Babeş–Bolyai Unversity
Department of Mathematics
PDEs 2021/21
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Partial Differential Equations (MLE0011)
Exercise sheet 1

1. The Variation of Constants Formula. Consider the ODE initial value problem

$$\begin{cases} u' = -au + f(t), & t > 0, \\ u(0) = u_0. \end{cases}$$

Prove that its solution can be represented as

$$u(t) = e^{-at}u_0 + \int_0^t e^{-(t-\tau)} f(\tau)\,\mathrm{d}\tau\,.$$

Revise the formula for the case of a nonzero initial time $u(t_0) = u_0$, with $t_0 \neq 0$.

2. Find the general solution of

$$v''(r) + \frac{1}{2}\,rv'(r) + \frac{1}{2}\,v(r) = 0.$$

3. Check that the hyperbolic trigonometric function $w(t) := \tanh(t)$ solves the nonlinear equation $w' = 1 - w^2$.

4. The Gauss integral[1]**.** Compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\,.$

5. A Gauss integral in the complex plane. Compute the complex integral $\displaystyle\int_{\mathrm{Im}\,z=a} e^{-z^2}\,\mathrm{d}z\,,$ where $z = x + iy \in \mathbb{C}$.

[1] also called Euler-Poisson integral.

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Exercise sheet 2

6. The parabolic[1] scaling. Check that if $u(t,x)$ solves the Heat equation $u_t = u_{xx}$ so does $\widetilde{u}(t,x) = u(\lambda^2 t, \lambda x)$.

7. Traveling wave solutions for the Heat equation. Does the Heat equation

$$u_t = u_{xx}, \quad t \geq 0,\ x \in \mathbb{R}$$

admit bounded traveling wave solutions $u(t,x) = U(x - ct)$, $c \in \mathbb{R}$ which are constant at infinity, i.e.,

$$u(t, -\infty) = \text{const} \quad \text{and} \quad u(t, \infty) = \text{const}$$

Motivate your answer.

8. Compute the solution of the nonhomogeneous transport IVP

$$\begin{cases} u_t = -u_x + 1, \quad t > 0,\ x \in \mathbb{R} \\ u(0,x) = e^{-x^2}, \end{cases}$$

and plot the solution at $t = 0, 1, 2$ (using some math software).

9. Compute the solution of the Wave equation IVP

$$\begin{cases} u_{tt} = \frac{1}{2} u_{xx}, \quad t > 0,\ x \in \mathbb{R} \\ u(0,x) = e^{-x^2}, \\ u_t(0,x) = 0 \end{cases}$$

and plot the solution at $t = 0, 1, 2$ (using some math software).

[1] In the early 1900s, french mathematician Jaques Hadamard proposed a classification for linear PDEs which distinguishes between **elliptic, parabolic** and **hyperbolic** equations depending on the coefficients of the equation. We will not go into further detail, but it is worth mentioning that the Heat Eq. is parabolic while the Wave Eq. is hyperbolic.

Consider the ODE initial value problem

$$\begin{cases} u' = -au + f(t), \quad a \in \mathbb{R},\ t > 0, \\ u(0) = u_0. \end{cases}$$

Prove that its solution can be represented as

$$u(t) = e^{-at}u_0 + \int_0^t e^{-a(t-\tau)} f(\tau)\,\mathrm{d}\tau\,.$$

Revise the formula for the case of a nonzero initial time $u(t_0) = u_0$, with $t_0 \neq 0$.

2. Find the general solution of

$$v''(r) + \frac{1}{2}\,rv'(r) + \frac{1}{2}\,v(r) = 0.$$

3. Check that the hyperbolic trigonometric function $w(t) := \tanh(t)$ solves the nonlinear equation $w' = 1 - w^2$.

4. The Gauss integral[2]. Compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$.

5. A Gauss integral in the complex plane. Compute the complex integral $\displaystyle\int_{\mathrm{Im}\,z=a} e^{-z^2}\,\mathrm{d}z$, where $z = x + iy \in \mathbb{C}$.

[2] also called Euler-Poisson integral.

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Partial Differential Equations (MLE0011)
Exercise sheet 3

10. The Transport equation. Compute the solution of the transport IVP with $c \in \mathbb{R}$

$$\begin{cases} u_t + cu_x = 0, & , \ x \in \mathbb{R}, t > 0, \\ u(0,x) = u_0(x). \end{cases}$$

using the Fourier Transform.
(Hint: don't forget about the **shift formula** $\mathcal{F}\left(u(x-x_0)\right)(y) = e^{-ix_0 y}\mathcal{F}\left(u(x)\right)(y)$.)

