1st Semester, 2021-2022

3rd Year, Math & CompSci

Mathematical Statistics

Seminar Exercises: Week 1

Exercise 1 (Properties of the Gamma function)**.** We define the **Gamma function** by:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

for every $a > 0$. Show the following properties:

1. For every $a > 0$ we have: $\Gamma(a+1) = a\Gamma(a)$;

2. For every $n \in \mathbb{N}$ we have: $\Gamma(n) = (n-1)!$;

3. For every $a > 0$ we have: $\Gamma(a) = 2\displaystyle\int_0^\infty x^{2a-1} e^{-x^2} dx$.

Exercise 2 (Properties of the Beta function)**.** We define the **Beta function** by:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

for every $a, b > 0$. Show the following properties:

1. $B(a,b) = B(b,a)$, for every $a, b > 0$;

2. $B(a,1) = 1/a$ and $B(1,b) = 1/b$, for every $a, b > 0$;

3. $B(a,b) = \dfrac{a-1}{b} B(a-1, b+1)$, for every $a > 1$, $b > 0$;

4. $B(a,b) = \dfrac{b-1}{a} B(a+1, b-1)$, for every $a > 0$, $b > 1$;

5. $B(a,b) = \dfrac{a-1}{a+b-1}B(a-1,b)$, for every $a > 1$, $b > 0$;

6. $B(a,b) = \dfrac{b-1}{a+b-1}B(a,b-1)$, for every $a > 0$, $b > 1$;

7. $B(k+1, n-k+1) = \dfrac{1}{(n+1)C_n^k} = \dfrac{k!(n-k)!}{(n+1)!}$ for $k, n \in \mathbb{N}$, $k \leq n$.

8. $\mathbf{B}(\mathbf{a}, \mathbf{b}) = \dfrac{\mathbf{\Gamma(a)\Gamma(b)}}{\mathbf{\Gamma(a+b)}}$, for every $a, b > 0$;

Proposition (Can be accepted without proof)**.** For every $a \in (0,1)$:

$$\Gamma(a) \cdot \Gamma(1-a) = B(a, 1-a) = \frac{\pi}{\sin(a\pi)}$$

Then:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx = \int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi}$$

Definition. Let X be a continuous random variable with probability density function (in short, **pdf**) f. Then we have the following notions:

- The **mean value** (also called **expected value** or **expectation**):

$$E(X) = \int_{\mathbb{R}} xf(x)dx$$

- The **variance**:

$$V(X) = E[(X - E(X))^2] = E(X^2) - E(X)^2$$

- The **moment of order** k for $k \in \mathbb{N}$:

$$\nu_k = E(X^k) = \int_{\mathbb{R}} x^k f(x)dx$$

Exercise 3. For $n \in \mathbb{N}^*$ let $X \sim \textit{Student } T(n)$, that is, its pdf is given by:

$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \forall\ x \in \mathbb{R}$$

Find the mean value and the variance of X.

Exercise 4. For $a, b > 0$ let $X \sim Gamma(a, b)$, that is, its pdf is given by:

$$f(x) = \begin{cases} 0, \ x \leq 0 \\ \dfrac{1}{b^a \Gamma(a)} x^{a-1} e^{-\frac{x}{b}}, \ x > 0 \end{cases}$$

Find the moments of order k of X for $k \in \mathbb{N}^*$

Exercise 5. For $a, b > 0$ let $X \sim Beta(a, b)$, that is, its pdf is given by:

$$f(x) = \begin{cases} 0, \ x \leq 0 \text{ or } x \geq 1 \\ \dfrac{1}{B(a,b)} x^{a-1} (1-x)^{b-1}, \ x \in (0,1) \end{cases}$$

Find the moments of order k of X for $k \in \mathbb{N}^*$