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Seminar W7 – 832

*A **point estimator** for the target parameter θ is a statistic:*

$$\overline{\theta} = \theta(X_1, X_2, \ldots, X_n)$$

We have the following notions:

- ***unbiased estimator**: $E(\overline{\theta}) = \theta$ (the **bias**: $B := E(\overline{\theta}) - \theta$);*
- ***absolutely correct estimator**: $E(\overline{\theta}) = \theta$, $\lim\limits_{n\to\infty} V(\overline{\theta}) = 0$;*
- ***consistent estimator**: $\overline{\theta} \xrightarrow{p} \theta$;*

*The **efficiency** of an absolutely correct estimator $\overline{\theta}$ is*

$$e(\overline{\theta}) = \frac{1}{I_n(\theta)V(\overline{\theta})}$$

*$\overline{\theta}$ is an **efficient estimator** for θ if $e(\overline{\theta}) = 1$*

- ***Fisher's (quantity of) information** relative to θ:*

$$I_n(\theta) = E\left(\left(\frac{\partial \ln L(X_1, X_2, \ldots, X_n; \theta)}{\partial \theta}\right)^2\right)$$

If the range of X does not depend on θ:

$$I_n(\theta) = -E\left(\frac{\partial^2 \ln L(X_1, X_2, \ldots, X_n; \theta)}{\partial \theta^2}\right)$$

or

$$I_n(\theta) = nI_1(\theta)$$

most of the time

- *The **likelihood function** of the sample $X_1, X_2, \ldots, X_n$:*

$$L(X_1, X_2, \ldots, X_n; \theta) = \prod_{i=1}^{n} f(X_i; \theta);$$

Exercise 1. Let $X \begin{pmatrix} -1 & 1 \\ \frac{1-\theta}{2} & \frac{1+\theta}{2} \end{pmatrix}$, where $\theta \in (0,1)$ is a parameter. Prove that the sample mean $\overline{X} = \frac{1}{n}\sum\limits_{j=1}^{n} X_j$, $n \in \mathbb{N}$, is an absolutely correct estimator of θ. Is this estimator efficient?

Sol.: $E(\overline{X}) = E\left(\frac{1}{n}\sum\limits_{j=1}^{n} X_j\right) = \frac{1}{n} \cdot E\left(\sum\limits_{j=1}^{n} X_j\right) = \frac{1}{n} \cdot \sum\limits_{j=1}^{n} E(X_j) =$

$= \frac{1}{n} \cdot n \cdot E(X_1) = E(X)$

$E(X) = \frac{1-\theta}{2} \cdot (-1) + \frac{1+\theta}{2} \cdot 1 = \frac{1+\theta}{2} - \frac{1-\theta}{2} = \theta$

$\Rightarrow E(\bar{X}) = E(X) = \theta \Rightarrow \bar{X}$ is an unbiased estimator for θ

$$V(\bar{X}) = V\left(\frac{1}{n} \cdot \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \cdot V\left(\sum_{i=1}^{n} X_i\right) \overset{X_i \text{ indep.}}{=} \frac{1}{n^2} \cdot n \cdot V(X_1) =$$

$$= \frac{1}{n} \cdot V(X)$$

$$V(X) = E(X^2) - E(X)^2$$

$$X \begin{pmatrix} -1 & 1 \\ \frac{1-\theta}{2} & \frac{1+\theta}{2} \end{pmatrix} \Rightarrow X^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$V(X) = 1 - \theta^2$$

$$V(\bar{X}) = \frac{1}{n} \cdot (1 - \theta^2) \Rightarrow \lim_{n \to \infty} V(\bar{X}) = 0 \Rightarrow \bar{X} \text{ absolutely correct estimator}$$

We need to find the efficiency of $\bar{X}$.

$$e(\bar{X}) = \frac{1}{I_n(\theta) \cdot V(\bar{X})}$$

Because the range of X is 2, hence it does not depend on θ, we have:

$$I_n(\theta) = n \cdot I_1(\theta) = -n \cdot E\left(\frac{\partial^2 \ln L(X_1; \theta)}{\partial \theta^2}\right)$$

$$L(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

$$I_n(\theta) = -n \cdot E\left(\frac{\partial^2 \ln f(x; \theta)}{\partial \theta^2}\right)$$

$$X \begin{pmatrix} -1 & 1 \\ \frac{1-\theta}{2} & \frac{1+\theta}{2} \end{pmatrix} \qquad f(x;\theta) = \frac{1+x\cdot\theta}{2}$$

$$\frac{\partial^2}{\partial\theta^2}\left(\ln\left(\frac{1+x\theta}{2}\right)\right) = \frac{\partial}{\partial\theta}\left(\frac{2}{1+x\theta}\cdot\frac{x}{2}\right) = \frac{\partial}{\partial\theta}\left(\frac{x}{1+x\theta}\right) =$$

$$= x\cdot(-1)\cdot\frac{1}{(1+x\theta)^2}\cdot x = -\frac{x^2}{(1+x\theta)^2}$$

$$I_n(\theta) = -n\cdot E\left(-\frac{x^2}{(1+x\theta)^2}\right) = n\, E\left(\frac{x^2}{(1+x\theta)^2}\right)$$

