5.7 Type II Errors, Power of a Test and the Neyman-Pearson Lemma

Recall that for a target parameter θ, we are testing

$$H_0: \quad \theta = \theta_0, \text{ versus one of}$$
$$H_1: \quad \begin{cases} \theta < \theta_0 \\ \theta > \theta_0 \\ \theta \neq \theta_0, \end{cases} \tag{5.1}$$

The "goodness" of a test is measured by the two probabilities of risk

$$\begin{aligned} \alpha &= P(\text{type I error}) &= P(\text{reject } H_0 \mid H_0) \\ \beta &= P(\text{type II error}) &= P(\text{note reject } H_0 \mid H_1). \end{aligned}$$

The smaller both of them are, the more reliable the test is. For some problems, a type I error is more dangerous, while for others, a significant type II error is unacceptable. In general, α is preset, at most 0.05 and the test is designed so that β is also small enough to be acceptable.

Type II Errors and Power of a Test

So far, type II errors were not discussed much. As we have seen in a few examples, the computation of β can be more difficult. The condition that H_1 is true *does not* specify an actual value for the unknown parameter and thus, does not identify a distribution, for which the probability can be computed. The simple condition that a parameter θ is less than, greater than or not equal to a value is not enough to help us compute the probability. However, if the alternate H_1 is also a *simple* hypothesis

$$H_1: \theta = \theta_1,$$

then β can be computed. Thus, β, unlike α, depends on the value specified in the alternative hypothesis,

$$\beta = \beta(\theta_1).$$

Example 5.1. Let us consider again the problem in Example 5.1. in Lecture 11 (or Example 5.2 in Lecture 10): The number of monthly sales at a firm is known to have a mean of 20 and a standard deviation of 4 and all salary, tax and bonus figures are based on these values. However, in times of economical recession, a sales manager fears that his employees do not average 20 sales per month, but less, which could seriously hurt the company. For a number of 36 randomly selected salespeople, it was found that in one month they averaged 19 sales. At the 5% significance level, does the data confirm or contradict the manager's suspicion?

Now let us find β for the test

$$\begin{aligned} H_0: &\quad \mu = \mu_0 = 20 \\ H_1: &\quad \mu = \mu_1 = 18 < 20, \end{aligned}$$

i.e. find $\beta(\mu_1)$.

Solution. We tested a left-tailed alternative for the mean

$$\begin{aligned} H_0: &\quad \mu = 20 \\ H_1: &\quad \mu < 20. \end{aligned}$$

The population standard deviation was given, $\sigma = 4$ and for a sample of size $n = 36$, the sample mean was $\overline{X} = 19$. For the test statistic

$$TS \;=\; Z \;=\; \frac{\overline{X} - \mu}{\dfrac{\sigma}{\sqrt{n}}} \;\in\; N(0,1),$$

the observed value was

$$Z_0 = \frac{\overline{X} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} \;=\; \frac{19 - 20}{\dfrac{4}{6}} \;=\; -1.5.$$

At the significance level $\alpha = 0.05$, we have determined the rejection region

$$\begin{aligned} RR \;&=\; \left\{ Z_0 = \frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \le z_{0.05} \right\} \;=\; \left\{ \frac{\overline{X} - 20}{\frac{4}{6}} \le -1.645 \right\} \\ &=\; \left\{ \overline{X} \le -1.645 \cdot \frac{4}{6} + 20 \right\} \;=\; \left\{ \overline{X} \le 18.9 \right\}. \end{aligned}$$

Then, in a similar fashion we compute

$$\beta(\mu_1) \;=\; P(\text{not reject } H_0 \mid H_1) \;=\; P\left(\overline{X} > 18.9 \mid \mu = \mu_1\right).$$

If the true value of μ is μ_1, then the statistic

$$Z_1 = \frac{\overline{X} - \mu_1}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{X} - 18}{\frac{4}{6}}$$

has a Standard Normal $N(0,1)$ distribution. Hence,

$$\begin{aligned} \beta(\mu_1) &= P\left(\overline{X} > 18.9 \mid \mu = \mu_1\right) \\ &= P\left(\frac{\overline{X} - 18}{\frac{4}{6}} > \frac{18.9 - 18}{\frac{4}{6}} \mid \mu = 18\right) \\ &= P(Z_1 > 1.35 \mid Z_1 \in N(0,1)) \\ &= 1 - P(Z_1 \leq 1.35 \mid Z_1 \in N(0,1)) \\ &= 1 - normcdf(1.35) \;=\; 0.0885. \end{aligned}$$

