14.12.21

Seminar W11 – 832

Exercise 1. Consider the following sample data for the weight (in kg) of the people in a certain city:

$$71.6, 88.7, 92.1, 72.0, 68.2, 79.9, 73.2, 75.3, 86.4, 82.6$$

Assume that the weight is a characteristic that follows the normal distribution. Using two-tailed tests accept or reject with the $\alpha = 5\%$ significance level (risk probability) the following hypotheses:

(a) the mean value of the weight is 75 kg, given that the standard deviation of the weight is 10 kg.

(b) the mean value of the weight is 75 kg, given that the standard deviation of the weight is unknown.

(c) the standard deviation of the weight is 5 kg.

Sol.: $W \sim N(\mu, \sigma)$

$$\begin{cases} H_0: & \mu = 75 \qquad \sigma = 10 \\ H_1: & \mu \neq 75 \end{cases}$$

σ known $\Rightarrow$ $TS = Z = \dfrac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \in N(0,1)$

$\overline{X} = 72.309 \qquad \sigma = 10 \qquad n = 10$

$$Z_0 = \frac{72.309 - 75}{\frac{10}{\sqrt{10}}} = -0.85094$$

$RR = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$

$z_{\frac{\alpha}{2}} = \text{norminv}\left(\frac{\alpha}{2}\right) = -1.96$

$z_{1-\frac{\alpha}{2}} = 1.96$

$z_0 \in RR$?

$-0.85\ldots \in (-\infty, -1.96] \cup [1.96, \infty)$? No!

$\Rightarrow$ We accept the hypothesis

(b) $H_0: \mu = 75$

$H_1: \mu \neq 75$

$RR = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$

σ unknown $\Rightarrow$ $TS = Z = \dfrac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}$

$\bar{X} = 72.309$ $n = 10$

$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2} = 23.312$

$\Rightarrow z_0 = \dfrac{72.309 - 75}{\frac{23.312}{\sqrt{10}}} = -0.36502$

$z_{\frac{\alpha}{2}} = \text{tinv}\left(\frac{\alpha}{2}, 9\right) = -2.2622$

$$z_{1-\frac{\alpha}{2}} = \text{tinv}(1-\tfrac{\alpha}{2}, 9) = 2.2622$$

$$RR = (-\infty, -2.2622] \cup [2.2622, \infty)$$

$z_0 \in RR$? No!

$\Rightarrow H_0$ is correct.

Parameter θ to be tested	Known	TS (test statistic)
μ	σ	$Z = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \in \mathcal{N}(0,1)$
μ		$Z = \frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}} \in \underline{T(n-1)}$ ←
σ		$Z = \frac{(n-1)s^2}{\sigma^2} \in \chi^2(n-1)$
p		$Z = \frac{\overline{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \in \mathcal{N}(0,1)$
$p_1 - p_2$		$Z = \frac{\overline{p}_1-\overline{p}_2-(p_1-p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1}+\frac{p_2(1-p_2)}{n_2}}} \in \mathcal{N}(0,1)$

(c) $\begin{cases} H_0 : \sigma = 5 \\ H_1 : \sigma \neq 5 \end{cases}$

$$Z = \frac{(n-1)\cdot s^2}{\sigma^2} \in \chi^2(n-1)$$

$$z_0 = \frac{9\cdot(23.-37)^2}{25} = 195.61$$

$$RR = (-\infty, z_{\frac{\alpha}{2}}] \cup [z_{1-\frac{\alpha}{2}}, \infty)$$

$$z_{\frac{\alpha}{2}} = \text{chi2inv}(\tfrac{\alpha}{2}, n-1) = 2.7004$$

$$z_{1-\frac{\alpha}{2}} = \text{chi2inv}(1-\tfrac{\alpha}{2}, n-1) = 19.023$$

$\Rightarrow z_0 \in RR$?

yes!

we reject $\sigma = 5$

$\Rightarrow \sigma \neq 5$

Exercise 2. In a pre-election poll, we are interested in the proportion p_A of people who plan to vote for candidate A against candidate B.

(a) Given that 530 persons out of a random sample of 1000 persons support A, is, at 5% significance level, candidate A favourite to win the elections? Find the corresponding p-value of the test.