$$X \begin{pmatrix} -1 & 1 \\ \frac{1-\theta}{2} & \frac{1+\theta}{2} \end{pmatrix} \Rightarrow \frac{x^2}{(1+x\theta)^2} \begin{pmatrix} \frac{1}{(1-\theta)^2} & \frac{1}{(1+\theta)^2} \\ \frac{1-\theta}{2} & \frac{1+\theta}{2} \end{pmatrix}$$

$$I_n(\theta) = n\cdot E\left(\frac{x^2}{(1+x\theta)^2}\right) = n\cdot\left(\frac{1-\theta}{2}\cdot\frac{1}{(1-\theta)^2} + \frac{1+\theta}{2}\cdot\frac{1}{(1+\theta)^2}\right) =$$

$$= \frac{1}{2}n\left(\frac{1}{1-\theta} + \frac{1}{1+\theta}\right) = \frac{1}{2}n\cdot\frac{1-\theta+(1+\theta)}{1-\theta^2} = n\cdot\frac{1}{1-\theta^2}$$

$$e(\overline{X}) = \frac{1}{I_n(\theta)\cdot V(\overline{X})} = \frac{1}{n\cdot\frac{2}{1-\theta^2}\cdot\frac{1}{n}\cdot(1-\theta^2)} = 1$$

$\Rightarrow \overline{X}$ is an efficient estimator for the parameter θ.

Exercise 2. Let $X \sim Unif([0, \theta])$, where $\theta > 0$ is a parameter. Consider the estimator $\bar{\theta} = c_n \cdot \max\{X_1, X_2, \ldots, X_n\}$, where $c_n \in \mathbb{R}$ depends only on $n \in \mathbb{N}$. Find c_n such that $\bar{\theta}$ is unbiased. Is $\bar{\theta}$ absolutely correct?

Sol.: $f_X(x) = \begin{cases} \frac{1}{\theta}, & x \in [0, \theta] \\ 0, & \text{otherwise} \end{cases}$ range $(X) = \theta$

$$\hat{\theta} := \max\{X_1, X_2, \ldots, X_n\}$$

$$E(\bar{\theta}) = c_n \cdot E(\hat{\theta})$$

$$E(\hat{\theta}) = \int_{\mathbb{R}} x \cdot f_{\hat{\theta}}(x)\, dx$$

$$F_{\hat{\theta}}(x) = P(\hat{\theta} \leq x) = P(\max(X_1, \ldots, X_n) \leq x) =$$

$$= P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) =$$

$$= P((X_1 \leq x) \cap (X_2 \leq x) \cap \cdots \cap (X_n \leq x)) =$$

$$= P(X_1 \leq x) \cdot P(X_2 \leq x) \cdot \ldots \cdot P(X_n \leq x) =$$

$$= P(X_1 \leq x)^n = P(X \leq x)^n = F_X(x)^n$$

$$f_X(x) = \begin{cases} \frac{1}{\theta}, & x \in [0, \theta] \\ 0, & \text{otherwise} \end{cases} \Rightarrow F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\theta}, & x \in [0, \theta] \\ 1, & x > \theta \end{cases}$$

$$F_{\hat\theta}(x) = F_X(x)^n = \begin{cases} 0, & x<0 \\ \frac{x^n}{\theta^n}, & x\in[0,\theta] \\ 1, & x>\theta \end{cases}$$

$$\Rightarrow \quad f_{\hat\theta}(x) = \begin{cases} 0, & x<0 \\ \frac{n x^{n-1}}{\theta^n}, & x\in[0,\theta] \\ 0, & x>\theta \end{cases}$$

$$E(\hat\theta) = \int_{\mathbb{R}} x \cdot f_{\hat\theta}(x)\, dx = \int_0^\theta x \cdot \frac{n x^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n}\int_0^\theta x^n\, dx =$$

$$= \frac{n}{\theta^n} \cdot \frac{1}{n+1} x^{n+1}\Big|_0^\theta = \frac{n}{\theta^n}\cdot\frac{1}{n+1}\theta^{n+1} = \frac{n\theta}{n+1}$$

$$\bar\theta = c_n \cdot \hat\theta$$

$$E(\bar\theta) = \theta \Rightarrow c_n \cdot E(\hat\theta) = \theta \Rightarrow c_n = \frac{\theta}{E(\hat\theta)} = \frac{\theta}{\frac{n\theta}{n+1}}$$

$$\Rightarrow c_n = \frac{n+1}{n}$$

$$\left(S = \sqrt{\frac{1}{n-1}\cdot\sum_{i=1}^n (X_i - \bar X)^2} \quad \text{sample deviation} \right)$$

