P1. Sa se stabileasca natura si suma seriilor:

a)

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$$

$s_n = \ln(1+\frac{1}{1}) + \ln\left(1+\frac{1}{2}\right) + ... + \ln\left(1+\frac{1}{n}\right) = \ln(2) - \ln(1) + \ln(3) - \ln(2) + ... + \ln(n+1) - \ln(n) = = \ln(n+1) - \ln(1) = \ln(n+1)$.

Atunci $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = +\infty$, deci seria este divergenta.

Solutia II:

$s_n = \ln(1+\frac{1}{1}) + \ln\left(1+\frac{1}{2}\right) + ... + \ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot ... \cdot\frac{n+1}{n}\right) = \ln(n+1) \to +\infty$ deci seria e divergenta.

b)

$$\sum_{n=1}^{\infty} arctg\frac{1}{n^2+n+1}$$

Folosim urmatoarea formula. $arctg\,(a) - arctg\,(b) = arctg\frac{a-b}{1+ab}$.

Incercam sa exprimam astfel $arctg\frac{1}{n^2+n+1} = arctg\frac{1}{n(n+1)+1}$ atunci putem spune ca $a = n+1$ si $b = n$.Asadar $arctg\frac{1}{n^2+n+1} = arctg\,(n+1) - arctg\,(n)$. Atunci

$s_n = arctg\frac{1}{1^2+1+1} + arctg\frac{1}{2^2+2+1} + ... + arctg\frac{1}{n^2+n+1} =$

$= arctg\,(2) - arctg\,(1) + arctg\,(3) - arctg\,(2) + ... + arctg\,(n+1) - arctg\,(n)$

$= arctg\,(n+1) - arctg\,(1) = arctg\,(n+1) - \frac{\pi}{4}$

Atunci $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(arctg\,(n+1) - \frac{\pi}{4}\right) = arctg\,(\infty) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

Deci $\sum\limits_{n=1}^{\infty} arctg\frac{1}{n^2+n+1} = \frac{\pi}{4}$.

c)

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$$

Atunci $u_n = \ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{n^2-1}{n^2}\right) = \ln\left(\frac{(n-1)(n+1)}{n^2}\right)$.

In aceste conditii $s_n = \ln\left(1-\frac{1}{2^2}\right) + \ln\left(1-\frac{1}{3^2}\right) + ... + \ln\left(1-\frac{1}{n^2}\right) =$

$=\ln\left(\frac{(2-1)(2+1)}{2^2}\cdot\frac{(3-1)(3+1)}{3^2}\cdot ... \cdot\frac{(n-1)(n+1)}{n^2}\right) = \ln\left(\frac{n+1}{2n}\right)$. Deci $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \ln\left(\frac{1}{2}\right)$.

Asadar $\sum\limits_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{1}{2}\right)$.

Solutia II:

$s_n = \sum \lim\limits_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \sum\limits_{k=2}^{n} \ln\frac{(k-1)(k+1)}{k^2} = \sum\limits_{k=2}^{n} \ln(k-1) + \sum\limits_{k=2}^{n} \ln(k+1) - 2\sum\limits_{k=2}^{n} \ln k =$

$=\sum\limits_{k=2}^{n} \ln k - \ln n + \sum\limits_{k=2}^{n} \ln k + \ln(n+1) - \ln 2 - 2\sum\limits_{k=2}^{n} \ln k = \ln\frac{n+1}{n} \to \ln\frac{1}{2} = -\ln 2.$

d)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}}$$

Observam ca $s_n = -1 + \frac{1}{2} - \frac{1}{2^2} + ... + \frac{(-1)^n}{2^{n-1}}$. Deci s_n este o progresie geometrica cu ratia $=-\frac{1}{2}$.

Atunci $s_n = \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}$ si asa $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$.

Deci $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}} = \frac{2}{3}$.

Solutia II;

se poate aplica si criteriul lui Leibnitz, dar asa nu putem determina explicit suma seriei.

e)

$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$$

Atunci $\frac{u_{n+1}}{u_n} = \frac{2(n+1)-1}{2^{(n+1)}} \cdot \frac{2^n}{2n-1} = \frac{2n+1}{2(2n-1)}$. Cu $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$ deci seria este convergenta.

$s_n = \sum_{k=1}^{n} \frac{2k-1}{2^k} = \sum_{k=1}^{n} \left(\frac{k}{2^{k=1}} - \frac{1}{2^k}\right) = \sum_{k=1}^{n} \frac{k}{2^{k-1}} - \sum_{k=1}^{n} \frac{1}{2^k} = \sum_{k=1}^{n} \frac{k}{2^{k-1}} - \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{2^{k-1}}$

Calculam acum $\sum_{k=1}^{n} \frac{k}{2^{k-1}} = \frac{1}{1} + \frac{2}{2} + \frac{3}{2^2} + ... + \frac{n}{2^{n-1}}$

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$\frac{1}{2} + \frac{1}{2}$

$\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2}$

...

$\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} + ... + \frac{1}{2^{n-1}}$

$=\left(1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}}\right) + \frac{1}{2}\left(1 + \frac{1}{2} + ... + \frac{1}{2^{n-2}}\right) + ... + \frac{1}{2^{n-2}} \cdot \left(1 + \frac{1}{2}\right) + \frac{1}{2^{n-1}} =$

$= \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} + \frac{1}{2} \cdot \frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}} + ... + \frac{1}{2^{n-2}} \left(\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}}\right) + \frac{1}{2^{n-1}} =$

In aceste conditii $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[2 \cdot \sum_{k=1}^{n} \frac{1}{2^k} - \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{2^{k-1}}\right] =$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \cdot \lim_{n\to\infty} \left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$ si atunci

$\sum_{n=1}^{\infty} \frac{2n-1}{2^n} = 1.$

Solutia II:

$u_n = \sum_{k=1}^{n} \frac{k}{2^k} = \sum_{k=1}^{n} \left(\sum_{l=0}^{n} \frac{1}{2^l} - \sum_{l=0}^{k-1} \frac{1}{2^l}\right) = \sum_{k=1}^{n} \left(\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - \frac{1-\left(\frac{1}{2}\right)^k}{1-\frac{1}{2}}\right) =$

$=2 \cdot \sum_{k=1}^{n} \left(1 - \left(\frac{1}{2}\right)^{n+1} - 1 + \left(\frac{1}{2}\right)^k\right) = 2 \cdot \left(-\frac{n}{2^{n+1}} + \frac{1}{2} \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k\right) =$

$=2 \cdot \left(-\frac{n}{2^{n+1}} + \frac{1}{2} \cdot \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = -\frac{n}{2^{n+1}} + 2 - \left(\frac{1}{2}\right)^{n-1}.$

Atunci $\sum\limits_{k=1}^{n} \frac{k}{2^{k-1}} = 2 \cdot \sum\limits_{k=1}^{n} \frac{k}{2^{k}} = -\frac{n}{2^{n+1}} + 4 - \left(\frac{1}{2}\right)^{n-1}.etc.$

f)

$$\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$$

Atunci $u_n = \frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left(\frac{1}{(3n-2)} - \frac{1}{(3n+1)}\right).$

Astfel $s_n = \frac{1}{3}\left(\frac{1}{(3-2)} - \frac{1}{(3+1)}\right) + \frac{1}{3}\left(\frac{1}{(3\cdot 2-2)} - \frac{1}{(3\cdot 2+1)}\right) + ... + \frac{1}{3}\left(\frac{1}{(3n-2)} - \frac{1}{(3n+1)}\right) =$
$= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$. Atunci $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) = \frac{1}{3}.$

Deci $\sum\limits_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{3}.$

Solutia II:

aplicand consecinta criteriul comparatiei, $\lim\limits_{n\to\infty} \frac{\frac{1}{(3n-2)(3n+1)}}{\frac{1}{n^2}} = \frac{1}{9} < 1$ deci cele doua serii au aceeasi natura.

Stiind ca seria $\sum\limits_{k=1}^{n} \frac{1}{n^2}$ e convergenta, rezulta ca si seria noastra e convergenta, dar in acest mod nu putem detrmina explicit suma seriei.

g)

$$\sum_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right)$$

$s_n = \left(\sqrt{1+2} - 2\sqrt{1+1} + \sqrt{1}\right) + \left(\sqrt{2+2} - 2\sqrt{2+1} + \sqrt{2}\right) + ... + \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right) =$
$= -\sqrt{2} + 1 + \sqrt{n+2} - \sqrt{n+1} = \left(1 - \sqrt{2}\right) + \left(\sqrt{n+2} - \sqrt{n+1}\right).$

Atuncti $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left[\left(1 - \sqrt{2}\right) + \left(\sqrt{n+2} - \sqrt{n+1}\right)\right] =$
$= \left(1 - \sqrt{2}\right) + \lim\limits_{n\to\infty} \left(\frac{n+2-n-2}{\sqrt{(n+2)} + \sqrt{(n+1)}}\right) = \left(1 - \sqrt{2}\right).$

Deci $\sum\limits_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right) = \left(1 - \sqrt{2}\right).$

h)

$$\sum_{n=1}^{\infty} \frac{1}{\left(n+\sqrt{2}\right)\left(n+\sqrt{2}+1\right)}$$

Atunci $u_n = \frac{1}{\left(n+\sqrt{2}\right)\left(n+\sqrt{2}+1\right)} = \frac{1}{\left(n+\sqrt{2}\right)} - \frac{1}{\left(n+\sqrt{2}+1\right)}.$

Deci $s_n = \left(\frac{1}{\left(1+\sqrt{2}\right)} - \frac{1}{\left(1+\sqrt{2}+1\right)}\right) + \left(\frac{1}{\left(2+\sqrt{2}\right)} - \frac{1}{\left(2+\sqrt{2}+1\right)}\right) + ... + \left(\frac{1}{\left(n+\sqrt{2}\right)} - \frac{1}{\left(n+\sqrt{2}+1\right)}\right).$

Asadar $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(\frac{1}{1+\sqrt{2}} - \frac{1}{n+\sqrt{2}}\right) = \frac{1}{1+\sqrt{2}}$si
$\sum\limits_{n=1}^{\infty} \frac{1}{\left(n+\sqrt{2}\right)\left(n+\sqrt{2}+1\right)} = \frac{1}{1+\sqrt{2}}.$

Se poate aplica si criteriul comparatiei, dar astfel nu determinam explicit suma seriei.

i)

$$\sum_{n=1}^{\infty} \frac{2n+1}{n\left(n+1\right)\left(n+2\right)}.$$

Atunci $u_n = \frac{2n+1}{n(n+1)(n+2)} = \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{n+1} - \frac{3}{2}\cdot\frac{1}{n+2}$.
Asadar $s_n = \left(\frac{1}{2}\cdot\frac{1}{1} + \frac{1}{1+1} - \frac{3}{2}\cdot\frac{1}{1+2}\right) + \left(\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2+1} - \frac{3}{2}\cdot\frac{1}{2+2}\right) + ... + \left(\frac{1}{2}\cdot\frac{1}{n} + \frac{1}{n+1} - \frac{3}{2}\cdot\frac{1}{n+2}\right) =$
$= \frac{1}{2}\cdot\left(\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}\right) + 1\cdot\left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n+1}\right) - \frac{3}{2}\cdot\left(\frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n+2}\right) =$
$= \frac{1}{2}\cdot\left(1 + \frac{1}{2}\right) + 1\cdot\left(\frac{1}{2} + \frac{1}{n+1}\right) - \frac{3}{2}\cdot\left(\frac{1}{n+1} + \frac{1}{n+2}\right) =$
$= \frac{3}{4} + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{n+1} - \frac{3}{2}\cdot\frac{1}{n+2}$
Deci $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(\frac{3}{4} + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{n+1} - \frac{3}{2}\cdot\frac{1}{n+2}\right) = \frac{5}{4}$.
Atunci $\sum\limits_{n=1}^{\infty} \frac{2n+1}{n(n+1)(n+2)} = \frac{5}{4}$.

Se poate aplica si criteriul comparatiei, dar astfel nu determinam explicit suma seriei.

j)

$$\sum_{n=2}^{\infty} \frac{1}{n\ln n}$$

Atunci $u_n = \frac{1}{n\ln n}$. Vom demonstra ca aceasta serie este divergenta, analizand fiecare caz u_n in parte.

