23.11.2021

Seminar Wg - 832

$X_1, \dots, X_n$ i.i.d. variables, $X_i \sim X$, X characteristic

We want to find estimators for the parameters of the law followed by X.

method of moments: $\overline{\nu_k} = \frac{1}{n}(X_1^k + \dots + X_n^k)$

$$\begin{cases} \overline{\nu_1} = \nu_1 \\ \overline{\nu_2} = \nu_2 \\ \vdots \\ \overline{\nu_k} = \nu_k \end{cases} \qquad \overline{\nu_1} = \overline{X}$$

method of maximal likelihood estimators: "minimising the likelihood function"

Exercise 1. Using the method of moments, find estimators for the parameter(s) of the distribution of the characteristic X, if:

(a) $X \sim Geo(p)$, $p \in (0,1)$;

(b) $X \sim Exp(\lambda)$, $\lambda > 0$;

(c) $X \sim Bino(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$;

(d) $X \sim Unif[a,b]$, $a < b$;

(e) $X \sim NBin(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$;

(f) $X \sim Gamma(a,b)$, $a, b > 0$;

(g) $X \sim Pareto(\alpha, \beta)$, $\alpha > 2$, $\beta > 0$;

Sol: (a)

We want to estimate p, so we just need 1 equation:

$$\overline{\nu_1} = \nu_1$$

$$\overline{X} = \nu_1$$

$$\nu_1 = E(X)$$

- *If $X \sim Geo(p)$, $p \in (0,1)$, then*

$$X \begin{pmatrix} k \\ p(1-p)^k \end{pmatrix}_{k \in \mathbb{N}}$$

$$E(X) = \frac{1-p}{p}, \ V(X) = \frac{1-p}{p^2}$$

$$\Rightarrow \overline{X} = \frac{1-p}{p}$$

$$\overline{X} = \frac{1-p}{p} \Rightarrow 1+\overline{X} = \frac{1}{p} \Rightarrow p = \frac{1}{1+\overline{X}} \Rightarrow \bar{p} = \frac{1}{1+\overline{X}}$$

The methods of moments (m.o.m) estimator for p is :

$$\bar{p} = \frac{1}{1+\overline{X}} = \frac{1}{1+\frac{1}{n}(x_1+\dots+x_n)}$$

(c) $X \sim \text{Bino}(n,p)$

$$\Rightarrow \begin{cases} \overline{\nu_1} = \nu_1 \\ \overline{\nu_2} = \nu_2 \end{cases}$$

$$\nu_1 = E(X) = np \qquad \nu_2 = E(X^2) = V(X) + E(X)^2 =$$

$$= np(1-p) + (np)^2$$

$$\Rightarrow \begin{cases} \overline{\nu_1} = np \\ \overline{\nu_2} = np(1-p) + (np)^2 \end{cases}$$ We just solve this system in terms of n and p

$$\Rightarrow \begin{cases} n = \frac{\overline{\nu_1}}{p} \\ \overline{\nu_2} = \frac{\overline{\nu_1}}{p} \cdot p \cdot (1-p) + \frac{\overline{\nu_1}^2}{p^2} \cdot p^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} n = \frac{\overline{\nu_1}}{p} \\ \overline{\nu_2} = \overline{\nu_1} \cdot (1-p) + \overline{\nu_1}^2 \end{cases} \Rightarrow \begin{cases} n = \frac{\overline{\nu_1}}{p} \\ 1-p = \frac{\overline{\nu_2} - \overline{\nu_1}^2}{\overline{\nu_1}} \end{cases}$$

$$\Rightarrow \bar{p} = 1 - \frac{\overline{\nu_2} - \overline{\nu_1}^2}{\overline{\nu_1}} \Rightarrow \bar{n} = \frac{\overline{\nu_1}}{1 - \frac{\overline{\nu_2} - \overline{\nu_1}^2}{\overline{\nu_1}}}$$

(b) $X \sim \mathcal{E}xp(\lambda)$

$$\bar{\nu}_1 = \nu_1$$

$\bar{\nu}_1 = \bar{X}$ $\qquad\qquad$ $\nu_1 = E(X) = \dfrac{1}{\lambda}$

$$\Rightarrow \bar{\lambda} = \frac{1}{\bar{X}}$$

(d) $X \sim Unif\,[a,b]$ $\qquad\qquad$ nü

$$\begin{cases} \bar{\nu}_1 = \nu_1 \\ \bar{\nu}_2 = \nu_2 \end{cases}$$

$V(X) = E(X^2) - E(X)^2$

$\nu_2 = E(X^2) = V(X) + E(X)^2$

$$\nu_1 = E(X) = \frac{a+b}{2}$$

$$\nu_2 = V(X) + E(X)^2 = \frac{(b-a)^2}{12} + \frac{(a+b)^2}{4} = \frac{4b^2+4a^2-4ab}{12}$$

