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**Université de Lille**
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École doctorale **École Graduée MADIS-631**

Unité de recherche **Centre de Recherche en Informatique, Signal et Automatique de Lille**

Thèse présentée par **Hector Kohler**

Soutenue le **1er décembre 2025**

En vue de l'obtention du grade de docteur de l'Université de Lille et de l'Inria

Discipline **Informatique**

Spécialité **Informatique et Applications**

# Interprétabilité, Arbres de Décision, et Prise de Décisions Séquentielle

| | | |
|---|---|---|
| **Thèse dirigée par** | Philippe Preux | directeur |
| | Riad Akrour | co-directeur |

**Composition du jury**

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|---|---|---|---|
| *Rapporteurs* | Olivier Buffet | chargé de recherche à l'Université de Lorraine | |
| | Aurélie Beynier | MCF au Sorbonne Universités | |
| *Examinateurs* | Lydia Boudjeloud-Assala | professeur à l'Université de Lorraine | président du jury |
| | Osbert Bastani | MCF à l'University of Pennsylvania | |
| *Invité* | Sonali Parbhoo | | |
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## Colophon

Mémoire de thèse intitulé « Interprétabilité, Arbres de Décision, et Prise de Décisions Séquentielle », écrit par Hector Kohler, achevé le 17 août 2025, composé au moyen du système de préparation de document LaTeX et de la classe yathesis dédiée aux thèses préparées en France.

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Thesis defended by **Hector Kohler**

Defended on **December 1, 2025**

In order to become Doctor from Université de Lille and from Inria

Academic Field **Computer Science**

Speciality **Computer Science and Applications**

# Interpretability, Decision Trees, and Sequential Decision Making

| | | |
|---|---|---|
| **Thesis supervised by** | Philippe Preux | Supervisor |
| | Riad Akrour | Co-Supervisor |

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| | Osbert Bastani | Associate Professor at University of Pennsylvania | |
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| | Riad Akrour | Junior Researcher at Inria | |

**Interprétabilité, Arbres de Décision, et Prise de Décisions Séquentielle**

**Résumé**

Dans cette thèse de doctorat, nous étudions des algorithmes d'apprentissage d'arbres de décision pour la classification et la prise de décision séquentielle. Les arbres de décision sont interprétables car les humains peuvent lire les opérations de l'arbre de décision depuis la racine jusqu'aux feuilles. Cela fait des arbres de décision le modèle de référence lorsque la vérification humaine est requise, comme dans les applications médicales. Cependant, les arbres de décision ne sont pas différentiables, ce qui les rend difficiles à optimiser, contrairement aux réseaux neuronaux qui peuvent être entraînés efficacement avec la descente de gradient. Les approches existantes d'apprentissage par renforcement interprétables apprennent généralement des arbres souples (non interprétables en l'état) ou sont ad hoc (entraînent un réseau neuronal puis entraînent un arbre à imiter le reseau). Cette apprentissage d'arbre indirect ne garantit pas de trouver des bonnes solutions pour le problème initial.

Dans la première partie de ce manuscrit, nous visons à apprendre directement des arbres de décision pour un processus de décision Markovien avec de l'apprentissage par renforcement. En pratique, nous montrons que cela revient à résoudre un problème de décision Markovien partiellement observable (PDMPO). La plupart des algorithmes d'apprentissage par renforcement existants ne sont pas adaptés aux PDMPOs. Ce parallèle entre l'apprentissage des arbres de décision et la résolution des PDMPOs nous aide à comprendre pourquoi, dans la pratique, il est souvent plus facile d'obtenir une politique experte non interprétable (un réseau neuronal) puis de la distiller en un arbre plutôt que d'apprendre l'arbre de décision à partir de zéro.

La deuxième contribution de ce travail découle de l'observation selon laquelle la recherche d'un classifieur (ou régresseur) arbre de décision peut être considérée comme l'ajout séquentiel de nœuds à un arbre afin de maximiser la précision des prédictions. Nous formulons donc l'induction d'arbres de décision comme la résolution d'un problème de décision Markovien et proposons un nouvel algorithme de pointe qui peut être entraîné à partir de données d'exemple supervisés et qui généralise bien à des données nouvelles.

Les travaux des parties précédentes reposent sur l'hypothèse que les arbres de décision sont un modèle interprétable que les humains peuvent utiliser dans des applications sensibles. Mais est-ce vraiment le cas ? Dans la dernière partie de cette thèse, nous tentons de répondre à des questions plus générales sur l'interprétabilité : pouvons-nous mesurer l'interprétabilité sans intervention humaine ? Et les arbres de décision sont-ils vraiment plus interprétables que les réseaux neuronaux ?

**Mots clés :** apprentissage par renforcement, arbres de décision, interprétabilité, méthodologie

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### Interpretability, Decision Trees, and Sequential Decision Making

**Abstract**

In this Ph.D. thesis, we study algorithms to learn decision trees for classification and sequential decision making. Decision trees are interpretable because humans can read through the decision tree computations from the root to the leaves. This makes decision trees the go-to model when human verification is required like in medicine applications. However, decision trees are non-differentiable making them hard to optimize unlike neural networks that can be trained efficiently with gradient descent. Existing interpretable reinforcement learning (RL) approaches usually learn soft trees (non-interpretable as is) or are ad-hoc (train a neural network then fit a tree to it) potentially missing better solutions.

In the first part of this manuscript, we aim to directly learn decision trees for a Markov decision process with reinforcement learning. In practice we show that this amounts to solving a partially observable Markov decision problem (POMDP). Most existing RL algorithms are not suited for POMDPs. This parallel between decision tree learning with RL and POMDPs solving help us understand why in practice it is often easier to obtain a non-interpretable expert policy–a neural network–and then distillate it into a tree rather than learning the decision tree from scratch.

The second contribution from this work arose from the observation that looking for a deicison tree classifier (or regressor) can be seen as sequentially adding nodes to a tree to maximize the accuracy of predictions. We thus formulate decision tree induction as sloving a Markov decision problem and propose a new state-of-the-art algorithm that can be trained with supervised example data and generalizes well to unseen data.

Work from the previous parts rely on the hypothesis that decision trees are indeed an interpretable model that humans can use in sensitive applications. But is it really the case? In the last part of this thesis, we attempt to answer some more general questions about interpretability: can we measure interpretability without humans? And are decision trees really more interpretable than neural networks?

**Keywords:** reinforcement learning, deicision trees, interpretability, methodology

# Sommaire

# Preliminary Concepts

## Interpretable Sequential Decision Making

### What is Sequential Decision Making?

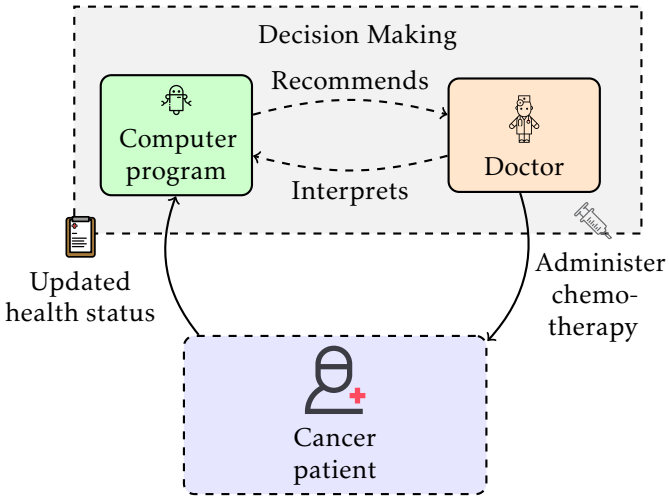



FIGURE 1 – Sequential decision making in cancer treatment. The AI system reacts to the patient's current state (tumor size, blood counts, etc.) and makes a recommendation to the doctor, who administers chemotherapy to the patient. The patient's state is then updated, and this cycle repeats over time.

In this manuscript, we study algorithms for sequential decision making. Humans engage in sequential decision making in all aspects of life. In medicine, doctors have to decide how much chemotherapy to administer based on the patient's current health [42]. In agriculture, agronomists have to decide when to fertilize based on the current soil and weather conditions to maximize plant growth [50]. In automotive settings, the autopilot system has to decide how to steer based on lidar and other sensors to maintain a safe trajectory [70]. These sequential decision making processes exhibit key similarities : an

agent takes actions based on current information to achieve a goal.

As computer scientists, we ought to design computer programs [64] that can help humans during these sequential decision making processes. For example, as depicted in Figure 1, a doctor could benefit from a program that would recommend the "best" treatment given the patient's state. Machine learning algorithms [128] output such helpful programs. For non-sequential decision making, when the doctor only takes one decision and does not need to react to the updated patient's health, e.g. making a diagnosis about cancer type, a program can be fitted to example data : given lots of patient records and the associated diagnoses, the program learns to make the same diagnosis a doctor would given the same patient record, this is *supervised* learning [87]. In the cancer treatment example, the doctor follows the patient over time and adapts treatment to the patient's changing health. In that case, machine learning—and in particular *reinforcement* learning (RL) [122]—can be used to teach the program how to take decisions that lead to recovery based on how the patient's health changes from one dose to another. Such machine learning algorithms train increasingly performant programs that are deployed to, e.g., identify digits in images [69], control tokamak fusion [32], or write the abstract of a scientific article [38].

However, the key problem behind this manuscript is that the computations performed by these programs cannot be understood and verified by humans : the programs are black-box. Next, we describe the notion of interpretability that is key to ensure safe deployment of computer programs trained with machine learning in critical sectors like medicine.

### What is Interpretability ?

Originally, the etymology of "interpretability" is the Latin "interpretabilis", meaning "that can be understood and explained". According to the Oxford English Dictionary, the first recorded use of the English word "interpretability" dates back to 1854, when the British logician George Boole (Figure 2) described the addition of concepts :

> I would remark in the first place that the generality of a method in Logic must very much depend upon the generality of its elementary processes and laws. We have, for instance, in the previous sections of this work investigated, among other things, the laws of that logical process of addition which is symbolized by the sign +. Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The

> expression x + y seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate ; that they embrace no individuals in common. And conditions analogous to this have been involved in those acts of conception from the study of which the laws of the other symbolical operations have been ascertained. The question then arises, whether it is necessary to restrict the application of these symbolical laws and processes by the same conditions of interpretability under which the knowledge of them was obtained. If such restriction is necessary, it is manifest that no such thing as a general method in Logic is possible. On the other hand, if such restriction is unnecessary, in what light are we to contemplate processes which appear to be uninterpretable in that sphere of thought which they are designed to aid ? [18, p. 48]

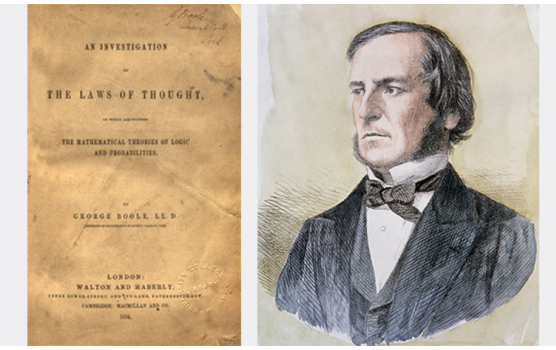
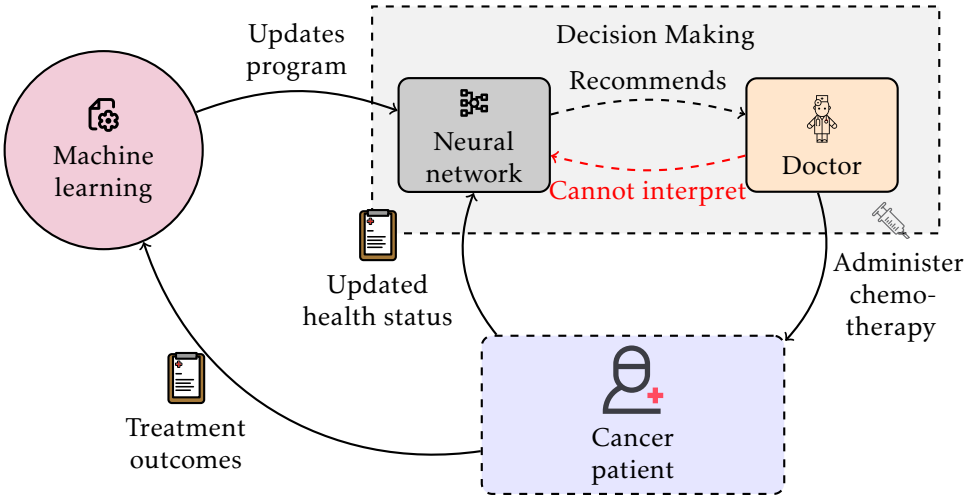


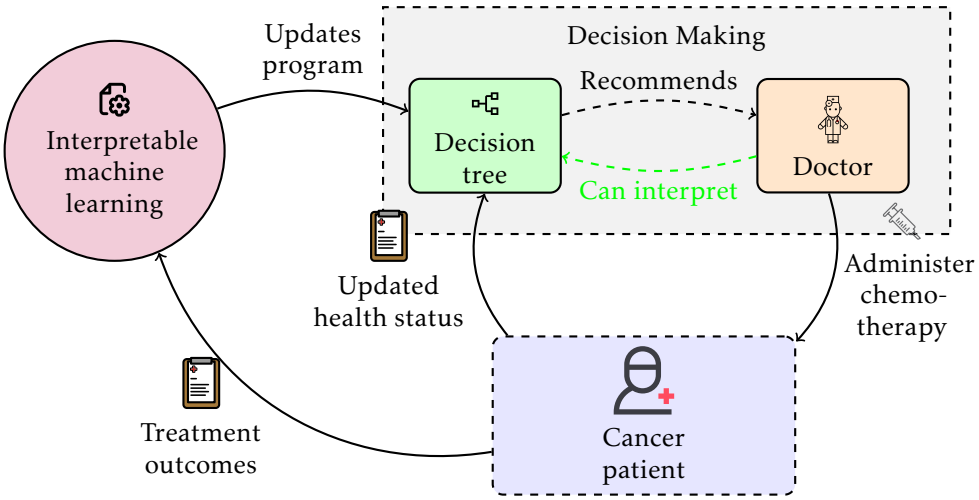

Figure 2 – British logician and philosopher George Boole (1815–1864) next to his book *The Laws of Thought* (1854), the oldest known record of the word "interpretability".

What is remarkable is that the first recorded occurrence of "interpretability" was in the context of logic and computation. Boole asked : *when can we meaningfully apply formal mathematical operations beyond the specific conditions under which we understand them ?* In Boole's era, the concern was whether logical operations like addition could be applied outside their original interpretable contexts—where symbols and their sum represent concepts that humans can understand (e.g., red + apples = red apples). Today, we face an analogous dilemma with machine learning algorithms : black-box programs like neural networks [108], which learn complex, unintelligible representations, are often deployed in contexts where computations should be understood by humans (e.g., in medicine [112]).

In Figure 3a, we illustrate how existing machine learning algorithms *could* be used in principle to help with cancer treatment. In truth, this should be prohibited without




(a) Black-box approach using neural networks




(b) Interpretable approach using decision trees

Figure 3 – Comparison of sequential decision making approaches in cancer treatment. Top : a black-box neural network approach where the doctor cannot interpret the AI's recommendations. Bottom : an interpretable decision tree approach where the doctor can understand and verify the AI's recommendations. Both systems learn from treatment outcomes to improve their recommendations over time.

some kind of transparency in the program's recommendation : why did the program recommend such a dosage ? In Figure 3b, we illustrate how machine learning *should* be used in practice. We would ideally want doctors to have access to computer programs that can recommend "good" treatments and whose recommendations are interpretable.

The key challenge of doing research on interpretability is the lack of formalism ; there is no *formal* definition of what constitutes an interpretable computer program. Hence, unlike for performance objectives, which have well-defined optimization targets (e.g., maximizing accuracy in supervised learning or maximizing rewards over time in reinforcement learning), it is not clear how to design machine learning algorithms to maximize interpretability. Despite this lack of formalism, the necessity of deploying interpretable programs has sparked many works, which we present next.

### What are existing approaches for learning interpretable programs ?

In this section we follow Sections 6 and 7 of [51] and Section 5 of [88]. Furthermore we now employ the term "model" to refer to "programs" to be consistent with the machine learning research conventions. Models are essentially mappings from inputs to outputs that can be trained with machine learning algorithms while programs might designate other types of computations like sorting.

Interpretable machine learning provides either local or global explanations [51]. Global methods output a model whose outputs can be interpreted without additional computations, e.g., a decision tree [20]. By contrast, local methods require additional computations but are agnostic to the model class : they can give an *approximate* interpretation of, e.g., neural network outputs. In Figure 4 we present the popular trade-off between interpretability and performance of different model classes.

Given a model, LIME (Local Interpretable Model-agnostic Explanations) [107] works by perturbing the input and learning a simple interpretable model locally to explain that particular prediction (see Figure 5). For each individual prediction, LIME provides explanations by identifying which features were most important for that specific decision. Hence, as stated above, LIME needs to learn one local surrogate model per output to be interpreted ; this requires substantial computation.

Global approaches are either direct or indirect [88]. Direct algorithms, such as decision tree induction [20], *directly* learn an interpretable model optimizing some objective (see Figure 4). One key challenge motivating this thesis is that decision tree induction is well-developed for supervised learning but not for reinforcement learning. To directly learn interpretable models for sequential decision making, one must design

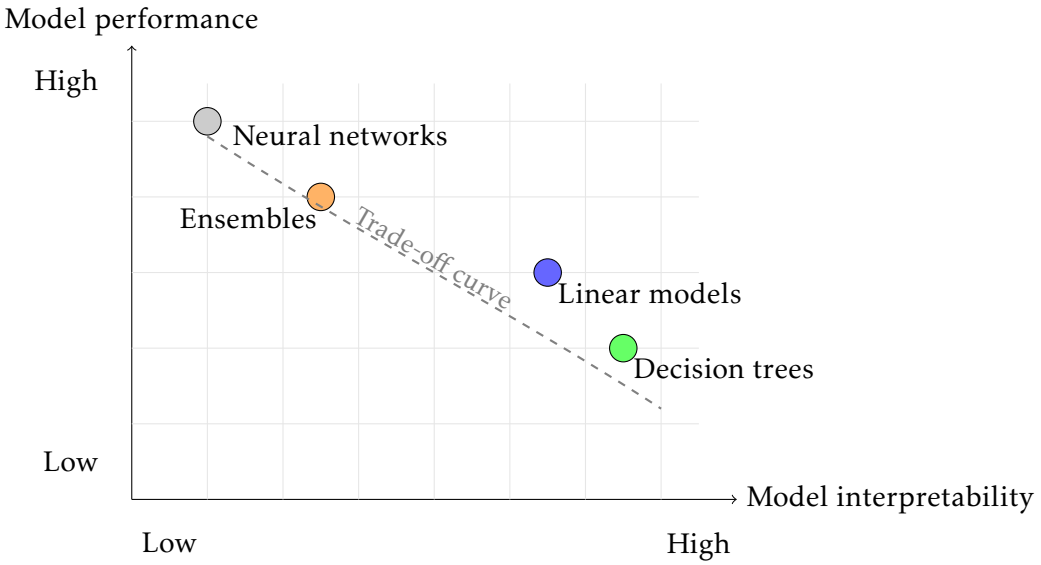



FIGURE 4 – The interpretability–performance trade-off in machine learning. Different model classes are positioned according to their typical interpretability and performance characteristics. The dashed line illustrates the general trade-off between these two properties.

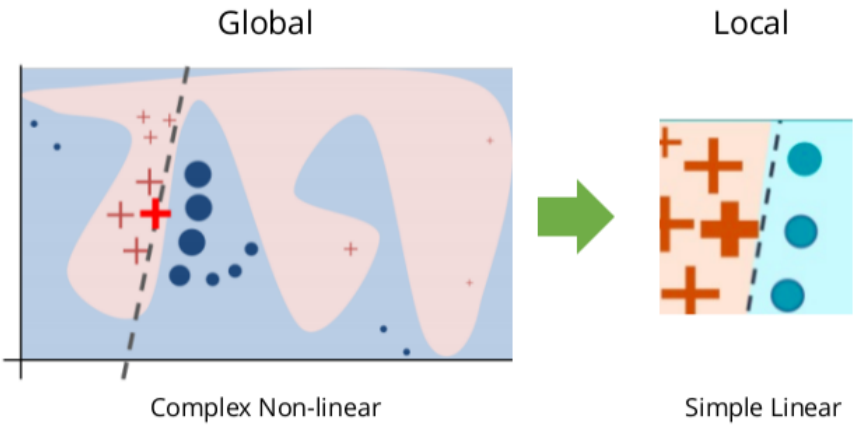



FIGURE 5 – Local Interpretable Model-agnostic Explanations [107] fit an interpretable linear model to data around the red cross prediction to be interpreted.

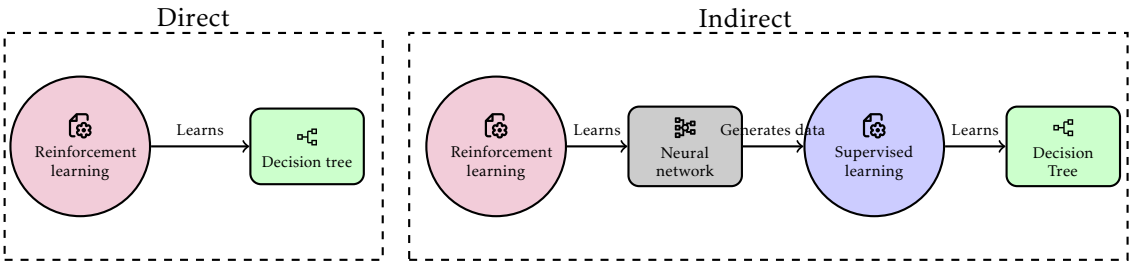



FIGURE 6 – Comparison of direct and indirect approaches for learning interpretable models in sequential decision making

new algorithms.

Most existing research has focused on developing indirect methods. Indirect methods for interpretable sequential decision making—sometimes called *post hoc* methods—begin by learning a non-interpretable model (e.g., reinforcement learning of a neural network model), and then use supervised learning to fit an interpretable model that emulates the black-box. This approach is called behavior cloning or imitation learning [97, 109], and many works on interpretable sequential decision rely on this indirect approach[11, 131]. However we believe this might be problematic.

Indeed, unlike direct methods that return interpretable models optimizing the desired objective, indirect methods learn an interpretable model to match the behavior of a black-box that itself optimizes the objective of interest. Hence, there is no guarantee that optimizing this surrogate objective yields the best interpretability–performance trade-offs. Figure 6 illustrates the key difference between these two approaches.

Verifiable Reinforcement learning via Policy Extraction, or VIPER [11], is a strong indirect method to learn decision tree models for sequential decision making. VIPER first trains a neural network model with reinforcement learning and then fit a decision tree to minimize the disagreement between the nerual network and the tree outputs given the same inputs. They show that decision tree models, in addition to being transparent, are also fast to verify in the formal sense of the term [139]. Programmatic models are an interpretable class that contains decision trees. Programmatically Interpretable Reinforcement Learning (PIRL) [131] synthesizes programs in a domain-specific language, also by imitating a neural network model.

Beyond direct and indirect learning, a complementary strategy is to train experts that are inherently easier to imitate and understand. This is achieved by adding interpretability-oriented regularization during training. In [106], authors regularize the neural netowk model during training such that indirect approaches will be biased towards more inter-

pretable trees.

Beyond transparency and verifiability, interpretability also supports detecting specification or reward misalignment in sequential decision making : by exposing the decision process of a model, one can identify goal misspecification or unintended shortcuts. Such shortcuts can be, for example, following the shadow of someone instead of actually following someone because for the model they lead to the same reactions. The learning of interpretable models for misalignment detection has been heavily studied by Quentin Delfosse contemporarily to this manuscript [34][114][33][35].

*Interpretable* decision making constrains the model class so that the computation is transparent by construction. On the other hand, *explainable* decision making, keeps black-box models and generates post hoc explanations of their decisions. Such explanations can take various forms : visual explanations with saliency maps [99], attribution such as SHAP[78], attention-based highlighting [113]. Causal approaches learn explicit causal models to support contrastive reasoning [79]. While useful for insight, these explanations are often subjective and might not be faithful to the underlying computations [6]. For safety-critical settings, this motivates our focus on models that are interpretable by design. Next we describe technical preliminaries useful to understand the content of this manuscript.

## Technical preliminaries

### What are decision trees ?

As the reader might have already guessed, we will put great emphasis as decision tree models as a mean to study interpretability. While other interpretable models might have other properties that the ones we will highlight through this thesis, one conjecture from [51] is that interpretable models are all hard to optimize or learn because they are non-differentiable in nature. This something that will be key in our study of decision tree models that we introduce next and that we illustrate in Figure 7.

**Definition 1** (Decision tree)**.** *A decision tree is a rooted tree $T = (V, E)$ where :*

- *Each internal node $v \in V$ is associated with a test function $f_v : \mathcal{X} \to \{0, 1\}$ that maps input features $x \in \mathcal{X}$ to a Boolean.*
- *Each edge $e \in E$ from an internal node corresponds to an outcome of the associated test function.*
- *Each leaf node $\ell \in V$ is associated with a prediction $y_\ell \in \mathcal{Y}$, where $\mathcal{Y}$ is the output space.*

— *For any input $x \in \mathcal{X}$, the tree defines a unique path from root to leaf, determining the prediction $T(x) = y_\ell$ where $\ell$ is the reached leaf.*

*The depth of a tree is the maximum path lentght from root to any leaf.*

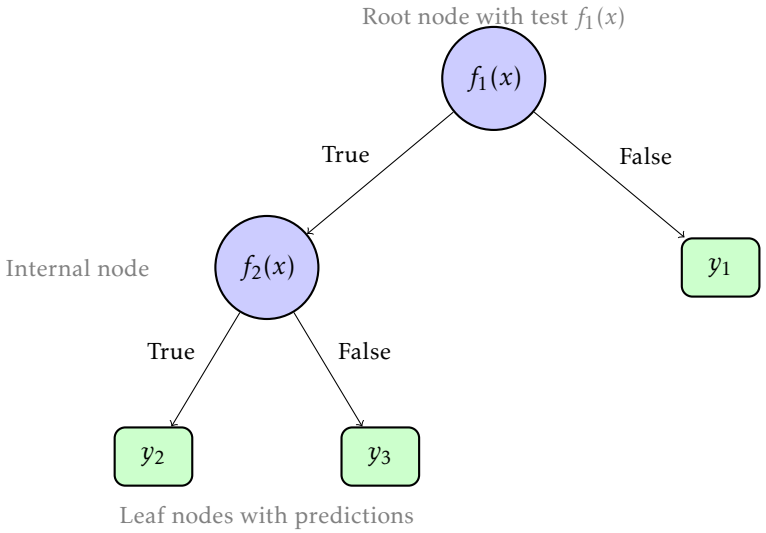



FIGURE 7 – A generic depth 2 decision tree with 3 nodes and 3 leaves. Internal nodes contain test functions $f_v(x) : \mathcal{X} \to \{0, 1\}$ that map input features to boolean values. Edges represent the outcomes of these tests (True/False), and leaf nodes contain predictions $y_\ell \in \mathcal{Y}$. For any input $x$, the tree defines a unique path from root to leaf.

### How to learn decision trees?

Training decision trees to optimize the supervised learning objective 2 is well studied.

**Definition 2** (Supervised learning)**.** *Assume that we have access to a set of $N$ examples denoted $\mathcal{E} = \{(x_i, y_i)\}_{i=1}^N$. Each datum $x_i$ is described by a set of $p$ features. $y_i \in \mathcal{Y}$ is the label associated with $x_i$. For a classification task $\mathcal{Y} = \{1, \ldots, K\}$ and for a regression task $\mathcal{Y} \subseteq \mathbb{R}$. The goal of supervised learning is to find a classifier (or regressor) $f : X \to y$ where $f$ is model in $F$, e.g. neural networks or decision tres.*

$$f^* = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \ell(y_i, f(x_i)) + \alpha C(f), \tag{1}$$

*where $C : \mathcal{F} \to \mathbb{R}$ is a regularization penalty.*

The Classification and Regression Trees (CART) algorithm [20] (Algorithm 1), developed by Leo Breiman and colleagues (Figure 8), is one of the most widely used methods

FIGURE 8 – The american statistician Leo Breiman (1928-2005) author of *Classification and Regression Trees* (1984)

for learning decision trees from supervised data. CART builds binary decision trees through a greedy, top-down approach that recursively partitions the feature space. At each internal node, the algorithm selects the feature and threshold that best splits the data according to a purity criterion such as the Gini impurity for classification or mean squared error for regression. CART uses threshold-based test functions of the form $f_v(x) = \mathbb{I}[x[\text{feature}] \leq \text{threshold}]$, where $\mathbb{I}[\cdot]$ is the indicator function. The key idea is to find splits that maximize the homogeneity of the resulting subsets. We use CART as well as other decision tree algorithms in this manuscript and rely on the scikit-learn implementations [95] for our experiments.

In particular, in the second part we will challenge decision tree algorithms that perform better than CART for the supervised learning objective. In the first and third parts, we study CART in conjunction with reinforcement learning as a means to obtain decision trees for sequential decision making.

In the next few sections we present the material related to sequential decision making.

### Markov decision processes and problems

Markov decision processes (MDPs) were first introduced in the 1950s by Richard Bellman [13]. Informally, an MDP models how an agent acts over time to achieve a goal. At every time step, the agent observes its current state (e.g., patient weight and tumor size) and takes an action (e.g., administers a certain amount of chemotherapy). The agent receives a reward that helps evaluate the quality of the action with respect to the

**Algorithme 1 :** CART for decision tree induction to optimize the supervised learning objective 2

**Data :** Training data $(X, y)$ where $X \in \mathbb{R}^{n \times p}$ and $y \in \{1, 2, \ldots, K\}^n$
**Result :** Decision tree $T$
**Function** `BuildTree(`$X, y$`)` **:**
  **if** *stopping criterion met* **then**
    **return** leaf node with prediction MajorityClass($y$)
  **end**
  $(feature, threshold) \leftarrow$ BestSplit$(X, y)$
  **if** *no valid split found* **then**
    **return** leaf node with prediction MajorityClass($y$)
  **end**
  Split data : $X_{left}, y_{left} = \{(x_i, y_i) : x_i[feature] \leq threshold\}$
    $X_{right}, y_{right} = \{(x_i, y_i) : x_i[feature] > threshold\}$
  $left\_child \leftarrow$ BuildTree$(X_{left}, y_{left})$
  $right\_child \leftarrow$ BuildTree$(X_{right}, y_{right})$
  **return** internal node with test function $f_v(x) = \mathbb{I}[x[feature] \leq threshold]$ and children $(left\_child, right\_child)$
**Function** `BestSplit(`$X, y$`)` **:**
  $best\_gain \leftarrow 0$
  $best\_feature \leftarrow None$
  $best\_threshold \leftarrow None$
  **for** *each feature* $f \in \{1, 2, \ldots, p\}$ **do**
    **for** *each unique value* $v$ *in* $X[:, f]$ **do**
      $y_{left} \leftarrow \{y_i : X[i, f] \leq v\}$
      $y_{right} \leftarrow \{y_i : X[i, f] > v\}$
      $gain \leftarrow \text{Gini}(y) - \frac{|y_{left}|}{|y|}\text{Gini}(y_{left}) - \frac{|y_{right}|}{|y|}\text{Gini}(y_{right})$
      **if** $gain > best\_gain$ **then**
        $best\_gain \leftarrow gain$
        $best\_feature \leftarrow f$
        $best\_threshold \leftarrow v$
      **end**
    **end**
  **end**
  **return** $(best\_feature, best\_threshold)$
**Function** `Gini(`$y$`)` **:**
  **return** $1 - \sum_{k=1}^{K} \left(\frac{|\{i : y_i = k\}|}{|y|}\right)^2$ `// Gini impurity`
**return** BuildTree$(X, y)$

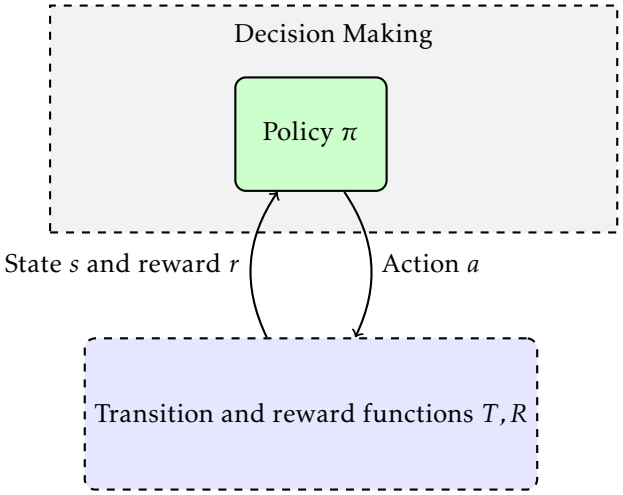



FIGURE 9 – Markov decision process

goal (e.g., tumor size decreases when the objective is to cure cancer). Finally, the agent transitions to a new state (e.g., the updated patient state) and repeats this process over time. Following Martin L. Puterman's book on MDPs[100], we formally define :

**Definition 3** (Markov decision process)**.** *An MDP is a tuple* $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$ *where :*

- $S$ *is a finite set of states* $s \in \mathbb{R}^n$ *representing all possible configurations of the environment.*
- $A$ *is a finite set of actions* $a \in \mathbb{Z}^m$ *available to the agent.*
- $R : S \times A \to \mathbb{R}$ *is the reward function that assigns a real-valued reward to each state-action pair.*
- $T : S \times A \to \Delta(S)$ *is the transition function that maps state-action pairs to probability distributions over next states, where* $\Delta(S)$ *denotes the probability simplex over* $S$*.*
- $T_0 \in \Delta(S)$ *is the initial distribution over states.*

Informally, the goal of a model output actions given states is to behave so as to obtain as much reward as possible over time. For example, in cancer treatment, the best outcome is to eliminate the patient's tumor as quickly as possible. We can formally this goal as an optimization problem as follows.

**Definition 4** (Markov decision problem)**.** *Given an MDP* $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$ *( 3), the objective of a model, aslo known as a policy* $\pi : S \to A$ *is to maximize the expected discounted*

*sum of rewards :*

$$J(\pi) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \mid s_0 \sim T_0, a_t = \pi(s_t), s_{t+1} \sim T(s_t, a_t)\right]$$

*where $\gamma \in (0, 1]$ is the discount factor that controls the trade-off between immediate and future rewards.*

Hence, algorithms presented in this manuscript aim to find solutions to Markov decision problems, i.e., to model the optimal policy : $\pi^\star = \underset{\pi}{\text{argmax}} J(\pi)$. For the rest of this text, by abuse of notation, we denote both a Markov decision process and the associated Markov decision problem by MDP.

### Example : a grid-world MDP

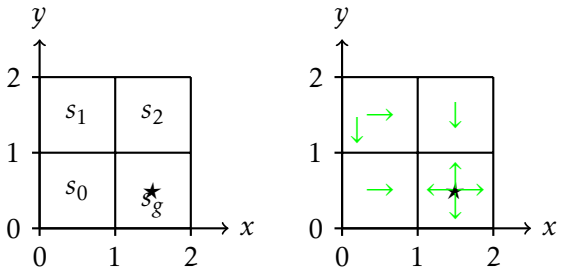



FIGURE 10 – A grid-world MDP (left) and optimal actions w.r.t. the objective 4 (right).

In Figure 10, we present a very simple MDP( 3). This MDP is essentially a grid where the starting state is chosen at random and the goal is to reach the bottom-right cell as fast as possible in order to maximize the objective 4. The state space is discrete with state labels representing 2D-coordinates. The actions are to move up, left, right, or down. The MDP transitions to the resulting cell. Only the bottom-right cell gives reward 1 and is an absorbing state, i.e., once in the state, the MDP stays in this state forever. The optimal actions that get to the goal as fast as possible in every state (cell) are presented in green.

Next we present the tools to find solutions to MDPs and retrieve such optimal policies.

### Exact solutions for Markov decision problems

It is possible to compute the exact optimal policy $\pi^\star$ using dynamic programming[13]. Indeed, one can leverage the Markov property to find for all states the best

action to take based on the reward of upcoming states.

**Definition 5** (Value of a state)**.** *The value of a state $s \in S$ under policy $\pi$ is the expected discounted sum of rewards starting from state s and following policy $\pi$ :*

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \mid s_0 = s, a_t = \pi(s_t), s_{t+1} \sim T(s_t, a_t)\right]$$

*Applying the Markov property gives a recursive definition of the value of s under policy $\pi$ :*

$$V^{\pi}(s) = R(s, \pi(s)) + \gamma \sum_{s' \in S} T(s, \pi(s), s')\, V^{\pi}(s')$$

*where $T(s, a, s')$ is the probability of transitioning to state $s'$ when taking action a in state s.*

**Definition 6** (Optimal value of a state)**.** *The optimal value of a state $s \in S$, $V^{\star}(s)$, is the value of state s when following the optimal policy : $V^{\pi^{\star}}(s)$.*

$$V^{\star}(s) = V^{\pi^{\star}}(s) = \max_{\pi} [J(\pi)]$$

**Definition 7** (Optimal value of a state–action pair)**.** *The optimal value of a state–action pair $(s, a) \in S \times A$, $Q^{\star}(s, a)$, is the value when taking action a in state s and then following the optimal policy.*

$$Q^{\star}(s, a) = R(s, a) + \gamma \sum_{s' \in S} T(s, a, s')\, V^{\star}(s')$$

Hence, objective 4 : $\pi^{\star} = \underset{\pi}{\operatorname{argmax}} \mathbb{E}[V^{\pi}(s_0) | s_0 \sim T_0]$. The well-known Value Iteration algorithm 2 solves this problem exactly.

More realistically, neither the transition kernel $T$ nor the reward function $R$ of the MDP are known ; e.g., the doctor cannot **know** how the tumor and the patient's health will change after a dose of chemotherapy, but can only **observe** the change. This distinction in available information parallels the distinction between dynamic programming and reinforcement learning (RL), described next.

### Reinforcement learning of approximate solutions to MDPs

Reinforcement learning algorithms popularized by Richard Sutton [122](Figure 11) don't **compute** an optimal policy but rather **learn** an approximate one based on sequences of observations $(s_t, a_t, r_t, s_{t+1})_t$. RL algorithms usually fall into two categories : value-based (cite) and policy search [120]. Examples of these approaches are shown in

**Algorithme 2 :** Value Iteration

**Data :** MDP $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$, convergence threshold $\theta$
**Result :** Optimal policy $\pi^*$
Initialize $V(s) = 0$ for all $s \in S$
**repeat**
  $\Delta \leftarrow 0$
  **for** *each state* $s \in S$ **do**
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a [R(s,a) + \gamma \sum_{s' \in S} T(s,a,s')V(s')]$ `// Bellman optimality update`
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
  **end**
**until** $\Delta < \theta$;
**for** *each state* $s \in S$ **do**
  $\pi^*(s) \leftarrow \arg\max_a [R(s,a) + \gamma \sum_{s' \in S} T(s,a,s')V(s')]$ `// Extract optimal policy`
**end**


(a) R. Bellman


(b) M.L. Puterman


(c) A. Barto


(d) R. Sutton

Figure 11 – The godfathers of sequential decision making. Andrew Barto and Richard Sutton are the ACM Turing Prize 2024 laureate and share an advisor advisee relationship.