Fixam $n \in N, n > 1$ si definim functia $f : [n, n+1] \to R$ prin

$$\forall x \in [n, n+1] \;\; f\left(x\right) = \ln\ln\left(x\right).$$

Aceasta functie este atat continua cat si derivabila, deci putem aplica teorema lui Lagrange pe intervalul $[n, n+1]$. Atunci $\exists\lambda \in \left]0,1\right[$ astfel incat

$$\ln\ln\left(n+1\right) - \ln\ln\left(n\right) = \left(\ln\ln\left(n+\lambda\right)\right)'\left[\left(n+1\right) - n\right]$$

$$\ln\ln\left(n+1\right) - \ln\ln\left(n\right) = \frac{1}{\left(n+\lambda\right)\ln\left(n+\lambda\right)}.$$

Pentru ca $\lambda \in \left]0,1\right[\Rightarrow n < n+\lambda$ si deci $\frac{1}{n+\lambda} < \frac{1}{n}$. Din aceleasi motive $\frac{1}{\ln(n+\lambda)} < \frac{1}{\ln(n)}$, asadar

$$\frac{1}{\left(n+\lambda\right)\ln\left(n+\lambda\right)} < \frac{1}{n\ln\left(n\right)}.$$

Pentru ca n a fost arbitrar, cele aratate mai sus sunt valabile $\forall n \in N, n > 1$, deci

$$\forall n \in N, \; n > 1 \;\; \ln\ln\left(n+1\right) - \ln\ln\left(n\right) < \frac{1}{n\ln\left(n\right)}.$$

Definim seria cu termenul general $v_n = \ln\ln\left(n+1\right) - \ln\ln\left(n\right)$ .Pentru ea

$s_n = \left(\ln\ln 3 - \ln\ln 2\right) + \left(\ln\ln 4 - \ln\ln 3\right) + ... + \left(\ln\ln\left(n+1\right) - \ln\ln\left(n\right)\right) = \ln\ln\left(n+1\right) - \ln\ln\left(2\right)$

iar $\lim\limits_{n\to\infty} s_n = +\infty$, asadar seria $\sum\limits_{n=2}^{\infty} v_n = \infty$. Deoarece $\forall n \in N, n > 1 \; v_n \leq u_n$ rezulta ca si seria $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln n}$ este divergenta si $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln n} = +\infty$.

Solutia II:

Fie $u_n = \frac{1}{n \ln n}$. Atunci $2^n \cdot u_{2^n} = 2^n \cdot \frac{1}{2^n \cdot \ln 2^n} = \frac{1}{\ln 2} \cdot \frac{1}{n}$. Conform criteriului de condensare al lui Cauchy, seria $\sum\limits_{n=1}^{\infty} u_n$ are aceeasi natura ca si $\sum\limits_{n=1}^{\infty} \frac{1}{n}$., deci e divergenta.

k)

$$\sum_{n=3}^{\infty} \frac{1}{n \cdot \ln(n) \cdot \ln\ln(n)}$$

Atunci $u_n = \frac{1}{n \ln(n) \ln\ln(n)}$. Vom demonstra ca aceasta serie este divergenta, analizand fiecare caz u_n in parte.

Fixam $n \in N, n > 2$ si definim functia $f : [n, n+1] \to R$ prin

$$\forall x \in [n, n+1] \;\; f(x) = \ln\ln\ln(x).$$

Aceasta functie este atat continua cat si derivabila, deci putem aplica teorema lui Lagrange pe intervalul $[n, n+1]$. Atunci $\exists \lambda \in]0, 1[$ astfel incat

$$\ln\ln\ln(n+1) - \ln\ln\ln(n) = (\ln\ln\ln(n+\lambda))' \left[(n+1) - n\right]$$

$$\ln\ln\ln(n+1) - \ln\ln\ln(n) = \frac{1}{(n+\lambda)\ln(n+\lambda)\ln\ln(n+\lambda)}.$$

Pentru ca $\lambda \in]0, 1[\Rightarrow n < n + \lambda$ si deci $\frac{1}{n+\lambda} < \frac{1}{n}$. Din aceleasi motive $\frac{1}{\ln(n+\lambda)} < \frac{1}{\ln(n)}$ si $\frac{1}{\ln\ln(n+\lambda)} < \frac{1}{\ln\ln(n)}$, asadar

$$\frac{1}{(n+\lambda)\ln(n+\lambda)\ln\ln(n+\lambda)} < \frac{1}{n\ln(n)\ln\ln(n)}.$$

Pentru ca n a fost arbitrar, cele aratate mai sus sunt valabile $\forall n \in N, n > 2$, deci

$$\forall n \in N, \;\; n > 1 \;\; \ln\ln\ln(n+1) - \ln\ln\ln(n) < \frac{1}{n\ln(n)\ln\ln(n)}.$$

Definim seria cu termenul general $v_n = \ln\ln\ln(n+1) - \ln\ln\ln(n)$ .Pentru ea

$s_n = (\ln\ln\ln 3 - \ln\ln\ln 2) + (\ln\ln\ln 4 - \ln\ln\ln 3) + ... + (\ln\ln\ln(n+1) - \ln\ln\ln(n)) = \ln\ln\ln(n+1) - \ln\ln\ln(2)$

iar $\lim\limits_{n\to\infty} s_n = +\infty$, asadar seria $\sum\limits_{n=2}^{\infty} v_n = \infty$. Deoarece $\forall n \in N, n > 2 \; v_n \leq u_n$ rezulta ca si seria $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln(n)\ln\ln(n)}$ este divergenta si $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln(n)\ln\ln(n)} = +\infty$.

Solutia II:

$u_n = \frac{1}{n \ln n \ln(\ln n)} \Rightarrow 2^n u_{2^n} = 2^n \frac{1}{2^n \ln 2^n \ln(\ln 2^n)} = \frac{1}{\ln 2} \cdot \frac{1}{n \ln(n \ln 2)} =$

$= \frac{1}{\ln 2} \cdot \frac{1}{n(\ln n + \ln\ln 2)}$. Deci $\sum\limits_{n=1}^{\infty} u_n$ are aceeasi natura cu seria $\sum\limits_{n=1}^{\infty} \frac{1}{n(\ln n + \ln\ln 2)}$.

Asadar $\lim\limits_{n\to\infty} \frac{\frac{1}{n(\ln n + \ln\ln 2)}}{\frac{1}{n \ln n}} = \lim\limits_{n\to\infty} \frac{\ln n}{\ln n + \ln\ln 2} = 1$. In consecinta seria $\sum\limits_{n=1}^{\infty} \frac{1}{n(\ln n + \ln\ln 2)}$ are aceeasi natura cu seria $\sum\limits_{n=1}^{\infty} \frac{1}{n \ln n}$, despre care am arata la $j)$ ca e divergenta.

P2. Stabiliti natura seriilor:

a)

$$\sum_{n=1}^{\infty} \frac{9+n}{2n+1}$$

Soluita e simpla. $\lim\limits_{n\to\infty} \frac{9+n}{2n+1} = \frac{1}{2} \neq 0$ deci seria este divergenta.

b)

$$\sum_{n=1}^{\infty} \frac{2^n+3^n}{2^{n+1}+3^{n+1}}$$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{2^n+3^n}{2^{n+1}+3^{n+1}} = \lim\limits_{n\to\infty} \frac{3^n\left[\left(\frac{2}{3}\right)^n+1\right]}{3^{n+1}\left[\left(\frac{2}{3}\right)^{n+1}+1\right]} = \frac{1}{3} \neq 0$, deci seria este divergenta.

c)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}-\sqrt{2n-1}}$$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{2n+1}-\sqrt{2n-1}} = \lim\limits_{n\to\infty} \frac{\sqrt{2n+1}+\sqrt{2n-1}}{2n+1-2n+1} = +\infty \neq 0$ deci seria este divergenta.

d)

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, deci seria este divergenta.

e)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\ln n}}.$$

$\forall n \in N, n > 1$ $\ln n < n$, deci si $\ln n < n^n$. Atunci $\forall n \in N, n > 1$ $\sqrt[n]{\ln n} < n$ si astfel $\frac{1}{\sqrt[n]{\ln}} > \frac{1}{n}$. Deoarece seria $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ e divergenta rezulta ca si seria $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{\ln n}}$ este divergenta.

Solutia II:

Aplicam criteriul radiculului pentru sirul $\sqrt[n]{\frac{1}{\ln n}}$. Conform criteriului radicalului, daca exista $\lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = l$ atunci $\lim\limits_{n\to+\infty} \sqrt[n]{a_n} = l$. In cazul nostru

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to+\infty} \frac{\ln n}{\ln n(1+\frac{1}{n})} = \lim_{n\to+\infty} \frac{\ln n}{\ln n + \ln(1+\frac{1}{n})} =$$

$$= \lim_{n\to+\infty} \frac{1}{1+\frac{\ln(1+\frac{1}{n})}{\ln n}} = 1.$$

Asadar $\lim\limits_{n\to+\infty} \sqrt[n]{\frac{1}{\ln n}} = 1$ deci $\lim\limits_{n\to+\infty} u_n = 1 \neq 0$, deci seria este divergenta.

Solutia III:
$\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{\ln n}} = 1 \neq 0$, deci seria e divergenta. Limita e 1 pentru ca

$$\lim_{n\to\infty} \ln \sqrt[n]{\ln n} = \lim_{n\to\infty} \frac{\ln(\ln n)}{n} = 0 => \lim_{n\to\infty} \sqrt[n]{\ln n} = 1.$$

f)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$$

Pentru oricare $n > 1$ $\sqrt[n]{n} < n$ si deci $\frac{1}{n} < \frac{1}{\sqrt[n]{n}}$. Deoarece $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ e divergenta rezulta ca si $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ este divergenta.

Solutia II:

$$\lim_{n\to\infty} \ln \sqrt[n]{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0 => \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

deci $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0$,rezulta ca si $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ este divergenta.

P3. Stabiliti natura seriilor:

a)

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

Pentru $n > 1,\ n < 2n - 1$ si deci $\frac{1}{n} < \frac{1}{2n-1}$. Deoarece $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ e divergenta rezulta ca si $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$ este divergenta.

Solutia II:
aplicand consecinta criteriului raportului, $\lim\limits_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \frac{1}{2} < 1$ deci $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$ are aceeasi naturca cu care $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ e divergenta.

b)

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Fie asadar $u_n = \frac{1}{(2n-1)^2}$. vom folosi o nou seria, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ despre care stim ca e convergenta si are termenul general $v_n = \frac{1}{n^2}$. Atunci $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{n^2}{(2n-1)^2} = \frac{1}{4} \in]0, +\infty[$ si deci cele doua serii au aceeasi natura, atunci si $\sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ este convegenta.

c)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}}.$$

Fie asadar $u_n = \frac{1}{\sqrt{4n^2-1}}$. vom folosi o nou seria, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ despre care stim ca e divergenta si are termenul general $v_n = \frac{1}{n}$. Atunci $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{n}{\sqrt{4n^2-1}} = \frac{1}{2} \in]0, +\infty[$ si deci cele doua serii au aceeasi natura, atunci si $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}}$ este divergenta.

d)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Fie asadar $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. vom folosi o nou seria, $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ despre care stim ca e divergenta si are termenul general $v_n = \frac{1}{\sqrt{n}}$. Atunci $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{2} \in]0, +\infty[$ si deci cele doua serii au aceeasi natura, atunci si $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ este divergenta.

Solutia II:
$\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum\limits_{k=1}^{\infty} \left(\sqrt{k+1}-\sqrt{k}\right) = \sqrt{n+1}-1 \to \infty$ deci seria e divergenta.

Solutia III:
$\frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2} \cdot \frac{1}{\sqrt{n+1}}$ si $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ e div, iar $\frac{1}{2} < 1 =>$ $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ e divergenta.

e)

$$\sum_{n=1}^{\infty} \frac{n \cos^2\left(\frac{n\pi}{3}\right)}{2^n}.$$

$\forall n \in N,\ \frac{n\cos^2\left(\frac{n\pi}{3}\right)}{2^n} < \frac{n}{2^n}$. (1)

Studiem acum seria $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$. $\lim\limits_{n\to\infty} \frac{v_{n+1}}{v_n} = \lim\limits_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim\limits_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n} = \frac{1}{2} < 1$ deci seria este convergenta. Atunci din (1) si $\sum\limits_{n=1}^{\infty} \frac{n\cos^2\left(\frac{n\pi}{3}\right)}{2^n}$ este convergenta.

f)

$$\sum_{n=2}^{\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}$$

$v_n = \frac{1}{n}$. $\lim\limits_{n\to\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} \cdot n = \frac{n^3\sqrt{1+\frac{1}{n}}}{n^{\frac{5}{3}}\sqrt{1-\frac{}{n}}} = +\infty$. Deoarece $\sum\limits_{n=1}^{\infty} v_n$ este divergenta $(T1.4, 3^0 b) =>$ ca si $\sum\limits_{n=2}^{\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}$ este divergenta.

g)

$$\sum_{n=1}^{\infty} \frac{1}{2^n - n + 1}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{2^n-n+1}{2^{n+1}-(n+1)+1} = \lim\limits_{n\to\infty} \frac{2^n\left(1-\frac{n}{2^n}+\frac{1}{2^n}\right)}{2^{n+1}\left(1-\frac{n}{2^{n+1}}\right)} = \frac{1}{2} < 1$, deci seria este convergenta.

h)

$$\sum_{n=1}^{\infty} \frac{1}{3^n + n^2 + n}$$

Fie $v_n = \frac{1}{n^2}$. $\forall n \in N \;\; n^2 < 3^n+n^2+n$ si deci $u_n < v_n$. Deoarece $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ este convergenta, rezulta ca si $\sum\limits_{n=1}^{\infty} \frac{1}{3^n+n^2+n}$ este convergenta.