$$= \frac{a^2+b^2-ab}{3}$$

$$\begin{cases} \bar{\nu}_1 = \dfrac{a+b}{2} \\ \bar{\nu}_2 = \dfrac{a^2+b^2-ab}{3} \end{cases}$$

$$\begin{cases} a = 2\bar{\nu}_1 - b \\ \bar{\nu}_2 = \dfrac{4\bar{\nu}_1^2 + b^2 - 4\bar{\nu}_1 b + b^2 - 2b\bar{\nu}_1 - b^2}{3} \end{cases}$$

$$\Rightarrow \begin{cases} a = 2\bar{\nu}_1 - b \\ 3\bar{\nu}_2 = 4\bar{\nu}_1^2 - 6b\bar{\nu}_1 + b^2 \end{cases}$$

$$b_{1,2} = \frac{6\bar{\vartheta}_1 \pm \sqrt{36\bar{\vartheta}_1^2 - 16\bar{\vartheta}_1^2 + 12\bar{\vartheta}_2}}{2} = 3\bar{\vartheta}_1 \pm \frac{\sqrt{5\bar{\vartheta}_1^2 + 3\bar{\vartheta}_2}}{2}$$

$$?/ \quad \bar{b} = \underbrace{(3-\sqrt{5})}_{\simeq 0.76}\bar{\vartheta}_1 \Rightarrow \bar{a} = 2\bar{\vartheta}_1 - (3-\sqrt{5})\bar{\vartheta}_1 = \underbrace{(\sqrt{5}-1)}_{\simeq 1.78}\bar{\vartheta}_1$$

~~Because we need $a < b$, this does not work~~

$$\Rightarrow \bar{b} = (3+\sqrt{5})\bar{\vartheta}_1 \Rightarrow \bar{a} = (-\sqrt{5}-1)\bar{\vartheta}_1$$

$$\vartheta_1 = E(X) = \frac{a+b}{2}$$

$$\vartheta_2 = V(X) + E(X)^2 = \frac{(b-a)^2}{12} + \frac{(a+b)^2}{4}$$

$$\begin{cases} \bar{\vartheta}_1 = \dfrac{a+b}{2} \\ \bar{\vartheta}_2 = \dfrac{(b-a)^2}{12} + \left(\dfrac{a+b}{2}\right)^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \bar{\vartheta}_1 = \dfrac{a+b}{2} \\ \bar{\vartheta}_2 = \dfrac{(b-a)^2}{12} + \bar{\vartheta}_1^2 \end{cases}$$

$$\Rightarrow (b-a)^2 = 12\bar{\vartheta}_2 - 12\bar{\vartheta}_1^2$$

$$\Rightarrow \left(\frac{b-a}{2}\right)^2 = 3\bar{\vartheta}_2 - 3\bar{\vartheta}_1^2$$

$$\Rightarrow \begin{cases} \dfrac{b-a}{2} = \sqrt{3\bar{\vartheta}_2 - 3\bar{\vartheta}_1^2} \\ \dfrac{a+b}{2} = \bar{\vartheta}_1 \end{cases}$$

$$\Rightarrow \quad \overline{a} = \overline{\nu_1} - \sqrt{3\overline{\nu_2} - 3\overline{\nu_1}^2}$$

$$\overline{b} = \overline{\nu_1} + \sqrt{3\overline{\nu_2} - 3\overline{\nu_1}^2}$$

Exercise 2. Using the maximum likelihood method, estimate the parameters of the distribution of the characteristic X, if:

~~(a) $X \sim Unif[a,b]$, $a < b$, for the sample data: $x_1, \dots, x_n \in [a,b]$.~~

(b) $X \sim Bino(10, p)$, $p \in (0,1)$, for the sample data: $x_1, \dots, x_n \in \{0, 1, \dots, 10\}$ not all zero and not all ten.

(c) $X \sim Exp(\lambda)$, $\lambda > 0$, for the sample data: $x_1, \dots, x_n > 0$.

(d) $X \sim Geo(p)$, $p \in (0,1)$, for the sample data: $x_1, \dots, x_n \in \mathbb{N}$ not all zero.