11. The Heat IVP. Let us still consider the IVP

$$\begin{cases} u_t = u_{xx}, \\ u(0) = u_0. \end{cases}$$

Find the solution of the IVP when:

a) $u_0(x) = e^{-ax^2}$ with $a > 0$;

b) $u_0(x) = H(1 - |x|)$ with H the Heaviside step function.

12. The Airy equation. Also called Stokes or linear Korteweg-deVries equation, the dispersive[1] equation that we are dealing with is

$$u_t = -u_{xxx}, \quad x \in \mathbb{R}, \ t > 0.$$

Use the Fourier transform approach to represent the solution of an IVP associated to Airy's equation.
(Hint: make use of the **Airy function**

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{z^3}{3} + xz\right) \mathrm{d}z\,,$$

a *special function* named after British astronomer George Biddell Airy.)

[1]Find out why this and quite a few other PDEs are called *dispersive.*

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Partial Differential Equations (MLE0011)
Exercise sheet 4

13. The Heat equation with diffusion coefficient $\mu \neq 1$. Find the representation formula for the solution of

$$\begin{cases} u_t = \mu u_{xx}, & \mu > 0,\ x \in \mathbb{R},\ t > 0 \\ u(0) = u_0, \end{cases}$$

and give an interpretation to the $\mu \ll 1$ small diffusion case.

(Hint: try to scale time and then apply the known, $\mu = 1$, formula.)

14. Positivity, contractivity and energy decay for the Heat eq. Consider the IVP for the Heat eq. (with $\mu = 1$) and assume that the initial temperature profile is continuous $u_0 \in L^2(\mathbb{R}) \cap C(\mathbb{R})$. Prove that

a) $u_0(x) \geq 0$ for all $x \in \mathbb{R}$ implies $u(t)(x) \geq 0$ for all $x \in \mathbb{R}$ and all $t > 0$;

b) $\|u(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$;

c) $\|u(t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}$;

d) $\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) \leq 0$ for $t > 0$, where $E(u) = \|u_x\|^2_{L^2(\mathbb{R})}$.

(Hint: use the fact that $\frac{\mathrm{d}}{\mathrm{d}t} \|w(t)\|^2_{L^2(\mathbb{R})} = 2\langle w(t), \frac{\mathrm{d}w}{\mathrm{d}t}(t)\rangle_{L^2(\mathbb{R})}$, with $\langle v, w\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} v(x)w(x)\,\mathrm{d}x$ the standard L^2 inner product.)

15. The Schrödinger equation (quantum mechanical free particle). Consider

$$\begin{cases} i\psi_t = -\psi_{xx}, & x \in \mathbb{R},\ t > 0, \\ \psi(0, x) = \psi_0(x). \end{cases}$$

Is the Schrödinger equation time-reversible? Represent the solution of the IVP[1] and check wether the representation formula is meaningful also for $t < 0$ or not.

(Hint: replace formally t in the heat equation and representation formula for the solution by it.)

Prove that

a) $\|\psi(t)\|_{L^2(\mathbb{R})} = \|\psi_0\|_{L^2(\mathbb{R})}$ and

b) $\|\psi(t)\|_{L^\infty(\mathbb{R})} = \dfrac{1}{\sqrt{4\pi|t|}} \|\psi_0\|_{L^1(\mathbb{R})}$.

(Hint: For a), one can use the same chain rule as in **14. d)** but for complex-valued functions and with the complex inner product, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\psi(t)\|^2_{L^2(\mathbb{R};\mathbb{C})} = 2\mathrm{Re} \int_{\mathbb{R}} \frac{\mathrm{d}\psi}{\mathrm{d}t}(t)\ \overline{\psi(t)}\,\mathrm{d}x$$

where $\psi = u + \mathrm{i}v$ while $\overline{\psi} = u - \mathrm{i}v$ is the complex conjugate and $\mathrm{Re}\, z$ denotes the real part of a complex number z.)

[1]Note that usually the models that we are dealing with involve real-valued functions $u(t, x) \in \mathbb{R}$. The Schrödinger equation is the notable exception. In this case $\psi(t, x) \in \mathbb{C}$.

