0.1 Serii de numere reale

Definiţia 1.1 SERIE DE NUMERE REALE= prechea ordonata $((u_n), (s_n))$

$$u_n \in R$$

$$s_n = u_1 + u_2 + ... + u_n, \ \forall n \in N$$

si se noteaza cu $\sum_{n=1}^{\infty} u_n = \sum_{n\geq 1} u_n = \sum_{n\in N} u_n = \sum_n u_n$.

Notatii:

- $\forall n \in N$:

u_n s.n. TERMENUL GENERAL al seriei

s_n s.n. SUMA PARTIALA DE RANG n a seriei

- (u_n) s.n. sirul termenilor seriei
- (s_n) s.n. sirul sumelor partiale.

Definiţia 1.2 Seria $\sum_n u_n$ s.n.CONVERGENTA daca sirul sumelor partiale, (s_n) e convergent.

Daca $\lim_{n\to\infty} s_n = s \in \overline{R}$ atunci $\sum_n u_n$ are suma s, si scriem $\sum_n u_n = s$.

Teorema 1.3 *(CRITERIUL GENERAL DE CONVERGENTA AL LUI CAUCHY)* $\sum_n u_n$ *e convergenta* $\Longleftrightarrow$

$$\forall \varepsilon > 0, \exists n_\varepsilon > 0 \ a.i. \ \forall n, p \in N, \ n \geq n_\varepsilon, \ |u_{n+1} + u_{n+2} + ... + u_{n+p}| < \varepsilon.$$

Teorema 1.4

$$\sum_n u_n \text{ e convergenta} \Longrightarrow \lim_{n\to\infty} u_n = 0.$$

Observaţia 1.5

a) Teorema de mai sus e in general folosita pentru a demonstra ca o serie nu este convergenta (vezi asemanarea cu criteriul lui Heine la siruri):adica, daca descoperim ca

$$\lim_{n\to\infty} u_n \neq 0 \Longrightarrow \sum_n u_n \text{ e divergenta.}$$

b) exista serii pentru care $\lim_{n\to\infty} u_n = 0$ dar nu sunt convergente, de exemplu, seria armonica

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Teorema 1.6

$$\forall m \in N,\ \sum_{n} u_n \textit{ e convergenta} \Longleftrightarrow \sum_{n=m}^{\infty} u_n \textit{ e convergenta.}$$

Teorema 1.7 *Daca* $\sum\limits_{n=1}^{\infty} u_n = s$ *si* $\sum\limits_{n=1}^{\infty} v_n = t \Longrightarrow \sum\limits_{n=1}^{\infty} (au_n + bv_n) = as + bt.$

Teorema 1.8 *Daca* $\sum\limits_{n} u_n$ *e convergenta, atunci* $\lim_{n\to\infty} r_n = 0$, *cand* $r_n = s - s_n$.

Exemple importante de serii:

a) SERIA GEOMETRICA

$$\sum_{n=1}^{\infty} q^{n-1},\ q \in R,\ fixat.$$

Atunci $s_n = 1 + q + ... + q^{n-1} = \begin{cases} \frac{1-q^n}{1-q} : q \neq 1 \\ 1 : q = 1 \end{cases}$

si

$$\lim_{n\to\infty} s_n = \begin{cases} \frac{1}{1-q} : |q| < 1 \\ -\infty : q \geq 1 \\ \nexists : q \leq -1 \end{cases}.$$

atunci, seria e convergenta, cu suma $\frac{1}{1-q}$ daca $|q| < 1$ si divergenta in rest.

b) SERIA ARMONICA

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \text{ deci e divergenta.}$$

Dem. se face usor cu criteriul lui Cauchy.

0.2 Serii cu termeni pozitivi

Definiţia 2.1 $\sum\limits_{n} u_n$ s.n. SERIE CU TERMENI POZITIVI daca $u_n > 0\ \forall n \in N$.

Teorema 2.2 *Daca* $\sum_n u_n$ *e o stp atunci*

$$\sum_n u_n \text{ e convergenta} \Longleftrightarrow (s_n) \text{ e marginit.}$$

(pt. ca daca avem o stp., (s_n) e stricit crescator).

Teorema 2.3 (*PRIMUL CRITERIU AL COMPARATIEI*)
Daca $\sum_n u_n$ si $\sum_n v_n$ sunt stp a.i.$\exists a > 0, \exists n_0 \in N$ cu

$$u_n \leq a v_n \ \forall n \geq n_0$$

atunci

$$\sum_n v_n \text{ convergenta} \Longrightarrow \sum_n u_n \text{ convergenta.}$$

$$\sum_n u_n \text{ divergenta} \Longrightarrow \sum_n v_n \text{ divergenta.}$$

Exemplul 2.4 $\sum_n \frac{1}{\sqrt{n}}$ e divergenta (comparatie cu $\sum_n \frac{1}{n}$).

