1

Are invers? $\rightarrow$ $\underbrace{3+5\sqrt{2} \in \mathbb{Q}(\sqrt{2})}$ $\Rightarrow (3+5\sqrt{2})(\underbrace{a+b\sqrt{2}}_{\text{invers}}) = 1$

$$3a + 3b\sqrt{2} + 5a\sqrt{2} + 10b = 1$$

$$\Rightarrow \begin{cases} 3a + 10b = 1 \Rightarrow * \\ 3b\sqrt{2} + 5a\sqrt{2} = 0 \Rightarrow \sqrt{2}(3b+5a) = 0 \\ \Rightarrow 3b + 5a = 0 \Rightarrow b = \frac{-5a}{3} \end{cases}$$

$$* \Rightarrow 3a + 10 \cdot \frac{-5a}{3} = 1 \quad | \cdot 3$$

$$9a - 50a = 3 \Rightarrow -41a = 3 \Rightarrow a = \frac{-3}{41}$$

$$\Rightarrow b = -5 \cdot \left(\frac{-\cancel{3}}{41}\right) \cdot \frac{1}{\cancel{3}}$$

$$= \frac{5}{41}$$

$\Rightarrow$ inversul lui $3+5\sqrt{2}$ este $a+b\sqrt{2}$, adică

$$-\frac{3}{41} + \frac{5}{41}\sqrt{2}.$$

② Este t.l.? → $f(x_1, x_2) = (-3x_1 + 4x_2, 2x_1 - x_2)$

$\Rightarrow$ TB. VERIF. COND: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$\Rightarrow f(\alpha(x_1, x_2) + \beta(y_1, y_2)) = (\underbrace{\alpha x_1 + \beta y_1}_{x_1}, \underbrace{\alpha x_2 + \beta y_2}_{x_2})$

$\Rightarrow = (-3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2), 2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2))$

$= (-3\alpha x_1 - 3\beta y_1 + 4\alpha x_2 + 4\beta y_2, 2\alpha x_1 + 2\beta y_1 - \alpha x_2 - \beta y_2)$

$= (-\alpha 3 x_1 + 4\alpha x_2, 2\alpha x_1 - \alpha x_2) + (-3\beta y_1 + 4\beta y_2, 2\beta y_1 - \beta y_2)$

$= \alpha \cdot (-3x_1 + 4x_2, 2x_1 - x_2) + \beta(-3y_1 + 4y_2, 2y_1 - y_2)$

$= \alpha f(x_1, x_2) + \beta f(y_1, y_2)$ „A".

$\Rightarrow$ baza canonică $\begin{cases} f(1,0) = (-3, 2) \\ f(0,1) = (4, -1) \end{cases} \Rightarrow [f]_e = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$

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③ Fie $[f]_e = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ în baza canonică. Find f, $\dim \operatorname{Ker} f$, $\dim \operatorname{Im} f$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \quad \underbrace{\begin{pmatrix} 2 \\ -4 \end{pmatrix}}_{f(1,0)} \quad \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{f(0,1)}$$

$\Rightarrow (2, -4) = f(1,0)$

$\Leftrightarrow$ baza canonică $e_1(1,0)$; $e_2(0,1)$

$(-1, 2) = f(0,1)$

$\dim([f]_e) = 2 \Leftrightarrow e \ \mathbb{R}^2$

$\dim \operatorname{Im} f =$ rangul matricii

$\Rightarrow f(2x_1 - x_2, -4x_1 + 2x_2)$

$\det([f]_e) = 0 \quad (2 \cdot 2 - 4 - 0) \Rightarrow \operatorname{rang}([f]_e) = 1$

$\Rightarrow \dim([f]_e) = \dim \operatorname{ker} f + \dim \operatorname{Im} f \Rightarrow 2 = 1 + 1$

(d) $f(x_1, x_2, x_3) = (-4x_1 - 18x_2 - 24x_3, -3x_1 - 7x_2 - 12x_3, 3x_1 + 9x_2 + 14x_3)$

$$\Rightarrow A = \begin{pmatrix} -4 & -18 & -24 \\ -3 & -7 & -12 \\ 3 & 9 & 14 \end{pmatrix}$$

$$\Rightarrow f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} -4 & -18 & -24 \\ -3 & -7 & -12 \\ 3 & 9 & 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(din definiția nucleului: ~~Kerf~~ $\operatorname{Ker} f = \{x \in V \mid f(x) = 0\}$)

$$\Rightarrow \begin{cases} -4x_1 - 18x_2 - 24x_3 = 0 \\ -3x_1 - 7x_2 - 12x_3 = 0 \\ 3x_1 + 9x_2 + 14x_3 = 0 \end{cases}$$

