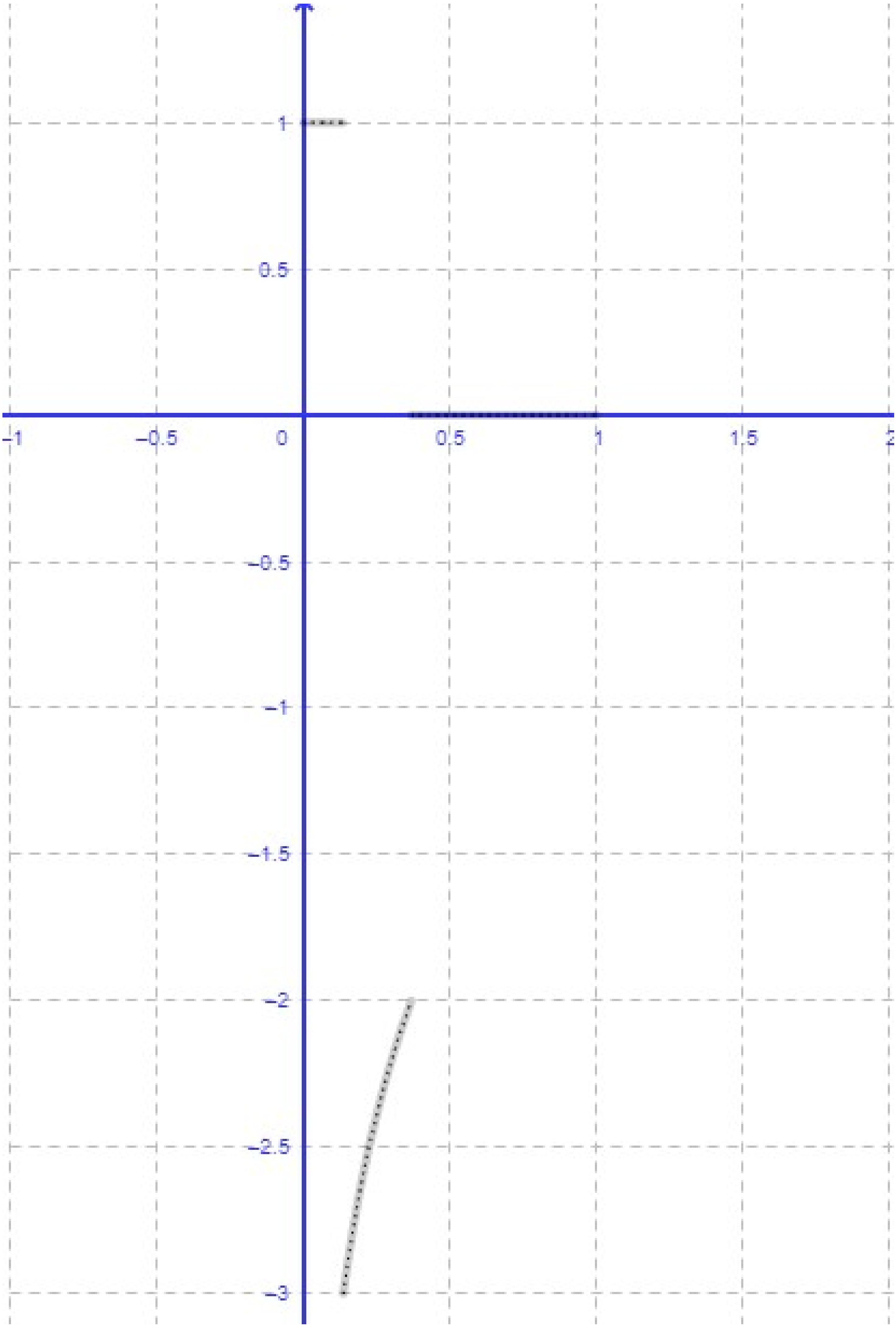
$$= \lim_{k \to \infty} k^{1/3} = \infty$$

