16.11.2021

Seminar W8 – 832

Exercise 1. Let $f(x;\theta) = e^{a(x)\alpha(\theta)+b(x)+\beta(\theta)}$, for x in the range of X, where θ is a parameter of X and a, α, b, β are measurable functions, be a probability density function of the (discrete or continuous) characteristic X. Prove that the statistic

$$S = S(X_1, \dots, X_n) = \sum_{i=1}^{n} a(X_i)$$

is sufficient for θ.

Sol: We will apply Fisher's factorization theorem:

$$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} e^{a(x_i)\cdot\alpha(\theta)+b(x_i)+\beta(\theta)} =$$

$$= \prod_{i=1}^{n} e^{a(x_i)\cdot\alpha(\theta)} \cdot \prod_{i=1}^{n} e^{b(x_i)} \cdot \prod_{i=1}^{n} e^{\beta(\theta)} =$$

$$= e^{\sum_{i=1}^{n} a(x_i)\cdot\alpha(\theta)} \cdot e^{\sum_{i=1}^{n} b(x_i)} \cdot e^{n\beta(\theta)} =$$

$$= \underbrace{e^{\alpha(\theta)\cdot S + n\beta(\theta)}}_{h(s;\theta)} \cdot \underbrace{e^{\sum_{i=1}^{n} b(x_i)}}_{g(x_1, x_2, \dots, x_n)}$$

Theorem (Fisher's Factorization Criterion). *A statistic*

$$S = S(X_1, X_2, \dots, X_n)$$

is sufficient for θ, if and only if the likelihood funnction

$$L(X_1, X_2, \dots, X_n; \theta) = \prod_{i=1}^{n} f(X_i;\theta)$$

can be factored into two nonnegative functions

$$L(x_1, x_2, \dots, x_n; \theta) = g(x_1, x_2, \dots, x_n) \cdot h(s;\theta),$$

where $s = S(x_1, x_2, \dots, x_n)$.

$\overset{\text{Fisher}}{\Longrightarrow}$ S sufficient statistic for θ

Exercise 3. Let $X \sim Unif[0,\theta]$, where $\theta > 0$ is a parameter.

(a) Prove that
$$S = \max\{X_1, \ldots, X_n\}$$
is a sufficient and complete statistic for θ.

(b) Show that
$$\overline{\theta} = \frac{n+1}{n}\max(X_1 \ldots, X_n)$$
is an unbiased estimator for θ.

(c) Find the MVUE of θ.

$X \sim Unif[0,\theta]$

$$f_X(x) = 1_{[0,\theta]} \cdot \frac{1}{\theta} = \begin{cases} \frac{1}{\theta}, & x \in [0,\theta] \\ 0, & \text{otherwise} \end{cases}$$

$$1_{[0,\theta]}(x) = \begin{cases} 1, & x \in [0,\theta] \\ 0, & \text{otherwise} \end{cases}$$

Sol.: $$L(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \left(1_{[0,\theta]}(x_i) \cdot \frac{1}{\theta} \right) =$$

$$= \frac{1}{\theta^n} \cdot \prod_{i=1}^{n} 1_{[0,\theta]}(x_i) = \frac{1}{\theta^n} \cdot 1_{x_1, x_2, \ldots, x_n \in [0,\theta]} = \frac{1}{\theta^n} \cdot 1_{\min(x_i) \geq 0 \text{ and } S \leq \theta} =$$

$$= \frac{1}{\theta^n} \cdot 1_{\min(x_i) \geq 0} \cdot 1_{S \leq \theta} = \underbrace{\frac{1}{\theta^n} \cdot 1_{S \leq \theta}}_{h(\theta, S)} \cdot \underbrace{1_{\min(x_i) \geq 0}}_{g(x_1, x_2, \ldots, x_n)}$$

We will now show that S is complete.

$$\forall \varphi : \quad E(\varphi(S)) = 0, \ \forall \theta \implies \underbrace{P(\varphi(S) = 0) = 1}_{\varphi \overset{a.s.}{=} 0}$$

Hint: If $X \sim Unif[0,\theta]$, then the pdf of $S = \max(X_1, X_2 \ldots, X_n)$ is:

$$f_S(x) = 1_{[0,\theta]} \cdot \frac{nx^{n-1}}{\theta^n} = \begin{cases} \frac{nx^{n-1}}{\theta^n}, & x \in [0,\theta] \\ 0, & \text{otherwise} \end{cases}$$

$$E(\varphi(S)) = \int_{\mathbb{R}} \varphi(x) \cdot f_S(x)\, dx = \int_0^\theta \varphi(x) \cdot \frac{n x^{n-1}}{\theta^n}\, dx$$

We know that $E(\varphi(S)) = 0, \forall \theta \Rightarrow \int_0^\theta \varphi(x) \cdot x^{n-1}\, dx = 0$