Solutia II:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3^n + n^2 + 1}{3^{n+1} + (n+1)^2 + 1} = \frac{1}{3} < 1 \Longrightarrow \text{ convergenta.}$$

i)

$$\sum_{n=1}^{\infty} \frac{\sqrt{7n}}{n^2 + 3n + 5}.$$

Fie $v_n = \frac{1}{n^{\frac{3}{2}}}$. Stim ca $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ este convergenta. $\lim\limits_{n\to\infty} \frac{\sqrt{7n}}{n^2+3n+5} \cdot \frac{1}{\frac{1}{n^{\frac{3}{2}}}} = \sqrt{7} \in \left]0, \infty\right[$. Atunci cele doua serii au aceeasi natura, deci sunt amandoua convergente, asa ca $\sum\limits_{n=1}^{\infty} \frac{\sqrt{7n}}{n^2+3n+5}$ este convergenta.

j)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt[3]{n-1}}.$$

Fie $v_n = \frac{1}{n}$. Stim ca $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ este divergenta. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}\sqrt[3]{n-1}} \cdot \frac{1}{\frac{1}{n}} = +\infty$. Atunci, (T1.d,3,b) din faptul ca $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ este divergenta va rezulta ca si $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt[3]{n-1}}$ este divergenta.

k)

$$\sum_{n=1}^{\infty} \frac{\ln^{10} n}{n^{1.1}}.$$

Datorita proprietatilor logaritmului stim ca exista un $n \in N$ suficient de mare astfel incat

$$\ln^{10} n < n^{0.09}$$

(daca studiem funcita cu derivata se observa usor lucrul rsp.). De aici rezulta ca

$$n^{1.09}\ln^{10} n < n^{1.1}$$

si astfel

$$\frac{\ln^{10} n}{n^{1.1}} < \frac{1}{n^{1.09}}.$$

Seria $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1.09}}$ este convergenta deoarece $1.09 > 1$ si astfel si seria $\sum\limits_{n=1}^{\infty} \frac{\ln^{10} n}{n^{1.1}}$ este convergenta.

l)

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{2}+\sqrt[3]{3}+...\sqrt[n]{n}}$$

$\forall i \in \{1, ..., n-1\}$ stim ca $\sqrt[n]{n} < \sqrt[i]{i}$. De aici rezulta ca $n\sqrt[n]{n} < 1+\sqrt{2}+\sqrt[3]{3}+...\sqrt[n]{n}$ si astfel

$$\frac{1}{1+\sqrt{2}+\sqrt[3]{3}+...\sqrt[n]{n}} < \frac{1}{n\sqrt[n]{n}} < \frac{1}{n^2}.$$

Despre seria $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ stim ca este convergenta, si utilizand criteriul comparatiei vom obtine ca si $\sum\limits_{n=1}^{\infty} \frac{1}{1+\sqrt{2}+\sqrt[3]{3}+...\sqrt[n]{n}}$ este convergenta.

m)

$$\sum_{n=1}^{\infty} \left(2-\sqrt{e}\right)\left(2-\sqrt[3]{e}\right)\cdot ... \cdot\left(2-\sqrt[n]{e}\right)$$

Tinem cont de urmatoarea inegalitate, pentru $\forall n \in N$

$$e < \left(1+\frac{1}{n+1}\right)^n$$

atunci

$$e^{\frac{1}{n}} < 1+\frac{1}{n+1}$$

si astfel $2-e^{\frac{1}{n}} > 2-1-\frac{1}{n+1} = 1-\frac{1}{n+1} = \frac{n}{n+1}$. Atunci

$$\left(2-\sqrt{e}\right)\left(2-\sqrt[3]{e}\right)\cdot ... \cdot\left(2-\sqrt[n]{e}\right) > \frac{2}{3}\cdot\frac{3}{4}\cdot ... \cdot\frac{n}{n+1} = \frac{2}{n+1}$$

Despre seria $\sum\limits_{n=1}^{\infty} \frac{2}{n+1}$ se poate arata cu usurinta ca este divergenta, astfel, din inegalitatea de mai sus rezulta ca si $\sum\limits_{n=1}^{\infty} (2-\sqrt{e})(2-\sqrt[3]{e})\cdot ... \cdot(2-\sqrt[n]{e})$ este tot o serie divergenta.

n)

$$\sum_{n=1}^{\infty} \frac{e^n}{n\left(1+2^n\right)}$$

Atunci $u_n = \frac{e^n}{n(1+2^n)}$ si $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{e^{n+1}}{(n+1)(1+2^{n+1})} \cdot \frac{n(1+2^n)}{e^n} = e \cdot \lim\limits_{n\to\infty} \frac{n}{n+1} \cdot \frac{(1+2^n)}{(1+2^{n+1})} =$
$= e \cdot \lim\limits_{n\to\infty} \frac{2^n\left(\left(\frac{1}{2}\right)^n+1\right)}{2^{n+1}\left(\left(\frac{1}{2}\right)^{n+1}+1\right)} = \frac{e}{2} > 1$ deci seria este divergenta.

Solutia II:

$$\lim_{n\to\infty} \frac{e^n}{n\left(1+2^n\right)} = \infty \text{ deoarece } \frac{e^n}{n\left(1+2^n\right)} > \frac{e^n}{2n \cdot 2^n} = \frac{1}{2} \cdot \frac{\left(\frac{e}{2}\right)^n}{n} \to \infty.$$

deci seria e divergenta.

o)

$$\sum_{n=1}^{\infty} \sin \frac{1}{n}$$

Definim $v_n = \frac{1}{n}$. Stim ca $\lim\limits_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$, deci $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1 \in]0, +\infty[$ si astfel cele doua serii au aceeasi natura. Deoarece $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ este divergenta rezulta ca si $\sum\limits_{n=1}^{\infty} \sin\frac{1}{n}$ este divergenta.

p)

$$\sum_{n=1}^{\infty} \frac{\sqrt[4]{n^2-1}}{\sqrt{n^4-1}}$$

$\lim\limits_{n\to\infty} \frac{\sqrt[4]{n^2-1}}{\sqrt{n^4-1}} \cdot \frac{1}{\frac{1}{n^2}} = 1 \in]0, +\infty[$, deci cele doua serii au aceeasi natura. Deoarece $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ este convergenta rezulta ca si $\sum\limits_{n=1}^{\infty} \frac{\sqrt[4]{n^2-1}}{\sqrt{n^4-1}}$ este convergenta.

q)

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2-1}}{\sqrt{n^3-1}}$$

Definim $v_n = \frac{1}{n^{\frac{3}{2}-\frac{2}{3}}} = \frac{1}{n^{\frac{5}{6}}}$. $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^2-1}}{\sqrt{n^3-1}} \cdot \frac{1}{\frac{1}{n^{\frac{5}{6}}}} = 1 \in]0, +\infty[$ si deci cele doua seii au aceeasi natura. $\frac{5}{6} < 1$ si deci $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{5}{6}}}$ este divergenta. Rezulta astfel ca si $\sum\limits_{n=1}^{\infty} \frac{\sqrt[3]{n^2-1}}{\sqrt{n^3-1}}$ este divergenta.

P4.

a)

$$\sum_{n=1}^{\infty} \frac{100^n}{n!}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \lim\limits_{n\to\infty} \frac{100}{n+1} = 0 < 1$, deci seria este convergenta.

b)

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \lim\limits_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$ deci seria este convergenta.

c)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$u_n = \frac{1\cdot 2\cdot \ldots \cdot n}{n\cdot n\cdot \ldots \cdot n} < \frac{2}{n^2}$ pentru $\forall n > 3$. Seria $\sum\limits_{n=1}^{\infty} \frac{2}{n^2}$ este convergenta, deci prin criteriul comparatiei si $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$ este convergenta.

d)

$$\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{2^{n+1}\cdot(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n\cdot n!} = 2 \cdot \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 2 \cdot \lim\limits_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^n =$
$= 2 \cdot \lim\limits_{n\to\infty} \left[\left(1 + \frac{-1}{n+1}\right)^{-(n+1)}\right]^{-\frac{n}{n+1}} = 2 \cdot e^{\lim\limits_{n\to\infty} \frac{-n}{n+1}} = \frac{2}{e} < 1$ deci seria este convergenta.

sau
$\lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$.

e)

$$\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n\cdot n!} = 3 \cdot \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 3 \cdot \lim\limits_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^n =$
$= 3 \cdot \lim\limits_{n\to\infty} \left[\left(1 + \frac{-1}{n+1}\right)^{-(n+1)}\right]^{-\frac{n}{n+1}} = 3 \cdot e^{\lim\limits_{n\to\infty} \frac{-n}{n+1}} = \frac{3}{e} > 1$ deci seria este divergenta.

Observatie:
Se observa ca seria $\sum\limits_{n=1}^{\infty} \frac{x^n\cdot n!}{n^n}$ este convergenta pentru $x < e$ si divergenta pentru $x > e$.

f)

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{((n+1)!)^2}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{(n!)^2} = \lim\limits_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}}$.

Pentru a calcula limita trecem la functii $\lim\limits_{x\to\infty} \frac{(x+1)^2}{2^{2x+1}} = \lim\limits_{x\to\infty} \frac{2(x+1)}{2\cdot 2^{2x+1}\cdot \ln 2} = \lim\limits_{x\to\infty} \frac{2}{2\cdot \ln 2\cdot 2\cdot \ln 2\cdot 2^{2n+1}} = 0 < 1$, deci seria este convergenta.

g)

$$\sum_{n=1}^{\infty} \frac{100\cdot 101\cdot \ldots \cdot (100+n)}{1\cdot 3\cdot \ldots \cdot (2n-1)}$$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{100\cdot 101\cdot \ldots \cdot(101+n)}{1\cdot 3\cdot \ldots \cdot (2n+1)} \cdot \frac{1\cdot 3\cdot \ldots \cdot(2n-1)}{100\cdot 101\cdot \ldots \cdot (100+n)} = \lim\limits_{n\to\infty} \frac{101+n}{2(n+1)} = \frac{1}{2} < 1$ deci seria este convergenta.

h)

$$\sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdot ... \cdot (4+3n)}{2 \cdot 6 \cdot ... \cdot (2+4n)}$$

atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{4\cdot7\cdot...\cdot(4+3(n+1))}{2\cdot6\cdot...\cdot(2+4(n+1))} \cdot \frac{2\cdot6\cdot...\cdot(2+4n)}{4\cdot7\cdot...\cdot(4+3n)} = \lim\limits_{n\to\infty} \frac{7+3n}{6+4n} = \frac{3}{4} < 1$ deci seria este convergenta.

k)

$$\sum_{n=1}^{\infty} \left(2-\sqrt[3]{2}\right)\left(2-\sqrt[5]{2}\right) \cdot ... \cdot \left(2-\sqrt[2n+1]{2}\right)$$

ne vom folosi de faptul ca $2 < e$ deci $\forall k \in \{1, ..., n\}$ este adevarat ca $\sqrt[2k+1]{2} < \sqrt[2k+1]{e}$ si deci

$\left(2-\sqrt[2k+1]{2}\right) > (2-\sqrt[2k+1]{e})$.