• $\lim\limits_{k \to \infty} \|u_k - u\|_{L^2} = \lim\limits_{k \to \infty} \|u_k\|_{L^2} \overset{\text{Poincaré ineq.}}{\leq} \lim\limits_{k \to \infty} \|u_k\|_{H_0^1} =$

$$= \lim_{k \to \infty} C^2 \int_\Omega (\nabla u_k)^2 dx = \lim_{k \to \infty} C^2 \cdot 2E(u_k) = 0. \quad (1)$$

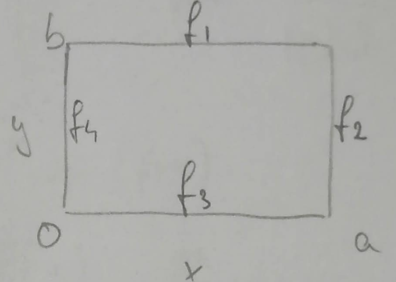
As we know $\lim\limits_{k \to \infty} \|u_k\|_{L}^2$ is nonnegative and $\overset{(1)}{\leq} 0 \Rightarrow \lim = 0$

• $\lim\limits_{k \to \infty} E(u_k) = \lim\limits_{k \to \infty} \frac{2\pi}{k^{1/3}} = 0$


1
0.5
−0.5
−1
−1.5
−2
−2.5
−3
-1
−0.5
0
0.5
1
1.5
2

30. $\Omega = (0, a) \times (0, b) \to$ RECTANGLE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{x=0} = u|_{x=a} = u|_{y=b} = 0 \\ u|_{y=0} = g(x) \end{cases}$$



$u|_{x=0} = u(0, y) = 0$

$u|_{x=a} = u(a, y) = 0$

$u|_{y=b} = u(x, b) = 0$

$u|_{y=0} = u(x, 0) = g(x)$

$$\begin{cases} f_1(x) = u(x, b), & y = b \\ f_2(y) = u(a, y), & x = a \\ f_3(x) = u(x, 0), & y = 0 \\ f_4(y) = u(0, y), & x = 0 \end{cases}$$

Laplace:

$\Delta u = u_{xx} + u_{yy}$, $\quad u_{xx} = \frac{\partial}{\partial x^2} u(x, y)$ ①

$\Delta u = 0$ $\quad u_{yy} = \frac{\partial}{\partial y^2} u(x, y)$ ②

$u(x, y) = A(x) \cdot B(y)$

$\hookrightarrow$ Fourier's separation of vars

We replace this into ① and ② $\Rightarrow$

$\Rightarrow u_{xx} = \frac{\partial}{\partial x^2} A(x) B(y) = A''(x)\, B(y)$

$u_{yy} = \frac{\partial}{\partial y^2} A(x) B(y) = A(x)\, B''(y)$

$\hookrightarrow \Delta u = A''(x) B(y) + A(x) B''(y) = 0 \; / : A(x) B(x)$ eigenvalue problem

$$\frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} = 0 \Leftrightarrow \frac{A''(x)}{A(x)} = -\frac{B''(y)}{B(y)} = \lambda$$

$$\begin{cases} -\dfrac{A''(x)}{A(x)} = \lambda & \Rightarrow A''(x) = -A(x)\lambda \Leftrightarrow r^2 = -\lambda r^0 \\ & \quad r^2 = -\lambda \Rightarrow \boxed{r = \pm i\sqrt{\lambda}} \\ \dfrac{B''(y)}{B(y)} = \lambda & \end{cases}$$

$$\Rightarrow A(x) = c_1 e^{\overset{=0}{\alpha x}} \cos(\beta x) + c_2 e^{\overset{=0}{\alpha x}} \sin(\beta x)$$

$$A(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x) \quad (3)$$

But we know from hypothesis that:

$$\begin{cases} u(0, y) = A(0)\, B(y) = 0 \Rightarrow A(0) = 0 \\ u(a, y) = A(a)\, B(y) = 0 \Rightarrow A(a) = 0 \\ u(x, b) = A(x)\, B(b) = 0 \Rightarrow B(b) = 0 \\ u(x, 0) = A(x)\, B(0) = g(x) \end{cases}$$

