16.11.2021

Seminar W8 - 831

Exercise 1. Let $f(x;\theta) = e^{a(x)\alpha(\theta)+b(x)+\beta(\theta)}$, for x in the range of X, where θ is a parameter of X and a, α, b, β are measurable functions, be a probability density function of the (discrete or continuous) characteristic X. Prove that the statistic

$$S = S(X_1, \dots, X_n) = \sum_{i=1}^{n} a(X_i)$$

is sufficient for θ.

Theorem (Fisher's Factorization Criterion). *A statistic*

$$S = S(X_1, X_2, \dots, X_n)$$

is sufficient for θ, if and only if the likelihood funnction

$$L(X_1, X_2, \dots, X_n; \theta) = \prod_{i=1}^{n} f(X_i;\theta)$$

can be factored into two nonnegative functions

$$L(x_1, x_2, \dots, x_n; \theta) = \underline{g(x_1, x_2, \dots, x_n)} \cdot \underline{h(s;\theta)},$$

where $s = S(x_1, x_2, \dots, x_n)$.

Sol: $L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} e^{a(x_i)\alpha(\theta)+b(x_i)+\beta(\theta)} =$

$$= \prod_{i=1}^{n} e^{a(x_i)\alpha(\theta)} \cdot \prod_{i=1}^{n} e^{b(x_i)} \cdot \prod_{i=1}^{n} e^{\beta(\theta)} =$$

$$= \underbrace{e^{\sum_{i=1}^{n} a(x_i)\cdot\alpha(\theta)}}_{= e^{\alpha(\theta)\cdot\left(\sum_{i=1}^{n} a(x_i)\right)}} \cdot e^{\sum_{i=1}^{n} b(x_i)} \cdot \underbrace{e^{\sum_{i=1}^{n} \beta(\theta)}}_{= e^{n\beta(\theta)}} =$$

$$= e^{\alpha(\theta)\cdot s} \cdot e^{\sum_{i=1}^{n} b(x_i)} \cdot e^{n\beta(\theta)} =$$

$$= \underbrace{e^{\alpha(\theta)\cdot s + n\beta(\theta)}}_{h(s;\theta)} \cdot \underbrace{e^{\sum_{i=1}^{n} b(x_i)}}_{g(x_1, x_2, \dots, x_n)}$$

$$f_X(x) = 1_{[0,\theta]} \cdot \frac{1}{\theta}$$

Exercise 3. Let $X \sim Unif[0,\theta]$, where $\theta > 0$ is a parameter.

(a) Prove that

$$S = \max\{X_1, \ldots, X_n\}$$

is a sufficient and complete statistic for θ.

(b) Show that

$$\overline{\theta} = \frac{n+1}{n}\max(X_1 \ldots, X_n)$$

is an unbiased estimator for θ.

(c) Find the MVUE of θ.

Hint: If $X \sim Unif[0,\theta]$, then the pdf of $S = \max(X_1, X_2 \ldots, X_n)$ is:

$$f_S(x) = 1_{[0,\theta]} \cdot \frac{nx^{n-1}}{\theta^n} = \begin{cases} \frac{nx^{n-1}}{\theta^n}, & x \in [0,\theta] \\ 0, & \text{otherwise} \end{cases}$$

Sol: (a) $L(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta) =$

$$= \prod_{i=1}^{n}\left(1_{[0,\theta]}(x_i) \cdot \frac{1}{\theta}\right) = \frac{1}{\theta^n} \cdot \underbrace{\prod_{i=1}^{n} 1_{[0,\theta]}(x_i)}_{= 1_{x_1, x_2, \ldots, x_n \in [0,\theta]}} =$$

$$= \frac{1}{\theta^n} \cdot 1_{\substack{\min(x_1, \ldots, x_n) \geq 0 \\ \max(x_1, \ldots, x_n) \leq \theta}} = \frac{1}{\theta^n} \cdot 1_{\min(x_1, \ldots, x_n) \geq 0} \cdot 1_{\max(x_1, \ldots, x_n) \leq \theta} =$$

$$= \underbrace{\frac{1}{\theta^n} \cdot 1_{S \leq \theta}}_{= h(S,\theta)} \cdot \underbrace{1_{\min(x_1, \ldots, x_n) \geq 0}}_{= g(x_1, \ldots, x_n)}$$

Fisher's factorisation criterion $\Rightarrow$ S is a sufficient statistic

We will now show that S is a complete statistic.

