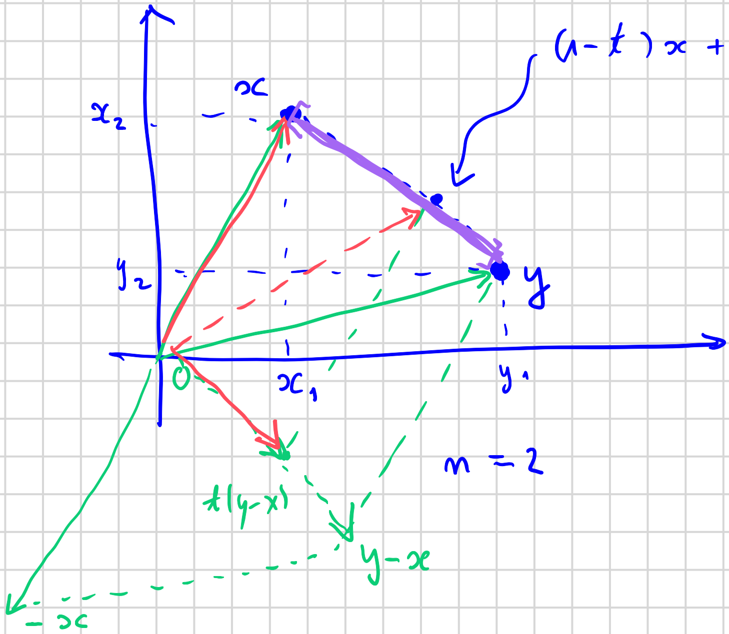
Seminar 2, G831

CONVEX SETS

Def. A set $S \subseteq \mathbb{R}^n$ is said to be convex if

$$\forall\, x, y \in S,\ \forall\, t \in [0,1],\ (1-t)\cdot x + t\cdot y \in S.$$

$$(1-t)x + ty = x + t\cdot(y-x).$$



Given $x, y \in \mathbb{R}^n$ it will be convenient to denote

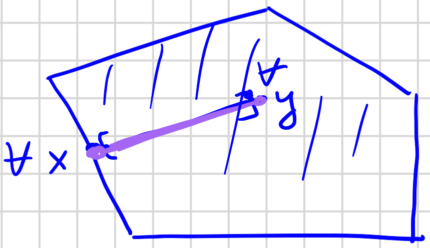
$$\boxed{[x,y] := \{(1-t)\cdot x + t\cdot y \mid t \in [0,1]\}}$$

the line-segment joining x and y.

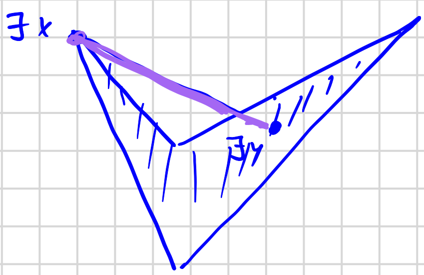
Notice that $[x,y]$ can degenerate into a singleton when $x = y$.