In acelasi timp stim ca $\forall n \in N$, $e < \left(1+\frac{1}{n+1}\right)^n$ si deci $\sqrt[n]{e} < \left(1+\frac{1}{n+1}\right)$, deci
$2-\sqrt[n]{e} > 2-\left(1+\frac{1}{n+1}\right) = 1-\frac{1}{n+1} = \frac{n}{n+1}$.
In acelasi timp stim ca $\sqrt[2k+1]{e} < \sqrt[k]{e}$, asadar $(2-\sqrt[2k+1]{e}) > (2-\sqrt[k]{e}) > \frac{k}{k+1}$.
Atunci $\forall n \in N$, $u_n = \left(2-\sqrt[3]{2}\right)\left(2-\sqrt[5]{2}\right) \cdot ... \cdot \left(2-\sqrt[2n+1]{2}\right) > \frac{1}{2} \cdot \frac{2}{3} \cdot ... \cdot \frac{n}{n+1} = \frac{1}{n+1}$.

Definim acum seria $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} \frac{1}{n+1}$, si seria $\sum\limits_{n=1}^{\infty} r_n = \frac{1}{n}$. Despre ultima stim ca este o serie divergenta si $\lim\limits_{n\to\infty} \frac{v_n}{r_n} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \in]0, +\infty[$, deci cele doua serii au aceeasi natura. In consecinta si $\sum\limits_{n=1}^{\infty} \frac{1}{n+1}$ este divergenta si stim ca $\forall n \in N$ $u_n > v_n$, atunci din criteriul comparatiei rezulta ca si $\sum\limits_{n=1}^{\infty} u_n$ este tot o serie divergenta.

l)

$$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

Deoarece $\forall n \in N$ este adevarate inegalitatatea $2+\frac{1}{n} > 2$ deci $\frac{1}{2+\frac{1}{n}} < \frac{1}{2}$ si

$$\frac{1}{\left(2+\frac{1}{n}\right)^n} < \frac{1}{2^n}$$

si datorita faptului ca $n \in N$ este deasemeana adevarata inegalitatea

$$\frac{n^2}{\left(2+\frac{1}{n}\right)^n} < \frac{n^2}{2^n}.$$

Definim astfel seria $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$, a carei natura o vom studia cu ajutorul criteriului raportului. Atunci $\lim\limits_{n\to\infty} \frac{v_{n+1}}{v_n} = \lim\limits_{n\to\infty} \left(\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right) = \frac{1}{2} < 1$ deci ea este convergenta.

Folosind acesta convergenta si criteriul comparatiei va rezulta ca si seria $\sum\limits_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$ este convergenta.

m)

$$\sum_{n=1}^{\infty} \frac{1}{(2n^2+n+1)^{\frac{n+1}{2}}}$$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \frac{1}{(2n^2+n+1)^{\frac{n+1}{2n}}} = \lim_{n\to\infty} \left(1 + \frac{-2n^2-n}{2n^2+n+1}\right)^{\frac{n+1}{2n}} = e^{\lim\limits_{n\to\infty} \frac{-2n^2-n}{2n^2+n+1} \cdot \frac{n+1}{2n}} = e^{-2} < 1$$

deci seria $\sum\limits_{n=1}^{\infty} \frac{1}{(2n^2+n+1)^{\frac{n+1}{2}}}$ este convergenta.

n)

$$\sum_{n=1}^{\infty} \left(\frac{1+2^3+...+n^3}{n^3} - \frac{n}{4}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left(\frac{1+2^3+...+n^3}{n^3} - \frac{n}{4}\right) = \lim_{n\to\infty} \left[\left(\frac{n(n+1)}{2}\right)^2 \cdot \frac{1}{n^3} - \frac{n}{4}\right] =$$

$$= \lim_{n\to\infty} \left(\frac{n^2\left(n^2+2n+1\right)}{4n^3} - \frac{n}{4}\right) = \lim_{n\to\infty} \frac{2n^3+n^2}{4n^3} = \frac{1}{2} < 1 \text{ deci seria este convergenta.}$$

0)

$$\sum_{n=1}^{\infty} \left(\sqrt[3]{n^3+n^2+1} - \sqrt[3]{n^3-n^2+1}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left(\sqrt[3]{n^3+n^2+1} - \sqrt[3]{n^3-n^2+1}\right) =$$

$$= \lim_{n\to\infty} \frac{n^3+n^2+1-(n^3-n^2+1)}{\sqrt[3]{(n^3+n^2+1)^2} + \sqrt[3]{(n^3+n^2+1)(n^3-n^2+1)} + \sqrt[3]{(n^3-n^2+1)^2}} = \frac{2}{3} < 1$$

deci seria este convergenta.

p)

$$\sum_{n=1}^{\infty} \frac{(3n)^2}{\sqrt{(16n^2+5n+1)^{n+1}}}$$

Ne folosim de faptul ca $\forall n \in N,\ n > 2\ \ (3n)^2 < (3n)^{\frac{n+1}{2}}$. Asadar

$$\frac{(3n)^2}{\sqrt{(16n^2+5n+1)^{n+1}}} \leq \left(\frac{3n}{16n^2+5n+1}\right)^{\frac{n+1}{2}}.$$

Folosind criteriul radicalului pentru seria $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} \left(\frac{3n}{16n^2+5n+1}\right)^{\frac{n+1}{2}}$ obtinem

$$\lim_{n\to\infty} \sqrt[n]{v_n} = \lim_{n\to\infty} \left(\frac{3n}{16n^2+5n+1}\right)^{\frac{n+1}{2n}} = \lim_{n\to\infty} \left(1 + \frac{-16n^2-2n-1}{16n^2+5n+1}\right)^{\frac{n+1}{2n}} =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-16n^2-2n-1}{16n^2+5n+1} \right)^{\frac{16n^2+5n+1}{-16n^2-2n-1}} \right]^{\frac{-16n^2-2n-1}{16n^2+5n+1}\cdot\frac{n+1}{2n}} = e^{-\frac{1}{2}} < 1,$$

deci seria $\sum\limits_{n=1}^{\infty} \left(\frac{3n}{16n^2+5n+1}\right)^{\frac{n+1}{2}}$ este convergenta, iar din crieteriul comparatiei rezulta ca si seria $\sum\limits_{n=1}^{\infty} \frac{(3n)^2}{\sqrt{(16n^2+5n+1)^{n+1}}}$ este convergenta.

Solutia II:
$u_n < \frac{(3n)^2}{(16n^2)^{\frac{n(n+1)}{2}}} = \frac{9}{4^{n+1}} \cdot n^{2-n-2} = \frac{9}{4^{n+1}} \cdot n^{1-n} => s_n < t_n$, cu $v_n = \frac{9}{4^{n+1}} \cdot \frac{1}{n^{n-1}}$,
$t_n = v_1 + ...v_{n.}$ $t_m < m \cdot v_m = \frac{9}{4^{m+1}} \cdot \frac{1}{m^m} \to 0 => \sum t_m$ e convergenta $\Longrightarrow \sum s_n$ e convergenta.

P5. Pentru fiecare a>0 studiati natura seriei:

a)

$$\sum_{n=1}^{\infty} \frac{1}{a^n+n}$$

Cazuri:
- daca $a > 1$ atunci, pentru un n suficient de mare este adevarata inegalitatea

$$a^n > n^2$$

(stuiul se poate face cu derivate). De aceea rezulta ca $a^n+1 > n^2+1$ si astfel $\frac{1}{n^2+1} < \frac{1}{a^n+n}$. Studiem acum seria $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+1}$ cu ajutorul criteriului raportului si a seriei $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ despre care am demostrat ca este convergenta. Astfel

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = 1 \in]0, +\infty[$$

deci cele doua serii au aceeasi natura, si astfel ajungem la concluzia ca seria $\sum\limits_{n=1}^{\infty} \frac{1}{a^n+n}$ este convergenta in acest caz.
- daca $a \leq 1$ atunci $a^n \leq 1$ si astfel $a^n + n \leq 1 + n$. Atunci $\frac{1}{1+n} \leq \frac{1}{a^n+n}$ si deoarece seria $\sum\limits_{n=1}^{\infty} \frac{1}{1+n}$ este divergenta, din criteriul comparatiei rezulta ca si seria $\sum\limits_{n=1}^{\infty} \frac{1}{a^n+n}$ este divergenta.

Concluzie: $\sum\limits_{n=1}^{\infty} \frac{1}{a^n+n}$
- convergenta cand $a > 1$
- divergenta cand $a \leq 1$.

b)

$$\sum_{n=1}^{\infty} \frac{a^n}{\sqrt{n!}}$$

Vom folosi criteriul raportului. $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\frac{a^{n+1}}{\sqrt{(n+1)!}}\cdot\frac{\sqrt{n!}}{a^n} = \lim\limits_{n\to\infty}\frac{a}{\sqrt{n+1}} = 0$, deci seria este convergenta.

c)

$$\sum_{n=1}^{\infty} a^{\ln n}$$

Vom incepe de la teorema care afirma ca daca o serie este convergenta, atunci $\lim\limits_{n\to\infty} u_n = 0$. Folosind principiile logicii matematice rezulta ca daca $\lim\limits_{n\to\infty} u_n \neq 0$, atunci seria este divergenta.

Incepe prin calcularea

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} a^{\ln n} = \begin{cases} +\infty \text{ daca } a>1 \\ 0 \text{ daca } a<1 \end{cases}.$$

In cazul in care $a=1$ nu ne putem exprima in privinta limitei respective, dar ne vom descurca si fara.

Cazuri:

- $a>1$, atunci $\lim\limits_{n\to\infty} a^{\ln n} = +\infty$ si astfel seria este divergenta.

- $a=1$ atunci seria devine $\sum\limits_{n=1}^{\infty} 1 = +\infty$ deci este divergenta.

- $a<1$, in acest caz $\lim\limits_{n\to\infty} a^{\ln n} = 0$, deci in acest caz seria ar putea fi convergenta, si trebuie analizata mai amplu. Incepem cu criteriul raportului. $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\frac{a^{\ln(n+1)}}{a^{\ln(n)}} = \lim\limits_{n\to\infty} a^{\ln(n+1)-\ln(n)}$
$= \lim\limits_{n\to\infty} a^{\ln\frac{n+1}{n}} = \lim\limits_{n\to\infty} a^{\ln\left(1+\frac{1}{n}\right)} = a^0 = 1$, deci nu obtinem nici o informatie valoroasa, trecand astfel la consecinta criteriului lui Raabe- Duhamel. Atunci $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim\limits_{n\to\infty} n\left(\frac{a^{\ln n}}{a^{\ln(n+1)}}-1\right) =$

$= \lim\limits_{n\to\infty} n\left(a^{\ln n-\ln(n+1)}-1\right) = \lim\limits_{n\to\infty}\frac{a^{\ln\left(\frac{n}{n+1}\right)}-1}{\frac{1}{n}} \overset{\frac{0}{0}}{=}$

$^{*}\!= \lim\limits_{n\to\infty}\frac{\left(a^{\ln\left(\frac{n}{n+1}\right)}\right)'}{\left(\frac{1}{n}\right)'} = \lim\limits_{n\to\infty}\frac{a^{\ln\left(\frac{n}{n+1}\right)}\cdot\left(\ln\left(\frac{n}{n+1}\right)\right)'\cdot\ln a}{-\frac{1}{n^2}} =$

$= \lim\limits_{n\to\infty}\frac{a^{\ln\left(\frac{n}{n+1}\right)}\cdot\frac{n+1}{n}\cdot\left(\frac{n}{n+1}\right)'\cdot\ln a}{-\frac{1}{n^2}} = \lim\limits_{n\to\infty}\frac{a^{\ln\left(\frac{n}{n+1}\right)}\cdot\frac{n+1}{n}\cdot\frac{n+1-n}{(n+1)^2}\cdot\ln a}{-\frac{1}{n^2}} =$

$= \ln a\cdot\lim\limits_{n\to\infty}\left(-\frac{n}{n+1}\right)\cdot a^{\ln\left(\frac{n}{n+1}\right)} = -\ln a$

sau, se putea ajunge la aceeasi concluzie cu mult mai usor, prin

$^{**}\!= \lim\limits_{n\to\infty}\frac{a^{\ln\left(\frac{n}{n+1}\right)}-1}{\frac{1}{n}} = \lim\limits_{n\to\infty}\frac{a^{\ln\left(\frac{n}{n+1}\right)}-1}{\ln\left(\frac{n}{n+1}\right)}\cdot\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} =$

$= \ln a\cdot\lim\limits_{n\to\infty}\frac{\ln\left(1-\frac{1}{n+1}\right)}{-\frac{1}{n+1}}\cdot\left(-\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = -\ln a.$

Asadar

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{a^{\ln n}}{a^{\ln(n+1)}}-1\right) = -\ln a$$

Avand in vedere cele de mai sus trebuie sa comparam $-\ln a$ cu 1. Deoarece $a \in]0,1[$ rezulta ca $\ln a < 0$ si astfel $-\ln a > 0$. Avem astfel ecuatia

$$-\ln a < 1 \ <=> \ln a > -1 <=> e^{\ln a} > e^{-1} <=> a > \frac{1}{e}$$

Asadar $-\ln a < 1$ daca $a > \frac{1}{e}$ si $-\ln a > 1$ daca $a < \frac{1}{e}$. Atunci, daca $a \in \left]\frac{1}{e}, 1\right[$ seria $\sum\limits_{n=1}^{\infty} a^{\ln n}$ este divergenta, din consecinta criteriului lui Raabe Duhamel, iar daca $a \in \left]0, \frac{1}{e}\right[$ seria $\sum\limits_{n=1}^{\infty} a^{\ln n}$ este convergenta.