Algorithms 3, 4 and 5. Q-learning and Sarsa compute an approximation of $Q^\star$ using temporal difference learning. Q-learning is *off policy*, it collects new transisitions with a random policy, e.g. epsilon-greedy, while Sarsa is on policy, it collects new transitions greedily w.r.t. the current q-values estimates. Policy gradient algorithms leverages the policy gradient theorem to approximate $\pi^\star$. The goal of an RL algorithm (also called agent) is to learn a policy that solves a Markov decision problem( 4) without access to transitions and rewards from the MDP. From now on, we refer to 4 as the RL objective.

Q-learning, Sarsa, and Policy gradients are known to converge to the optimal value or (locally) optimal policy under some conditions. The books from Puterman [100], or Sutton and Barto [122], offer a great overview of MDPs and algorithms to solve them. There are many other ways to learn policies such as simple random search [80] or model-based reinforcement learning [121]. However, not many algorithms consider the learning of policies that can be easily understood by humans which we discuss next and that is the core of this manuscript.

**Algorithme 3 :** Value-based RL (Q-Learning)

**Data :** MDP $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$, learning rate $\alpha$, exploration rate $\epsilon$
**Result :** Policy $\pi$
Initialize $Q(s,a) = 0$ for all $s \in S, a \in A$
**for** *each episode* **do**
  Initialize state $s_0 \sim T_0$
  **for** *each step t* **do**
    Choose action $a_t$ using $\epsilon$-greedy : $a_t = \arg\max_a Q(s_t, a)$ with prob. $1 - \epsilon$
    Take action $a_t$, observe $r_t = R(s_t, a_t)$ and $s_{t+1} \sim T(s_t, a_t)$
    $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha[r_t + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t)]$
    $s_t \leftarrow s_{t+1}$
  **end**
**end**
$\pi(s) = \arg\max_a Q(s,a)$ `// Extract greedy policy`

## Deep reinforcement learning for large or continuous state spaces

Reinforcement learning has also been successfully combined with function approximations [123] to solve MDPs with large discrete state spaces or continuous state spaces ($S \subset \mathbb{R}^\varkappa$). Such continuous states MDP can be formalized as factored MDPs[19] :

**Definition 8** (Factored Markov decision process)**.** *A factored MDP is an MDP[***def:mdp***] where the state space is a hyperrectangle :* $S \subseteq \mathbb{R}^n$.

**Algorithme 4 :** Value-based RL (Sarsa)

**Data :** MDP $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$, learning rate $\alpha$, exploration rate $\epsilon$
**Result :** Policy $\pi$
Initialize $Q(s,a) = 0$ for all $s \in S, a \in A$
**for** *each episode* **do**
  Initialize state $s_0 \sim T_0$
  Choose action $a_0$ using $\epsilon$-greedy : $a_0 = \arg\max_a Q(s_0, a)$ with prob. $1 - \epsilon$
  **for** *each step t* **do**
    Take action $a_t$, observe $r_t = R(s_t, a_t)$ and $s_{t+1} \sim T(s_t, a_t)$
    Choose action $a_{t+1}$ using $\epsilon$-greedy : $a_{t+1} = \arg\max_a Q(s_{t+1}, a)$ with prob. $1 - \epsilon$
    $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha[r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)]$
    $s_t \leftarrow s_{t+1}$
    $a_t \leftarrow a_{t+1}$
  **end**
**end**
$\pi(s) = \arg\max_a Q(s,a)$ `// Extract greedy policy`

**Algorithme 5 :** Policy Gradient RL (REINFORCE)

**Data :** MDP $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$, learning rate $\alpha$, policy parameters $\theta$
**Result :** Policy $\pi_\theta$
Initialize policy parameters $\theta$
**for** *each episode* **do**
  Generate trajectory $\tau = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots)$ following $\pi_\theta$
  **for** *each time step t in trajectory* **do**
    $G_t \leftarrow \sum_{k=t}^{T} \gamma^{k-t} r_k$ `// Compute return`
    $\theta \leftarrow \theta + \alpha G_t \nabla_\theta \log \pi_\theta(a_t|s_t)$ `// Policy gradient update`
  **end**
**end**

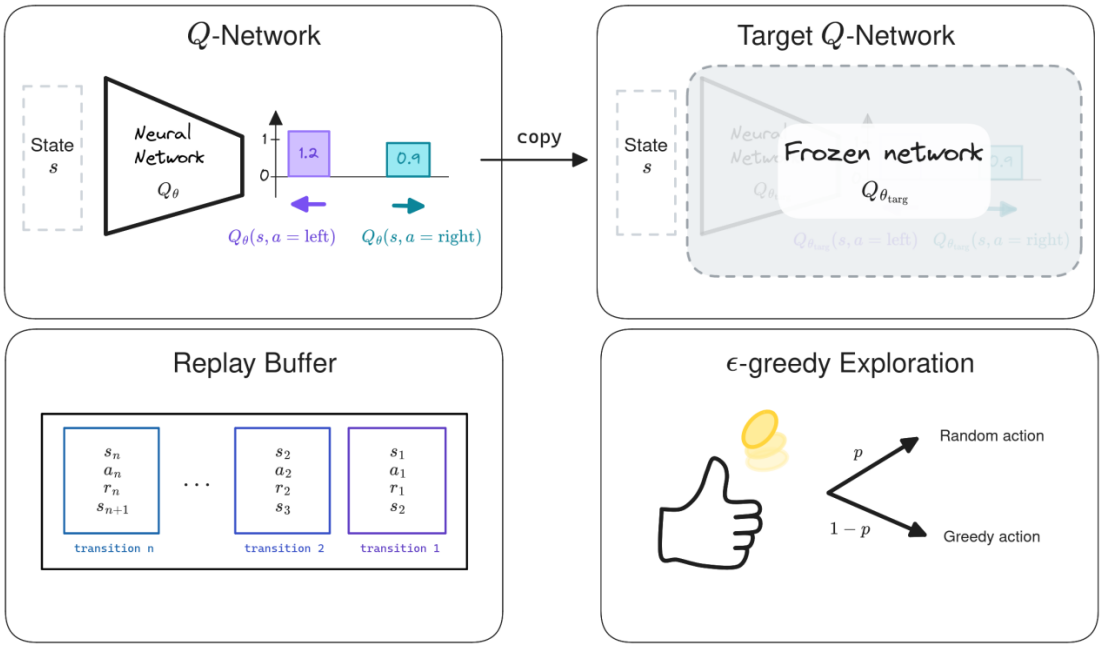



FIGURE 12 – DQN tricks added to the vanilla Q-learning (Alg. 3). Courtesy of Antonin Raffin.

From now on, all the MDPs we study in the this manuscript are factored unless stated otherwise. We also consider that the Example MDP(10) is continuous with two state dimensions. In the rest of this manuscript, unless stated otherwise, we write $\boldsymbol{s}$ a state vector in a continuous space. Note that discrete states MDP can be encoded into factored MDP by one-hot encoding individual states.

Deep Q-Networks (DQN) [89], described in Algorithm 6 was a breakthrough in modern reinforcement learning. Authors successfully extended the Q-learning (Algorithm 3) to the function approximation setting by introduction target networks to mitigate distributional shift in the td error and replay buffer to increase sample efficiency (see Figure **??** for the tricks used to adap Q-learning to the function approximation setting). DQN achieved super-human performance on a set of Atari games.

Proximal Policy Optimization (PPO) [111], described in Algorithm 7, is an actor-critic algorithm[120] optimizing neural network policies. Actor-critic algorithms are instances of policy gradient algorithms where the cumulative discounted rewards–the returns–are also estimated with a neural network. Actor-critic are not as simple efficient as DQN but is known to work well in a variety of domains including robot control in simulation among others.

**Algorithme 6 :** Deep Q-Network (DQN)

**Data :** MDP $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$, learning rate $\alpha$, exploration rate $\epsilon$, Q-network parameters $\theta$, update frequency $C$
**Result :** Policy $\pi$
Initialize Q-network parameters $\theta$ and target network parameters $\theta^- = \theta$
Initialize replay buffer $\mathcal{B} = \emptyset$
**for** *each episode* **do**
  Initialize state $s_0 \sim T_0$
  **for** *each step* $t$ **do**
    Choose action $a_t$ using $\epsilon$-greedy : $a_t = \arg\max_a Q_\theta(\boldsymbol{s}_t, a)$ with prob. $1 - \epsilon$
    Take action $a_t$, observe $r_t = R(s_t, a_t)$ and $\boldsymbol{s}_{t+1} \sim T(\boldsymbol{s}_t, a_t)$
    Store transition $(\boldsymbol{s}_t, a_t, r_t, \boldsymbol{s}_{t+1})$ in $\mathcal{B}$
    Sample random batch $(\boldsymbol{s}_i, a_i, r_i, \boldsymbol{s}_{i+1}) \sim \mathcal{B}$
    $y_i = r_i + \gamma \max_{a'} Q_{\theta^-}(\boldsymbol{s}_{i+1}, a')$ // Compute target
    $\theta \leftarrow \theta - \alpha \nabla_\theta (Q_\theta(\boldsymbol{s}_i, a_i) - y_i)^2$ // Update Q-network
    **if** $t$ mod $C = 0$ **then**
      $\theta^- \leftarrow \theta$ // Update target network
    **end**
    $\boldsymbol{s}_t \leftarrow \boldsymbol{s}_{t+1}$
  **end**
**end**
$\pi(\boldsymbol{s}) = \arg\max_a Q_\theta(\boldsymbol{s}, a)$ // Extract greedy policy

**Algorithme 7 :** Proximal Policy Optimization (PPO)

**Data :** MDP $\mathcal{M} = \langle S, A, R, T, T_0 \rangle$, learning rate $\alpha$, policy parameters $\theta$, clipping parameter $\epsilon$, value function parameters $\phi$
**Result :** Policy $\pi_\theta$
Initialize policy parameters $\theta$ and value function parameters $\phi$
**for** *each episode* **do**
  Generate trajectory $\tau = (\boldsymbol{s}_0, a_0, r_0, \boldsymbol{s}_1, a_1, r_1, \ldots)$ following $\pi_\theta$
  **for** *each time step t in trajectory* **do**
    $G_t \leftarrow \sum_{k=t}^{T} \gamma^{k-t} r_k$ // Compute return
    $A_t \leftarrow G_t - V_\phi(\boldsymbol{s}_t)$ // Compute advantage
    $r_t(\theta) \leftarrow \frac{\pi_\theta(a_t|\boldsymbol{s}_t)}{\pi_{\theta_{old}}(a_t|\boldsymbol{s}_t)}$ // Compute probability ratio
    $L_t^{CLIP} \leftarrow \min(r_t(\theta)A_t, \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)A_t)$ // Clipped objective
    $\theta \leftarrow \theta + \alpha \nabla_\theta L_t^{CLIP}$ // Policy update
    $\phi \leftarrow \phi + \alpha \nabla_\phi (G_t - V_\phi(\boldsymbol{s}_t))^2$ // Value function update
  **end**
  $\theta_{old} \leftarrow \theta$ // Update old policy
**end**

In this manuscript we study those two deep reinforcement learning algorithms for various problems and use their stable-baselines3 implementations[105].

### Imitation learning : a baseline (indirect) interpretable reinforcement learning method

Unlike PPO or DQN for neural networks, there does not exist an algorithm that trains decision tree policies to optimize the RL objective (4). In fact, we will show in the first part of the manuscript that training decision trees that optimize the RL objective is very difficult.

Hence, many interpretable reinforcement learning approaches first train a neural network policy with, e.g., DQN or PPO to optimize the RL objective (4), and then fit, e.g., a decision tree using CART (Alg.1) to optimize the supervised learning objective (2) with the neural policy's actions as targets. This approach is known as imitation learning and is essentially training a policy to optimize the objective :

**Definition 9** (Imitation learning)**.** *Given an MDP $\mathcal{M}$ (3) expert policy $\pi^*$ and a policy class $\Pi$, the imitation learning objective is to find a policy $\pi \in \Pi$ that minimizes the expected action disagreement with the expert :*

$$\min_{\pi \in \Pi} \mathbb{E}_{s \sim \rho(s)} [\mathcal{L}(\pi(s), \pi^*(s))] \tag{2}$$

*where $\rho(s)$ is the state distribution in $\mathcal{M}$ induced by the expert policy and $\mathcal{L}$ is a loss function measuring the disagreement between the learned policy's action $\pi(s)$ and the expert's action $\pi^*(s)$.*

There are two main imitation learning methods used for interpretable reinforcement learning. DAgger ([131]; Algorithm 8) is a straightforward way to fit a decision tree policy to optimize the imitation learning objective9. VIPER ([11]; Algorithm 9) was designed specifically for interpretable reinforcement learning. VIPER reweights the transitions collected by the neural network expert by a function of the state–action value4. The authors of VIPER showed that decision tree policies fitted with VIPER tend to have the same RL objective value as DAgger trees while being more interpretable (shallower or with fewer nodes) and sometimes outperform DAgger trees. DAgger and VIPER are two strong baselines for decision tree learning in MDPs, but they optimize a surrogate objective only, even though in practice the resulting decision tree policies often achieve high RL objective value.

We use the two algorithms extensively throughout the manuscript.

Next we show how to learn a decision tree policy for the Example MDP 10.

**Algorithme 8 :** DAgger[109]

**Input :** Expert policy $\pi^*$, MDP $M$, policy class $\Pi$
**Output :** Fitted student policy $\hat{\pi}_i$
Initialize dataset $\mathcal{D} \leftarrow \emptyset$;
Initialize $\hat{\pi}_1$ arbitrarily from $\Pi$;
**for** $i \leftarrow 1$ **to** $N$ **do**
  **if** $i = 1$ **then** $\hat{\pi} \leftarrow \pi^*$ ;
  **else** $\pi_i \leftarrow \hat{\pi}_i$ ;
  Sample transitions from $M$ using $\hat{\pi}$;
  Collect dataset $\mathcal{D}_i \leftarrow \{(\boldsymbol{s}, \pi^*(\boldsymbol{s}))\}$ of states visited by $\hat{\pi}$;
  $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}_i$;
  Fit classifier/regressor $\hat{\pi}_{i+1}$ on $\mathcal{D}$;
**end**
**return** $\hat{\pi}$;

**Algorithme 9 :** VIPER[11]

**Input :** Expert policy $\pi^*$, Expert Q-function $Q^*$, MDP $M$, policy class $\Pi$
**Output :** Fitted student policy $\hat{\pi}_i$
Initialize dataset $\mathcal{D} \leftarrow \emptyset$;
Initialize $\hat{\pi}_1$ arbitrarily from $\Pi$;
**for** $i \leftarrow 1$ **to** $N$ **do**
  **if** $i = 1$ **then** $\hat{\pi} \leftarrow \pi^*$ ;
  **else** $\pi_i \leftarrow \hat{\pi}_i$ ;
  Sample transitions from $M$ using $\hat{\pi}$;
  Weight each transition by $w(\boldsymbol{s}) \leftarrow V^{\pi^*}(\boldsymbol{s}) - \min_a Q^{\pi^*}(\boldsymbol{s}, a)$;
  Collect dataset $\mathcal{D}_i \leftarrow \{(\boldsymbol{s}, \pi^*(\boldsymbol{s}), w(\boldsymbol{s}))\}$ of states visited by $\hat{\pi}$;
  $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}_i$;
  Fit classifier/regressor $\hat{\pi}_{i+1}$ on the weighted dataset $\mathcal{D}$;
**end**
**return** $\hat{\pi}$;

## Your first decision tree policy

Now the reader should know how to train decision tree classifiers or regressors for supervised learning using CART4. The reader should also know what an MDP is and how to compute or learn policies that optimize the RL objective 4 with (deep) reinforcement learning4. Finally, the reader should now know how to obtain a decision

tree policy for an MDP through imitation learning by first using RL to get an expert policy and then fitting decision trees to optimize the supervised learning objective2, using the expert's actions as labels. Note that, in theory, there is no guarantee that such decision tree policies also maximize the RL objective ; they optimize only the imitation learning objective9.

In this section we present the first decision tree policies of this manuscript obtained using DAgger or VIPER after learning an expert policy and expert Q-function for the grid-world MDP 10 with Q-learning3. Recall the optimal policies for the grid-world, taking the green actions in each state in Figure10. Among the optimal policies, the ones that take action to go left or up in the goal state can be problematic for imitation learning algorithms.

Indeed, we know that for this grid-world MDP there exists a decision tree policy that is optimal and very interpretable (depth-1) : go right if the $x$-label of the state is greater than 1 and go left otherwise. This tree takes exactly the same actions in the same states as some of the optimal policy from Figure10.

In Figure13, we present a depth-1 decision tree policy that is optimal w.r.t. the RL objective and a depth-1 tree that is suboptimal. Indeed, Figure3.3 shows that the optimal depth-1 tree achieves exactly the same RL objective value as the optimal policies from Figure10, independently of the discount factor $\gamma$.

Now a fair question is : can DAgger or VIPER retrieve such an optimal depth-1 tree given access to an expert optimal policy from Figure10?

We start by running the standard Q-learning algorithm as presented in Algorithm3 with $\epsilon = 0.3$, $\alpha = 0.1$ over 10,000 time steps. The careful reader might wonder how ties are broken in the argmax operation from Algorithm3. While Sutton and Barto break ties by index value in their book[122] (the greedy action is the  action with smallest index), we show that the choice of tie-breaking greatly influences the performance of subsequent imitation learning algorithms. Indeed, depending on how actions are ordered in practice, Q-learning may be biased toward some optimal policies rather than others. While this does not matter for one who just wants to find an optimal policy, in our example of finding the optimal depth-1 decision tree policy, it matters *a lot*.

In the left plot of Figure15, we see that Q-learning, independently of how ties are broken, consistently converges to an optimal policy over 100 runs (random seeds). However, in the right plot of Figure15, where we plot the proportion over 100 runs of optimal decision trees returned by DAgger or VIPER at different stages of Q-learning, we observe that imitating the optimal policy obtained by breaking ties at random consistently yields more optimal trees than breaking ties by indices. What actually

happens is that the most likely output of Q-learning when ties are broken by indices is the optimal policy that goes left in the goal state, which cannot be perfectly represented by a depth-1 decision tree, because there are three different actions taken and a binary tree of depth $D = 1$ can only map to $2^D = 2$ labels.

Despite this negative result, we still find that VIPER almost always finds the optimal depth-1 decision tree policy in terms of the RL objective4 when ties are broken at random. However this sheds light on the sub-optimality of indirect imitation w.r.t the RL objective ( 4).

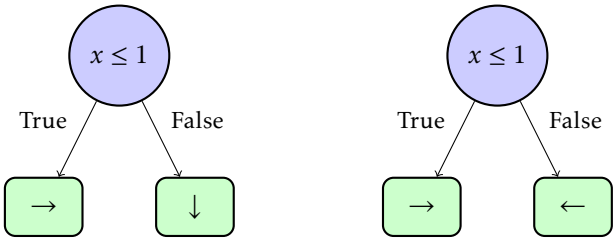



Figure 13 – Left, a sub-optimal depth-1 decision tree policy. On the right, an optimal depth-1 decision tree policy.

Now that the reader has seen how to get an interpretable policy for an MDP, we believe it is ready to dive into the contributions of this thesis.

## Outline of the thesis

Throughout our thesis, we make the assumption that constraing models (e.g. policies or classifiers) to decision trees is engouh ensuring interpretability. In this thesis we study different decision tree learning algorithms in different settings. In the first part of the manuscript, we show that direct decision tree learning methods 6 struggle to find decision tree policies even for very simple sequential decision making problems. For that, we first reproduce the work from[125] that presents a formalism for learning decision tree policies that optimize 4 directly using reinforcement learning, and then make connexions with hardness results from the partially observable MDPs (POMDPs)[119, 41] litterature. We defer the introduction of POMDPs to later sections. In the second part of the manuscript, we formulate decision tree induction for supervised learning as solving a sequential decision making problem. By formalizing decision tree induction for objective2 as solving an MDP 3, we design novel algorithms that achieve very good performances.

In the last part of the text, we lift our assumption about decision tree learning gua-

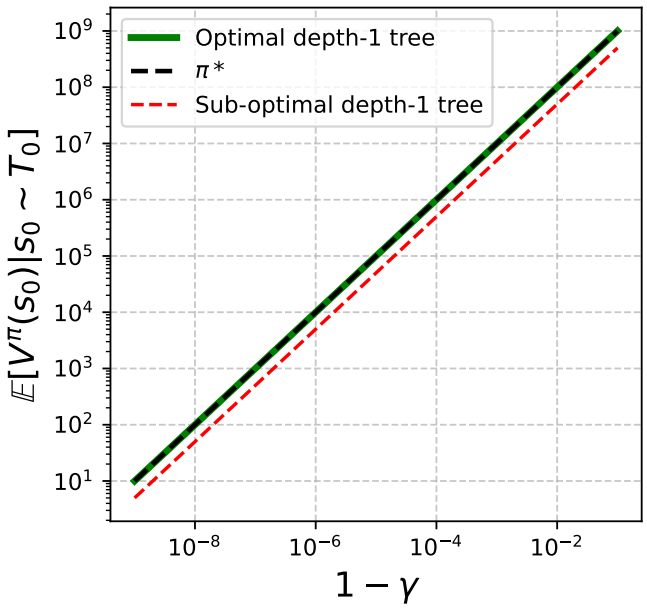



FIGURE 14 – The objective (4) values of the optimal policies from Figure 10 and of the decision tree policies from Figure 13.

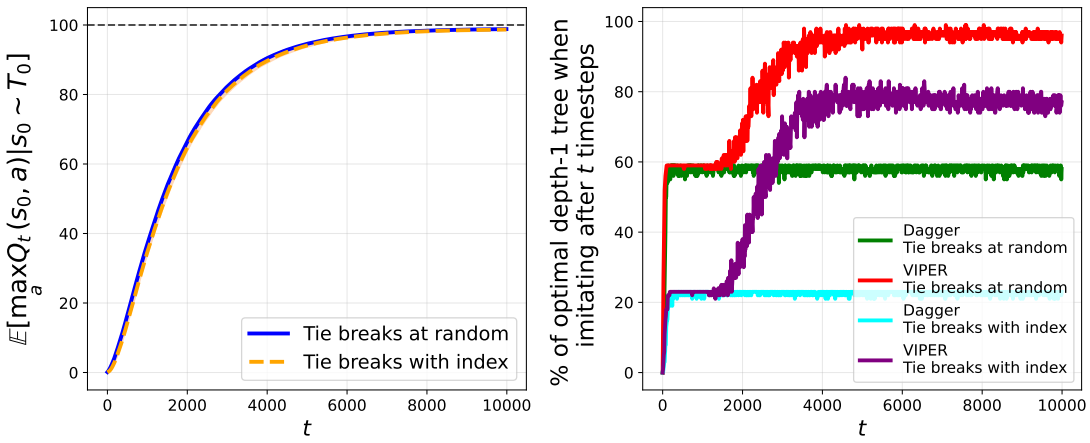



FIGURE 15 – Left, sample complexity curve of Q-learning with default hyperparameters on the $2 \times 2$ grid-world MDP over 100 random seeds. Right, performance of indirect interpretable methods when imitating the greedy policy with a tree at different Q-learning stages.

ranteeing interpretability and study other model classes. In particular, we leverage the simplicity of *indirect* methods **??** to imitate neural network experts with models from 4 and and perform a large-scale empirical study of the interpretability-performances trade-offs on various sequential decision making tasks. In Table 1 we summarize the outline of this manuscript in terms of learning objective ( 2, **??**, or 9) and model class (Figure 4).

| | **Decision Trees** | **Linear** | **Ensembles** | **Neural Networks** |
|---|---|---|---|---|
| Supervised Learning | Part II | | Part II | Part II |
| Reinforcement Learning | Part I, III | Part III | | Part I, III |
| Imitation Learning | Part I, III | Part III | | Part III |

Tableau 1 – Summary of objectives and model classes studied in this manuscript.

We summarize our results as follows :

1. Direct reinforcement learning of decision tree policies is hard because it involves POMDPs.
2. One can use Dynamic Programming in MDPs to induce highly performing decision tree classifiers and regressors.
3. In practice, controlling MDPs with interpretable policies does not necessarily decrease performances.

# Première partie

# A Difficult Problem : Direct Interpretable Reinforcement Learning

Chapitre 1

# Introduction

In the first part of the manuscript, we show that direct reinforcement learning of decision tree policies for MDPs, i.e. learning a decision tree that optimizes (4) is often very difficult. In particular, we provide some insights as to why it is so difficult and show that indirect imitation of a neural network policy (Sec. 4), while not optimizing (4) directly, often yields very good tree policies.

This first part of the manuscript is organised as follows. In this Chapter we present, Topin et. al. [125] framework for direct reinforcement learning of decision tree policies. In Chapter 2, we reproduce experiments from Topin et. al. where we compare direct deep reinforcement learning (Sec. 4) of decision tree policies to indirect imitation of a neural network policy with a decision tree for the simple CartPole MDP. In Chapter 3, we show that the direct approach proposed by Topin et. al. is equivalent to learning a deterministic memoryless policy for partially observable MDP (POMDP)[119, 41] and show that this might be the biggest source of hardness. In Chapter 4, we further support this claim by constructing special instances of Topin et. al. in which the POMDPs are virtually fully observable, and show that in those cases, direct reinforcement learning of decision tree works well.

## 1.1 Learning Decision Tree policies for MDPs

There already exist algorithms that return decision tree policies for MDPs. Those algorithms either learn parametric trees or non-parametric trees.

Parametric trees are not “grown” from the root by iteratively adding internal or leaf nodes (c.f. Fig. 7), but are rather “optimized” : the depth, internal nodes arrengement,

and state-features to consider in each nodes are fixed *a priori* and only the thresholds of each nodes are learned similar to doing gradient descent on neural network weights. As the reader might have guessed, those parametric trees are advantageous in that they can be learned with the policy gradient [120]. Both [115], [136] and [85] use PPO [111] to optimize such differentiable decision trees. In particular, [85] explicitely studies the gap in performances between their direct optimization and the indirect imitation. While those methods optimize the RL objective (4) directly and train high-performing policies w.r.t the downstream tasks, in general a user cannot know *a priori* what a "good" tree policy structure should be for a particular MDP : either the specified structure is too deep and pruning will be required after training or the tree structure is not expressive enough to encode a good policy, i.e. parametric trees cannot trade off interpretability and performances during training. Furthermore, authors from [85] show that extra stabilizing tricks, such as adaptive batch sizes, are required during training to outperform indirect imitation in terms of RL objective (4).

Non-parametric trees are the standard model in supervised learning [20, 103, 102] and offer good trade-off between interpretability and performances when optimizing (2). On the other hand, to the best of our knowledge, there exists onlye one work studying non-parametric trees to optimize a trade-off between interpretability and the RL objective (4) : Topin et. al. [125].

Given an MDP for which one wants to learn a decision tree policy, Topin et. al. introduced iterative bouding Markov decision processes (IBMDPs) that are an augmented version of this base MDP with more state features, more actions, additional rewards that trade-off between the RL objective and number of added nodes, and additional transitions. Authors showed that using RL to learn certain policies in IBMDPs is equivalent to growing a tree that optimizes a trade-off between interpretability and the RL objective (4) w.r.t the MDP of interest.

There also exists more specialized approaches that can return decision tree policies only for small problem classes. In [82], authors prove that for maze-like MDPs, there always exist an optimal decision tree policy w.r.t 4 and provide an algorithm to find it. Finally, in [134], authors study decision tree policies for planning in MDPs, i.e., when the transitions and rewards are known. In the next section we present IBMDPs as introduced in Topin et. al.[125].

## 1.2 Iterative Bounding Markov Decision Processes

The key thing to know about IBMDPs is that they are, as their name suggests, MDPs. Hence, IBMDPs admits an optimal deterministic Markovian policy for problem (4). In this part we will assume that all the MDP we consider are factored MDPs **??** with a finite set of actions, so we use bold fonts for states and observations as they are vector-valued. However all our results generalize to discrete states (in $\mathbb{Z}^m$) MDP that we can factor using one-hot encodings. Given an MDP for which we want to learn a decision tree policy, that we call the base MDP, the states in an associated IBMDP are concatenations of the base MDP states and some observations. Those observations are information about the base states that are refined–"iteratively bounded"– at each step and represent a subspace of the base MDP state space. Actions available in an IBMDP are the actions of the base MDP, that change the state of the latter, and *information gathering* actions that change the observation part of the IBMDP state. Now, taking base actions in an IBMDP is rewarded like in the base MDP, this ensures that the base objective, e.g. balancing the pole or treating cancer, is still encoded in the IBMDP reward. When taking an information gathering action, the reward is an arbitrary value such that optimizing (4) in the IBMDP is equivalent to optimizing some trade-off between 4 in the base MDP and interpretability.

Before showing how to get decision tree policies from IBMDP policies, we give a formal definition of IBMDPs following Topin et. al. [125].

**Definition 10** (Iterative Bounding Markov decision process)**.** *Given a factored MDP $\mathcal{M}$ $\langle S, A, R, T, T_0\rangle$ (**??**), an associated iterative bouding Markov decision process $\mathcal{M}_{IB}$ is a tuple :*

$$\langle \overbrace{S \times O}^{\text{State space}}, \underbrace{A \cup A_{info}}_{\text{Action space}}, \overbrace{(R, \zeta)}^{\text{Reward function}}, \underbrace{(T_{info}, T, T_0)}_{\text{Transitions}} \rangle$$

- *$S$ is the base MDP factored state space. A state $\boldsymbol{s} = (s_1, \ldots, s_n) \in S$ has $n$ bounded features $s_i \in [L_i, U_i]$ where $\infty < L_i \leq U_i < \infty \forall 1 \leq i \leq n$.*
- *$O$ are the observations in an IBMDP. They are partial information about the base MDP state features. The set of observations is the current features bounds : $O \subsetneq S^2 = [L_1, U_1] \times \cdots \times [L_n, U_n] \times [L_1, U_1] \times \cdots \times [L_n, U_n]$. So the complete IBMDP state space is $S \times O$, the concatenations of base state features and observations.*
- *$A$ is the base MDP actions set.*
- *$A_{info}$ are information gathering actions (IGAs) of the form $\langle i, v\rangle$ where $i$ is a state*

*feature index* $1 \leq i \leq n$ *and* $v$ *is a real number. So the complete action space of an IBMDP is the set of base MDP actions and information gathering actions* $A \cup A_{info}$.

— $R : S \times A \to \mathbb{R}$ *is the base MDP reward function.*

— $\zeta$ *is a reward signal for taking an information gathering action. So the IBMDP reward function is to get a reward from the base MDP if the action is a base MDP action or to get* $\zeta$ *if the action is an IGA action.*

— $T_{info} : S \times O \times (A_{info} \cup A) \to \Delta(S \times O)$ *is the transition kernel of IBMDPs : Given some observation* $\boldsymbol{o}_t = (L'_1, U'_1, \ldots, L'_n, U'_n) \in O$ *and state features* $\boldsymbol{s}_t = (s'_1, s'_2, \ldots, s'_n)$ *if an IGA* $\langle i, v \rangle$ *is taken, the new observation is :*

$$\boldsymbol{o}_{t+1} = \begin{cases} (L'_1, U'_1, \ldots, L'_i, \min\{v, U'_i\}, \ldots, L''_n, U'_n) \text{ if } s_i \leq v \\ (L'_1, U'_1, \ldots, \max\{v, L'_i\}, U'_i, \ldots, L''_n, U'_n) \text{ if } s_i > v \end{cases}$$

*If a base action* $a_t \in A$ *is taken, the new observation is reset to the default state bounds* $(L_1, U_1, \ldots, L_n, U_n)$ *and the state features change according to the base MDP transitition kernel :* $\boldsymbol{s}_{t+1} \sim T(\boldsymbol{s}_t, a_t)$. *At initialization, the base state features are drawn from the base MDP* $T_0$ *and the observation is always set to the default state features bounds* $\boldsymbol{o}_0 = (L_1, U_1, \ldots, L_n, U_n)$.

Now remains to extract a decision tree policy for MDP $\mathcal{M}$ from a policy for an associated IBMDP $\mathcal{M}_{IB}$.

### 1.2.1 From Policies to Trees

One can notice that information gathering actions (10) resemble the Boolean functions that make up internal decision tree nodes (c.f. Figure 7). Indeed, a policy taking actions in an IBMDP essentially builds a tree by taking sequences of IGAs (internal nodes) and then a base action in the base MDP (leaf node) and repeats this process over time. In particular, the IGA rewards $\zeta$ can be seen as a regularization or a penalty for interpretabiliy : if $\zeta$ is very low compared to base rewards, a policy will try to act in the base MDP as often as possible, i.e. build shallow trees that short paths between root and leaves.

Authors from [125] show that not all IBMDP policies are decision tree policies. In particular, they show that only deterministic policies depending solely on the observation part of the IBMDP states are guaranteed to correspond to decision tree policies for the base MDP. The intuition is that if one trains a policy $\pi : S \to A \cup A_{info}$, the policy might learn to rely only on state features of the base MDP $\boldsymbol{s}$ and take only base actions

**Algorithme 10 :** Extract a Decision Tree Policy

**Data :** Deterministic partially observable policy $\pi_{po}$ for IBMDP
$\langle S \times O, A \cup A_{info}, (R, \zeta), (T_{info}, T, T_0) \rangle$ and observation IBMDP
$\boldsymbol{o} = (L'_1, U'_1, \ldots, L'_n, U'_n)$
**Result :** Decision tree policy $\pi_{\mathcal{T}}$ for MDP $\langle S, A, R, T, T_0 \rangle$
**Function** `Subtree_From_Policy(`$\boldsymbol{o}, \pi_{po}$`)` **:**
  $a \leftarrow \pi_{po}(\boldsymbol{o})$
  **if** $a \in A_{info}$ **then**
    **return** Leaf_Node(action : $a$) `// Leaf if base action`
  **end**
  **else**
    $\langle i, v \rangle \leftarrow a$ `// Splitting action is feature and value`
    $\boldsymbol{o}_L \leftarrow \boldsymbol{o}$; $\boldsymbol{o}_R \leftarrow \boldsymbol{o}$
    $\boldsymbol{o}_L \leftarrow (L'_1, U'_1, \ldots, L'_i, v, \ldots, L'_n, U'_n)$; $\boldsymbol{o}_R \leftarrow (L'_1, U'_1, \ldots, v, U'_i, \ldots, L'_n, U'_n)$
    $child_L \leftarrow$ Subtree_From_Policy($\boldsymbol{o}_L, \pi_{po}$)
    $child_R \leftarrow$ Subtree_From_Policy($\boldsymbol{o}_R, \pi_{po}$)
    **return** Internal_Node(feature : $i$, value : $v$, children : $(child_L, child_R)$)
  **end**

(no IGAs) which would simply be any policy for the base MDP.

**Proposition 1** (Deterministic partially observable IBMDP policies are decision trees)**.** *Given a factored MDP $\mathcal{M}$ $\langle S, A, R, T, T_0 \rangle$ (**??**) and an associated IBMDP $\mathcal{M}_{IB}$ $\langle S \times O, A \cup A_{info}, (R, \zeta), (T_{info}, T, T_0) \rangle$ (10), a deterministic partially observable policy $\pi_{po} : O \to A \cup A_{info}$ is a decision tree policy $\pi_{\mathcal{T}} : S \to A$ for the base MDP $\mathcal{M}$.*

*Démonstration.* (Sketch) Algorithm 10 that takes as input a deterministic partially observable policy (1) for an IBMDP $\mathcal{M}_{IB}$ $\langle S \times O, A \cup A_{info}, (R, \zeta), (T_{info}, T, T_0) \rangle$ (10), returns a decision tree policy $\pi_{\mathcal{T}}$ (**??**) for $\mathcal{M}$ $\langle S, A, R, T, T_0 \rangle$ and always terminates unless the deterministic partially observable policy only takes IGAs. □

While the connexions with partially observable MDPs [119, 41] is obvious, we defer the implications to Chapter 3 as this connexion was not make in the original IBMDP paper [125]. Next we present an example of an IBMDP policy that is a decision tree for the base MDP.

### 1.2.2 Example : an IBMDP for a grid world

For the sake of example, we re-formulate the example MDP (10) as a factored MDP with a finite number of vector valued states ($x, y$-coordinates). The states are

$S = \{(0.5, 0.5), (0.5, 1.5), (1.5, 1.5), (1.5, 0.5)\} \subsetneq [0, 2] \times [0, 2]$. The actions are the cardinal directions $A = \{\rightarrow, \leftarrow, \downarrow, \uparrow\}$ that shift the states by one as long as the coordinates remain in the grid. The reward for taking any action is 0 except when in the bottom right state $(1.5, 0.5)$ which is an absorbing state : once in this state, you stay there forever. Optimal deterministic tabular policies were presented for this MDP in Example (10).

Suppose an associated IBMDP (10) with two IGAs :

— $\langle x, 1 \rangle$ that tests if $x \leq 1$

— $\langle y, 1 \rangle$ that tests if $y \leq 1$

The initial observation is always the grid bounds $\boldsymbol{o}_0 = (0, 2, 0, 2)$ because a state in the grid world is always in $[0, 2] \times [0, 2]$. There are only finitely many observations since with those two IGAs there are only nine possible observations that can be attained from $\boldsymbol{o}_0$ following the IBMDP transitions (10). For example when the IBMDP initial state features are $\boldsymbol{s}_0 = (0.5, 1.5)$, and taking first $\langle x, 1 \rangle$ then $\langle y, 1 \rangle$ the corresponding observations are first $\boldsymbol{o}_{t+1} = (0, 1, 0, 2)$ and then $\boldsymbol{o}_{t+2} = (0, 1, 1, 2)$. The full observation set is $O = \{(0, 2, 0, 2), (0, 1, 0, 2), (0, 2, 0, 1), (0, 1, 0, 1), (1, 2, 0, 2), (1, 2, 0, 1), (1, 2, 1, 2), (0, 1, 1, 2), (0, 2, 1, 2)\}$. The transitions and rewards are given by definition (10).

In Figure (1.1) we show a trajectory in this IBMDP.

## 1.3 Summary

In this chapter, we presented the approach of Topin et. al. [125] to find decision tree policies that directly trade off interpretability and the RL objective (4) rather than the surrogate imitation loss (9). To achieve that Topin et. al. showed that partially observable deterministic policy in some MDP, an IBMDP (10).