Teorema 2.5 (*AL DOILEA CRITERIU AL COMPARATIE*)
Daca $\sum_n u_n$ *si* $\sum_n v_n$ *sunt stp a.i.*$\exists \lim_{n\to\infty} \frac{u_n}{v_n} \in [0, +\infty]$ *atunci:*

$$1.\ \lim_{n\to\infty} \frac{u_n}{v_n} \in [0, +\infty] \Longrightarrow \sum_n u_n \text{ si } \sum_n v_n \text{ au aceeasi natura;}$$

$$2.\ \lim_{n\to\infty} \frac{u_n}{v_n} = 0 \Longrightarrow \begin{array}{l} a) \sum_n v_n \text{ convergenta} \Longrightarrow \sum_n u_n \text{ convergenta;} \\ b) \sum_n u_n \text{ divergenta} \Longrightarrow \sum_n v_n \text{ divergenta;} \end{array}$$

$$3.\ \lim_{n\to\infty} \frac{u_n}{v_n} = +\infty \Longrightarrow \begin{array}{l} a) \sum_n u_n \text{ convergenta} \Longrightarrow \sum_n v_n \text{convergenta;} \\ b) \sum_n v_n \text{ divergenta} \Longrightarrow \sum_n u_n \text{ divergenta;} \end{array}$$

Teorema 2.6 (*AL TREILEA CRITERIU AL COMPARATIE*)
Daca $\sum_n u_n$ *si* $\sum_n v_n$ *sunt stp a.i.*

$$\exists n_0 \in N \text{ a.i. } \frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}, \ \forall n \geq n_0$$

atunci:

$$1. \sum_n v_n \text{ convergenta} \Longrightarrow \sum_n u_n \text{ convergenta.}$$

$$2. \sum_n u_n \text{ divergenta} \Longrightarrow \sum_n v_n \text{ divergenta.}$$

Exemplul 2.7 $\sum\limits_n \frac{1}{n^2}$ e convergenta (comparatie cu $\sum\limits_n \frac{1}{n(n+1)}$).

Teorema 2.8 *(CRITERIUL RAPORTULUI, AL LUI D'ALAMBERT)* $\sum\limits_n u_n$ *e o stp. Atunci:*

1. *Daca* $\exists q \in [0,1[,\ \exists n_0 \in N$ *a.i.* $\frac{u_{n+1}}{u_n} \leq q,\ \forall n \geq n_0 \implies \sum\limits_n u_n$ *e convergenta.*

2. *Daca* $\exists n_0 \in N$ *a.i.* $\frac{u_{n+1}}{u_n} \geq 1,\ \forall n \geq n_0 \implies \sum\limits_n u_n$ *e divergenta.*

Propoziţia 2.9 *(CONSECINTA CRITERIULUI RAPORTULUI)*
$\sum\limits_n u_n$ *e o stp a.i.* $\exists \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$. *Atunci*

1. *Daca* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1 \implies \sum\limits_n u_n$ *e convergenta.*

2. *Daca* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1 \implies \sum\limits_n u_n$ *e divergenta.*

Exemplul 2.10 $\sum\limits_n \frac{2^n n!}{n^n}$ e convergenta.

Exemplul 2.11 $\sum\limits_n \frac{3^n n!}{n^n}$ e divergenta.

Observaţia 2.12 daca $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$ nu putem aplica consecinta criteriului raportului (de ex. $\sum\limits_n \frac{1}{n}$ e divergenta si $\sum\limits_n \frac{1}{n(n+1)}$ e convergenta).

Teorema 2.13 (*CRITERIUL RADACINII, AL LUI CAUCHY*)
$\sum\limits_n u_n$ *e o stp. Atunci:*

1. *Daca* $\exists q \in [0,1[,\ \exists n_0 \in N$ *a.i.* $\sqrt[n]{u_n} \leq q\ \forall n \geq n_0 \implies \sum\limits_n u_n$ *e convergenta.*

2. *Daca* $\exists n_0 \in N$ *a.i.* $\sqrt[n]{u_n} \geq 1,\ \forall n \geq n_0 \implies \sum\limits_n u_n$ *e divergenta.*

Teorema 2.14 (*CONSECINTA CRITERIULUI RADACINII*)
$\sum\limits_n u_n$ *e o stp a.i.*$\exists \lim_{n\to\infty} \sqrt[n]{u_n}$. *Atunci:*

$$
\begin{aligned}
1.\ \textit{Daca}\ \lim_{n\to\infty} \sqrt[n]{u_n} &< 1 \implies \sum_n u_n\ \textit{e convergenta.} \\
2.\textit{Daca}\ \lim_{n\to\infty} \sqrt[n]{u_n} &> 1 \implies \sum_n u_n\ \textit{e divergenta.}
\end{aligned}
$$

Exemplul 2.15 $\sum\limits_n \frac{n^r}{2^n}, r > 0$ e convergenta.