Ramane de studiat cazul in care $a = \frac{1}{e}$ caz in care si cu criteriul lui Raabe Duhamel nu obtinem nici o concluzie. Analizam seria asa cum e ea, deci $\sum\limits_{n=1}^{\infty} a^{\ln n} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^{\ln n} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ deci ajungem din nou la divergenta.

Asadar, seria $\sum\limits_{n=1}^{\infty} a^{\ln n}$ este:
- convergenta cand $a \in \left]0, \frac{1}{e}\right[$
- divergenta cand $a \in [\frac{1}{e}, +\infty[$.

Solutia II:

$$a^{\ln n} = e^{\ln a \ln n} = \left(e^{\ln n}\right)^{\ln a} = n^{\ln a}$$

si $\sum\limits_{n=1}^{\infty} n^{\ln a} = \sum\limits_{n=1}^{\infty} \frac{1}{n^{-\ln a}}$ care e convergenta $<=> \ -\ln a > 1 \ <=> \ln a < -1$ deci daca $0 < a < \frac{1}{e}$.

d)

$$\sum_{n=1}^{\infty} \frac{a^n}{n^n}$$

Putem folosii criteriul radicalului.

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left[\left(\frac{a}{n}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a}{n} = 0 < 1 \text{ deci seria } \sum_{n=1}^{\infty} \frac{a^n}{n^n} \text{ este divergenta.}$$

rezolvarea problemei cu ajutorul raportului, desi posibila, dupa cum se va vedea mai jos, este mai laborioasa:

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{a^{n+1}}{(n+1)^{(n+1)}} \cdot \frac{n^n}{a^n} = a \cdot \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} =$

$= a \cdot \lim\limits_{n\to\infty} \left[\left(1 - \frac{1}{n+1}\right)^{-(n+1)}\right]^{-\frac{n}{n+1}} \cdot \lim\limits_{n\to\infty} \frac{1}{n+1} = a \cdot e^{-1} \cdot 0 = 0$ deci seria $\sum\limits_{n=1}^{\infty} \frac{a^n}{n^n}$ este divergenta.

e)

$$\sum_{n=1}^{\infty} \left(\frac{n^2+n+1}{n^2} a\right)^n$$

Folosim criteriul radicalului.

$\lim\limits_{n\to\infty} \sqrt[n]{u_n} = \lim\limits_{n\to\infty} \left[\left(\frac{n^2+n+1}{n^2}a\right)^n\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left(\frac{n^2+n+1}{n^2}a\right) = a.$

Astfel:

-daca $a > 1$ seria $\sum\limits_{n=1}^{\infty} \left(\frac{n^2+n+1}{n^2}a\right)^n$ este divergenta

-daca $a < 1$ seria $\sum\limits_{n=1}^{\infty} \left(\frac{n^2+n+1}{n^2}a\right)^n$ este convergenta.

- ramane de studiat cazul in care $a = 1$.

Vom calcula $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \left(\frac{n^2+n+1}{n^2}\right)^n = \lim\limits_{n\to\infty} \left[\left(1 + \frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}}\right]^{\frac{n+1}{n^2}\cdot n} = e^1 = e$

Deoarece $\lim\limits_{n\to\infty} u_n = e \neq 0$ rezulta ca seria $\sum\limits_{n=1}^{\infty} \left(\frac{n^2+n+1}{n^2}a\right)^n$ este divergenta.

Deci:

- convergenta petru $a < 1$
- divergenta pentru $a \geq 1$.

f)

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n + a^n}$$

Vom folosi criteriul raportului.

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{3^{n+1}}{2^{n+1}+a^{n+1}} \cdot \frac{2^n+a^n}{3^n} = 3 \cdot \lim\limits_{n\to\infty} \frac{2^n+a^n}{2^{n+1}+a^{n+1}}$ si incepem studiul pe cazuri

- daca $a = 2$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 3 \cdot \lim\limits_{n\to\infty} \frac{2^{n+1}}{2^{n+2}} = \frac{3}{2} > 1$, deci seria va fi divergenta.

- daca $a < 2$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 3 \cdot \lim\limits_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \frac{1+\left(\frac{a}{2}\right)^n}{1+\left(\frac{a}{2}\right)^{n+1}} = \frac{3}{2} > 1$, deci seria este divergenta

- daca $a > 2$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 3 \cdot \lim\limits_{n\to\infty} \frac{a^n}{a^{n+1}} \cdot \frac{\left(\frac{2}{a}\right)^n+1}{\left(\frac{2}{a}\right)^{n+1}+1} = \frac{3}{a}$ si ajungem din nou la o discutie:

i) $3 = a$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ si nu ne spune nimic, dar putem sa analizam seria initiala care va deveni $\sum\limits_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$. Atunci $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{3^n}{2^n+3^n} = \lim\limits_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n+1} = 1 \neq 0$ deci seria nu poate fi convergenta, fiind astfel divergenta.

ii) $3 > a$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$ si astfel seria este divergenta

iii) $3 < a$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$ si seria este convergenta.

Concluzie:

- convergenta pentru $a \in]3, +\infty[$
- divergenta pentru $a \in]0, 3]$.

g)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^a},\ a \in R$$

In primul rand vom calcula $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^a} =$
$= \lim\limits_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{1}{n^a} = \lim\limits_{n\to\infty} \frac{1}{n^{a+\frac{1}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \lim\limits_{n\to\infty} n^{-a-\frac{1}{2}}.$

Ajungem acum la o discutie.

- daca $-a-\frac{1}{2} > 0$ atunci $\lim\limits_{n\to\infty} u_n = +\infty$ si deci seria nu poate fi convergenta, fiind astfel divergenta
- daca $-a-\frac{1}{2} < 0$ atunci $\lim\limits_{n\to\infty} u_n = 0$ si in acest caz, seria ar putea fi convergenta, dar trebuie studiata mai indeaproape. Incepem cu criteriul raportului.

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{(n+1)^a} \cdot \frac{n^a}{\sqrt{n+1}-\sqrt{n}} =$
$= \lim\limits_{n\to\infty} \frac{n+2-(n+1)}{\sqrt{n+2}+\sqrt{n+1}} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{n+1-n} \cdot \left(\frac{n}{n+1}\right)^a = 0$

P6. Stabiliti natura seriilor

a)

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}.$$

Vom folosi criteriul raportului.

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{1}{2n+3} \cdot (2n+1) \cdot \frac{(2n)!!}{(2n-1)!!} =$
$= \lim\limits_{n\to\infty} \frac{(2n+1)^2}{(2n+2)(2n+3)} = 1$, deci nu obtinem nimic, continuam cu consecinta criteriului lui Raabe-Duhamel

$\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{4n^2+10n+6-\left(4n^2+4n+1\right)}{4n^2+4n+1}\right) =$
$= \lim\limits_{n\to\infty} n \cdot \frac{6n+5}{4n^2+4n+1} = \frac{6}{4} = \frac{3}{2} > 1$, deci seria este convergenta.

b)

$$\sum_{n=1}^{\infty} \frac{1}{n!} \cdot \left(\frac{n}{e}\right)^n$$

Folosim criteriul raportului.

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{1}{(n+1)!} \cdot \left(\frac{n+1}{e}\right)^{(n+1)} \cdot (n!) \cdot \left(\frac{e}{n}\right)^n = \frac{1}{e} \cdot \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \frac{1}{e} \cdot e = 1$ deci nu putem face nici o afirmatie. Trecem astfel la criteriul lui Raabe Duhamel.

$\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{e}{\left(1+\frac{1}{n}\right)^n} - 1\right).$

Pentru a putea calcula limita definim functia

$f: R \to R$ prin $f(x) = \frac{1}{x}\left(\frac{e}{(1+x)^x} - 1\right),\ \forall x \in R.$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1}{x}\left(\frac{e}{(1+x)^x} - 1\right) \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{\left(\frac{e}{(1+x)^{\frac{1}{x}}}-1\right)'}{x'} =$
$= \lim\limits_{x\to 0} e\cdot\left((1+x)^{-x}\right)' = e\cdot\lim\limits_{x\to 0}\left[(1+x)^{-\frac{1}{x}} \cdot \left(-\frac{1}{x}\right)' \cdot \ln(1+x) + \left(-\frac{1}{x}\right) \cdot (1+x)^{-\frac{1}{z}-1} \cdot (1+x)'\right] =$
$= e \cdot \lim\limits_{x\to 0}\left[\frac{1}{x^2} \cdot \ln(1+x) \cdot \frac{1}{(1+x)^{\frac{1}{x}}} - \frac{1}{x(1+x)} \cdot \frac{1}{(1+x)^{\frac{1}{x}}}\right] =$
$= e \cdot \lim\limits_{x\to 0} \frac{1}{(1+x)^{\frac{1}{x}}} \cdot \left[\frac{\ln(1+x)}{x^2} - \frac{1}{x(x+1)}\right] = e \cdot \frac{1}{e} \cdot \lim\limits_{x\to 0}\left[\frac{\ln(1+x)}{x^2} - \frac{1}{x(x+1)}\right] =$

$= \lim\limits_{x\to 0} \frac{1}{x^2}\left[\ln(1+x) - \frac{x}{x+1}\right] \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{\frac{1}{1+x} - \frac{1+x-x}{(x+1)^2}}{2x} = \lim\limits_{x\to 0} \frac{\frac{x+1-1}{(x+1)^2}}{2x} =$
$= \lim\limits_{x\to 0} \frac{1}{2} \cdot \frac{1}{(x+1)^2} = \frac{1}{2} < 1.$

Atunci $\lim\limits_{n\to\infty} n\left(\frac{e}{\left(1+\frac{1}{n}\right)^n} - 1\right) = \frac{1}{2} < 1$, si in consecinta seria $\sum\limits_{n=1}^{\infty} \frac{1}{n!}\cdot\left(\frac{n}{e}\right)^n$ este divergenta.

P7. Pentru fiecare $a > 0$ studiati natura seriilor:

a)

$$\sum_{n=1}^{\infty} \frac{n!}{a(a+1)\cdot\ldots\cdot(a+n)}$$

Vom folosi criteriul raportului.
$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{a(a+1)\cdot\ldots\cdot(a+n)(a+n+1)} \cdot \frac{a(a+1)\cdot\ldots\cdot(a+n)}{n!} = \lim\limits_{n\to\infty} \frac{n+1}{a+n+1} = 1$ deci nu obtinem nici o informatie. Trecem la consecinta criteriului lui Raabe-Duhamel
$\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{a+n+1}{n+1} - 1\right) = \lim\limits_{n\to\infty} n \cdot \frac{a}{n+1} = a.$
Ajungem astfel la o discutie:
- daca $a > 1$ atunci seria este convergenta
- daca $a < 1$ atunci seria este divergenta
- daca $a = 1$ din criteriul mai sus mentionat nu putem trage nici o concluzie, asa ca revenim la seria initiala. Ea va fi $\sum\limits_{n=1}^{\infty} \frac{n!}{1\cdot(1+1)\cdot\ldots\cdot(1+n)} = \sum\limits_{n=1}^{\infty} \frac{n!}{(n+1)!} = \sum\limits_{n=1}^{\infty} \frac{1}{n+1}$. Despre aceasta serie se poate demonstra usor cu ajutorul raportului si a seriei $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ care este divergenta, ca si $\sum\limits_{n=1}^{\infty} \frac{1}{n+1}$ este divergenta.