The maximum likelihood method

Step 1: write $L(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} f_{X_i}(x)$

(a) $L(x_1, \dots, x_n; a, b) = \prod_{i=1}^{n} \frac{1}{b-a} = \frac{1}{(b-a)^n}$

- *If $X \sim Unif[a,b]$, $a, b \in \mathbb{R}$, $a < b$, then:*

$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$

$$E(X) = \frac{a+b}{2},\ V(X) = \frac{(b-a)^2}{12}$$

Step 2: In order to find the estimator, we need to impose the condition that $L(x_1, \dots, x_n; \theta)$ is maximal

→ This can either be done directly or by imposing the condition

$$\frac{\partial \ln(L(x_1, \dots, x_n; \theta_1, \dots, \theta_m))}{\partial \theta_i} = 0 \qquad i = \overline{1, m}$$

(a) $L(x_1, \dots, x_n; a, b) = \frac{1}{(b-a)^n}$

The maximum is achieved $\overline{b} = \max(x_1, \dots, x_n)$
$\overline{a} = \min(x_1, \dots, x_n)$

- *If $X \sim Exp(\lambda)$, $\lambda > 0$ then:*

$$f_X(x) = \lambda e^{-\lambda x} \cdot 1_{[0,\infty)}(x)$$

$E(X) = \frac{1}{\lambda}$, $V(X) = \frac{1}{\lambda^2}$

(c) $X \sim Exp(\lambda)$

$$L(x_1, \dots, x_n; \lambda) = \prod_{i=1}^{n} \left(\lambda e^{-\lambda x_i}\right) =$$

$$= \lambda^n \cdot e^{-\lambda(x_1 + \dots + x_n)}$$

$$\ln L = n \ln(\lambda) - \lambda(x_1 + \dots + x_n)$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - (x_1 + \dots + x_n)$$

Using the method of maximal likelihood:

$$\frac{n}{\lambda} = x_1 + \dots + x_n$$

$$\Rightarrow \lambda = \frac{n}{x_1 + \dots + x_n} \Rightarrow \overline{\lambda} = \frac{n}{x_1 + \dots + x_n} = \frac{1}{\overline{X}}$$

(d) $X \sim Geo(p)$

- *If $X \sim Geo(p)$, $p \in (0,1)$, then*

$$X \begin{pmatrix} k \\ p(1-p)^k \end{pmatrix}_{k \in \mathbb{N}}$$

$$E(X) = \frac{1-p}{p},\ V(X) = \frac{1-p}{p^2}$$

$$L(x_1, \dots, x_n; p) = \prod_{i=1}^{n} p(1-p)^{x_i} = p^n \cdot (1-p)^{x_1 + \dots + x_n}$$

$$\ln L(x_1, \dots, x_n; p) = n \ln(p) + (x_1 + \dots + x_n) \cdot \ln(1-p)$$

$$\frac{\partial \ln L}{\partial p} = \frac{n}{p} - \frac{x_1 + \dots + x_n}{1-p}$$

$$\Rightarrow \frac{n}{p} = \frac{x_1 + \dots + x_n}{1-p}$$

$$\Rightarrow \frac{1}{p} = \frac{\overline{X}}{1-p} \Rightarrow \frac{1-p}{p} = \overline{X} \Rightarrow \overline{X} + 1 = \frac{1}{p} \Rightarrow$$

$$\Rightarrow \overline{p} = \frac{1}{\overline{X}+1}$$

(b) $X \sim Bino(10, p)$

- *If $X \sim Bino(n,p)$, $n \in \mathbb{N}^*$, $p \in (0,1)$, then:*

$$X \begin{pmatrix} k \\ C_n^k p^k (1-p)^{n-k} \end{pmatrix}_{k \in \overline{0,n}}$$

$$E(X) = np,\ V(X) = np(1-p)$$

$$L(x_1, \dots, x_m; p) = \prod_{i=1}^{m} C_{10}^{x_i} \cdot p^{x_i} \cdot (1-p)^{10-x_i} =$$

$$= \underbrace{\prod_{i=1}^{m} C_{10}^{x_i}}_{=:K} \cdot p^{x_1 + \dots + x_m} \cdot (1-p)^{10m - (x_1 + \dots + x_m)}$$

$$\ln L = \ln k + (x_1 + \dots + x_n) \cdot \ln(p) + (10n - (x_1, \dots + x_n)) \cdot \ln(1-p)$$

$$\frac{\partial \ln L}{\partial p} = 0$$

$$\Rightarrow \frac{x_1 + \dots + x_n}{p} - \frac{10n - (x_1 + \dots + x_n)}{1-p} = 0$$

$$\Rightarrow \frac{\overline{x}}{p} - \frac{10 - \overline{x}}{1-p} = 0 \Rightarrow \frac{\overline{x}}{p} = \frac{10 - \overline{x}}{1-p} = \frac{10}{p+(1-p)} = 10$$

$$\Rightarrow p = \frac{1}{10} \cdot \overline{x} \Rightarrow \overline{p} = \frac{1}{10} \cdot \overline{x}$$