Formula lui Taylor

Definition 1 *Fie $D \subseteq \mathbb{R}, x_0 \in \mathbb{R}$ si $f : D \to \mathbb{R}$ derivabila de n ori in x_0. Functia polinomiala $T_n f : \mathbb{R} \to \mathbb{R}$*

$$T_n f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

*se numeste **Polinomul lui Brooke-Taylor de grad n**, atasat functiei f si punctului x_0. Functia $r_n f : D \to \mathbb{R}$*

$$r_n f(x) = f(x) - T_n f(x), \forall x \in D$$

*se numeste **restul Taylor de grad n** atasat functiti f in punctul x_0.*

$$f = T_n f + r_n f$$

*se numeste **formula lui Taylor de ordin n**.*

Theorem 2 *(teorema lui Taylor-Young) Fie $I \subseteq \mathbb{R}$ un interval, $x_0 \in i$ si $f : I \to \mathbb{R}$. Daca f e de n ori derivabila in x_0 atunci exista o fnctie $\alpha_n f : I \to \mathbb{R}$ astfel incat:*

1. $(\alpha_n f)(x_0) = 0$;
2. $\alpha_n f$ *e continua in* x_0;
3. $\forall x \in X$

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + (x-x_0)^n (\alpha_n f)(x).$$

Theorem 3 *(teorema lui Taylor) Fie $I \subseteq \mathbb{R}$ un interval, $x_0 \in i$ si $f : I \to \mathbb{R}$. Daca f e de $n+1$ ori derivabila in x_0. Atunci*

$$\forall p \in \mathbb{N}, p \geq 1 \text{ si } x \in I \backslash \{x_0\} \ \exists c \text{ intre } x \text{ si } x_0 \text{ astfel incat}$$

$$(r_n f)(x) = \frac{(x-x_0)^p (x-c)^{n-p+1}}{n! p} f^{n+1}(c).$$

Remark 4 *Deoarece c e cuprins intre x si x_0, putem defini*

$$\theta := \frac{c-x_0}{x-x_0}, \text{ deci } \theta \in]0,1[\text{ si } c = x_0 + \theta(x-x_0).$$

Restul se poate exprima, printre altele, si intr-una din urmatoarele forme:

$$(r_n f)(x) = \frac{(x-x_0)^{n+1}(1-\theta)^{n-p+1}}{n! p} f^{(n+1)}(x_0 + \theta(x-x_0)) \quad \textbf{\textit{Schlömilch-Roche}};$$

$$(r_n f)(x) = \frac{(x-x_0)^{n+1}(1-\theta)^{n}}{n!} f^{(n+1)}(x_0 + \theta(x-x_0)) \quad \textbf{\textit{Cauchy}};$$

$$(r_n f)(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \theta(x-x_0)) \quad \textbf{\textit{Lagrange}}.$$

Remark 5 *Formula lui Taylor de ordin n corespunzatoare lui f in punctul $x_0 = 0$, cu restul lui Lagrange se numeste* ***formula lui Maclaurin****:*

$$f(0) = f(0) + \frac{x}{1!}f'(0) + ... + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x) \ \ cu\ \theta \in]0,1[.$$

Exemple

- $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$, iar $f^{(k)}(x) = e^x$ cu k intreg

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!}... + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}$$

- $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$, iar $f^{(k)}(x) = \sin(x + \frac{k\pi}{2})$ cu k intreg

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!}\cos(\theta x), \ \text{cu } \theta \in]0,1[.$$

- $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos x$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + (-1)^n \frac{x^{2n}}{(2n)!}\cos(\theta x), \ \text{cu } \theta \in]0,1[.$$

- $f :]-1, \infty[\to \mathbb{R}, f(x) = \ln(1+x)$, iar $f^{(k)}(x) = (-1)^k \frac{(k-1)!}{(1+x)^k}$ cu k intreg

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \frac{x^{n+1}}{(n+1)}\frac{1}{(1+\theta x)^{n+1}}, \ \text{cu } \theta \in]0,1[.$$

Caracterizari ale punctelor de optim cu ajutorul derivatelor de ordin superior

Theorem 6 *Fie $I \subseteq \mathbb{R}$ un interval, $x_0 \in I$, si $f : I \to \mathbb{R}$ o functie de $n \geq 2$ ori derivabila in x_0 astfel incat:*

(i) $f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0$,

(ii) $f^{(n)}(x_0) \neq 0$.

Atunci:

1. *x_0 e un punct de* ***minim local*** *al lui f relativ la I daca*

   $$n \text{ e par si } f^{(n)}(x_0) > 0;$$

2. *x_0 e un punct de* ***maxim local*** *al lui f relativ la I daca*

   $$n \text{ e par si } f^{(n)}(x_0) < 0;$$

3. *x_0 e un punct de* ***optim local*** *al lui f relativ la I daca si numai daca*

   $$n \text{ e par }.$$