DOBRE DELIA
GRUPA 131

Tema 2

1) $S = \{A \in M_n(\mathbb{R}) \mid \det A \in [-1,1] \setminus \{0\}\}$

Fie $A, B \in S$.

$$\left.\begin{array}{l} \det(A \cdot B) = \det A \cdot \det B \\ \det A,\ \det B \in [-1,1] \setminus \{0\} \end{array}\right\} \Rightarrow \det(A \cdot B) \in [-1,1] \setminus \{0\}$$

$\Rightarrow A \cdot B \in S$

Înmulțirea matricilor este asociativă, în general, deci $(S, \cdot)$ asociativă.

Știm I_n = elementul neutru în înmulțirea matricilor.

$\det I_n = 1 \in [-1,1] \setminus \{0\} \Rightarrow I_n \in S$

$\Rightarrow S$ monoid.

$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$.

$S \subseteq GL_n(\mathbb{R})$

$S \leq GL_n(\mathbb{R}) \Leftrightarrow \forall A \in S,\ A^{-1} \in S$

Fie $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\det A = \frac{1}{4} \in [-1,1] \setminus \{0\}$.

$A^{-1} = \frac{1}{\det A} \cdot A^* = 4 \cdot A^*$

$A^t = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$, $A^* = \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 2 & -\frac{1}{4} \\ 0 & 2 \end{pmatrix}$, $\det A^{-1} = 4$

$\Rightarrow \exists A \in S,\ A^{-1} \notin S \Rightarrow S \not\leq GL_n(\mathbb{R})$

2) $(M_2(\mathbb{Z}_5), +)$ grup comutativ

Fie $A, B \in M_2(\mathbb{Z}_5)$

$A = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, B = \begin{pmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{pmatrix} \Rightarrow A+B = \begin{pmatrix} \widehat{a+e} & \widehat{b+f} \\ \widehat{c+g} & \widehat{d+h} \end{pmatrix}$

$\Rightarrow A+B \in M_2(\mathbb{Z}_5)$

$\forall \hat{a}, \hat{b}, \hat{c} \in \mathbb{Z}_5, (\hat{a}+\hat{b})+\hat{c} = \widehat{a+b}+\hat{c} = \widehat{a+b+c} = \hat{a}+(\widehat{b+c}) =$

$= \hat{a}+(\hat{b}+\hat{c}) \Rightarrow (\mathbb{Z}_5, +)$ asoc. $\Rightarrow (M_2(\mathbb{Z}_5), +)$ asoc.

$\forall \hat{a}, \hat{b} \in \mathbb{Z}_5, \hat{a}+\hat{b} = \widehat{a+b} = \widehat{b+a} = \hat{b}+\hat{a}$

$\Rightarrow (\mathbb{Z}_5, +)$ comutativ $\Rightarrow (M_2(\mathbb{Z}_5), +)$ comutativ

$\exists O_2 \in M_2(\mathbb{Z}_5)$ a.î. $\forall A \in M_2(\mathbb{Z}_5)$

$A + O_2 = \begin{pmatrix} \hat{a}+\hat{0} & \hat{b}+\hat{0} \\ \hat{c}+\hat{0} & \hat{d}+\hat{d} \end{pmatrix} = \begin{pmatrix} \widehat{a+0} & \widehat{b+0} \\ \widehat{c+0} & \widehat{d+0} \end{pmatrix} = \underbrace{\begin{pmatrix} \widehat{0+a} & \widehat{0+b} \\ \widehat{0+c} & \widehat{0+d} \end{pmatrix}}_{= O_2+A} =$

$= \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} = A$

$\forall A \in M_2(\mathbb{Z}_5), \exists -A \in M_2(\mathbb{Z}_5)$ a.î. $A+(-A) = -A+A = O_2$

$A+(-A) = \begin{pmatrix} \hat{a}+(-\hat{a}) & \hat{b}+(-\hat{b}) \\ \hat{c}+(-\hat{c}) & \hat{d}+(-\hat{d}) \end{pmatrix} = \begin{pmatrix} \widehat{a+n-a} & \widehat{b+n-b} \\ \widehat{c+n-c} & \widehat{d+n-d} \end{pmatrix}$

$= \begin{pmatrix} \hat{n} & \hat{n} \\ \hat{n} & \hat{n} \end{pmatrix} = \begin{pmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{0} \end{pmatrix} = O_2$

$\Rightarrow (M_2(\mathbb{Z}_5), +)$ grup abelian (1)

$(M_2(\mathbb{Z}_5), \cdot)$ semigrup comutativ

Fie $A, B \in M_2(\mathbb{Z}_5)$

$A \cdot B = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \cdot \begin{pmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{pmatrix} = \begin{pmatrix} \widehat{ae}+\widehat{bg} & \widehat{af}+\widehat{bh} \\ \widehat{ce}+\widehat{dg} & \widehat{cf}+\widehat{dh} \end{pmatrix} = \begin{pmatrix} \widehat{ae+bg} & \widehat{af+bh} \\ \widehat{ce+dg} & \widehat{cf+dh} \end{pmatrix}$

