Seminar 6 Matrix games

Ex 1 Solve the game whose pay-off matrix is:

$$C = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Solution First we compute $\underline{v}$ and $\overline{v}$ (the so-called lower value and upper value of the game)

$$\left.\begin{array}{l} \alpha_1 = \min\{2, 0\} = 0 \\ \alpha_2 = \min\{-1, 1\} = -1 \end{array}\right\} \Rightarrow \underline{v} = \max\{\alpha_1, \alpha_2\} = 0.$$

$$\left.\begin{array}{l} \beta_1 = \max\{2, -1\} = 2 \\ \beta_2 = \max\{0, 1\} = 1 \end{array}\right\} \Rightarrow \overline{v} = \min\{\beta_1, \beta_2\} = 1$$

According to L10 we know that:

$$0 = \underline{v} \leq v \leq \overline{v} = 1$$

hence $v \in [0, 1]$

Since $\underline{v} \neq \overline{v}$, the game has no saddle points.

Thus, Theorem 11.1 does not apply

On the other hand, we are not sure wether

$v \overset{?}{>} 0$

(we just know that $v \in [0, 1]$)

In order to see Theorem 11.2, which requires that the game's value is positive (>0),

we will add a convenient constant $k \in \mathbb{R}$ to all elements of C, such that

$\underline{v} + k = 0$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

For instance, let

$k = 1$

We obtain a new pay-off matrix

$$\hat{C} = C + (k) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

$\hat{\underline{v}} = \underline{v} + k = 0 + 1 = 1$

$\hat{\overline{v}} = \overline{v} + k = 1 + 1 = 2$

$\hat{v} = v + k = v + 1 \in [1, 2] \Rightarrow \hat{v} > 0$

La examen ne dă matrice aranjată fără translatare

$\Rightarrow$ we can apply th 11.2 to the new game, whose pay-off matrix is $\hat{C}$.

We associate to $\hat{C}$ two optimization problems.

$$(P1)\begin{cases} \text{Minimize } u_1 + u_2 \\ 3 \cdot u_1 + 0 \cdot u_2 \geq 1 \\ 1 \cdot u_1 + 2u_2 \geq 1 \\ u_1, u_2 \geq 0 \end{cases} \quad \text{and} \quad (P2)\begin{cases} \text{Maximise } v_1 + v_2 \\ 3 \cdot v_1 + 1 \cdot v_2 \leq 1 \\ 0 \cdot v_1 + 2 \cdot v_2 \leq 1 \\ v_1, v_2 \geq 0 \end{cases}$$

Next we solve (P1) and (P2)

In order to use the SIMPLEX Algorithm.
we have to transform (P1) and (P2) in standard form

$$(P_1\ st)\ \begin{cases} \text{Minimize } u_1+u_2 \\ 3u_1 - u_3 = 1 \\ u_1+2u_2-u_4 = 1 \\ u_1,\dots,u_4 \geq 0 \end{cases} \iff$$

$$\iff \begin{cases} \text{Minimize } u_1+u_2 \\ -3u_1+u_3 = -1 \\ -u_1-2u_2+u_4 = -1 \\ u_1,\dots,u_4 \geq 0 \end{cases}$$

and

$$(P_2\ st)\ \begin{cases} \text{Minimize } -v_1-v_2 \\ 3v_1+v_2+v_3 = 1 \\ 2v_2+v_4 = 1 \\ v_1,\dots,v_4 \geq 0 \end{cases}$$

Let us solve (P_1 st)

$n = 4, m = 2, c = (c_1, c_2, c_3, c_4)$

$$c_1 = 1,\quad c_2 = 1,\quad c_3 = 0,\quad c_4 = 0$$

$$A = \begin{array}{cccc} \begin{pmatrix} -3 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccc} A^1 & A^2 & A^3 & A^4 \end{array} \end{array} ; \quad b \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Let $B = (A^3, A^4)$

		0	0			
	1	A^3	A^4	dfb		
1	A^1	$-3<0$	$-1<0$	-1	≤ 0	$-\frac{1}{-3}$ *
1	A^2	0	$-2<0$	-1	≤ 0	$-\frac{1}{0}$
	dfb	-1	-1	0		
		*				

→ dfb ⇒ we apply DSA

2	A^1		dfb	dfb
dfb				

2	A^1	A^4	dfb
A^3	$-\frac{1}{3}$	$-\frac{1}{3} < 0$	$-\frac{1}{3}$
A^2	0	$-2 < 0$	-1
$\uparrow$ fb	$\frac{1}{3}$	$-\frac{2}{3} < 0$	$\frac{1}{3}$

