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Seminar W10 – 837

Exercise 3. A vending machine contains the following numbers of bills:

1 RON	5 RON	10 RON
25	15	10

Assume that each client pays the vending machine with only one bill. Find 95% confidence intervals for:

(a) the mean value of the amount of money paid by a client;

(b) the standard deviation of the amount of money paid by a client;

(c) the proportion of clients that pay the vending machine with a 5 RON bill.

confidence interval

$100(1-\alpha)\ \%$

$\Rightarrow 100(1-\alpha) = 95$

$\Rightarrow 1-\alpha = \frac{95}{100}$

$\Rightarrow \alpha = 0.05$

- *One population, $X \sim \mathcal{N}(\mu, \sigma)$ or $n > 30$, unknown variance:*

$$\mu \in \left[\overline{X} - z_{1-\frac{\alpha}{2}}\frac{S}{\sqrt{n}}, \overline{X} - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right],\quad z_\beta = tinv(\beta, n-1)$$

$n = 50 > 30 \Rightarrow$ we can use the above confidence interval

$$\overline{X} = \frac{25\cdot 1 + 15\cdot 5 + 10\cdot 10}{50} = \frac{200}{50} = 4$$

sample standard deviation

$$s = \sqrt{\frac{1}{n-1}\cdot\sum_{i=1}^{n}(X_i - \overline{X})^2} = \sqrt{\frac{1}{49}\cdot 25\cdot(1-4)^2 + 15(5-4)^2 + 10(10-4)^2} =$$

$$= \sqrt{\frac{1}{49}(25\cdot 9 + 15 + 360)} = \frac{1}{7}\cdot\sqrt{(225+375)} = \frac{1}{7}\cdot\sqrt{600} =$$

$$= \frac{10\sqrt{6}}{7}$$

$$z_\beta = tinv(\beta, n-1)$$

$$\alpha = 0.05$$

$$z_{\frac{\alpha}{2}} = z_{0.025} = -2.0096$$

$$z_{1-\frac{\alpha}{2}} = z_{0.975} = 2.0096$$

- *One population, $X \sim \mathcal{N}(\mu, \sigma)$ or $n > 30$, unknown variance:*

$$\mu \in \left[\overline{X} - z_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}}, \overline{X} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}\right], \; z_\beta = tinv(\beta, n-1)$$

$$\mu \in \left[4 - 2.0096 \cdot \frac{10\sqrt{6}}{7\sqrt{50}}, 4 + 2.0096 \cdot \frac{10\sqrt{6}}{7\sqrt{50}}\right)$$

$$\Rightarrow \mu \in [3.0055, 4.9945]$$

(b) $$\sigma \in \left[\sqrt{\frac{(n-1)\cdot s^2}{z_{1-\frac{\alpha}{2}}}}, \sqrt{\frac{(n-1)s^2}{z_{\frac{\alpha}{2}}}}\right]$$

$$z_{\frac{\alpha}{2}} = chi2inv\left(\frac{\alpha}{2}, n-1\right) = chi2inv(0.025, 49) = 31.555$$

$$z_{1-\frac{\alpha}{2}} = chi2inv\left(1-\frac{\alpha}{2}, n-1\right) = chi2inv(0.975, 49) = 70.222$$

$$s = \frac{10\sqrt{6}}{7}$$

$$\Rightarrow \sigma \in \left[\frac{\sqrt{49}\cdot s}{\sqrt{z_{1-\frac{\alpha}{2}}}}, \frac{\sqrt{49}\cdot s}{\sqrt{z_{\frac{\alpha}{2}}}}\right) = \left(\frac{7\cdot\frac{10\sqrt{6}}{7}}{\sqrt{70.222}}, \frac{7\cdot\frac{10\sqrt{6}}{7}}{\sqrt{31.555}}\right]$$

$$= [2.92, 4.36]$$

(c)

- *One proportion, $n > 30$:*

$$p \in \left[\overline{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}, \overline{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}\right], \ z_\beta = norminv(\beta)$$

p = "the proportion of people that pay the vending machine with a 5 RON bill, from the Jurassic period, to the end of days (or the vending mashine)"

