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Seminar W7 - 831

A ***point estimator*** *for the target parameter* θ *is a statistic:*

$$\overline{\theta} = \theta(X_1, X_2, \ldots, X_n)$$

We have the following notions:

- ***unbiased estimator:*** $E(\overline{\theta}) = \theta$ *(the* ***bias:*** $B := E(\overline{\theta}) - \theta$*);*
- ***absolutely correct estimator:*** $E(\overline{\theta}) = \theta,\ \lim_{n\to\infty} V(\overline{\theta}) = 0$*;*
- ***consistent estimator:*** $\overline{\theta} \xrightarrow{p} \theta$*;*
- *The* ***efficiency*** *of an absolutely correct estimator* $\overline{\theta}$ *is*

$$e(\overline{\theta}) = \frac{1}{I_n(\theta)V(\overline{\theta})}$$

$\overline{\theta}$ *is an* ***efficient estimator*** *for* θ *if* $e(\overline{\theta}) = 1$

- ***Fisher's (quantity of) information*** *relative to* θ*:*

$$I_n(\theta) = E\left(\left(\frac{\partial \ln L(X_1, X_2, \ldots, X_n; \theta)}{\partial \theta}\right)^2\right)$$

If the range of X *does not depend on* θ*:*

$$I_n(\theta) = -E\left(\frac{\partial^2 \ln L(X_1, X_2, \ldots, X_n; \theta)}{\partial \theta^2}\right)$$

or

$$I_n(\theta) = nI_1(\theta) \quad \to -E\left(\frac{\partial^2 \ln L(x;\theta)}{\partial \theta^2}\right)$$

- *The* ***likelihood function*** *of the sample* $X_1, X_2, \ldots, X_n$*:*

$$L(X_1, X_2, \ldots, X_n; \theta) = \prod_{i=1}^{n} f(X_i; \theta);$$

Exercise 4. Let $X \sim N(\mu, \sigma)$. For a random sample $X_1, X_2, \ldots, X_n$ we consider the estimator $\overline{s} = \frac{1}{n}\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n}\left|X_i - \mu\right|$. Show that it is an absolutely correct estimator for σ and find its efficiency.

Sol: $X \sim N(\mu, \sigma) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

We have to show that $E(\overline{s}) = \sigma$ and $\lim_{n\to\infty} V(\overline{s}) = 0$

$E(\overline{s}) = E\left(\frac{1}{n}\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n}|X_i - \mu|\right) = \frac{1}{n}\cdot\sqrt{\frac{\pi}{2}}\cdot E\left(\sum_{i=1}^{n}|X_i - \mu|\right) =$

$= \frac{1}{n}\cdot\sqrt{\frac{\pi}{2}}\cdot\sum_{i=1}^{n} E(|X_i - \mu|) = \frac{1}{n}\cdot\sqrt{\frac{\pi}{2}}\cdot n\cdot E(|X - \mu|) =$

$$= \sqrt{\frac{\pi}{2}} \cdot E(|X-\mu|)$$

$$X \sim N(\mu, \sigma) \Rightarrow Y = X-\mu \sim N(0, \sigma)$$

$$E(|Y|) = ? \quad , \quad f_Y(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-x^2}{2\sigma^2}}$$

$$E(|Y|) = \int_{\mathbb{R}} |x| \cdot f_Y(x)\, dx = \int_{-\infty}^{0} (-x) \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-x^2}{2\sigma^2}} dx +$$

$$+ \int_0^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{\frac{-x^2}{2\sigma^2}} dx = \frac{2}{\sqrt{2\pi}\cdot\sigma} \int_0^{\infty} x\, e^{\frac{-x^2}{2\sigma^2}} dx =$$

$$= \frac{2}{\sqrt{2\pi}\cdot\sigma} \cdot \int_0^{\infty} \left(-e^{\frac{-x^2}{2\sigma^2}} \cdot \sigma^2\right)' dx = \frac{2}{\sqrt{2\pi}\cdot\sigma} \cdot \left(-e^{\frac{-x^2}{2\sigma^2}} \cdot \sigma^2\right)\Big|_0^{\infty} =$$

$$= \frac{2}{\sqrt{2\pi}\cdot\sigma} \cdot \sigma^2 = \frac{2\sigma}{\sqrt{2\pi}} = \sigma\sqrt{\frac{2}{\pi}}$$