$\frac{-\frac{1}{3}}{-\frac{1}{3}} = 1$

$\frac{-1}{-2} = \frac{1}{2}$ *

*

3	A^1	A^2	dfb
A^3			
A^4			
$\uparrow$ fb	$\frac{1}{3} \geq 0$	$\frac{1}{3} \geq 0$	$\frac{2}{3}$

pfb, hence optimal

An optimal sol (P_1 *) is:

$$u^0{*} = (u_1^0, u_2^0, u_3^0, u_4^0) = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, 0\right)$$

The optimal value (min) is

$$\alpha_{00} = \frac{2}{3}$$

We deduce that an optimal sol of initial problem (P_1) is:

$$\mu^0 = (u^1, u^2) = \left(\frac{1}{3}, \frac{1}{3}\right)$$

and the optimal value (min) is:

$$u_1^0 + u_2^0 = \alpha_{00} = \frac{2}{3}$$

According to theorem 11.2, we have:

$$\hat{w} = \frac{1}{u_1^0 + u_2^0} = \frac{1}{2/3} = \frac{3}{2}$$

Remark: $\hat{w} = \frac{3}{2} \in [1, 2]$

An optimal strategy of Player 1 is:

$$\boxed{\hat{x}^0 = \hat{w} \cdot u^0} = \frac{3}{2}\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

In what concerns the initial game with the pay-off matrix C, we have:

$$w = \hat{w} - k = \frac{3}{2} - 1 = \frac{1}{2} \in [0, 1]$$

An optimal strategy of Player 1 is:

$$x^0 = \hat{x}^0 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

- We solve P2 st:

$n = 4, m = 2$

$$C = (\underset{\underset{-1}{\shortparallel}}{c_1}, \underset{\underset{-1}{\shortparallel}}{c_2}, \underset{\underset{0}{\shortparallel}}{c_3}, \underset{\underset{0}{\shortparallel}}{c_4})$$

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $B = (A^3, A^4)$

	0	0		
	1	A^3	A^4	dfb
-1	A^1	$3>0$	$0>0$	$1>0$ *
-1	A^2	$1>0$	$2>0$	$5>0$
	rfb	$1\geq 0$	$1\geq 0$	0

rfb ⇒ we apply P.S.A

$\frac{1}{3}$ $\frac{1}{4}$

*

2	A^1	A^4	dfb	
A^3	$\frac{1}{3}$	-0	$-\frac{1}{3}$	
A^2	$\frac{1}{3}>0$	$2>0$	$\frac{2}{3}$	≥ 0 *
rfb	$\frac{1}{3}$	1	$-\frac{1}{3}$	

$\frac{\frac{1}{3}}{\frac{1}{3}} = 1$

3	A^1	A^2	dfb
A^3			$-\frac{1}{3} \leq 0$
A^4			$-\frac{1}{3} \leq 0$
rfb	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{2}{3}$

} dfb hence optimal

An optimal sol of $(P_2\ st)$ is:

$$v^\circ st = (v_1^\circ, v_2^\circ, v_3^\circ, v_4^\circ)$$
$$v_1^\circ = \frac{1}{6},\quad v_2^\circ = \frac{1}{2},\quad v_3^\circ = 0,\quad v_4^\circ = 0$$

The optimal value is:

$$\alpha_{00} = -\frac{2}{3}$$

An optimal sol of (P_2) is:

$$v^\circ = (v_1^\circ, v_2^\circ) = \left(\frac{1}{6}, \frac{1}{2}\right)$$

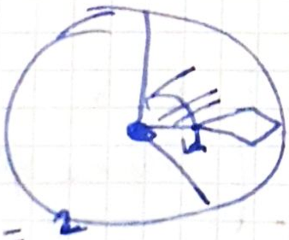
$$\hat{w} = \frac{1}{v_1^\circ + v_2^\circ} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

An optimal strategy of Player 2 is:

$$\hat{y}^\circ = \hat{w} \cdot v^\circ = \frac{5}{3}\left(\frac{1}{5}, \frac{2}{5}\right) = \left(\frac{1}{4}, \frac{3}{4}\right)$$

$$w = \hat{w} - k = \frac{5}{3} - 1 = \frac{1}{2}$$

$$y^\circ = \hat{y}^\circ = \left(\frac{1}{4}, \frac{3}{4}\right)$$



$$y_1^\circ = \frac{1}{4}$$

$$y_2^\circ = \frac{2}{3}$$