We have to show that $P(\varphi(S) = 0) = 1$

Let $g(\theta) = \int_0^\theta \varphi(x) \cdot x^{n-1}$

Given that $g(\theta) = 0, \forall \theta \Rightarrow g'(\theta) = 0 \Rightarrow \varphi(S) \cdot S^{n-1} = 0$

$$\Rightarrow P(\varphi(S) = 0) = 1$$

(b) Show that $\bar{\theta} = \frac{n+1}{n} \underbrace{\max(X_1, \ldots, X_n)}_{S}$ is an unbiased estimator for θ. So we have to show that $E(\bar{\theta}) = \theta$

$$f_S(x) = 1_{[0,\theta]} \cdot \frac{n x^{n-1}}{\theta^n}$$

$$E(\bar{\theta}) = E\left(\frac{n+1}{n} \cdot S\right) = \frac{n+1}{n} \cdot E(S) = \frac{n+1}{n} \cdot \int_{\mathbb{R}} x \cdot 1_{[0,\theta]}(x) \frac{n x^{n-1}}{\theta^n}\, dx =$$

$$= \frac{n+1}{n} \int_0^\theta \frac{n x^n}{\theta^n}\, dx = \frac{n+1}{\theta^n} \int_0^\theta x^n\, dx = \frac{n+1}{\theta^n} \cdot \frac{1}{n+1} x^{n+1} \Big|_0^\theta = \frac{\theta^{n+1}}{\theta^n} = \theta$$

$\Rightarrow \bar{\theta}$ is an unbiased estimator.

(c) Find the MVUE of θ

Theorem 2.9 (Lehmann-Scheffé).
Let $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ be an unbiased estimator for θ and $S = S(X_1, \ldots, X_n)$ be a sufficient and complete statistic for θ. Then the estimator

$$\overline{\theta} \quad = \quad \overline{\theta}(X_1, \ldots, X_n) \quad = \quad E(\hat{\theta} \mid S) \tag{2.5}$$

is a MVUE.

Sol. : $S = \max(X_1, \ldots, X_n)$ is a sufficient and complete statistic

$\overline{\theta} = \frac{n+1}{n} \cdot \max(x_1, \ldots, x_n)$ is an unbiased estimator for θ

Lehmann-Scheffé
$\Rightarrow$ $\tilde{\theta} = E(\overline{\theta} \mid S)$ is an MVUE

$\tilde{\theta} = E(\overline{\theta} \mid S) = E\left(\overline{\theta} \,\middle|\, \frac{n}{n+1} \cdot \overline{\theta}\right) = \overline{\theta}$

$\Rightarrow$ $\overline{\theta}$ is an MVUE

Exercise 4. Let $X \sim Unid(\theta)$, where $\theta \in \mathbb{N}^*$ is a parameter.

1. Prove that
$$S = \max\{X_1, \ldots, X_n\}$$
is a sufficient and complete statistic for θ.

2. Show that
$$\overline{\theta} = \frac{S^{n+1} - (S-1)^{n+1}}{S^n - (S-1)^n}$$
is an unbiased estimator for θ.

3. Find the MVUE of θ.

<u>Hint:</u> If $X \sim Unid(\theta)$, then the pdf of $S = \max(X_1, X_2 \ldots, X_n)$ is:

$$f_S(x) = 1_{\{1,\ldots,\theta\}} \cdot \left(\left(\frac{x}{\theta}\right)^n - \left(\frac{x-1}{\theta}\right)^n\right)$$

$X \sim Unid(\theta) \Rightarrow X \begin{pmatrix} k \\ \frac{1}{\theta} \end{pmatrix}_{k=\overline{1,\theta}}$ $\quad f(x;\theta) = 1_{\{1,2,\ldots,\theta\}} \cdot \frac{1}{\theta}$

$L(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} 1_{\{1,2,\ldots,\theta\}}(x_i) \cdot \frac{1}{\theta} =$

$$= 1_{\min(x_i) \geq 1} \cdot 1_{\max(x_i) \leq \theta} \cdot \frac{1}{\theta^n} = \underbrace{\frac{1}{\theta^n} \cdot 1_{S \leq \theta}}_{h(S;\theta)} \cdot \underbrace{1_{\min(x_i) \geq 1}}_{g(x_1, \dots, x_n)}$$

We will now see if S is complete

$$f_S(x) = 1_{\{1, \dots, \theta\}} \cdot \left(\left(\frac{x}{\theta}\right)^n - \left(\frac{x-1}{\theta}\right)^n \right)$$

$$E(\varphi(S)) = \sum_{k=1}^{\theta} \varphi(k) \cdot \left[\left(\frac{k}{\theta}\right)^n - \left(\frac{k-1}{\theta}\right)^n \right]$$

We know that $E(\varphi(S)) = 0, \ \forall \theta$

$$\Rightarrow \sum_{k=1}^{\theta} \varphi(k) \cdot \left(k^n - (k-1)^n\right) = 0$$

$P(\varphi(S) = 0)$

For $\theta = 1 \Rightarrow \varphi(1) \cdot 1 = 0 \Rightarrow \varphi(1) = 0$

$\theta = 2 \Rightarrow \varphi(1) \cdot 1 + \varphi(2) \cdot (2^n - 1) = 0$

$\Rightarrow \varphi(2) = 0$

We prove by induction that $\varphi(x) = 0, \ \forall x$

$$\Rightarrow P(\varphi(S) = 0) = 1$$