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Partial Differential Equations (MLE0011)
Exercise sheet 5

16. Positive nonlinear feed-back and finite time blow-up.
Solve the IVP $u' = u^2$, $u(0) = u_0 > 0$ and show that finite time blow-up occurs for any $u_0 > 0$.a

17. The logistic (Verhulst) population model.
Consider the logistic equation $u' = u(1-u)$.

a) Find stationary (equilibrium) solutions and study their stability.

b) Solve the associated IVP with $u(0) = u_0$ and plot on the same plot (using a computer) solutions originating from $u_0 = -0.1,\ 0.1,\ 0.5,\ 1.2$.

c) Find a Lyapunov function for this model.

18. A generalized logistic model. Consider $u' = u(1-u^2)$.

a) Find a Lyapunov function for this model.

b) Find stationary (equilibrium) solutions and study their stability.

19. The harmonic oscillator (or ideal pendulum). Rewrite $u'' = -\omega^2$ as a first order system of equations and

a) solve the IVP with $u(0) = u_0$ and $v(0) := u'(0) = v_0$;

b) find equilibria and study their stability.

20. The damped harmonic oscillator. Let $a > 0$. Rewrite $u'' = -au' - \omega^2$ as a first order system of equations and

a) solve the IVP with $u(0) = u_0$ and $v(0) := u'(0) = v_0$;

b) find a Lyapunov function for this model (is it strict?).

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Partial Differential Equations (MLE0011)
Exercise sheet 6

21. Remains of the day (of equations on unbounded domains[1]): Radial solutions of the Laplace Equation. Find all radial solutions $u(x) = v(|x|)$ of Laplace's equation

$$\Delta u = 0 \quad \text{on} \quad \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\},$$

where $|x|^2 = x_1^2 + \ldots + x_n^2$ is the Euclidean norm (squared) of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. (Hint: distinguish the cases $n = 2$ and $n \geq 3$. No Fourier Transform required, just solve an ODE for v.)

22. The Laplacian in spherical and polar coordinates[2]. Find the expression of the Laplacian Δ in

a) spherical coordinates (ρ, φ, θ) in $\mathbb{R}^3$;

b) polar coordinates (ρ, φ) in $\mathbb{R}^2$.

23. Green's formulae. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1.

a) If $u \in C^1(\overline{\Omega})$ and $v \in C^2(\Omega)$, then

$$\int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, \mathrm{d}\sigma = \int_{\Omega} (u \Delta v + \nabla u \cdot \nabla v) \, \mathrm{d}x \,. \tag{G1}$$

b) If $u, v \in C^2(\overline{\Omega})$, then

$$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - vu \frac{\partial u}{\partial \nu} \right) \mathrm{d}\sigma = \int_{\Omega} (u \Delta v - v \Delta u) \, \mathrm{d}x \,. \tag{G2}$$

24. The Dirichlet principle ((i)⇔(ii)). Consider the Dirichlet problem

$$\begin{cases} -\Delta u = f, & \text{in} \quad \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{D}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded and of class C^1 while $f \in C(\overline{\Omega})$ and $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$, with

$$C_0^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \right\}.$$

Prove that (i)⇔(ii)

(i) u is a classical solution of (D) (i.e., $u \in C^2(\overline{\Omega})$ and (D) holds);

(ii) u satisfies the *variational identity*

$$\int_{\Omega} (\nabla u(x) \cdot \nabla v(x) - f(x) v(x)) \, \mathrm{d}x = 0 \quad \text{for all} \quad v \in C_0^1(\overline{\Omega}).$$

(Hint: use Green's identity (G1).)

[1] Starting with Week 8 we consider PDEs on bounded spatial domains, e.g., $x \in \Omega = (0, 1) \subset \mathbb{R}$.

[2] Check the MS Teams Files for a graphical depiction of these coordinate systems. To avoid misunderstandings, **please use the same notations as I do.**

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Partial Differential Equations (MLE0011)
Exercise sheet 7

25. Poincaré's Inequality.
Prove Poincaré's inequality (see **[Precup, Theorem 3.13**, p. 38-39]).

26. Dirichlet's principle for $-\Delta u + c(x)u$.
Prove a **[Precup, Theorem 3.12]** type result for the Dirichlet problem

$$\begin{cases} -\Delta u + c(x)u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

What are the necessary conditions (regularity, positivity, etc.) on $c(x)$?