$\Rightarrow +4 \cdot \frac{5x_3}{3} + 18x_3 - 24x_3 = 0 \mid \cdot 3$ *

$\Rightarrow -3x_1 + 7x_3 - 12x_3 = 0$

$\Rightarrow -3x_1 - 5x_3 = 0 \Rightarrow x_1 = \frac{-5x_3}{3}$

$/ \quad 2x_2 + 2x_3 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

~~* $4 \cdot 5x_3 - 18x_3 = 0 \Rightarrow x_3 = 0$~~

Not. $x_3 = \alpha \Rightarrow x_2 = -\alpha$ și $x_1 = \frac{-5\alpha}{3}$

$$\Rightarrow \operatorname{Ker}(f) = \left\{ \begin{pmatrix} -\frac{5\alpha}{3} \\ -\alpha \\ \alpha \end{pmatrix} \right\}$$ subspațiu de dim 1

bază: fie $\alpha = 1 \Rightarrow \begin{pmatrix} -\frac{5}{3} \\ -1 \\ 1 \end{pmatrix}$

(5) $f(x_1, x_2) = (-3x_2, x_1 + 2x_2, 4x_1) \to$ T. L. ?

Pt. a fi T.L.: $f(\alpha \cancel{x} + \beta \cancel{y}) = \alpha f(x) + \beta f(y)$

$\Rightarrow f(\alpha(x_1, x_2) + \beta(y_1, y_2)) = (\underbrace{\alpha x_1 + \beta y_1}_{x_1}, \underbrace{\alpha x_2 + \beta y_2}_{x_2})$

$= (-3\alpha x_2 - 3\beta y_2, \alpha x_1 + \beta y_1 + 2\alpha x_2 + 2\beta y_2, 4\alpha x_1 + 4\beta y_1)$

$= (\alpha \cdot (\underbrace{-3x_2, x_1 + 2x_2, 4x_1}_{f(x_1, x_2)}) + \beta(\underbrace{-3y_2, y_1 + 2y_2, 4y_1}_{f(y_1, y_2)})$

$= \alpha f(x) + \beta f(y)$ „A".

$\Rightarrow$ det. Im f și o bază:

bază canonică $\begin{cases} f(1,0) = (0, 1, 4) \\ f(0,1) = (-3, 2, 0) \end{cases}$ $\Rightarrow [f]_e = \begin{pmatrix} 0 & -3 \\ 1 & 2 \\ 4 & 0 \end{pmatrix}$

Im f

$\Rightarrow \det([f]_e) = 0 - (-3) \neq 0 \Rightarrow \text{rang}([f]_e) = 2$

$\Rightarrow$ bază pentru Im f: $\begin{pmatrix} 0 & -3 \\ 1 & 2 \\ 4 & 0 \end{pmatrix}$

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⑥ Există o T.L. $f: \mathbb{R}^3 \to \mathbb{R}^2$ a.î.:

$$f(1,0,3) = (1,1) \text{ și } f(-2,0,-6) = (2,1) \text{ ?}$$

Obs. că $-2 \cdot f(1,0,3) = f(-2,0,-6)$ // combinație liniară

$\Rightarrow$ Este adev. și $-2 \cdot (1,1) \overset{?}{=} (2,1)$? „~~[illegible]~~"

$\Rightarrow$ NU.

Deci, $\nexists$

⑦ Să se arate că vectorii $v_1 = (1, 2, -1)$, $v_2 = (3, 2, 4)$ și $v_3 = (-1, 2, -6)$ sunt liniar dependenți:

v_1, v_2, v_3 l.d $\Leftrightarrow \alpha v_1 + \beta v_2 + \gamma v_3 = (0, 0, 0)$, α, β, γ nu toți nuli

(*)

$\Rightarrow \alpha(1, 2, -1) + \beta(3, 2, 4) + \gamma(-1, 2, -6) = (0, 0, 0)$

$$\Rightarrow \begin{cases} \alpha & 3\beta & -\gamma & = 0 \\ 2\alpha & 2\beta & 2\gamma & = 0 \\ -\alpha & 4\beta & -6\gamma & = 0 \end{cases}$$

$$\Rightarrow \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ -1 & 4 & -6 \end{vmatrix} = -12 + 8 - 6 - 2 + 36 - 8 = 0$$

$\Rightarrow \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \neq 0 \Rightarrow$ minor principal $\Rightarrow \alpha$ și β nec. princ.

$$\Rightarrow \begin{cases} \alpha + 3\beta = \gamma \\ 2\alpha + 2\beta = -2\gamma \end{cases}$$

Fie $\gamma = 1 \Rightarrow \begin{cases} \alpha + 3\beta = 1 & \Rightarrow \alpha = 1 - 3\beta \\ 2\alpha + 2\beta = -2 & \Rightarrow 2 - 6\beta + 2\beta = -2 \end{cases}$

$-4\beta = -4 \Rightarrow \beta = 1 \Rightarrow \alpha = -2$

$\Rightarrow -2 v_1 + 1 \cdot v_2 + 1 \cdot v_3 = (0, 0, 0) \Rightarrow v_1, v_2, v_3$ l.d.

nenuli (*)

(8) $f(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3,\ x_1 - x_2 + x_3)$ T.L?