Atunci:
- convergenta pentru $a \in]1, +\infty[$
- divergenta pentru $a \in]0, 1]$.

b)

$$\sum_{n=1}^{\infty} a^{-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)}$$

Folosim criteriul raportului.
$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{a^{-\left(1+\frac{1}{2}+\ldots+\frac{1}{n+1}\right)}}{a^{-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)}} = \lim\limits_{n\to\infty} a^{-\frac{1}{n+1}}$ si discutam cazurile posibile:
- daca $a = 1$ atunci $\sum\limits_{n=1}^{\infty} a^{-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)} = \sum\limits_{n=1}^{\infty} 1 = +\infty$ deci e divergenta
- daca $a \neq 1$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} a^{-\frac{1}{n+1}} = 1$ deci nu obtinem nici o informatie

In aceasta situatie apelam la consecinta criteriului lui Raabe-Duhamel
$\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{1}{a^{-\frac{1}{n+1}}} - 1\right) = \lim\limits_{n\to\infty} n \cdot \frac{1-a^{-\frac{1}{n+1}}}{a^{-\frac{1}{n+1}}} =$
Pentru a calcula aceasta limita definim functia
$-\frac{1}{n+1} = x \Rightarrow -\frac{1}{x} = n+1 \Rightarrow -\frac{1}{x} - 1 = n \Rightarrow -\frac{1+x}{x} = n$

$f : R\setminus\{0\} \to R$ prin $\forall x \in R\setminus\{0\}$, $f(x) = -\frac{1+x}{x} \cdot \frac{1-a^x}{a^x}$
$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1+x}{a^x} \cdot \frac{a^x-1}{z} = \ln a.$

Asadar $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{1}{a^{-\frac{1}{n+1}}} - 1\right) = \ln a$. Acum incepem din nou discutia;

- daca $\ln a < 1 <=> a < e$ atunci seria este divergenta
- daca $\ln a > 1 <=> a > e$ atunci seria este convergenta.
- daca $a = e$ atunci criteriul lui Raabe-Duhamel nu decide, si natura seriei trebuie studiata prin alte mijloace.

Solutia II:
$\lim\limits_{n\to\infty} n\left(\frac{1}{a^{-\frac{1}{n+1}}} - 1\right) = \lim\limits_{n\to\infty} n\left(a^{\frac{1}{n+1}} - 1\right) = \lim\limits_{n\to\infty} \frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n+1}} \cdot \frac{n}{n+1} = \ln a \cdot 1$
concluziile sunt identice.

c)

$$\sum_{n=1}^{\infty} \frac{a^n \cdot n!}{n^n}$$

Folosim criteriul raportului
$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{a^{(n+1)}\cdot(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{a^n \cdot n!} = \lim\limits_{n\to\infty} a \cdot \left(\frac{n}{n+1}\right)^n = a \cdot \frac{1}{e} = \frac{a}{e}$

- daca $a < e$ atunci seria este convergenta
- daca $a > e$ atunci seria este divergenta
- daca $a = e$ nu putem spune inca nimic.

In acest caz seria devine $\sum\limits_{n=1}^{\infty} \frac{e^n n!}{n^n}$. Vom folosi exercitiul P6b) unde am demonstrat ca seria $\sum\limits_{n=1}^{\infty} \frac{1}{n!} \cdot \left(\frac{e}{n}\right)^n$ este divergenta. Atunci $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{e^n n!}{n^n} \cdot \frac{n^n}{n! e^n} = 1 \in]0, +\infty[$ si utilizand criteriul raportului ajungem la concluzia ca cele doua serii au aceeasi natura, deci seria $\sum\limits_{n=1}^{\infty} \frac{e^n \cdot n!}{n^n}$ este divergenta.

Concluzie:
- convergenta pentru $a \in]0, e[$
- divergenta pentru $a \in [e, +\infty[$.

P8. Daca $\alpha, \beta, \gamma, x \in]0, +\infty[$ stabiliti natura seriei

$$\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)\,\beta(\beta+1)\cdot\ldots\cdot(\beta+n-1)}{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1)} x^n$$

Incepem cu criteriul raportului
$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n)\beta(\beta+1)\cdot\ldots\cdot(\beta+n)}{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n)} \cdot x^{n+1} \cdot x^{-n} \cdot \frac{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1)}{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)\beta(\beta+1)\cdot\ldots\cdot(\beta+n-1)} =$
$= \lim\limits_{n\to\infty} \frac{(\alpha+n)(\beta+n)}{(\gamma+n)} \cdot x = +\infty > 1$, deci seria este divergenta.

!!!! O imbunatatire a exercitiului

$$\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)\,\beta(\beta+1)\cdot\ldots\cdot(\beta+n-1)}{(n!)\,\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1)} x^n$$

si se numeste **seria hipergeometrica a lui Gauss**.

Aici incepem din nou cu criteriul raportului:
$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n)\beta(\beta+1)\cdot\ldots\cdot(\beta+n)}{((n+1)!)\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n)}\cdot x^{n+1}x^{-n}\cdot\frac{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1)}{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)\beta(\beta+1)\cdot\ldots\cdot(\beta+n-1)}=$
$=\lim\limits_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(\gamma+n)}\cdot\frac{1}{n+1}\cdot x=\lim\limits_{n\to\infty}\frac{x\left(n^2+(\alpha+\beta)n+\alpha\beta\right)}{n^2+(\gamma+1)n+\gamma}=x$
-daca $x<1$ atunci seria este convergenta
- daca $x>1$ atunci seria este divergenta
- daca $x=1$ atunci nu obtinem nici o cocluzie pentru ca $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=1$ si trebuie sa trecem la criteriul lui Raabe-Duhamel
$\lim\limits_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim\limits_{n\to\infty}n\left(\frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}-1\right)=$
$=\lim\limits_{n\to\infty}n\cdot\frac{\left(n^2+(\gamma+1)n+\gamma\right)-\left(n^2+(\alpha+\beta)n+\alpha\beta\right)}{n^2+(\alpha+\beta)n+\alpha\beta}=\lim\limits_{n\to\infty}n\cdot\frac{(\gamma+1-\alpha-\beta)n+\gamma-\alpha\beta}{n^2+(\alpha+\beta)n+\alpha\beta}=\gamma+1-\alpha-\beta$
atunci

- daca $\gamma+1-\alpha-\beta>1$ seria este convergenta
- daca $\gamma+1-\alpha-\beta<1$ seria este divergenta.
- daca $\gamma+1-\alpha-\beta=1$ criteriul lui Raabe Duhamel nu decide.

Deci seria este:
- convergenta cand $x<1$ sau $x=1$ si $\gamma-\alpha-\beta>0$
- divergenta cand $x>1$ sau $x=1$ si $\gamma-\alpha-\beta<0$
-daca $x=1$ si $\gamma-\alpha-\beta=0$ trebuie sa ne reintoarcem la seria initiala.

P9. Sa se stabileasca natura seriilor: (Pe parcursul problemelor de la acest exercitiu vom incerca sa aplicam teorema lui Gottfried-Wilhelm-Leibnitz)

a)

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)^{n+1}}{n^{n+2}}$$

Studiem monotonia sirului (u_n).
$\frac{u_{n+1}}{u_n}=\frac{(n+2)^{n+2}}{(n+1)^{n+3}}\cdot\frac{n^{n+2}}{(n+1)^{n+1}}=\left(\frac{n+2}{n+1}\cdot\frac{n}{n+1}\right)^{n+2}<1$ deci (u_n) este descrescator.
pentru ca $n\,(n+2)<(n+1)^2$.
$\lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{n^{n+2}}=\lim\limits_{n\to\infty}\frac{1}{n}\cdot\left(1+\frac{1}{n}\right)^{n+1}=0\cdot\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{n+1}{n}}=0\cdot e=0.$

Din cele de mai sus si criteriul anterior amintit rezulta ca seria $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)^{n+1}}{n^{n+2}}$ este convergenta.

b)

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)^{n-1}}{n^{n+1}}$$

$\frac{u_{n+1}}{u_n}=\frac{(n+2)^{n}}{(n+1)^{n+2}}\cdot\frac{n^{n+1}}{(n+1)^{n-1}}=\left(\frac{n+2}{n+1}\cdot\frac{n}{n+1}\right)^{n+1}<1$ deci sirul (u_n) este descrescator

$\lim\limits_{n\to\infty} \frac{(n+1)^{n-1}}{n^{n+1}} = \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^{n-1} \cdot \frac{1}{n^2} = 0 \cdot \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{n-1} = 0 \cdot e = 0$
deci seria $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{(n+1)^{n-1}}{n^{n+1}}$ este convergenta.

c)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{3^n}$$

$\frac{u_{n+1}}{u_n} = \frac{2n+3}{3^{n+1}} \cdot \frac{3^n}{2n+1} = \frac{1}{3} \cdot \frac{2n+3}{2n+1} < 1$ pentru $\forall n \in N$ deci sirul (u_n) este descrescator
$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{2n+1}{3^n} = 0$
Pentru a demonstra acest lucru definim functia
$f : R \to R$ prin $f(x) = \frac{2x+1}{3^x}$ pentru $\forall x \in R$.Atunci $\lim\limits_{z\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{2z+1}{3^x} \overset{\frac{\infty}{\infty}}{=}$ $\lim\limits_{x\to\infty} \frac{2}{3^x \ln 3} = 0$, deci $\lim\limits_{n\to\infty} u_n = 0$, si astfel seria $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{3^n}$ este convergenta.

Solutia II;
sau $0 < \frac{2n+1}{3^n} = \frac{2n+1}{(1+2)^n} < \frac{2n+1}{1+n\cdot 2+\frac{n(n-1)}{2}\cdot 2^2} = \frac{2n+1}{2n(n-1)+2n+1} \to 0$ deci seria e convergenta.

d)

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$$

$\frac{u_{n+1}}{u_n} = \frac{\ln n}{\ln(n+1)} < 1$ pentru $\forall n \geq 2$, deci sirul (u_n) este descrescator
$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\ln n} = 0$, deci seria $\sum\limits_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ este convergenta.

e)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$$

$\frac{u_{n+1}}{u_n} = \frac{n(n+1)}{(n+1)(n+2)} = \frac{n}{n+2} < 1 \ \forall n \geq 2$ deci sirul (u_n) este descrescator
$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{n(n+1)} = 0$, asadar seria $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ este convergenta.

f)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n(n+1)}}$$

$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n(n+1)}}{\sqrt{(n+1)(n+2)}} = \sqrt{\frac{n}{n+2}} < 1 \ \forall n \geq 2$ deci sirul (u_n) este descrescator
$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} = 0$, asadar seria $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n(n+1)}}$ este convergenta.

g)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+\sqrt{5}}$$

$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n+1}}{(n+2)+\sqrt{5}} \cdot \frac{n+\sqrt{5}}{\sqrt{n}} = \sqrt{\frac{n+1}{n}} \cdot \frac{n+\sqrt{5}}{n+2+\sqrt{5}} < 1 \ \forall n \in N$ deci sirul (u_n) este descrescator

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{n+\sqrt{5}} = \lim\limits_{n\to\infty} \frac{1}{n^{\frac{1}{2}}\left(1+\frac{\sqrt{5}}{n}\right)} = 0$, asadar seria $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+\sqrt{5}}$ este convergenta.

h)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-1)!!}{(2n)!!}$$

$\frac{u_{n+1}}{u_n} = \frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{(2n)!!}{(2n-1)!!} = \frac{2n+1}{2n+2} < 1 \ \forall n \in N$ deci sirul (u_n) este descrescator

Dorim sa calculam $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{(2n-1)!!}{(2n)!!}$ si pentru acest lucru folosim inegalitatea

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}} \ \forall n \in N$$

care se demonstreaza prin inductie matematica. Astfel $0 < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}}$ pentru $\forall n \in N$ si deoarece $\lim\limits_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$, din criteriul clestelui rezulta ca si $\lim\limits_{n\to\infty} \frac{(2n-1)!!}{(2n)!!} = 0$, asadar seria $\sum\limits_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$ este convergenta.

P10. Stabiliti natura seriilor:

In aceste exercitii vom folosi criteriul lui Abel-Dirichlet

a)

$$\sum_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \frac{n^{100}}{2^n}$$

Avem $\sum\limits_{n=1}^{\infty} (-1)^{\frac{n(n+!)}{2}}$ o serie cu termneul general de forma (se verifica usor)

$$u_n = (-1)^{\frac{n(n+1)}{2}} = \begin{cases} 1 & n = 4k \\ -1 & n = 4k+1 \\ -1 & n = 4k+2 \\ 1 & n = 4k+3 \end{cases} \ \forall n \in N$$

De aceea termenul general al sirului sumelor partiale

$$s_n = \begin{cases} -1: & n = 4k \\ -2 & n = 4k+1 \\ -1 & n = 4k+2 \\ 0 & n = 4k+3 \end{cases} \ \forall n \in N.$$

deci in mod evident sirul sumelor partiale (s_n) este marginit.