27. Hadamard's counterexample (ill-posedness[1]). Consider the family of Boundary Value Problems

$$\begin{cases} \Delta u = 0, \quad \text{in } \mathbb{R}^2 \\ \\ u = 0 \text{ and } \frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1), \quad \text{on } M = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}. \end{cases}$$

a) Chack that $u_n(x_1, x_2) = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2)$ are solutions of the *family of BVPs* with different data (indexed by n) on M.

b) What happens when $n \to \infty$ (and the second part of the boundary condition converges to $\partial u/\partial x_2 = 0$)? Is the problem well-posed?

28. Courant's counterexample. Consider the Dirichlet problem on the unit disc $\{\Omega = x \in \mathbb{R}^2 : |x| < 1\}$, i.e.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and the sequence of functions (in polar coordinates)

$$u_k(r, \phi) = \begin{cases} k^{1/3}, & \text{for } 0 < r < e^{-2k}, \\ -k^{-2/3}(k - \ln r), & \text{for } e^{-2k} < r < e^{-k}, \\ 0, & \text{for } e^{-k} < r < 1. \end{cases}$$

a) Prove that $u \equiv 0$ is the unique solution of the Dirichlet problem.

b) Plot the function $f : [0, 1] \to \mathbb{R},\ f(r) = u_1(r, 0)$.

c) Compute explictly the Dirichlet energy of u_k, that is $E(u_k) = 2\pi \displaystyle\int_0^1 \left|\frac{\partial u_k}{\partial r}\right|^2 r\,\mathrm{d}r$.

d) Study the limits

$$\lim_{k\to\infty} u_k(0, 0), \quad \lim_{k\to\infty} \|u_k - u\|_{L^2} \quad \text{and} \quad \lim_{k\to\infty} E(u_k).$$

[1] Jaques Hadamard introduces the concept of a **well-posed** BVP, i.e., a problem which satisfies the minimal requirements that one encounters in applications: solutions **exist, are unique** and **depend continuously on the problem's data**. The last, and least trivial property, means that by changing the data of the problem (domain, BCs, source terms, etc.) only very little, the solution will only change very little. Problems which are **not** well-posed are called **ill-posed**.

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Partial Differential Equations (MLE0011)
Exercise sheet 8

29. Separation of variables I: The Laplace equation on a disk. Use Fourier's separation of variables method to solve the BVP (in polar coordinate, i.e., $u = u(\rho, \varphi)$)

$$\begin{cases} \Delta u = 0, \quad \text{for } \rho < 2 \\ u|_{\rho=2} = (\sin\varphi)^2. \end{cases}$$

30. Separation of variables II: The Laplace equation on a rectangle. Let $\Omega = (0, a) \times (0, b)$. Use Fourier's separation of variables method to solve the BVP

$$\begin{cases} \Delta u = 0, \quad \text{in } \Omega \\ u|_{x=0} = u|_{x=a} = u|_{y=b} = 0, \\ u|_{y=0} = g(x) \end{cases}$$

31. The Heat equation: heat sources. Consider the mixed problem for the nonhomogeneous heat equation with a source f

$$\begin{cases} u_t = u_{xx} + f(t, x), \quad \text{in } (0, \infty) \times \Omega \\ u(t, 0) = u(t, 1) = 0, \\ u(0, x) = u_0(x), \end{cases}$$

with u_0 and f given such that a unique solution u exists at all times and $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for all $t \geq 0$. Find a representation formula for u using the eigenvalues and eigenfunctions $\{e_n; \lambda_n\}$.

As usual, we work on a one-dimensional domain $\Omega = (0, 1)$ where the eigenvalue problem for the Laplace operator (in this case the simple second derivative)

$$\begin{cases} -u_{xx} = \lambda u, \quad \text{in } \Omega = (0, 1) \\ u(0) = u(1), \end{cases}$$

has an unbounded sequence of positive eigenvalues $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$ and an $L^2(\Omega)$ orthonormal basis of eigenfunctions (solutions)

$$e_n(x) = \sqrt{2}\sin(n\pi x), \quad \text{with } \langle e_i, e_j \rangle_{L^2(\Omega)} = \begin{cases} 1, \text{ for } i = j, \\ 0, \text{ for } i \neq j. \end{cases}$$

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Partial Differential Equations (MLE0011) Exercise sheet 9