(b) Estimate the minimum number n_A of persons that must support A out of a random sample of 900 persons, in order to conclude that candidate A is favourite to win the elections, at 5% significance level.

(a) $p = ? = \dfrac{\text{\# people in Romania that would vote for A}}{\text{total number of voters}}$

$\overline{p} = \dfrac{530}{1000} = 0.53$ $\alpha = 0.05$

Parameter θ to be tested	*Known*	*TS (test statistic)*
μ	σ	$Z = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \in \mathcal{N}(0,1)$
μ		$Z = \frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}} \in T(n-1)$
σ		$Z = \frac{(n-1)s^2}{\sigma^2} \in \chi^2(n-1)$
p		$Z = \frac{\overline{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \in \mathcal{N}(0,1)$ ←
$p_1 - p_2$		$Z = \frac{\overline{p}_1-\overline{p}_2-(p_1-p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1}+\frac{p_2(1-p_2)}{n_2}}} \in \mathcal{N}(0,1)$

$$\Rightarrow TS = z = \frac{\overline{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

$$\begin{cases} H_0 : & p = 0.5 \\ H_1 : & p > 0.5 \end{cases}$$

(b) *For **significance testing**: Let F_Z be the cdf of the test statistic Z. Find the P-value:*

$$P = \begin{cases} F_Z(Z_0), \text{ for a left-tailed test} \\ 1 - F_Z(Z_0), \text{ for a right-tailed test} \\ 2 \cdot \min\{F_Z(Z_0), 1 - F_Z(Z_0)\}, \text{ for a two-tailed test} \end{cases}$$

If $P \leq \alpha$, then the hypothesis H_0 is rejected, otherwise it is accepted.

right tailed test $\Rightarrow$

$\Rightarrow P = 1 - F_Z(Z_0)$

$$z = \frac{\bar{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \in N(0,1)$$

$$\bar{p} = 0.53$$

$$z_0 = \frac{0.53 - 0.5}{\sqrt{\frac{0.5 \cdot 0.5}{1000}}} = \frac{0.03}{\frac{0.5}{10\sqrt{10}}} = \frac{0.3}{0.5} \cdot \sqrt{10} = \frac{3}{5} \cdot \sqrt{10} = 1.8974$$

right tailed test $\Rightarrow$

$\Rightarrow P = 1 - F_Z(Z_0) = 1 - \text{normcdf}(1.8974, 0, 1) = 0.02888\ldots$

We compare P to α. If $P \leq \alpha$, then we reject the hypothesis

$$\alpha = 0.05$$

$P \leq \alpha \Rightarrow H_0$ is rejected $\Rightarrow p > 0.5 \Rightarrow$ A wins the election

(b) Estimate the minimum number n_A of persons that must support A out of a random sample of 900 persons, in order to conclude that candidate A is favourite to win the elections, at 5% significance level.

$$z = \frac{\bar{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \in N(0,1) \qquad n = 900 \qquad \alpha = 0.05$$

$$\bar{p} = ?$$

$$\left.\begin{array}{l} H_0: \quad p = 0.5 \\ H_1: \quad p > 0.5. \end{array}\right\} \Rightarrow \text{right-tailed test}$$

We can use whichever method we like, let's go for hypothesis testing (z-test)

Right-tailed test $\Rightarrow$ $RR = [z_{1-\alpha}, \infty)$

$$z_{1-\alpha} = \text{norminv}(0.95) = 1.649$$

We reject the hypothesis $\Leftrightarrow$ $z_0 \geq 1.649$

$$z = \frac{\bar{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \qquad z_0 = \frac{\bar{p} - 0.5}{\sqrt{\frac{0.5 \cdot 0.5}{900}}} = \frac{\bar{p} - 0.5}{\frac{0.5}{30}}$$

We reject the hypothesis $\Leftrightarrow$ $z_0 \geq 1.649$ $\Leftrightarrow$ $\bar{p} - 0.5 \geq \frac{0.5}{30} \cdot 1.649$

$\Leftrightarrow$ $\bar{p} \geq 0.5 + \frac{0.5}{30} \cdot 1.649.$ $\Leftrightarrow$ $\bar{p} \geq 0.52741$

$\Rightarrow$ in our sample, we need to have at least

$$900 \cdot 0.52741 \simeq 474.67$$