Characterization subgroup

Theorem 2.19. (Teorema de caracterizare a subgrupului)
Let $(G, \cdot)$ be a group and $H \subseteq G$. The following statements are equivalent:
1) H is a subgroup of $(G, \cdot)$.
2)The following conditions hold for H:
α) $H \neq \emptyset$;
β) $h_1, h_2 \in H \Rightarrow h_1 h_2 \in H$;
γ) $h \in H \Rightarrow h^{-1} \in H$.
3) The following conditions hold for H:
α) $H \neq \emptyset$;
δ) $h_1, h_2 \in H \Rightarrow h_1 h_2^{-1} \in H$.

Definition 2.23. Let $(G, *)$, $(G', \perp)$ be two groups. A map $f : G \to G'$ is called **homomorphism** if

$$f(x_1 * x_2) = f(x_1) \perp f(x_2),\ \forall\ x_1, x_2 \in G.$$

A bijective homomorphism is called **isomorphism**. A homomorphism of $(G, *)$ into itself is called **endomorphism** of $(G, *)$. An isomorphism al lui $(G, *)$into itself is called **automorphism** of $(G, *)$. If there exists an isomorphism $f : G \to G'$, we say that the groups $(G, *)$ and $(G', \perp)$ are isomorphic and we denote this by $G \simeq G'$ or $(G, *) \simeq (G', \perp)$.

Theorem 2.25. Let $(G, \cdot)$ and $(G', \cdot)$ be groups, and let 1 and $1'$, respectively, be the identity element of $(G, \cdot)$ and $(G', \cdot)$, respectively. If $f : G \to G'$ is a group homomorphism, then:
(i) $f(1) = 1'$;
(ii) $[f(x)]^{-1} = f(x^{-1}), \forall x \in G$.

Definition 2.31. Let $f : G \to G'$ be a group homomorphism. Then the set

$$\mathrm{Ker} f = \{x \in G \mid f(x) = 1'\}$$

is called the **kernel** of the homomorphism f.

CHARA SUBRING

[redacted] **characterization theorem for subrings**:

Theorem 2.45. Let $(R, +, \cdot)$ be a ring and $A \subseteq R$. The following conditions are equivalent:

1) A is a subring of $(R, +, \cdot)$.

2) The following conditions hold for A:

 α) $A \neq \emptyset$;

 β) $\alpha_1, \alpha_2 \in A \Rightarrow a_1 - a_2 \in A$;

 γ) $\alpha_1, a_2 \in A \Rightarrow a_1 a_2 \in A$.

3) The following conditions hold for A:

 α) $A \neq \emptyset$;

 β') $a_1, a_2 \in A \Rightarrow a_1 + a_2 \in A$;

 β'') $a \in A \Rightarrow -a \in A$;

 γ) $a_1, a_2 \in A \Rightarrow a_1 a_2 \in A$.

CARACT SUBFIELD

Next result provide us with a **characterization theorem for subfields**.

Theorem 2.48. Let $(K, +, \cdot)$ be a field and $A \subseteq K$. The following conditions are equivalent: 1) A is a subfield of $(K, +, \cdot)$.

2) The following conditions hold for A:

 α) $|A| \geq 2$;

 β) $a_1, a_2 \in A \Rightarrow a_1 - a_2 \in A$;

 γ) $a_1, a_2 \in A$; $a_2 \neq 0 \Rightarrow a_1 a_2^{-1} \in A$;

3) The following conditions hold for A:

 α) $|A| \geq 2$;

 β') $a_1, a_2 \in A \Rightarrow a_1 + a_2 \in A$;

 β'') $a \in A \Rightarrow -a \in A$;

 γ') $a_1, a_2 \in A \Rightarrow a_1 a_2 \in A$;

 γ'') $a \in A$; $a \neq 0 \Rightarrow a^{-1} \in A$.

Definition 2.50. Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be rings and $f : R \to R'$. Then f is called a **(ring) homomorphism** if

$$f(x + y) = f(x) + f(y)\,,\ \forall x, y \in R$$

$$f(x \cdot y) = f(x) \cdot f(y)\,,\ \forall x, y \in R.$$

The notions of **(ring) isomorphism**, **endomorphism** and **automorphism** are defined as usual.

Definition 2.53. Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be rings with identity elements 1 and $1'$ respectively and let $f : R \to R'$ be a ring homomorphism. Then f is called a **unitary homomorphism** if $f(1) = 1'$.

Definition 3.1. A **vector space over** K (or a K-**vector space**) is an abelian group $(V, +)$ together with an external operation

$$\cdot : K \times V \to V\,, \quad (k, v) \mapsto k \cdot v\,,$$

satisfying the following axioms:

(L_1) $k \cdot (v_1 + v_2) = k \cdot v_1 + k \cdot v_2$;
(L_2) $(k_1 + k_2) \cdot v = k_1 \cdot v + k_2 \cdot v$;
(L_3) $(k_1 \cdot k_2) \cdot v = k_1 \cdot (k_2 \cdot v)$;
(L_4) $1 \cdot v = v$,

for any $k, k_1, k_2 \in K$ and any $v, v_1, v_2 \in V$.

Hence we have the following **characterization theorem for subspaces**.

Theorem 3.8. Let V be a vector space over K and let $S \subseteq V$. The following conditions are equivalent:
1) $S \leq_K V$.
2) The following conditions hold for S:
α) $S \neq \emptyset$;
β) $\forall x, y \in S\,,\ x + y \in S$;
γ) $\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$
3) The following conditions hold for S:
α) $S \neq \emptyset$;
δ) $\forall k_1, k_2 \in K\,,\ \forall x, y \in S\,,\ k_1 x + k_2 y \in S\,.$

Definition 3.25. Let V and V' be vector spaces over K. The map $f : V \to V'$ is called a **(vector space) homomorphism** or a **linear map** (or a **linear transformation**) if

$$f(x + y) = f(x) + f(y)\,,\ \forall x, y \in V\,,$$

$$f(kx) = kf(x),\ \ \forall k \in K,\ \forall x \in V\,.$$

The notions of **(vector space) isomorphism**, **endomorphism** and **automorphism** are defined as usual.

Theorem 3.27. Let V and V' be vector spaces over K and $f : V \to V'$. Then f is a linear map if and only if

$$f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)\,,\ \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in V.$$