Note that: Throughout these exercises we work on a one-dimensional domain $\Omega = (0,1)$. Furthermore, the corresponding eigenvalue problem for the Laplace operator (in this case the simple second derivative)

$$\begin{cases} -u_{xx} = \lambda u, & \text{in } \Omega = (0,1) \\ u(0) = u(1), \end{cases}$$

has an unbounded sequence of positive eigenvalues $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$ and an $L^2(\Omega)$ orthonormal basis of eigenfunctions (solutions)

$$e_n(x) = \sqrt{2}\sin(n\pi x), \quad \text{with } \langle e_i, e_j\rangle_{L^2(\Omega)} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

32. The Wave equation: Energy conservation. Consider the mixed problem for the wave equation

$$\begin{cases} u_{tt} = u_{xx}, & \text{in } (0,\infty)\times\Omega \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = v_0(x). \end{cases}$$

Prove that $\frac{1}{2}\|u_t(t)\|_{L^2} + \frac{1}{2}\|u_x(t)\|_{L^2} = \frac{1}{2}\|v_0\|_{L^2} + \frac{1}{2}\|u_0\|_{L^2}$ for all $t \geq 0$.
(**Hint:** Use the fact that the ODE governing the evolution of each component

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} = -\lambda_n u_n$$

has the energy conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v_n(t)^2 + \frac{\lambda_n}{2}u_n(t)^2\right) = 0$$

where $v_n(t) := \frac{\mathrm{d}u_n}{\mathrm{d}t}(t)$.)

33. A Damped Wave equation: two types of damping. Consider the mixed problem for the damped wave equation with two types of damping

$$\begin{cases} u_{tt} + u_t - u_{txx} = u_{xx}, & \text{in } (0,\infty)\times\Omega \\ u(t,0) = u(t,1) = 0, \\ u_t(t,0) = u_t(t,1) = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = v_0(x). \end{cases}$$

Give a representation formula for the solution[1].
(**Hint:** observe that in terms of the new unknown $q := u_t - u_{xx}$ the equation reduces to an easily solvable ODE (for each $x \in \Omega$)

$$q_t + q = 0,$$

from which q can be computed. Then, in order to find u, one has to solve the heat equation with source term $u_t = u_{xx} + q$, see exercise **31**.)

34. The Heat equation: Neumann boundary conditions. Consider the mixed problem for the Heat equation where only the Boundary Condition (BC) has been replaced[2]

$$\begin{cases} u_t = u_{xx}, & \text{in } (0,\infty)\times\Omega \\ u_x(t,0) = u_x(t,1) = 0, & \text{for all } t > 0 \\ u(0,x) = u_0(x) \geq 0, & \text{for all } x \in \Omega. \end{cases}$$

a) Prove that the total heat in the rod remains constant at all times, i.e.

$$Q(t) := \int_\Omega u(t,x)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x =: Q_0, \quad \text{for all } t \geq 0$$

(**Hint:** Check that $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\mathrm{d}x = 0$, by using the equation and boundary condition.)

[1] assuming it exists and it has the desired regularity

[2] If we imagine the model as describing the temperature evolution in a heated metal rod, the Neumann, or "no-flux", condition describes the situation in which the rod's ends are insulated and there is no heat flux (no heat loss) through them. By contrast, the Dirichlet condition $u = 0$ at the ends of the rod describes the case in which the rod's ends are in contact with two cold sources (ice blocks) and thus, kept at zero temperature, in which case heat form the rod is lost by transfer to the cold sources.

b) An easy computation shows that the unique equilibrium state of the system is the constant function $u^*(x) = Q_0$ for all $x \in \Omega$ (the same temperature has been reached at all points). Prove that the state converges exponentially to equilibrium

$$\|u(t) - u^*\|_{L^2(\Omega)} \leq e^{-\omega t}.$$

(**Hint:** One has to use both the standard *Poincaré inequality*, applied to u_x in order to obtain the exponential decay of $\|u_x(t) - u_x^*\|_{L^2(\Omega)} = \|u_x(t)\|_{L^2(\Omega)}$ and the so called *Poincaré-Wirtinger inequality*[3] to reach the desired result.)