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Remark 5.2. Let us take a closer look at the computation of α and β in the previous example. We used the fact that the variable

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

has a $N(0,1)$ distribution. So, when the true value of μ is $\mu_0 = 20$, then

$$Z_0 = Z(\mu = \mu_0) \in N(0,1)$$

and when the value is $\mu_1 = 18$, then

$$Z_1 = Z(\mu = \mu_1) \in N(0,1).$$

However, in the end, we expressed the error probabilities α and β, by looking at the distribution of $\overline{X}$ *by itself*, not its reduced version. In other words, we used the fact that, when the true value of μ is $\mu_0 = 20$, then

$$\overline{X} \in N(\mu_0, \sigma/\sqrt{n}) \text{ and } \alpha = P(\overline{X} \leq 18.9),$$

while when the true value is $\mu_1 = 18$, then

$$\overline{X} \in N(\mu_1, \sigma/\sqrt{n}) \text{ and } \beta = P(\overline{X} > 18.9).$$

This can be seen graphically in Figure 1.

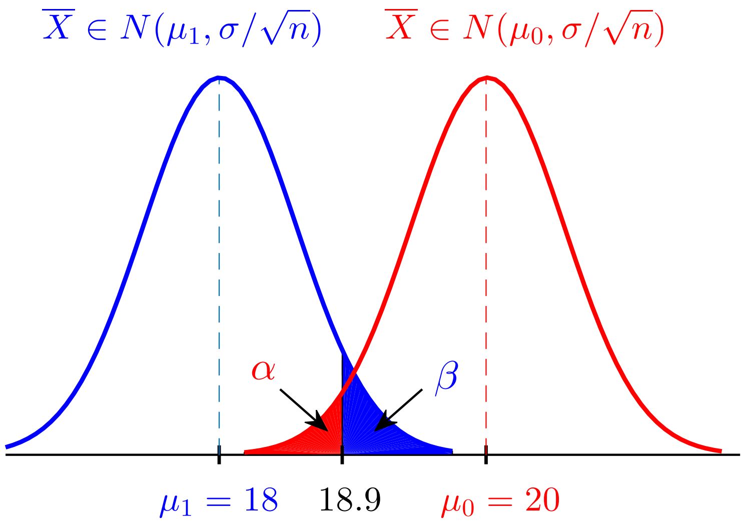

Fig. 1: Type I and type II errors

In order to have a better control over β, we introduce the following notion.

Definition 5.3. *The **power of a test** on a parameter θ, unknown, is the probability of rejecting the null hypothesis*

$$\pi(\theta^*) \quad = \quad P(\text{reject } H_0 \mid \theta = \theta^*) \quad = \quad P(TS \in RR \mid \theta = \theta^*), \tag{5.2}$$

when the true value of the parameter is $\theta = \theta^*$.

Notice that the power of a test is, usually, a function of the parameter θ, because the alternative hypothesis includes a set of parameter values.
Indeed, if the null hypothesis is true, i.e. $\theta = \theta_0$, then

$$\pi(\theta_0) \quad = \quad P(TS \in RR \mid \theta = \theta_0) \quad = \quad P(\text{reject } H_0 \mid H_0) \quad = \quad \alpha. \tag{5.3}$$

For any *other* true value (in the alternative hypothesis H_1) $\theta = \theta_1 \neq \theta_0$,

$$\begin{aligned} \pi(\theta_1) \quad &= \quad P(\text{reject } H_0 \mid \theta = \theta_1) \quad = \quad P(\text{reject } H_0 \mid H_1) \\ &= \quad 1 - P(\text{not reject } H_0 \mid H_1) \quad = \quad 1 - \beta(\theta_1). \end{aligned} \tag{5.4}$$

So, basically, the power of a test is the probability of rejecting a *false* null hypothesis. The larger the

power is, the smaller β is, which is what we want in a test. Then we can state a hypothesis testing problem the following way:
For a parametric test where both hypotheses are simple

$$\begin{aligned} H_0 : &\quad \theta = \theta_0 \\ H_1 : &\quad \theta = \theta_1, \end{aligned}$$

we preset $\alpha = \pi(\theta_0)$ and we determine a rejection region RR for which

$$\pi(\theta_1) \; = \; 1 - \beta(\theta_1)$$

is *the largest possible*. Such a test is called a **most powerful test**.