$$E(S) = \sqrt{\frac{\pi}{2}} \cdot \sigma\sqrt{\frac{2}{\pi}} = \sigma \Rightarrow S \text{ unbiased estimator of } \sigma$$

$$V(S) = V\left(\frac{1}{n}\sqrt{\frac{\pi}{2}} \sum_{i=1}^{n} |X_i - \mu|\right) = \left(\frac{1}{n} \cdot \sqrt{\frac{\pi}{2}}\right)^2 \cdot$$

$$\cdot V\left(\sum_{i=1}^{n} |X_i - \mu|\right) = \frac{1}{n^2} \cdot \frac{\pi}{2} \cdot \sum_{i=1}^{n} V(|X_i - \mu|) =$$

$$= \frac{1}{n^2} \cdot \frac{\pi}{2} \cdot n \cdot V(|X_1 - \mu|) = \frac{\pi}{2n} \cdot V(|Y|) = \frac{\pi}{2n} \cdot \left(E(|Y|^2) -\right.$$

$$\left. - E(|Y|)^2\right) = \frac{\pi}{2n} \cdot \left(E(Y^2) - E(|Y|)^2\right) =$$

$$= \frac{\pi}{2n} \cdot \left(E(Y^2) - \sigma^2 \cdot \frac{2}{\pi} \right)$$

$$Y \sim N(0, \sigma) \quad , \quad f_Y(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2\sigma^2}}$$

$$E(Y^2) = \int_{\mathbb{R}} x^2 \cdot f_Y(x)\,dx = \int_{\mathbb{R}} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2\sigma^2}}\,dx =$$

the integrand is even

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot 2\int_0^\infty x^2 \cdot e^{\frac{-x^2}{2\sigma^2}}\,dx$$

$$\int_0^\infty x \cdot \left(e^{\frac{-x^2}{2\sigma^2}}\right)' \cdot (-\sigma^2)\,dx =$$

$$= -\sigma^2 \cdot \left[\left(x\,e^{\frac{-x^2}{2\sigma^2}} \right)\Big|_0^\infty - \int_0^\infty e^{\frac{-x^2}{2\sigma^2}}\,dx \right] =$$

$$= \sigma^2 \int_0^\infty e^{\frac{-x^2}{2\sigma^2}}\,dx \overset{\frac{x}{\sqrt{2}\sigma} = y}{=} \sigma^2 \cdot \int_0^\infty e^{-y^2} \cdot \sqrt{2} \cdot \sigma\,dy =$$

$$= \sigma^3 \cdot \sqrt{2} \cdot \underbrace{\int_0^\infty e^{-y^2}\,dy}_{\frac{1}{2}\int_{\mathbb{R}} e^{-x^2}\,dy = \frac{1}{2}\cdot\sqrt{\pi}}$$

$$\Rightarrow E(Y^2) = \frac{2}{\sigma\sqrt{2\pi}} \cdot \sigma^3 \cdot \sqrt{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \sigma^2$$

$$V(\bar{s}) = \frac{\pi}{2n} \cdot \left(\sigma^2 - \sigma^2 \cdot \frac{2}{\pi} \right) = \sigma^2 \cdot \frac{\pi - 2}{2n}$$

$\lim_{n\to\infty} V(\bar{s}) = 0 \Rightarrow \bar{s}$ is an absolutely correct estimator.

$$e(\bar{s}) = \frac{1}{I_n(\sigma) \cdot V(\bar{s})}$$

$X \sim \mathcal{N}(\mu, \sigma) \Rightarrow$ the range of X does not depend on the parameter that we wish to estimate $\Rightarrow$ we can calculate $I_n(\sigma)$ using the formulas:

$$I_n(\sigma) = n \cdot I_1(\sigma) = n \cdot \left(-E\left(\frac{\partial^2 \ln L(X;\sigma)}{\partial \sigma^2} \right) \right)$$

$$L(X;\sigma) = f_X$$

$$\ln L(X;\sigma) = \ln \left(\frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(X-\mu)^2}{2\sigma^2}} \right) =$$

$$= \frac{-(X-\mu)^2}{2\sigma^2} - \ln(\sqrt{2\pi}\,\sigma)$$

$$\frac{\partial^2 \ln L(X;\sigma)}{\partial \sigma^2} = \frac{\partial}{\partial \sigma} \left(-\frac{(X-\mu)^2}{2} \cdot (-2) \cdot \frac{1}{\sigma^3} - \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \sqrt{2\pi} \right) =$$

