Seminar 4 Primal simplex algorithm (PSA)

Ex 1 Solve the following problem by using (PSA)

$$(P)\begin{cases} \text{Minimize } f(x) = 2x_1 - x_2 + x_3 \\ \text{s.t} \\ \left.\begin{array}{l} -x_1 + 2x_3 + x_4 = 0 \\ 3x_1 + x_2 - x_3 = 2 \\ x_1, \dots, x_4 \geq 0 \end{array}\right] (S) \end{cases}$$

Solution: Problem (P) is given in standard form (see Lecture 7)

$$(P)\begin{cases} \text{Minimize } f(x) = \langle c, x \rangle = c_1 x_1 + \dots + c_n x_n \\ Ax = b \\ x \geq 0_n \text{ (componentwise)}, \quad x = (x_1, \dots, x_n) \end{cases}$$

$$0_n = (0, \dots, 0)$$

We have

$n = 4$

$$c = (\underset{\substack{\parallel \\ 2}}{c_1}, \underset{\substack{\parallel \\ -1}}{c_2}, \underset{\substack{\parallel \\ 1}}{c_3}, \underset{\substack{\parallel \\ 0}}{c_4})$$

$$A = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 3 & 1 & -1 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\quad A^1 \quad A^2 \quad A^3 \quad A^4$$

$$\text{rank } A = \underset{\substack{\parallel \\ 2}}{m} < \underset{\substack{\parallel \\ 4}}{n}$$

Consider the basis

$B = (A^4, A^2)$

Then the $\det [A^4, A^2] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Hence rank $A = 2$

The simplex tableau associated to B is:

→ B is a dual feasible basis (d.f.b)

		0	−1	
	1	A^4	A^2	Test d.f.b
2	A^1	−1	3	−5
1	A^3	2	−1	0
	b	$0 \geq 0$	$2 \geq 0$	−2

$A^1 = (-1)A^4 + 3A^2$

$[(-1)\cdot 0 + 3\cdot(-1)] - 2$

$[2\cdot 0 - 1(-1)] - 1 = 0$

Test p.f.b = primal fisability of the basis

B is a primal feasible basis (p.f.b)

(bază primal admisibilă)

⇒ we can use P.SA

$$d_{i,0} = \sum_{j \in B_B} \alpha_{ij} c_j - c_i$$

$$d_{0,b} = \sum_{j \in B_B} \alpha_{0,j} c_j$$

Since b is both p.f.b and d.f.b, it follows that B is an optimal basis

An optimal solution of problem (P) is

$x^0 = x^B = (x_1^0, x_2^0, x_3^0, x_4^0)$

if L7: $x_1^0 = 0$, $x_2^0 = 2$, $x_3^0 = 0$, $x_4^0 = 0$

The optimal value (min) of f on S is:

$\alpha_{0,0} = -2$

of L7

Checking: $f(x^0) = 2x_1^0 - x_2^0 + x_3^0 = 2 \cdot 0 - 2 + 0 = -2$

$f(x) = 2x_1 - x_2 + x_3$

$x^0 = (0, 2, 0, 0)$

Solve the Resource allocation Problem with the following data (see Seminar 1) by means of P.S.A.

Resources \ Products	P_1	P_2	Available amount of resources ~~the requested amount of nutrients~~
R_1	1	2	6
R_2	0	1	4
R_3	3	0	9
Unit cost	1	3	

$$\begin{cases} \text{Maximize } f(x) = 1x_1 + 3x_2 \\ \text{s.t.} \\ 1x_1 + 2x_2 \geq 6 \\ 0x_1 + 1x_2 \geq 4 \\ 3x_1 + 0x_2 \geq 9 \\ x_1, x_2 \geq 0 \end{cases}$$

This problem has no standard form (because of "$\leq$" type constraints)

