|| Statistică Matematică ||

Seminar 1

The Gamma and Beta functions

Ex 1: $\boxed{\text{For } a>0,\ \Gamma(a)=\int_0^\infty x^{a-1}\cdot e^{-x}\,dx.}$

Prove that: 1. for $a>0$ we have $\Gamma(a+1)=a\Gamma(a)$

2. for $m\in\mathbb{N}$ $\Gamma(m)=(m-1)!$

3. for $\forall\, a>0$ $\Gamma(a)=2\int_0^\infty x^{2a-1}\cdot e^{-x^2}\,dx$

1. We know $\Gamma(a)=\int_0^\infty x^{a-1}\cdot e^{x}\,dx \rightarrow \Gamma(a+1)=\int_0^\infty x^{(a+1)-1}\cdot e^{x}\,dx$

$a\cdot\Gamma(a)=a\int_0^\infty x^{a-1}\cdot e^{-x}\,dx \rightarrow \Gamma(a+1)=\int_0^\infty x^{a}\cdot \underbrace{e^{-x}}_{(-e^{-x})'}\,dx$

$\rightarrow = -e^{-x}\cdot x^{a}\Big|_0^\infty - \int_0^\infty a x^{(a-1)}\cdot e^{-\cancel{x}}\,dx$

$= \overbrace{-\ln\frac{x^a}{e^{x}}}^{=0} + a\int_0^\infty x^{a-1}\cdot e^{-x}\,dx$

$= a\,\Gamma(a)$

2. $\Gamma(m)=(m-1)\,\Gamma(m-1)=(m-1)(m-2)\,\Gamma(m-3)=\ldots$

$=(m-1)!\,\Gamma(1)$

$\Gamma(1)=\int_0^\infty e^{-x}\,dx = e^{-x}\Big|_0^\infty = 1 \Rightarrow \Gamma(m)=(m-1)!\cdot 1$

$\Gamma(m)=(m-1)!$

3. $\forall a>0, \Gamma(a) = 2 \int_0^\infty x^{2a-1} \cdot e^{-x^2} dx$

$$\cancel{= 2 \int_0^\infty x^{a-1} \cdot e^{-x} dx}$$

$$\Gamma(a) = \int_0^\infty x^{a-1} \cdot e^{-x} dx \underset{=}{\overset{x=y^2}{=}} \int_0^\infty y^{2(a-1)} e^{-y} \, 2y \, dy$$

$$= 2 \int_0^\infty x^{2a-1} \cdot e^{-x^2} dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} \cdot e^{-x} dx$$

$$= 2 \int_0^\infty e^{-x^2} dx = \underline{\underline{\int_{\mathbb{R}} e^{-x^2} dx}} = \ldots = \sqrt{\pi}$$

Ex2: The Beta Function:

$$\boxed{\text{For } a,b>0 : B(a,b) = \int_0^1 x^{a-1} \cdot (1-x)^{b-1} dx}$$

1. $\forall a,b>0 : B(a,b) = B(b,a)$
2. $\forall a,b>0 \quad B(a,1) = \frac{1}{a}, \; B(1,b) = \frac{1}{b}$

3. $B(a,b) = \frac{a-1}{b} \cdot B(a-1, b+1)$, $a>1, b>0$
4. $B(a,b) = \frac{b-1}{a} \cdot B(a+1, b-1)$, $a>0, b>1$

5. $B(a,b) = \frac{a-1}{a+b-1} \cdot B(a-1,b)$, $a>1, b>0$
6. $B(a,b) = \frac{b-1}{a+b-1} \cdot B(a,b-1)$, $a>0, b>1$

7. $B(k+1, m-k) = \frac{1}{(m+1) \cdot C_m^k} = \frac{k! \cdot (m-k)!}{(m+1)!}$

8. $\boxed{B(a,b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}, \quad a,b>0}$

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• Proving 8 ~ by myself ~

We know that $\Gamma(a) = \int_0^\infty x^{a-1} \cdot e^{-x}\, dx$

$$\left.\begin{array}{l}\Gamma(a) = \int_0^\infty x^{a-1} \cdot e^{-x}\, dx \\ \Gamma(b) = \int_0^\infty x^{b-1} \cdot e^{-x}\, dx\end{array}\right\} \to \Gamma(a+b) = \int_0^\infty x^{(a+b-1)} \cdot e^{-x}\, dx$$

$$\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} = \frac{\int_0^\infty x^{a-1} \cdot e^{-x}\, dx \cdot \int_0^\infty x^{b-1} \cdot e^{-x}\, dx}{\int_0^\infty x^{(a+b)-1} \cdot e^{-x}\, dx} = (*)$$

$$(*) = \int_0^\infty x^{a-1} \cdot x^{b-1} \cdot e^{-x} \cdot \cancel{x^{a+b}} \cdot e^{-x} = \int_0^\infty x^{a+b-2} \cdot e^{-2x}$$