Thus, a set $S \subseteq \mathbb{R}^n$ is convex if and only if

$$\forall\, x, y \in S,\ [x,y] \subseteq S.$$

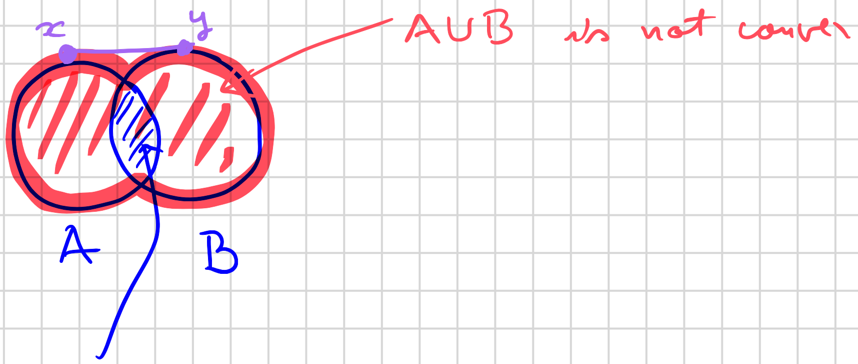


S is convex



S is not convex

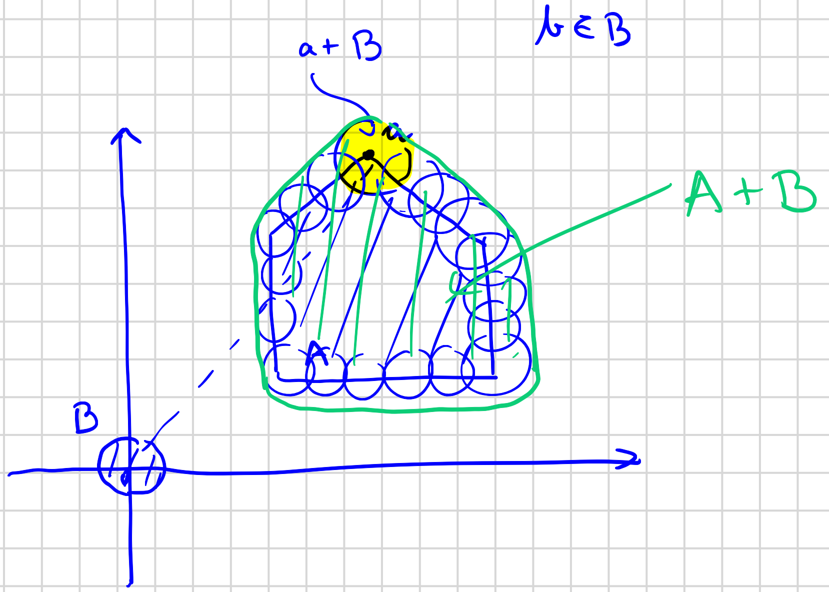
Remark: It is easy to see that the intersection of a family of convex sets in $\mathbb{R}^n$ is always convex (see L3). However, the union of convex sets is not necessarily convex. For instance,



In Ex 3 in OT-S2 we use the following notations.

For $A, B \subseteq \mathbb{R}^n$, the Minkowski sum of A and B is:

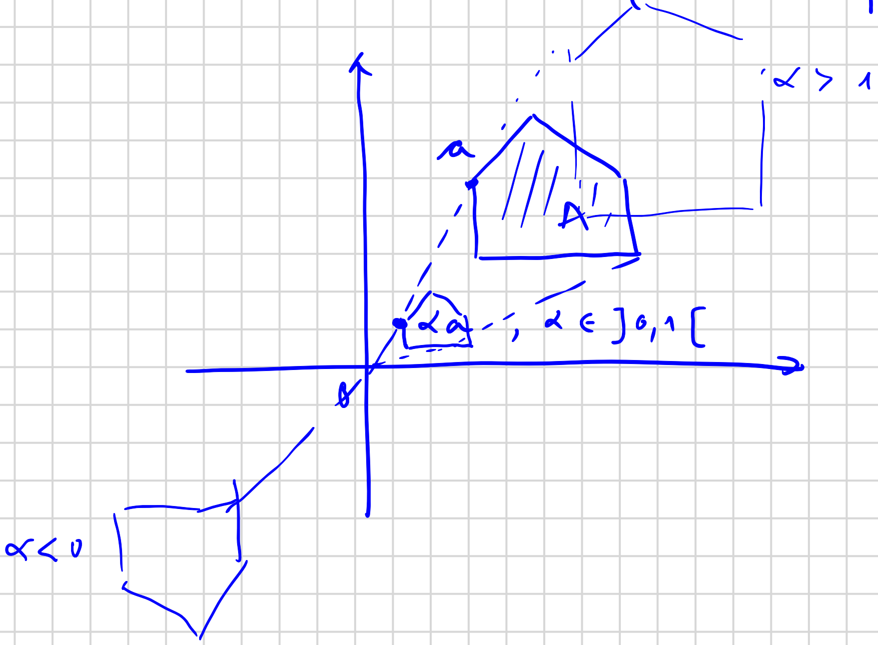
$$A+B := \{a+b \mid a \in A, b \in B\}$$
$$= \{x \in \mathbb{R}^n \mid \exists a \in A, \exists b \in B \text{ s.t. } x = a+b\}$$
$$= \bigcup_{a \in A} (a+B)$$
$$= \bigcup_{b \in B} (A+b)$$