In order to apply the P.S.A. we will transform it into an equivalent problem whose constraints are in standard form

$$(P) \begin{cases} \text{Minimize } g(x) = -f(x) = -x_1 - 3x_2 \\ \left.\begin{array}{l} x_1 + 2x_2 + x_3 = 6 \\ x_2 + x_4 = 4 \\ 3x_1 + x_5 = 9 \\ x_1, \dots, x_5 \geq 0 \end{array}\right] (S) \end{cases}$$

$n = 5 \quad m = 3, \quad c = (c_1, c_2, c_3, c_4, c_5) = (-1, -3, 0, 0, 0)$

$$A = \begin{array}{ccccc} & & & & \\ \left(\begin{array}{ccccc} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}\right) \\ A^1 \quad A^2 \quad A^3 \quad A^4 \quad A^5 \end{array} ; \quad b = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$$

Let $B = (A^3, A^4, A^5)$

		0	0	0	
	1	A^3	A^4	A^5	dfb
-1	A^1	$1>0$	0	$3>0$	$1>0$
-3	A^2	$2>0$	$1>0$	0	$3>0$ *
	rfb	$6 \geq 0$	$4 \geq 0$	$9 \geq 0$	0

$\frac{6}{2}$ * $\quad \frac{4}{2}$ $\quad$ / $\longrightarrow$ B is a A.f.b, hence we apply P.S.A.

2	A^2	A^4	A^5	dfb
A^1	$\frac{1}{2}$			$-\frac{1}{2}$
A^3	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{3}{2}$
Affy	3	1	9	-9

$-\frac{1}{2} \leq 0$, $\frac{1 \cdot 2 - 1 \cdot 3}{2}$ Gauss Jordan

$-\frac{3}{2} \leq 0$ $\Longrightarrow$ (A^2, A^4, A^5) is d.f.b.

Hence an optimal basis

Dederborn - muzen

An optimal sol of (P) is $x^0_{\text{standard}} = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0)$ with $x_1^0 = 0$, $x_2^0 = 3$, $x_3^0 = 0$, $x_4^0 = 1$, $x_5^0 = 9$

$\min g = \alpha_{00} = 9 \Rightarrow \max f = 9$

The sol of R.A.P is

$x^0 = (x_1^0, x_2^0) = (0, 3)$

We observe that (P) is given in standard form

$$(P)\begin{cases} \text{Minimize } f(x) = 3x_1 + 2x_2 + x_3 \\ -x_1 - 2x_3 + x_5 = -6 \\ -3x_2 - x_3 + x_4 = -4 \\ x_1, \dots x_3 \geq 0 \end{cases} \quad (S)$$

$n = 5$, $m = 2$

$$c = (c_1, c_2, c_3, c_4, c_5) \quad c_1 = 3,\ c_2 = 2,\ c_3 = 1,\ c_4 = 0,\ c_5 = 0$$

$$A = \begin{pmatrix} -1 & 0 & -2 & 0 & 1 \\ 0 & -3 & -1 & 1 & 0 \end{pmatrix} ; \quad b = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

$$\begin{array}{ccccc} A^1 & A^2 & A^3 & A^4 & A^5 \end{array}$$

Let $B = (A^5, A^4$

		0	0		
	1	A^5	A^4	Test d.f.b	
3	A^1	$-1<0$	0	$-3 \leq 0$	$0\cdot(-1)+0\cdot 0-3=-3$ $\frac{-3}{-1}$
2	A^2	0	$-3<0$	$-2 \leq 0$	$0\cdot 0+0\cdot(-3)-2=-2$
1	A^3	$-2<0$	$-1<0$	$-1 \leq 0$	$-2\cdot 0-1\cdot 0-1=-1$ $-\frac{1}{2}$ *
	Test r.f.b	$-6<0$	$-4<0$	0	$-6\cdot 0-4\cdot 0=0$

*

If no negative w_i => set empty

Calculate d.f.b or r.f.b

=> B is d.f.b => we apply the Dual Simplex Alg

2	A^3	A^4	Dist d.f.b
A^1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$
A^2	0	-3	-2
A^5	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
r.f.b	3	-1	3