$$= (X-\mu)^2 \cdot (-3) \cdot \frac{1}{\sigma^4} + \frac{1}{\sigma^2}$$

$$I_n(\sigma) = n \cdot I_1(\sigma) = -n \cdot E\left(\frac{1}{\sigma^2} - \frac{3}{\sigma^4} \cdot (X-\mu)^2 \right) =$$

$$= -n \left(\frac{1}{\sigma^2} - \frac{3}{\sigma^4} \cdot \underbrace{E((X-\mu)^2)}_{= V(X) = \sigma^2} \right) = -n \cdot \frac{-2}{\sigma^2} = \frac{2n}{\sigma^2}$$

$$e(\bar{S}) = \frac{1}{I_n(\sigma) \cdot V(\bar{S})} = \frac{1}{\frac{2n}{\sigma^2} \cdot \sigma^2 \cdot \frac{\pi-2}{2n}} = \frac{1}{\pi - 2} \neq 1 \Rightarrow \bar{S} \text{ is not}$$

an efficient estimator for σ.

Exercise 5. Prove that the sample moment of order 2:

$$\overline{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

is a consistent estimator of the variance $V(X)$. Deduce that the sample standard deviation is a consistent estimator of the standard deviation of $\sigma = \sqrt{V(X)}$.

Hint: For a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, almost sure convergence implies convergence in probability:

$$X_n \overset{a.s.}{\to} X \Longrightarrow X_n \overset{p}{\to} X$$

Sol: We have to show that $\overline{\mu_2} \xrightarrow{P} V(X)$. Instead we will prove a stronger result, that is $\overline{\mu_2} \xrightarrow{a.s.} X$

We will use:

- ***The Strong Law of Large Numbers (SLLN):***

  If $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables with $X_n \sim X$, then

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} E(X)$$

$$\overline{\mu_2} = \frac{1}{n}\cdot\sum_{i=1}^{n}\left(X_i - \underbrace{\overline{X}}_{\frac{1}{n}(x_1+\dots+x_n)}\right)^2$$

Using SLLN: $\frac{1}{n}\cdot\sum_{i=1}^{n} X_i \xrightarrow{a.s.} E(X) =: m$

$$\overline{\mu_2} = \frac{1}{n}\cdot\sum_{i=1}^{n}\left(X_i - m + m - \overline{X}\right)^2 =$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i - m)^2 + \frac{1}{n}\sum_{i=1}^{n}(m-\overline{X})^2 + \frac{2}{n}\cdot\sum_{i=1}^{n}(X_i - m)(m-\overline{X})$$

$$\frac{1}{n}\sum_{i=1}^{n}\overbrace{(X_i - m)^2}^{=:Y_i} = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow[\text{(due to the SLLN)}]{a.s.} E(Y_1) =$$

$$= E\left((X-m)^2\right) = V(X)$$

$$\frac{1}{n}\sum_{i=1}^{n}(m-\overline{X})^2 = \frac{1}{n}\cdot n\cdot(m-\overline{X})^2 = (m-\overline{X})^2 \xrightarrow{a.s} 0,$$

using the SLLN

$$\frac{2}{n}\sum_{i=1}^{n}(X_i-m)(m-\overline{X}) = \frac{2(m-\overline{X})}{n}\cdot\sum_{i=1}^{n}(X_i-m) =$$

$$= \underbrace{2(m-\overline{X})}_{\xrightarrow{a.s} 0}\cdot\underbrace{\frac{1}{n}\cdot\sum_{i=1}^{n}(X_i-m)}_{\xrightarrow{a.s} E(X-m) = E(X)-m = m-m = 0}$$

$\Rightarrow \overline{\mu_2} \xrightarrow{a.s} V(X) \Rightarrow \overline{\mu_2} \xrightarrow{P} V(X) \Rightarrow \overline{\mu_2}$ is a consistent estimator for $V(X)$

$$\overline{S} = \sqrt{\frac{1}{n-1}\cdot\sum_{i=1}^{n}(X_i-\overline{X})^2} = \sqrt{\frac{n}{n-1}\cdot\overline{\mu_2}}$$

$$\overline{\mu_2} \xrightarrow{a.s.} V(X) \Rightarrow \overline{S} \xrightarrow{a.s} \sqrt{V(X)} = \sigma$$