[3]The *Poincaré-Wirtinger inequality* is

$$\|w - \overline{w}\|_{L^2(\Omega)} \leq \|w_x\|_{L^2(\Omega)}, \quad \text{for all } w \in H^1(\Omega) = \{u \in L^2 : u_x \in L^2\}$$

with $\overline{w} := \frac{1}{|\Omega|} \int_\Omega w \, \mathrm{d}x$. In our case, the measure of Ω is just $|\Omega| = 1$.

$$u' = -au + f(t)$$

$$\underbrace{u' + au}_{} = f(t)$$

construct a derivative (of a product) here

$$\left(e^{at}u(t)\right)' = a e^{at} u(t) + e^{at} u'(t) = e^{at}(u' + au)$$

So we need to multiply by e^{at}

$$u' + au = f(t) \quad | \; e^{at}$$

$$\left(e^{at}u(t)\right)' = \cancel{f}\; e^{at} f(t) \quad \Big| \int_0^t d\tau$$

$$e^{at}u(t) - e^{0}\underbrace{u(0)}_{=u_0} = \int_0^t e^{a\tau} f(\tau)\, d\tau \quad \Big| \; e^{-at}$$

$$u(t) = e^{-at}u_0 + \int_0^t e^{-a(t-\tau)} f(\tau)\, d\tau$$

related to Lecture 2, § 2.1.

(2) $$v''(r) + \underbrace{\frac{1}{2} r v'(r) + \frac{1}{2} v(r)}_{\text{this is just } \frac{1}{2}(r v(r))'} = 0$$

So $$v''(r) + \frac{1}{2}(r v(r))' = 0$$ and we can integrate

(*) $$v'(r) + \frac{1}{2}(r v(r)) = C_0$$ again a non homog. 1st order ODE

$$v' + \frac{1}{2} r v = 0$$ solve homog. eq. first.

$$\frac{dv}{v} = -\frac{1}{2} r dr$$ separation of variables $\left(v' = \frac{dv}{dr}\right)$ and then integrate

$$\int \frac{dv}{v} = -\frac{1}{2}\int r dr, \quad \ln v = -\frac{1}{4}r^2 + \ln C_1$$

$$v_0(r) = C_1 e^{-\frac{r^2}{4}}.$$

Now, variation of const. meth.: $v_p(r) = C_1(r) e^{-\frac{r^2}{4}}$ in (*)

$$C_1'(r) e^{-\frac{r^2}{4}} + \cancel{C_1(r)\left(-\frac{r}{2}\right) e^{-\frac{r^2}{4}}} + \cancel{\frac{1}{2} r C_1(r) e^{-\frac{r^2}{4}}} = C_0$$

$$C_1'(r) = e^{r^2/4} C_0 \qquad C_1(r) = C_0 \int e^{r^2/4} dr$$

2.10.15 EXEMPLU. Să calculăm integrala lui Gauss,

$$I = \int_0^\infty e^{-x^2}\, dx.$$

Cu substituţia $x = t\,y$, obţinem:

$$\begin{aligned}
I &= \int_0^\infty e^{-x^2}\, dx = t\int_0^\infty e^{-t^2 y^2}\, dy;\\
e^{-t^2} I &= t\,e^{-t^2}\int_0^\infty e^{-t^2 y^2}\, dy;\\
I^2 &= \int_0^\infty t\,e^{-t^2}\int_0^\infty e^{-t^2 y^2}\, dy\, dt = \int_0^\infty\int_0^\infty t\,e^{-t^2(1+y^2)}\, dt\, dy\\
&= \frac{1}{2}\int_0^\infty \frac{1}{1+y^2}\, dy = \frac{\pi}{4}.
\end{aligned}$$

Rezultă

$$\boxed{\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.}$$

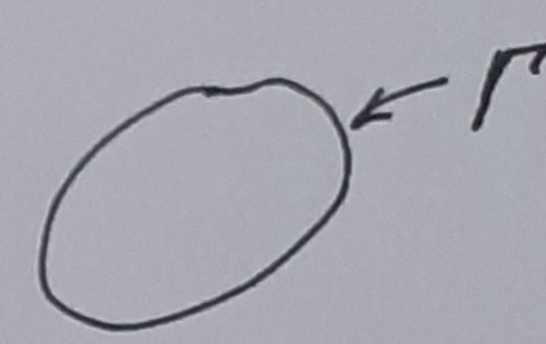
(5) related to Lecture 4 ($\mathcal{F}(e^{-x^2})$)