Now we replace in (3) $\Rightarrow A(0) = c_1 \cos(0) + c_2 \sin(0) = 0$

$$\boxed{c_1 = 0}$$

$$\Rightarrow A(x) = c_2 \sin(\sqrt{\lambda}\, x)$$

$$\Rightarrow A(a) = c_2 \sin(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \sqrt{\lambda}\, a = k\pi, \ \forall k \in \mathbb{N}$$

$$\Rightarrow \lambda_k = \left(\frac{k\pi}{a}\right)^2, \ \forall k \in \mathbb{N}$$

$\Rightarrow A_k(x) = c_2 \sin\left(\frac{k\pi}{a} x\right) \rightarrow$ solution for $A''(x) = -A(x)\lambda$

Now that we have A_k solution and $\lambda_k \Rightarrow B_k$ solution for $B''(y) = B(y)\lambda$

$B''(y) = B(y)\lambda \Leftrightarrow r^2 = \lambda_k$

$r^2 = \left(\frac{k\pi}{a}\right)^2 \Rightarrow r = \pm \frac{k\pi}{a}$

$\Rightarrow B_k(y) = c_3 e^{\frac{k\pi}{a}y} + c_4 e^{-\frac{k\pi}{a}y}$

This is not the best representation for what we need:

* we know $B(b) = 0 \Leftrightarrow c_3 e^{\frac{k\pi}{a}b} + c_4 e^{-\frac{k\pi}{a}b} = 0$

we cannot get much from this

*

We can instead use cosh and sinh

* $\cosh = \frac{e^x + e^{-x}}{2}$, $\sinh = \frac{e^x - e^{-x}}{2}$

~~cos h~~

$\cosh' = \sinh$, $\sinh' = \cosh$

$\cosh(0) = 1$, $\sinh(0) = 0$

*

$\Rightarrow B_k(y) = c_3 \cosh\left(\frac{k\pi}{a}y\right) + c_4 \sin\left(\frac{k\pi}{a}y\right)$, $\forall k \in \mathbb{N}$

Recall that $u(x,y) = A(x)B(x)$

We found A_k and B_k, $\forall k \in \mathbb{N}$, so the solution will be a sum:

$$u(x,y) = \sum_{k=1}^{\infty} A_k(x) B_k(x)$$

$$u(x,y) = \sum_{k=1}^{\infty} c_2 \sin\left(\frac{k\pi}{a}x\right)\left(c_3 \cosh\left(\frac{k\pi}{a}y\right) + c_4 \sin\left(\frac{k\pi}{a}y\right)\right)$$

Now that we got our function, we have to also check that it respects our initial boundry conditions.

For $u(0, y) = 0$ we already checked that $u(x, y) = 0$
$u(a, y) = 0$ (just plug in 0 and a for x $\Rightarrow$ that $\overset{c_2 \cdot \sin(\ldots)}{\cancel{\sin}} = 0$
$\Rightarrow$ the whole sum is 0) ✓

For $u(x, b) = 0 \Leftrightarrow B(b) = 0$

$$u(x, b) = \sum_{k=1}^{\infty} c_2 \sin\left(\frac{k\pi}{a} x\right) \underbrace{\left(c_3 \cosh\left(\frac{k\pi}{a} b\right) + c_4 \sinh\left(\frac{k\pi}{a} b\right)\right)}_{B(b) = 0}$$

$$\Rightarrow \sum_{k=1}^{\infty} c_2 \sin(\ldots) \cdot 0 = 0$$

Lastly, $u(x, 0) = g(x)$
Here we will use the cosh, sinh properties:

$$u(x, 0) = \sum_{k=1}^{\infty} c_2 \sin\left(\frac{k\pi}{a} x\right)\left(c_3 \cosh\left(\frac{k\pi}{a} \cdot 0\right) + c_4 \sinh\left(\frac{k\pi}{a} \cdot 0\right)\right)$$

$$u(x, 0) = \sum_{k=1}^{\infty} c_2 \sin\left(\frac{k\pi}{a} x\right)\left(c_3 \cdot 1 + c_4 \cdot 0\right)$$

$$u(x, 0) = \sum_{k=1}^{\infty} c_2 \cdot c_3 \cdot \sin\left(\frac{k\pi}{a} x\right) = g(x)$$

