Seminar 3, G 831, 22 Mar 2022

Let $f_1 : S = \mathbb{R} \longrightarrow \mathbb{R}$, $f_1(x) = x^2 - 1$

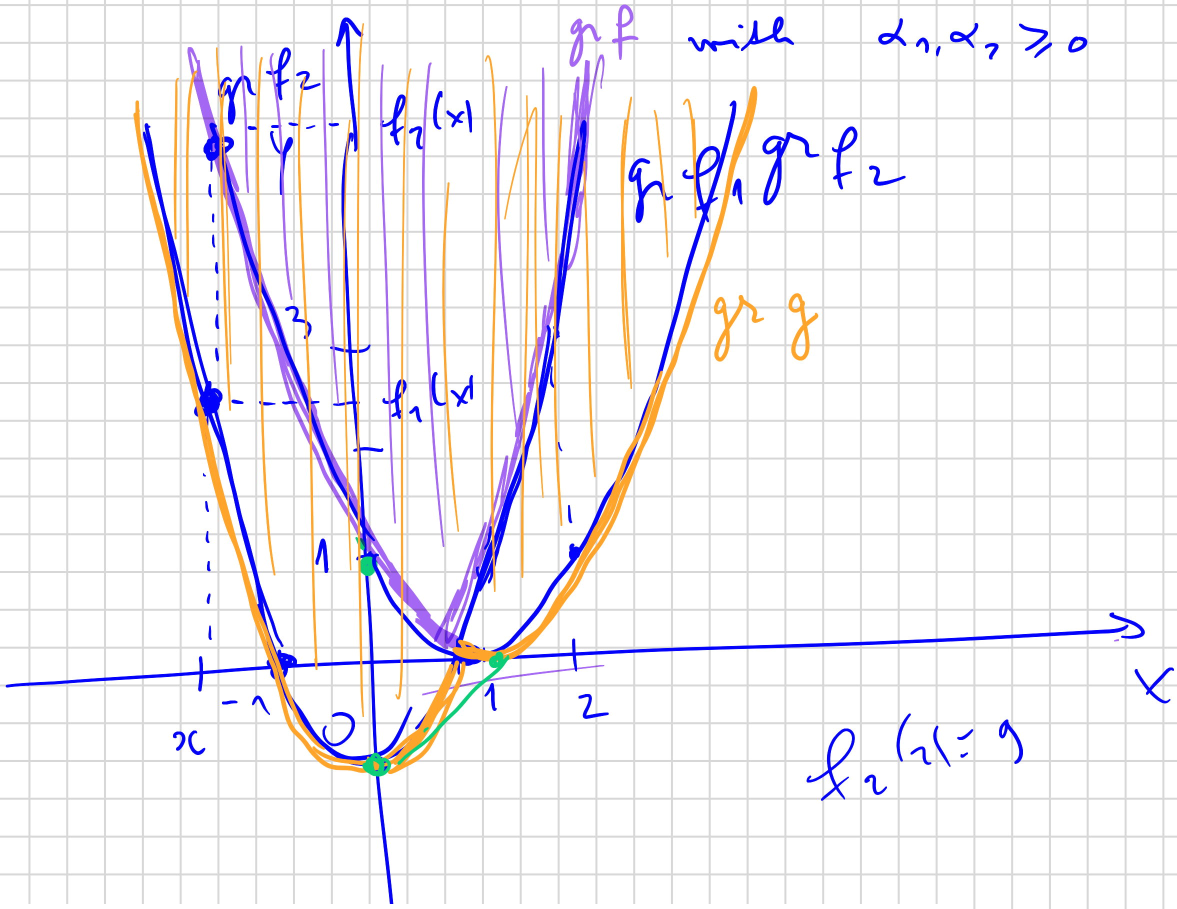
and $f_2 : S = \mathbb{R} \longrightarrow \mathbb{R}$, $f_2(x) = x^2 - 2x + 1$

Consider the functions

$\boxed{f : S = \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = \max\{f_1(x), f_2(x)\}}$

$\boxed{g : S = \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) = \min\{f_1(x), f_2(x)\}}$

$h : S = \mathbb{R} \longrightarrow \mathbb{R}, \quad h(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$ with $\alpha_1, \alpha_2 \geq 0$



Exercise 1 Let $M \subseteq \mathbb{R}^n$ be a nonempty convex set, let $f_i : M \to \mathbb{R}$, $i = \overline{1,k}$, be convex functions, and let $\alpha_i \geq 0$, $i = \overline{1,k}$ ($k \in \mathbb{N}^*$). Define the functions $f, g, h : M \to \mathbb{R}$ for every $x \in M$ by:

$$\begin{aligned} f(x) &:= \max\{f_1(x), \ldots, f_k(x)\}, \\ g(x) &:= \min\{f_1(x), \ldots, f_k(x)\}, \\ h(x) &:= \alpha_1 f_1(x) + \cdots + \alpha_k f_k(x). \end{aligned}$$

a) Show that $\operatorname{epi} f = \bigcap_{i=1}^k \operatorname{epi} f_i$.

b) Prove that f is convex.

c) Study the convexity of g and h.