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① $B(a,b) = \int_0^1 x^{a-1} \cdot (1-x)^{b-1}\, dx \xrightarrow{y=1-x} \int_0^1 (1-y)^{a-1} \cdot y^{b-1} (-1)\, dy$

$$= \int_0^1 y^{b-1} (1-y)^{a-1}\, dy = B(b,a)$$

② $B(a,1) = \int_0^1 x^{a-1}\, dx = \frac{x^a}{a}\Big|_0^1 = \frac{1}{a}$

$B(1,b) = \int_0^1 (1-x)^{b-1}\, dx = -\frac{(1-x)^b}{b}\Big|_0^1 = \frac{1}{b}$

③ $B(a,b) = \int_0^1 x^{a-1} \underbrace{(1-x)^{b-1}}_{\frac{-(1-x)^b}{b}}\, dx = -\frac{x^{a-1} \cdot (1-x)^b}{b}\Big|_0^1 + \int_0^1 (a-1) \cdot x^{a-2} \cdot \frac{(1-x)^b}{b}\, dx$

$$= \frac{a-1}{b} \int_0^1 x^{a-2} \cdot (1-x)^b\, dx = \frac{a-1}{b} \cdot B(a-1, b+1)$$

④ ~ same as above ~

⑤ $B(a,b) = \frac{a-1}{a+b-1} \cdot B(a-1,b)$, $a>1, b>1$

$B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$

$= \int_0^1 x^{a-1} (1-x)^{b-2} (1-x) dx$

by myself

$= \int_0^1 x^{a-1} (1-x)^{b-2} dx - \int_0^1 x^{a} (1-x)^{b-2} dx$

$= B(a,b-1) - B(a+1,b-1) =$

⑥ $B(a,b-1) = \frac{a}{b-1} - B(a,b)$

$\Rightarrow (1+\frac{a}{b-1}) B(a,b) = B(a,b-1) \Rightarrow B(a,b) = \frac{b-1}{a} B(a,b$

5 is the same

⑧ $B(a,b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}$

$\Gamma(a) \cdot \Gamma(b) = \int_0^\infty x^{a-1} e^{-x} dx \cdot \int_0^\infty x^{b-1} \cdot e^{-x} dx$

$= \int_0^\infty x^{a-1} \cdot e^{-x} dx \cdot \int_0^\infty y^{b-1} \cdot e^{-y} dy$

$= \int_0^\infty \int_0^\infty x^{a-1} \cdot y^{b-1} \cdot e^{-x-y} dx\, dy$

$\Gamma(a+b) = \int_0^\infty u^{a+b-1} \cdot e^{-u} du$

$B(a,b) = \int_0^1 v^{a-1} \cdot (1-v)^{b-1} dv$

$u := x+y$

$v := \frac{x}{x+y}$ $\Rightarrow$ $x = uv$

$y = u - x = u - uv$

$(x,y) \in (0,\infty) \times (0,\infty)$

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$\Rightarrow (u,v) \in (0,\infty) \times (0,1)$

$dx\,dy = |J_{u,v}| \cdot du\, dv$, $J_{u,v} = \begin{vmatrix} \frac{\partial}{\partial u} x & \frac{\partial}{\partial v} x \\ \frac{\partial}{\partial u} y & \frac{\partial}{\partial v} y \end{vmatrix}$

$J_{u,v} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -\cancel{vu} - u + \cancel{uv}$

$= -u$

$$\Gamma(a) \cdot \Gamma(b) = \int_0^\infty \int_0^\infty x^{a-1} \cdot y^{b-1} \cdot e^{-x-y}\, dx\, dy$$

$$= \int_0^\infty \int_0^1 \underbrace{(uv)^{a-1}}_{dx} \cdot \underbrace{(u-uv)^{b-1}}_{dy} e^{-u} \cdot u \cdot du\, dv$$

$$= \int_0^\infty \int_0^1 u^{a+b-1} \cdot e^{-u}\, v^{a-1} (1-v)^{b-1}\, du\, dv$$

$$= \underbrace{\int_0^\infty u^{a+b-1} \cdot e^{-u}\, du}_{\Gamma(a+b)} + \underbrace{\int_0^1 v^{a-1} (1-v)^{b-1} dv}_{B(u,v)}$$

Result (can be taken without the proofs)

$\forall a > 0 \quad \Gamma(a) \cdot \Gamma(1-a) = \frac{\pi}{\sin(a\pi)}$

For $a = \frac{1}{2}$: $\Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{\sin \frac{\pi}{2}} = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_{\mathbb{R}} e^{-x^2} dx$$