$\overline{p}$ = "the proportion of 5 RON notes currently in the vending mashine" = the sample proportion

$$\overline{p} = \frac{15}{50} = 0.3$$

$$z_{\frac{\alpha}{2}} = norminv(0.025) = -1.96$$

$$z_{1-\frac{\alpha}{2}} = norminv(0.975) = 1.96$$

$$p \in \left[0.3 - 1.96 \cdot \sqrt{\frac{0.3 \cdot 0.7}{50}}, 0.3 + 1.96 \cdot \sqrt{\frac{0.3 \cdot 0.7}{50}}\right)$$

$$\Rightarrow p \in [0.17298, 0.4202] \quad \Rightarrow \quad p \in [17\%, 42\%)$$

$$z_\beta = P(Z \le \beta) = F_Z(\beta)$$

Exercise 4. In an orange juice factory, cans are filled by a machine according to the normal distribution.

(a) Consider a random sample of 100 cans that contain a total amount of 24.8 liters of orange juice. Find a 95% confidence interval for the mean value of the amount of orange juice in a can, given that the standard deviation for the filling machine is 5 ml;

(b) Estimate the minimum number of cans in the sample to obtain a 95% confidence interval for the mean value m of the amount of orange juice in a can with a marginal error less than 1 ml, given that the standard deviation for the filling machine is 5 ml;

(c) Find a 95% confidence interval for the proportion of cans that contain between 249 ml and 251 ml, given the following sample:

251.2	250.2	249.6	247.2	250.4	250.2	251.4	251.0	250.3
250.4	251.3	250.5	249.1	248.4	251.3	250.5	250.2	251.7
250.0	250.6	250.0	250.1	247.7	249.7	249.2	249.8	249.3
249.5	249.5	249.7						

- *One population, $X \sim \mathcal{N}(\mu, \sigma)$ or $n > 30$, known variance σ^2:*

$$\mu \in \left[\overline{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right], \ z_\beta = norminv(\beta)$$

$$\overline{X} = \frac{24.8\,l}{100} = 0.248\,l = 248\,ml$$

$$\sigma = 5\,ml$$

$$z_{\frac{\alpha}{2}} = norminv(0.025) = -1.96$$

$$z_{1-\frac{\alpha}{2}} = norminv(0.975) = 1.96$$

$$\mu \in \left[248 - 1.96 \cdot \frac{5}{10}, 248 + 1.96 \cdot \frac{5}{10}\right]$$

$$\Rightarrow \mu \in [248 - 0.98, 248 + 0.98]$$

$$\Rightarrow \mu \in [247.02, 248.98]$$

(b) marginal error = half of the length of the confidence interval

The variance is known, so the CI is:

$$\mu \in \left[\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \ \overline{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right]$$

The marginal error is:

$$e = \frac{1}{2}\left(\left(\overline{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) - \left(\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right)\right) =$$

$$= \frac{1}{2} \cdot \frac{\sigma}{\sqrt{n}} \cdot \left(z_{1-\frac{\alpha}{2}} - z_{\frac{\alpha}{2}}\right) = \frac{1}{2} \cdot \frac{5}{\sqrt{n}} \cdot (1.96 - (-1.96)) =$$

$= \frac{5}{\sqrt{n}} \cdot 1.96 = \frac{9.8}{\sqrt{n}}$

$\Rightarrow \frac{9.8}{\sqrt{n}} < 1 \Rightarrow \sqrt{n} > 9.8 \Rightarrow n > 96.040$

$\Rightarrow$ the minimum size of the sample is $n = 97$

(c)

- *One proportion, $n > 30$:*

$$p \in \left[\overline{p} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}, \overline{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}\right], \; z_\beta = norminv(\beta)$$

$\alpha = 0.05$ $\quad z_{\frac{\alpha}{2}} = -1.96$ $\quad z_{1-\frac{\alpha}{2}} = 1.96$

$\overline{p} = \frac{22}{30}$ $\quad n = 30$

$\Rightarrow p \in \left[\frac{22}{30} - 1.96 \cdot \sqrt{\frac{\frac{22}{30} \cdot \frac{8}{30}}{30}}, \frac{22}{30} + 1.96 \cdot \sqrt{\frac{\frac{22}{30} \cdot \frac{8}{30}}{30}}\right)$

$p \in \left[\frac{22}{30} - 1.96 \cdot \frac{1}{30}\sqrt{\frac{88}{15}}, \frac{22}{30} + 1.96 \cdot \frac{1}{30} \cdot \sqrt{\frac{88}{15}}\right]$