$\frac{-2}{-3}$

$\frac{-1/2}{-1/2} = 1$

- Divide everything by pivot on same column as pivot
- Divide everything by the opposite of pivot on the same row

3	A^3	A^2	d.f.b
A^1			
A^4			
A^5			
r.f.b	3	$\frac{1}{3}$	$\frac{11}{3}$

d.f.b, according to DSA

(A^3, A^2) is p.f.b, hence optimal basis

An optimal sol of (P) is

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0) = (0, \tfrac{1}{3}, 3, 0, 0)$$

The optimal value (min) of f on S is:

Remark $f(x^0) = 3x_1^0 + 2x_2^0 + x_3^0$
$= 3.0 + 2\frac{1}{3} + 3 = \frac{2}{3} + 3 = \frac{11}{3}$

Ex 2 Solve the foll

$$(P)\begin{cases} \text{Minimise } f(x) = \alpha^2 x_1 + (\alpha-1)^2 x_2 \\ \left.\begin{array}{l} -x_1 \quad + x_3 = 0 \\ -2x_1 - x_2 \quad + x_4 = -5 \\ x_1, \dots, x_5 \geq 0 \end{array}\right] (S) \end{cases}$$

α is a parameter

Solution:

(P) is given in standard form

$n = 4$, $m = 2$

$$c = (\underset{\substack{\shortparallel \\ \alpha^2}}{c_1}, \underset{\substack{\shortparallel \\ (\alpha-1)^2}}{c_2}, \underset{\substack{\shortparallel \\ 0}}{c_3}, \underset{\substack{\shortparallel \\ 0}}{c_4})$$

$$A \begin{pmatrix} -1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \quad b \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$A^1 \quad A^2 \quad A^3 \quad A^4$$

$B = (A^3, A^4)$

		$\overset{0}{A^3}$	$\overset{0}{A^4}$	$\alpha R b$
α^2	A^1	-1	$-2 \leq 0$	$-\alpha^2 \leq 0$
$\alpha^2-2\alpha+1$	A^2	0	$-1 < 0$	$-\alpha^2+2\alpha-1 \leq 0$
	$1Rb$	$\boxed{0}$	$\boxed{-5}$	0

B is $\alpha R b \Rightarrow \frac{-\alpha^2}{-2} \Rightarrow$ we can apply $\frac{-\alpha^2+2\alpha-1}{-1}$ D.F.A.

Simplexul primal nu merge pt că $-5<0$

$-\alpha^2+2-1 = -(-\alpha-1)^2 \leq 0$

Alegem cel mai mic nr din sectorul orizontal

$$\frac{-\alpha^2}{-2} = \frac{\alpha^2}{2}$$

$$\frac{-\alpha^2+2\alpha-1}{-1} = \alpha^2-2\alpha+1$$

$$\frac{\alpha^2}{2} \leq \alpha^2-2\alpha+1$$

$$\Leftrightarrow \alpha^2 \leq 2\alpha^2-4\alpha+2 \Rightarrow \alpha^2-4\alpha+2 \geq 0$$

$$\Delta = (-4)^2-4\cdot 2 = 16-8 = 8 > 0$$

$$\alpha_{1,2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

α	$-\infty$		$2-\sqrt{2}$		$2+\sqrt{2}$		$+\infty$
	+ + +		0	− −	0	+ + +	

Case 1) $\alpha \in (-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$

1	A^3	A^4	dfb
A^1	-1	-2	$-\alpha^2$
A^2	0	-1	$-\alpha^2+\alpha-1$
rfb	0	-5	0

$\frac{\alpha^2}{2}$ * (in Case 1)

$\alpha^2-2\alpha+1$

2	A^3	A^1	dfb
A^4			
A^2			
rfb	$\frac{5}{2}$	$\frac{5}{2}$	

$\geq 0 \geq 0$

$\Rightarrow (A^3, A^1)$ is rfb, hence an optimal basis

An optimal sol of (P) in Case 1 is:

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0) = \left(\tfrac{5}{2}, 0, \tfrac{5}{2}, 0\right)$$

and the optimal value (min) of f on S is

$$\alpha_{00} = \frac{5\alpha^2}{2}$$

Case 2 :

1	A^3	A^4	dfb
A^1	-1	-2	$-\alpha^2$
A^2	0	-1	$\alpha^2+2\alpha-1$
rfb	0	-5	0

* Case 2

2	A^3	A^2	dfb
A^1			
A^4			
nbb	0	5	$-5\alpha^2+10\alpha-5$

$\Downarrow$

(A^3, A^2) is pbb, hence optimal basis

$\Rightarrow$ An optimal sol of (P) in Case 2 is

$$x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$$
$$\ \ \shortparallel \quad\ \shortparallel \quad\ \shortparallel \quad\ \shortparallel$$
$$\ \ 0 \quad\ 5 \quad\ 0 \quad\ 0$$

The optimal value of f is $\alpha_{00} = -5\alpha^2+10\alpha-5$

Ex 3 _ Apply the dual simplex algorithm to the foll pb.

$$(P)\begin{cases} \text{Minimize } f(x) = -x_2 - x_3 + 3x_5 - 3x_6 \\ 2x_1 + x_2 + x_3 + x_5 + 3x_6 = 0 \\ x_1 + x_3 + x_4 - x_5 = -2 \\ x_1, \dots x_6 \geq 0 \end{cases} \quad (S)$$

(P) is in std form

$n = 6$, $m = 2$

$$C = (c_1, c_2, c_3, c_4, c_5, c_6)$$
$$\quad\ \ 0 \quad -1 \quad -1 \quad 0 \quad 3 \quad -3$$

$$A = \left(\begin{array}{cccccc} 2 & 1 & 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 1 & -1 & 0 \end{array}\right), \; b = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$A^1 \quad A^2 \quad A^3 \quad A^4 \quad A^5 \quad A^6$

		-1	0	
	1	A^2	A^4	dfb
0	A^1	2	1	$-2 \le 0$
-1	A^3	1	1	$0 \le 0$
3	A^5	1	$-1 < 0$	$-4 \le 0$
-3	A^6	3	0	$0 \le 0$
	pfb	0	$-2 < 0$	0
			*	

$\Rightarrow$ B- is DFB

We apply D. FA

$\frac{-4}{-1}$ *

2	A^2	A^5	dfb
A^1	$3 \not< 0$		
A^3	$2 \not< 0$		
A^4	$1 \not< 0$		
A^6	$3 \not< 0$		
pfb	$-2 < 0$		

$\Rightarrow$ (P) has no feasible sol $\Rightarrow S = \emptyset$

Seminar 6 Matrice gauss

Ex 1 Solve the game whose pay-off matrix is:

$$C = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Solution First we compute $\underline{v}$ and $\overline{v}$ (the so-called lower value and upper value of the game)

$$\left.\begin{array}{l} \alpha_1 = \min\{2, 0\} = 0 \\ \alpha_2 = \min\{-1, 1\} = -1 \end{array}\right\} \Rightarrow \underline{v} = \max\{\alpha_1, \alpha_2\} = 0.$$

$$\left.\begin{array}{l} \beta_1 = \max\{2, -1\} = 2 \\ \beta_2 = \max\{0, 1\} = 1 \end{array}\right\} \Rightarrow \overline{v} = \min\{\beta_1, \beta_2\} = 1$$

According to L10 we know that:

$$0 = \underline{v} \le v \le \overline{v} = 1$$

hence $v \in [0, 1]$

Since $\underline{v} \neq \overline{v}$, the game has no saddle points.