11. a) Heat IVP: $\begin{cases} u_t = u_{xx} \\ u(0) = u_0 \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt} u(t) = \frac{d^2}{dx^2} u(t) \overset{\shortparallel}{=} \Delta u(t) \\ u(0) = u_0 \end{cases}$

First, we apply the Fourier trans~~formation~~ to the IVP

$$\mathcal{F}\left(\frac{d}{dt} u(t)\right)(y) = \frac{d}{dt} \mathcal{F}(u)(y)$$

$$\mathcal{F}\left(\frac{d^2}{dx^2} u(t)\right)(y) = \frac{d^2}{dx^2} \mathcal{F}(u)(y) = -|y|^2 \mathcal{F}(u)(y)$$

Not: $\hat{u}(t) = \mathcal{F}(u(t))(y)$

$$\mathcal{F}(u(0))(y) = \mathcal{F}(u_0)(y)$$

We replace in the initial IVP

$$\begin{cases} \frac{d}{dt} \hat{u}(t) = \frac{d^2}{dx^2} \hat{u}(t) \\ \hat{u}(0) = \hat{u}_0 \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt} \hat{u}(t) = -|y|^2 \hat{u}(t) \\ \hat{u}(0) = \hat{u}_0 \end{cases}$$

And solve $\hat{u}(t) = e^{-|y|^2 t} \hat{u}_0$

$$\mathcal{F}(u(t))(y) = e^{-|y|^2} + \mathcal{F}(u_0)(y)$$

We apply $\mathcal{F}^{-1}$, and by the convolution theorem:

$$\mathcal{F}^{-1} \Big| \ \mathcal{F}(f * g) = \frac{1}{(2\pi)^{\frac{n}{2}}} F(f) F(g)$$

$$\Rightarrow \mathcal{F}^{-1} \Big| \ \mathcal{F}(f) \cdot \mathcal{F}(g) = \frac{1}{(2\pi)^{\frac{n}{2}}} f * g.$$

The preimage of $e^{-|y|^2 t}$ would be:

$$\mathcal{F}^{-1}\left[e^{-|y|^2 t}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|y|^2 t} e^{-iyx} \, dy =$$

$$= \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \quad \text{heat Kernel}$$

Compute the solution of the nonhomogeneous IVP

$$\begin{cases} u_t = -u_x + 1, & t > 0, x \in \mathbb{R} \\ u(0,x) = e^{-x^2} \end{cases}$$

$$\Rightarrow \begin{cases} u_t + u_x = 1 = \cancel{f(t,x)} = f(t,x) \\ u(0,x) = e^{-x^2} = g(x) \end{cases}$$