Ramane de arata ca sirul $\left(\frac{n^{100}}{2^n}\right)$ este descrescator cu limita 0.

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{100}}{2^{n+1}} \cdot \frac{2^n}{n^{100}} = \frac{1}{2} \cdot \left(1+\frac{1}{n}\right)^{100} < 1 \ \forall n \in N$ deci sirul este descrescator.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^{100}}{2^n} = 0$

b)

$$\sum_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \sin\frac{\pi}{n\sqrt{n+1}}$$

Prima parte coincide cu cea de la $a)$ in ceea ce priveste seria $\sum\limits_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}}$.

Ramane de studiat sirul $\left(\sin\frac{\pi}{n\sqrt{n+1}}\right)$.

Studiem functia $f : [1, +\infty[\to R$ definita prin $\forall x \in [1, +\infty[,\ f(x) = \frac{\pi}{x\sqrt{x+1}}$

$f'(x) = \pi \cdot \left(x^{-1} \cdot (1+x)^{-\frac{1}{2}}\right)' = \pi \cdot \left(-\frac{1}{x^2} \cdot (1+x)^{-\frac{1}{2}} + \frac{1}{x} \cdot \left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}}\right) =$

$= -\pi\left(\frac{1}{x^2\sqrt{1+x}} + \frac{1}{2x(1+x)^{\frac{3}{2}}}\right) < 0$ deci functia f este strict descrescatoare pe $[1, +\infty[$. si deoarece sinusul este o functie crescatore pe intervalul $]-\frac{\pi}{2}, \frac{\pi}{2}[$, rezulta ca functia $(\sin \circ f)(x)$ este descrescatoare pe $[1, +\infty[$.

$f(1) = \sin\frac{\pi}{\sqrt{2}}$

$f(2) = \sin\frac{\pi}{2\sqrt{3}}$

...

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{\pi}{x\sqrt{x+1}} = 0,$

deci $\forall n \in N,\ \sin\frac{\pi}{n\sqrt{n+1}} \geq \lim\limits_{x\to\infty} (\sin \circ f)(x) = 0$

Pentru a-i calcula limita ne folosim de faptul ca cu cat x este mai mare cu atat f descreste spre 0.

Atunci se cunoaste ca incepand de la un n suficient de mare

$$0 \leq \sin\frac{\pi}{n\sqrt{n+1}} < \frac{\pi}{n\sqrt{n+1}}$$

Deoarece $\lim\limits_{n\to\infty} \frac{\pi}{n\sqrt{n+1}} = 0$, din criteriul clestelui rezulta ca $\lim\limits_{n\to\infty} \sin\frac{\pi}{n\sqrt{n+1}} = 0$.

Asadar conditiile din criteriul mai sus mentionat sunt indeplinite si astfel seria $\sum\limits_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \sin\frac{\pi}{n\sqrt{n+1}}$ este convergenta.

Solutia II:

$-x \leq \sin x \leq x$ pentru $\forall x \in R_+ \Longrightarrow$

$$0 < --\frac{\pi}{n\sqrt{n+1}} \leq \sin\frac{\pi}{n\sqrt{n+1}} \leq \frac{\pi}{n\sqrt{n+1}} \to 0$$

P11. Pentru fiecare $a \in R$, stabiliti natura seriilor:

(si in aceste probleme vom folosi criteriul lui Abel Dirichlet de convergenta a seriilor)

a)

$$\sum_{n=1}^{\infty} \cos(na) \sin\frac{a}{n}$$

Distingem astfel cele doua siruri:

$u_n = \cos(na)\,,\ \forall n \in N$
$a_n = \sin\left(\frac{a}{n}\right)\,,\ \forall n \in N.$

Aum analiza sirul sumelor partiale pentru seria $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \cos(na)$

Deci

$$s_n = \cos(a) + \cos(2a) + ... + \cos(na)$$

atunci $2 \cdot \sin\frac{a}{2} \cdot s_n = 2 \cdot \sin\frac{a}{2} \cdot (\cos(a) + \cos(2a) + ... + \cos(na))$
si stiind ca $2\sin x \cdot \cos y = \sin(x-y) + \sin(x+y)$

$s_n = \frac{\left(\sin\left(\frac{a}{2}-a\right)+\sin\left(\frac{a}{2}+a\right)\right)+\left(\sin\left(\frac{a}{2}-2a\right)+\sin\left(\frac{a}{2}+2a\right)\right)+...+\left(\sin\left(\frac{a}{2}-(n-1)a\right)+\sin\left(\frac{a}{2}+(n-1)a\right)\right)+\left(\sin\left(\frac{a}{2}-na\right)+\sin\left(\frac{a}{2}+na\right)\right)}{2\sin\frac{a}{2}} =$
$= \frac{\left(-\sin\frac{a}{2}+\sin\frac{3a}{2}\right)+\left(-\sin\frac{3a}{2}+\sin\frac{5a}{2}\right)+...+\left(-\sin\frac{(2n-3)a}{2}+\sin\frac{(2n-1)a}{2}\right)+\left(-\sin\frac{(2n-1)a}{2}+\sin\frac{(2n+1)a}{2}\right)}{2\sin\frac{a}{2}} =$
$= \frac{\sin\frac{(2n+1)a}{2}-\sin\frac{a}{2}}{2\sin\frac{a}{2}} = \frac{2\sin\frac{(2n+1)a-a}{2}\cos\frac{(2n+1)a+a}{2}}{2\sin\frac{a}{2}} = \frac{\sin(na)\cos((n+1)a)}{\sin\frac{a}{2}}.$

Evident sirul sumelor partiale $s_n = \frac{\sin(na)\cos((n+1)a)}{\sin\frac{a}{2}}$ pentru $\forall n \in N$ este marginit, avand valorile cuprinse intre $\frac{1}{\sin\frac{a}{2}}$ si $-\frac{1}{\sin\frac{a}{2}}$.

Analizam acum sirul (a_n) cu $a_n = \sin\frac{a}{n}$, $\forall n \in N$.

Deoarece functia $f : [1,+\infty[\to R$ definita prin $\forall x \in [1,+\infty[\ f(x) = \frac{a}{x}$ este strict descrescatoare iar sinusul este o functie crescatoare pe $]-\frac{\pi}{2},\frac{\pi}{2}[$, va rezulta ca, incepand de la un n suficient de mare sirul $\left(\sin\frac{a}{n}\right)$ este descrecscator.Deoarece $\lim\limits_{x\to\infty}\frac{a}{x} = 0$, va rezulta din monotonie ca $\forall n \in N$ $\sin\frac{a}{n} > 0$ De asemenea, de la un n suficient de mare

$$0 < \sin\frac{a}{n} < \frac{a}{n}$$

$\lim\limits_{n\to\infty}\frac{a}{n} = 0$, si astfel din criteriul clestelui $\lim\limits_{n\to\infty}\sin\frac{a}{n} = 0$.

Atunci conditiile din teroema lui Abel-Dirichlet sunt indeplinite, si seria $\sum\limits_{n=1}^{\infty}\sin(na)\sin\frac{a}{n}$ este convergenta.

b)

$$\sum_{n=1}^{\infty}\sin(na)\sin\frac{a}{n}$$

Distingem astfel cele doua siruri:
$u_n = \sin(na)\,,\ \forall n \in N$
$a_n = \sin\left(\frac{a}{n}\right)\,,\ \forall n \in N.$

Aum analiza sirul sumelor partiale pentru seria $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \sin(na)$

Deci

$$s_n = \sin(a) + \sin(2a) + ... + \sin(na)$$

atunci $2 \cdot \sin\frac{a}{2} \cdot s_n = 2 \cdot \sin\frac{a}{2} \cdot (\cos(a) + \cos(2a) + ... + \cos(na))$
si stiind ca $2\sin x \cdot \sin y = \cos(x-y) - \cos(x+y)$

$s_n = \frac{\left(\cos\left(\frac{a}{2}-a\right)-\cos\left(\frac{a}{2}+a\right)\right)+\left(\cos\left(\frac{a}{2}-2a\right)-\cos\left(\frac{a}{2}+2a\right)\right)+...+\left(\cos\left(\frac{a}{2}-(n-1)a\right)-\cos\left(\frac{a}{2}+(n-1)a\right)\right)+\left(\cos\left(\frac{a}{2}-na\right)-\cos\left(\frac{a}{2}+na\right)\right)}{2\sin\frac{a}{2}} =$
$= \frac{\left(\cos\frac{a}{2}-\cos\frac{3a}{2}\right)+\left(\cos\frac{3a}{2}-\cos\frac{5a}{2}\right)+...+\left(\cos\frac{(2n-3)a}{2}-\cos\frac{(2n-1)a}{2}\right)+\left(\cos\frac{(2n-1)a}{2}-\cos\frac{(2n+1)a}{2}\right)}{2\sin\frac{a}{2}} =$

$= \frac{\cos\frac{a}{2} - \cos\frac{(2n+1)a}{2}}{2\sin\frac{a}{2}} = \frac{2\sin\frac{(2n+1)a-a}{2}\sin\frac{(2n+1)a+a}{2}}{2\sin\frac{a}{2}} = \frac{\sin(na)\sin((n+1)a)}{\sin\frac{a}{2}}$.

Evident sirul sumelor partiale $s_n = \frac{\sin(na)\sin((n+1)a)}{\sin\frac{a}{2}}$ pentru $\forall n \in N$ este marginit, avand valorile cuprinse intre $\frac{1}{\sin\frac{a}{2}}$ si $-\frac{1}{\sin\frac{a}{2}}$.

Analizam acum sirul (a_n) cu $a_n = \sin\frac{a}{n}$, $\forall n \in N$, ca si in cazul exemplului anterior

Deoarece functia $f : [1, +\infty[\to R$ definita prin $\forall x \in [1, +\infty[\ f(x) = \frac{a}{x}$ este strict descrescatoare iar sinusul este o functie crescatoare pe $]-\frac{\pi}{2}, \frac{\pi}{2}[$, va rezulta ca, incepand de la un n suficient de mare sirul $\left(\sin\frac{a}{n}\right)$ este descrecscator.Deoarece $\lim\limits_{x\to\infty}\frac{a}{x} = 0$, va rezulta din monotonie ca $\forall n \in N\ \sin\frac{a}{n} > 0$ De asemenea, de la un n suficient de mare

$$0 < \sin\frac{a}{n} < \frac{a}{n}$$

$\lim\limits_{n\to\infty}\frac{a}{n} = 0$, si astfel din criteriul clestelui $\lim\limits_{n\to\infty}\sin\frac{a}{n} = 0$.

Atunci conditiile din teroema lui Abel-Dirichlet sunt indeplinite, si seria $\sum\limits_{n=1}^{\infty}\cos(na)\sin\frac{a}{n}$ este convergenta.

c)

$$\sum_{n=1}^{\infty}(-1)^{\left[\frac{n}{4}\right]}\ln\frac{n+a^2}{n}$$

Definim cele doua siruri

$u_n = (-1)^{\left[\frac{n}{4}\right]}\ \forall n \in N$

$a_n = \ln\frac{n+a^2}{n},\ \forall n \in N$.

Atunci termenii sirului u_n se distribuie astfel

$(1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, ..., 1, 1, 1, 1, -1, -1, -1, -1)$,

deci

$$u_n = \begin{cases} 1 : k = par \text{ si } n \in \{4k, 4k+1, 4k+2, 4k+3\} \\ -1 : k = impar \text{ si } n \in \{4k, 4k+1, 4k+2, 4k+3\} \end{cases}$$

atunci

$$s_n = \sum_{i=1}^{n} u_i = \begin{cases} 1 : n = 4k \\ 2 : n = 4k+1 \\ 3 : n = 4k+2 \\ 4 : n = 4k+3 \\ 3 : n = 4k+4 \\ 2 : n = 4k+5 \\ 1 : n = 4k+6 \\ 0 :: n = 4k+7 \end{cases}$$

astfel se dovedeste ca sirul sumelor partiale ale seriei $\sum\limits_{n=1}^{\infty}u_n$ este marginit.

$\frac{a_{n+1}}{a_n} = \left(\ln\frac{n+1+a^2}{n+1}\right)\cdot\left(\frac{1}{\ln\left(\frac{n+a^2}{n}\right)}\right) = \frac{\ln\left(1+\frac{a^2}{n+1}\right)}{\ln\left(1+\frac{a}{n}^n\right)} < 1, \forall n \in N$, deci sirul (a_n) este descrescator.