The promise of Topin et. al. is that optimizing (4) in an IBMDP trades off naturally the base MDP rewards and the interpretability of the policy through the reward signal $\zeta$.

We can thus write an interpretable RL objective as follows :

**Definition 11** (Interpretable RL objective)**.** *Given a factored MDP $\mathcal{M}\langle S, A, R, T, T_0 \rangle$ for which we want an interpretable policy, e.g. a decision tree, a discount factor $\gamma \in (0, 1]$, some interpretability penalty $\zeta$, and a set of information gathering actions $A_{info}$, we solve :*

$$\pi^{\star}_{po} = \underset{\pi_{po}}{argmax} \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R((\boldsymbol{s}_t, \boldsymbol{o}_t), a_t) \mid \boldsymbol{s}_0 \sim T_0, a_t = \pi_{po}(\boldsymbol{o}_t), \boldsymbol{s}_{t+1} \sim T(\boldsymbol{s}_t, a_t), \boldsymbol{o}_{t+1} \sim T(\boldsymbol{o}_t, a_t)\right]$$
$$= \underset{\pi_{po}}{\text{argmax}} \mathbb{E}[V^{\pi_{po}}(s_0, o_0) | s_0 \sim T_0]$$

*With $V^{\pi_{po}}$ the value function (5) of partially observable deterministic policy $\pi_{po} : O \rightarrow A \cup A_{info}$ (1) in the IBMDP $\mathcal{M}_{IB} \langle S \times O, A \cup A_{info}, (R, \zeta), (T_{info}, T, T_0) \rangle$ (10).*

After optimizing objective (11) with, e.g. reinforcement learning, we thus use Algorithm 10 to obtain a decision tree policy that trades off between interpretability and the RL objectvie (4) for our MDP of intereset. This is exactly what we do in the next chapters.

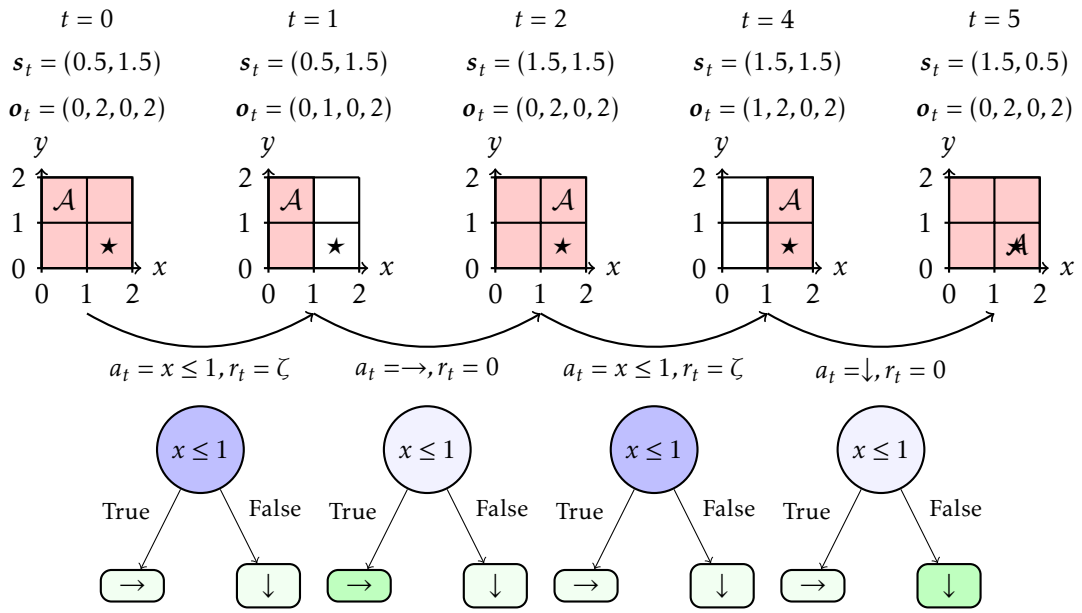



FIGURE 1.1 – An IBMDP trajectory when the base MDP is 2×2 grid world, and equivalent decision tree policy traversal. $\mathcal{A}$ indicates the current state $\boldsymbol{s}_t$ in the grid. The pink obstructions of the grid represent the current partial observations $\boldsymbol{o}_t$ of the state features. When the pink covers the whole grid, the information contained in the observation is "the current state could be anywhere in the grid". The more Information gathering actions are taken, the more refined the bounds on the current state features get. At $t = 0$, the state features are $\boldsymbol{s}_0 = (0.5, 1.5)$. The initial observation is always the base MDP default feature bounds, here $\boldsymbol{o}_0 = (0, 2, 0, 2)$ because the base states are in $[0, 2] \times [0, 2]$. The first action is an IGA that tests the feature $x$ of the states against the value 1 and the reward $\zeta$. This transition corresponds to going through an internal node in a decision tree policy as illustrated in the figure. At $t = 1$, after gathering the information that the $x$-value of the current base state is below 1, the observation is updated with the refined state bounds $\boldsymbol{o}_1 = (0, 1, 0, 2)$, i.e. the pink area shrinks, and the base state features remain unchanged. The agent then takes a base action that is to move right. This gives a reward 0, resets the observation to the original bounds, and changes the base state to $\boldsymbol{s}_2 = (1.5, 1.5)$. And the trajectory continues like this until the agent reaches the absorbing base state $\boldsymbol{s}_5 = (1.5, 0.5)$.

Chapitre 2

# Direct Deep Reinforcement Learning of Decision Tree Policies

In this chapter, we compare deep reinforcement learning of decision tree policies (Chaper 1) to imitation learning of decision tree policies (Sec. 4) for the CartPole MDP [10].

In particular, we attempt to reproduce the results from [125, Table 1] in which authors constraint the solution space of decision tree policies to depth-2 trees. In the original results, authors find that deep reinforcement learning to solve the interpretable rl objective (11) and Dagger or VIPER (Sec. 4) to solve the imitation learning objective (9) find similar decision trees. We find that imitation learning, despite not directly optimizing the RL objective (4) for CartPole, outperforms deep RL that optimizes (11) even though the deep RL approach directly optimizes the RL objective for CartPole (up to some trade-off with interpretability).

## 2.1 Reproducing "Iterative Bounding MDPs : Learning Interpretable Policies via Non-Interpretable Methods"

### 2.1.1 IBMDP formulation

Given a a factored base MDP $\mathcal{M}\langle S, A, R, T, T_0\rangle$ (**??**), in order to define an IBMDP $\mathcal{M}_{IB}\langle S\times O, A\cup A, (R,\zeta), (T, T_0, T_{info})\rangle$ (10), the user needs to provide the set of information gathering actions $A_{info}$ and the reward $\zeta$ for taking those. Authors of [125] propose to parametrize the set of IGAs with $i \times p$ actions $\langle v_k, i\rangle$ with $v_k$ depending on the current

observation $\boldsymbol{o}_t = (L'_1, U'_1, \ldots, L'_i, U'_i, \ldots, L'_n, U'_n) : v_k = \frac{k(U'_i - L'_i)}{p+1}$. This parametric IGAs space keeps the discrete IBMDP action space at a reasonable size while providing a learning algorithm with varied IGAs to try.

For example, if we define an IBMDP with $p = 3$ for the grid world from Example (cite), the grid world action space is augmented with six IGAs. At $t = 0$, recall that $\boldsymbol{o}_0 = (0, 2, 0, 2)$, so if an IGA is taken, e.g. $\langle v_2, 2\rangle$, the effective IGA is $\langle v_2 = \frac{k(2-0)}{3+1}, i\rangle = \langle 1, 2\rangle$ which in turn effectively corresponds to an internal decision tree node $y \leq 1$. If the current state $y$-feature value is 0.5, then the next observation at $t = 1$ is $\boldsymbol{o}_1 = (0, 2, 0, 1)$. At $t = 2$ if $a_t = \langle v_2, 2\rangle$ again, it would be effectively $\langle v_2 = \frac{k(1-0)}{3+1}, i\rangle = \langle 0.5, 2\rangle$. This would give the next observation at $t = 2$ $\boldsymbol{o}_2 = (0, 2, 0, 0.5)$ and so on ….

Furthermore, author propose to regularize the learned decision tree policy with a maximum depth parameter $D$. Unfortunately, authors did not describe how they implemented the depth control in their work, hence we have to try different approaches to reproduce their results.

To control the tree depth during learning in the IBMDP, we can either give negative reward for taking $D$ IGAs in a row, or we could terminate the trajectory. The penalization approaches can break the MDP formalism because the reward function now depends on time while it should only depend on states and actions (3). Similarly, the termination approach requires a transition function that depends on time which also breaks the Makrov property.

We actually find that when $p+1$, the IBMDP information gathering space parameter, is a prime number, then as a direct consequence of the *Chinese Reminder Theorem*, the current tree depth is directly encoded in the current observation $\boldsymbol{o}_t$. Hence, when $p+1$ is prime, we can control the depth through either transitions or rewards without tracking the time.

We will try various $\zeta$, various $p$, and various depth control in our experiments but first we describe the reinforcement learning algorithms used in [125].

### 2.1.2 Modified Deep Reinforcement Learning algorithms

Authors of [125] use two deep reinforcement learning baselines to which they apply some modifications in order to learn partially observable policies as required by proposition (1) and objective (11).

The first algorithm is a modified proximal policy optimization algorithm (PPO)([111], Alg. 7) that we present in Algorithm 12. Authors modify the standard PPO and train a neural network policy $O \rightarrow A \cup A_{info}$ while the neural network value function is

$S \times O \to A \cup A_{info}$.

The second deep reinforcement learning algorithm used is the deep Q-networks algorithm (DQN)([89], Alg. 6) that we present in Algorithm 11. A similar modification is done to DQN to return a partially observable policy. The trained $Q$-function is approximated with a neural network $O \to\to \mathbb{R}^{|A \cup A_{info}|}$ rather than $S \times O \to \mathbb{R}^{|A \cup A_{info}|}$. In this modified DQN, the temporal difference error target for the $Q$-function $O \to A \cup A_{info}$ is approximated by a neural network $S \times O \to A \cup A_{info}$ that is in turn trained by bootstrapping the temporal difference error with itself.

Those two variants of DQN and PPO have first been introduced in [96] for robotics tasks with partially observable components, under the name "asymmetric" actor-critic. Asymmetric RL algorithms that have policy and value estimates using different information from a POMDP [119, 41] were later studied theoretically to solve POMDPs in Baisero's work [8, 7]. The connexions from Deep RL in IBMDPs for objective 11 is absent from [125] and we defer their connexions to direct interpretable reinforcement learning to the next Chapter as our primary goal is to reproduce [125] *as is*.

Next, we present the precise experimental setup we use to reproduce [125, Table 1] in order to study direct deep reinforcement learning of decision tree policies for the CartPole MDP.

## 2.2 Experimental setup

### 2.2.1 (IB)MDP

We use the exact same MDP and associated IBMDPs for our experiments as [125] except mentioned otherwise.

**MDP** The problem is to optimize (4) with a decision tree policy for the CartPole MDP [10]. At each time step a learning algorithm observes the cart position velocity and the pole angle and angular velocity, and can take action to push the cart left or right. While the cart is roughly balanced, i.e., while the cart angle remains in some fixed range, the agent gets a positive reward. If the cart is out of balance; the MDP transitions to an absorbing terminal state and gets 0 reward forever. Like in [125], we use the gymnasium `CartPole-v0` implementation [126] of the CartPole MDP in which trajectories are truncated after 200 timesteps making the maximum cumulative reward, i.e. the optimal value of objective 4, to be 200. The state features of the CartPole MDP are in $[-2, 2] \times [-2, 2] \times [-0.14, 0.14] \times [-1.4, 1.4]$.

**Algorithme 11 :** Modified Deep Q-Network (DQN)

**Data :** IBMDP $\mathcal{M}_{IB}\langle S \times O, A \cup A, (R, \zeta), (T, T_0, T_{info})\rangle$, learning rate $\alpha$, exploration rate $\epsilon$, partially observable Q-network parameters $\theta$, Q-network parameters $\phi$, replay buffer $\mathcal{B}$, update frequency $C$

**Result :** Partially pbservable deterministic policy $\pi_{po}$

Initialize partially observable Q-network parameters $\theta$

Initialize Q-network parameters $\phi$ and target network parameters $\phi^- = \phi$

Initialize replay buffer $\mathcal{B} = \emptyset$

**for** *each episode* **do**

  Initialize state $\boldsymbol{s}_0 \sim T_0$

  Initialize state $\boldsymbol{o}_0 = (L_1, U_1, \ldots, L_n, U_n)$

  **for** *each step t* **do**

    Choose action $a_t$ using $\epsilon$-greedy : $a_t = \arg\max_a Q_\theta(\boldsymbol{o}_t, a)$ with prob. $1 - \epsilon$

    Take action $a_t$, observe $r_t$

    Store transition $(\boldsymbol{s}_t, \boldsymbol{o}_t, a_t, r_t, \boldsymbol{s}_{t+1})$ in $\mathcal{B}$

    Sample random batch $(\boldsymbol{s}_i, \boldsymbol{o}_i, a_i, r_i, \boldsymbol{s}_{i+1}) \sim \mathcal{B}$

    $a' = \underset{a}{\arg\max} Q_\theta(\boldsymbol{o}_i, a)$

    $y_i = r_i + \gamma Q_{\phi^-}(\boldsymbol{s}_{i+1}, a')$ // Compute target

    $\phi \leftarrow \phi - \alpha \nabla_\phi (Q_\phi(\boldsymbol{s}_i, a_i) - y_i)^2$ // Update Q-network

    $\theta \leftarrow \theta - \alpha \nabla_\theta (Q_\theta(\boldsymbol{o}_i, a_i) - y_i)^2$ // Update partially observable Q-network

    **if** $t \bmod C = 0$ **then**

      $\theta^- \leftarrow \theta$ // Update target network

    **end**

    $\boldsymbol{s}_t \leftarrow \boldsymbol{s}_{t+1}$

    $\boldsymbol{o}_t \leftarrow \boldsymbol{o}_{t+1}$

  **end**

**end**

$\pi_{po}(\boldsymbol{o}) = \arg\max_a Q_\theta(\boldsymbol{o}, a)$ // Extract greedy policy

**Algorithme 12 :** Proximal Policy Optimization (PPO)

**Data :** IBMDP $\mathcal{M}_{IB}\langle S \times O, A \cup A, (R, \zeta), (T, T_0, T_{info})\rangle$, learning rate $\alpha$, policy parameters $\theta$, clipping parameter $\epsilon$, value function parameters $\phi$
**Result :** Partially observable stochastic policy $\pi_{po_\theta}$
Initialize policy parameters $\theta$ and value function parameters $\phi$
**for** *each episode* **do**
  Generate trajectory $\tau = (\boldsymbol{s}_0, \boldsymbol{o}_0, a_0, r_0, \boldsymbol{s}_1, \boldsymbol{o}_1, a_1, r_1, \ldots)$ following $\pi_\theta$
  **for** *each timestep t in trajectory* **do**
    $G_t \leftarrow \sum_{k=t}^{T} \gamma^{k-t} r_k$ // Compute return
    $A_t \leftarrow G_t - V_\phi(\boldsymbol{s}_t)$ // Compute advantage
    $r_t(\theta) \leftarrow \frac{\pi_{po_\theta}(a_t|\boldsymbol{o}_t)}{\pi_{po_{\theta\,old}}(a_t|\boldsymbol{o}_t)}$ // Compute probability ratio
    $L_t^{CLIP} \leftarrow \min(r_t(\theta)A_t, \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)A_t)$ // Clipped objective
    $\theta \leftarrow \theta + \alpha \nabla_\theta L_t^{CLIP}$ // Policy update
    $\phi \leftarrow \phi + \alpha \nabla_\phi (G_t - V_\phi(\boldsymbol{s}_t))^2$ // Value function update
  **end**
  $\theta_{old} \leftarrow \theta$ // Update old policy
**end**

**IBMDP** Authors define the associated IBMDP with $\zeta = -0.01$ and parametric information gathering action space defined by $p = 3$. In addition we also try $\zeta = 0.01$ and $p = 2$. The discount factor used by the authors is $\gamma = 1$.

We potentially differ from the original paper setting in the way we handle maximum depth limiation. Indeed authors restrain the learning of policies to be equivalent to depth-2 trees but don't detail how they do so. We hence try two different approaches as mentioned in the previous secion : terminating trajectories if the agent takes too much information gathering in a row or simply giving a reward of $-1$ to the agent everytime it takes an information gathering action past the depth limit. We will also try IBMDPs where we do not limit the maximum depth for completeness.

### 2.2.2 Baselines

**Modified DQN** as mentioned above, authors use the modified version of DQN from Algorithm 11. We use the exact same hyperparameters for modified DQN as the authors when possible. We use the same layers width (128) and number of hidden layers (2), the same exploration strategy ($\epsilon$-greedy with linarly decreasing value $\epsilon$ between 0.5 and 0.05 during the first 10% of the training), the same replay buffer size ($10^6$) and the same number of transitions to be collected randomly before doing value updates ($10^5$). We also try to use more exploartion during training (change the initial $\epsilon$ value to 0.9).

We use the same optimizer (RMSprop (cite) with hyperparmeter 0.95 and learning rate $2.5 \times 10^{-4}$) to update the $Q$-networks.

Authors did not share what DQN implementation they used so we use the stable-baselines3 one [105]. Authors did not share what activations they used so we try both tanh() and relu().

**Modified PPO** for the modified PPO algorithm (Alg. |refalg :mod-ppo), we can exactly match the authors hyperparameters since they use the open source stable-baselines3 implementation of PPO.

We match training budgets : we train modified DQN on 1 million timesteps and modified PPO on 4 million timesteps.

**DQN and PPO** We also benchmark the standard DQN and PPO when learning IBMDP policies $\pi : S \times O \to A \cup A_{info}$ and when learning standard $\pi : S \to A$ policies directly in the CartPole MDP.

We summarize hyperpameters for the IBMDP and for the learning algorithms in Tables 2.1, 2.2 and 2.3.

Tableau 2.1 – IBMDP hyperparameters. We try 12 different IBMDPs. In green we highlight the hyperparameters from the original paper and in red we highlight the hyperparameter names for which author do not give information.

| **Hyperparameter** | **Values** |
|---|---|
| Discount factor $\gamma$ | 1 |
| Information gathering actions parameter $p$ | 2, 3 |
| Information gathering actions rewards $\zeta$ | -0.01, 0.01 |
| Depth control | Done signal, negative reward, none |

**Indirect methods** We also compare modified RL algorithm to imitation learning (Sec. 4). To do so, we use VIPER or Dagger (Algs 9 [109], 9 [11]) to imitate greedy neural network policies obtained with standard DQN learning directly on CartPole. And we use Dagger to imitate neural network policies obtained with the standard PPO learning directly on CartPole.

For each indirect method, we imitate the neural network experts by fitting decision trees on 10000 expert transitions using the CART (Alg. 1 [20]) implementation from scikit-learn [95] with default hyperparameters and maximum depth of 2 like in [125].

Tableau 2.2 – (Modified) DQN trained on $10^6$ timesteps. This gives four different instantiation of (modified) DQN. Hyperparameters not mentioned are stable-baselines3 default. In green we highlight the hyperparameters from the original paper and in red we highlight the hyperparameter names for which author do not give information.

| **Hyperparameter** | **Values** |
|---|---|
| Buffer size | $10^6$ |
| Random transitions before learning | $10^5$ |
| Epsilon stard | 0.9, 0.5 |
| Epsilon end | 0.05 |
| Exploration fraction | 0.1 |
| Optimizer | RMSprop ($\alpha = 0.95$) |
| Learning rate | $2.5 \times 10^{-4}$ |
| Networks architectures | [128, 128] |
| Networks activation | tanh(), relu() |

Tableau 2.3 – (Modified) PPO trained on $4 \times 10^6$ timesteps. This gives two different instantiation of (modified) PPO. Hyperparameters not mentioned are stable-baselines3 default. In green we highlight the hyperparameters from the original paper and in red we highlight the hyperparameter names for which author do not give information.

| **Hyperparameter** | **Values** |
|---|---|
| Steps between each policy gradient steps | 512 |
| Number of minibatch for policy gradient updates | 4 |
| Networks architectures | [64, 64] |
| Networks activations | tanh(), relu() |

### 2.2.3 Metrics

The key metric of this section is performance when controlling the CartPole, i.e, the average *undiscounted* cumulative reward of a policy on 100 trajectories (objective 4 with $\gamma = 1$).

For modified RL algorithms that learn a partially observable policy (or $Q$-function) in an IBMPD, we periodically extract the policy (or $Q$-function) and use Alg.10 to extract a decision tree for the CartPole MDP. We then evaluate the tree on 100 independent trajectories in the MDP and report the mean undiscounted cumulative reward.

For standard RL applied to IBMDPs, since we can't deploy learned policies directly to the base MDP as the state dimensions mismatch (such policies are $S \times O \to A \cup A_{info}$ but the MDP states are in $S$), we periodically evaluate those IBMDP policies in a copy of the training IBMDP in which we fix $\zeta = 0$ ensuring that the copied IBMDP undiscounted

cumulative rewards only correspond to rewards from the base CartPole MDP (non-zero rewards in the IBMDP only occur when a reward from the base MDP is given, i.e. when $a_t \in A$ in the IBMDP (c.f. Def. 10)). Similarly, we do 100 trajectories of the extracted policies in the copied IBMDP and report the average undiscounted cumulative reward.

For RL applied directly to the base MDP we can just periodically extract the learned policies and evaluate them on 100 CartPole trajectories.

Since imitation learning baselines train offline, i.e, on a fixed dataset, their performances cannot be reported on the same axis as RL baselines. For that reason, during the training of a standard RL baseline, we periodically extract the trained neural policy/$Q$-function that we consider as the expert to imitate. Those experts are then imitated with VIPER or Dagger using 10 000 newly generated transitions and the fitted decision tree policies are then evaluated on 100 CartPole trajectories. We do not report the imitation learning objective values (9) during VIPER or Dagger training. Every single combination of IBMDP and Modified RL hyperparameters is run 20 times. For standard RL on either an IBMDP or an MDP with use the paper's original hyperparameters when they were spicified, with depth control using negative rewards, tanh() activations, and we repeat this training 20 times.

Next, we present our results when reproducing [125, Table 1].

## 2.3 Results

### 2.3.1 How well do modified Deep RL baselines learn in IBMDPs?

On Figure 2.3.1, we observe that modified DQN can learn in IBMDPs–the curves have an increasing trend–but we also observe that modified DQN finds poor decision tree policies for CartPole in average–the curves flatten at the end of the x-axis and have low y-values–. In, particular, among all the learning curves that could possibly correspond to the original paper's modified DQN, the learning curve with highest final y-value is converging to decision tree policies for CartPole high poor performances.

On Figure 2.3.1, we observe that modified PPO finds decision tree policies with almost 150 cumulative rewards towrads the end of training. The performance difference with modified DQN could be because we trained longer, like in the original paper.

However it could also be because DQN-like algorithm with those hyperpameters struggle to learn in CartPole (IB)MDPs. Indeed, we notice that for DQN-like baselines, learning seems difficult in general indepedently of the setting.

On Figures 2.3.1 and 2.3.1, we observe that standard RL baselines (RL + IBMDP and

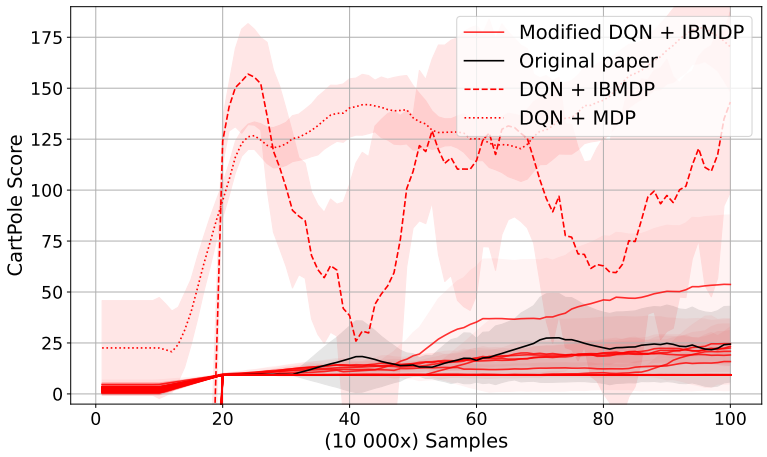



FIGURE 2.1 – Variations of modified DQN and DQN (c.f. 2.2), on different CartPole IBMDPs (c.f. 2.1). We give different line-styles for the learning curves for DQN applied directly on CartPole and DQN applied on the IBMDP. Since there are multiple possible candidates for the original paper hyperparameters, we choose to color the (modified DQN variant, IBMDP variant) pair that resulted in the best decision tree policy on CartPole among the instances that could match the original paper's. Shaded areas represent the confidence interval at 95% at each measure on the y-axis.

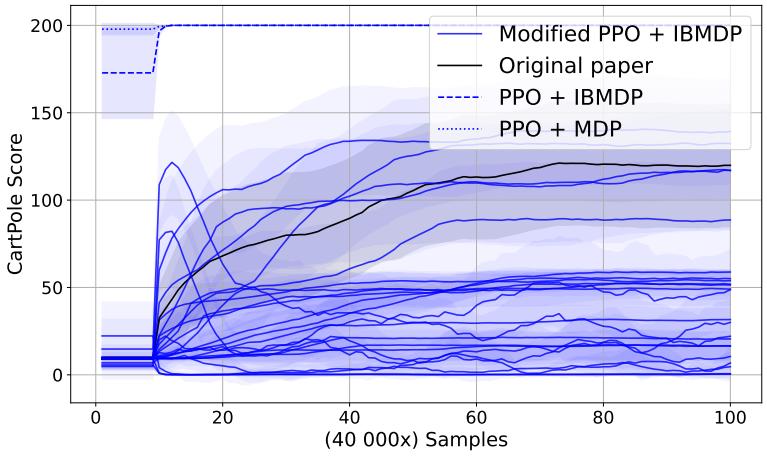



FIGURE 2.2 – Variations of modified PPO and PPO (c.f. 2.3), on different CartPole IBMDPs (c.f. 2.1). We give different line-styles for the learning curves for PPO applied directly on CartPole and DQN applied on the IBMDP. Since there are multiple possible candidates for the original paper hyperparameters, we choose to color the (modified PPO variant, IBMDP variant) pair that resulted in the best decision tree policy on CartPole among the instances that could match the original paper's. Shaded areas represent the confidence interval at 95% at each measure on the y-axis.

RL + MDP), learn better CartPole policies in average than their modified counterparts that learn partially observable policies (1). on Figure 2.3.1, it is clear that for the standard PPO baselines, learning is super efficient and algorithms learn optimal policies with reward 200 in few thousands steps.

In Tables 2.4 and 2.5 we report the top-5 hyperparameters for Modified RL baselines when learning partially observable IBMDP policies in terms of extracted decision tree policies performances in CartPole control.

Tableau 2.4 – Top 5 Hyperparameter Configurations for modified DQN + IBMDP, bold font represent the original paper hyperparameters.

| Rank | $p$ | Depth control | Activation | Exploration | $\zeta$ | Final Performance |
|---|---|---|---|---|---|---|
| 1 | 3 | termination | tanh() | 0.9 | 0.01 | 53 |
| 2 | 2 | termination | tanh() | 0.5 | -0.01 | 24 |
| **3** | **3** | **termination** | tanh() | **0.5** | **-0.01** | **24** |
| 4 | 2 | termination | tanh() | 0.5 | 0.01 | 23 |
| 5 | 2 | termination | tanh() | 0.9 | -0.01 | 22 |

Tableau 2.5 – Top 5 Hyperparameter Configurations for modified PPO + IBMDP, bold font represent the original paper hyperparameters.

| Rank | $p$ | Depth Control | Activation | $\zeta$ | Final Performance |
|---|---|---|---|---|---|
| 1 | 3 | reward | relu() | 0.01 | 139 |
| 2 | 3 | done | relu() | 0.01 | 132 |
| **3** | **3** | **reward** | tanh() | **-0.01** | **119** |
| 4 | 3 | reward | relu() | -0.01 | 117 |
| 5 | 3 | reward | tanh() | 0.01 | 116 |

### 2.3.2 What decision tree policies does direct reinforcement learning return for CartPole ?

On Figure 2.3, we isolate the best performing algorithms instantiations that learn decision tree policies for CartPole. We compare the best modified DQN or modified PPO to imitation learning baselines that use the surrogate imitation objective ( 9) to find CartPole decision tree policies. We find that despite having poor performances in *average*, the modified deeo reinforcement learning baselines can find very good decision tree policies as shown by the min-max shaded areas on the left of Figure 2.3 and the corresponding estimated density of final trees performances. However this is not

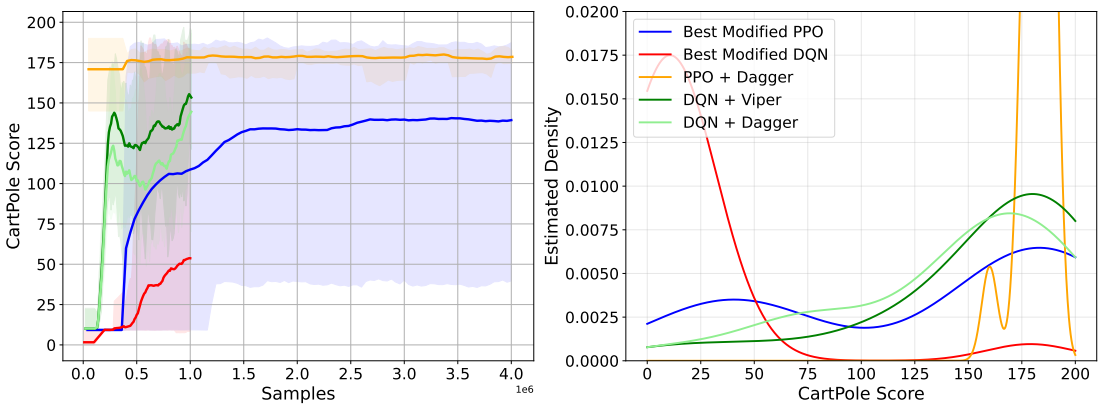



FIGURE 2.3 – (left) Mean performance of the best–w.r.t to the RL objective ( 4) for CartPole–modified RL + IBMDP combination. Shaded areas representing the min and max performance over the 20 seeds during training. (right) Corresponding scores distribution of the final decision tree policies performances w.r.t to the RL objective ( 4) for CartPole.

desirable, a user typically wants an algorithm that can consistantly find good decision tree policies. As shown by the estimated densities, indirect methods consistently find good decision tree policies (the higher modes of distributions are on the right of the plot). On the other hand, the final trees returned by direct RL methods seem equally distributed on both extremes of the scores.

On Figure 2.4, we present the best decision tree policies for CartPole returned by modified DQN and modified PPO. We used Algorithm 10 to extract 20 trees from the 20 partially observable policies returned by the modified deep reinforcement learning algorithms over the 20 training seeds. We then plot the best tree for each baseline. Those trees get an everage reward of roughly 175. Similarly, we plot a representative tree for imitation learning baseline as well as a tree that is optimal for CartPole w.r.t ( 4) obtained with VIPER. Unlike for direct methods, the trees returned by imitation learning are extremely similar across seeds. In particular they often only vary in the scalar value used in the root node but in general have the same structure and test the angular velocity. On the other hand the most frequent trees across seeds returned by modified RL baselines are “trivial” decision tree policy that either repeat the same base action forever or repeat the same IGA (Def. 10) forever.

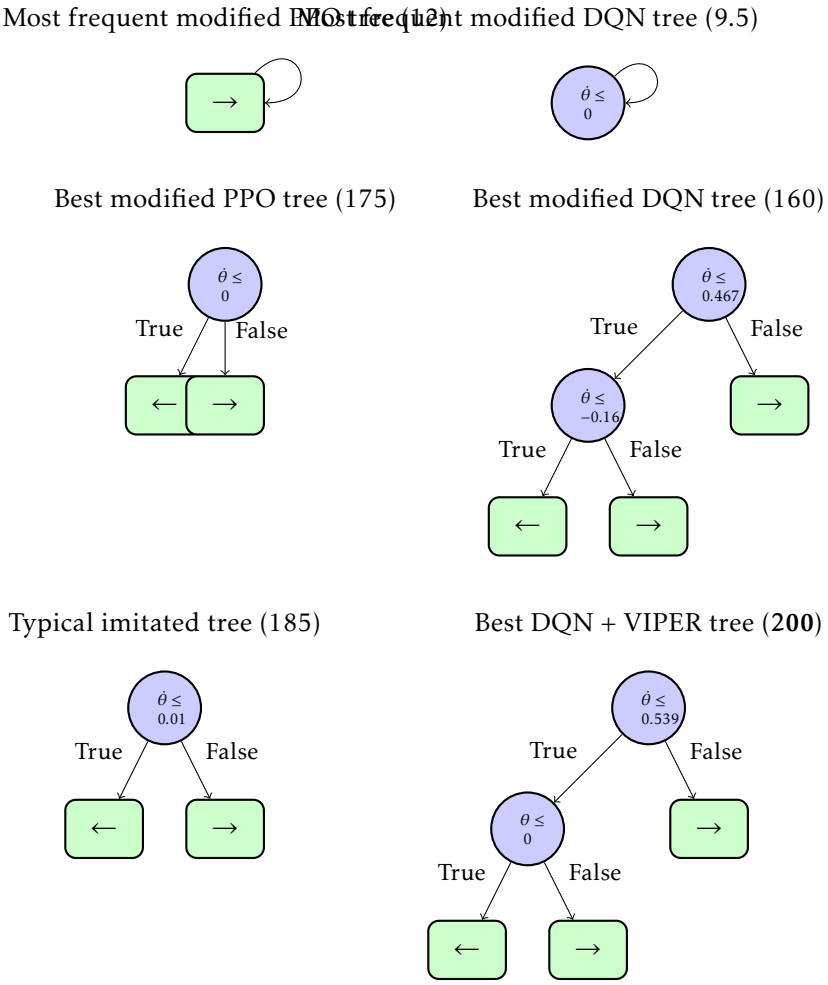



FIGURE 2.4 – Trees obtained by Deep RL in IBMDPs against trees obtained with imitation (CartPole cumulative rewards). $\theta$ and $\dot{\theta}$ are respectively the angle and the angular velocity of the pole

## 2.4 Discussion

We have shown that compared to learning non-interpretable neural network policies for the base MDP or some associated IBMDP, reinforcement learning of partially observable policies in IBMDP is less efficient (c.f. Figures 2.3.1 and 2.3.1). As a consequence, only a handful of modified RL runs are able to learn decision tree policies that are on par with imitated trees (c.f. Figure 2.3).

In the next chapter, we highlight the conncexions between direct interpretable RL (Def. 11) and POMDPs to get insights on the hardness of direct reinforcement learning of decision trees.

Chapitre 3

# Limits of Direct Reinforcement Learning of Decision Tree Policies

From the previous Chapter 2 we know that to directly learn decision trees that directly optimize the RL objective 4 for an MDP, one can learn a deterministic partially observable policy for an IBMDP (Definitions 10 and 11 and Proposition 1). Such problems are classical instances of partially observable Markov decision processes (POMDPs) [119, 41]. This connexion with POMDPs brings novel insights to direct reinforcement learning of decision tree policies. In this chapter, all the decision processes presented have a finite number of vector-valued states and observations. Hence we will use bold fonts for states and observations but can still use summations rather than integrals when required.

### 3.0.1 Partially Observable IBMDPs

A POMDP is an MDP where the current state is hidden ; only some information about the current state is observable.

**Definition 12** (Partially Observable Markov Decision Processes)**.** *A partially observable Markov decision process is a tuple* $\langle X, A, O, T, T_0, \Omega, R\rangle$ *where :*

- *X is the state space (like in the definition of MDPs (3)).*
- *A is a finite set of actions (like in the definition of MDPs (3)).*
- *O is a set of observations.*
- $T : X \times A \to \Delta X$ *is the transition kernal, where* $T(boldsymbolx_t, a, \boldsymbol{x}_{t+1}) = P(\boldsymbol{x}_t|\boldsymbol{x}_{t+1}, a)$ *is the probability of transitioning to state* $\boldsymbol{x}_t$ *when taking action a in state* $\boldsymbol{x}$

— $T_0$ *: is the intial distribution over states.*
— $\Omega : X \to \Delta O$ *is the observation kernel, where* $\Omega(\boldsymbol{o}, a, \boldsymbol{x}) = P(\boldsymbol{o}|\boldsymbol{x}, a)$ *is the probability of observing* $\boldsymbol{o}$ *in state* $\boldsymbol{x}$
— $R : X \times A \to \mathbb{R}$ *is the reward function, where* $R(\boldsymbol{x}, a)$ *is the immediate reward for taking action a in state* $\boldsymbol{x}$

*Note that* $\langle X, A, R, T, T_0 \rangle$ *defines an MDP (cite).*

Let us define explicitely a partially observable iterative bounding Markove decision process (POIBMDP). It is essentially an IBMDP extended with an observation space and an observation kernel :

**Definition 13** (Partially Observable Iterative Bounding Markov Decision Processes)**.** *a partially observable iterative bounding Markov decision process* $\mathcal{M}_{POIB}$

$$\langle \overbrace{S \times \underbrace{O}_{Observations}}^{full\ states}, \underbrace{A \cup A_{info}}_{Actionspace}, \overbrace{(R, \zeta)}^{Rewards}, \underbrace{(T_{info}, T, T_0)}_{Transitions}, \Omega \rangle$$

*Note that* $\langle S \times O, A \cup A, (R, \zeta), (T, T_0, T_{info}) \rangle$ *is an IBMDP 10. The transition kernel* $\Omega$ *maps state features and observations to observations,* $\Omega : S \times O \to O$*, with* $P(\boldsymbol{o}|(\boldsymbol{s}, \boldsymbol{o})) = 1$

One can see POIBMDPs as particular instances of POMDPs where the observation kernel simply applies a mask over some features of the full state. This setting has other names in the litterature. For example, POIBMDPs are Mixed Observability MDPs [5] with base MDP state features as the *hidden variables* and feature bounds as *visible* variables. POIBMDPs can also be seen as non-stationary MDPs (N-MDPS) [116] in which there is one different transition kernel per base MDP state : these are called Hidden-Mode MDPs [29].

Following [116] we can write the value of a deterministic partially observable policy $\pi : O \to A \cup A_{info}$ in observation $\boldsymbol{o}$.