We use the new variable $z = z(s) := u(t+s, x+cs)$

$$\frac{dz}{ds}(s) = u_t(t+s), x+cs) + u_x(t+s, x+cs) \cdot c$$

$$= f(t+s), x+cs)$$

In our case, $c = 1$

$$\Rightarrow \frac{dz}{ds}(s) = u_t(t+s, x+s) + u_x(t+s, x+s)$$

$$= f(t+s, x+s)$$

We write u and g using z

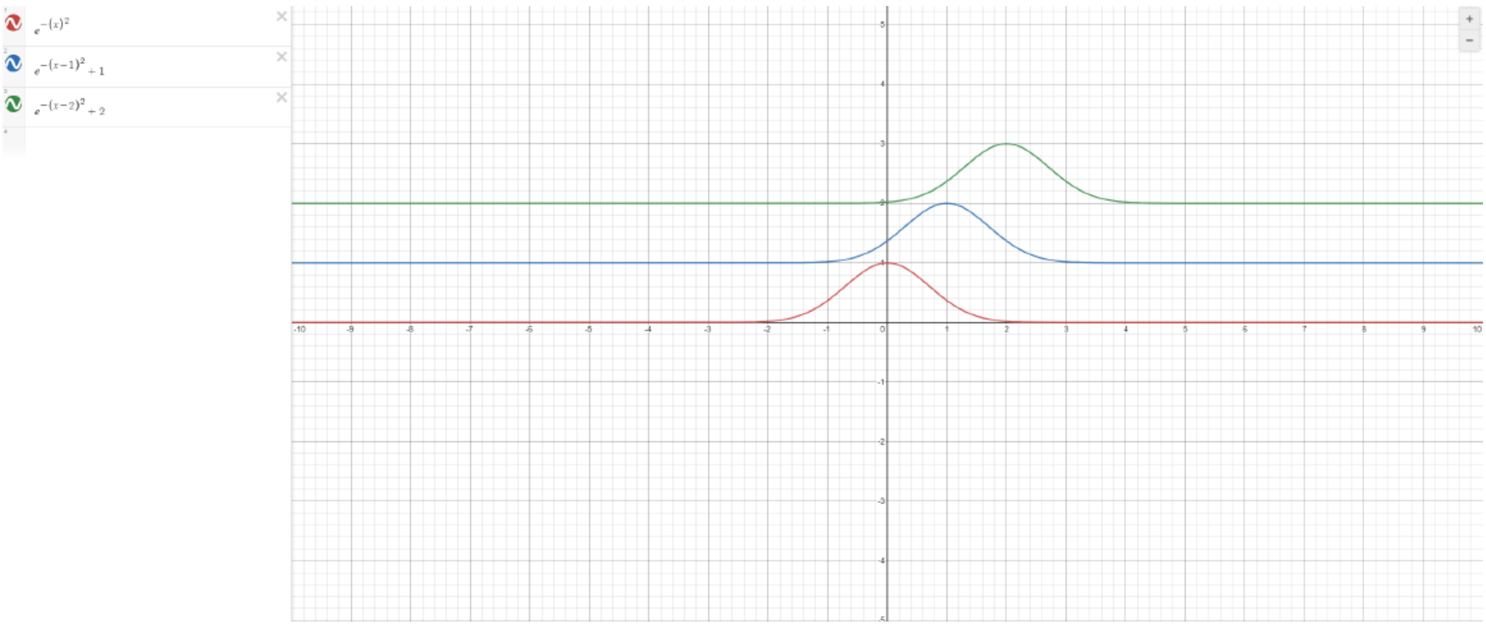
$$u(t,x) - \underbrace{g(x-t)}_{\substack{u(0,x-t) \\ = z(-t)}} = z(0) - z(-t) = u(s,x) - u(0, x-t)$$

$$= \int_{-t}^{0} \frac{dz}{ds}(s)\, ds = f(t+s, x+s)$$

$\Rightarrow \quad u(t,x) = g(x-t) + \int_0^t f(s, x+t-s)\, ds$

$u(t,x) = e^{-(x-t)^2} + \cancel{\int_0^t} 1$

$$\boxed{u(t,x) = e^{-(x-t)^2} + t}$$


$e^{-(x)^2}$
$e^{-(x-1)^2}+1$
$e^{-(x-2)^2}+2$

$$\begin{cases} u'' = -au' - w^2 \\ u(0) = u_0 \\ u'(0) = v_0 \end{cases}$$

$u' = z \Rightarrow u'' = z'$; $z = z(t)$; $u = u(t)$

$\Rightarrow z' = -az - w^2$ - linear nonhomog.