Exemplul 2.16 $\sum_n \left(\sqrt[3]{n^3+3n^2+1} - \sqrt[3]{n^3-n^2+1}\right)^n$ e divergenta.

Observaţia 2.17 Daca $\lim_{n\to\infty} \sqrt[n]{u_n} = 1$ criteriul radacinii nu ne ofera nici o conluzie.

Teorema 2.18 (*CRITERIUL LUI E.E. KUMMER*)
$\sum_n u_n$ *e o stp. Atunci:*

1. *Daca* $\exists\,(a_n) \subseteq R_+, \exists r > 0,\ \exists n_0 \in N$ *a.i.*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} > r\ ,\ \forall n \geq n_0 \Longrightarrow \sum_n u_n \textit{ e convergenta.}$$

2. *Daca* $\exists\,(a_n) \subseteq R_+$ *a.i.* $\sum_n \frac{1}{a_n}$ *e divergenta si* $\exists n_0 \in N$ *a.i.*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} \leq 0\ ,\ \forall n \geq n_0 \Longrightarrow \sum_n u_n \textit{ e divergenta.}$$

Teorema 2.19 (*CRITERIUL LUI J.L. RAABE-J.DUHAMEL*)
$\sum_n u_n$ *e o stp. Atunci:*

1. *Daca* $\exists q \in]1, +\infty[,\ \exists n_0 \in N$ *a.i.*

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq q\ ,\ \forall n \geq n_0 \Longrightarrow \sum_n u_n \textit{ e convergenta.}$$

2. *Daca* $\exists n_0 \in N$ *a.i.*

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq 1\ ,\ \forall n \geq n_0 \Longrightarrow \sum_n u_n \textit{ e divergenta.}$$

Teorema 2.20 (*CONSECINTA CRITERIULUI LUI RAABE-DUHAMEL*)
$\sum_n u_n$ *e o stp a.i.* $\exists \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$*Atunci:*

$$1.\textit{Daca } \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 \Longrightarrow \sum_n u_n \textit{ e convergenta.}$$

$$2.\ \textit{Daca } \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1 \Longrightarrow \sum_n u_n \textit{ e divergenta.}$$

Exemplul 2.21 $\sum_n \frac{n!}{a(a+1)\ldots(a+n-1)}$ e convergenta $\Longleftrightarrow a > 2$.

Teorema 2.22 (*CRITERIUL CONDENSARII, A LUI A.L.CAUCHY*)

$\sum_n u_n$ *o stp a.i.* (u_n) *e descrescator. Atunci*

$$\sum_n u_n \text{ si } \sum_n 2^n u_{2^n} \text{ au aceeasi natura.}$$

Exemplul 2.23 SERIA ARMONICA GENERALIZATA

$$\sum_N \frac{1}{n^a}$$

ea are acceasi natura cu $\sum_n 2^n \frac{1}{(2^n)^a} = \sum_n \left(\frac{1}{2^{a-1}}\right)^n$,

$$\begin{aligned} \text{convergenta} \iff & \quad a > 1 \\ \text{divergenta} \iff & \quad a \leq 1. \end{aligned}$$

0.3 Serii cu termeni oarecare

Teorema 3.1 (*CRITERIUL LUI ABEL-DIRICHLET*)
$\sum_n u_n$ *a.i.* (s_n) *e marginit. Atunci*

$$\text{Daca } (a_n) \subseteq R \text{ e un sir } \begin{array}{l} -\text{ descrescator} \\ -\ \lim_{n\to\infty} a_n = 0 \end{array} \implies \text{seria } \sum_n a_n u_n \text{ e convergenta.}$$

Definiţia 3.2 Se numeste SERIE ALTERNATA o serie de forma

$$\sum_n (-1)^n u_n, \text{ cu } u_n > 0 \ \forall n \in N.$$

Teorema 3.3 (*TEOREMA LUI LEIBNIZ*)

$$\text{Daca sirul } (u_n) \text{ e } \begin{array}{l} -\text{ descrescator} \\ -\ \lim_{n\to\infty} u_n = 0 \end{array} \implies \text{seria alternata } \sum_n (-1)^n u_n \text{ e convergenta.}$$

Exemplul 3.4 SERIA ARMONICA ALTERNATA

$$\sum_n \frac{(-1)^{n+1}}{n}$$

e convergenta, pentru ca $\frac{1}{n} \to 0$.

Definiţia 3.5 $\sum_n u_n$ s.n. ABSOLUT CONVERGENTA daca $\sum_n |u_n|$ e convergenta.

Teorema 3.6 *Daca o serie de numere reale e abosolut convergenta* $\implies$ *e convergenta.*

Definiţia 3.7 O serie s.n. SEMICONVERGENTA daca e convergenta, dar nu e si absolut convergenta.

Exemplul 3.8 $\sum_n \frac{(-1)^{n+1}}{n}$ e semiconvergenta.