**Definition 14** (Partially observable value function)**.** *In a POIBMDP (13), the expected cumulative discounted reward of a deterministic partially observable policy* $\pi : O \to A \cup A_{info}$ *starting from observation o is* $V^\pi(\boldsymbol{o})$ *:*

$$V^\pi(\boldsymbol{o}) = \sum_{(s, \boldsymbol{o}') \in S \times O} P^\pi((\boldsymbol{s}, \boldsymbol{o}')|\boldsymbol{o}) V^\pi((\boldsymbol{s}, \boldsymbol{o}'))$$

*with* $P^\pi((\boldsymbol{s}, \boldsymbol{o}')|\boldsymbol{o})$ *the asymptotic occupancy distribution (see [116, Section 4] for the full*

*defintion) of the complete POIBMDP state* $(\boldsymbol{s}, \boldsymbol{o}')$ *given the partial observation o and* $V^{\pi}((s, o'))$ *the classical state-value function defined in (cite).*

The asymptotic occupancy distribution is the probability of a policy $\pi$ to arrive in $(s, \boldsymbol{o}')$ while observing $\boldsymbol{o}$ in some trajectory. We can re-write the direct interpretable RL objective (11) in terms of POIBMDPs :

**Definition 15** (Revised direct interpretable RL objective)**.** *Given an MDP* $\mathcal{M}$ *(cite) and an associated POIBMDP* $\mathcal{M}_{POIB}$ *(13), the direct interpretable RL objective becomes :*

$$\pi^{\star} = \underset{\pi}{\operatorname{argmax}} J(\pi) = \underset{\pi}{\operatorname{argmax}} V^{\pi}(\boldsymbol{o}_0) \tag{3.1}$$

*With* $\pi$ *a determintic partially observable policy* $\pi : O \to A \cup A_{info}$*. There is no expectation over possible initial observation in the above objective function as there is only one initial observation in a POIBMDP :* $\boldsymbol{o}_0 = (L_1, U_1, \ldots, L_n, U_n)$ *(c.f. Def.10).*

This revised objective is just a re-writing of (11) making explicit the POMDP model. In this Chapter, we use reinforcement learning to train decision tree policies for MDPs by seeking deterministic partially observable policies for POIBMDPs (13), i.e. by solving (15). We summarized the approach in Figure 3.1

We will attempt to *learn* an optimal depth-1 tree policy w.r.t the RL objective (4) for the $2 \times 2$ grid world from Example 10. One of those two optimal depth-1 tree is given in Figure 3.2. The other optimal depth-1 tree is to go right when $y \leq 1$ and down otherwise We formulate this as solving the (revised) direct interpretble RL objective 15 where the base MDP is the grid world and the POIBMDP is obtained from the IBMDP of Example 1.1.

We choose $\gamma$ and $\zeta$ in the POIBMDP such that the *optimal* partially observable deterministic policy, i.e. the solution to (15), corresponds exactly to the decision tree of depth 1 from Figure 13. This depth 1 tree is in turn optimal in the grid world MDP w.r.t to the base RL objective (4). Next we present some insights about the solution space of (15).

## 3.1 Constructing POIBMDPs which optimal solutions are the depth-1 tree

Because we know all the base states, all the observations, all the actions, all the rewards and all the transitions of our POIBMDP, we can compute exactly the values of

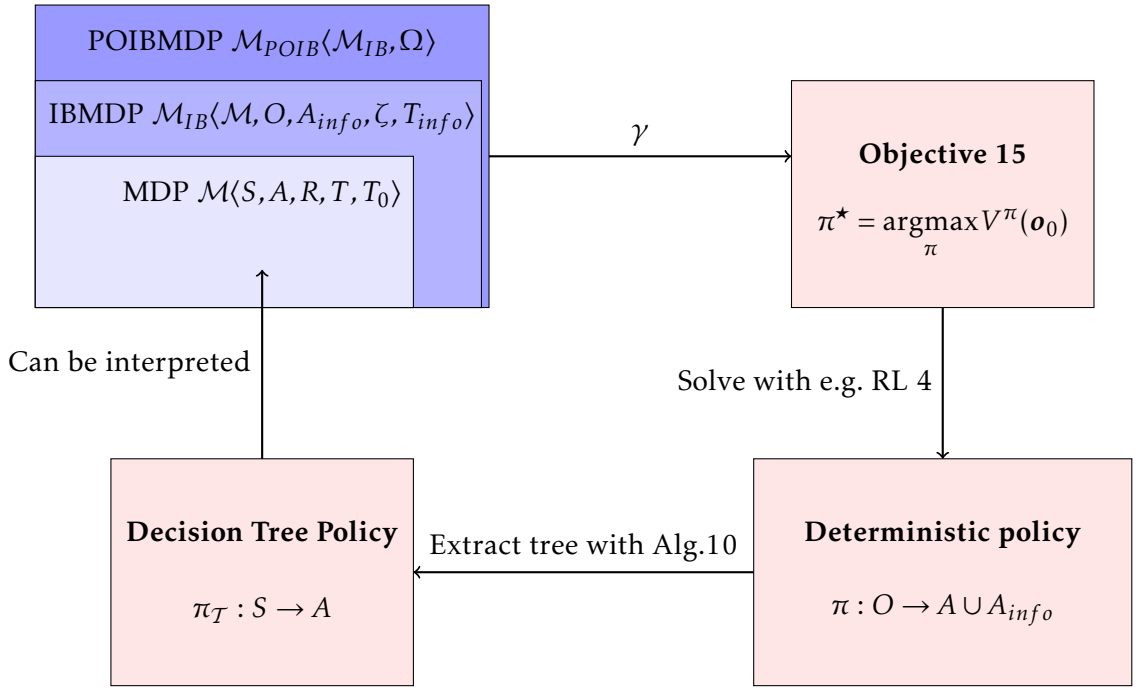



FIGURE 3.1 – A formal framework to learn decision tree policies for MDPs that directly optimize a trade-off between the RL objective 4 and interpretability. This framework relies on learning a partially observable deterministic policy in a POIBMDP 13.

different partially observable deterministic policies given $\zeta$ the reward for IGAs and $\gamma$ the discount factor. Each of those policies can be one of the following trees illustrated in Figure 3.2 :

- $\pi_{\mathcal{T}_0}$ : a depth-0 tree equivalent to always taking the same base action
- $\pi_{\mathcal{T}_1}$ : a depth-1 tree equivalent alternating between an IGA and a base action
- $\pi_{\mathcal{T}_u}$ : an unbalanced depth-2 tree that sometimes takes two IGAs then a base action and sometimes a an IGA then a base action
- $\pi_{\mathcal{T}_2}$ : a depth-2 tree that alternates between taking two IGAs and a base action
- an inifinite "tree" that only takes IGAs

Furthermore, because from [116] we know that for POMDPs, stochastic policies can sometimes get better expected discounted rewards than deterministic policies, we also compute the value of the stochastic policy that alternates between two base actions : $\rightarrow$ and $\downarrow$. Those two base actions always lead to the goal state (c.f. Figure 10) in expectation.

We detail the calculations for the detpth-1 decision tree objective value 15 and defer the calculations for the other policies to the Appendix B.1.

**Proposition** 2 (Depth-1 decision tree objective value)**.** *The objective value of the best*

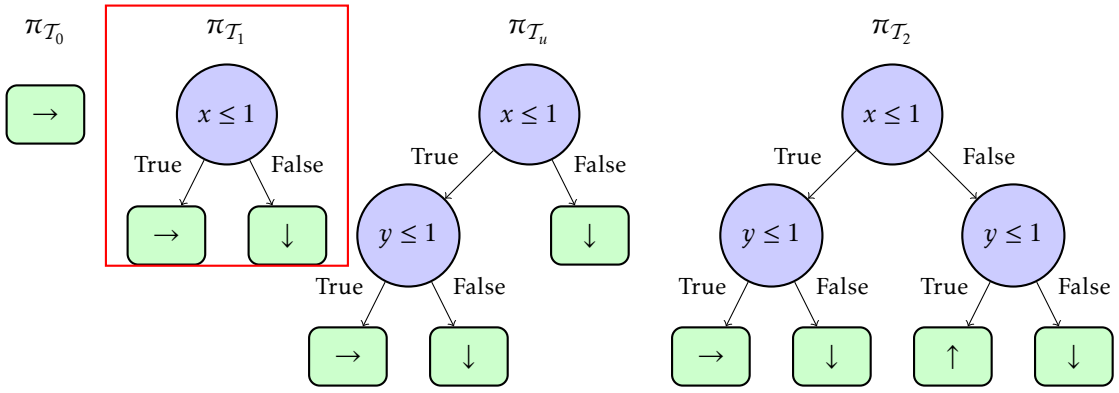



FIGURE 3.2 – For each decision tree structure, e.g., depth-1 or unbalanced depth-2, we illustrate a decision tree which maximizes the RL objective (4).

*depth-1 decision tree from Figure 3.2 is* $V^{\pi_{\mathcal{T}_1}}(o_0) = \frac{4\zeta+\gamma+2\gamma^3+\gamma^5}{4(1-\gamma^2)}$.

*Démonstration.* $\pi_{\mathcal{T}_1}$ has one root node that tests $x \leq 1$ (respectively $y \leq 1$) and two leaf nodes $\rightarrow$ and $\downarrow$. To compute $V^{\pi}_{\mathcal{T}_1}(\boldsymbol{o}_0)$, we compute the values of $\pi_{\mathcal{T}_1}$ in each of the possible startin states $(\boldsymbol{s}_0, \boldsymbol{o}_0), (\boldsymbol{s}_1, \boldsymbol{o}_0), (\boldsymbol{s}_2, \boldsymbol{o}_0), (\boldsymbol{s}_g, \boldsymbol{o}_0)$ and compute the expectation over those. At inititalization, when the base state is $\boldsymbol{s}_g = (1.5, 0.5)$, the depth-1 decision tree policy cycles between taking an information gathering action $x \leq 1$ and moving down to get a positive reward for which it gets the returns :

$$\begin{aligned} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_g, \boldsymbol{o}_0) &= \zeta + \gamma + \gamma^2\zeta + \gamma^3 \dots \\ &= \sum_{t=0}^{\infty} \gamma^{2t}\zeta + \sum_{t=0}^{\infty} \gamma^{2t+1} \\ &= \frac{\zeta + \gamma}{1-\gamma^2} \end{aligned}$$

At inititialization, in either of the base states $\boldsymbol{s}_0 = (0.5, 0.5)$ and $\boldsymbol{s}_2 = (1.5, 1.5)$, the value of the depth-1 decision tree policy is the return when taking one information gathering action $x \leq 1$, then moving right or down, then following the policy from the goal state $\boldsymbol{s}_g$ :

$$\begin{aligned} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_0, o_0) &= \zeta + \gamma 0 + \gamma^2 V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_g, o_0) \\ &= \zeta + \gamma^2 V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_g, o_0) \\ &= V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_2, o_0) \end{aligned}$$

Similarly, the value of the best depth-1 decision tree policy in state $\boldsymbol{s}_1 = (0.5, 1.5)$ is the

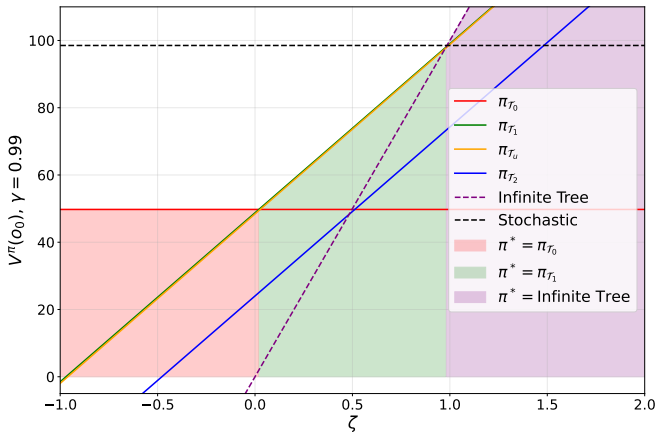



FIGURE 3.3 – POIBMDP objective values of different policies as functions of $\zeta$. Shaded areas show the optimal policies in different ranges of $\zeta$ values.

value of taking one information gathering action then moving right to $\boldsymbol{s}_2$ then following the policy in $\boldsymbol{s}_2$ :

$$\begin{aligned} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_1, \boldsymbol{o}_0) &= \zeta + \gamma 0 + \gamma^2 V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_2, \boldsymbol{o}_0) \\ &= \zeta + \gamma^2 V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_2, o_0) \\ &= \zeta + \gamma^2(\zeta + \gamma^2 V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_g, o_0)) \\ &= \zeta + \gamma^2\zeta + \gamma^4 V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_g, o_0) \end{aligned}$$

Since the probability of being in any base states at initialization given that the agent observe $\boldsymbol{o}_0$ is the probability of being in any base states at initialization, we can write :

$$\begin{aligned} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{o}_0) &= \frac{1}{4} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_g, \boldsymbol{o}_0) + \frac{2}{4} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_2, \boldsymbol{o}_0) + \frac{1}{4} V^{\pi_{\mathcal{T}_1}}(\boldsymbol{s}_1, \boldsymbol{o}_0) \\ &= \frac{1}{4}\frac{\zeta + \gamma}{1-\gamma^2} + \frac{2}{4}(\zeta + \gamma^2 \frac{\zeta + \gamma}{1-\gamma^2}) + \frac{1}{4}(\zeta + \gamma^2\zeta + \gamma^4 \frac{\zeta + \gamma}{1-\gamma^2}) \\ &= \frac{1}{4}\frac{\zeta + \gamma}{1-\gamma^2} + \frac{2}{4}(\frac{\zeta + \gamma^3}{1-\gamma^2}) + \frac{1}{4}(\frac{\zeta + \gamma^5}{1-\gamma^2}) \\ &= \frac{4\zeta + \gamma + 2\gamma^3 + \gamma^5}{4(1-\gamma^2)} \end{aligned}$$

□

We can now plot, in Figure 3.3, the POIBMDP objective values of the different policies corresponding to trees for the grid world MDP as functions of $\zeta$ when we fix

$\gamma = 0.99$. When $\gamma = 0.99$ on Figure 3.3, despite objective values being very similar for the depth-1 and unbalanced depth-2 tree, we now know from the green shaded area a depth-1 tree is the optimal deterministic partially observable POIBMDP policy for $0 < \zeta < 1$.

Interestingly, two POMDP challenges described in [116] can already be observed in Figure 3.3. First, there is a whole range of $\zeta$ values for which the stochastic policy is optimal. Second, for e.g. $\zeta = 0.5$, while a depth-1 tree is the optimal deterministic partially observable policy, the value of state $(\boldsymbol{s}_2, \boldsymbol{o}_0) = (1.5, 1.5, 0, 2, 0, 2)$ is not maximized by this policy but by the sub-optimal policy that always goes down.

We can now define a POIBMDP with the grid wolrd (10) as the base MDP, with IGAs as in the IBMDP from Exapmle 1.1, with $\gamma = 0.99$ and $0 < \zeta < 1$ and verify if RL can retrieve the optimal depth-1 decision tree in this very controlled experiment.

### 3.1.1 Reinforcement Learning in PO(IB)MDPs

In general, the policy that maximizes the expected discounted cumulative reward in a POMDP maps "belief states" or observations histories [41] to actions, i.e., those policies are not solutions to our problem since we require that policies depend only on the current observation. If we did not have this constraint, we could apply any standard RL algorithm to solve POIBMDPs by seeking such policies because both histories and belief states are sufficient statistic for POMDPs full states [41, 66].

The particular, the problem of finding the optimal deterministic partially observable policies for POMDPs is NP-HARD, even with full knowledge of transitions and rewards [75, Section 3.2].

It means that, there is no reason to believe that any algorithm for solving it must enumerate all possible policies and take the best one. For even moderate-sized POMDPs, a brute-force approach will take a very long time since there are $|A|^{|O|}$ policies.

Hence it is interesting to study reinforcement learning for finding the best deterministic partially observable policy since it would not search the whole solution space. However applying RL to our revised interpretable RL objective (15) is non-trivial.

In [116, Fact 2], authors show that the optimal partially observable policy can be stochastic, hence policy gradient algoriothms [120] are to prohibit since we want a *deterministic* policy. Furthermore, the optimal deterministic patially observable policy might not maximize all the values of all observations simulataneously [116, Fact 5] which makes difficult to use TD-learning to learn policies. Indeed, doing a TD-learning update of one partially observable value 14 with, e.g. Q-learning, can change the value of

*all* other observations in a unctrollable manner because of the depedence in $P^\pi((s, \boldsymbol{o}')|\boldsymbol{o})$.

Despite those hardness results, empirical results of applying RL to POMDPs by naively replacing $\boldsymbol{x}$ by $\boldsymbol{o}$ in Q-learning or Sarsa, has already demonstrated successful in practice [76]. More recently, the framework of Baisero et. al. called asymmetric RL [8, 7] has also shown promising results to learn POMDP solutions. Asymmetric RL trains a full-state-dependent model and a history-dependent (or observation-dependent) model informed by the former. The history-dependent (or observation-dependent) model can thus be deplyed in the POMDP after training since it does not require access to the hidden full state to output actions. In Algorithms 13 and 14 we present asymmetric Q-learning and asymmetric Sarsa. Given a POMDP, both train an observation-dependet Q-function $Q : O \times A \to \mathbb{R}$ and a state-dependent Q-function $U : X \times A \to \mathbb{R}$.

In [60], authors introduce a policy search algorithm 4 that learns a (stochastic) policy $\pi : O \to \Delta A$ and a critic $V : X \to \mathbb{R}$ using Monte Carlo estimates to guide policy improvement. We write this algorithm that we call JSJ (for the authors name Jaakkola, Singh, Jordan) in Algorithm 15. JSJ is equivalent to a tabular asymmetric policy gradient algorithm (c.f. Alg. 5).

Until recently, the benefits of asymmetric RL over standard RL was only shown empirically and only for history-dependent models. The work of Gaspard Lambrechts [67] proves that some asymmetric RL algorithms learn better history-dependent **or** observation-dependent policies for solving POMDPs. This is exactly what we wish for. However, those algorithms are intractable in practice because they require estimation of the quantity $P^\pi((s, \boldsymbol{o}')|\boldsymbol{o})$ (14). We leave it to future work to use those algorithms that combine asymmetric RL and estimation of future visitations since those results are contemporary to the writing of this manuscript.

Note that, in the previous chapter, modified DQN (11) and modified PPO (12) are respectively asymmetric DQN and asymmetric PPO from [8, 7].

In the next section, we use (asymmetric) RL to learn decision tree policies for the grid world MDP (10).

## 3.2 Results

Unfortunately, our results are negative and show that (asymmetric) reinforcement learning fails for the aforementioned problem. Let us understand why.

**Algorithme 13 :** Asymmetric Q-Learning

**Data :** POMDP $\mathcal{M}_{po} = \langle X, O, A, R, T, T_0, \Omega \rangle$, learning rates $\alpha_u$, $\alpha_q$, exploration rate $\epsilon$
**Result :** $\pi : O \to A$
Initialize $U(x, a) = 0$ for all $x \in X, a \in A$
Initialize $Q(o, a) = 0$ for all $o \in O, a \in A$
**for** *each episode* **do**
  Initialize state $x_0 \sim T_0$
  Initialize observation $\boldsymbol{o}_0 \sim \Omega(x_0)$
  **for** *each step* $t$ **do**
    Choose action $a_t$ using $\epsilon$-greedy : $a_t = \arg\max_a Q(\boldsymbol{o}_t, a)$ with prob. $1 - \epsilon$
    Take action $a_t$, observe $r_t = R(x_t, a_t)$, $x_{t+1} \sim T(x_t, a_t)$, and $\boldsymbol{o}_{t+1} \sim \Omega(x_{t+1})$
    $y \leftarrow r + \gamma U(x_{t+1}, a' Q(\boldsymbol{o}_{t+1}, a'))$ // `TD target`
    $U(x_t, a_t) \leftarrow (1 - \alpha_u) U(x_t, a_t) + \alpha_u y$
    $Q(\boldsymbol{o}_t, a_t) \leftarrow (1 - \alpha_q) Q(\boldsymbol{o}_t, a_t) + \alpha_q y$
    $x_t \leftarrow x_{t+1}$
    $\boldsymbol{o}_t \leftarrow \boldsymbol{o}_{t+1}$
  **end**
**end**
$\pi(o) = \arg\max_a Q(o, a)$ // `Extract greedy policy`

### 3.2.1 Experimental Setup

**Baselines :** we consider two groups of RL algorithms. The first group is standard tabular RL naively applied to POIBMDPs ; Q-learning, Sarsa, and Policy Gradient with a softmax policy (c.f. Sec. 4). In theory the Policy Gradient algorithm should not be a good candidate for our problem since it searches for stochastic policies that we showed can be better than our seeked depth-1 decision tree policy (c.f. Figure (cite)).

In addition to the traditional tabular RL algorithms above, we also apply asymmetric Q-learning, asymmetric Sarsa, and JSJ (Algorithms 13, 14 and 15). We use at least 200 000 POIBMDP time steps per experiment. Each experiment, i.e an RL algorithm learning in a POIBMDP, is repeated 100 times.

**Hyperparameters :** For all baselines we use, when applicable, exploration rates $\epsilon = 0.3$ and learning rates $\alpha = 0.1$.

**Metrics :** we will consider two metrics. First, the sub-optimality gap during training of the learned partially observable policy value with respect to the optimal deterministic partially observable POIBMP policy : $|V^{\pi\star}(\boldsymbol{o}_0) - V^{\pi}(\boldsymbol{o}_0)|$ Because we know the whole

**Algorithme 14 :** Asymmetric Sarsa

**Data :** POMDP $\mathcal{M}_{po} = \langle X, O, A, R, T, T_0, \Omega \rangle$, learning rates $\alpha_u$, $\alpha_q$, exploration rate $\epsilon$

**Result :** $\pi : O \to A$

Initialize $U(x, a) = 0$ for all $x \in X, a \in A$

Initialize $Q(o, a) = 0$ for all $o \in O, a \in A$

**for** *each episode* **do**

  Initialize state $x_0 \sim T_0$

  Initialize observation $\boldsymbol{o}_0 \sim \Omega(x_0)$

  Choose action $a_0$ using $\epsilon$-greedy : $a_0 = \arg\max_a Q(\boldsymbol{o}_0, a)$ with prob. $1 - \epsilon$

  **for** *each step t* **do**

    Take action $a_t$, observe $r_t = R(x_t, a_t)$, $x_{t+1} \sim T(x_t, a_t)$, and $\boldsymbol{o}_{t+1} \sim \Omega(x_{t+1})$

    Choose action $a_{t+1}$ using $\epsilon$-greedy : $a_{t+1} = \arg\max_a Q(\boldsymbol{o}_{t+1}, a)$ with prob. $1 - \epsilon$

    $y \leftarrow r + \gamma U(x_{t+1}, a_{t+1})$ // TD target using actual next action

    $U(x_t, a_t) \leftarrow (1 - \alpha_u) U(x_t, a_t) + \alpha_u y$

    $Q(\boldsymbol{o}_t, a_t) \leftarrow (1 - \alpha_q) Q(\boldsymbol{o}_t, a_t) + \alpha_q y$

    $x_t \leftarrow x_{t+1}$

    $\boldsymbol{o}_t \leftarrow \boldsymbol{o}_{t+1}$

    $a_t \leftarrow a_{t+1}$

  **end**

**end**

$\pi(o) = \arg\max_a Q(o, a)$ // Extract greedy policy

**Algorithme 15 :** JSJ algorithm. Uses Monte Carlo estimates of the average reawrd value functions to perform policy imporvements

**Data :** POMDP $\mathcal{M}_{po} = \langle X, O, A, R, T, T_0, \Omega \rangle$, learning rate $\alpha$, policy parameters $\theta$, number of trajectories $N$
**Result :** Stochastic partially observable policy $\pi_\theta : O \to \Delta A$
Initialize policy parameters $\theta$
Initialize $Q(o, a) = 0$ for all observations $o$ and actions $a$
**for** *each episode* **do**
  **for** $i = 1$ *to* $N$ **do**
    Generate trajectory $\tau_i = (\boldsymbol{s}_0, a_0, r_0, \boldsymbol{s}_1, a_1, r_1, \ldots, \boldsymbol{s}_T)$ following $\pi_\theta$
    **for** *each timestep t in trajectory* $\tau_i$ **do**
      $G_t \leftarrow \sum_{k=t}^{T} \gamma^{k-t} r_k$ // Compute return
      Store $(\boldsymbol{o}_t, a_t, G_t)$ for later averaging
    **end**
  **end**
  **for** *each unique observation-action pair* $(o, a)$ **do**
    $Q(o, a) \leftarrow \frac{1}{|\{(o,a)\}|} \sum_{(o,a,G)} G$ // Monte Carlo estimate
  **end**
  **for** *each observation o* **do**
    **for** *each action a* **do**
      $\pi_1(a|o) \leftarrow 1.0$ if $a =_{a'} Q(o, a')$, 0.0 otherwise // Deterministic policy from Q-values
      $\pi(a|o) \leftarrow (1 - \alpha)\pi(a|o) + \alpha \pi_1(a|o)$ // Policy improvement step
    **end**
  **end**
  Reset $Q(o, a) = 0$ for all observations $o$ and actions $a$ // Reset for next episode
**end**

POIBMDP model that we can represent exactly as tables ; and because we know for each $\zeta$ the POIBMDP objective value of the optimal partially observable policy (c.f. Figure) ; we can report the *exact* sub-optimality gaps.

Second, we consider the distribution of the learned trees over the 100 training seeds. Indeed, since for every POIBMDP we run each algorihtm 100 times, at the end of training we get 100 partially observable deterministic policies (we compute the greedy policy for stochastic policies returned by JSJ and Policy Gradient), from which we can extract the equivalent 100 decision tree policies using Algorithm 10 and we can count which one are of e.g. depth-1. This helps understand which trees RL algorithms tend to learn.

### 3.2.2 Can (asymmetric) RL retrieve optimal deterministic partially observable POIBMDP policies ?

In Figure 3.4, we plot the sub-optimality gaps–averaged over 100 seeds–of learned policies during training. We do so for 200 different POIBMDPs where we change the reward for information gathering actions : we sample 200 $\zeta$ values uniformily in $[-1, 2]$. In Figure 3.4, a different color represents a different POIBMDP.

Recall from Figure 3.3 that for :

— $\zeta \in [-1, 0]$, the optimal deterministic partiall observable policy is a depth-0 tree
— $\zeta \in ]0, 1[$, the optimal deterministic partiall observable is a depth-1 tree
— $\zeta \in [1, 2]$, the optimal deterministic partiall observable is a "inifnite" tree that contains infinite number of internal nodes.

We observe that, despite all experiments converging, independently of the $\zeta$ values, not all algorithms in all POIBMDPs fully minimize the sub-optimality gap. In particular, all algorithms seem to consistantly minimze the gap, i.e. learn the optimal policy or Q-function, only for $\zeta \in [1, 2]$ (all the yellow lines go to 0). However, we are interested in the range $\zeta \in ]0, 1[$ where the optimal decision tree is not taking the same action forever. In that range, no baseline consistently minimizes the sub-optimality gap.

In Figure 3.5, we plot the distributions over the final learned trees in function of $\zeta$ from the above runs. For example, in Figure 3.5, in the top left plot, when learning 100 times in a POIBMP with $\zeta = 0.5$, Q-learning returned almost 100 times a depth 0 tree. Again, on none of those subplots do we see a high rate of learned depth-1 trees for $\zeta \in ]0, 1[$. It is alerting that the most frequent learned trees are the depth-0 trees for $\zeta \in ]0, 1[$ because such trees are way more sub-optimal w.r.t to (15) than e.g. the depth-2 unbalanced trees (c.f. Figure 3.3). One interpretation of this phenomenon is that the learning in POIBMDPs is very difficult and so agents tend to converge to trivial policies,

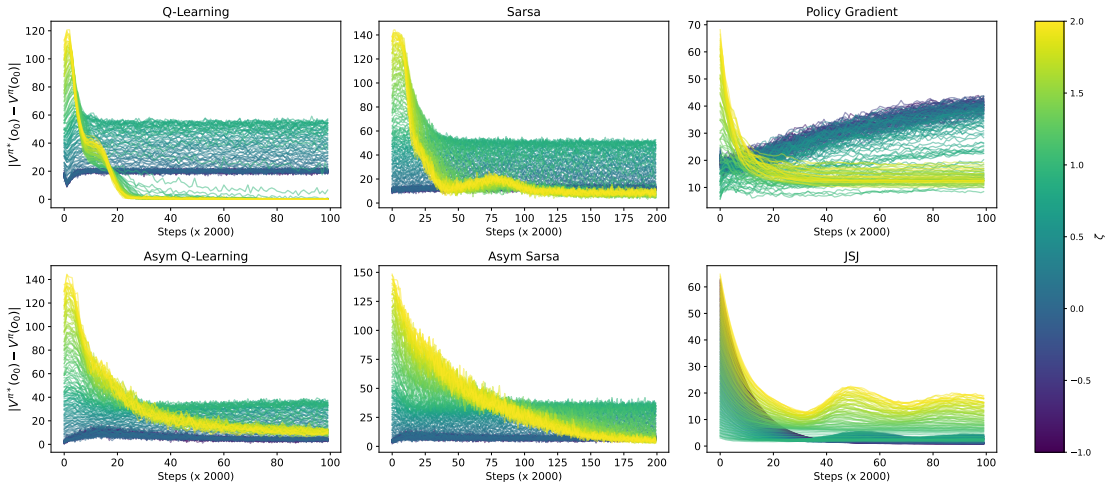



FIGURE 3.4 – (Asymmetric) reinforcement learning in POIBMDPs. In each subplot, each single line is colored by the value of $\zeta$ in the corresponding POIBMDP in which learning occurs. Each single learning curve represent the sub-optimality gap averaged over 100 seeds.

e.g., repeating the same base action.

However, on the positive side, we observe that asymmetric versions of Q-learning and Sarsa have found the optimal policy, the depth-1 decision tree, more frequently throught the optimality range $]0,1[$ than their symmetric counter-parts for $\zeta \in ]0,1[$. Next, we quantify how difficult it is to do RL to learn partially observable policies in POIBMDPs.

### 3.2.3 How difficult is it to learn in POIBMDPs ?

In this section we run the same (asymmetric) reinforcement learning algorithms to optimize either the RL objective (4) in MDPs (3) or IBMDPs (10), or the interpretable RL objective in POIBMDPs 15. This essentially results in three disctint problems :

1. Learning an optimal deterministic state-dependent policy in MDPs.
2. Learning an optimal deterministic full-state-dependent policy in IBMDPs.
3. Learning an optimal deterministic partially observable policy in POIBMDPs.

In order to see how difficult each of these three problems is, we can run a *big* number of experiments on each and compare success rates. To make success rates comparable we consider a unique instance for each of those problems. Problem 1., is learning one of the optimal policy from Figure 10 for the grid world from Example 10 with $\gamma = 0.99$.

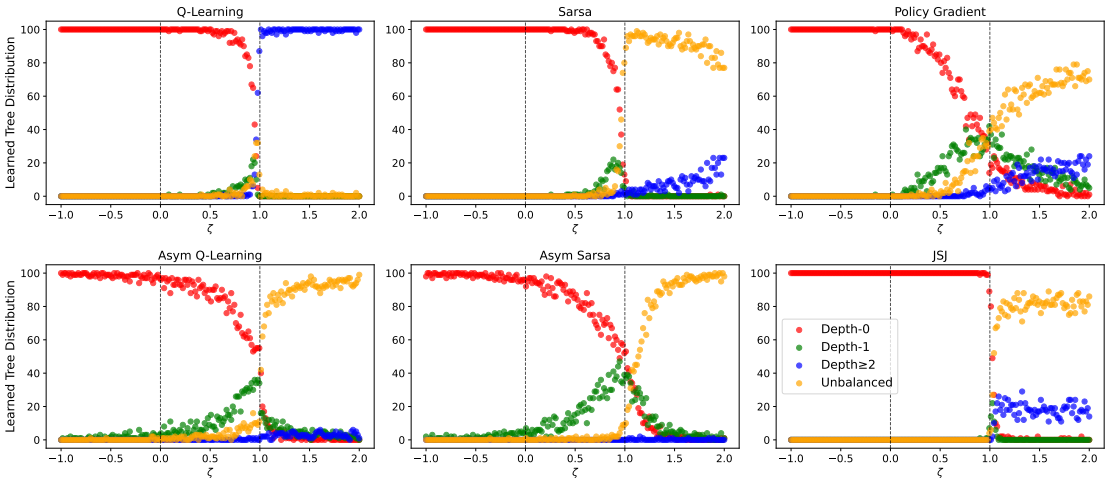



FIGURE 3.5 – Distributions of final tree policies learned across the 100 seeds. For each $\zeta$ value, there are four colored points. Each point represent the share of depth-0 trees (red), depth-1 trees (green), unbalanced depth-2 trees (orange) and depth-2 trees (blue).

Problem 2., is learning one deterministic optimal policy for the IBMDP from Figure 1.1 with $\gamma = 0.99$ and $\zeta = 0.5$. This is similar to the previous chapter's experiments where we applied DQN or PPO to the IBMDP for CartPole without constraining the search to partially observable policies (see e.g. Fig.2.3.1). Problem 3. is what has been done in the previous section where in addition of fixing $\gamma = 0.99$ we also fix $\zeta = 0.5$.

We use the six (asymmetric) RL algoroithms from the previous section and try a wide set of hyperparameters and additional learning tricks (optimistic Q-function, eligibility traces, entropy regularization and $\epsilon$-decay, all are described in [122]). We only provide the detailed hyperparameters for asymmetric Sarsa and a an overall summary for all the algorithms. The complete detailed lists of hyperparameters are given in the Appendix B.2. Furhtermore, the careful reader might notice that there is no point running asymmetric RL on MDPs or IBMDPs when the problem does not require partial observability. Hence, we only run asymmetric RL for POIBMDPs and otherwise run all other RL algorithms and all problems.

Each unique hyperparameters combination for a given algorithm on a given problem is run 10 times on 1 million learning steps. For example, for asymmetric Sarsa, we run a total of $10 \times 768 = 7680$ experiments for learning partially observable deterministic policies for a POIBMDP (c.f. Table 3.1). To get a success rate, we can simply divide the number of learned depth 1 tree by 7680 (recall that for $\gamma = 0.99$ and $\zeta = 0.5$, the optimal policy is a depth-1 tree (e.g. Figure 3.2) as per Figure 3.3).

Tableau 3.1 – Asymmetric Sarsa Hyperparameter Space (768 combinations each run 10 times)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate $U$ | 0.001, 0.005, 0.01, 0.1 | learning rate for the full-state-dependent Q-function |
| Learning Rate $Q$ | 0.001, 0.005, 0.01, 0.1 | learning rate for the partial observation dependent Q-function |
| Optimistic | True, False | Optimistic initialization |

Tableau 3.2 – Summary of RL baselines Hyperparameters

| Algorithm | Problem | Total Hyperparameter Combinations |
|---|---|---|
| Policy Gradient | PO/IB/MDP | 420 |
| JSJ | POIBMDP | 15 |
| Q-learning | PO/IB/MDP | 192 |
| Asym Q-learning | POIBMDP | 768 |
| Sarsa | PO/IB/MDP | 192 |
| Asym Sarsa | POIBMDP | 768 |

The key observations from Figure 3.6 is that reinforcement learning a partially observable determninistic policy in a POIBMDP, is way harder than learning Q-function of policies that have access to all the state information. For example, Q-learning only finds the optimal solution to (15) in only 3% of the experiments while the same algorithms to optimize the standard RL objective (4) in an MDP or IBMDP found the optimal solutions 50% of the time. Even though asymmetry seems to increase performances ; learning a decision tree policy for a simple grid world directly with RL using the framework of POIBMDP originally developed in [125] seem way to difficult and costly as successes might require a million steps for such a seemingly simple problem. An other difficulty in practice that we did not cover here, is the choice of information gathering actions. For the grid world MDP, choosing good IGAs ($x \leq 1$ and $y \leq 1$) is simple but what about more complicated MDPs : how to instantiate the (PO)IBMDP action space such that internal nodes in resulting trees are useful for predictions ?

To go further, on Figure 3.7 we re-run experiments from Figure 3.4 and Figure 3.5 using the top performing hyperparameters for asymmetric Q-learning (given in Appendix B.9). While those hyperparemeters resulted in asymmetric Q-learning returning 10

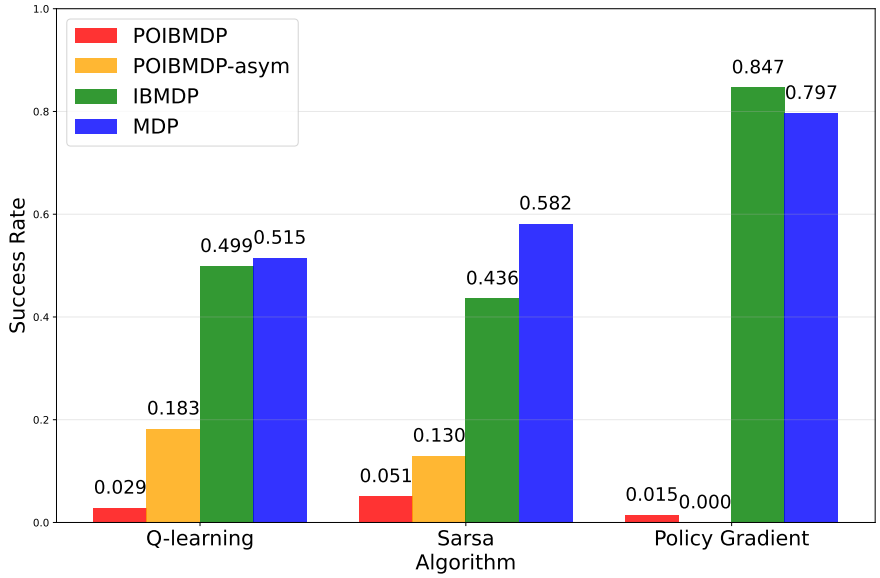



FIGURE 3.6 – Success rates of different (asymmetric) RL algorithms over thousands of runs when applied to learning partially observable deterministic policies in a POIBMDP or learning deterministic policies in associated MDP and IBMDP.

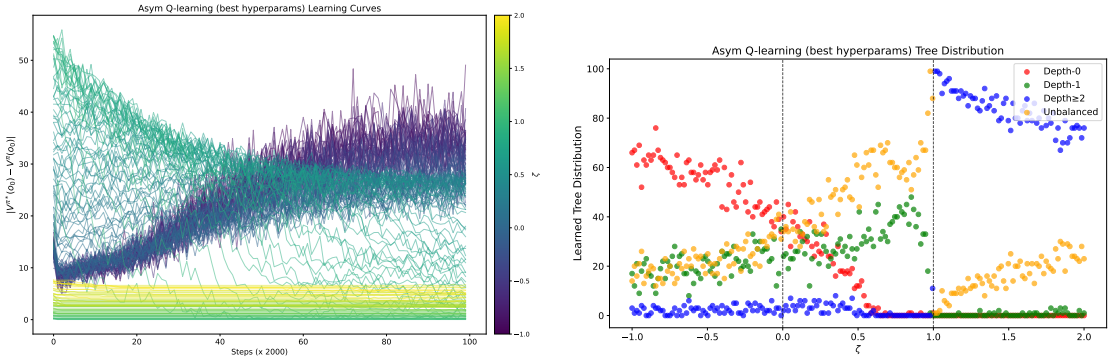
of out 10 times an optimal depth 1 tree, the performances didn't transfer. On Figure 3.7 depsite higher success rates in the region $\zeta \in ]0, 1[$ compared to Figure 3.5.