Pt. a fi T.L: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$\Rightarrow$ Fie $x = (x_1, x_2, x_3)$ şi $y = (y_1, y_2, y_3) \in V$ a.î.:

$$\Rightarrow f(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)) = (\underbrace{\alpha x_1 + \beta y_1}_{x_1}, \underbrace{\alpha x_2 + \beta y_2}_{x_2}, \underbrace{\alpha x_3 + \beta y_3}_{x_3})$$

$$= (\underbrace{\alpha x_1 + \beta y_1}_{x_1} + \underbrace{\alpha x_2 + \beta y_2}_{x_2} + \underbrace{2\alpha x_3 + 2\beta y_3}_{2x_3}, \underbrace{\alpha x_1 + \beta y_1}_{x_1} \underbrace{- \alpha x_2 - \beta y_2}_{-x_2} + \underbrace{\alpha x_3 + \beta y_3}_{x_3})$$

$$= \alpha(\underbrace{x_1 + x_2 + 2x_3, x_1 - x_2 + x_3}_{f(x_1, x_2, x_3)}) + \beta(\underbrace{y_1 + y_2 + 2y_3, y_1 - y_2 + y_3}_{f(y_1, y_2, y_3)})$$

$= \alpha f(x) + \beta f(y)$ „A".

- Baza canonică: $e = (e_1, e_2, e_3)$

$f(e_1) = f(1, 0, 0) = (1, 1)$

$f(e_2) = f(0, 1, 0) = (1, -1)$

$f(e_3) = f(0, 0, 1) = (2, 1)$

$$\Rightarrow [f]_e = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array}$$

⑨ $a = ?$ a.î. $v_1 = (a, 1, 1)$; $v_2 = (1, a, 1)$ și $v_3 = (1, 1, a)$ să formeze o bază a lui $\mathbb{R}^3$.

$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \neq 0$ pt. a fi bază

$\Rightarrow a^3 + 1 + 1 - a - a - a \neq 0$

$\Rightarrow a^3 - 3a + 2 \neq 0$

$a(a^2 - 3) \neq -2 \quad \begin{cases} a \neq -2 \\ a^2 \neq 1 \Rightarrow a \neq \pm 1 \end{cases}$

dar pt. $a = -1$ ar fi: $-1((-1)^2 - 3) = -1 \cdot (-2) = 2 \neq -2$

$\Rightarrow a \in \mathbb{R} \setminus \{-2, 1\}$.

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(10) Să se arate că matricile:

$E_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $E_2 = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$, $E_3 = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ și $E_4 = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

formează o bază și să se scrie coord. lui $A = \begin{pmatrix}-2 & 3\\ 4 & -2\end{pmatrix}$.

$\Rightarrow$ Formează o bază iff $\forall X = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in M_2(\mathbb{R}), \exists\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ a.î.:

$X = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_5$

$$\Rightarrow \alpha_1 \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \alpha_2 \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + \alpha_3 \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + \alpha_4 \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a = -2 & \Rightarrow \alpha_1 = -5 \\ \alpha_2 + \alpha_3 + \alpha_4 = b = 3 & \Rightarrow \alpha_2 = -1 \\ \alpha_3 + \alpha_4 = c = 4 & \Rightarrow \alpha_3 = 6 \\ \alpha_4 = d = -2 \end{cases} \Rightarrow \boxed{A = -5E_1 - E_2 + 6E_3 - 2E_4.}$$

11 Se consideră urm. subspații generate astfel:

$S = \langle u_1, u_2 \rangle$ cu $u_1 = (1, 1, 0, 0)$

$u_2 = (1, 0, 1, 1)$

$T = \langle v_1, v_2 \rangle$ cu $v_1 = (0, 0, 1, 1)$

$v_2 = (0, 1, 1, 0)$

Dați bază și dim pt. S, T, $S+T$ și $S \cap T$.

$\Rightarrow \dim S = \text{rang}(u_1, u_2) = \left(\underbrace{\boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}}}_{\neq 0} \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \right) = 2$ și (u_1, u_2) bază în S

-//- pt. T

$$\dim(S+T) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{=\!=}{\xrightarrow{L_2 = L_2 - L_1}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{C_2 = C_2 -}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 0 + 0 - \cancel{1} + \cancel{1} - 1 = -1 \neq 0$$

$\Rightarrow \dim(S+T) = 4$. $\Rightarrow$ baza canonică $\overset{\dim \mathbb{R}^4}{}$ bază pt. $S+T$

$\dim(S \cap T) = ?$ Știm formula: $\dim S + \dim T = \dim(S \cap T) + \dim(S+T)$

$\Rightarrow 2 + 2 = \dim(S \cap T) + 4$

$\Rightarrow \dim(S \cap T) = 0 \Rightarrow S \cap T = \{(0, 0, 0, 0)\}$

$\Rightarrow \emptyset$ bază pt. $S \cap T$.