$\Rightarrow A \cdot B \in M_2(\mathbb{Z}_5)$

$\forall \hat{a}, \hat{b}, \hat{c} \in \mathbb{Z}_5, (\hat{a}\cdot\hat{b})\cdot\hat{c} = \widehat{a \cdot b \cdot c} = \hat{a} \cdot \widehat{b c} = \hat{a} \cdot (\hat{b} \cdot \hat{c})$

$\Rightarrow (\mathbb{Z}_5, \cdot)$ asociativă $\Rightarrow (M_2(\mathbb{Z}_5), \cdot)$ asociativă

Fie $A=\begin{pmatrix}\hat{1} & \hat{2}\\ \hat{3} & \hat{4}\end{pmatrix}$, $B=\begin{pmatrix}\hat{0} & \hat{2}\\ \hat{3} & \hat{1}\end{pmatrix} \in M_2(\mathbb{Z}_5)$

$A\cdot B=\begin{pmatrix}\hat{1} & \hat{4}\\ \hat{2} & \hat{0}\end{pmatrix}$, $B\cdot A=\begin{pmatrix}\hat{1} & \hat{3}\\ \hat{1} & \hat{0}\end{pmatrix}$

$A\cdot B \neq B\cdot A \Rightarrow \exists\ A, B$ a.î. $A\cdot B\neq B\cdot A \Rightarrow$ „$\cdot$" necomutativă

$\Rightarrow (M_2(\mathbb{Z}_5), \cdot)$ semigrup necomutativ. (2)

$(A+B)\cdot C = A\cdot C + B\cdot C$

$A=\begin{pmatrix}\hat{a}_1 & \hat{a}_2\\ \hat{a}_3 & \hat{a}_4\end{pmatrix}$ $B=\begin{pmatrix}\hat{b}_1 & \hat{b}_2\\ \hat{b}_3 & \hat{b}_4\end{pmatrix}$ $C=\begin{pmatrix}\hat{c}_1 & \hat{c}_2\\ \hat{c}_3 & \hat{c}_4\end{pmatrix}$

$\widehat{a_1+b_1}\cdot\hat{c}_1 + \widehat{a_2+b_2}\cdot\hat{c}_3 = \widehat{a_1c_1+b_1c_1+a_2c_3+b_2c_3}$

$= \widehat{a_1c_1} + \widehat{a_2c_3} + \widehat{b_1c_1} + \widehat{b_2c_3}$

Analog celelalte elemente

$\Rightarrow$ „$\cdot$" distributivă față de „+" (3)

(1), (2), (3) $\Rightarrow (M_2(\mathbb{Z}_5), +, \cdot)$ inel necomutativ

$(M_2(\mathbb{Z}_5), +, \cdot)$ corp $\Leftrightarrow (M_2(\mathbb{Z}_5)\setminus\{O_2\}, \cdot)$ grup

$\exists\ I_2 \in M_2(\mathbb{Z}_5)$ a.î. $\forall\ A \in M_2(\mathbb{Z}_5)$

$A\cdot I_2 = \begin{pmatrix}\hat{a} & \hat{b}\\ \hat{c} & \hat{d}\end{pmatrix}\cdot\begin{pmatrix}\hat{1} & \hat{0}\\ \hat{0} & \hat{1}\end{pmatrix} = \begin{pmatrix}\hat{a}\cdot\hat{1}+\hat{b}\cdot\hat{0} & \hat{a}\cdot\hat{0}+\hat{b}\cdot\hat{1}\\ \hat{c}\cdot\hat{1}+\hat{d}\cdot\hat{0} & \hat{c}\cdot\hat{0}+\hat{d}\cdot\hat{1}\end{pmatrix} = \begin{pmatrix}\hat{a} & \hat{b}\\ \hat{c} & \hat{d}\end{pmatrix} = A$

$\forall\ A\in M_2(\mathbb{Z}_5)$, $\exists\ A^{-1}\in M_2(\mathbb{Z}_5)$ a.î. $A\cdot A^{-1} = A^{-1}\cdot A = I_2 \Leftrightarrow$

$\Leftrightarrow \forall\ A \in M_2(\mathbb{Z}_5)$, $\det A \neq 0$

Fie $A=\begin{pmatrix}\hat{1} & \hat{0}\\ \hat{1} & \hat{0}\end{pmatrix} \neq O_2$

$\det A = \hat{1}\cdot\hat{0} - \hat{0}\cdot\hat{1} = \hat{0}$

$\Rightarrow \exists\ A\in M_2(\mathbb{Z}_5)$ a.î. $\det A = 0$

$\Rightarrow (M_2(\mathbb{Z}_5)\setminus\{O_2\}, \cdot)$ nu e grup

$\Rightarrow (M_2(\mathbb{Z}_5), +, \cdot)$ nu e corp

3.) Exemplu de spațiu vectorial de dimensiune 5:

${}_{\mathbb{Q}}\mathbb{Q}^5$

Exemplu de subspațiu de dimensiune 2:

$(a, b, 0, 0, 0)$, $a, b \in \mathbb{Q}$