The Neyman-Pearson Lemma (NPL)

Most powerful tests cannot always be found. The following result gives a procedure for finding a most powerful test, when both hypotheses tested are simple.

Lemma 5.4 (Neyman-Pearson (NPL))**.** *Let X be a characteristic with pdf $f(x;\theta)$, with $\theta \in A \subset \mathbb{R}$, unknown. Suppose we test on θ the simple hypotheses*

$$\begin{aligned} H_0 : &\quad \theta = \theta_0 \\ H_1 : &\quad \theta = \theta_1, \end{aligned}$$

based on a random sample $X_1, \ldots, X_n$. Let $L(\theta) = L(X_1, \ldots, X_n; \theta)$ denote the likelihood function of this sample. Then for a fixed $\alpha \in (0,1)$, a most powerful test is the test with rejection region given by

$$RR \;\; = \;\; \left\{ \frac{L(\theta_1)}{L(\theta_0)} \geq k_\alpha \right\}, \tag{5.5}$$

where the constant $k_\alpha > 0$ depends only on α and the sample variables.

Example 5.5. Suppose X_1 represents a single observation from a probability density given by

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Find the NPL most powerful test that at the 5% significance level tests

$$\begin{aligned} H_0: &\quad \theta = 1 \quad\ (= \theta_0) \\ H_1: &\quad \theta = 30 \quad (= \theta_1). \end{aligned}$$

Also, find β for that test.

Solution. Since our sample has size 1, we have

$$\frac{L(\theta_1)}{L(\theta_0)} \;=\; \frac{f(X_1;\theta_1)}{f(X_1;\theta_0)} \;=\; \frac{30X_1^{29}}{1} \;=\; 30X_1^{29}.$$

So the rejection region given by the NPL is

$$RR \;=\; \{30X_1^{29} \geq k_\alpha\} \;=\; \{X_1 \geq K_\alpha\},$$

where $K_\alpha = \left(\dfrac{1}{30}k_\alpha\right)^{1/29}$.
We find the value of K_α from

$$\begin{aligned} \alpha \;&=\; P(X_1 \in RR \mid H_0) \;=\; P(X_1 \geq K_\alpha \mid \theta = 1) \\ &=\; \int_{K_\alpha}^{1} dx \;=\; 1 - K_\alpha, \end{aligned}$$

i.e. $K_\alpha \;=\; 1 - \alpha \;=\; 0.95.$
So, of all tests for testing H_0 versus H_1, based on a sample of size 1, the observation X_1, at the significance level $\alpha = 0.05$, the most powerful test has rejection region

$$RR \;=\; \{X_1 \geq 0.95\}.$$

For this test,

$$\begin{aligned} \beta(\theta_1) \;&=\; P(X_1 < K_\alpha \mid \theta = 30) \;=\; \int_0^{K_\alpha} 30x^{29}\,dx \\ &=\; x^{30}\Big|_0^{K_\alpha} \;=\; (K_\alpha)^{30} \;=\; (1-\alpha)^{30} \;=\; 0.166 \end{aligned}$$

and the power is

$$\pi(\theta_1) = 1 - \beta(\theta_1) = 0.834.$$

Note that the error probability β that we obtained is *unacceptably large*, but considering that the estimation was based on a sample of size *one*, we cannot expect too much accuracy.

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Remark 5.6. Notice that the rejection region and, hence, the most powerful test we found in Example 5.5, depend on the value stated in H_1. For a different value of θ_1, we would have found a *different* rejection region. That is usually the case. However, sometimes, a test obtained with the NPL actually maximizes the power for *every* value in H_1, i.e. even if H_1 is not a simple hypothesis. Such a test is called a **uniformly most powerful test**.