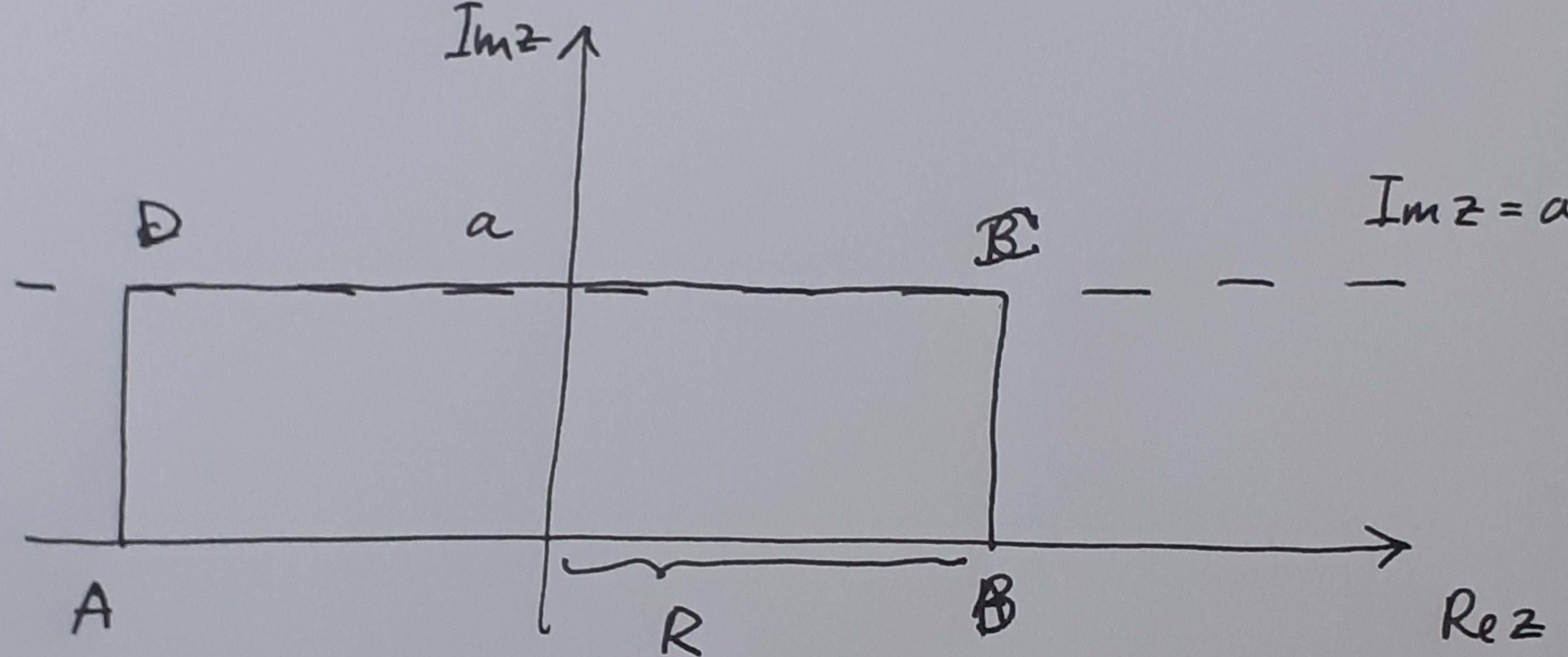
$$\int_{\operatorname{Im} z = a} e^{-z^2}\,dz = ? \qquad z = x + iy \in \mathbb{C}$$

IDEA: use the fact that $\displaystyle\int_{\operatorname{Im} z = 0} e^{-z^2}\,dz = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ ← Gauss

and Cauchy's Theorem

$$\int_\Gamma f(z)\,dz = 0$$





$$\int_{ABCDA} e^{-z^2}\,dz = 0 \quad \text{(Cauchy's Thm)}$$

but due to symmetry $\displaystyle\int_{BC} e^{-z^2}\,dz = -\int_{DA} e^{-z^2}\,dz$

Hence $\displaystyle 0 = \int_{ABCDA} = \int_{AB} + \cancel{\int_{BC}} + \int_{CD} + \cancel{\int_{DA}}$

Cauchy ↗

leads to $\displaystyle\int_{AB} = \int_{DC}$ so $\displaystyle\int_{AB\ (\text{on } \operatorname{Im} z = 0)} e^{-z^2}\,dz = \int_{CD\ (\text{on } \operatorname{Im} z = a)} e^{-z^2}\,dz$

if $R \to \infty$ we get $\displaystyle\int_{\operatorname{Im} z = 0} e^{-z^2}\,dz = \int_{\operatorname{Im} z = a} e^{-z^2}\,dz$

1.