## 3.3 Conclusion

In this Chapter, we have shown that direct learning of decision tree policies for MDPs can be reduced to learning deterministic partially observable policies in POMDPs that we called POIBMDPs. By crafting a POIBMDP for which we know exactly the optimal deterministic partially observable policy, we were able to benchmark the sub-optimality of solutions learned with (asymmetric) reinforcement learning.

Across our experiments, we found that no algorithm could consistently learn a depth-1 decision tree policy for a grid world MDP despite it being optimal both in the POIBMDP and in the base MDP.

In the next chapter, we find that RL can find optimal deterministic partially observable policies for a special class of POIBMDPs that we believe makes for a convincing argument as to why learning in POIBMDP, i.e. direct learning of decision tree policies that optimize the RL objective (4) is so difficult.




(a) Learning curves for asymmetric Q-learning with good hyperparameters.

(b) Trees distributions for asymmetric Q-learning with good hyperparameters

FIGURE 3.7 – Analysis of the top-performing asymmetric Q-learning instantiation. (left) Learning curves, and (right) tree distributions across different POIBMDP configurations.

Chapitre 4

# When transitions in POIBMDPs are uniform, Reinforcement Learning works

In this section, we show that for a special class of POIBMDPs (cite), reinforcement learning such as Q-learning or Sarsa can retrieve optimal deterministic partially observable policies, i.e we can do direct deicision tree policy learning for MDPs. This class of POIBMDPs are those which base MDP have uniform transitions, i.e. $T(s,a,s') = \frac{1}{|S|}$ in (cite). Supervised learning problems (cite) can be formulated as MDPs with such uniform transitions. Indeed a supervised learning problem (cite) can be formulated as an MDP (cite) where, actions are class labels, states are training data, the reward at every step is 1 if the correct label was predicted and 0 otherwise, and the transitions are uniform : the next state is given by uniformly sampling a new training datum (similarly to contextual bandits (cite)). This implies that learning partially observable deterministic policies in POIBMDPs where the base MDP is a supervised learning task is equivalent to doing decision tree induction to optimize the supervised learning objective (cite). If RL does work for such fully observable POIBMDPs (they are just MDPs (cite)), this would mean that : 1. the difficulty of direct learning of decision tree policies for *any* MDP with POIMDPs is most likely due to the partial observability of the latter, and 2., we can design new decision tree induction algorithms by solving MDPs.

Let us show that, POIBMDPs associated with a supervised learning problems formulated as an MDP, are also MDPs.

Let us define such supervised learning MDPs in the context of a classification task.

**Definition 16** (Classification Markov Decision Process). *Given a set of N examples denoted* $\mathcal{E} = \{(x_i, y_i)\}_{i=1}^N$ *where each datum* $x_i$ *is described by a set of p features and* $y_i \in \mathbb{Z}^m$ *is the label associated with* $x_i$*, a classification Markov decision Process is an MDP (cite)* $\langle S, A, R, T, T_0 \rangle$ *where :*

- *the state space is* $S = \{x_i\}_{i=1}^N$*, the set of data features*
- *the action space is* $A = \mathbb{Z}^m$*, the set of unique labels*
- *the reward function is* $R : S \times A \to \{0, 1\}$ *with* $R(s = x_i, a) = 1_{a=y_i}$
- *the transition function is* $T : S \times A \to \Delta S$ *with* $T(s, a, s') = \frac{1}{N} \quad \forall s, a, s'$
- *the initial distribution is* $T_0(s_0 = s) = \frac{1}{N}$

One can be convinced that policies that maximize the RL objective (cite) in classification MDPs (cite) are classifiers that maximize the prediction accuracy because $\sum_{i=1}^N 1_{\pi(x_i)=y_i} = \sum_{i=1}^N R(x_i, \pi(x_i))$. We defer the formal proof in the next part of the manuscript in which we extensively study supervised learning problems.

In Figure (cite) we give an example of such classification MDP with 4 data in the training set and 2 classes :

$$\mathcal{X} = \{(0.5, 0.5), (0.5, 1.5), (1.5, 1.5), (1.5, 0.5)\}$$
$$y = \{0, 0, 1, 1\}$$

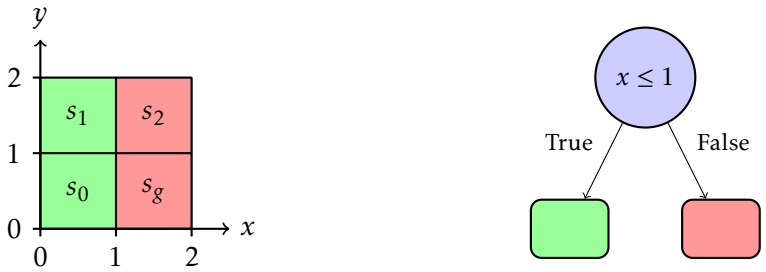



FIGURE 4.1 – Classification MDP optimal actions. In this classification MDP, there are four data to which to assign either a green or red label. On the right, there is the unique optimal depth-1 tree for this particular classification MDP. This depth-1 tree also maximizes the accuracy on the corresponding classification task.

Now let us show that associated POIBMDPs are in fact MDPs. We show this by construction.

**Definition 17** (Classification POIBMDP). *Given a classification MDP (cite)* $\langle \{x_i\}_{i=1}^N, \mathbb{Z}^m, R, T, T_0 \rangle$*, and an associated POIBMDP (cite)* $\langle S, O, A, A_{info}, R, \zeta, T_{info}, T, T_0 \rangle$*, a classification POIBMDP*

*is an MDP :*

$$\langle \overbrace{O}^{\text{State space}}, \underbrace{\mathbb{Z}^m, A_{info}}_{\text{Action space}}, \overbrace{R, \zeta}^{\text{Reward function}}, \underbrace{\mathcal{P}, \mathcal{P}_0}_{\text{Transition kernels}} \rangle$$

- *$O$ is the set of possible observations in $[L_1, U_1] \times \cdots \times [L_d, U_d] \times [L_1, U_1]$ $times \cdots \times$ $[L_d, U_d]$ where $L_j$ is the minimum value of feature $j$ over all data $x_i$ and $U_j$ the maximum*
- *$\mathbb{Z}^m \cup A_{info}$ is action space : actions can be label assignments in $\mathbb{Z}^m$ or bounds refinement in $A_{info}$*
- *The reward for assigning label $a \in \mathbb{Z}^m$ when observing some observation $o = (L'_1, U'_1, \ldots, L'_d, U'_d)$ is the proportion of training data satistifying the bounds and having label $a$ : $R(o, a) = \frac{|\{x_i : L'_j \leq x_{ij} \leq U'_j \forall i, j\} \cap \{x_i : y_i = a \forall i\}|}{|\{x_i : L'_j \leq x_{ij} \leq U'_j \forall i, j\}|}$. The reward for taking an information gathering action that refines bounds is $\zeta$*
- *The transition kernel is $\mathcal{P} : O \times (\mathbb{Z}^m \cup A_{info}) \to \Delta O$ where :*
  - *For $a \in \mathbb{Z}^m$ : $\mathcal{P}(o, a, (L_1, U_1, \ldots, L_d, U_d)) = 1$ (reset to full bounds)*
  - *For $a = (k, v) \in A_{info}$ : from $o = (L'_1, U'_1, \ldots, L'_d, U'_d)$, the MDP will transit to $o_{left} = (L'_1, U'_1, \ldots, L_k, v, dots, L'_d, U'_d)$ (resp. $o_{right} = (L'_1, U'_1, \ldots, U'_k, v, dots, L'_d, U'_d)$) with probability $\frac{|\{x_i : L'_j \leq x_{ij} \leq U'_j \forall j \wedge x_{ik} \leq v\}|}{|\{x_i : L'_j \leq x_{ij} \leq U'_j \forall j\}|}$ (resp. $\frac{|\{x_i : L'_j \leq x_{ij} \leq U'_j \forall j \wedge x_{ik} > v\}|}{|\{x_i : L'_j \leq x_{ij} \leq U'_j \forall j\}|}$)*

We have constructed classification POIBMDPs that are POIBMDP for learning decision tree classifiers. Next, we apply RL to learn deterministic policies $O \to A$ in classification POIBMDPs.

## 4.1 How well can RL baselines learn in Classification POIBMDPs ?

Similarly to the previous chapter, we are interested in a very simple classification POIBMDP. We study classification POIBMDPs associated with the example classification MDP from Figure (cite).

We construct Classification POIBMDPs with $\gamma = 0.99$, 200 values of $\zeta \in [0, 1]$ and IGAs $x \leq 1$ and $y \leq 1$. Since Classification POIBMDPs are MDPs, we do not need to analyze asymmetric RL and JSJ baselines (cite).

Fortunately this time, compared to general POIBMDPs, RL can be used to retrieve optimal policies in classification POIBMDPs equivalent to decision tree classifiers. We observe on Figure (cite) that both Q-learning and Sarsa consistently minimize the

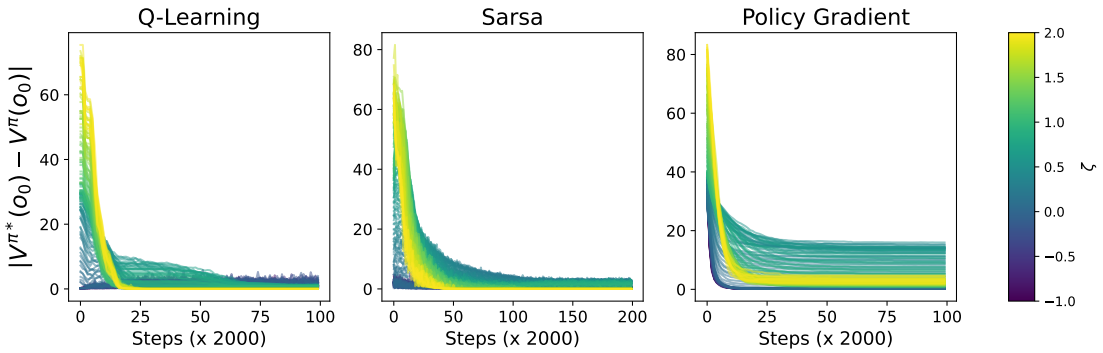



FIGURE 4.2

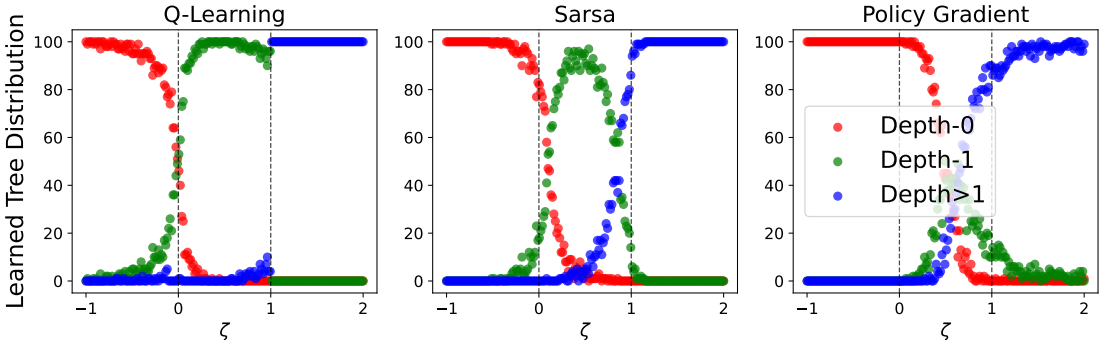



FIGURE 4.3

sub-optimality gap. Hence they are able to retrieve the optimal detph-1 decision tree classifier from Figure (cite).

In this part of the manuscript, we have highlighted the challenges of directly learning decision tree policies for MDPs. In particular in Chapter (cite), we showed that when we tried to reproduce existing baselines for direct RL of decision tree policies (cite), those often performed worse in terms of RL objective than the indirect approaches VIPER or Dagger even though the latter optimize the imitation learning objective (cite).

We further analyzed the failure mode of direct learning of decision tree policies by making connexions with POMDPs. In Chapter (cite), we showed that learning decision tree policies for MDPs that optimize the RL objective could be explicitely formulated as learning a deterministic partially observable (also known as memoryless or reactive) policy in a specific POMDP that we called POIBMDP. We showed that both RL and asymmetric RL, a class of algorithms specifically designed for POMDPs, were unable to consistently retrieve optimal depth 1 decision tree policy for a very small $2 \times 2$ grid world.

Finally, in this Chapter we showed that RL in fully observable POIBMDPs, i.e. POIBMDPs that are just MDPs, could retrieve optimal decision tree policies, we conjecture that most of the difficulty for the general POIBMDP case arise from the partial observability. This provides one clear avenue for future research : developing algorithm tailored for POIBMDPs.

However it is hard to encourage such research when indirect methods like VIPER or Dagger already optimize the RL objective pretty well and often require few samples if an neural network expert policy is already available.

In the next part of this manuscript, we extend the ideas of classification POIBMDPs (cite) and show how to design competitive decision tree induciton algorithms for the supervised learning objective using techniques from MDPs.

## Deuxième partie

# An easier problem : Learning Decision Trees for MDPs that are Classification tasks

Chapitre 5

# DPDT-intro

In supervised learning, decision trees are valued for their interpretability and performance. While greedy decision tree algorithms like CART remain widely used due to their computational efficiency, they often produce sub-optimal solutions with respect to a regularized training loss. Conversely, optimal decision tree methods can find better solutions but are computationally intensive and typically limited to shallow trees or binary features. We present Dynamic Programming Decision Trees (DPDT), a framework that bridges the gap between greedy and optimal approaches. DPDT relies on a Markov Decision Process formulation combined with heuristic split generation to construct near-optimal decision trees with significantly reduced computational complexity. Our approach dynamically limits the set of admissible splits at each node while directly optimizing the tree regularized training loss. Theoretical analysis demonstrates that DPDT can minimize regularized training losses at least as well as CART. Our empirical study shows on multiple datasets that DPDT achieves near-optimal loss with orders of magnitude fewer operations than existing optimal solvers. More importantly, ex-

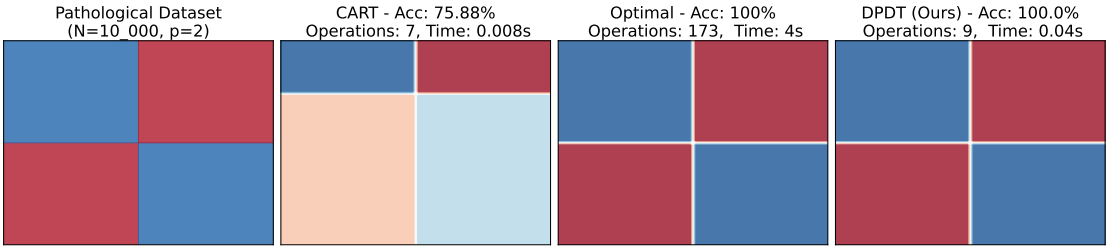



FIGURE 5.1 – Pathological dataset and learned depth-2 trees with their scores, complexities, runtimes, and decision boundaries.

tensive benchmarking suggests statistically significant improvements of DPDT over both CART and optimal decision trees in terms of generalization to unseen data. We demonstrate DPDT practicality through applications to boosting, where it consistently outperforms baselines. Our framework provides a promising direction for developing efficient, near-optimal decision tree algorithms that scale to practical applications.

## 5.1 Introduction

Decision trees [103, 102, 20] are at the core of various machine learning applications. Ensembles of decision trees such as tree boosting [48, 47, 27, 98] are the state-of-the-art for supervised learning on tabular data [53]. Human users can make sense of decision trees predictions [39, 74, 22] which allows for real-world applications when safety or trust is critical [110]. More recently, decision trees have been used to model sequential decision policies with imitation learning [**kohler2024interpretable**, 11] or directly with reinforcement learning (RL) [**marton2024sympolsymbolictreebasedonpolicy**, 125, 136, 135].

To motivate the design of new decision tree induction algorithms, Figure 5.1 exhibits a dataset for which existing greedy algorithms are suboptimal, and optimal algorithms are computationally expensive. The dataset is made up of $N = 10^4$ samples in $p = 2$ dimensions that can be perfectly labeled with a decision tree of depth 2. When running CART [20], greedily choosing the root node yields a suboptimal tree. This is because greedy algorithms compute locally optimal splits in terms of information gain. In our example, the greedy splits always give two children datasets which themselves need depth 2 trees to be perfectly split. On the other hand, to find the root node, an optimal algorithm such as [86] iterates over all possible splits, that is, $N \times p = 20,000$ operations to find one node of the solution tree.

In this work, we present a framework for designing non-greedy decision tree induction algorithms that optimize a regularized training loss nearly as well as optimal methods. This is achieved with orders of magnitude less operations, and hence dramatic computation savings. We call this framework "Dynamic Programming Decision Trees" (DPDT). For every node, DPDT heuristically and dynamically limits the set of admissible splits to a few good candidates. Then, DPDT optimizes the regularized training loss with some depth constraints. Theoretically, we show that DPDT minimizes the empirical risk at least as well as CART. Empirically, we show that on all tested datasets, DPDT can reach 99% of the optimal regularized train accuracy while using thousands times less operations than current optimal solvers. More importantly, we follow [53] methodology

to benchmark DPDT against both CART and optimal trees on hard datasets. Following the same methodology, we compare boosted DPDT [46] to boosted CART and to some deep learning methods and show clear superiority of DPDT.

## 5.2 Related Work

To learn decision trees, greedy approaches like CART [20] iteratively partition the training dataset by taking splits optimizing a local objective such as the Gini impurity or the entropy. This makes CART suboptimal with respect to training losses [92]. But CART remains the default decision tree algorithm in many machine learning libraries such as [95, 27, 62, 142] because it can scale to very deep trees and is very fast. To avoid overfitting, greedy trees are learned with a maximal depth or pruned a posteriori [20, chapter 3]. In recent years, more complex optimal decision tree induction algorithms have shown consistent gains over CART in terms of generalization capabilities [15, 132, 36].

Optimal decision tree approaches optimize a regularized training loss while using a minimal number of splits [15, 3, 133, 86, 36, 37, 72, 26]. However, direct optimization is not a convenient approach, as finding the optimal tree is known to be NP-Hard [59]. Despite the large number of algorithmic tricks to make optimal decision tree solvers efficient [36, 86], their complexity scales with the number of samples and the maximum depth constraint. Furthermore, optimal decision tree induction algorithms are usually constrained to binary-features dataset while CART can deal with any type of feature. When optimal decision tree algorithms deal with continuous features, they can usually learn only shallow trees, e.g. Quant-BnB [86] can only compute optimal trees up to depth 3. `PySTreeD`, the latest optimal decision tree library [72], can compute decision trees with depths larger than three but uses heuristics to binarize a dataset with continuous features during a pre-processing step. Despite their limitations to binary features and their huge computational complexities, encouraging practical results for optimal trees have been obtained [91, 71, 30, 73]. Among others, they show that optimal methods under the same depth constraint (up to depth four) find trees with 1–2% greater test accuracy than greedy methods.

In this work, we only consider the induction of nonparametric binary depth-constrained axis-aligned trees. By nonparametric trees, we mean that we only consider tree induction algorithms that optimize both features and threshold values in internal nodes of the tree. This is different from the line of work on Tree Alternating Optimization (TAO) algorithm [24, 141, 25] that only optimizes tree nodes threshold values for fixed nodes

features similarly to optimizing neural network weights with gradient-based methods.

There exist many other areas of decision tree research [77] such as inducing non-axis parallel decision trees [93, 61], splitting criteria of greedy trees [73], different optimization of parametric trees [94, 98], or pruning methods [43, 90].

Our work is not the first to formulate the decision tree induction problem as solving a Markov decision process (MDP) [40, 49, 125, 26]. Those works formulate tree induction as solving a partially observable MDP and use approximate algorithms such as Q-learning [49] or deep Q-learning [125] to solve them in an online fashion one datum from the dataset at a time. In a nutshell, our work, DPDT that we present next, is different in that it builds a stochastic and fully observable MDP that can be explicitly solved with dynamic programming. This makes it possible to solve exactly the decision tree induction problem.

Chapitre 6

# DPDT-paper

## 6.1 Decision Trees for Supervised Learning

Let us briefly introduce some notations for the supervised classification problem considered in this paper. We assume that we have access to a set of $N$ examples denoted $\mathcal{E} = \{(x_i, y_i)\}_{i=1}^{N}$. Each datum $x_i$ is described by a set of $p$ features. $y_i \in \mathcal{Y}$ is the label associated with $x_i$.

A decision tree is made of two types of nodes : split nodes that are traversed, and leaf nodes that finally assign a label. To predict the label of a datum $x$, a decision tree $T$ sequentially applies a series of splits before assigning it a label $T(x) \in \mathcal{Y}$. In this paper, we focus on binary decision trees with axis-aligned splits as in [20], where each split compares the value of one feature with a threshold.

Our goal is to learn a tree that generalizes well to unseen data. To avoid overfitting, we constrain the maximum depth $D$ of the tree, where $D$ is the maximum number of splits that can be applied to classify a data. We let $\mathcal{T}_D$ be the set of all binary decision trees of depth $\leq D$. Given a loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$, we induce trees with a regularized training loss defined by :

$$
\begin{aligned}
T^* &= \underset{T \in \mathcal{T}_D}{\operatorname{argmin}} \; \mathcal{L}_\alpha(T), \\
T^* &= \underset{T \in \mathcal{T}_D}{\operatorname{argmin}} \; \frac{1}{N} \sum_{i=1}^{N} \ell(y_i, T(x_i)) + \alpha C(T),
\end{aligned}
\tag{6.1}
$$

where $C : \mathcal{T} \to \mathbb{R}$ is a complexity penalty that helps prevent or reduce overfitting

such as the number of nodes [20, 86], or the expected number of splits to label a data[91]. The complexity penalty is weighted by $\alpha \in [0,1]$. For supervised classification problems, we use the 0–1 loss : $\ell(y_i, T(x_i)) = 1_{\{y_i \neq T(x_i)\}}$. Please note while we focus on supervised classification problems in this paper, our framework extends naturally to regression problems.

We now formulate the decision tree induction problem 6.1 as finding the optimal policy in an MDP.

## 6.2 Decision Tree Induction as an MDP

Given a set of examples $\mathcal{E}$, the induction of a decision tree is made of a sequence of decisions : at each node, we must decide whether it is better to split (a subset of) $\mathcal{E}$, or to create a leaf node.

This sequential decision-making process corresponds to a Markov Decision Problem (MDP) [100] $\mathcal{M} = \langle S, A, R_\alpha, P, D \rangle$. A state is a pair made of a subset of examples $X \subseteq \mathcal{E}$ and a depth $d$. Then, the set of states is $S = \{(X,d) \in P(\mathcal{E}) \times \{0,\dots,D\}\}$ where $P(\mathcal{E})$ denotes the power set of $\mathcal{E}$. $d \in \{0,\dots,D\}$ is the current depth in the tree. An action $A$ consists in creating either a split node, or a leaf node (label assignment). We denote the set of candidate split nodes $\mathcal{F}$. A split node in $\mathcal{F}$ is a pair made of one feature $i$ and a threshold value $x_{ij} \in \mathcal{E}$. So, we can write $A = \mathcal{F} \cup \{1,\dots,K\}$. From state $s = (X,d)$ and a splitting action $a \in \mathcal{F}$, the transition function $P$ moves to the next state $s_l = (X_l, d+1)$ with probability $p_l = \frac{|X_l|}{|X|}$ where $X_l = \{(x_i, y_i) \in X : x_i \leq x_{ij}\}$, or to state $s_r = (X \setminus X_l, d+1)$ with probability $1 - p_l$. For a class assignment action $a \in \{1,\dots,K\}$, the chain reaches an absorbing terminal state with probability 1. The reward function $R_\alpha : S \times A \to \mathbb{R}$ returns $-\alpha$ for splitting actions and the proportion of misclassified examples of $X$ $-\frac{1}{|X|}\sum_{(x_i,y_i)\in X} \ell(y_i, a)$ for class assignment actions. $\alpha \in [0,1]$ controls the accuracy-complexity trade-off defined in the regularized training objective 6.1. The horizon $D$ limits tree depth to $D$ by forbidding class assignments after $D$ MDP transitions.

The solution to this MDP is a deterministic policy $\pi : S \to A$ that maximizes $J_\alpha(\pi) = \mathbb{E}\left[\sum_{t=0}^{D} R_\alpha(s_t, \pi(s_t))\right]$, the expected sum of rewards where the expectation is taken over transitions $s_{t+1} \sim P(s_t, \pi(s_t))$ starting from initial state $s_0 = (\mathcal{E}, 0)$. Any such policy can be converted into a binary decision tree through a recursive extraction function $E(\pi, s)$ that returns, either a leaf node with class $\pi(s)$ if $\pi(s)$ is a class assignment, or a tree with root node containing split $\pi(s)$ and left/right sub-trees $E(\pi, s_l)/E(\pi, s_r)$ if $\pi(s)$ is a split. The final decision tree $T$ is obtained by calling $E(\pi, s_0)$ on the initial state $s_0$.

**Proposition 3** (Objective Equivalence)**.** *Let $\pi$ be a deterministic policy of the MDP and $\pi^*$ be an optimal deterministic policy. Then $J_\alpha(\pi) = -\mathcal{L}_\alpha(E(\pi, s_0))$ and $T^* = E(\pi^*, s_0)$ where $T^*$ is a tree that optimizes Eq. 6.1.*

This proposition is key as it states that the return of any policy of the MDP defined above is equal to the regularized training accuracy of the tree extracted from this policy. A consequence of this proposition is that when all possible splits are considered, the optimal policy will generate the optimal tree in the sense defined by Eq. (6.1). The proof is given in the Appendix 6.6.

## 6.3 Algorithm

We now present the Dynamic Programming Decision Tree (DPDT) induction algorithm. The algorithm consists of two essential steps. The first and most computationally expensive step constructs the MDP presented in Section 6.2. The second step solves it to obtain a policy that maximizes Eq.6.2 and that is equivalent to a decision tree. Both steps are now detailed.

### 6.3.1 Constructing the MDP

An algorithm constructing the MDP of section 6.2 essentially computes the set of all possible decision trees of maximum depth $D$ which decision nodes are in $\mathcal{F}$. The transition function of this specific MDP is a directed acyclic graph. Each node of this graph corresponds to a state for which one computes the transition and reward functions. Considering all possible splits in $\mathcal{F}$ does not scale. We thus introduce a state-dependent action space $A_s$, much smaller than $A$ and populated by a splits generating function. In Figure **??**, we illustrate the MDP constructed for the classification of a toy dataset using some arbitrary splitting function.

### 6.3.2 Heuristic splits generating functions

A split generating function is any function $\phi$ that maps an MDP state, i.e., a subset of training examples, to a split node. It has the form $\phi : S \to P(\mathcal{F})$, where $P(\mathcal{F})$ is the power set of all possible split nodes in $\mathcal{F}$. For a state $s \in S$, the state-dependent action space is defined by $A_s = \phi(s) \cup \{1, \dots, K\}$.

When the split generating function does not return all the possible candidate split nodes given a state, solving the MDP with state-dependent actions $A_s$ is not guaranteed to yield the minimizing tree of Eq. 6.1, as the optimization is then performed on the

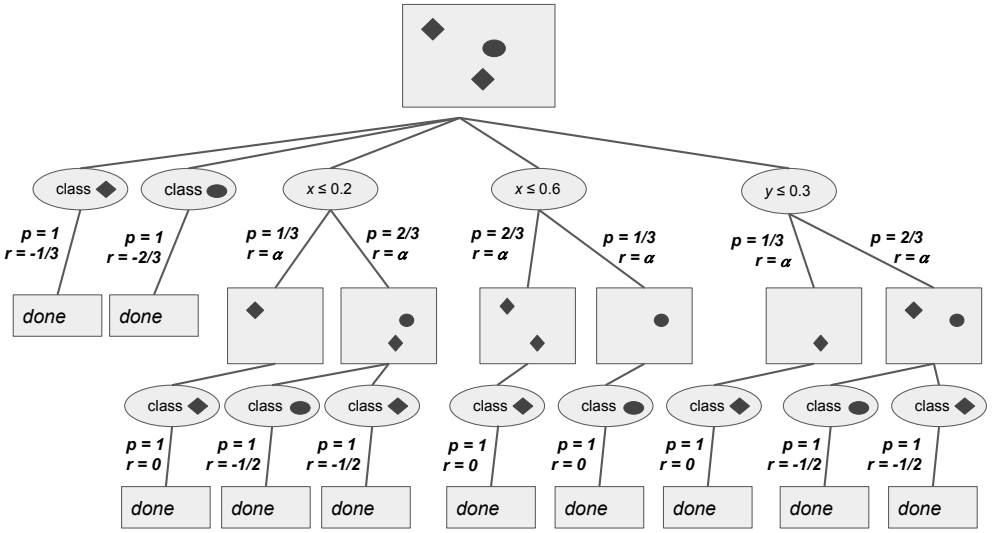



FIGURE 6.1 – Schematics of the MDP to learn a decision tree of depth 2 to classify a toy dataset with three samples, two features (x,y), and two classes (oval, diamond) and using an arbitrary splits generating function.

subset of trees of depth smaller or equal to $D$, $\mathcal{T}_D$. We now define some interesting split generating functions and provide the time complexity of the associated decision tree algorithms. The time complexity is given in big-O of the number of candidate split nodes considered during computations.

**Exhaustive function.** When $\mathcal{F} \subseteq \phi(s), \forall s \in S$, the MDP contains all possible splits of a certain set of examples. In this case, *the optimal MDP policy is the optimal decision tree of depth at most D*, and the number of states of the MDP would be $O((2Np)^D)$. Solving the MDP for $A_s = \phi(s)$ is equivalent to running one of the optimal tree induction algorithms [132, 15, 72, 86, 133, 36, 3, 37, 71, 26]

**Top $B$ most informative splits.** [17] proposed to generate splits with a function that returns, for any state $s = (X, d)$, the $B$ most informative splits over $X$ with respect to some information gain measure such as the entropy or the Gini impurity. The number of states in the MDP would be $O((2B)^D)$. *When $B = 1$, the optimal policy of the MDP is the greedy tree.* In practice, we noticed that the returned set of splits lacked diversity and often consists of splits on the same feature with minor changes to the threshold value.

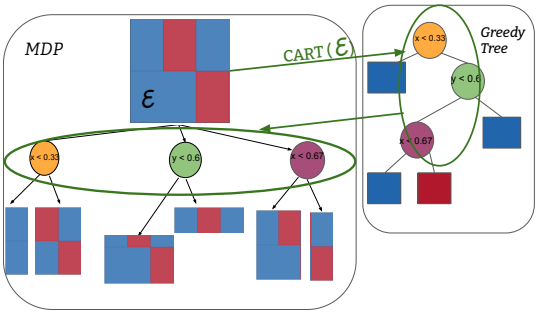



FIGURE 6.2 – How CART is used in DPDT to generate candidate splits given the example data in the current state.

**Calls to CART** Instead of returning the most informative split at each state $s = (X, d)$, we propose to find the most discriminative split, i.e. the feature splits that best predicts the class of data in $X$. We can do this by considering the split nodes of the greedy tree. In practice, we run CART on $s$ and use the returned nodes as $\phi(s)$. We control the number of MDP states by constraining CART trees with a maximum number of nodes $B : \phi(s) = nodes(\text{CART}(s, max\_nodes = B))$ The number of MDP states would be $O((2B)^D)$.*When* $B = 1$, *the MDP policy corresponds to the greedy tree.* The process of generating split nodes with calls to CART is illustrated in Figure 6.2.

### 6.3.3 Dynamic Programming to solve the MDP

**Algorithme 16 :** DPDT

**Data :** Dataset $\mathcal{E}$, max depth $D$, split function $\phi()$,
split function parameter $B$, regularizing term $\alpha$
**Result :** Tree $T$
$\mathcal{M} \leftarrow build\_mdp(\mathcal{E}, D, \phi(), B)$
`// Backward induction`
$Q^*(s,a) \leftarrow R_\alpha^{\mathcal{M}}(s,a) + \sum_{s'} P^{\mathcal{M}}(s,a,s') \max_{a' \in A_{s'}^{\mathcal{M}}} Q^*(s',a') \forall s, a \in \mathcal{M}$
`// Get the optimal policy`
$\pi^*(s) =_{a \in A_s^{\mathcal{M}}} Q^*(s,a) \forall s \in \mathcal{M}$
`// Extracting tree from policy`
$T \leftarrow E(\pi^*, s_0^{\mathcal{M}})$

After constructing the MDP with a chosen splits generating function, we solve for the optimal policy using dynamic programming. Starting from terminal states and working backward to the initial state, we compute the optimal state-action values using Bellman's

optimality equation [**BELLMAN1958228**], and then deducing the optimal policy.

From now on, we write DPDT to denote Algorithm 16 when the split function is a call to CART. We discuss key bottlenecks when implementing DPDT in subsequent sections. We now state theoretical results when using DPDT with the CART heuristic.

### 6.3.4 Performance Guarantees of DPDT

We now show that : 1) DPDT minimizes the loss from Eq. 6.1 at least as well as greedy trees and 2) there exists problems for which DPDT has strictly lower loss than greedy trees. As we restrict the action space at a given state $s$ to a subset of all possible split nodes, DPDT is not guaranteed to find the tree minimizing Eq. 6.1. However, we are still guaranteed to find trees that are better or equivalent to those induced by CART :

**Theorem 1** (MDP solutions are not worse than the greedy tree)**.** *Let $\pi^*$ be an optimal deterministic policy of the MDP, where the action space at every state is restricted to the top B most informative or discriminative splits. Let $T_0$ be the tree induced by CART and $\{T_1, \dots, T_M\}$ all the sub-trees of $T_0$,* [1] *then for any $\alpha > 0$,*

$$\mathcal{L}_\alpha(E(\pi^*, s_0)) \leq \min_{0 \leq i \leq M} \mathcal{L}_\alpha(T_i)$$

*Démonstration.* Let us first define $C(T)$, the expected number of splits performed by tree $T$ on dataset $\mathcal{E}$. Here $T$ is deduced from policy $\pi$, i.e. $T = E(\pi, s_0)$. $C(T)$ can be defined recursively as $C(T) = 0$ if $T$ is a leaf node, and $C(T) = 1 + p_l C(T_l) + p_r C(T_r)$, where $T_l = E(\pi, s_l)$ and $T_r = E(\pi, s_r)$. In words, when the root of $T$ is a split node, the expected number of splits is one plus the expected number of splits of the left and right sub-trees of the root node. □

It is known that the greedy tree of depth 2 fails to perfectly classify the XOR problem as shown in Figure 5.1 and in [92, 91]. We aim to show that DPDT is a cheap way to alleviate the weaknesses of greedy trees in this type of problems. The following theorem states that there exist classification problems such that DPDT optimizes the regularized training loss strictly better than greedy algorithms such as CART, ID3 or C4.5.

**Theorem 2** (DPDT can be strictly better than greedy)**.** *There exists a dataset and a depth D such that the DPDT tree $T_D^{DPDT}$ is strictly better than the greedy tree $T_D^{greedy}$ , i.e, $\mathcal{L}_{\alpha=0}(T_D^{greedy}) > \mathcal{L}_{\alpha=0}(T_D^{DPDT})$.*

1. These sub-trees are interesting to consider since they can be returned by common postprocessing operations following a call to CART, that prune some of the nodes from $T_0$. Please see [43] for a review of pruning methods for decision trees.

The proof of this theorem is given in the next section.

### 6.3.5 Proof of Improvement over CART

In this section we construct a dataset for which the greedy tree of depth 2 fails to accurately classify data while DPDT with calls to CART as a splits generating function guarantees a strictly better accuracy. The dataset is the XOR pattern like in Figure 5.1. We will first show that greedy tree induction like CART chooses the first split at random and the second split in between the two columns or rows. Then we will quantify the misclassification of the depth-2 greedy tree on the XOR gate. Finally we will show that using the second greedy split as the root of a tree and then building the remaining nodes greedily, i.e. running DPDT with the CART heuristic, strictly decreases the misclassification.

**Definition 18** (XOR dataset)**.** *Let us defined the XOR dataset as* $\mathcal{E}_{XOR} = \{(X_i, Y_i)\}_{i=1}^{N}$. $X_i = (x_i, y_i) \sim \mathcal{U}([0,1]^2)$ *are i.i.d 2-features samples.* $Y_i = f(X_i)$ *are alternating classes with* $f(x,y) = (\lfloor 2x \rfloor + \lfloor 2y \rfloor) \bmod 2$.

**Lemma 1.** *The first greedy split is chosen at random on the XOR dataset from definition 18.*

*Démonstration.* Let us consider an arbitrary split $x = x_v$ parallel to the y-axis. The results apply to splits parallel to the x-axis because the XOR pattern is the same when rotated 90 degrees. The split $x_v$ partitions the dataset into two regions $R_{left}$ and $R_{right}$. Since the dataset has two columns and two rows, any rectangular area that spans the whole height $[0,1)$ has the same proportion of class 0 samples and class 1 samples from definition 18. So in both $R_{left}$ and $R_{right}$ the probabilities of observing class 0 or class 1 at random are $\frac{1}{2}$. Since the class distributions in left and right regions are independent of the split location, all splits have the same objective value when the objective is a measure of information gain like the entropy or the Gini impurity. Hence, the first split in a greedily induced tree is chosen at random. □

**Lemma 2.** *When the first split is greedy on the XOR dataset from definition 18, the second greedy splits are chosen perpendicular to the first split at* $y = \frac{1}{2}$

*Démonstration.* Assume without loss of generality due to symmetries, that the first greedy split is vertical, at $x = x_v$, with $x_v <= \frac{1}{2}$. This split partitions the unit square into $R_{left} = [0, x_v) \times [0,1)$ and $R_{right} = [x_v, 1) \times [0,1)$. The split $y = \frac{1}{2}$ further partitions $R_{left}$ into $R_{left-down}$ and $R_{left-up}$ with same areas $x_v \times y = \frac{x_v}{2}$. Due to the XOR pattern, there

are only samples of class 0 in $R_{left-down}$ and only samples of class 1 in $R_{left-up}$. Hence the the split $y = \frac{1}{2}$ maximizes the information gain in $R_{left}$, hence the second greedy split given an arbitrary first split $x = x_v$ is necessarily $y = \frac{1}{2}$. □

**Definition 19** (Forced- Tree)**.** *Let us define the forced-tree as a greedy tree that is forced to make its first split at* $y = \frac{1}{2}$.

