(3)

$A \in R^{N \times N}$ - a symmetric adjency matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$$

where $a_{ij} = 1$, if nodes i and j are connected
$a_{ij} = 0$, otherwise

e - a column vector of N elements all equal to 1.

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

a) k_i - a degree of the node i

$$k = A \cdot e = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + \ldots + a_{1N} \\ a_{21} + a_{22} + \ldots + a_{2N} \\ \vdots \\ a_{N1} + a_{N2} + \ldots + a_{NN} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix}$$

- the number of connections between node i and others

b) L - the total number of links

$$L = \frac{1}{2} \cdot \sum_{i=1}^{N} deg(v_i) = \frac{1}{2} \cdot \sum_{i=1}^{N} k_i = \frac{1}{2} \cdot \sum_i (A\,e)$$

c) N - the matrix whose element n_{ij} is equal to the number of common neighbors of nodes i and j

$B = A^k, \quad k \in N$

b_{ij} - the number of paths of length k between nodes i and j

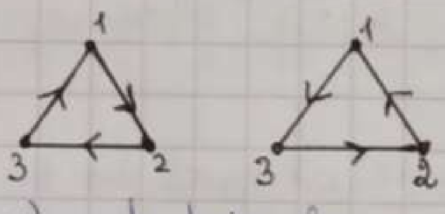
$N = A^2$ - the number of paths of length 2 between nodes => the number of nodes that are neighbor for this both nodes.

d) T - the number of triangles present in the network

A^3 - the number of paths of length 3 between nodes

$tr(A^3)$ - the number of cycles of length 3 (triangles)

But:



1) each triangle is counted twice: in one direction and in another one

2) each triangle is counted three times: for each vertex.

So we need to divide by $2 \cdot 3 = 6$:

$$T = \frac{1}{6} tr(A^3)$$

e) A connected graph is a graph that has only 1 component; there is a path from any node to any other node in the graph

$B = A^k$, $k < n$, $k \in N$; b_{ij} - the number of paths of length k between nodes i and j.

$B_2 = A^2$, $b_{ij}^{(2)}$ - the number of paths of length 2 between nodes i and j

$B_3 = A^3$, $b_{ij}^{(3)}$ - the number of paths of length 3 between nodes i and j.

...

$B_{k-1} = A^{n-1}$, $b_{ij}^{(n-1)}$ - the number of paths of length $n-1$ between nodes i and j.

On the other hand, we have $b_{ij}^{(k)} = 0$ where there is no path of length k between nodes i and j.

Let's add all the matrices:

$C = A + A^2 + A^3 + \dots + A^{n-1}$, where

$c_{ij} = 0$ - only when there is no path of any length between nodes i and j.

So, if we have element $c_{ij} = 0$, there is no path of any length between nodes i and j ~~nodes i and j~~. Therefore, a graph isn't connected.

As a result,

A graph is connected $\Longleftrightarrow$ there are no zero elements in the matrix $C = A + A^2 + \dots + A^{n-1}$