$n=2$

Given $\alpha \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$ we can define

$$\alpha \cdot A := \{\alpha \cdot a \mid a \in A\}$$
$$= \{x \in \mathbb{R}^n \mid \exists a \in A \text{ s.t. } x = \alpha \cdot a\}$$



Remark:

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$\alpha, \beta \in \mathbb{R}$

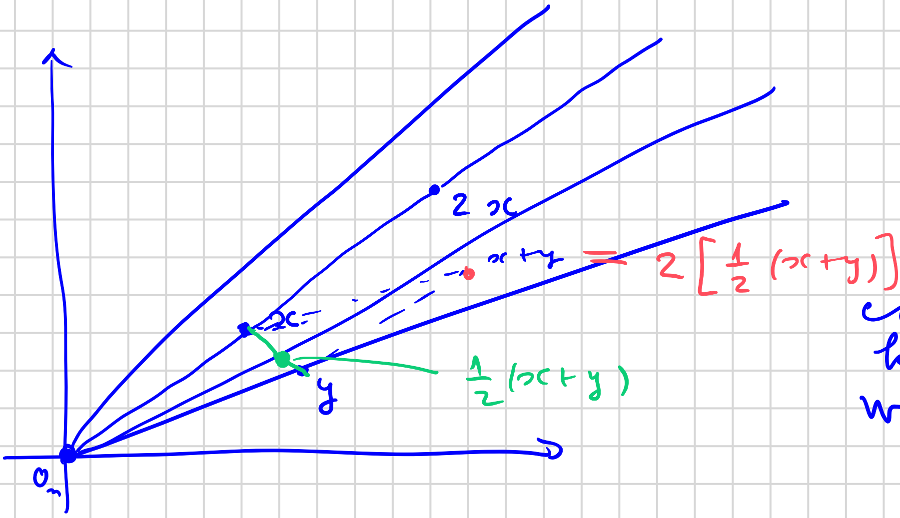
$$(\alpha + \beta) \cdot x = ((\alpha+\beta)\cdot x_1, \dots, (\alpha+\beta)\cdot x_n)$$

$$\alpha \cdot x + \beta \cdot x = (\alpha x_1, \dots, \alpha x_n) + (\beta x_1, \dots, \beta x_n)$$
$$= (\alpha x_1 + \beta x_1, \dots, \alpha x_n + \beta x_n)$$

So,
$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$

In Ex 5: a set $C \subseteq \mathbb{R}^n$ is a cone if

$$\begin{cases} C \neq \emptyset \\ \mathbb{R}_+ \cdot C \subseteq C \end{cases} \iff \begin{cases} 0_n \in C \\ \alpha \cdot x \in C, \ \forall \alpha \in \mathbb{R}, \alpha \geq 0, \forall x \in C \end{cases}$$



$n=2$

cones are also known as wedges

If we denote

$$\mathrm{ray}(0_n, x) := \{\alpha \cdot x \mid \alpha \in \mathbb{R}, \alpha \geq 0\},$$

then C is a cone if and only if

$$\forall x \in C, \quad \mathrm{ray}(0_n, x) \subseteq C$$

Exercise (Exam type):

Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Prove that the set

$$H := \{x \in \mathbb{R}^n \mid \langle c, x \rangle = \lambda\}$$

is convex, where

$$\langle c, x \rangle = c_1 x_1 + \dots + c_n x_n$$

stands for the inner product.

Solution:

We have to show that

$$\forall x, y \in H, \ \forall t \in [0,1], \quad (1-t)x + ty \overset{?}{\in} H$$

Indeed, let $x = (x_1, \dots, x_n) \in H$ and $y = (y_1, \dots, y_n) \in H$

Then, $\begin{cases} \langle n, x \rangle = \lambda & \big| \cdot (1-t) \\ \langle n, y \rangle = \lambda & \big| \cdot t \end{cases}$

$\Rightarrow \quad (1-t) \cdot \langle n, x \rangle + t \cdot \langle n, y \rangle = (1-t)\lambda + t\lambda$

$\Rightarrow \quad \langle n, (1-t)x + ty \rangle = \lambda$

$\Rightarrow \quad (1-t)x + ty \in H$ ✓