S=S$_0$ + s_1 where s_0 is the solution of the homogenous system and s_1 a particular solution

u'=-au

u(0)=u_0

$s_1(t) = u(t)\phi(t)$.

Solution of the homogenous system: u(t)=$e^{-ta}c$,

and if we impose u(0)=u$_0$ => $u(t) = e^{-ta}u_0$.

$s_1'(t) = -as_1(t) + f(t)$

$(u(t)\phi(t))$'$= -au(t)\phi(t)) + f(t)$ => $u'(t)\phi(t) + u(t)\phi'(t) = -au(t)\phi(t) + f(t)$ <=>

$-au(t)\phi(t) + u(t)\phi'(t) = -au(t)\phi(t) + f(t)$ => $u(t)\phi'(t) = f(t)$ => $\phi'(t) = \frac{1}{u_0}e^{ta}f(t)$

$\phi(t) = \int_0^t \frac{1}{u_0}e^{sa}f(s)ds$. Here, we hon't need a constant because we need only a particular solution.

That means $\phi(t) = \int_0^t \frac{1}{u_0}e^{sa}f(s)ds$.

$s_1(t) = u(t)\phi(t) = e^{-ta}u_0\left[\int_0^t \frac{1}{u_0}e^{sa}f(s)ds\right]$

$= \int_0^t e^{(s-t)a}f(s)ds + u(t) = \int_0^t e^{(s-t)a}f(s)ds$.

Hence, the general solution is

$$S = u(t) + u(t)\phi(t) = \int_0^t e^{(s-t)a}f(s)ds + e^{-ta}u_0$$

2. The general solution of $v''(r) + \frac{1}{2}rv'(r) + \frac{1}{2}v(r) = 0$.

$\frac{1}{2}rv'(r) + \frac{1}{2}v(r) = \frac{1}{2}\left(rv(r)\right)'$.

Our equation become: $v''(r) + \frac{1}{2}\left(rv(r)\right)' = 0 \int$

$v'(r) + \frac{1}{2}rv(r) = c$

The homogenous equation: $v'(r)+\frac{1}{2}rv(r) = 0 => \frac{v'(r)}{v(r)} = \frac{-r}{2} => (\ln v(r))' = \frac{-r}{2}$

$v(r) = e^{-\frac{r^2}{4}}C$

We have the solution of the homogenous equation, and proceed as in the previous execise.

3.

$\tanh(t) = \frac{\sinh(t)}{\cosh(t)}$

$\sinh(t) = \frac{e^t - e^{-t}}{2}; \cosh(t) = \frac{e^t + e^{-t}}{2} => \tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$.

$(\sinh(t))' = \cosh(t)$

$(\cosh(t))' = \sinh(t)$

4.

$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy$

$x = r\cos t; y = r\sin t$
$J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} = r.$
$I^2 = \int_{t=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r dr dt = \int_{t=0}^{2\pi} \int_{r=0}^{+\infty} -\frac{1}{2} \left(e^{-r^2}\right)_{\partial r} dr dt$
$= \int_{t=0}^{2\pi} -\frac{1}{2} e^{-r^2}|_0^{\infty} = \int_{t=0}^{2\pi} \frac{1}{2} = \pi.$

5.
Compute the integral $\int_{imz=a} e^{-z^2} dz.$

We consider the contour ABCDA.
From complex analysis, the integral over a closed contour is 0.
$\int_{ABCDA} e^{-z^2} dz = 0 =>$
$\int_{AB} e^{-z^2} dz + \int_{BC} e^{-z^2} dz + \int_{CD} e^{-z^2} dz + \int_{DA} e^{-z^2} dz = 0$
We observe that $\int_{DA} e^{-z^2} dz = -\int_{BC} e^{-z^2} dz$
$\int_{AB} e^{-z^2} dz = \int_{DC} e^{-z^2} dz$
Take A=(-R,0) and B=(R,0)
We will obtain that
$\int_{DC} e^{-z^2} dz = \int_{[-R,R]} e^{-z^2} dz$
D,C are arbitrary, so we can choose them to belong to segment y=a. In this case
$\int_{DC} e^{-z^2} dz = \int_{\mathrm{Im}\, z=a} e^{-z^2} dz$
$\sqrt{\pi} = \int_{[-R,R]} e^{-z^2} dz = \int_{\mathrm{Im}\, z=a} e^{-z^2} dz.$