Thus, Theorem 11.1 does not apply

On the other hand, we are not sure wether $v \overset{?}{>} 0$

(we just know that $v \in [0, 1]$)

In order to see Theorem 11.2, which requires that the game's value is positive (>0), we will add a convenient constant $k \in \mathbb{R}$ to all elements of C, such that

$\underline{v} + k = 0$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

For instance, let

$k = 1$

We obtain a new pay-off matrix

$$\hat{C} = C + (k) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

$\hat{\underline{v}} = \underline{v} + k = 0 + 1 = 1$

$\hat{\overline{v}} = \overline{v} + k = 1 + 1 = 2$

$\hat{v} = v + k = v + 1 \in [1,2] \Rightarrow \hat{v} > 0$

La examen ne dă matrice aranjată fără translatare

$\Rightarrow$ we can apply th 11.2 to the new game, whose pay-off matrix is $\hat{C}$.

We associate to $\hat{C}$ two optimization problems.

$$(P1) \begin{cases} \text{Minimize } u_1 + u_2 \\ 3 \cdot u_1 + 0 \cdot u_2 \geq 1 \\ 1 \cdot u_1 + 2 u_2 \geq 1 \\ u_1, u_2 \geq 0 \end{cases} \quad \text{and} \quad (P2) \begin{cases} \text{Maximise } v_1 + v_2 \\ 3 \cdot v_1 + 1 \cdot v_2 \leq 1 \\ 0 \cdot v_1 + 2 \cdot v_2 \leq 1 \\ v_1, v_2 \geq 0 \end{cases}$$

Next we solve (P1) and (P2)

In order to use the SIMPLEX Algorithm.
we have to transform (P1) and (P2) in standard form

$$(P_1\ st)\ \begin{cases} \text{Minimize } u_1 + u_2 \\ 3u_1 - u_3 = 1 \\ u_1 + 2u_2 - u_4 = 1 \\ u_1, \dots, u_4 \geq 0 \end{cases} \iff$$

$$\iff \begin{cases} \text{Minimize } u_1 + u_2 \\ -3u_1 + u_3 = -1 \\ -u_1 - 2u_2 + u_4 = -1 \\ u_1, \dots, u_4 \geq 0 \end{cases}$$

and

$$(P_2\ st)\ \begin{cases} \text{Minimize } -v_1 - v_2 \\ 3v_1 + v_2 + v_3 = 1 \\ 2v_2 + v_4 = 1 \\ v_1, \dots, v_4 \geq 0 \end{cases}$$

Let us solve (P_1 st)

$n=4, m=2, c=(c_1, c_2, c_3, c_4)$
$\ \| \quad \| \quad \| \quad \|$
$\ 1 \quad 1 \quad 0 \quad 0$

$$A = \begin{pmatrix} -3 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix} ; \quad b \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$
$$\quad A^1 \quad A^2 \quad A^3 \quad A^4$$

Let $B = (A^3, A^4)$

		0	0			
	1	A^3	A^4	dfb		
1	A^1	$-3<0$	$-1<0$	-1	≤ 0	$\frac{-1}{-3}$ *
1	A^2	0	$-2<0$	-1	≤ 0	$-\frac{1}{0}$
	dfb	-1	-1	0		
		*				

→ dfb ⇒ we apply DSA

2	A^1		dfb	dfb
dfb				

2	A^1	A^4	$d f b$
A^3	$-\frac{1}{3}$	$-\frac{1}{3} < 0$	$-\frac{1}{3}$
A^2	0	$-2 < 0$	-1
$\uparrow f b$	$\frac{1}{3}$	$-\frac{2}{3} < 0$	$\frac{1}{3}$

$\frac{-\frac{1}{3}}{-\frac{1}{3}} = 1$

$\frac{-1}{-2} = \frac{1}{2}$ *

3	A^1	A^2	$d f b$
A^3			
A^4			
$\uparrow f b$	$\frac{1}{3} \geq 0$	$\frac{1}{3} \geq 0$	$\frac{2}{3}$

$\uparrow f b$, hence optimal

An optimal sol $(P_1 *)$ is:

$$u^0 * = (u_1^0, u_2^0, u_3^0, u_4^0) = \left(\frac{1}{3}, \frac{1}{3}, 0, 0\right)$$