**Lemma 3.** *The forced-tree of depth 2 has a 0 loss on the XOR dataset from definition 18 while, with probability* $1 - \frac{1}{|\mathcal{E}_{XOR}|}$*, the greedy tree of depth 2 has strictly positive loss.*

*Démonstration.* This is trivial from the definition of the forced tree since if we start with the split $y = \frac{1}{2}$, then clearly CART will correctly split the remaining data. If instead the first split is some $x_v \neq \frac{1}{2}$ then CART is bound to make an error with only one extra split allowed. Since the first split is chosen at random, from Lemma 6.3.5, there are only two splits ($x = \frac{1}{2}$ and $y = \frac{1}{2}$) out of $2|\mathcal{E}_{XOR}|$ that do not lead to sub-optimality. □

We can now formally prove theorem 2.

*Démonstration.* By definition of DPDT, all instances of DPDT with the CART nodes parameter $B \geq 2$ include the forced-tree from definition 19 in their solution set when applied to the XOR dataset (definition 18). We know from lemma 3 that with high probability, the forced-tree of depth 2 is strictly more accurate than the greedy tree of depth 2 on the XOR dataset. Because we know by proposition 3 that DPDT returns the tree with maximal accuracy from its solution set, we can say that DPDT depth-2 trees are strictly better than depth-2 greedy trees returned by e.g. CART on the XOR dataset. □

### 6.3.6 Practical Implementation

The key bottlenecks lie in the MDP construction step of DPDT (Section 6.2). In nature, all decision tree induction algorithms have time complexity exponential in the number of training subsets per tree depth $D$ : $O((2B)^D)$, e.g., CART has $O(2^D)$ time complextiy. We already saw that DPDT saves time by not considering all possible tree splits but only $B$ of them. Using state-dependent split generation also allows to generate more or less candidates at different depths of the tree. Indeed, the MDP state $s = (X, d)$ contains the current depth during the MDP construction process. This means that one can control DPDT's time complexity by giving multiple values of maximum nodes : given $(B_1, B_2, ..., B_D)$, the splits generating function in

Algorithm 16 becomes $\phi(s_i) = \phi(X_i, d = 1) = nodes(\text{CART}(s, max\_nodes = B_1))$ and $\phi(s_j) = \phi(X_j, d = 2) = nodes(\text{CART}(s, max\_nodes = B_2))$.

Similarly, the space complexity of DPDT is exponential in the space required to store training examples $\mathcal{E}$. Indeed, the MDP states that DPDT builds in Algorithm 16 are training samples $X \subseteq \mathcal{E}$. Hence, the total space required to run DPDT is $O(Np(2B)^D)$ where $Np$ is the size of $\mathcal{E}$. In practice, one should implement DPDT in a depth first search manner to obtain a space complexity linear in the size of training set : $O(DNp)$. In practice DPDT builds the MDP from Section 6.2 by starting from the root and recursively splitting the training set while backpropagating the $Q$-values. This is possible because the MDP we solve has a (not necessarily binary) tree structure (see Figure **??**) and because the $Q$-values of a state only depend on future states.

We implemented DPDT [2] following scikit-learn API [23] with depth-first search and state-depth-dependent splits generating.

## 6.4 Empirical Evaluation

In this section, we empirically demonstrate strong properties of DPDT trees. The first part of our experiments focuses on the quality of solutions obtained by DPDT for objective Eq.6.1 compared to greedy and optimal trees. We know by theorems 1 and 2 that DPDT trees should find better solutions than greedy algorithms for certain problems ; but what about real problems ? After showing that DPDT can find optimal trees by considering much less solutions and thus performing orders of magnitude less operations, we will study the generalization capabilities of the latter : do DPDT trees label unseen data accurately ?

### 6.4.1 DPDT optimizing capabilities

From an empirical perspective, it is key to evaluate DPDT training accuracy since optimal decision tree algorithms against which we wish to compare ourselves are designed to optimize the regularized training loss Eq.6.1.

#### Setup

**Metrics :** we are interested in the regularized training loss of algorithms optimizing Eq.6.1 with $\alpha = 0$ and a maximum depth $D$. We are also interested in the number of key operations performed by each baseline, namely computing candidate split nodes for

2. https://github.com/KohlerHECTOR/DPDTreeEstimator

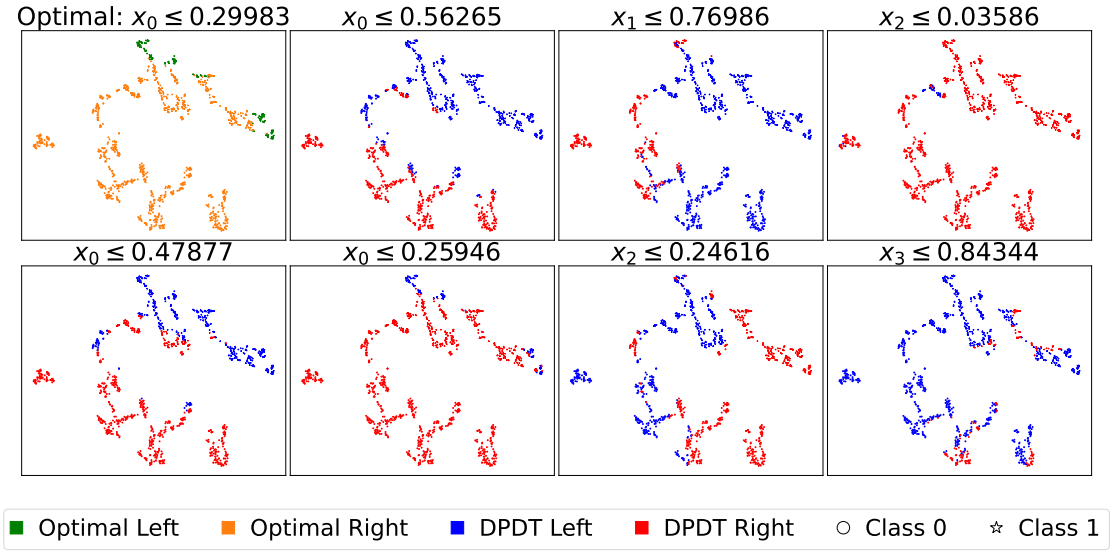



FIGURE 6.3 – Root splits candidate obtained with DPDT compared to the optimal root split on the Bank dataset. Each split creates a partition of $p$-dimensional data that we projected in the 2-dimensional space using t-SNE.

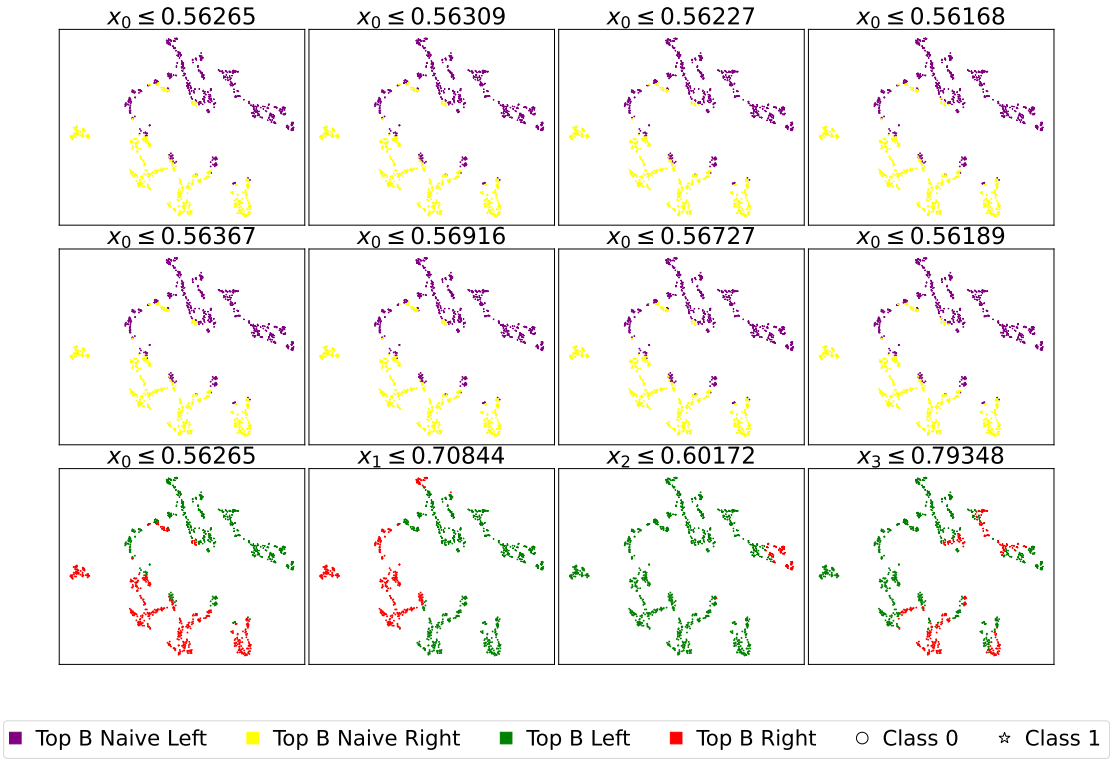



FIGURE 6.4 – Root splits candidate obtained with Top-B[17] on the Bank dataset. Each split creates a partition of $p$-dimensional data that we projected using t-SNE.

Tableau 6.1 – Comparison of train accuracies of depth-3 trees and number of operations on classification tasks. For DPDT and Top-B, "light" configurations have split function parameters (8, 1, 1) "full" have parameters (8, 8, 8). We also include the mean train accuracy over 5 deep RL runs. **Bold** values are optimal accuracies and blue values are the largest non-optimal accuracies.

| | | | Accuracy | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Opt | Greedy | DPDT | | Top-B | | Deep RL | Opt | Greedy |
| **Dataset** | N | p | Quant-BnB | CART | light | full | light | full | Custard | Quant-BnB | CART |
| room | 8103 | 16 | **0.992** | 0.968 | 0.991 | **0.992** | 0.990 | **0.992** | 0.715 | $10^6$ | 15 |
| bean | 10888 | 16 | **0.871** | 0.777 | 0.812 | 0.853 | 0.804 | 0.841 | 0.182 | $5{\cdot}10^6$ | 15 |
| eeg | 11984 | 14 | **0.708** | 0.666 | 0.689 | 0.706 | 0.684 | 0.699 | 0.549 | $2{\cdot}10^6$ | 13 |
| avila | 10430 | 10 | **0.585** | 0.532 | 0.574 | **0.585** | 0.563 | 0.572 | 0.409 | $3{\cdot}10^7$ | 9 |
| magic | 15216 | 10 | **0.831** | 0.801 | 0.822 | 0.828 | 0.807 | 0.816 | 0.581 | $6{\cdot}10^6$ | 15 |
| htru | 14318 | 8 | **0.981** | 0.979 | 0.979 | 0.980 | 0.979 | 0.980 | 0.860 | $6{\cdot}10^7$ | 15 |
| occup. | 8143 | 5 | **0.994** | 0.989 | 0.991 | **0.994** | 0.990 | 0.992 | 0.647 | $7{\cdot}10^5$ | 13 |
| skin | 196045 | 3 | **0.969** | 0.966 | 0.966 | 0.966 | 0.966 | 0.966 | 0.612 | $7{\cdot}10^4$ | 15 |
| fault | 1552 | 27 | **0.682** | 0.553 | 0.672 | 0.674 | 0.672 | 0.673 | 0.303 | $9{\cdot}10^8$ | 13 |
| segment | 1848 | 18 | **0.887** | 0.574 | 0.812 | 0.879 | 0.786 | 0.825 | 0.137 | $2{\cdot}10^6$ | 7 |
| page | 4378 | 10 | **0.971** | 0.964 | 0.970 | 0.970 | 0.964 | 0.965 | 0.902 | $10^7$ | 15 |
| bidding | 5056 | 9 | **0.993** | 0.981 | 0.985 | **0.993** | 0.985 | **0.993** | 0.810 | $3{\cdot}10^5$ | 13 |
| raisin | 720 | 7 | **0.894** | 0.869 | 0.879 | 0.886 | 0.875 | 0.883 | 0.509 | $4{\cdot}10^6$ | 15 |
| rice | 3048 | 7 | **0.938** | 0.933 | 0.934 | 0.937 | 0.933 | 0.936 | 0.519 | $2{\cdot}10^7$ | 15 |
| wilt | 4339 | 5 | **0.996** | 0.993 | 0.994 | 0.995 | 0.994 | 0.994 | 0.984 | $3{\cdot}10^5$ | 13 |
| bank | 1097 | 4 | **0.983** | 0.933 | 0.971 | 0.980 | 0.951 | 0.974 | 0.496 | $6{\cdot}10^4$ | 13 |

subsets of the training data. We disregard running times as solvers are implemented in different programming languages and/or using optimized code : operations count is more representative of an algorithm efficiency. We also qualitatively compare different decision trees root splits to some optimal root split.

**Baselines :** we benchmark DPDT against greedy trees and optimal trees. For greedy trees we compare DPDT to CART [20]. For optimal trees we compare DPDT to Quant-BnB [86] which is the only solver specialized for depth 3 trees and continuous features. We also consider the non-greedy baseline Top-B [17]. Ideally, DPDT should have training accuracy close to the optimal tree while performing a number of operations close to the greedy algorithm. Furthermore, comparing DPDT to Top-B brings answers to which heuristic splits are better to consider.

We use the CART algorithm implemented in `scikit-learn` [95] in `CPython` with a maximum depth of 3. Optimal trees are obtained by running the `Julia` implementation of the Quant-BnB solver from [86] specialized in depth 3 trees for datasets with conti-

nuous features. We use a time limit of 24 hours per dataset. DPDT and Top-B trees are obtained with Algorithm 16 implemented in pure `Python` and the calls to CART and Top-B most informative splits generating functions from Section 6.2 respectively. We also include Custard, a deep RL baseline [125]. Custard fits a neural network online one datum at a time rather than solving exactly the MDP from Section 6.2 which states are sets of data. Similarly to DPDT, Custard neural network policy is equivalent to a decision tree. We implement Custard with the DQN agent from `stable-baselines3` [105] and train until convergence.

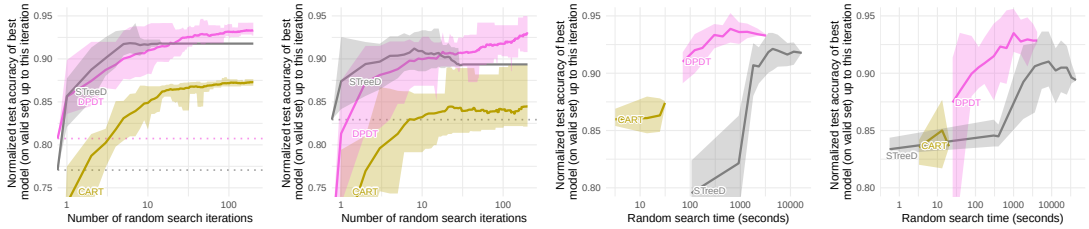
**Datasets :** we us the same datasets as the Quant-BnB paper [86].

**Observations**

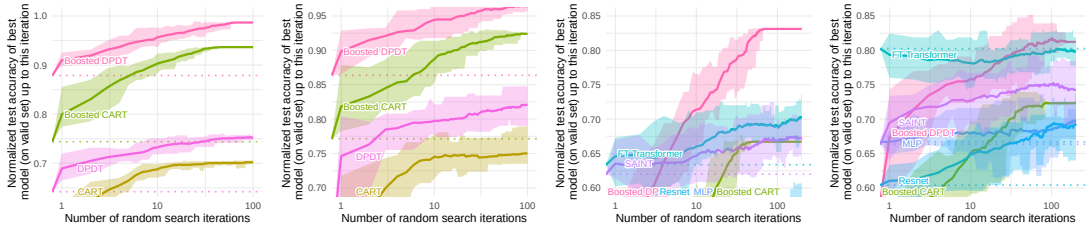
**Near-optimality.** Our experimental results demonstrate that unlike Deep RL, DPDT and Top-B approaches consistently improve upon greedy solutions while requiring significantly fewer operations than exact solvers. Looking at Table 6.1, we observe several key patterns : first, light DPDT with 16 candidate root splits consistently outperforms the greedy baseline in all datasets. This shows that in practice DPDT can be strictly netter than CART outside of theorem 2 assumptions. Second, when comparing DPDT to Top-B, we see that DPDT generally achieves better accuracy for the same configuration. For example, on the bean dataset, full DPDT reaches 85.3% accuracy while full Top-B achieves 84.1%. This pattern holds on most datasets, suggesting that DPDT is more effective than selecting splits based purely on information gain.

Third, both approaches achieve impressive computational efficiency compared to exact solvers. While optimal solutions require between $10^4$ to $10^8$ operations, DPDT and Top-B typically need only $10^2$ to $10^4$ operations, a reduction of 2 to 4 orders of magnitude. Notably, on several datasets (room, avila, occupancy, bidding), full DPDT matches or comes extremely close to optimal accuracy while requiring far fewer operations. For example, on the room dataset, full DPDT achieves the optimal accuracy of 99.2% while reducing operations from $1.34 \times 10^6$ to $1.61 \times 10^4$. These results demonstrate that DPDT provides an effective middle ground between greedy approaches and exact solvers, offering near-optimal solutions with reasonable computational requirements. While both DPDT and Top-B improve upon greedy solutions, DPDT CART-based split generation strategy appears to be particularly effective at finding high-quality solutions.

**DPDT splits** To understand why the CART-based split generation yields more accurate DPDT trees than the Top-B heuristic, we visualize how splits partition the feature space




(a) Single Tree Numerical (b) Single Tree Categorical (c) Single Tree Numerical (d) Single Tree Categorical




(e) Boosting vs Single Tree Num. (f) Boosting vs Single Tree Cat. (g) Boosting vs Neural Numerical (h) Boosting vs Neural Categorical

FIGURE 6.5 – Benchmark on medium-sized datasets. Dotted lines correspond to the score of the default hyperparameters, which is also the first random search iteration. Each value corresponds to the test score of the best model (obtained on the validation set) after a specific number of random search iterations (a, b) or after a specific time spent doing random search (c, d), averaged on 15 shuffles of the random search order. The ribbon corresponds to the minimum and maximum scores on these 15 shuffles.

(Figures 6.3, 6.4). We run both DPDT with splits from CART and DPDT with the Top-B most informative splits on the bank dataset. We use t-SNE to create a two-dimensional representations of the dataset partitions given by candidates root splits from CART and Top-B. The optimal root split for the depth-3 tree for bank–obtained with Quant-BnB–is shown on Figure 6.3 in the top-left subplot using green and orange colors for the resulting partitions. On the same figure we can see that the DPDT split generated with CART $x_0 \leq 0.259$ is very similar to the optimal root split. However, on Figure 6.4 we observe that no Top-B candidate splits resemble the optimal root and that in general Top-B split lack diversity : they always split along the same feature. We tried to enforce diversity by getting the most informative split *per feature* but no candidate split resembles the optimal root.

### 6.4.2 DPDT generalization capabilities

The goal of this section is to have a fair comparison of generalization capabilities of different tree induction algorithms. Fairness of comparison should take into account the number of hyperparameters, choice of programming language, intrinsic purposes of each algorithms (what are they designed to do?), the type of data they can read (categorical features or numerical features). We benchmark DPDT using [53]. We choose this benchmark because it was used to establish XGBoost [27] as the SOTA tabular learning model.

#### Setup

**Metrics :** We re-use the code from [53] [3]. It relies on random searches for hyper-parameter tuning [14]. We run a random search of 100 iterations per dataset for each benchmarked tree algorithms. To study performance as a function of the number $n$ of random search iterations, we compute the best hyperparameter combination on the validation set on these $n$ iterations (for each model and dataset), and evaluate it on the test set. Following [53], we do this 15 times while shuffling the random search order at each time. This gives us bootstrap-like estimates of the expected test score of the best tree found on the validation set after each number of random search iterations. In addition, we always start the random searches with the default hyperparameters of each tree induction aglorithm. We use the test set accuracy (classification) to measure model performance. The aggregation metric is discussed in details in [53, Section 3].

**Datasets :** we use the datasets curated by [53]. They are available on `OpenML` [129] and described in details in [53, Appendix A.1]. The attributes in these datasets are either numerical (a real number), or categorical (a symbolic values among a finite set of possible values). The considered datasets follow a strict selection [53, Section 3] to focus on core learning challenges. Some datasets are very large (millions of samples) like Higgs or Covertype [138, 16]. To ensure non-trivial learning tasks, datasets where simple models (e.g. logistic regression) performed within 5% of complex models (e.g. ResNet [52], HistGradientBoosting [95]) are removed. We use the same data partitioning strategy as [53] : 70% of samples are allocated for training, with the remaining 30% split between validation (30%) and test (70%) sets. Both validation and test sets are capped at 50,000 samples for computational efficiency. All algorithms and hyperparameter combinations were evaluated on identical folds. Finally, while we focus on classification

3. https ://github.com/leogrin/tabular-benchmark

Tableau 6.2 – Hyperparameters importance comparison. A description of the hyperparameters can be found in the scikit-learn documentation : https://scikit-learn.org/stable/modules/generated/sklearn.tree.DecisionTreeClassifier.html.

| Hyperparameter | DPDT (%) | CART (%) | STreeD (%) |
|---|---|---|---|
| min_samples_leaf | 35.05 | 33.50 | 50.50 |
| min_impurity_decrease | 24.60 | 24.52 | - |
| cart_nodes_list | 15.96 | - | - |
| max_features | 11.16 | 18.06 | - |
| max_depth | 7.98 | 10.19 | 0.00 |
| max_leaf_nodes | - | 7.84 | - |
| min_samples_split | 2.67 | 2.75 | - |
| min_weight_fraction_leaf | 2.58 | 3.14 | - |
| max_num_nodes | - | - | 27.51 |
| n_thresholds | - | - | 21.98 |

datasets in the main text, we provide results for regression problems in table 6.5 in the appendix.

**Baselines :** we benchmark DPDT against CART and STreeD when inducing trees of depth at most 5. We use hyperparameter search spaces from [65] for CART and DPDT. For DPDT we additionally consider eight different splits functions parameters configurations for the maximum nodes in the calls to CART. Surprisingly, after computing the importance of each hyperparameter of DPDT, we found that the maximum node numbers in the calls to CART are only the third most important hyperparametrer behind classical ones like the minimum size of leaf nodes or the minimum impurity decrease (Table 6.2). We use the CPython implementation of STreeD [4]. All hyperparameter grids are given in table 6.7 in the appendix.

**Hardware :** experiments were conducted on a heterogeneous computing infrastructure made of AMD EPYC 7742/7702 64-Core and Intel Xeon processors, with hardware allocation based on availability and algorithm requirements. DPDT and CART random searches ran for the equivalent of 2-days while PySTreeD ran for 10-days.

**Observations**

**Generalization** In Figure 6.5, we observe that DPDT learns better trees than CART and STreeD both in terms of generalization capabilities and in terms of computation

4. PySTreeD : https://github.com/AlgTUDelft/pystreed

Tableau 6.3 – Depth-10 decision trees for the KDD 1999 cup dataset.

| Model | Test Accuracy (%) | Time (s) | Memory (MB) |
|---|---|---|---|
| DPDT-(4,) | **91.30** | 339.85 | 5054 |
| DPDT-(4,4,) | **91.30** | 881.07 | 5054 |
| CART | 91.29 | 25.36 | 1835 |
| GOSDT-$\alpha = 0.0005$ | 65.47 | 5665.47 | 1167 |
| GOSDT-$\alpha = 0.001$ | 65.45 | 5642.85 | 1167 |

cost. On Figures 6.5a and 6.5b, DPDT obtains best generalization scores for classification on numerical and categorical data after 100 iterations of random hyperparameters search over both CART and STreeD. Similarly, we also present generalization scores as a function of compute time (instead of random search iterations). On Figures 6.5c and 6.5d, despite being coded in the slowest language (`Python` vs. `CPython`), our implementation of DPDT finds the best overall model before all STreeD random searches even finish. The results from Figure 6.5 are appealing for machine learning practitioners and data scientists that have to do hyperparameters search to find good models for their data while having computation constrains.

Now that we have shown that DPDT is extremely efficient to learn shallow decision trees that generalize well to unseen data, it is fair to ask if DPDT can also learn deep trees on very large datasets.

**Deeper trees on bigger datasets.** We also stress test DPDT by inducing deep trees of depth 10 for the KDD 1999 cup dataset [5]. The training set has 5 million rows and a mix of 80 continuous and categorical features representing network intrusions. We fit DPDT with 4 split candidates for the root node (DPDT-(4,)) and with 4 split candidates for the root and for each of the internal nodes at depth 1 (DPDT-(4,4,)). We compare DPDT to CART with a maximum depth of 10 and to GOSDT [6] [**McTavish_Zhong_Achermann_Karimalis_Chen**
with different regularization values $\alpha$. GOSDT first trains a tree ensemble to binarize a dataset and then solve for the optimal decision tree of depth 10 on the binarized problem. In Table 6.3 we report the test accuracy of each tree on the KDD 1999 cup test set. We also report the memory peak during training and the training duration (all experiments are run on the same CPU). We observe that DPDT can improve over CART even for deep trees and large datasets while using reasonable time and memory. Furthermore, Table 6.3 highlights the limitation of optimal trees for practical problems when the dataset is not binary. We observed that GOSDT could not find a good binarization of the dataset

5. `http://kdd.ics.uci.edu/databases/kddcup99/kddcup99.html`
6. Code available at : `https://github.com/ubc-systopia/gosdt-guesses`

even when increasing the budget of the tree ensemble up to the point where most of the computations are spent on fitting the ensemble (see more details about this phenomenon in [**McTavish_Zhong_Achermann_Karimalis_Chen_Rudin_Seltzer_2022**]). In table 6.6 in the appendix, we also show that DPDT performs better than optimal trees for natively binary datasets. In the next section we study the performance of boosted DPDT trees.

## 6.5 Application of DPDT to Boosting

In the race for machine learning algorithms for tabular data, boosting procedures are often considered the go-to methods for classification and regression problems. Boosting algorithms [46, 48, 47] sequentially add weak learners to an ensemble called strong learner. The development of those boosting algorithms has focused on what data to train newly added weak learners [48, 47], or on efficient implementation of those algorithms [27, 98]. We show next that Boosted-DPDT (boosting DPDT trees with AdaBoost [46]) improves over recent deep learning algorithms.

### 6.5.1 Boosted-DPDT

We benchmark Boosted-DPDT with the same datasets, metrics, and hardware as in the previous section on single-tree training. Second, we verify the competitiveness of Boosted-DPDT with other models such as deep learning ones (SAINT [118] and other deep learning architectures from [52]).

On Figures 6.5e and 6.5f we can notice 2 properties of DPDT. First, as in any boosting procedure, Boosted-DPDT outperforms its weak counterpart DPDT. This serves as a sanity check for boosting DPDT trees. Second, it is clear that boosting DPDT trees yields better models than boosting CART trees on both numerical and categorical data. Figures 6.5g and 6.5h show that boosting DPDT trees using the default AdaBoost procedure [46] is enough to learn models outperforming deep learning algorithms on datasets with numerical features and models in the top-tier on datasets with categorical features. This show great promise for models obtained when boosting DPDT trees with more advanced procedures.

### 6.5.2 (X)GB-DPDT

We also boost DPDT trees with Gradient Boosting and eXtreme Gradient Boosting [47, 48, 27](X(GB)-DPDT). For each dataset from [53], we trained (X)GB-DPDT

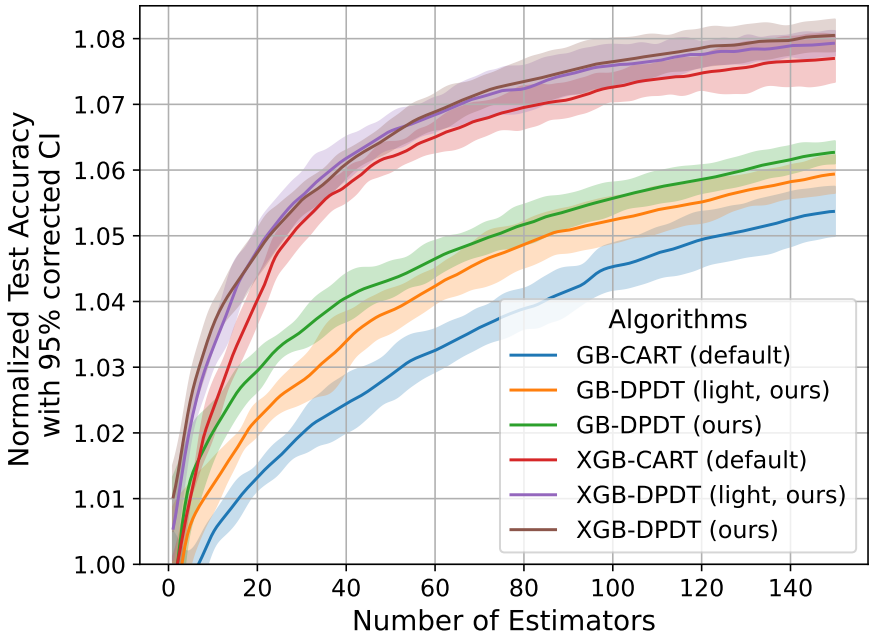



FIGURE 6.6 – Aggregated mean test accuracies of Gradient Boosting models as a function of the number of single trees.

models with 150 boosted single DPDT trees and a maximum depth of 3 for each. We evaluate two DPDT configurations for the single trees : light (DPDT-(4, 1, 1)) and the default (DPDT-(4,4,4)). We compare (X)GB-DPDT to (X)GB-CART : 150 boosted CART trees with maximum depth of 3 and default hyperparameters for each. All models use a learning rate of 0.1. For each each dataset, we normalize all boosted models scores by the accuracy of a single depth-3 CART decision tree and aggregate the results : the final curves represent the mean performance across all datasets, with confidence intervals computed using 5 different random seeds.

Figure 6.6 shows that similarly to simple boosting procedures like AdaBoost, more advanced ones like (eXtreme) Gradient Boosting yields better models when the weak learners are DPDT trees rather than greedy trees. This is a motivation to develop efficient implementation of (eXtreme) Gradient Boosting with DPDT as the weak learning algorithm to perform extensive benchmarking following [53] and potentially claim the state-of-the-art.

## 6.6 Proof of Propostion 3

For the purpose of the proof, we overload the definition of $J_\alpha$ and $\mathcal{L}_\alpha$, to make explicit the dependency on the dataset and the maximum depth. As such, $J_\alpha(\pi)$ becomes $J_\alpha(\pi, \mathcal{E}, D)$ and $\mathcal{L}_\alpha(T)$ becomes $\mathcal{L}_\alpha(T, \mathcal{E})$. Let us first show that the relation $J_\alpha(\pi, \mathcal{E}, 0) = -\mathcal{L}_\alpha(T, \mathcal{E})$ is true. If the maximum depth is $D = 0$ then $\pi(s_0)$ is necessarily a class assignment, in which case the expected number of splits is zero and the relation is obviously true since the reward is the opposite of the average classification loss. Now assume it is true for any dataset and tree of depth at most $D$ with $D \geq 0$ and let us prove that it holds for all trees of depth $D+1$. For a tree $T$ of depth $D+1$ the root is necessarily a split node. Let $T_l = E(\pi, s_l)$ and $T_r = E(\pi, s_r)$ be the left and right sub-trees of the root node of $T$. Since both sub-trees are of depth at most $D$, the relation holds and we have $J_\alpha(\pi, X_l, D) = \mathcal{L}_\alpha(T_l, X_l)$ and $J_\alpha(\pi, X_r, D) = \mathcal{L}_\alpha(T_r, X_r)$, where $X_l$ and $X_r$ are the datasets of the "right" and "left" states to which the MDP transitions—with probabilities $p_l$ and $p_r$—upon application of $\pi(s_0)$ in $s_0$, as described in the MDP formulation. Moreover, from the definition of the policy return we have

$$
\begin{aligned}
J_\alpha(\pi, \mathcal{E}, D+1) &= -\alpha + p_l * J_\alpha(\pi, X_l, D) + p_r * J_\alpha(\pi, X_r, D) \\
&= -\alpha - p_l * \mathcal{L}_\alpha(T_l, X_l) - p_r * \mathcal{L}_\alpha(T_r, D) \\
&= -\alpha - p_l * \left( \frac{1}{|X_l|} \sum_{(x_i, y_i) \in X_l} \ell(y_i, T_l(x_i)) + \alpha C(T_l) \right) \\
&\quad - p_r * \left( \frac{1}{|X_r|} \sum_{(x_i, y_i) \in X_r} \ell(y_i, T_r(x_i)) + \alpha C(T_r) \right) \\
&= -\frac{1}{N} \sum_{(x_i, y_i) \in X} \ell(y_i, T(x_i)) - \alpha(1 + p_l C(T_l) + p_r C(T_r)) \\
&= -\mathcal{L}(T, \mathcal{E})
\end{aligned}
$$

## 6.7 Additional Experiments and Hyperparameters

In this section we provide additional experimental results. In Table 6.5, we compare DPDT trees to CART and STreeD trees using 50 train/test splits of regression datasets from [53]. All algorithms are run with default hyperparameters.

The configuration of DPDT is (4, 4, 4) or (4,4,4,4,4). STreeD is run with a time limit of 4 hours per tree computation and on binarized versions of the datasets. Both for depth-3 and depth-5 trees, DPDT outperforms other baselines in terms of train and test accuracies. Indeed, because STreeD runs on "approximated" datasets, it performs pretty poorly.

In Table 6.6, we compare DPDT(4, 4, 4, 4, 4) to additional optimal decision tree baselines on datasets with **binary features**. The optimal decision tree baselines run with default hyperparameters and a time-limit of 10 minutes. The results show that even on binary datasets that optimal algorithms are designed to handle well ; DPDT outperforms other baselimes. This is likely because optimal trees are slow and/or don't scale well to depth 5.

In Table 6.4 compare DPDT to lookahead depth-3 trees when optimizing Eq.6.1. Unlike the other greedy approaches, lookahead decision trees [**norton**] do not pick the split that optimizes a heuristic immediately. Instead, they pick a split that sets up the best possible heuristic value on the following split. Lookahead-1 chooses nodes at depth $d < 3$ by looking 1 depth in the future : it looks for the sequence of 2 splits that maximizes the information gain at depth $d + 1$. Lookahead-2 is the optimal depth-3 tree and Lookahead-0 would be just building the tree greedily like CART. The conclusion are roughly the same as for Table 6.1. Both lookahead trees and DPDT [7] are in Python which makes them slow but comparable.

We also provide the hyperparameters necessary to reproduce experiments from section 6.4.2 and 6.5.1 in Table 6.7.