$z' + az + w^2 = 0.$

$z' + az = -w^2$

I. $z' + az = 0.$

$z' = -az$

$$\frac{dz}{dt} = -az.$$

$$\int \frac{dz}{z} = \int -a \cdot dt \rightarrow \ln z = -at + C.$$

$$\Rightarrow z_0 = e^{-at} \cdot C$$

II. $z_p = e^{-at} \cdot C(t)$

$$z'_p = -ae^{-at} \cdot C(t) + C'(t) \cdot e^{-at}$$

$$\Rightarrow \cancel{-ae^{-at} \cdot C(t)} + C'(t) \cdot e^{-at} + \cancel{a \cdot e^{-at} \cdot C(t)} = w^2$$

$$C'(t) \cdot e^{-at} = w^2$$

$$C'(t) = \frac{w^2}{e^{-at}} \Rightarrow C(t) = \int \frac{w^2}{e^{-at}} = w^2 \int \frac{1}{e^{-at}} = w^2 \int e^{at} =$$

$$C(t) = w^2 \cdot \frac{e^{at}}{2} \Rightarrow \boxed{z_p} = w^2 \cdot \frac{e^{at}}{2} \cdot \cancel{e^{-at}} = \boxed{\frac{w^2}{2}}$$

$\Rightarrow z = z_p + z_0 =$

$$= \frac{w^2}{2} + e^{-at} \cdot C$$

$$z = \frac{w^2}{2} + e^{-at} \cdot C_2$$

$$u' = z = \frac{w^2}{2} + e^{-at} \cdot C_2 \Big| \int_t$$

$$u = \frac{w^2}{2} + C_1 + \frac{e^{-at}}{-a} C_2$$

$$\boxed{u = \frac{w^2}{2} + C_1 - \frac{e^{-at} C_2}{a}}$$

$$u(0) = u_0$$

$$u(0) = \frac{w^2}{2} + C_1 - \frac{C_2}{a} = u_0.$$

$$u'(0) = v_0.$$

$$u'(0) = \frac{w^2}{2} + e^{-at} \cdot C_2 = \frac{w^2}{2} + C_2 = v_0$$

$$\Rightarrow C_2 = v_0 - \frac{w^2}{2}$$

$$\Rightarrow \frac{w^2}{2} + C_1 - \left(v_0 - \frac{w^2}{2}\right) \cdot \frac{1}{a} = u_0$$

$$\frac{w^2}{2} + C_1 - \frac{v_0}{a} + \frac{w^2}{2a} = u_0.$$

$$C_1 = u_0 + \frac{v_0}{2} - \frac{w^2}{2} - \frac{w^2}{2a}$$

$$\Rightarrow u = \cancel{\frac{w^2}{2}} + u_0 + \frac{v_0}{2} - \cancel{\frac{w^2}{2}} - \frac{w^2}{2a} - \frac{e^{-at}}{a} \cdot \left(v_0 - \frac{w^2}{2}\right)$$

$$u = u_0 + \frac{v_0}{2} - \frac{w^2}{2a} - \frac{v_0 \cdot e^{-at}}{a} + \frac{w^2 \cdot e^{-at}}{2a}.$$

5) Liapunov function

$$u'' = -au' - \omega^2$$

We can write the eq. as the following system:

$$\begin{cases} v = u' \\ v' = -au' - \omega^2 \end{cases}$$

We have the general form of Lyapunov function:

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = -av - \nabla E(u) \end{cases}$$

$$\Rightarrow \nabla E(u) = \omega^2$$

$$\frac{\partial}{\partial u} E(u) = \omega^2 \Rightarrow E(u) = \int \omega^2 = \omega^2 \cdot u.$$

$$\Rightarrow \boxed{E(u) = \omega^2 \cdot u}$$

$$\Rightarrow \tilde{E} = \frac{1}{2} v^2 + u \cdot \omega^2$$

$$\tilde{E}(t) = \frac{1}{2} (u'(t))^2 + u(t) \cdot \omega^2$$

Savina Ruxandra

16. $u' = u^2$

$u(0) = u_0 > 0$

Solve the IVP and show that finite blow-up occures $\forall u_0 > 0$

Solution

$$u' = u^2 \Leftrightarrow \frac{du}{dx} = u^2 \Leftrightarrow \frac{du}{u^2} = dx \Leftrightarrow$$