Solution: A proof based on the epigraph is described in the file OT-S3.pdf.

Alternatively, we can prove the convexity of f by definition.

Let $x^1, x^2 \in M$ and $t \in [0,1]$. We have to show that

$$f\left((1-t)x^1 + t x^2\right) \overset{?}{\leq} (1-t) f(x^1) + t \cdot f(x^2) \qquad (*)$$

We know that all functions $f_1, \ldots, f_k : M \to \mathbb{R}$ are convex. This means that:

$$\begin{cases} f_1\left((1-t)x^1 + t x^2\right) \leq (1-t) f_1(x^1) + t f_1(x^2) \\ - - - - - - - - - - - - \\ f_k\left((1-t)x^1 + t x^2\right) \leq (1-t) f_k(x^1) + t f_k(x^2) \end{cases} \qquad (1)$$

Observe that, by definition of f, we have:

$$\begin{cases} f_1(x^1) \leq \max\{f_1(x^1), \ldots, f_k(x^1)\} = f(x^1) \Big| \cdot (1-t) \geq 0 \\ \vdots \qquad \vdots \\ f_k(x^1) \leq \max\{f_1(x^1), \ldots, f_k(x^1)\} = f(x^1) \Big| \cdot (1-t) \geq 0 \end{cases} \qquad (2)$$

and also

$$\begin{cases} f_1(x^2) \leq \max\{f_1(x^2), \ldots, f_k(x^2)\} = f(x^2) \Big| \cdot t \geq 0 \\ \vdots \qquad \vdots \\ f_k(x^2) \leq \max\{f_1(x^2), \ldots, f_k(x^2)\} = f(x^2) \Big| \cdot t \geq 0 \end{cases} \qquad (3)$$

Summing up, we get

$$\begin{cases} (1-t) f_1(x^1) + t f_1(x^2) \leq (1-t) f(x^1) + t f(x^2) \\ - - - - - - - - - \\ (1-t) f_k(x^1) + t f_k(x^2) \leq (1-t) f(x^1) + t f(x^2) \end{cases} \qquad (4)$$

By (1) and (4) we obtain:

$$\begin{cases} f_1\left((1-t)x^1 + t x^2\right) \leq (1-t) f(x^1) + t f(x^2) \\ - - - - - \\ f_k\left((1-t)x^1 + t x^2\right) \leq (1-t) f(x^1) + t f(x^2) \end{cases}$$

$$\Rightarrow \max\left\{ f_1\big((1-t)x^1+tx^2\big), \dots, f_k\big((1-t)x^1+tx^2\big)\right\} \le (1-t)f(x^1)+t f(x^2)$$

$$\Rightarrow f\big((1-t)x^1+tx^2\big) \le (1-t)f(x^1)+t f(x^2).$$

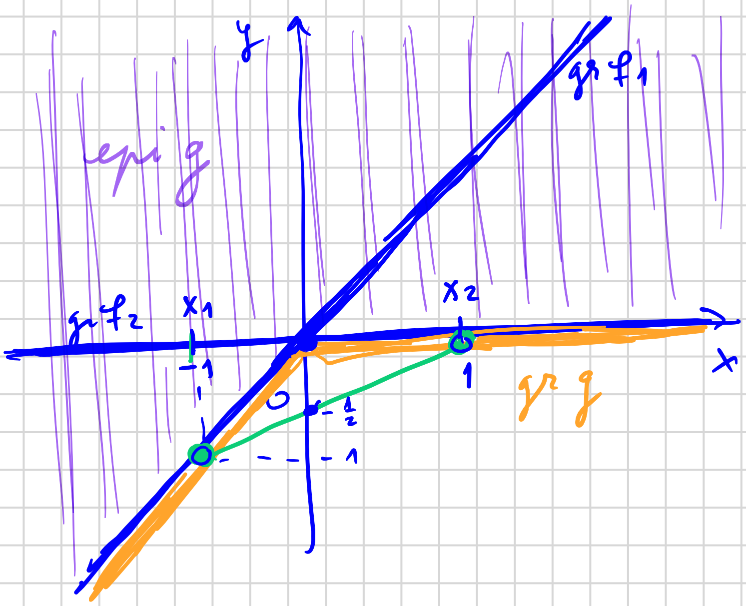
def. of f

So, f is convex.

The minimum of several convex functions is not convex, in general

$$g(x) = \min\{f_1(x), \dots, f_k(x)\}$$

For instance, let $k=2$, $f_1(x)=x$, $f_2(x)=0$, $\forall x \in S = \mathbb{R}$



For $x_1=-1$, $x_2=1$ and $t=\frac{1}{2}$ we obtain:

$$(1-t)x_1+tx_2 = \tfrac{1}{2}\cdot(-1)+\tfrac{1}{2}\cdot 1 = 0$$

$$g\big((1-t)x_1+tx_2\big) = g(0) = 0$$

$$(1-t)g(x_1)+t g(x_2) = \tfrac{1}{2}\cdot(-1)+\tfrac{1}{2}\cdot 0 = -\tfrac{1}{2}$$

So, $g\big((1-t)x_1+tx_2\big) \not\le (1-t)g(x_1)+t g(x_2)$, which shows that g is not convex.