7. https://github.com/KohlerHECTOR/DPDTreeEstimator

Tableau 6.4 – Train accuracies of depth-3 trees (with number of operations). Lookahead trees are trained with a time limit of 12 hours.

| **Dataset** | **DPDT** | **Lookahead-1** |
|---|---|---|
| avila | 57.22 (1304) | OoT |
| bank | 97.99 (699) | 96.54 (7514) |
| bean | 85.30 (1297) | OoT |
| bidding | 99.27 (744) | 98.12 (20303) |
| eeg | 69.38 (1316) | 69.09 (10108) |
| fault | 67.40 (1263) | 67.20 (32514) |
| htru | 98.01 (1388) | OoT |
| magic | 82.81 (1451) | OoT |
| occupancy | 99.31 (1123) | 99.01 (15998) |
| page | 97.03 (1243) | 96.44 (16295) |
| raisin | 88.61 (1193) | 86.94 (9843) |
| rice | 93.44 (1367) | 93.24 (37766) |
| room | 99.23 (1196) | 99.04 (5638) |
| segment | 87.88 (871) | 68.83 (24833) |
| skin | 96.60 (1300) | 96.61 (1290) |
| wilt | 99.47 (862) | 99.31 (36789) |

Tableau 6.5 – Mean train and test scores (with standard errors) for regression datasets over 50 cross-validation runs.

| | **Depth 3** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Dataset | DPDT | | Optimal | | CART | | DPDT | | |
| | Train | Test | Train | Test | Train | Test | Train | Test | |
| nyc-taxi | 39.0 ± 0.0 | 38.9 ± 0.2 | 33.8 ± 0.0 | 33.8 ± 0.1 | 39.0 ± 0.0 | 38.9 ± 0.2 | 45.8 ± 0.0 | 45.7 ± 0.2 | 33. |
| medical_charges | 95.2 ± 0.0 | 95.2 ± 0.0 | 90.1 ± 0.0 | 90.1 ± 0.1 | 95.2 ± 0.0 | 95.2 ± 0.0 | 97.7 ± 0.0 | 97.7 ± 0.0 | 90. |
| diamonds | 93.0 ± 0.0 | 92.9 ± 0.1 | 90.1 ± 0.0 | 90.1 ± 0.1 | 92.7 ± 0.0 | 92.6 ± 0.1 | 94.2 ± 0.0 | 94.0 ± 0.1 | 90. |
| house_16H | 39.9 ± 0.1 | 38.1 ± 2.5 | 32.8 ± 0.1 | 29.4 ± 1.6 | 35.8 ± 0.1 | 35.8 ± 1.9 | 59.4 ± 0.1 | 35.2 ± 4.1 | 32. |
| house_sales | 67.0 ± 0.0 | 66.0 ± 0.4 | 65.1 ± 0.0 | 64.4 ± 0.4 | 66.8 ± 0.0 | 66.1 ± 0.4 | 77.6 ± 0.0 | 76.1 ± 0.3 | 65. |
| superconduct | 73.1 ± 0.0 | 72.7 ± 0.5 | 70.9 ± 0.0 | 70.5 ± 0.5 | 70.4 ± 0.0 | 69.7 ± 0.5 | 83.0 ± 0.0 | 81.7 ± 0.4 | 70. |
| houses | 51.7 ± 0.0 | 50.7 ± 0.7 | 48.5 ± 0.1 | 47.3 ± 0.7 | 49.5 ± 0.0 | 48.4 ± 0.7 | 69.1 ± 0.0 | 67.6 ± 0.5 | 48. |
| Bike_Sharing | 55.2 ± 0.0 | 54.7 ± 0.5 | 45.1 ± 0.1 | 44.8 ± 0.7 | 48.1 ± 0.0 | 47.9 ± 0.5 | 65.2 ± 0.0 | 63.3 ± 0.5 | 45. |
| elevators | 48.0 ± 0.0 | 46.8 ± 1.1 | 40.2 ± 0.1 | 38.2 ± 1.0 | 46.8 ± 0.0 | 45.5 ± 1.2 | 65.6 ± 0.0 | 61.2 ± 1.0 | 40. |
| pol | 72.2 ± 0.0 | 71.3 ± 0.6 | 67.8 ± 0.1 | 67.5 ± 0.9 | 72.0 ± 0.0 | 71.2 ± 0.8 | 93.3 ± 0.0 | 92.4 ± 0.3 | 67. |
| MiamiHousing2016 | 62.3 ± 0.0 | 60.4 ± 0.8 | 60.8 ± 0.0 | 58.3 ± 0.8 | 62.3 ± 0.0 | 60.6 ± 0.8 | 79.8 ± 0.0 | 77.5 ± 0.5 | 60. |
| Ailerons | 63.5 ± 0.0 | 62.6 ± 0.7 | 61.6 ± 0.0 | 60.3 ± 0.7 | 63.5 ± 0.0 | 62.6 ± 0.7 | 76.0 ± 0.0 | 72.9 ± 0.6 | 61. |
| Brazilian_houses | 90.7 ± 0.0 | 90.3 ± 0.8 | 89.2 ± 0.0 | 89.4 ± 0.8 | 90.7 ± 0.0 | 90.4 ± 0.8 | 97.6 ± 0.0 | 96.6 ± 0.4 | 89. |
| sulfur | 72.5 ± 0.1 | 66.6 ± 2.2 | 35.7 ± 0.1 | 19.1 ± 6.7 | 72.0 ± 0.1 | 68.0 ± 2.2 | 89.0 ± 0.0 | 68.4 ± 6.7 | 35. |
| yprop_41 | 6.3 ± 0.0 | 2.3 ± 0.7 | 3.6 ± 0.0 | 1.5 ± 0.4 | 6.2 ± 0.0 | 2.1 ± 0.8 | 13.2 ± 0.0 | 1.2 ± 1.7 | 3. |
| cpu_act | 93.4 ± 0.0 | 92.0 ± 0.6 | 89.0 ± 0.0 | 86.5 ± 1.9 | 93.4 ± 0.0 | 92.0 ± 0.6 | 96.5 ± 0.0 | 94.7 ± 0.5 | 89. |
| wine_quality | 27.9 ± 0.0 | 23.3 ± 0.9 | 25.2 ± 0.0 | 23.7 ± 0.8 | 27.7 ± 0.0 | 24.5 ± 0.8 | 37.4 ± 0.0 | 26.7 ± 1.0 | 25. |
| abalone | 46.3 ± 0.0 | 39.6 ± 1.6 | 42.5 ± 0.0 | 40.4 ± 1.4 | 43.3 ± 0.0 | 39.2 ± 1.2 | 58.6 ± 0.0 | 44.7 ± 1.8 | 42. |

TABLEAU 6.6 – Train/test accuracies of different decision tree induction algorithms. All algorithms induce trees of depth at most 5 on 8 classification datasets. A time limit of 10 minutes is set for OCT-type algorithms. The values in this table are averaged over 3 seeds giving 3 different train/test datasets.

| Datasets | | | | Train Accuracy depth-5 | | | | | Test | |
|---|---|---|---|---|---|---|---|---|---|---|
| Names | Samples | Features | Classes | DPDT | OCT | MFOCT | BinOCT | CART | DPDT | OCT |
| balance-scale | 624 | 4 | 3 | 90.9% | 71.8% | 82.6% | 67.5% | 86.5% | 77.1% | 66.9% |
| breast-cancer | 276 | 9 | 2 | 94.2% | 88.6% | 91.1% | 75.4% | 87.9% | 66.4% | 67.1% |
| car-evaluation | 1728 | 6 | 4 | 92.2% | 70.1% | 80.4% | 84.0% | 87.1% | 90.3% | 69.5% |
| hayes-roth | 160 | 9 | 3 | 93.3% | 82.9% | 95.4% | 64.6% | 76.7% | 75.4% | 77.5% |
| house-votes-84 | 232 | 16 | 2 | 100.0% | 100.0% | 100.0% | 100.0% | 99.4% | 95.4% | 93.7% |
| soybean-small | 46 | 50 | 4 | 100.0% | 100.0% | 100.0% | 76.8% | 100.0% | 93.1% | 94.4% |
| spect | 266 | 22 | 2 | 93.0% | 92.5% | 93.0% | 92.2% | 88.5% | 73.1% | 75.6% |
| tic-tac-toe | 958 | 24 | 2 | 90.8% | 68.5% | 76.1% | 85.7% | 85.8% | 82.1% | 69.6% |

TABLEAU 6.7 – Hyperparameter search spaces for tree-based models. More details about the hyperparamters meaning are given in [65].

| **Parameter** | **CART** | **Boosted-CART** | **DPDT** | **Boosted-DPDT** | **STreeD** |
|---|---|---|---|---|---|
| *Common Tree Parameters* | | | | | |
| max_depth | {5 : 0.7, 2,3,4 : 0.1} | {2 : 0.4, 3 : 0.6} | {5 : 0.7, 2,3,4 : 0.1} | {2 : 0.4, 3 : 0.6} | 5 |
| min_samples_split | {2 : 0.95, 3 : 0.05} | {2 : 0.95, 3 : 0.05} | {2 : 0.95, 3 : 0.05} | {2 : 0.95, 3 : 0.05} | – |
| min_impurity_decrease | {0.0 : 0.85, 0.01,0.02,0.05 : 0.05} | {0.0 : 0.85, 0.01,0.02,0.05 : 0.05} | {0.0 : 0.85, 0.01,0.02,0.05 : 0.05} | {0.0 : 0.85, 0.01,0.02,0.05 : 0.05} | – |
| min_samples_leaf | $\mathcal{Q}(\log\text{-}\mathcal{U}[2,51])$ | $\mathcal{Q}(\log\text{-}\mathcal{U}[2,51])$ | $\mathcal{Q}(\log\text{-}\mathcal{U}[2,51])$ | $\mathcal{Q}(\log\text{-}\mathcal{U}[2,51])$ | $\mathcal{Q}(\log\text{-}\mathcal{U}[2,51])$ |
| min_weight_fraction_leaf | {0.0 : 0.95, 0.01 : 0.05} | {0.0 : 0.95, 0.01 : 0.05} | {0.0 : 0.95, 0.01 : 0.05} | {0.0 : 0.95, 0.01 : 0.05} | – |
| max_features | {"sqrt" : 0.5, "log2" : 0.25, 10000 : 0.25} | {"sqrt" : 0.5, "log2" : 0.25, 10000 : 0.25} | {"sqrt" : 0.5, "log2" : 0.25, 10000 : 0.25} | {"sqrt" : 0.5, "log2" : 0.25, 10000 : 0.25} | – |
| *Model-Specific Parameters* | | | | | |
| max_leaf_nodes | {32 : 0.85, 5,10,15 : 0.05} | {8 : 0.85, 5 : 0.05, 7 : 0.1} | – | – | – |
| cart_nodes_list | – | – | 8 configs (uniform) | 5 configs (uniform) | – |
| learning_rate | – | $\log\mathcal{N}(\ln(0.01), \ln(10))$ | | $\log\mathcal{N}(\ln(0.01), \ln(10))$ | |
| n_estimators | – | 1000 | – | 1000 | – |
| max_num_nodes | – | – | – | – | {3,5,7,11, 17,25,31} (uniform) |
| n_thresholds | – | – | – | – | {5,10,20,50} (uniform) |
| cost_complexity | – | – | – | – | 0 |
| time_limit | – | – | – | – | 1800 |

Chapitre 7

# Conclusion

## 7.1 Conclusion

In this paper, we introduced Dynamic Programming Decision Trees (DPDT), a novel framework that bridges the gap between greedy and optimal decision tree algorithms. By formulating tree induction as an MDP and employing adaptive split generation based on CART, DPDT achieves near-optimal training loss with significantly reduced computational complexity compared to existing optimal tree solvers. Furthermore, we prove that DPDT can learn strictly more accurate trees than CART.

Most importantly, extensive benchmarking on varied large and difficult enough datasets showed that DPDT trees and boosted DPDT trees generalize better than other baselines. To conclude, we showed that DPDT is a promising machine learning algorithm.

The key future work would be to make DPDT industry-ready by implementing it in `C` and or making it compatible with the most advanced implementation of e.g. XGBoost.

## 7.2 What about imitation ?

## Troisième partie

# Beyond Decision Trees : what can be done with other Interpretable Policies ?

Chapitre 8

# Imitation and Evaluation

## 8.1 Intro

There exist applications of reinforcement learning like medicine where policies need to be "interpretable" by humans. User studies have shown that some policy classes might be more interpretable than others. However, it is costly to conduct human studies of policy interpretability. Furthermore, there is no clear definition of policy interpretabiliy, i.e., no clear metrics for interpretability and thus claims depend on the chosen definition. We tackle the problem of empirically evaluating policies interpretability without humans. Despite this lack of clear definition, researchers agree on the notions of "*simulatability*" : policy interpretability should relate to how humans understand policy actions given states. To advance research in interpretable reinforcement learning, we contribute a new methodology to evaluate policy interpretability. We distillate expert neural networks policies into small programs that we use as baselines. We then show that using our methodology to evaluate the baselines interpretability leads to similar conclusions as user studies. Most importantly, we show that there is no policy class that better trades off interpretability and performance across tasks.

There is increasing research in developing reinforcement learning algorithms that return "interpretable" policies such as trees, programs, first-order logic, or linear maps [**kohler2024interpretableeditableprogrammatictree**, 11, 131, 81, 33, 88, 51]. Indeed, interpretability has been useful for different applications : policy verification [11], misalignment detection [34, 85] and features importance analysis [137, 4, 1].

User studies have established the common beliefs that decision trees are more "interpretable" than linear maps, oblique trees (trees where nodes are tests of linear combina-

tions of features), and multi-layer perceptrons (MLPs) [44, 45, 84, 130]. Furthermore, for a fixed class of models, humans give different values of interpretability to models with different numbers of parameters [68]. However, survey works argue that every belief about interpretability needs to be verified with user studies and that interpretability evaluations are grounded to a specific set of users, to a specific application, and to a specific definition of interpretability [**rigorous**, **mythos**]. For example, [**mythos**] claims that depending on the notion of *simulatability* studied, MLPs can be more interpretable than trees, since deep trees can be harder for a human to read than compact MLPs. Hence, even with access to users it would be difficult to research interpretability. More realistically, since the cost of user studies is high (time, variety of subjects required, ethics, etc.), designing proxies for interpretability in machine learning has become an important open problem in both supervised [**rigorous**] and reinforcement learning [51].

In this work, we propose a methodology to evaluate the interpretability of reinforcement learning without human evaluators, by measuring inference times and memory consumptions of policies as programs. We show that those measures constitute adequate proxies for the notions of "*simulatability*" described in [**mythos**], which relates the interpretability of policy to humans ability to understand the inference of actions given states. In addition to the contributions summarized next, we open source some of the interpretable baselines to be be used for future interpretability research and teaching [1].

1. https://anonymous.4open.science/r/interpretable-rl-zoo-4DCC/README.md

Chapitre 9

# Evaluation

**Oblique decision trees.** One can imitate oracles with programs that make tests of linear combinations of features. Many oracles learn oblique or more complex decision rules over an MDP state space. This is illustrated in Figure 9.3 where a PPO neural oracle creates oblique partitions of the state-space for the Pong environments. Programs that test only individual features would fail to fit this partition (see Figure 9.3). We thus modify CART breiman, an algorithm returning axes-parallel trees for regression and supervised classification problems, for it to return oblique decision trees.

In addition to single feature tests, our oblique trees consider linear combinations of two features with weights 1 and −1, e.g., for MDP states $s_i \in \mathbb{R}^p$, the oblique features values are $s_i^{oblique} = \{s_{i1}-s_{i0}, s_{i2}-s_{i0}, \ldots, s_{ip}-s_{i0}, \ldots, s_{ip-1}-s_{ip}\} \in \mathbb{R}^{p^2}$. For example, using an oracle dataset with $n$ state-actions pairs : $(\bar{S}, \bar{A} = \pi^*(\bar{S})) \subsetneq \mathbb{R}^{n\cdot(p+\dim(A))}$, we obtain oblique decision trees by fitting $(\bar{S}, \bar{S}^{oblique}, \bar{A} = \pi^*(\bar{S})) \subsetneq \mathbb{R}^{n\cdot(p(p+1)+\dim(A))}$. Given $\bar{S}$, computing

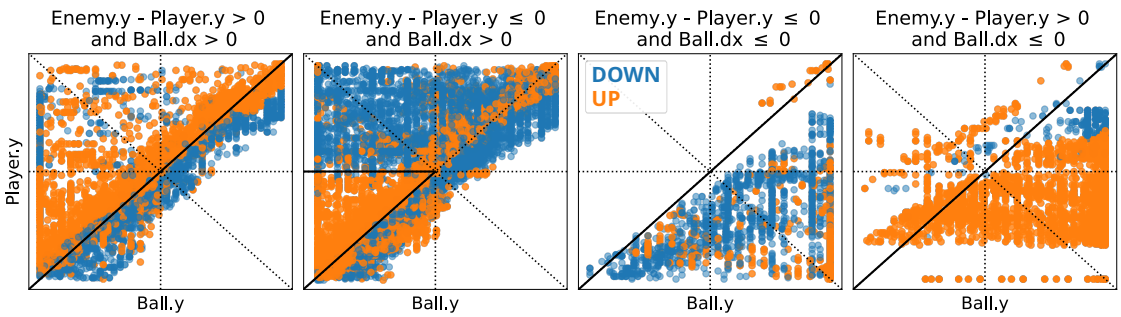



FIGURE 9.1 – **Oracle decision rules are oblique** illustrated on PPO for different state space partitions of the Pong environment. Decisions boundaries are both oblique and parallel.

$\bar{S}^{oblique}$ can be done efficiently by computing the values of the lower (or upper) triangles in the $\bar{S} \otimes \bar{S} - (\bar{S} \otimes \bar{S})^T$ tensor (excluding the diagonals) (see line **??** of Algorithm **??**). We further demonstrate the superiority of oblique trees in our experimental evaluation on a diverse set of RL tasks.

[7]r0.55

| MDP | Ast. | Box. | Free. | Kang. | Pong | Sea. | SpaceI. | Ten. |
|---|---|---|---|---|---|---|---|---|
| Full | 100 | 8 | 48 | 196 | 12 | 172 | 176 | 16 |
| Simplified | 90 | 8 | 22 | 28 | 8 | 54 | 164 | 16 |

**The complexity** of building the tree (line **??** of algorithm **??**) is $O(pn\log_2(n))$ when no maximum tree depth is given, and $O(pnD)$ with a maximum tree depth of $D$ complexcart. In particular, at iteration $i$ of our algorithm the complexity of building the tree is $O(p(p+1)itD)$, as rollouts of $t$ MDP transitions are aggregated (line **??**) and oblique features are added to states (line **??**). This means that at each iteration $i$, the cost of computing an oblique tree is $p+1$ times the cost of computing an axes-parallel tree. In our algorithm we pass $K$ the maximum number of leaf nodes as an argument. A tree with $K$ leaf nodes has $2K-1$ total nodes and a depth of at most $D = K-1$.

### 9.0.1 Real life use case of tree programs for fertilization of soils (Q3)

[18]r0.53 Pythonannexes/programmes/play$_d$ssat$_o$blique$_6$6.2$_c$opy$_h$uman.py In this last experiment, we distill a human expert policy for soil fertilization on the gym_DSSAT gym environment [**gautron2023learning**]. Here, an RL agent has to learn to manage a crop, based on an accurate simulated mechanistic model of plant growth. We consider the task that consists in optimizing plant nitrogen absorption while penalizing the application of fertilizer to minimize the economical and the environmental costs. We extract an 's Python program, depicted in Figure 9.0.1. This program outputs the exact same actions as the human heuristic given the soil state and obtain the same cumulative reward in average (corresponding to an accuracy of 100%). It also provides an interpretation of the human expert heuristic that delivers a certain amount of fertilizer ($\{27, 35, 54\}$) after $\{39, 45, 80\}$ days after seeding, respectively). The feature importance coincides with agronomic principles and have been validated by an expert from the *Consultative Group on International Agricultural Research*. The nitrogen requirements of corn vary depending on the growth stage. They are important during the vegetative phase (plant growth) and the reproductive phase (from flowering to grain filling). This is why it is essential to consider the number of days after planting and the growth stage of the corn, as nitrogen requirements are highest during grain filling.

## 9.1 Methodology Overview

In this section, we explain our methodology for the evaluation of interpretability. Our approach consists of three main steps : (1) obtaining deep neural network policies trained with reinforcement learning that obtain high cumulative rewards, (2) distilling those policies into less complex ones to use as baselines (3) after parsing baselines from different classes into a common comparable language, we evaluate the interpretability of the policies using proxy metrics for *simulatability*.

**Deep Neural Network Policies** In reinforcement learning, an agent learns how to behave in an environment to maximize a cumulative reward signal [122]. The environment is defined by a Markov decision process (MDP) $M = (\mathcal{S}, \mathcal{A}, T, R)$, where $\mathcal{S}$ is the state space, $\mathcal{A}$ is the action space, $T : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$ is the state-transition function, $R : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ is the reward function. At each time step $t$, an agent observes the current state $s_t$ and chooses an action $a_t$ according to its policy $\pi$. The agent executes $a_t$, receives reward $r_t$, and observes the next state $s'_{t+1}$. The goal is to find an optimal policy $\pi^*$ that maximizes the expected discounted future return : $\pi^* = \text{argmax}_\pi Q^\pi(s, a) = \text{argmax}\, \mathbb{E}[r + \gamma Q^\pi(s', a)]$, with $\gamma$ a discount factor in $[0, 1)$. For large or continuous state spaces like the MDPs we consider in this work, MLPs are used to represent $Q^\pi$ or $\pi$. While these MLPs can be trained efficiently to obtain high cumulative rewards [111, 89], they are too complex for interpretability considerations.

**Distilling into Interpretable Policies** To obtain interpretable policies, we distill the complex neural networks into simpler models using imitation learning, as described in Algorithm 17. This approach transforms the reinforcement learning task into a sequence of supervised learning problems.

Algorithm 17 inputs an environment, that simulates taking steps in an MDP, an expert policy to imitate, also called a teacher, and an (interpretable) policy class to fit, also called student. The hyperparameters of Algorithm 17 are : the number of times we fit a student policy, the total number of samples to be collected, and whether or not to use importance sampling. At each iteration of Algorithm 17 the student policy is fitted to a dataset of states collected with the expert at iteration 1 or with the previously fitted student (see Line 17). The actions are always given by the expert (see Line 17). When using importance sampling, the states are further re-weighted by the worst state-action value possible in the given state. When the number of iteration is 1, Algorithm 17 is behavior cloning [97]. When we use importance sampling, Algorithm 17 is $Q$-

**Algorithme 17 :** Imitate Expert [97, 109, 11]

**Input :** Expert policy $\pi^*$, MDP $M$, policy class $\Pi$, number of iterations $N$, total samples $S$, importance sampling flag $I$
**Output :** Fitted student policy $\hat{\pi}_i$
Initialize dataset $\mathcal{D} \leftarrow \emptyset$;
Initialize $\hat{\pi}_1$ arbitrarily from $\Pi$;
**for** $i \leftarrow 1$ **to** $N$ **do**
  **if** $i = 1$ **then** $\pi_i \leftarrow \pi^*$ ;
  **else** $\pi_i \leftarrow \hat{\pi}_i$ ;
  Sample $S/N$ transitions from $M$ using $\pi_i$;
  **if** $I$ *is True* **then** $w(s) \leftarrow V^{\pi*}(s) - \min_a Q^{\pi*}(s,a)$ ;
  **else** $w(s) \leftarrow 1$ ;
  Collect dataset $\mathcal{D}_i \leftarrow \{(s, \pi^*(s), w(s))\}$ of states visited by $\pi_i$ and expert actions;
  $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}_i$;
  Fit classifier/regressor $\hat{\pi}_{i+1}$ on $\mathcal{D}$;
**end**
**return** $\hat{\pi}_N$;

DAgger [11]. In other cases, Algorithm 17 is DAgger [109].

**Measuring Policy Interpretability** After obtaining interpretable policy baselines using Algorithm 17, we use two metrics to evaluate policy interpretability without requiring human intervention. Those metrics are proxies for the notion of *simulatability* from [**mythos**] that gives insights on how a human being would read a policy to understand how actions are inferred. In particular, *simulatability* admits two sub-definitions. The first one is a measure of how difficult it is for a human to reproduce the computations of the policy to infer actions given states. The second one measures how difficult it is for a human to read through the entire policy. [**mythos**] argues that this nuance is key when measuring interpretability because a tree is not read entirely to compute a single action and because there is no consensus on what is easier for a human to read between an MLP and a tree.

*1. Policy Inference Time :* to measure how a human would compute the action of a policy given a state at each environment step, we measure policy step inference time in seconds.

*2. Policy Size :* to measure how easily a human can read the entire policy, we measure its size in bytes. While this correlates with inference time for MLPs and linear models, tree-based policies may have large sizes but quick inference because they do not traverse all decision paths at each step.

As these measurements depend on many technical details (programming language, the compiler if any, the operating system, versions of libraries, the hardware it is executed on, etc), to ensure fair comparisons, we translate all student policies into a simple representation that mimics how a human being "reads" a policy. We call this process of standardizing policies language “unfolding”. In Figure 9.4, 9.5, and 9.6, we present some unfolded policy programs. Other works have distilled neural networks into programs [131] or even directly learn programmatic policies [101] from scratch. However, those works directly consider programs as a policy class and could compare a generic program (not unfolded, with, e.g., while loops or array operations) to, e.g, a decision tree [127]. We will discuss later on the limitations of unfolding policies in the overall methodology.

## 9.2 Computing Baseline Policies

### 9.2.1 Setup

All the experiments presented next run on a dedicated cluster of Intel Xeon Gold 6130 (Skylake-SP), 2.10GHz, 2 CPUs/node, 16 cores/CPU with a timeout of 4 hours per experiment. Codes to reproduce our results are given in the supplementary material. In the future, we will open source a python library with all the tools of our methodology. Using Algorithm 17, we distill deep neural network expert policies into less complex policy classes.

| Policy Class | Parameters | Training Algorithm |
|---|---|---|
| Linear Policies | Determined by state-action dimensions | Linear/Logistic Regression |
| Decision Trees | [4, 8, 16, 64, 128] nodes | CART (2× nodes maximum leaves) |
| Oblique Decision Trees | [4, 8, 16, 64, 128] nodes | CART (2× nodes maximum leaves) |
| ReLU MLPs | [2×2, 4×4, 8×8, 16×16] weigths | Adam optimization (500 iterations) |

Tableau 9.1 – Summary of baseline policy classes parameters and fitting algorithms (used in Line 17).

**Policy classes** We consider four policy classes for our baselines. We choose those policy classes because there exist efficient algorithms to fit them with supervised data which is a required step of imitation learning in Line 17. We consider linear policies that have been shown to be able to solve Mujoco tasks [81]. We fit linear policies to expert policies using simple linear (logistic) regressions with scikit-learn [95] default implementation. We also consider decision trees [**cart**] and oblique decision trees [**oblique**]. (Oblique) Decision trees are often considered the most interpretable model class in machine lear-

ning [**mythos**] and reinforcement learning [**IBMDP**, 11, 51, 88]. We train trees using the default CART [**cart**] implementation of scikit-learn with varying numbers of parameters (number of nodes in the tree). We also consider MLPs with ReLU activations [57] with varying number of parameters (total number of weights). This class of policy is often considered the least interpretable and is often used in deep reinforcement learning [56, 28, 58]. We train ReLU MLPs using the default scikit-learn implementation of Adam optimization [63] with 500 iterations. The 15 baseline policy classes that we consider are summarized in Appendix 9.1.

**Neural network experts** We do not train new deep reinforcement learning agents [89, 111, 56] but rather re-use ones available at the stables-baselines3 zoo [104]. Depending on the environments described next, we choose neural network policies from different deep reinforcement learning agents. Some may argue that during the imitation learning, ReLU MLPs baselines may obtain better performance because they are often from the same class as the expert they imitate unlike trees. But this is not of our concern as we do not benchmark the imitation learning algorithms. Furthermore, it is important to note that not all experts are compatible with all the variants of imitation learning Algorithm 17. Indeed, SAC experts [56] are not compatible with $Q$-DAgger [11] because it only works for continuous actions ; and PPO experts, despite working with discrete actions do not compute a $Q$-function necessary for the re-weighting in $Q$-DAgger.

**Environments** We consider common environments in reinforcement learning research. We consider the classic control tasks from gymnasium [126], MuJoCo robots from [124], and Atari games from [12]. For Atari games, since the state space is frame pixels that can't be interpreted, we use the object-centric version of the games from [35]. In Appendix 9.3 we give the list of environments we consider in our experiments with their state-action spaces as well as a cumulative reward threshold past which an environment is consider "solved".

### 9.2.2 Ablation study of imitation learning

In this section, we present the results of the expert distillation into smaller policies. For each environment, we fit all the policy classes. To do so, we run different instantiations of Algorithm 17 multiple times with different total sample sizes. For each environment and each imitation learning variant, we summarize the number of times we fit all the baselines to an expert and which expert we use. The number of runs and imitation algorithm variants of Algorithm 17 are summarized in Appendix 9.4. After

running the imitation learnings, we obtain roughly 40000 baseline policies (35000 for classic control, 5000 thousands for MuJoCo and 400 for OCAtari). A dataset with all the baselines measurements is given in the supplementary material.

**What is the best imitation algorithm?** Even though the focus of our work is to evaluate trained policies, we still provide some insights on the best way to obtain interpretable policies from experts. Using the reinforcement learning evaluation library rliable [2], we plot on Figure 9.7 the interquartile means (IQM, an estimator of the mean robust to outliers) of the baseline policies cumulative rewards averaged over 100 episodes. For each imitation algorithm variant, we aggregate cumulative rewards over environments and policy classes. We normalize the baselines cumulative rewards between expert and random agent cumulative rewards.

The key observation is that for tested environments (Figures 9.7a,9.7b), Behavior Cloning is not an efficient way to train baseline policies compared to DAgger. This is probably because Behavior Cloning trains a student policy to match the expert's actions on states visited by the expert while DAgger trains a student to take the expert's actions on the states visited by the student [109]. An other observation is that the best performing imitation algorithms for MuJoCo (DAgger, Figure 9.7b) and OCAtari ($Q$-Dagger, Figure 9.7c) obtain baselines that in average cannot match well the performances of the experts. However baseline policies almost always match the expert on simple tasks like classic control (Figure 9.7a).

**What is the best policy class in terms of reward?** We also wonder if there is a policy class that matches expert performances more often than others across environments. For that we plot performance profiles of the different policy classes obtained with a fixed expert and fixed imitation learning algorithm. In particular, for each environments group we use the baseline policies obtained from the best performing imitation learning algorithm from Figure 9.7. From Figure 9.8 we see that on classic control environments, MLPs tend to perform better than other classes while on OCAtari games, trees tend to perform better than other classes. Now we move on to interpretability evaluation of our programmatic policies.

## 9.3 Measuring Policy Interpretability

### 9.3.1 From Policy to Program

In this section, we compute the step inference times, as well as the policy size for both the folded and unfolded variant of each policy obtained for classic control environments with DAgger-100K. To unfold policies, we convert them into Python programs formatted with PEP 8 (comparing other unfolding formats such as ONNX `https://github.com/onnx/onnx` is left to future work). We ensure that all policies operations are performed sequentially and compute the metrics for each policy on 100 episodes using the same CPUs.

**Is it necessary to unfold policies to compute interpretability metrics?** We see on Figure 9.9 that folded policies of the same class almost always give similar interpretability values (dotted lines) despite having very different number of parameters. Hence, measuring folded policies interpretability would contradict established results from user studies such as, e.g., trees of different sizes have different levels of interpretability [68].

**Is there a best policy class in terms of interpretability?** User studies from [45, 84, 130] show that decision trees are easier to understand than models involving mathematical equations like oblique trees, linear maps, and MLPs. However, [**mythos**] states that for a human wanting to have a global idea of the inference of a policy, a compact MLP can be more interpretable than a very deep decision tree. In Figure 9.9, we show that inference speed and memory size of programs help us capture those nuances : policy interpretability does not only depend on the policy class but also on the metric choice. Indeed, when we measure interpretability with inference times, we do observe that trees are more interpretable than MLPs. However, when measuring interpretability with policy size, we observe that MLPs can be more interpretable than trees for similar number of parameters. Because there seem to not be a more interpretable policy class across proxy metrics, we will keep studying both metrics at the same time.

### 9.3.2 Interpretability-performance trade-offs

Now that we trained baseline policies and validated the proposed methodology, we use the latter to tackle open problems in interpretable reinforcement learning. For each environment, we fix the imitation learning algorithm and save the best baseline policy of each class in terms of episodic rewards after unfolding them. Each single Python

| Environment Attributes | Importance for Step inference | Importance for Policy size |
|---|---|---|
| States dimension | **80.87** | **35.52** |
| Expert episodes lengths | 11.39 | 9.28 |
| Episode reward of random | 2.26 | 4.75 |
| Expert episode reward | 1.51 | 16.80 |
| Episode reward to solve | 1.41 | 14.26 |
| Actions dimension | 1.41 | 2.02 |
| Expert reward - Solve reward | 1.15 | 17.37 |

Tableau 9.2 – Environment attributes importance to predict interpretability using either of our metrics.

policy is then **run again on the same dedicated CPU** for 100 new environment episodes (similarly to choosing a classifier with validation score and reporting the test score in the context of supervised learning).

**Is it possible to compute interpretable policies for high-dimensional environments?** [51] claim that computing an interpretable policy for high dimensional MDPs is difficult since it is similar to program synthesis which is known to be NP-hard [55]. Using our measures of interpretability, we can corroborate this claim. On Figure 9.10, we can indeed observe that some relatively interpretable policies can solve Pong (20 state dimensions) or HalfCheetah (17 state dimensions) while for very high-dimensional environments like Seaquest (180 state dimensions), no baseline can solve the game.

**For what environment are there good interpretable policies?** We fitted a random forest regressor [21] to predict the interpretability values of our baseline policies using environment attributes. In Table 9.2 we report the importance of each environment attribute when it comes to accurately predicting interpretability scores. We show that as hinted previously, the states dimensions of the environment is determining to predict the interpretability of good policies. Unsurprisingly, expert attributes also influence interpretability : for the environments where there is a positive large gap between expert and threshold rewards, the task could be considered easy and vice-versa.

**How does interpretability influence performance?** [81, 83] show the existence of linear and tree policies respectively that solve MuJoCo and continuous maze environments respectively; essentially showing that there exist environments for which policies more interpretable than deep neural networks can still compete performance-wise. Our evaluation indeed shows the existence of such environments. On Figure 9.10 we

observe that on, e.g., LunarLander, increasing policy interpretability up to a certain point does not decrease reward. Actually, we can observe that for Pong a minimum level of interpretability is required to solve the game. Indeed, as stated in [44], optimizing interpretability can also be seen as regularizing the policy which can increase generalization capabilities. The key observation is that the policy class achieving the best interpretability-performance trade-off depends on the problem. Indeed, independent of the interpretability proxy metric, we see on Figure 9.10 that for LunarLander it is an MLP that achieves the best trade-off while for Pong it is a tree. Next, we compare our proxies for interpretability with another one ; the verification time of policies used in [11, 9].

### 9.3.3 Verifying interpretable policies

[9] states that the cost of formally verifying properties of MLPs scales exponentially with the number of the parameters. Hence, they propose to measure interpretability of a policy as the computations required to verify properties of actions given state subspaces, what they call local explainability queries [54]. Before [9], [11] also compared the time to formally verified properties of trees to the time to verify properties of MLPs to evaluate interpretability. In practice, this amounts to passing states and actions bounds and solving the SAT problem of finding a state in the state bounds for which the policy outputs an action in the action bounds. For example, for the LunarLander problem, a query could be to verify if when the y-position of the lander is below some threshold value, i.e, when the lander is close to the ground, there exists a state such that the tested policy would output the action of pushing towards the ground : if the solver outputs "SAT", then there is a risk that the lander crashes.

Designing interesting queries covering all risks is an open problem, hence to evaluate the verification times of our baseline policies, we generate 500 random queries per environment by sampling state and action subspaces uniformily. Out of those queries we only report the verification times of "UNSAT" queries since to verify that, e.g., the lander does not crash we want the queries mentioned above to be "UNSAT". We also only verify instances of ReLU MLPs using [139] for this experiment as verifying decision trees requires a different software [31] for which verification times would not be comparable.

On Figure 9.11, we can observe that verification time decreases exponentially with MLP interpretability, both memory and inference speed, as shown in [9]. This is another good validation of our proposed methodology as well as a motivation to learn interpretable policies.

## 9.4 Experimental details

In this section we give all the experimental details necessary to reproduce our results.

| **Classic** | **MuJoCo** | **OCAtari** |
|---|---|---|
| CartPole (4, 2, **490**) | Swimmer (8, 2, **300**) | Breakout (452, 4, **30**) |
| LunarLander (8, 4, **200**) | Walker2d (17, 6, **2000**) | Pong (20, 6, **14**) |
| LunarLanderContinuous (8, 2, **200**) | HalfCheetah (17, 6, **3000**) | SpaceInvaders (188, 6, **680**) |
| BipedalWalker (24, 4, **250**) | Hopper (11, 3, **2000**) | Seaquest (180, 18, **2000**) |
| MountainCar (2, 3, **90**) | | |
| MountainCarContinuous (2, 1, **-110**) | | |
| Acrobot (6, 3, **-100**) | | |
| Pendulum (3, 1, **-400**) | | |

Tableau 9.3 – Summary of considered environments (dimensions of states and number or dimensions of actions, **reward thresholds**). The rewards thresholds are obtained from gymnasium [126]. For OCAtari environments, we choose the thresholds as the minimum between the DQN expert from [104] and the human scores. We also adapt subjectively some thresholds that we find too restrictive especially for MuJoCo (for example, the PPO expert from [104] has 2200 reward on Hopper while the default threshold was 3800).

## 9.5 All interpretability-performance trade-offs

In this appendix we provide the interpretability-performance trade-offs of all the tested environments. All the measures come from the experiment from Section 9.3.2.

| Envs | BC 50K | BC 100K | DAgger 50K | DAgger 100K | $Q$ 50K | $Q$-DAgger 100K |
|---|---|---|---|---|---|---|
| Classic | 50 (PPO, DQN) | 50 (PPO, DQN) | 50 (PPO, DQN) | 50 (PPO, DQN) | 50 (DQN) | 50 (DQN) |
| OCAtari | 0 | 0 | 0 | 5 (DQN) | 0 | 5 (DQN) |
| Mujoco | 10 (SAC) | 10 (SAC) | 10 (SAC) | 10 (SAC) | 0 | 0 |

Tableau 9.4 – Repetitions of each imitation learning algorithm on each environment. We specify which deep reinforcement learning agent from the zoo [104] uses as experts in parentheses.

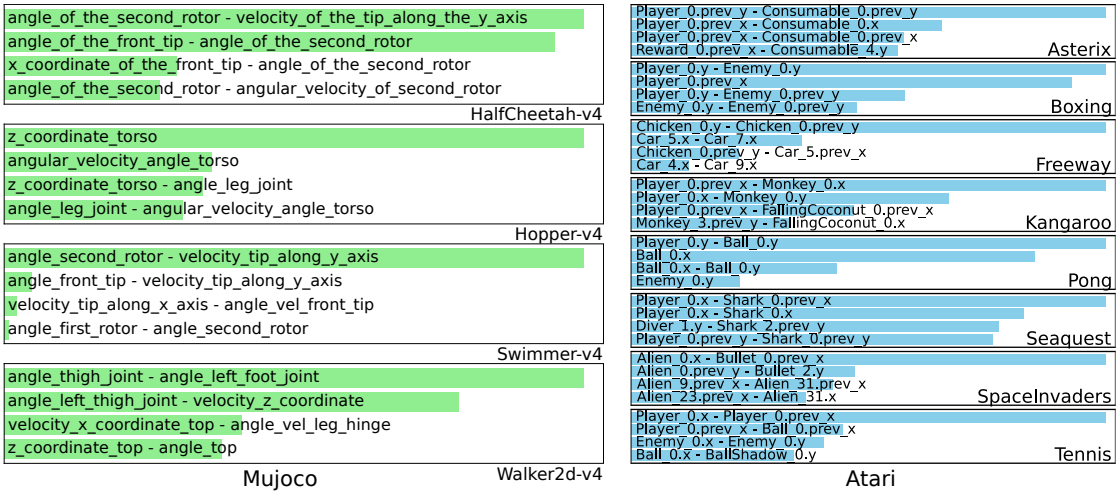



FIGURE 9.2 – **Oracle decision rules are oblique** illustrated on PPO for different state space partitions of the Pong environment. Decisions boundaries are both oblique and parallel.