Example 5.7. Let $X_1, \dots, X_n$ be a random sample drawn from a Normal $N(\mu, \sigma)$ distribution, with $\mu \in \mathbb{R}$ unknown and $\sigma > 0$ known. At the significance level $\alpha \in (0, 1)$, find the most powerful right-tailed test for testing

$$\begin{aligned} H_0 :& \quad \mu = \mu_0 \\ H_1 :& \quad \mu > \mu_0. \end{aligned}$$

Solution. First we use the NPL to find a most powerful test for a *simple* alternative, i.e.

$$\begin{aligned} H_0 :& \quad \mu = \mu_0 \\ H_1 :& \quad \mu = \mu_1 > \mu_0. \end{aligned}$$

We have the Normal pdf

$$f(x; \mu) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2} \Big), \ \forall x \in \mathbb{R}.$$

The likelihood function is

$$\begin{aligned} L(\mu) &= \prod_{i=1}^{n} f(X_i; \mu) \\ &= \Big(\frac{1}{\sigma\sqrt{2\pi}} \Big)^n \exp\Big(- \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \Big). \end{aligned}$$

Then, by the NPL, we find

$$\frac{L(\mu_1)}{L(\mu_0)} = \exp\Big(\frac{1}{2\sigma^2} \Big[\sum_{i=1}^{n} (X_i - \mu_0)^2 - \sum_{i=1}^{n} (X_i - \mu_1)^2 \Big] \Big) \ \geq \ k_\alpha,$$

or, taking the logarithm $\ln$ (which is an increasing function) on both sides,

$$\begin{aligned} \frac{1}{2\sigma^2}\Big[\sum_{i=1}^n (X_i-\mu_0)^2 - \sum_{i=1}^n (X_i-\mu_1)^2\Big] &\geq \ln k_\alpha, \\ \sum_{i=1}^n X_i^2 - 2\mu_0\sum_{i=1}^n X_i + n\mu_0^2 - \left(\sum_{i=1}^n X_i^2 - 2\mu_1\sum_{i=1}^n X_i + n\mu_1^2\right) &\geq 2\sigma^2 \ln k_\alpha. \end{aligned}$$

After cancellations and using $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, we have

$$2n\overline{X}(\mu_1-\mu_0) \geq 2\sigma^2 \ln k_\alpha + n(\mu_1^2-\mu_0^2).$$

Since $\mu_1 > \mu_0$, we get

$$\overline{X} \geq \frac{\sigma^2 \ln k_\alpha}{n(\mu_1-\mu_0)} + \frac{\mu_1+\mu_0}{2} = K_\alpha.$$

Then we use the test statistic $TS = \overline{X}$, for which we found the rejection region

$$RR = \{\overline{X} \geq K_\alpha\}.$$

But

$$\begin{aligned} \alpha &= P\Big(\overline{X} \geq K_\alpha \mid \mu = \mu_0\Big) \\ &= P\Big(\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \geq \frac{K_\alpha-\mu_0}{\sigma/\sqrt{n}} \mid \mu=\mu_0\Big) \\ &= P\Big(Z_0 \geq \frac{K_\alpha-\mu_0}{\sigma/\sqrt{n}} \mid Z_0 \in N(0,1)\Big) \\ &= P\Big(Z_0 \geq z_{1-\alpha}\Big), \end{aligned}$$

since $Z_0 = \dfrac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \in N(0,1)$. Then we must have

$$\frac{K_\alpha-\mu_0}{\sigma/\sqrt{n}} = z_{1-\alpha}, \quad K_\alpha = \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}},$$

so K_α is *independent* of μ_1. Then the test with $RR = \{\overline{X} \geq K_\alpha\}$ is a *uniformly* most powerful test for testing

$$\begin{aligned} H_0 : &\quad \mu = \mu_0 \\ H_1 : &\quad \mu > \mu_0, \end{aligned}$$

at the significance level α.

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Remark 5.8. In a similar manner, we can find a uniformly most powerful test for the left-tailed case

$$\begin{aligned} H_0 : &\quad \mu = \mu_0 \\ H_1 : &\quad \mu < \mu_0. \end{aligned}$$