$\lim\limits_{n\to\infty}a_n = \lim\limits_{n\to\infty}\ln\frac{n+a^2}{n} = 0$, deci conditiile din criteriul lui Abel-Dirichlet sunt satisfacute si astfel seria

$\sum\limits_{n=1}^{\infty}(-1)^{\left[\frac{n}{4}\right]}\ln\frac{n+a^2}{n}$ este convergenta.

P12. Pentru fiecare $a,b\in R,\ a>0,b>0$ **stabiliti natura seriei**

a)

$$\sum_{n=1}^{\infty}\frac{a\,(2a+1)\,(3a+1)\cdot...\cdot(na+1)}{b\,(2b+1)\,(3b+1)\cdot...\cdot(nb+1)}$$

-daca $a=b$, $\sum\limits_{n=1}^{\infty}\frac{a(2a+1)(3a+1)\cdot...\cdot(na+1)}{b(2b+1)(3b+1)\cdot...\cdot(nb+1)}=\sum\limits_{n=1}^{\infty}\frac{a(2a+1)(3a+1)\cdot...\cdot(na+1)}{a(2a+1)(3a+1)\cdot...\cdot(na+1)}=\sum\limits_{n=1}^{\infty}1$ care este o serie divergenta

- daca $a\neq b$ atunci incepem cu criteriul raportului
$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{a(2a+1)(3a+1)\cdot...\cdot(na+1+a)}{b(2b+1)(3b+1)\cdot...\cdot(nb+1+b)}\cdot\frac{b(2b+1)(3b+1)\cdot...\cdot(nb+1)}{a(2a+1)(3a+1)\cdot...\cdot(na+1)}=\lim\limits_{n\to\infty}\frac{na+1+a}{nb+1+b}=\frac{a}{b}$
asadar, utilizand criteriul raportului putem extrage concluziile
- daca $a<b$ atunci seria este convergena
- daca $a>b$ atunci seria este divergena.
Concluzii:
- convergenta daca $a<b$
- divergenta daca $a\geq b$.

b)

$$\sum_{n=1}^{\infty}\frac{a^n}{a^n+b^n}$$

-daca $a=b$ atunci seria devine $\sum\limits_{n=1}^{\infty}\frac{1}{2}=\frac{1}{2}\cdot\sum\limits_{n=1}^{\infty}1$ care este divergenta
- daca $a\neq b$ incepem cu criteriul raportului
$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{a^{n+1}}{a^{n+1}+b^{n+1}}\cdot\frac{a^n+b^b}{a^n}=a\cdot\lim\limits_{n\to\infty}\frac{a^n+b^n}{a^{n+1}+b^{n=1}}$
- daca $a<b$ atunci $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=a\cdot\lim\limits_{n\to\infty}\frac{a^n+b^n}{a^{n+1}+b^{n=1}}=a\cdot\lim\limits_{n\to\infty}\frac{b^n\left(\left(\frac{a}{b}\right)^n+1\right)}{b^{n+1}\left(\left(\frac{a}{b}\right)^{n+1}+1\right)}=\frac{a}{b}<1$ deci seria este convergenta in acest caz
- daca $a>b$ atunci $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=a\cdot\lim\limits_{n\to\infty}\frac{a^n+b^n}{a^{n+1}+b^{n=1}}=a\cdot\lim\limits_{n\to\infty}\frac{a^n\left(1+\left(\frac{b}{a}\right)^n\right)}{a^{n+1}\left(1+\left(\frac{b}{a}\right)^{n+1}\right)}=1$ si nu putem afirma nimic.
Continuamcu consecinta criteriului lui Raabe-Duhamel
$\lim\limits_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim\limits_{n\to\infty}n\left(\frac{a^{n+1}+b^{n+1}}{a^{n+1}+ab^n}-1\right)=\lim\limits_{n\to\infty}n\left(\frac{b^n(b-a)}{a^{n+1}+ab^n}\right)=\lim\limits_{n\to\infty}n\cdot\frac{b^n(b-a)}{b^na\left(\left(\frac{a}{b}\right)^n+1\right)}=$
$=\frac{b-a}{a}\lim\limits_{n\to\infty}\frac{n}{\left(\frac{a}{b}\right)^n+1}\overset{\frac{\infty}{\infty}}{\cong}\frac{b-a}{2}\lim\limits_{n\to\infty}\frac{1}{\left(\frac{a}{b}\right)^n\ln\frac{a}{b}}=\frac{b-a}{2}\cdot0=0$, limita trebuie facuta corect cu trecerea la functii.., oricum, concluzia e ca e 0, deci <1 si astfel seria este divergenata in acest caz.
Concluzii:
- convergenta daca $a<b$
- divergenta daca $a\geq b$.

c)

$$\sum_{n=1}^{\infty} \frac{a\,(a+1)\cdot ...\cdot (a+n-1)}{n!}\cdot \frac{1}{n^b},\ a>0 \text{ si } b\in R$$

Incepem cu criteriul raportului
$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{a(a+1)\cdot...\cdot(a+n-1)(a+n)}{(n+1)!} \cdot \frac{1}{(n+1)^b} \cdot (n)^b \cdot \frac{n!}{a(a+1)\cdot...\cdot(a+n-1)} =$
$= \lim\limits_{n\to\infty} \frac{a+n}{n+1} \cdot \left(\frac{n}{n+1}\right)^b = 1 \cdot \lim\limits_{n\to\infty} \left(1-\frac{1}{n+1}\right)^b = 1 \cdot \lim\limits_{n\to\infty} \left[\left(1-\frac{1}{n+1}\right)^{-(n+1)}\right]^{-\frac{b}{n+1}} =$
$= 1 \cdot e^0 = 1$ deci nu obtinem nimic concret. Trecem la consecinta criteriului lui Raabe-Duhamel.
$\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim\limits_{n\to\infty} n\left(\frac{(n+1)^{b+1}}{(a+n)n^b}-1\right) = \lim\limits_{n\to+\infty} n\left(\frac{\left(1+\frac{1}{n}\right)^{b+1}n^{b+1}}{(a+n)n^b}-1\right) =$
$= \lim\limits_{n\to+\infty} \frac{n}{a+n}\left(n\left(1+\frac{1}{n}\right)^{b+1}-a-n\right) = \lim\limits_{n\to+\infty} \frac{n}{a+n}\left(\frac{(1+\frac{1}{n})^{b+1}-1}{\frac{1}{n}}-a\right) = b+1-a$
$=$
- daca $b+1-a>1 <=> b>a$ atunci seria este convergenta
- daca $b<a$ atunci seria este divergenta,
- daca $a=b$ trebuie analizata din nou seria, in forma ei initiala.

d)

$$\sum_{n=1}^{\infty} \frac{2^n}{a^n+b^n}$$

- daca $a=b$ atunci seria devine $\sum\limits_{n=1}^{\infty} \frac{2^n}{2\cdot a^n} = \frac{1}{2}\sum\limits_{n=1}^{\infty} \left(\frac{2}{a}\right)^n$.
incercam cu criteriul radacinii. $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = \lim\limits_{n\to\infty} \frac{2}{a} = \frac{2}{a} \neq 0$ deci seria este divergenta
- daca $a\neq b$ incercam cu criteriul raportului
$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{2^{n+1}}{a^{n+1}+b^{n+1}} \cdot \frac{a^n+b^n}{2^n} = 2\cdot \lim\limits_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}}$

i) daca $a<b$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 2\cdot \lim\limits_{n\to\infty} \frac{b^n\left(\left(\frac{a}{b}\right)^n+1\right)}{b^{n+1}\left(\left(\frac{a}{b}\right)^{n+1}+1\right)} = \frac{2}{b}$, iar seria este convergenta daca $2<b$ si divergena daca $b<2$, ramane de studia $b=2$.

ii) daca $a>b$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 2\cdot \lim\limits_{n\to\infty} \frac{a^n\left(1+\left(\frac{b}{a}\right)^n\right)}{a^{n+1}\left(1+\left(\frac{b}{a}\right)^{n+1}\right)} = \frac{2}{a}$,iar seria este convergenta daca $2<a$ si divergena daca $a<2$, ramane de studiat $a=2$.

iii) daca $b=2$ si $a<b$ atunci seria devine $\sum\limits_{n=1}^{\infty} \frac{2^n}{2^n+a^n}$

iv) daca $a=2$ si $a>b$ atunci seria devine $\sum\limits_{n=1}^{\infty} \frac{2^n}{2^n+b^n}$

Studiem seria $\sum\limits_{n=1}^{\infty} \frac{2^n}{2^n+a^n}$. Atunci pentru ea $\lim\limits_{n\to\infty} \frac{2^n}{2^n+a^n} = \begin{cases} 1 \text{ daca } a<2 \\ 0 \text{ daca } a>2 \\ \frac{1}{2} \text{ daca } a=2 \end{cases}$ atunci pentru cazul $iii)$ seria

$\sum\limits_{n=1}^{\infty} \frac{2^n}{2^n+a^n}$ ar putea fi convergenta numai in cazul in care $a > 2$, caz imposibil pentru ca $2 = b > a$.

Cazul $iv)$ ar putea fi o convergenta atunci cand $b > 2$, caz imposibil pentru ca $2 = a > b$.

Concluzii:

- convergenta pentru ($a < b$ si $b > 2$) sau ($b < a$ si $a > 2$)
- divergenta pentru $a = b$ sau ($a < b \ \leq 2$) sau ($b < a \ \leq 2$) .

e)

$$\sum_{n=1}^{\infty} \frac{a^n b^n}{a^n + b^n}$$

-daca $a = b$ atunci seria devine $\sum\limits_{n=1}^{\infty} \frac{a^{2n}}{2a^n} = \frac{1}{2} \cdot \sum\limits_{n=1}^{\infty} a^n$, observam ca in cazul de fata termenii sunt in progresie geometrica, si stim ca seria este convergenta daca $a < 1$ si divergenta daca $a \geq 1$

- daca $a \neq b$ incepem cu criteriul raportului

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{a^{n+1}b^{n+1}}{a^{n+1}+b^{n+1}} \cdot \frac{a^n+b^n}{a^n b^n} = ab \cdot \lim\limits_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}}$

si obtinem o discutie asemanatoarea cu cea de la exercitiul precedent

i) daca $a < b$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = ab \cdot \lim\limits_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}} = ab \cdot \lim\limits_{n\to\infty} \frac{b^n\left(\left(\frac{a}{b}\right)^n+1\right)}{b^{n+1}\left(\left(\frac{a}{b}\right)^{n+1}+1\right)} =$

$= ab \cdot \frac{1}{b} = a$, deci daca $a < 1$ atunci seria e convergenta, iar daca $a > 1$, seria e divergenta.
Cazul $a = 1$ ramane de studiat mai tarziu

ii) daca $a > b$ atunci $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = ab \cdot \lim\limits_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}} = ab \cdot \lim\limits_{n\to\infty} \frac{a^n\left(1+\left(\frac{b}{a}\right)^n\right)}{a^{n+1}\left(1+\left(\frac{b}{a}\right)^{n+1}\right)} = ab \cdot \frac{1}{a} = b$,
asadar daca $b < 1$ seria este convergenta iar daca $b > 1$ seria este divergenta. Cazul $b = 1$ ramane de studiat mai tarziu.

iii) daca $a = 1$ si $a < b$ atunci seria devine $\sum\limits_{n=1}^{\infty} \frac{b^n}{1+b^n}$

iv) daca $b = 1$ si $a > b$ atunci seria devine $\sum\limits_{n=1}^{\infty} \frac{a^n}{1+a^n}$.

Analizam acum seria $\sum\limits_{n=1}^{\infty} \frac{a^n}{1+a^n}$.

Evaluam $\lim\limits_{n\to\infty} \frac{a^n}{1+a^n} = \begin{cases} 0 \text{ daca } a < 1 \\ 1 \text{ daca } a > 1 \\ \frac{1}{2} \text{ daca } a = 1 \end{cases}$.

Pentru $iii)$ un caz plauzibil de convergenta ar fi atunci cand $b < 1$ ceea ce este imposibil pentru ca noi lucram in cazul in care $1 = a < b$.

Pentru cazul $iv)$ un caz plauzibil de convergena ar fi atunci cand $a < 1$ ceea ce este imposibil pentru ca noi lucram in cazul in care $1 = b < a$.

Concluzii:

- convergenta daca $a = b < 1$ sau ($a < b$ si $a < 1$) sau ($a > b$ si $b < 1$)
- divergenta daca $a = b \geq 1$ sau ($1 \leq a < b$) sau ($1 \leq b < a$).