$$\Leftrightarrow \int \frac{du}{u^2} = \int dx \Leftrightarrow -\frac{1}{u} = x + c \Leftrightarrow$$

$$\Leftrightarrow u(x) = \frac{1}{-x+c} \quad \forall c \in \mathbb{R}$$

$$u(0) = u_0 \Leftrightarrow \frac{1}{c} = u_0 \Rightarrow c = \frac{1}{u_0}$$

$$u(x) = \frac{1}{-x + \frac{1}{u_0}}$$

blow-up if $\exists X_{max} < \infty$ and $\| u(x, u_0) \| \xrightarrow{x \to X_{max}} \infty$

We see that for $x_0 \to \frac{1}{u_0} \Rightarrow u(x_0) \to \infty$ for finite $x_0 \to \frac{1}{u_0}$

$\Rightarrow u(x)$ blows up in finite time $\forall u_0 > 0$

Homework

Sofian Răzvan

(*) Consider the logistic equation $u' = u(1-u)$

a) find stationary (equilibrium) solutions and study their stability

$$f(u) = u(1-u)$$

$$f(u) = 0 \Leftrightarrow u(1-u) = 0 \Leftrightarrow \begin{cases} u_1 = 0 \\ 1-u = 0 \Leftrightarrow u_2 = 1 \end{cases} \Rightarrow u \in \{0, 1\}$$

$u=0$ <u>unstable</u> because when pop. is $0 \Rightarrow$ it can not grow and the solutions above it go ~~tow~~ towards $u=1$ which is <u>stable</u> (the saturation level)

b) $u(0) = u_0$

$$u' = u(1-u) \Leftrightarrow \frac{du}{dt} = u(1-u) \Leftrightarrow \frac{du}{u(1-u)} = dt \Leftrightarrow$$

$$\Leftrightarrow \int \frac{du}{u(1-u)} = \int dt \overset{1}{\Leftrightarrow}$$

$$\frac{1}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u} \Leftrightarrow \quad 1 = (1-u)A + uB \Leftrightarrow 1 = u(-A+B) + A \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} B-A = 0 \Leftrightarrow B-1 = 0 \Leftrightarrow B = 1 \\ A = 1 \end{cases}$$

$$\Rightarrow \frac{1}{u(1-u)} = \frac{1}{u} + \frac{1}{1-u}$$

$$\overset{1}{\Leftrightarrow} \int \frac{1}{u} du + \int \frac{1}{1-u} du = t + C \Leftrightarrow \cancel{\ln u + \ln(1-u) = t + C \Leftrightarrow}$$

$$\cancel{\Leftrightarrow \ln u \cdot (1-u) = t + C \Leftrightarrow u(1-u) = e^{t+C}}$$

$$\Leftrightarrow \ln u - \ln(u-1) = t + C \Leftrightarrow \frac{u}{u-1} = e^{t+C} \quad (2)$$

$$u = e^{t+c} \cdot (u-1)$$

$$\Leftrightarrow u = u \cdot e^{t+c} - e^{t+c} \quad / :u$$

$$\Leftrightarrow 1 = e^{t+c} - \frac{e^{t+c}}{u} \Leftrightarrow$$

$$\Leftrightarrow 1 - e^{t+c} = -\frac{e^{t+c}}{u}$$

$$\Leftrightarrow \frac{e^{t+c} - 1}{e^{t+c}} = \frac{1}{u} \Rightarrow \boxed{u(t) = \frac{e^{t+c}}{e^{t+c} - 1}}$$

$$u(0) = u_0 \Rightarrow \frac{e^{0+c}}{e^{0+c} - 1} = e_0 \overset{2}{\Leftrightarrow} e^{c} = \frac{u_0}{1-u_0} \Leftrightarrow \boxed{c = \ln\left(\frac{u_0}{1-u_0}\right)}$$

Tolcan Cezar 823

18.

b) $u(1-u^2) = u(1-u)(1+u) = 0 \Rightarrow u=0,\ u=1,\ u=-1$ ~~pu equilib~~

equilibrium points