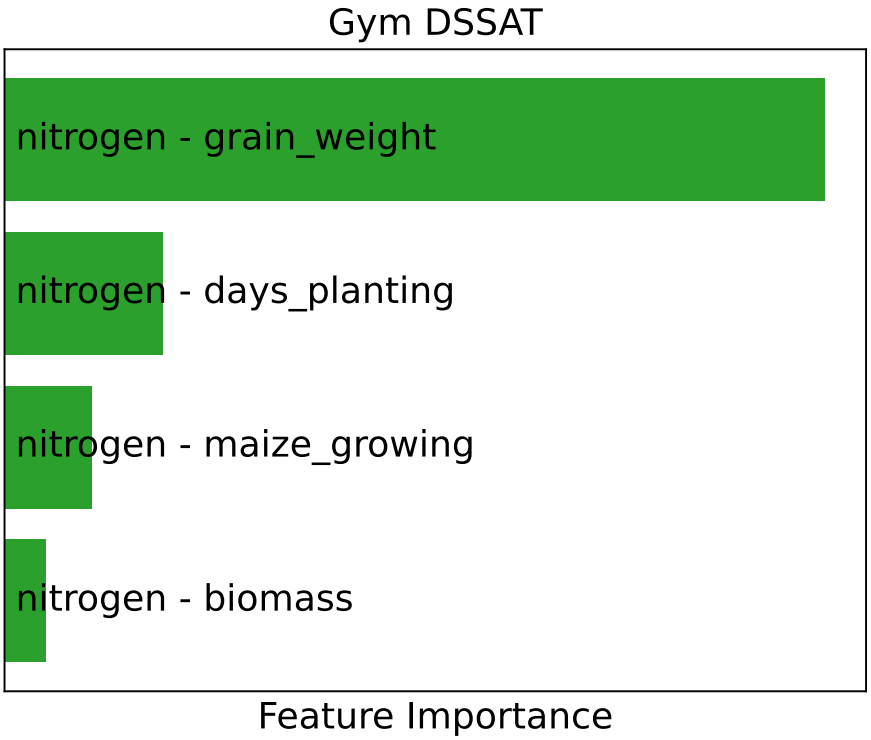



FIGURE 9.3 – **Oracle decision rules are oblique** illustrated on PPO for different state space partitions of the Pong environment. Decisions boundaries are both oblique and parallel.

```python
import gymnasium as gym

env = gym.make("MountainCar")
s, _ = env.reset()
done = False
while not done:
    y0 = 0.969*s[0]-30.830*s[
    y1 = -0.205*s[0]+22.592*s
    y2 = -0.763*s[0]+8.237*s[
    max_val = y0
    action = 0
    if y1 > max_val:
        max_val = y1
        action = 1
    if y2 > max_val:
        action = 2
    s, r, terminated, truncat
    infos = env.step(action)
    done = terminated or trun
```

FIGURE 9.4 – Unfolded linear policy interacting with an environment.

```python
def play(x):
    h_layer_0_0 = 1.238*x[0]+0.971*x[1]
                  +0.430*x[2]+0.933
    h_layer_0_0 = max(0, h_layer_0_0)
    h_layer_0_1 = -1.221*x[0]+1.001
                  *x[1]-0.423*x[2]
                  +0.475
    h_layer_0_1 = max(0, h_layer_0_1)
    h_layer_1_0 = -0.109*h_layer_0_0
                  -0.377*h_layer_0_1
                  +1.694
    h_layer_1_0 = max(0, h_layer_1_0)
    h_layer_1_1 = -3.024*h_layer_0_0
                  -1.421*h_layer_0_1
                  +1.530
    h_layer_1_1 = max(0, h_layer_1_1)

    h_layer_2_0 = -1.790*h_layer_1_0
                  +2.840*h_layer_1_1
                  +0.658
    y_0 = h_layer_2_0
    return [y_0]
```

FIGURE 9.5 – Tiny ReLU MLP for Pendulum.

```python
def play(x):
    if (x[3] - x[4]) <= -0.152:
        if (x[2] - x[3]) <= 0.049:
            return 3
        else:
            if x[5] <= 0.030:
                return 2
            else:
                if (x[2] - x[4]) <=
                    return 3
                else:
                    #Fire main eng
                    return 2
    else:
        if (x[0] - x[6]) <= -0.840
            return 0
        else:
            if (x[0] - x[4]) <= -0
                # Fire right engin
                return 3
            else:
                # Fire left engine
                return 1
```

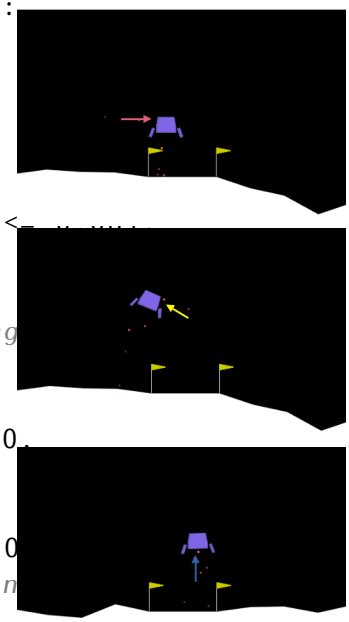
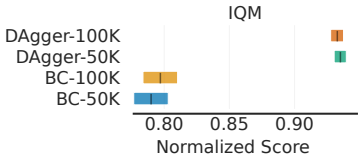
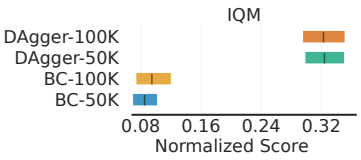


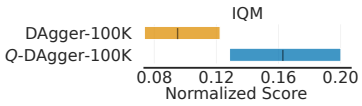
Figure 9.6 – An unfolded oblique tree policy's actions obtaining 250 rewards on Lunar Lander.




(a) Classic control, PPO expert




(b) MuJoCo, SAC expert




(c) OCAtari, DQN expert

Figure 9.7 – Performance of imitation learning variants of Algorithm 17 on different environments. We plot the 95% stratified bootstrapped confidence intervals around the IQMs.

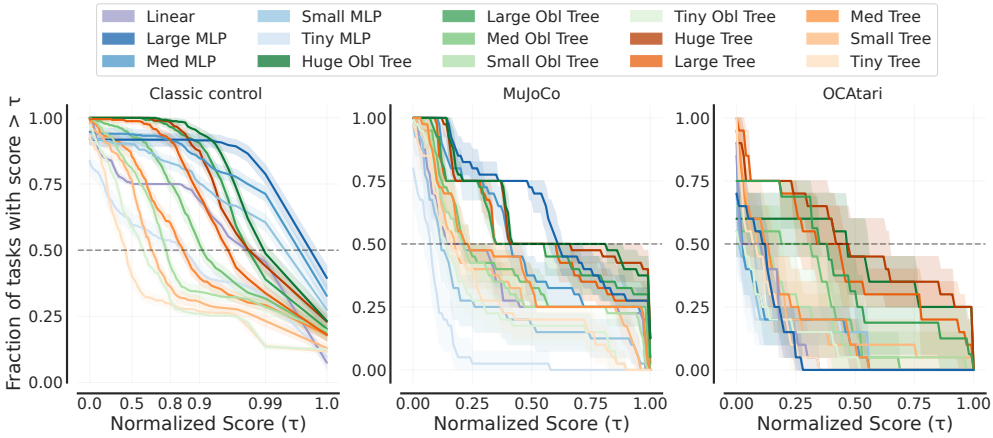



FIGURE 9.8 – Performance profiles of different policy classes on different environments.

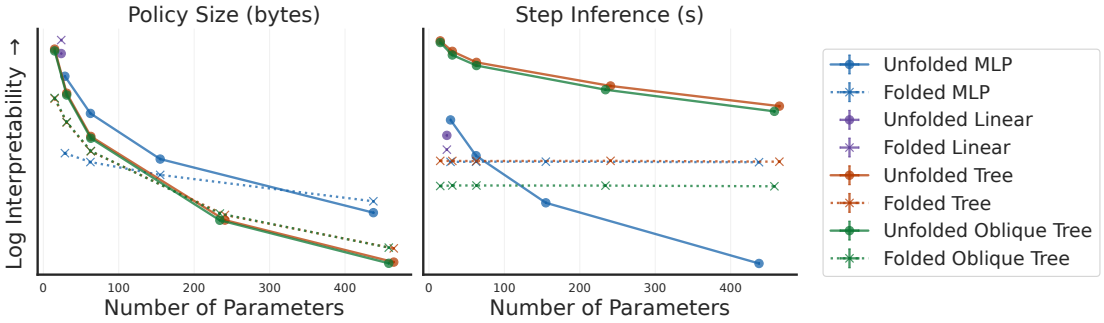



FIGURE 9.9 – Policies interpretability on classic control environments. We plot 95% stratified bootstrapped confidence intervals around means in both axes. In each sub-plot, interpreatbility is measured with either bytes or inference speed.

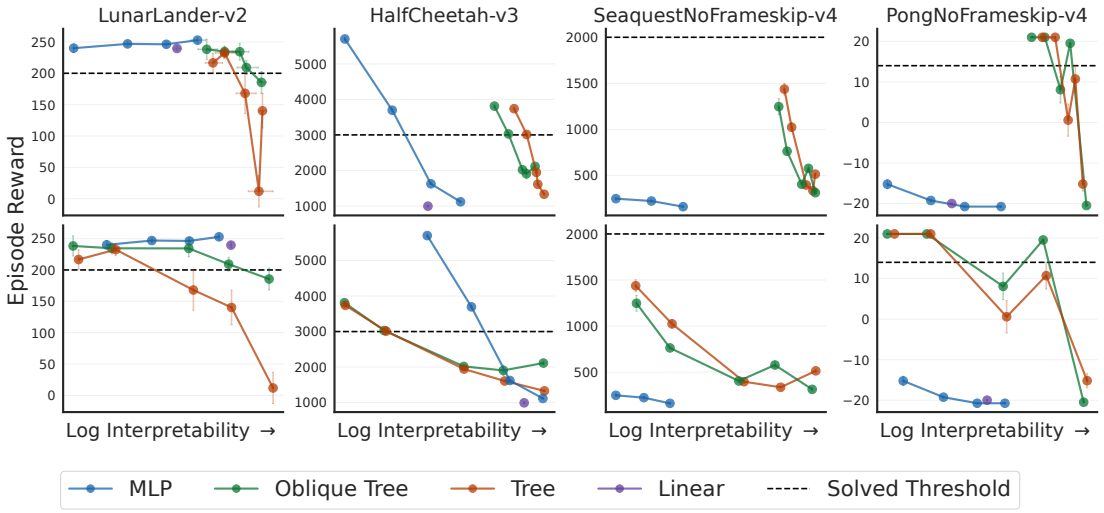



FIGURE 9.10 – Interpretability-Performance trade-offs. Top row, interpretability is measured with step inference times. Bottom row, the interpretability is measured with policy size. We plot 95% bootstrapped confidence intervals around means on both axes.

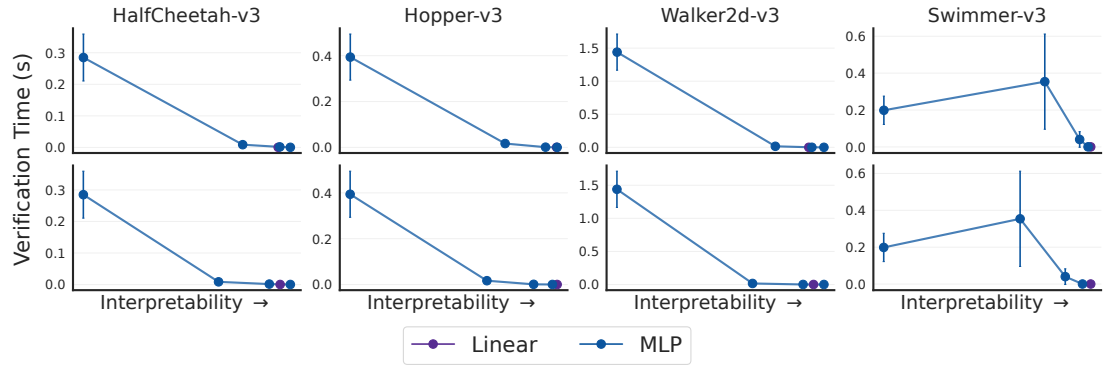



FIGURE 9.11 – Verification time as a function of policy interpretability. Top row, interpretability is measured with step inference times. Bottom row, the interpretability is measured with policy size. We plot 95% confidence intervals around means on both axes.

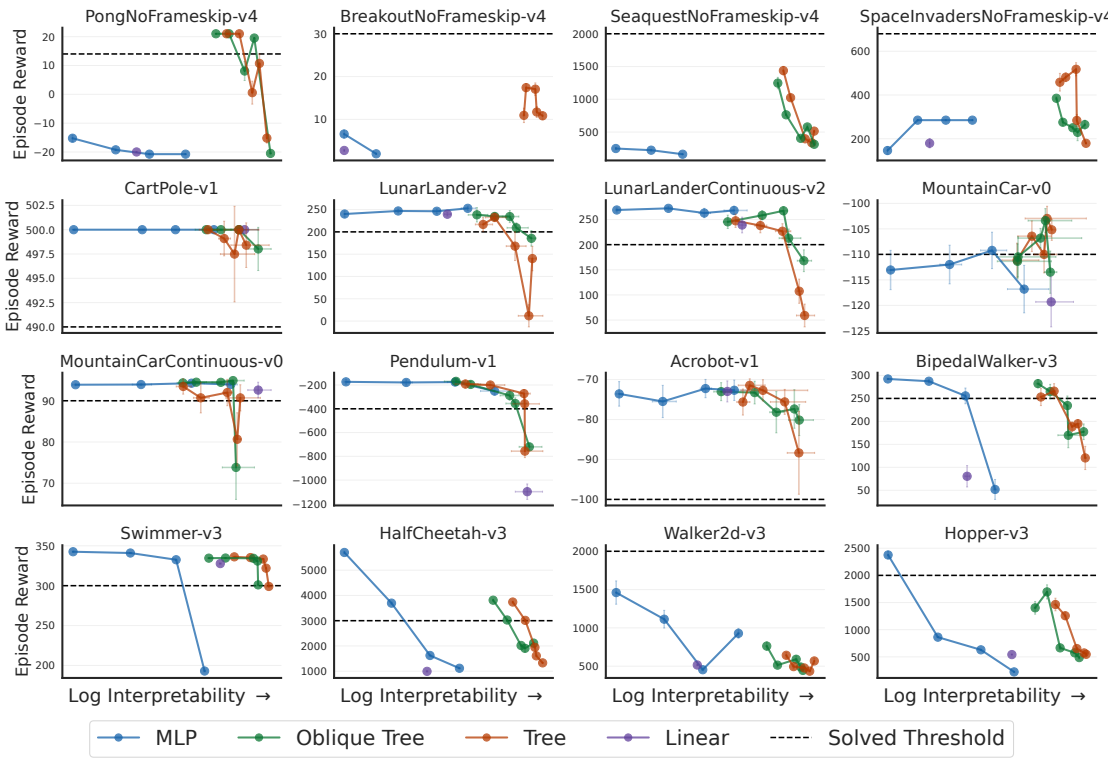



FIGURE 9.12 – Trade-off Cumulative Reward vs. Step Inference Time

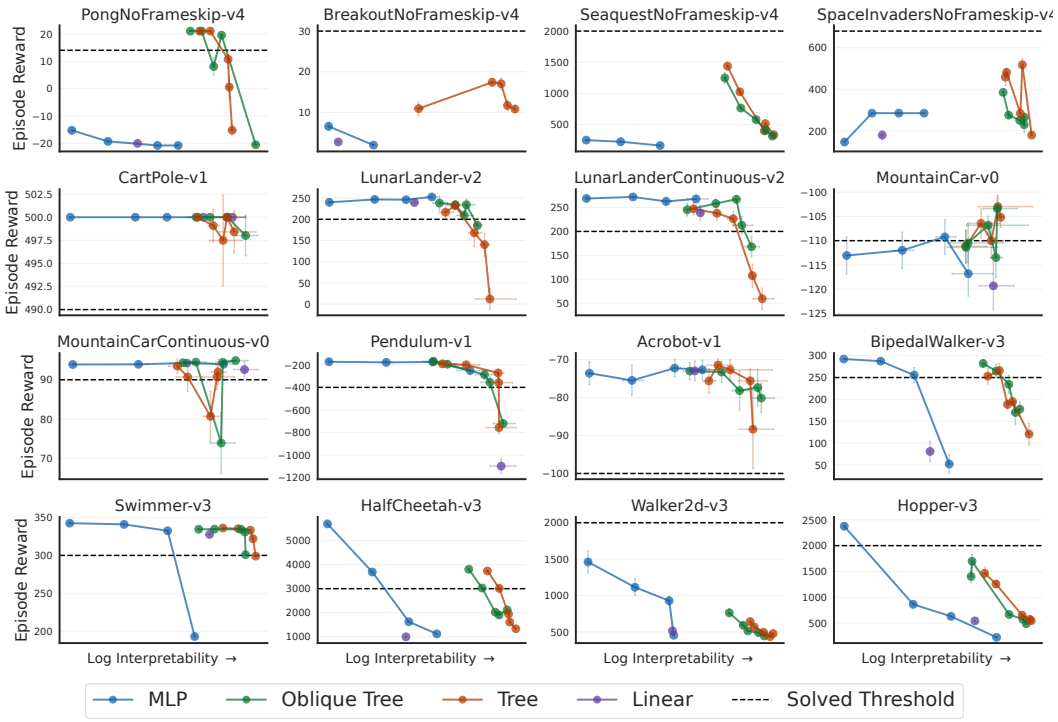



FIGURE 9.13 – Trade-off Cumulative Reward vs. Episode Inference Time

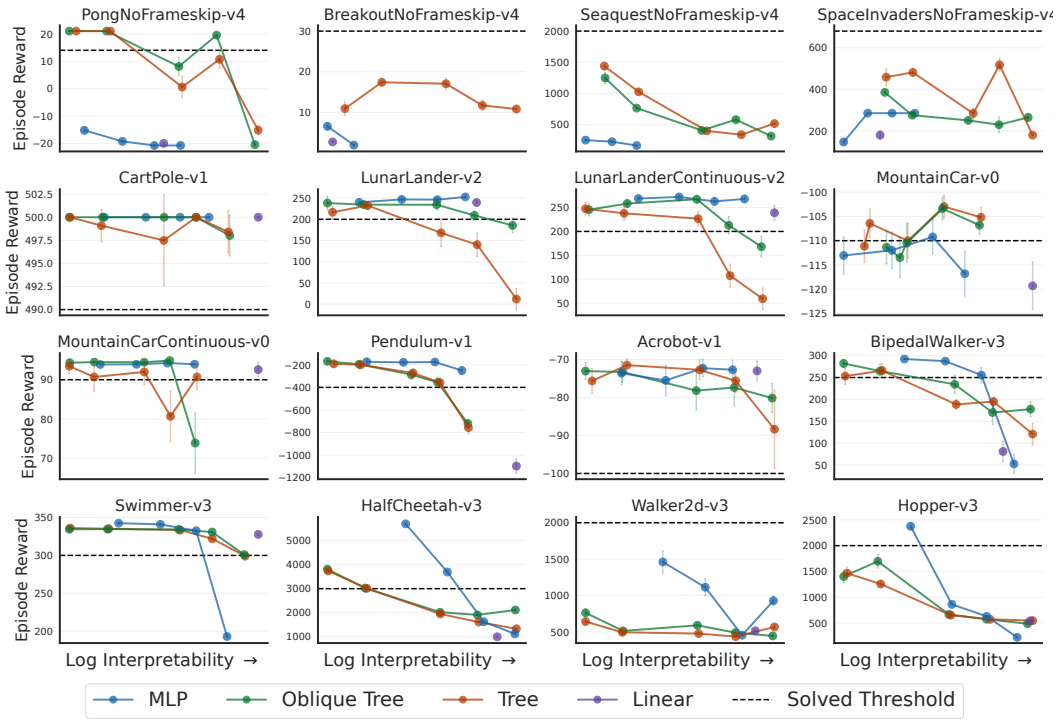



FIGURE 9.14 – Trade-off Cumulative Reward vs. Policy Size

Chapitre 10

# Conclusion Imitation

## 10.1 Limitations and conclusions

We have shown that our proposed methodology provides researchers with a sound way of evaluating policy interpretability. In particular, we have shown that unfolding policies in a common language such as Python is a key component of our methodology to ensure that interpretability depends on the policy complexity (c.f. Figure 9.9). Furthermore, we were able to show that the proxies we use for interpretability leads to similar conclusions from user studies of interpretability or from other empirical evaluations of interpretability (c.f. Figures 9.9, 9.10, and 9.11). Using the proposed methodology, we were able to illustrate the trade-offs between episodic reward and interpretability of policies from different classes (c.f. Figure 9.10) and showed the crucial need of our methodology as there is no better off policy class across tasks and metrics (c.f. Figures 9.8, 9.9, and 9.10).

A nice property of our methodology is that it is independent of the learning algorithm of the interpretable policy. We chose imitation learning but it could have been a random search in the policies parameters space [81]. Furhtermore, there sould be no limitation to use our methodology to evaluate the interpretability of arbitrary compositions of linear policies, trees and oblique trees, and MLPs, such as the hybrid policies from [114]. However, the unfolded version of policies with loops which lengths depend on the state would change between step, hence, the policy size metric value will change during episodes. This is not necessarily a strong limitation but would require more work on the unfolding procedures as well as on defining episodic metrics.

In the future, it would be interesting to compare episodic to averaged measures of

interpretability. Indeed, we additionally show in Appendix 9.13 the interpretability-performance trade-offs using the inference time summed over entire episodes as the measure of interpretability. Even though using episodic inference does not change the trade-offs compared to step inference time, it is important to discuss this nuance in future work since a key difference between supervised learning and reinforcement learning interpretability could be that human operators would read policies multiple times until the end of a decision process. Using episodic metrics for interpretability is not as straightforward as someone would think as for some MDPs, e.g. Acrobot, the episodes lengths depend on the policy. We also did not evaluate the role of sparsity in the interpretability of linear and MLP policies even thought this could greatly influence the inference time. In the future it would be interesting to apply our methodologies to policies obtained with e.g. [117]. Moving away from evaluation, we also believe that our interpretable baselines can be used to train hierarchical agents [140] using our baselines as options. We hope that our methodology as well as the provided baselines will pave the way to a more rigorous science of interpretable reinforcemeent learning.

# Conclusion générale

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Annexe A

# Programmes informatiques

Les listings suivants sont au cœur de notre travail.

Listing A.1 – Il est l'heure

```
#include <stdio.h>
int heures, minutes, secondes;

/*****************************************************/
/*                                                   */
/*            print_heure                            */
/*                                                   */
/*   But:                                            */
/*       Imprime l'heure                             */
/*                                                   */
/*   Interface:                                      */
/*       Utilise les variables globales              */
/*           heures, minutes, secondes               */
/*                                                   */
/*****************************************************/

void print_heure(void)
{
  printf("Il est %d heure",heures);
  if (heures > 1) printf("s");
  printf(" %d minute",minutes);
  if (minutes > 1) printf("s");
  printf(" %d seconde",secondes);
  if (secondes > 1) printf("s");
  printf("\n");
}
```

Listing A.2 – Factorielle

```
1 int factorielle(int n)
2 {
3   if (n > 2) return n * factorielle(n - 1);
4   return n;
5 }
```

Annexe B

# Appendix I

## B.1 Tree value computations

**Depth-0 decision tree :** has only one leaf node that takes a single base action indefinitely. For this type of tree the best reward achievable is to take actions that maximize the probability of reaching the objective $\rightarrow$ or $\downarrow$. In that case the objective value of such tree is : In the goal state $G = (1, 0)$, the value of the depth-0 tree $\mathcal{T}_0$ is :

$$\begin{aligned} V_G^{\mathcal{T}_0} &= 1 + \gamma + \gamma^2 + \dots \\ &= \sum_{t=0}^{\infty} \gamma^t \\ &= \frac{1}{1-\gamma} \end{aligned}$$

In the state $(0, 0)$ when the policy repeats going right respectively in the state $(0, 1)$ when the policy repeats going down, the value is :

$$\begin{aligned} V_{S_0}^{\mathcal{T}_0} &= 0 + \gamma V_g^{\mathcal{T}_0} \\ &= \gamma V_G^{\mathcal{T}_0} \end{aligned}$$

In the other states the policy never gets positive rewards ; $V_{S_1}^{\mathcal{T}_0} = V_{S_2}^{\mathcal{T}_0} = 0$. Hence :

$$\begin{aligned} J(\mathcal{T}_0) &= \frac{1}{4}V_G^{\mathcal{T}_0} + \frac{1}{4}V_{S_0}^{\mathcal{T}_0} + \frac{1}{4}V_{S_1}^{\mathcal{T}_0} + \frac{1}{4}V_{S_2}^{\mathcal{T}_0} \\ &= \frac{1}{4}V_G^{\mathcal{T}_0} + \frac{1}{4}\gamma V_G^{\mathcal{T}_0} + 0 + 0 \\ &= \frac{1}{4}\frac{1}{1-\gamma} + \frac{1}{4}\gamma\frac{1}{1-\gamma} \\ &= \frac{1+\gamma}{4(1-\gamma)} \end{aligned}$$

**Unbalanced depth-2 decision tree :** the unbalanced depth-2 decision tree takes an information gathering action $x \leq 0.5$ then either takes the $\downarrow$ action or takes a second information $y \leq 0.5$ followed by $\rightarrow$ or $\downarrow$. In states $G$ and $S_2$, the value of the unbalanced tree is the same as for the depth-1 tree. In states $S_0$ and $S_1$, the policy takes two information gathering actions before taking a base action and so on :

$$V_{S_0}^{\mathcal{T}_u} = \zeta + \gamma\zeta + \gamma^2 0 + \gamma^3 V_G^{\mathcal{T}_1}$$

$$\begin{aligned} V_{S_1}^{\mathcal{T}_u} &= \zeta + \gamma\zeta + \gamma^2 0 + \gamma^3 V_{S_0}^{\mathcal{T}_u} \\ &= \zeta + \gamma\zeta + \gamma^2 0 + \gamma^3(\zeta + \gamma\zeta + \gamma^2 0 + \gamma^3 V_G^{\mathcal{T}_1}) \\ &= \zeta + \gamma\zeta + \gamma^3\zeta + \gamma^4\zeta + \gamma^6 V_G^{\mathcal{T}_1} \end{aligned}$$

We get :

$$\begin{aligned} J(\mathcal{T}_u) &= \frac{1}{4}V_G^{\mathcal{T}_u} + \frac{1}{4}V_{S_0}^{\mathcal{T}_u} + \frac{1}{4}V_{S_1}^{\mathcal{T}_u} + \frac{1}{4}V_{S_2}^{\mathcal{T}_u} \\ &= \frac{1}{4}V_G^{\mathcal{T}_1} + \frac{1}{4}(\zeta + \gamma\zeta + \gamma^3 V_G^{\mathcal{T}_1}) + \frac{1}{4}(\zeta + \gamma\zeta + \gamma^3\zeta + \gamma^4\zeta + \gamma^6 V_G^{\mathcal{T}_1}) + \frac{1}{4}V_{S_2}^{\mathcal{T}_1} \\ &= \frac{1}{4}(\frac{\zeta+\gamma}{1-\gamma^2}) + \frac{1}{4}(\frac{\gamma\zeta + \gamma^4 + \zeta - \gamma^2\zeta}{1-\gamma^2}) + \frac{1}{4}(\zeta + \gamma\zeta + \gamma^3\zeta + \gamma^4\zeta + \gamma^6 V_G^{\mathcal{T}_1}) + \frac{1}{4}V_{S_2}^{\mathcal{T}_1} \\ &= \frac{1}{4}(\frac{\zeta+\gamma}{1-\gamma^2}) + \frac{1}{4}(\frac{\gamma\zeta + \gamma^4 + \zeta - \gamma^2\zeta}{1-\gamma^2}) + \frac{1}{4}(\frac{\zeta + \gamma\zeta - \gamma^2\zeta - \gamma^5\zeta + \gamma^6\zeta + \gamma^7}{1-\gamma^2}) + \frac{1}{4}V_{S_2}^{\mathcal{T}_1} \\ &= \frac{1}{4}(\frac{\zeta+\gamma}{1-\gamma^2}) + \frac{1}{4}(\frac{\gamma\zeta + \gamma^4 + \zeta - \gamma^2\zeta}{1-\gamma^2}) + \frac{1}{4}(\frac{\zeta + \gamma\zeta - \gamma^2\zeta - \gamma^5\zeta + \gamma^6\zeta + \gamma^7}{1-\gamma^2}) + \frac{1}{4}(\frac{\zeta+\gamma^3}{1-\gamma^2}) \\ &= \frac{\zeta(4 + 2\gamma - 2\gamma^2 - \gamma^5 + \gamma^6) + \gamma + \gamma^3 + \gamma^4 + \gamma^7}{4(1-\gamma^2)} \end{aligned}$$

**The balanced depth-2 decision tree :** alternates in every state between taking the two available information gathering actions and then a base action. The value of the policy

in the goal state is :

$$\begin{aligned} V_G^{\mathcal{T}_2} &= \zeta + \gamma\zeta + \gamma^2 + \gamma^3\zeta + \gamma^4\zeta + \dots \\ &= \sum_{t=0}^{\infty}\gamma^{3t}\zeta + \sum_{t=0}^{\infty}\gamma^{3t+1}\zeta + \sum_{t=0}^{\infty}\gamma^{3t+2} \\ &= \frac{\zeta}{1-\gamma^3} + \frac{\gamma\zeta}{1-\gamma^3} + \frac{\gamma^2}{1-\gamma^3} \end{aligned}$$

Following the same reasoning for other states we find the objective value for the depth-2 decision tree policy to be :

$$\begin{aligned} J(\mathcal{T}_2) &= \frac{1}{4}V_G^{\mathcal{T}_2} + \frac{2}{4}V_{S_2}^{\mathcal{T}_2} + \frac{1}{4}V_{S_1}^{\mathcal{T}_2} \\ &= \frac{1}{4}V_G^{\mathcal{T}_2} + \frac{2}{4}(\zeta + \gamma\zeta + \gamma^2 0 + \gamma^3 V_G^{\mathcal{T}_2}) + \frac{1}{4}(\zeta + \gamma\zeta + \gamma^2 0 + \gamma^3\zeta + \gamma^4\zeta + \gamma^5 0 + \gamma^6 V_G^{\mathcal{T}_2}) \\ &= \frac{\zeta(3+3\gamma) + \gamma^2 + \gamma^5 + \gamma^8}{4(1-\gamma^3)} \end{aligned}$$

**Infinite tree :** we also consider the infinite tree policy that repeats an information gathering action forever and has objective : $J(\mathcal{T}_{\text{inf}}) = \frac{\zeta}{1-\gamma}$

**Stochastic policy :** the other non-trivial policy that can be learned by solving a partially observable IBMDP is the stochastic policy that guarantees to reach $G$ after some time : fifty percent chance to do $\rightarrow$ and fifty percent chance to do $\downarrow$. This stochastic policy has objective value :

$$\begin{aligned} V_G^{\text{stoch}} &= \frac{1}{1-\gamma} \\ V_{S_0}^{\text{stoch}} &= 0 + \frac{1}{2}\gamma V_G^{\text{stoch}} + \frac{1}{2}\gamma V_{S_1}^{\text{stoch}} \\ V_{S_2}^{\text{stoch}} &= 0 + \frac{1}{2}\gamma V_G^{\text{stoch}} + \frac{1}{2}\gamma V_{S_1}^{\text{stoch}} = V_{S_0}^{\text{stoch}} \\ V_{S_1}^{\text{stoch}} &= 0 + \frac{1}{2}\gamma V_{S_2}^{\text{stoch}} + \frac{1}{2}\gamma V_G^{\text{stoch}} = \frac{1}{2}\gamma V_{S_0}^{\text{stoch}} + \frac{1}{2}\gamma V_G^{\text{stoch}} \end{aligned}$$

Solving these equations :

$$\begin{aligned}
V_{S_1}^{\text{stoch}} &= \frac{1}{2}\gamma V_{S_0}^{\text{stoch}} + \frac{1}{2}\gamma V_G^{\text{stoch}} \\
&= \frac{1}{2}\gamma(\frac{1}{2}\gamma V_G^{\text{stoch}} + \frac{1}{2}\gamma V_{S_1}^{\text{stoch}}) + \frac{1}{2}\gamma V_G^{\text{stoch}} \\
&= \frac{1}{4}\gamma^2 V_G^{\text{stoch}} + \frac{1}{4}\gamma^2 V_{S_1}^{\text{stoch}} + \frac{1}{2}\gamma V_G^{\text{stoch}} \\
V_{S_1}^{\text{stoch}} - \frac{1}{4}\gamma^2 V_{S_1}^{\text{stoch}} &= \frac{1}{4}\gamma^2 V_G^{\text{stoch}} + \frac{1}{2}\gamma V_G^{\text{stoch}} \\
V_{S_1}^{\text{stoch}}(1 - \frac{1}{4}\gamma^2) &= (\frac{1}{4}\gamma^2 + \frac{1}{2}\gamma) V_G^{\text{stoch}} \\
V_{S_1}^{\text{stoch}} &= \frac{\frac{1}{4}\gamma^2 + \frac{1}{2}\gamma}{1 - \frac{1}{4}\gamma^2} V_G^{\text{stoch}} \\
&= \frac{\gamma(\frac{1}{4}\gamma + \frac{1}{2})}{1 - \frac{1}{4}\gamma^2} \cdot \frac{1}{1-\gamma} \\
&= \frac{\gamma(\frac{1}{4}\gamma + \frac{1}{2})}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)}
\end{aligned}$$

$$\begin{aligned}
V_{S_0}^{\text{stoch}} &= \frac{1}{2}\gamma V_G^{\text{stoch}} + \frac{1}{2}\gamma V_{S_1}^{\text{stoch}} \\
&= \frac{1}{2}\gamma \cdot \frac{1}{1-\gamma} + \frac{1}{2}\gamma \cdot \frac{\gamma(\frac{1}{4}\gamma + \frac{1}{2})}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)} \\
&= \frac{\frac{1}{2}\gamma}{1-\gamma} + \frac{\frac{1}{2}\gamma^2(\frac{1}{4}\gamma + \frac{1}{2})}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)} \\
&= \frac{\frac{1}{2}\gamma(1 - \frac{1}{4}\gamma^2) + \frac{1}{2}\gamma^2(\frac{1}{4}\gamma + \frac{1}{2})}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)} \\
&= \frac{\frac{1}{2}\gamma - \frac{1}{8}\gamma^3 + \frac{1}{8}\gamma^3 + \frac{1}{4}\gamma^2}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)} \\
&= \frac{\frac{1}{2}\gamma + \frac{1}{4}\gamma^2}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)} \\
&= \frac{\gamma(\frac{1}{2} + \frac{1}{4}\gamma)}{(1 - \frac{1}{4}\gamma^2)(1-\gamma)}
\end{aligned}$$

$$
\begin{aligned}
J(\mathcal{T}_{\text{stoch}}) &= \frac{1}{4}(V_G^{\text{stoch}} + V_{S_0}^{\text{stoch}} + V_{S_1}^{\text{stoch}} + V_{S_2}^{\text{stoch}}) \\
&= \frac{1}{4}\left(\frac{1}{1-\gamma} + 2 \cdot \frac{\gamma(\frac{1}{2}+\frac{1}{4}\gamma)}{(1-\frac{1}{4}\gamma^2)(1-\gamma)} + \frac{\gamma(\frac{1}{4}\gamma+\frac{1}{2})}{(1-\frac{1}{4}\gamma^2)(1-\gamma)}\right) \\
&= \frac{1}{4}\left(\frac{1}{1-\gamma} + \frac{2\gamma(\frac{1}{2}+\frac{1}{4}\gamma)+\gamma(\frac{1}{4}\gamma+\frac{1}{2})}{(1-\frac{1}{4}\gamma^2)(1-\gamma)}\right) \\
&= \frac{1}{4}\left(\frac{1}{1-\gamma} + \frac{\gamma+\frac{1}{2}\gamma^2+\frac{1}{4}\gamma^2+\frac{1}{2}\gamma}{(1-\frac{1}{4}\gamma^2)(1-\gamma)}\right) \\
&= \frac{1}{4}\left(\frac{1}{1-\gamma} + \frac{\frac{3}{2}\gamma+\frac{3}{4}\gamma^2}{(1-\frac{1}{4}\gamma^2)(1-\gamma)}\right) \\
&= \frac{1}{4}\left(\frac{1-\frac{1}{4}\gamma^2+\frac{3}{2}\gamma+\frac{3}{4}\gamma^2}{(1-\frac{1}{4}\gamma^2)(1-\gamma)}\right) \\
&= \frac{1}{4}\left(\frac{1+\frac{3}{2}\gamma+\frac{1}{2}\gamma^2}{(1-\frac{1}{4}\gamma^2)(1-\gamma)}\right) \\
&= \frac{1+\frac{3}{2}\gamma+\frac{1}{2}\gamma^2}{4(1-\frac{1}{4}\gamma^2)(1-\gamma)}
\end{aligned}
$$

## B.2 Hyperparameters

Tableau B.1 – PG Hyperparameter Space (140 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Learning Rate (lr) | 0.001, 0.005, 0.01, 0.05, 0.1 | Policy gradient step size |
| Entropy Regularization (tau) | -1.0, -0.1, -0.01, 0.0, 0.01, 0.1, 1.0 | Entropy regularization coefficient |
| Temperature (eps) | 0.01, 0.1, 1.0, 10 | Softmax temperature |
| Episodes per Update (n_steps) | 20, 200, 2000 | Number of episodes per policy update |

Tableau B.2 – PG-IBMDP Hyperparameter Space (140 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Learning Rate (lr) | 0.001, 0.005, 0.01, 0.05, 0.1 | Policy gradient step size |
| Entropy Regularization (tau) | -1.0, -0.1, -0.01, 0.0, 0.01, 0.1, 1.0 | Entropy regularization coefficient |
| Temperature (eps) | 0.01, 0.1, 1.0, 10 | Softmax temperature |
| Episodes per Update (n_steps) | 10, 100, 1000 | Number of episodes per policy update |

Tableau B.3 – QL Hyperparameter Space (192 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate (lr_o) | 0.001, 0.005, 0.01, 0.1 | Observation Q-learning rate |
| Optimistic | True, False | Optimistic initialization |

Tableau B.4 – QL-Asym Hyperparameter Space (768 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate (lr_o) | 0.001, 0.005, 0.01, 0.1 | Observation Q-learning rate |
| Learning Rate (lr_v) | 0.001, 0.005, 0.01, 0.1 | State-action Q-learning rate |
| Optimistic | True, False | Optimistic initialization |

Tableau B.5 – QL-IBMDP Hyperparameter Space (192 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate (lr_v) | 0.001, 0.005, 0.01, 00.1 | State-action Q-learning rate |
| Optimistic | True, False | Optimistic initialization |

Tableau B.6 – SARSA Hyperparameter Space (192 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate (lr_o) | 0.001, 0.005, 0.01, 0.1 | Observation SARSA learning rate |
| Optimistic | True, False | Optimistic initialization |

Tableau B.7 – SARSA-Asym Hyperparameter Space (768 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate (lr_o) | 0.001, 0.005, 0.01, 0.1 | Observation SARSA learning rate |
| Learning Rate (lr_v) | 0.001, 0.005, 0.01, 0.1 | State-action SARSA learning rate |
| Optimistic | True, False | Optimistic initialization |

Tableau B.8 – SARSA-IBMDP Hyperparameter Space (192 combinations)

| Hyperparameter | Values | Description |
|---|---|---|
| Epsilon Schedules | (0.3, 1), (0.3, 0.99), (1, 1) | Initial exploration and decrease rate |
| Epsilon Schedules | (0.1, 1), (0.1, 0.99), (0.3, 0.99) | Initial exploration and decrease rate |
| Lambda | 0.0, 0.3, 0.6, 0.9 | Eligibility trace decay |
| Learning Rate (lr_v) | 0.001, 0.005, 0.01, 0.1 | State-action SARSA learning rate |
| Optimistic | True, False | Optimistic initialization |

| Hyperparameter | Asym Q-learning (10/10) | Asym Sarsa (10/10) | PG (4/10) |
|---|---|---|---|
| epsilon_start | 1.0 | 1.0 | - |
| epsilon_decay | 0.99 | 0.99 | - |
| batch_size | 1 | 1 | - |
| lambda_ | 0.0 | 0.0 | - |
| lr_o | 0.01 | 0.1 | - |
| lr_v | 0.1 | 0.005 | - |
| optimistic | False | False | - |
| lr | - | - | 0.05 |
| tau | - | - | 0.1 |
| eps | - | - | 0.1 |
| n_steps | - | - | 2000 |

Tableau B.9 – Best hyperparameters for each algorithm on the POIBMDP problem

# Table des matières