For $\bar\theta$ to be absolutely correct, we need $E(\bar\theta) = \theta$ and

$$\lim_{n\to\infty} V(\bar\theta) = 0$$

$$\bar{\theta} = \frac{n+1}{n} \cdot \hat{\theta}$$

$$V(\bar{\theta}) = \left(\frac{n+1}{n}\right)^2 \cdot V(\hat{\theta})$$

$$V(\hat{\theta}) = E(\hat{\theta}^2) - E(\hat{\theta})^2$$

$$E(\hat{\theta}^2) = \int_{\mathbb{R}} x^2 \cdot f_{\hat{\theta}}(x)\, dx$$

$$f_{\hat{\theta}}(x) = \begin{cases} 0, & x < 0 \\ \frac{n x^{n-1}}{\theta^n}, & x \in [0, \theta] \\ 0, & x > \theta \end{cases}$$

$$E(\hat{\theta}^2) = \int_0^\theta x^2 \cdot \frac{n x^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n} \cdot \int_0^\theta x^{n+1}\, dx =$$

$$= \frac{n}{\theta^n} \cdot \frac{1}{n+2} x^{n+2} \Big|_0^\theta = \frac{n}{\theta^n} \cdot \frac{1}{n+2} \cdot \theta^{n+2} = \frac{\theta^2 \cdot n}{n+2}$$

$$V(\hat{\theta}) = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = n\theta^2 \left(\frac{1}{n+2} - \frac{n}{(n+1)^2}\right) =$$

$$= n\theta^2 \cdot \frac{(n+1)^2 - n(n+2)}{(n+2)\cdot(n+1)^2} = n\theta^2 \cdot \frac{1}{(n+2)\cdot(n+1)^2}$$

$$V(\bar{\theta}) = \left(\frac{n+1}{n}\right)^2 \cdot n\theta^2 \cdot \frac{1}{(n+2)(n+1)^2}$$

$$\Rightarrow \lim_{n\to\infty} V(\bar{\theta}) = 0$$

Exercise 5. Prove that the sample moment of order 2:

$$\overline{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

is a consistent estimator of the variance $V(X)$. Deduce that the sample standard deviation is a consistent estimator of the standard deviation of $\sigma = \sqrt{V(X)}$.

Hint: For a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, almost sure convergence implies convergence in probability:

$$X_n \overset{a.s.}{\to} X \Longrightarrow X_n \overset{p}{\to} X$$

Sol. : We have to show that $\overline{\mu_2} \xrightarrow{p} V(X)$

We will show that $\overline{\mu_2} \xrightarrow{a.s.} V(X)$

- ***The Strong Law of Large Numbers (SLLN):***

  If $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables with $X_n \sim X$, then

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} E(X)$$

$$\overline{\mu_2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \underbrace{\overline{X}}_{= \frac{1}{n}(X_1+\dots+X_n)})^2$$

$X_1, \dots, X_n$ i.i.d $\Rightarrow$ $\overline{X} \xrightarrow{a.s.} E(X) =: m$

$$\overline{\mu_2} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - m + m - \overline{X}\right)^2 =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[(X_i - m)^2 + (m-\overline{X})^2 + 2\cdot(X_i - m)(m-\overline{X})\right] =$$

$$= \frac{1}{n}\cdot\sum_{i=1}^{n}(X_i - m)^2 + \frac{1}{n}\sum_{i=1}^{n}(m-\overline{X})^2 + \frac{2}{n}\sum_{i=1}^{n}(X_i - m)(m-\overline{X})$$

by the SLLN

$$\frac{1}{n}\sum_{i=1}^{n}\underbrace{(X_i-m)^2}_{=:Y_i} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 \xrightarrow{a.s.} E(Y_1^2) =$$

$$= E((X-m)^2) = E((X-E(X))^2) = V(X)$$

$$\frac{1}{n}\sum_{i=1}^{n}(m-\overline{X})^2 = \frac{1}{n}\cdot n\cdot(m-\overline{X})^2 =$$

$= (m-\overline{X})^2 \xrightarrow{a.s.} 0$, because $\overline{X} \xrightarrow{a.s.} m$ (by the SLLN)

$$\frac{2}{n}\sum_{i=1}^{n}(X_i-m)\cdot(m-\overline{X}) = \frac{2\cdot(m-\overline{X})}{n}\cdot\sum_{i=1}^{n}(X_i-m) =$$

$$= 2\cdot\underbrace{(m-\overline{X})}_{\xrightarrow{a.s.} 0}\cdot\underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i-m)}_{\xrightarrow{a.s.} E(X_i-m) = E(X_i)-m = E(X)-m=0}$$

$$\Rightarrow \overline{\mu_2} \xrightarrow{a.s.} V(X) \overset{hint}{\Longrightarrow} \overline{\mu_2} \xrightarrow{P} V(X) \Rightarrow$$

$\Rightarrow \overline{\mu_2}$ is a consistent estimator for $V(X)$

$$\overline{S} = \sqrt{\frac{1}{n-1}\cdot\sum_{i=1}^{n}(X_i-\overline{X})^2} = \sqrt{\frac{n}{n-1}\cdot\overline{\mu_2}}$$

$$\overline{\mu_2} \xrightarrow{P} \underset{\sigma^2}{\underset{\|}{V(X)}} \Rightarrow \overline{S} \xrightarrow{P} \sqrt{\sigma^2} = \sigma$$