Homework KOHAN ALEXANDRU

33. A Damped Wave Equation: two types of damping. Consider the mixed problem for the damped wave equation with two types of damping

$$\begin{cases} u_{tt} + u_t - u_{txx} = u_{xx}, & \text{in } (0,\infty)\times\Omega \\ u(t,0) = u(t,1) = 0, & \\ u_t(t,0) = u_t(t,1) = 0 & \\ u(0,x) = u_0(x) & \\ u_t(0,x) = v_0(x) & \end{cases}$$

Give a representation formula for the solution.

Damped / Dissipative Systems

$a > 0$

$\tilde{E}$ strict Lyap

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}\|v(t)\|^2}_{\text{Kin Energy}} + \underbrace{E(u(t))}_{\text{Potential Energy}} \Big) = \underbrace{-a\|v(t)\|^2}_{\text{dissipated energy}}$$

$$(D)\begin{cases} \dfrac{d}{dt} u = v \\ \dfrac{d}{dt} v = -a v - \nabla E(u) \end{cases}$$

Damped Wave Eq

$$u_{tt} = -u_t + \Delta u$$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & i \\ \Delta & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{cases} u_t - u_{xx} = u_{txx} - u_{tt} \\ u(t,0) = u(t,1) = 0 \\ u_t(t,0) = u_t(t,1) = 0 \\ u(0,x) = u_0(x) \\ u_t(0,x) = v_0(x) \end{cases}$$

$$g := u_t - u_{xx} = u_{txx} - u_{tt}$$

Denote $g(t,x) := u_t - u_{xx} = u_{txx} - u_{tt}$

Then, observe that:

$g_t + g = 0 \Rightarrow g_t = u_{tt} - (u_t)_{xx}$

$g' + g = 0 \qquad r + 1 = 0 \Rightarrow r = -1$

Therefore $\dfrac{\partial g}{\partial t} + g = 0 \Rightarrow$

$\Rightarrow g(t,x) = e^{-t_0} \cdot c\, u_0(x)$

Therefore, we should solve

$u_t - u_{xx} = e^{-t} \cdot c$

$g(0,x) = u_t(0,x) - u_{xx}(0,x)$

$c = v_0(x) - u_{xx}(0,x)$

$g(t,x) = e^{-t} \cdot c$

$c = g(0,x)$

Notice that $\begin{cases} u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \quad (*) \\ u_t - u_{xx} = e^{-t} \cdot c \end{cases}$

$$u_t - u_{xx} = g = e^{-t} \cdot g(0, x)$$

$$\Omega = (0, 1)$$

$$\begin{cases} -u_{xx} = \lambda u, & \text{in } \Omega = (0,1) \\ u(0) = u(1) \end{cases}$$

$$\lambda_n = n^2 \pi^2, \quad n = 1, 2, \dots$$

$$e_n(x) = \sqrt{2} \sin(n \pi x)$$

$$\langle e_i, e_j \rangle_{L^2(\Omega)} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

$$u_n'(t) = -(n\pi)^2 u_n(t) + f_n(t) \quad \text{(ODE)}$$

From exe 31:

$$u_k'(t) = -(k\pi)^2 u_k(t) + e^{-t} \cdot g(0)_k$$

$$g(0)_k = \int g(0, x) \cdot \sqrt{2} \cdot \sin(k\pi x)\, dx$$

$$\begin{cases} u' = -a \cdot u + f(t) \\ u(0) = u_0 \end{cases} \Rightarrow$$

Problem 1, Seminary 1

$$\Rightarrow u(t) = e^{-at} u_0 + \int_0^t e^{-(t-\tau)} f(\tau)\, d\tau$$

$$u_k(t) = e^{(k\pi)^2 t} u_0 + \int_0^t e^{-(t-\tau)} \cdot e^{-t} \cdot g(0)_k \, d\tau$$