The optimal value (min) is

$\alpha_{00} = \frac{2}{3}$

We deduce that an optimal sol of initial problem (P_1) is:

$\mu^0 = (u^1, u^2) = \left(\frac{1}{3}, \frac{1}{3}\right)$

and the optimal value (min) is:

$u_1^0 + u_2^0 = \alpha_{00} = \frac{2}{3}$

According to theorem 11.2, we have:

$$\hat{w} = \frac{1}{u_1^0 + u_2^0} = \frac{1}{2/3} = \frac{3}{2}$$

Remark: $\hat{w} = \frac{3}{2} \in [1, 2]$

An optimal strategy of Player 1 is:

$$\boxed{\hat{x}^0 = \hat{w} \cdot u^0} = \frac{3}{2}\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

In what concerns the initial game with the pay-off matrix C, we have:

$$w = \hat{w} - k = \frac{3}{2} - 1 = \frac{1}{2} \in [0, 1]$$

An optimal strategy of Player 1 is:

$$x^0 = \hat{x}^0 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

- We solve P2 *:

$n = 4, m = 2$

$$C = (\underset{\underset{-1}{\shortparallel}}{c_1}, \underset{\underset{-1}{\shortparallel}}{c_2}, \underset{\underset{0}{\shortparallel}}{c_3}, \underset{\underset{0}{\shortparallel}}{c_4})$$

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $B = (A^3, A^4)$

1	A^3	A^4	dfb	
	0	0		
-1 A^1	$3>0$	$0>0$	$1>0$	*
-1 A^2	$1>0$	$2>0$	$1>0$	
rfb	$1\geq 0$	$1\geq 0$	0	

rfb ⇒ we apply P.S.A

$\frac{1}{3}$ $\frac{1}{1}$

*

2	A^1	A^4	dfb	
A^3	$\frac{1}{3}$	-1	$-\frac{1}{3}$	
A^2	$\frac{1}{3}>0$	$2>0$	$\frac{2}{3}$	≥ 0 *
rfb	$\frac{1}{3}$	1	$-\frac{1}{3}$	

$\frac{\frac{1}{3}}{\frac{1}{3}} = 1$

3	A^1	A^2	dfb
A^3			$-\frac{1}{3} \leq 0$
A^4			$-\frac{1}{3} \leq 0$
rfb	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{2}{3}$

} dfb hence optimal

An optimal sol of $(P_2\ st)$ is:

$$v^0 st = (v_1^0, v_2^0, v_3^0, v_4^0)$$
$$v_1^0 = \frac{1}{6},\quad v_2^0 = \frac{1}{2},\quad v_3^0 = 0,\quad v_4^0 = 0$$

the optimal value is:

$$\alpha_{00} = -\frac{2}{3}$$

An optimal sol of (P_2) is:

$$v^0 = (v_1^0, v_2^0) = \left(\frac{1}{6}, \frac{1}{2}\right)$$

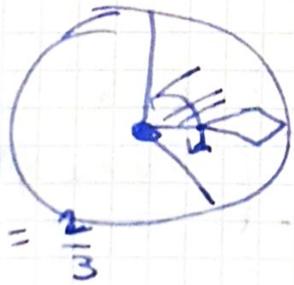
$$\hat{w} = \frac{1}{v_1^0 + v_2^0} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

An optimal strategy of Player 2 is:

$$\hat{y}^0 = \hat{w} \cdot v^0 = \frac{5}{3}\left(\frac{1}{5}, \frac{2}{5}\right) = \left(\frac{1}{4}, \frac{3}{4}\right)$$

$$w = \hat{w} - k = \frac{5}{3} - 1 = \frac{1}{2}$$

$$y^0 = \hat{y}^0 = \left(\frac{1}{4}, \frac{3}{4}\right)$$



$y_1^0 = \frac{1}{4}$

$y_2^0 = \frac{2}{3}$