Let φ be a (measurable) function. We have to show that

if $\forall \theta \in \mathbb{R}_+ : \quad E(\varphi(S)) = 0$, then $P(\varphi(S) = 0) = 1$

We know that $f_S(x) = 1_{[0,\theta]} \cdot \dfrac{n x^{n-1}}{\theta^n}$

$$E(\varphi(S)) = \int_{\mathbb{R}} \varphi(x) f_S(x)\, dx = \int_0^\theta \varphi(x) \cdot \frac{n x^{n-1}}{\theta^n}\, dx =$$

$$= \frac{n}{\theta^n} \int_0^\theta \varphi(x) \cdot x^{n-1}\, dx$$

We know that $\dfrac{n}{\theta^n} \displaystyle\int_0^\theta \varphi(x) \cdot x^{n-1}\, dx = 0,\ \forall \theta \in \mathbb{R}_+ \Rightarrow$

$$\Rightarrow \int_0^\theta \varphi(x) \cdot x^{n-1}\, dx = 0, \quad \forall \theta \in \mathbb{R}_+$$

$$F(\theta) := \int_0^\theta \varphi(x)\, x^{n-1}\, dx \Rightarrow \forall x \in \mathbb{R} : \quad \varphi(x) \cdot x^{n-1} = 0$$

$$\Rightarrow P(\varphi(S) = 0) = 1$$

- ***complete*** *for the family of probability distributions $(f(x;\theta))_{\theta \in A}$ if for every measurable function ϕ we have the implication:*

$$\underbrace{E(\phi(S)) = 0, \forall \theta \in A}_{\text{assume this is true}} \Rightarrow \underbrace{P(\phi(S) = 0) = 1, \forall \theta \in A}_{\text{prove this}}$$

(b) We will prove that $\bar{\theta} = \frac{n+1}{n} \underbrace{\max(X_1, \dots, X_n)}^{S}$ is an unbiased estimator, that is, we will show that $E(\bar{\theta}) = \theta$

$$E(\bar{\theta}) = E\left(\frac{n+1}{n} \cdot S\right) = \frac{n+1}{n} \cdot E(S)$$

We know that $f_S(x) = 1_{[0,\theta]}(x) \cdot \frac{nx^{n-1}}{\theta^n}$

$$E(S) = \int_{\mathbb{R}} x \cdot f_S(x)\, dx = \int_0^\theta x \cdot \frac{nx^{n-1}}{\theta^n}\, dx =$$

$$= \frac{n}{\theta^n} \int_0^\theta x^n\, dx = \frac{n}{\theta^n} \cdot \frac{1}{n+1} x^{n+1} \Big|_0^\theta = \frac{n}{n+1} \cdot \frac{1}{\theta^n} \cdot \theta^{n+1} =$$

$$= \frac{n}{n+1} \cdot \theta$$

$$\Rightarrow E(\bar{\theta}) = \frac{n+1}{n} \cdot E(S) = \frac{n+1}{n} \cdot \frac{n}{n+1} \theta = \theta$$

(c) $S = \max(X_1, \dots, X_n)$ sufficient and complete

$\bar{\theta}$ is an unbiased estimator of θ

Lehmann-Scheffé $\Rightarrow$ $\tilde{\theta} = E(\bar{\theta} \mid S)$ is an MVUE

$$\bar{\theta} = \frac{n+1}{n} \cdot S \Rightarrow \tilde{\theta} = E\left(\bar{\theta} \,\Big|\, \frac{n}{n+1} \bar{\theta}\right) = \bar{\theta}$$

$\Rightarrow \bar{\theta}$ is an MVUE