Seminar 3

Exercise 1 Let $M \subseteq \mathbb{R}^n$ be a nonempty convex set, let $f_i\colon M \to \mathbb{R}$, $i = \overline{1,k}$, be convex functions, and let $\alpha_i \geq 0$, $i = \overline{1,k}$ ($k \in \mathbb{N}^*$). Define the functions $f, g, h\colon M \to \mathbb{R}$ for every $x \in M$ by:

$$\begin{aligned} f(x) &:= \max\{f_1(x), \dots, f_k(x)\}, \\ g(x) &:= \min\{f_1(x), \dots, f_k(x)\}, \\ h(x) &:= \alpha_1 f_1(x) + \cdots + \alpha_k f_k(x). \end{aligned}$$

a) Show that $\operatorname{epi} f = \bigcap_{i=1}^k \operatorname{epi} f_i$.
b) Prove that f is convex.
c) Study the convexity of g and h.

Solution. a) The equality follows from the fact that for $x \in M$ and $\lambda \in \mathbb{R}$ the inequality $f(x) \leq \lambda$ holds if and only if $f_i(x) \leq \lambda$, for every $i \in \{1, \dots, k\}$.

b) By Theorem 5.5 of Lecture 5 (characterization of convex functions by means of their epigraph), the sets $\operatorname{epi} f_i$, $i = \overline{1,k}$, are convex. Thus their intersection is also convex. Also, the equality from (a) yields the convexity of f by Theorem 5.5 of Lecture 5.

c) The function g is not necessarily convex, as the following example shows:

Let $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ be the functions defined by $f_1(x) := x$ and $f_2(x) := 0$, for every $x \in \mathbb{R}$. Then $g = \min\{f_1, f_2\}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} x & \text{if} \quad x \leq 0 \\ 0 & \text{if} \quad x > 0. \end{cases}$$

Since $g(\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1) = 0 > -\frac{1}{2} = \frac{1}{2} g(-1) + \frac{1}{2} g(1)$, g is not convex.

We are going to show that function h is convex. Let $x^1, x^2 \in M$ and $t \in [0,1]$. By convexity of f_i, $i = \overline{1,k}$, we get

$$f_i((1-t)x^1 + tx^2) \leq (1-t) f_i(x^1) + t f_i(x^2),$$

for every $i \in \{1, \dots, k\}$. Multiplying the above inequality by the nonnegative number α_i, and summing up the resulting inequalities, we get that $h((1-t)x^{+}tx^2) \leq (1-t)h(x^1) + th(x^2)$. This shows that h is convex. $\square$

Exercise 2 Let $\emptyset \neq M_1 \subseteq \mathbb{R}$ and $\emptyset \neq M_2 \subseteq \mathbb{R}^n$ be convex sets, and let $g\colon M_1 \to \mathbb{R}$ and $h\colon M_2 \to \mathbb{R}$ be functions such that $h(M_2) \subseteq M_1$. Prove that:

a) If g is convex and nondecreasing, and h is convex, then $g \circ h$ is convex.

b) If g is convex and nonincreasing, and h is concave, then $g \circ h$ is convex.

Proof. Put $f := g \circ h\colon M_2 \to \mathbb{R}$.

a) Consider $x^1, x^2 \in M_2$ and $t \in [0,1]$. Then, by convexity of h, we obtain

$$h((1-t)x^1 + tx^2) \leq (1-t)h(x^1) + th(x^2). \tag{3.1}$$

Since the set M_2 is convex and $h(M_2) \subseteq M_1$, we obtain that $h((1-t)x^1+tx^2), h(x^1), h(x^2) \in M_1$. By convexity of M_1, it follows that $(1-t)h(x^1)+th(x^2) \in M_1$. Using now the fact that g is convex and nondecreasing, we obtain from (3.1) that

$$g(h((1-t)x^1 + tx^2)) \leq g((1-t)h(x^1) + th(x^2)) \leq (1-t)g(h(x^1)) + tg(h(x^2)),$$

hence $f((1-t)x^{+}tx^2) \leq (1-t)f(x^1) + tf(x^2)$. This yields the convexity of f.

b) The conclusion follows by a similar argument as above. □

Exercise 3 Let $M \subseteq \mathbb{R}^n$ be a nonempty convex set, let $h\colon M \to \mathbb{R}$ be a convex function, and let $f\colon M \to \mathbb{R}$ be the map defined for all $x \in M$ by $f(x) := [h(x)]^2$.

a) Prove that if $h(M) \subseteq \mathbb{R}_+$, then the map f is convex.

b) Show that if h is affine, then f is convex (even if $h(M) \not\subseteq \mathbb{R}_+$).

c) Is f convex even if $h(M) \not\subseteq \mathbb{R}_+$?

Solution. a) The map $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(x) = x^2$ is convex and increasing, and $f = g \circ h$. The convexity of f follows now from Exercise 2 (a).

b) Let $x^1, x^2 \in M$ and $t \in [0,1]$. Since h is affine, we have that

$$h((1-t)x^1 + tx^2) = (1-t)h(x^1) + th(x^2),$$

hence

$$\begin{aligned} f((1-t)x^1 + tx^2) &= (1-t)^2 f(x^1) + 2(1-t)th(x^1)h(x^2) + t^2 f(x^1) \\ &= (1-t)f(x^1) + tf(x^2) - t(1-t)(h(x^1) - h(x^2))^2 \\ &\leq (1-t)f(x^1) + tf(x^2). \end{aligned}$$

Thus f is convex.

c) If $h(M) \not\subseteq \mathbb{R}_+$, then f may be not convex. For example, let $h\colon \mathbb{R} \to \mathbb{R}$ be defined for all $x \in \mathbb{R}$ by $h(x) = x^2 - 1$. This map is convex by Exercise 1, since $h = f_1 + f_2$, where f_1 is the square of the identity function (notice that f_1 is convex, according to (b)), and f_2 is the constant function -1 (which is obviously convex). However, $f = h^2\colon \mathbb{R} \to \mathbb{R}$ is not convex, because $\frac{9}{16} = f((1-\frac{1}{2})0 + \frac{1}{2} \cdot 1) \not\leq (1-\frac{1}{2})f(0) + \frac{1}{2}f(1) = \frac{1}{2}$. □